TWO ESSAYS IN MICROECONOMICS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Economics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

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Abstract

This dissertation consists of two essays. The first essay is a study of strategic firm competition in a differentiated product environment. We develop a tractable spatial model of oligopolistic competition in which firms endogenously determine both franchise/product locations and prices. Remarkably, we find that firms are completely unsuccessful at exploiting endogenous product-specific heterogeneity, whenever, it is the sole source of heterogeneity: while ex-post consumer heterogeneity ensures positive gross profits, competition for market share results in socially excessive product lines and zero net profits. We then introduce exogenous taste heterogeneity, so that consumers also differ in their ex-ante preferences over product lines. We prove that price competition due to the endogenous spatial heterogeneity drives profits below what they would be with only taste heterogeneity. Finally, we introduce multiple product lines, and show that when the product costs differ across product lines, firms earn positive profits as long as consumer preferences over product lines are not perfectly correlated.

The second essay is a study of optimal voting rules. Society tastes for government policy vary over time, as society itself changes. Ceteris paribus, having a legislature that can freely tailor policy to reflect these changing tastes is good. However, the composition of a legislature may not always be reflective of society. In particular, the views of the median legislator may sometimes be rather different than those of the median citizen in society: An unchecked legislature can sometimes implement bad policy. The legislative process itself, by choosing more extreme agenda setters, may generate less representative outcomes. We consider both the possibility that the proposer of policy each period is the median member of the legislature and the standard assumption that the proposer is a randomly selected legislator. A proposal is adopted only if it wins approval from a sufficient fraction of the legislature against a status quo corresponding to the policy in the previous period. Building in more inertia amounts to requiring a larger supermajority for approval. Somewhat surprisingly, it is possible that increasing the probability of drawing a less representative legislature reduces the optimal supermajority. Also, building a source of moderacy into who proposes legislation (i.e., the proposer is the median legislator, rather than a randomly selected member of the legislature) may make it optimal to increase the supermajority.

To Lys.

Acknowledgments

This thesis would not have been possible without the support of many people. Many thanks to my adviser, Dan Bernhardt, who went above and beyond the call of duty in helping me complete this work. This small note does little justice to his contribution to this work. I would have been lost without him.

My appreciation also extends to my committee members, George Deltas, Stefan Krasa and Mattias Polborn, for their guidance and support.

I am grateful for the collegiate atmosphere I experienced with the students in the University of Illinois Economics Department that made my time in Champaign-Urbana so rewarding. Particular mention must be made of Marcus Casey, Michael Dorsch, Neus Herranz, Dennis O'Dea, Adam Stephenson, Harry Tsang and Jungmo Yoon.

I wish to thank the secretaries in the economics department of the University of Illinois Economics Department for assisting me in many different ways. Carol Hartman and Toni Wendler deserve special mention.

Finally, thanks to my wife, Lys, who endured this long process with me, always offering support and love and my children, Alex and Seb, for keeping me sane.

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Chapter 1

Competitive Franchising

1.1 Introduction

What happens when firms compete using both product locations and prices? More concretely, how does Wendy's compete against Burger King, or Coca-Cola compete against PepsiCo, and how successful will they be at exploiting endogenous product-specific heterogeneity to extract profits?

These questions get at the heart of competition between firms. However, these questions remain unanswered by the profession due to the intractability of endogenizing product location and pricing in standard spatial models, where optimal pricing hinges sensitively on the specific details about the locations of each and every product variety or franchise.

We develop a novel spatial structure to get at these questions. We first explore an environment in which consumers receive *firm-specific* location shocks, in which a consumer's location relative to one firm's product line/franchise network is uncorrelated with his location relative to another firm's product line. The other features of our economy are standard. First, firms choose product locations and prices. Then, consumers receive spatial location shocks and choose where to shop given the prices and distances from product locations.

In our base model, the only source of consumer heterogeneity is the spatial heterogeneity that firms endogenously introduce via their location choices. We prove that when consumers are distributed uniformly across spatial locations, firms optimally spread their products evenly and set the same price at each product location. Prices reflect only the average properties of the two networks—summarized by each firm's concentration of product locations.

With these results in hand, we characterize equilibrium product/franchise concentration and pricing. The ex-post heterogeneity in the distances consumers must travel to each firm's franchise locations, reduces the elasticity of demand, and hence price competition. As a result, firms price above marginal cost and generate positive gross profits, i.e., profits before consideration of franchise location costs. However, we derive a stark result for net profits: competition for market share via franchise concentration *completely exhausts* the profits generated by the ex-post heterogeneity. That is, the gross profits from product sales just cover the costs of establishing the franchises. We then prove that this qualitative finding extends when there is additional heterogeneity between firms so that one firm has a "better product", one that, ceteris paribus, all consumers prefer, and/or has lower costs of franchise development: in equilibrium, the "disadvantaged" firm earns zero net profits.

From a social welfare perspective, this competition for market share results in over-provision of franchises—the competitive equilibrium features more franchises than a social planner would choose. Because higher franchise concentrations imply lower prices, an immediate implication is that firm profits are lower and consumer surplus is higher in the competitive equilibrium, than they would be were franchise locations chosen by a social planner.

We then introduce additional exogenous heterogeneity in consumer preferences between firms and determine how competition on endogenous spatial dimensions spills over to affect the profits that firms derive from exogenous taste heterogeneity. Specifically, we allow for consumer heterogeneity along a non-spatial dimension—so that, for example, ex-ante, some consumers prefer Wendy's hamburgers, while others prefer Burger King's.

It is immediate that, in equilibrium, firms can exploit exogenous heterogeneity in tastes to extract positive net profits. What is surprising is the extent of competition on the endogenous spatial dimension: the resulting intensified price competition drives profits *below* what they would be were the spatial dimension absent so that consumers were distinguished solely by their tastes for one firm's product. Indeed, in the neighborhood of zero taste heterogeneity, endogenous spatial heterogeneity causes firms to compete away fully three-quarters of the possible profit gain from introducing slight taste heterogeneity. The profit loss due to competition on product location *grows* as taste heterogeneity increases up to the point that there is so much exogenous taste heterogeneity that the price in the economy without the endogenous spatial dimension is the same as that when the spatial dimension is present. At this point, firms extract no benefits from the franchises that they establish: *all* franchise establishment costs are, in essence, wasted.

We conclude our analysis by extending our model to a multi-product line setting in which each firm is associated with two product lines (say sodas and juices). Correlation in taste across product lines implies a consumer whose most preferred product from one firm is a soda is more likely to most prefer a soda from the other firm. While a preference for Coke could reveal a likely preference for sodas over juices, we maintain the assumption that this preference for Coke reveals nothing about preferences for Diet Pepsi versus Pepsi.

Correlation alone does not change our findings: with perfectly-positively correlated preferences, so that a consumer either prefers sodas from both firms or prefers juices, intra-firm competition decomposes to two competitions, one over soda and one over juice. Our earlier analysis then implies that firms compete all profits away. However, matters are different when (a) preferences are imperfectly correlated so that a consumer who prefers a soda from one firm may prefer a juice from the other firm **and** (b) product provision costs differ across product lines, so that, say juices are more expensive to develop than sodas. Firms still compete away all profits from their more expensive product line, but they earn positive profits from their less expensive products. Specifically, firms profit from consumers who prefer its inexpensive product, but the other firm's expensive product. The intuition for this result is that because a firm stocks its inexpensive product line more extensively, sometimes that line will compete against itself for some consumers, rather than against the other firm's expensive product line—some consumers will prefer more than one product on a firm's inexpensive line to any of the other firm's products. Firms internalize this own product line competition, and do not expand their inexpensive product lines to the same extent. In turn, this reduces the intensity of price competition, with the result that firms extract strictly positive profits. Interestingly, while profits go to zero as the correlation in preferences across product lines goes to one, profits are *not* globally decreasing in this correlation: raising the probability that a consumer who likes one firm's inexpensive product likes an expensive product from the other firm lowers profits once this probability is high enough.

The paper's outline is as follows. We next highlight our methodology and discuss related research. In section 1.2 we show that an optimal response to any franchise network structure of the other firm features identical pricing at each franchise and franchise locations that equate market shares of each franchise. We then assume these features and treat the number of franchises of a firm as a continuous variable, focusing on the *concentration* of franchises. In section 1.3 we first develop our core continuous model with two symmetric firms. We proceed to consider an asymmetric duopoly setting and then a symmetric N-firm setting. The section concludes by contrasting equilibrium outcomes with that preferred by a social planner. Section 1.4 explores how heterogeneous consumer tastes affect outcomes. Section 1.5 investigates competition between product lines. Section 1.6 concludes. All proofs are in Section 1.7.

Methodology and Related Research. Using spatial concepts to model economic phenomena, and market structure in particular, dates back to Hotelling [1929]. Subsequent notable contributions include Lancaster [1971], d'Aspremont et al. [1979] and Salop [1979]. Existing research on competition in product lines have largely focused on the simpler problem of endogenizing the range of appeal for a single product, supposing that firms provide a product characterized by an interval $[a, b]$ (see Dewan et al. [2000, 2003] or Alexandrov [2008]). Other related research includes Bernhardt et al. [2007], which models product customization, where firms invest in technologies that consumers can use to imperfectly tailor the product to their preferences; and Bernhardt and Massoud [2005], which models the design of ATM networks.

The issue of franchise location and pricing has remained open primarily because solving for equilibrium outcomes when franchise location and pricing is endogenous is infeasible in standard models. To ensure that a posited set of (price, location) strategies is an equilibrium, one must verify that *no* deviation in location or prices can raise profits—and in standard spatial settings, payoffs are not quasi-concave or continuous for all possible location choices, so that pure strategy equilibria typically do not exist.[1] Only recently, did Vogel [2008] make the breakthrough to solve for the equilibrium locations and prices when firms have a single product and heterogeneous marginal costs of production. His key insight was that one did not need to fully characterize off-equilibrium mixed-strategy outcomes to determine

[1] de Palma et al. [1985] obtain existence by adding heterogeneous consumer tastes that are orthogonal to the spatial dimension, using a multinomial logit specification: with sufficient consumer heterogeneity, equilibria in pure strategies exist when firms compete simultaneously over both price and location.

the equilibrium path. While Vogel's model must confront asymmetries, each firm still chooses a single location, rather than a product line. In particular, there is no clear way to extend his approach, and optimal pricing with asymmetries across local franchise markets will hinge sensitively on the precise details of all franchise locations.

Our spatial model remains tractable *even in asymmetric settings in which firms are heterogeneous along multiple dimensions*. The source of this tractability is the coarse information conveyed to a firm by the nature of a consumer's preference for its most preferred product about the intensity of preferences for its competitor's products: the consumer's distance to its preferred firm A product can convey information about *which* firm B product is closest (juice or soda), but it reveals nothing about the intensity of those preferences. Concretely, if Diet Pepsi is a consumer's most preferred Pepsi product, differences in the distance to Diet Pepsi convey no information about preferences for Coke vs. Sprite. While this abstraction is a simplification that does not perfectly describe actual preferences; so too is the standard spatial assumption that a consumer's preference for Diet Pepsi relative to Pepsi *exactly* determines the preference for Coke relative to Sprite.

1.2 Discrete Model

Our core model develops a spatial oligopoly game between two firms, A and B. The firms compete to provide a product to a measure 1 of consumers. Each firm is associated with its own spatial circle with circumference of length L along which consumers are uniformly distributed. The firm must choose where on its spatial circle to establish its franchises or products—we use the terms franchise and product location interchangeably. Consumers must travel to a franchise location to purchase a product, incurring a linear travel cost of Td from traveling distance d, where $T > 0$.[2] The cost to a firm of establishing a franchise at any point on its spatial circle is $F > 0$. Hence, the total cost to firm $j \in \{A, B\}$ of establishing n_j franchises is $n_j F$. The marginal cost of providing the good is constant and normalized to 0. Firms seek to maximize profit.

We index firm j's franchises by $1, 2, \dots n_j$, and let $N_j = \{1, \dots, n_j\}$. We define l_{j_i} to be the location of the i^{th} franchise of firm j. Without loss of generality we normalize the location of franchise j_1 to $l_{j_1} = 0$ and order franchises so that $l_{j_i} < l_{j_{i+1}}$. One can interpret franchise locations as the store locations (e.g., of Wendy's franchises) in a franchise network or as the characteristics locations (e.g., of Coca-Cola soft drink flavors) of a firm's product line. Franchise i of firm j charges price p_{j_i} for its product. A strategy for firm j is a *franchise profile* $S_j = [n_j, \{l_{j_i}, p_{j_i}\}_{i=1}^{n_j}]$ that specifies the number of franchises, each franchise location, and the price set by each franchise. The set of possible franchise profiles for firm j is Σ_j.

A consumer receives utility V from the homogeneous good that the two firms sell. We assume that V is large enough that, in equilibrium, all consumers purchase the good. After firms simultaneously choose franchise profiles,

[2]Our central results extend when consumers incur quadratic travel costs, Td^2.

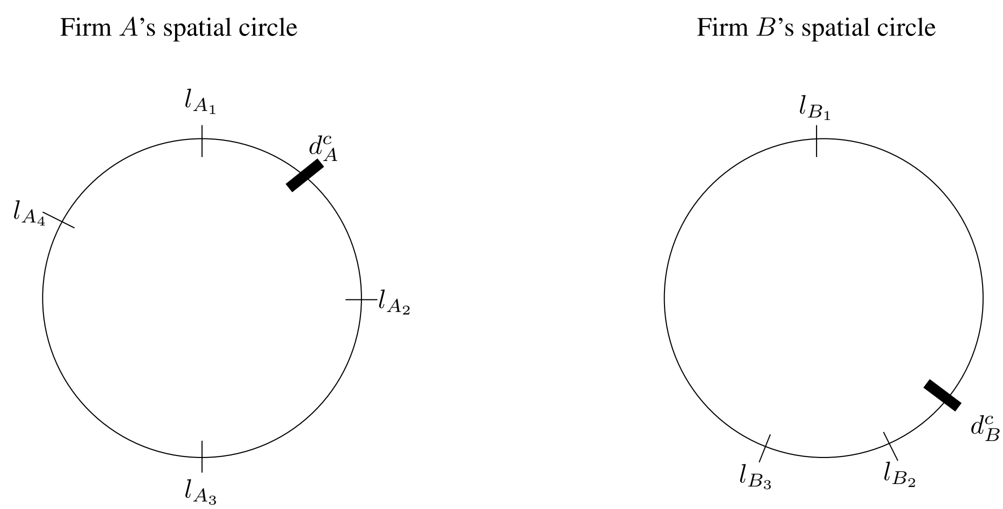


Figure 1.1: Consumer c's location shock, d_A^c for firm A is independent of his location shock for B. l_{j_i} is the location of firm j's i^{th} franchise.

consumers receive firm-specific location shocks, d_A and d_B. For firm j, a given consumer c is equally likely to be located at any point on firm j's circle, and c's location on firm A's spatial circle is uncorrelated with his location on firm B's spatial circle. Figure 1.1 shows a potential location realization for consumer c. These location shocks could reflect geographical differentiation, or product characteristic differentiation with associated dis-utility from not consuming at one's most preferred point in the characteristic space. The location shocks are easiest to interpret in characteristic space: for example, a consumer may prefer Diet Coke to Sprite, i.e., be closer to Diet Coke than to Sprite in characteristic space, but be equally likely to prefer Pepsi or Diet Pepsi.

We define $\delta_j^c(S_j, S_{-j})$ to be an indicator function that takes on the value 1 if consumer c purchases from a firm j franchise and is 0 otherwise. Consumer c maximizes utility when

$$\delta_j^c(S_j, S_{-j}) = \begin{cases} 1 & \text{if} \quad \min_{i \in N_j}\{p_{j_i} + T|l_{j_i} - d_j^c|\} < \min_{i \in N_{-j}}\{p_{-j_i} + T|l_{-j_i} - d_{-j}^c|\} \\ 0 & \text{if} \quad \min_{i \in N_j}\{p_{j_i} + T|l_{j_i} - d_j^c|\} > \min_{i \in N_{-j}}\{p_{-j_i} + T|l_{-j_i} - d_{-j}^c|\}, \end{cases}$$

where d_j^c is consumer c's location shock for firm j.

Given franchise profiles (S_A, S_B), let $y_{j_i}(d_j, S_A, S_B)$ be the conditional probability that a consumer with location shock d_j purchases from franchise j_i and let $Y_{j_i}(S_A, S_B)$ be the expected measure of consumers who purchase from franchise j_i. Explicit solutions for $y_{j_i}(d_j, S_A, S_B)$ and $Y_{j_i}(S_A, S_B)$ are in the Appendix. Then firm j's profits are

$$\pi_j(S_A, S_B) = \sum_{i=1}^{n_j} p_{j_i} Y_{j_i}(S_A, S_B) - n_j F, \qquad j \in \{A, B\}.$$

Equilibrium. An equilibrium is a collection of (i) franchise profiles, $S_j^* = [n_j^*, \{l_{j_i}^*, p_{j_i}^*\}_{i=1}^{n_j}]$, $j \in \{A, B\}$, and (ii) a set of demand functions for each consumer c, $\delta_j^{c*}(S_A, S_B)$, such that

- Franchise profiles maximize profit $\pi_j(S_j^*, S_{-j}^*) \geq \pi_j(\hat{S}_j, S_{-j}^*) \quad \forall \hat{S}_j \in \Sigma_j, \quad j \in \{A, B\}$ given subsequent optimization by almost all consumers, and

- Almost every consumer maximizes utility.

We first characterize how a firm's own franchises compete with each other for consumers. To do this, we develop the notion of franchise j_i's *service area*—the set of optimizing consumers who, *if* they purchase from firm j, will do so at franchise j_i. In any equilibrium, each of firm j's franchises must be patronized by some customers (else the costly franchise ought not be built); however, among consumers who purchase from firm j, only those who are sufficiently nearby franchise j_i will patronize it. Accordingly, we let $a_{j_{i,i+1}}(S_j)$ be the identity of the consumer who is indifferent between purchasing from firm j at franchises j_i and j_{i+1}:

$$a_{j_{i,i+1}}(S_j) = \frac{p_{j_{i+1}} - p_{j_i}}{2T} + \frac{l_{j_{i+1}} + l_{j_i}}{2}, \qquad \forall i \in N_j, \quad j \in \{A, B\},$$

where $l_{j_1} = 0$ and $l_{j_{n_j+1}} = L$ (the position of the first franchise from the viewpoint of the last franchise). Any optimizing consumer located outside of $[a_{j_{i-1,i}}(S_j), a_{j_{i,i+1}}(S_j)]$ who purchases from firm j can derive a higher payoff by patronizing a firm j franchise other than j_i (in particular, patronizing either franchise j_{i+1} or j_{i-1}).

Definition 1. *Franchise j_i is* **isolated** *if*

$$y_{j_i}(a_{j_{i-1,i}}(S_j), S_A, S_B) = y_{j_i}(a_{j_{i,i+1}}(S_j), S_A, S_B) = 0.$$

Definition 2. *Franchise j_i is* **connected** *if*

$$\min\{y_{j_i}(a_{j_{i-1,i}}(S_j), S_A, S_B), y_{j_i}(a_{j_{i,i+1}}(S_j), S_A, S_B)\} > 0.$$

Franchise j_i is isolated if j_i does not compete against other firm j franchises for market share. In particular, if franchise j_i is isolated, then an individual located at $a_{j_{i-1,i}}(S_j)$, who is indifferent between purchasing from j_i and j_{i-1}, prefers with probability one to purchase from the other firm. If a firm's franchises are isolated, then its franchises only compete for customers against the other firm, and not against each other. In contrast, franchise j_i is connected if, in addition to competing against the other firm, it competes for customers with an adjacent franchise, j_{i-1} or j_{i+1}. That is, franchise j_i is connected if there is a strictly positive probability that a consumer who is indifferent between patronizing j_i and a neighboring franchise strictly prefers those alternatives to patronizing *any* of the other firm's franchises. We first establish an important result for how a firm's franchises compete against each other.

Lemma 1. *In firm j's best response, either all of its franchises are isolated or all of its franchises are connected.*

The intuition for this result is that if a firm had a mix of isolated and connected franchises then it would be able to earn higher profits by bringing its isolated franchises marginally closer together and spreading its other franchises marginally further apart, even if it did not change its prices at different franchises. Bringing isolated franchises marginally closer would not affect their market shares because these franchises do not compete against each other for customers; while the market shares of the remaining franchises would grow because their service areas increase. But then the firm's profits would be higher, a contradiction of the premise that the mix of isolated and connected franchises was optimal.

Lemmas 2 and 3 characterize the implications of Lemma 1 for pricing and location.

Lemma 2. *Suppose firm j's best response to S_{-j} has only isolated franchises. Then firm j's best response features identical pricing at each franchise and equal market shares.*

If each firm j franchise is isolated, then each has the same demand. Therefore, charging the same price at each franchise, and capturing the same market share, is a best response.

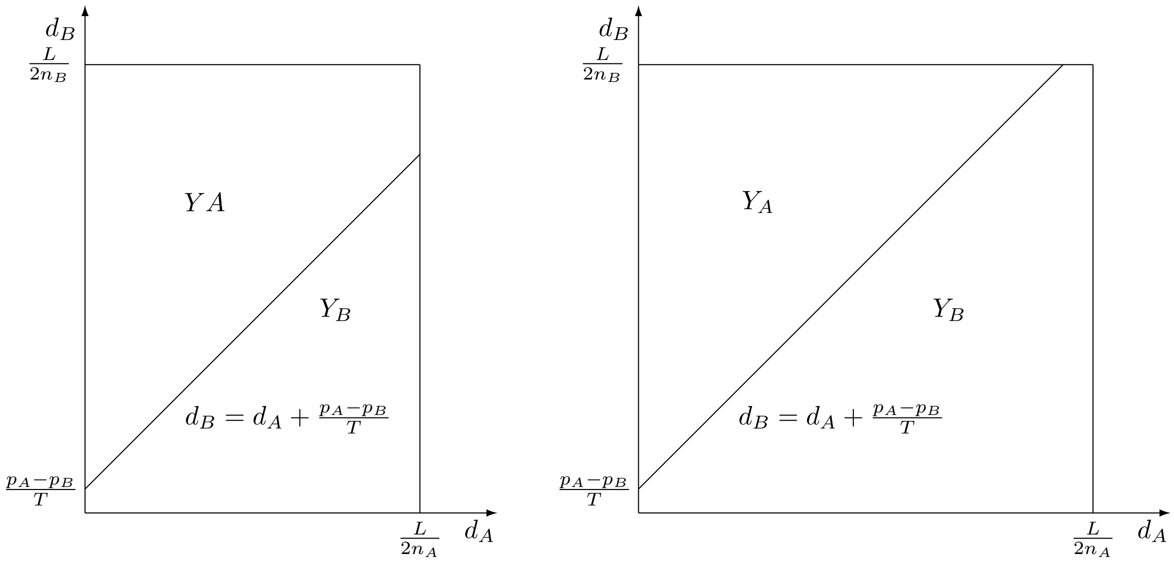
Now consider a firm with connected franchises. The analogous result to Lemma 2 is that this firm does best to space its franchises equally, and set the same price at each franchise. To prove this we first show that equal spacing and identical pricing solves the firm's first-order conditions for profit maximization; an exhaustive numerical analysis then indicates that this strategy is the unique best response.

Lemma 3. *Suppose firm j's best response to S_{-j} has only connected franchises. Then firm j's best response spaces franchises at equal distances and sets identical prices.*

These lemmas reveal that our model delivers key empirical features of the franchise industry. In particular, an optimizing firm sets the same price for its product at each franchise, regardless of the structure of the competing firm's franchise network. Of course, the optimal *level* of this price reflects the competing network. A corollary is that without loss of generality, we can restrict attention to strategies that feature franchise profiles with uniform pricing and equidistant franchise spacing, and consider demand for a representative firm j franchise. We now do this, treating the number of franchises as a continuous variable and focusing on a firm's choice of franchise concentration. As L gets large, this approximation approaches the outcome for an integer number of franchises. Because we now focus on a representative firm j franchise, we use d_j^c to measure the distance of consumer c from a firm j franchise.

1.3 Continuous Model

Symmetric Duopoly. In a symmetric setting, without loss of generality, we can assume that $p_A \geq p_B$. We first prove that if $p_A \geq p_B$, then, in equilibrium, firm B's franchises are isolated, competing only for the market share from firm A franchises, and not cannibalizing market share from its own franchise family. In turn, this will imply that firm B cannot earn positive profits.



1.2.1: Market shares when firm B franchises are isolated. 1.2.2: Market shares when firm A franchises are isolated.

Figure 1.2: Market shares when $p_A \geq p_B$. Area Y_j denotes the expected market share for firm j. The density of the area is $4n_A n_B / L^2$.

Figure 1.2 illustrates the market shares captured by each firm under the two possible scenarios (under the maintained assumption that $p_A \geq p_B$ in equilibrium). In each graph, area Y_j captures firm j's expected market share and from the firms' perspective, consumers are uniformly distributed over the graph with a density of $4n_A n_B / L^2$. Consider a consumer c who receives a location shock pair that puts him on the edge of each of the representative franchise's service areas, i.e., $(d_A^c, d_B^c) = (L/(2n_A), L/(2n_B))$. Figure 1.2.1 illustrates the case where

$$\frac{L}{2n_B} > \frac{p_A - p_B}{T} + \frac{L}{2n_A}. \tag{1.1}$$

Then this marginal consumer prefers to purchase from firm A, implying that firm B franchises are isolated. Figure 1.2.2 illustrates the other possibility, i.e., where

$$\frac{L}{2n_B} < \frac{p_A - p_B}{T} + \frac{L}{2n_A}. \tag{1.2}$$

Then the marginal consumer prefers to purchase from firm B, implying that the firm A franchises are isolated. Hence, if firm B franchises are isolated, i.e., if equation (1.1) holds, then

$$\begin{aligned} Y_A &= \frac{4n_A n_B}{L^2} \int_0^{\frac{L}{2n_A}} \int_{d_A + \frac{p_A - p_B}{T}}^{\frac{L}{2n_B}} dd_B dd_A \\ &= 1 - \frac{n_B}{2n_A} - 2n_B \frac{p_A - p_B}{LT}, \text{ and} \qquad (1.3) \\ Y_B &= 1 - Y_A = \frac{n_B}{2n_A} + 2n_B \frac{p_A - p_B}{LT}. \qquad (1.4) \end{aligned}$$

If, instead, firm A franchises are isolated, i.e., if equation (1.2) holds, then

$$\begin{aligned} Y_A &= \frac{4n_A n_B}{L^2} \int_0^{\frac{L}{2n_A} - \frac{p_A - p_B}{T}} \int_{d_A + \frac{p_A - p_B}{T}}^{\frac{L}{2n_B}} dd_B dd_A \\ &= \frac{n_A}{2n_B} \left(1 - 2n_B \frac{p_A - p_B}{LT}\right)^2 \qquad (1.5) \\ Y_B &= 1 - Y_A = 1 - \frac{n_A}{2n_B} \left(1 - 2n_B \frac{p_A - p_B}{LT}\right)^2. \qquad (1.6) \end{aligned}$$

Lemma 4 below shows that if $p_A^* \geq p_B^*$, then, in equilibrium, we can restrict attention to an environment where firm B franchises are isolated, the case illustrated in Figure 1.2.1.

Lemma 4. *If $p_A^* \geq p_B^*$ then $Y_A^* \geq Y_B^*$ and firm B franchises are isolated in equilibrium.*

The intuition is that the marginal reduction in a firm's market share due to raising its price is the same for both firms. To see this, let $p_d = p_A - p_B$. Then

$$\frac{\partial Y_A}{\partial p_A} = \frac{\partial Y_A}{\partial p_d} = \frac{\partial(1 - Y_B)}{\partial p_d} = -\frac{\partial Y_B}{\partial p_d} = \frac{\partial Y_B}{\partial p_B}.$$

The first-order conditions for profit maximization of each firm with respect to price gives

$$Y_A^* = -p_A^* \frac{\partial Y_A^*}{\partial p_A^*} \text{ and } Y_B^* = -p_B^* \frac{\partial Y_B^*}{\partial p_B^*}.$$

Substituting for $\frac{\partial Y_A}{\partial p_A} = \frac{\partial Y_B}{\partial p_B}$ yields

$$\frac{Y_A^*}{Y_B^*} = \frac{p_A^*}{p_B^*}.$$

Therefore, $p_A^* \geq p_B^*$ implies $Y_A^* \geq Y_B^*$. The proof reveals that a necessary condition for this is that the extreme consumer, i.e., the consumer who is located $\frac{L}{2n_A}$ from franchise A and $\frac{L}{2n_B}$ from franchise B, purchases from firm A; that is, firm B has isolated franchises.

We now derive the consequences for equilibrium firm profits.

Proposition 1. *In the unique equilibrium, the two firms earn zero profits, the franchise concentration for both firms is* $n_j^* = \frac{1}{2}\sqrt{\frac{LT}{2F}}$ *and the price set at each franchise is* $p_j^* = \sqrt{\frac{FLT}{2}}$.

To understand why firms earn zero profits in equilibrium, first consider the perspective of the weakly smaller firm B that charges $p_B \leq p_A$. From Lemma 4, B's franchises must be isolated in equilibrium. Suppose that firm B earns positive profit from its outlets (on average). In an asymmetric equilibrium, B's franchises compete only with those of the larger firm A. That is, when the smaller firm B increases its franchise concentration marginally, B only steals customers away from firm A, and *not* from its own established franchises. Since marginal intra-firm competition is zero and customers served per franchise is constant, adding franchises must increase B's profit: over the range where a firm's franchises are isolated, profits are linearly increasing in the number of franchises, exhibiting "constant returns to scale". But this contradicts the premise that the weakly smaller firm B is optimizing, because it would then increase profits by increasing its franchise concentration.

Now consider the possibility of an asymmetric equilibrium in which the larger firm A earns strictly positive profits. But then a price deviation by firm B to $p_B = p_A^*$, must generate strictly positive profits, because its market share per franchise is strictly higher. But then this deviation gives firm B strictly positive profit, a contradiction, as we just showed that the smaller firm must earn zero profit.

Finally, consider the possibility of a symmetric equilibrium in which both firms earn strictly positive profits. But then each firm has an incentive to increase franchise concentration marginally: The within-firm cannibalization of market share from its other franchises due to increasing franchise concentration slightly is arbitrarily small and second order, whereas the "new franchise" gains a market share and profit that is first order. That is, each firm's franchises are "almost" isolated. It follows that the firms compete profits down to zero: in equilibrium, firms fail to exploit the ex-post heterogeneity in consumers that lead them to prefer one firm's franchise to another's. Even though the firms earn positive gross (of franchise establishment costs) profits due to this heterogeneity, in equilibrium, the cost of establishing franchises just offsets these profits. We next establish the robustness of these results.

Asymmetric costs and preferences. We now relax the symmetrical properties of the economic environment to allow for

1. Firm specific heterogeneity in costs of establishing franchises, $F_A \neq F_B$.

2. Firm specific marginal costs of production, $c_i \geq 0$.

3. Consumers with preferences for one firm's product: consumers derive a common utility $V + a$ from firm A's product and V from firm B, where a could be positive or negative.

4. Firm specific heterogeneity in the dimensions of a firm's spatial environment, $L_A \neq L_B$.

Proposition 2. *At least one firm earns zero profit in equilibrium.*

To expand on this result consider a setting with a clearly-identifiable *disadvantaged* firm, say firm B. Specifically, assume $F_B \geq F_A$, $c_B \geq c_A = 0$, $L_B \geq L_A$ and $a \geq 0$, where at least one of these inequalities is strict.

Proposition 3. *If $F_B \geq F_A$, $c_B \geq c_A = 0$, $L_B \geq L_A$ and $a \geq 0$, with one inequality strict, then in the unique equilibrium*

- *firm B earns zero profits and firm A earns positive profits,*
- $p_B^* = \sqrt{\frac{F_B L_B T}{2}} + c_B < p_A^* - a$*, and*
- n_A^* *is the largest root of* $-8F_A(2\alpha + 3\beta)n_A^3 + 4F_A L_A T n_A^2 + 4L_A T(\alpha + \beta)n_A - L_A^2 T^2$*, where* $\alpha = a + c_B$ *and* $\beta = \sqrt{2F_B L_B T}$.

In equilibrium, the disadvantaged firm's franchises are isolated. This implies that the disadvantaged firm scales up franchise concentration to the point where its profits are zero, setting the price given in Proposition 3. The advantaged firm exploits its preferred product and/or better franchise technology to earn positive profits. A numerical analysis verifies the expected comparative statics: Firm A's profits rise with a, c_B, L_B and F_B and fall with F_A and L_A. More interestingly, A's profits fall with T: the reduction in price competition due to increased travel costs is more than offset by the increase in franchise provision.

We next show in Proposition 4 that the finding that symmetric duopolists earn zero profit in the competitive equilibrium extends to a symmetric N firm setting in which each consumer receives N uncorrelated firm-specific location shocks.

Proposition 4. *In the symmetric equilibrium with N firms, firms earn zero profits.*

Again, the smallest firm's franchises are isolated, and do not compete against each other. Therefore, the smallest firm can scale up its franchises, as it earns the same profit per franchise. But then, the smallest firm—and hence each firm in a symmetric setting—must earn zero profits in equilibrium. In a symmetric setting, an increase in the number of firms N leads to smaller market shares for each franchise. As a result, firms reduce franchise concentration until the market share for each franchise returns to its "original" level. Because firms earn zero profit in equilibrium and market share per franchise does not change, the optimal price remains unchanged.

Social Planner's Problem. We now return to a two firm setting[3] and compare the competitive equilibrium outcome with the solution to a social planner's problem, in which the social planner maximizes total (consumer plus producer) surplus by choosing franchise concentration for each firm; and then given this concentration choice, firms compete for customers by setting price. To make the comparison to the symmetric competitive equilibrium meaningful, we require

[3]The analogue of the social planner Proposition 5 below extends generally to N firms.

that the social planner establish the same franchise concentration for each firm. Because V is large enough that all consumers purchase in equilibrium, prices just transfer surplus from consumers to firms, and hence do not affect total social surplus. It follows that the social planner seeks to minimize the sum of travel and franchise establishment costs,

$$\frac{4n_A n_B}{L^2}\int_0^{\frac{L}{2n_A}}\left(\int_0^{d_A+\frac{p_A-p_B}{T}} Td_B dd_B dd_A + \int_{d_A+\frac{p_A-p_B}{T}}^{\frac{L}{2n_B}} Td_A dd_B\right) dd_A + F(n_A+n_B).$$

Proposition 5. *The competitive concentration of franchises exceeds the socially optimal concentration.*

It follows from Proposition 5 that at the social optimum, firms earn strictly positive profits. Intuitively, the competitive over-provision of franchises results from the efforts of firms to compete for greater market share. The social planner internalizes this externality: the social planner does not care about the market share of individual firms, but the firms do.

1.4 Additional Taste Heterogeneity

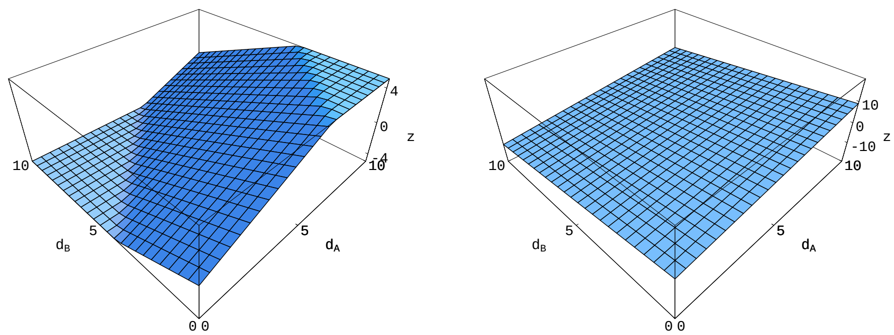
We now investigate how outcomes are affected when, in addition to the endogenous contestable spatial consumer heterogeneity, consumers also differ exogenously in their intrinsic taste for each firm's product. For example, in a franchise setting, some consumers may like Wendy's hamburgers more than Burger King's, while other consumers have the opposite preference. So, too, in a product line setting, some consumers may prefer the marketing or branding by one firm (e.g., Nike's swoosh) that is common to that firm's product line, while other consumers prefer another firm's branding.

Specifically, we suppose that in the population of consumers, the relative valuation z of firm A is uniformly distributed on $[-m, m]$, where $m > 0$. A consumer with a relative valuation of z gains an additional value (in dollar terms) of $z/2$ from purchasing firm A's good and loses $z/2$ from purchasing firm B's good. As a result, consumer preferences will differ due to both (i) the endogenous spatial distance between a consumer's location and a firm's product locations, and (ii) to the exogenous differences in their relative tastes for firm A's product line. The magnitude of m captures the importance of the exogenous taste heterogeneity relative to the endogenous spatial heterogeneity.

Now when consumers make their purchases they consider their relative preferences (or dispreferences) for firm A: for almost every consumer, $\delta_B^c(z, S_A, S_B) = 1$ if and only if

$$V + \frac{z}{2} - p_A - Td_A^c \geq V - \frac{z}{2} - p_B - Td_B^c.$$

Ex ante, the probability a consumer shops at a firm A franchise is $Prob(d_A^c - d_B^c - \frac{z}{T} \leq \frac{p_B - p_A}{T})$. In Figures 1.3.1 and 1.3.2, the area above the plane shows firm A's market share when taste heterogeneity is small and large, respectively.



1.3.1: Market shares when m is small. 1.3.2: Market shares when m is large.

Figure 1.3: Market shares with heterogeneous consumers. The cube represents the distribution of consumer's firm locations shocks (relative to a franchise location) and taste preferences. The plane represents the set of location and taste shocks for which consumers are indifferent between purchasing from firm A and firm B.

Proposition 6. *In a symmetric firm setting with heterogeneity in consumer tastes, i.e., $m > 0$, firms earn strictly positive profits. If $m \leq \sqrt{2FLT}$ then*

$$p^* = \frac{2FLT}{\sqrt{8FLT} - m}, \qquad n^* = \frac{1}{2}\sqrt{\frac{LT}{2F}}, \text{ and } \qquad \pi^* = \frac{m\sqrt{2FLT}}{4(\sqrt{8FLT} - m)}.$$

If $m > \sqrt{2FLT}$ then

$$p^* = m, \qquad n^* = \frac{1}{2}\sqrt{\frac{LT}{2F}}, \text{ and } \qquad \pi^* = \frac{1}{2}\left(m - \sqrt{\frac{FLT}{2}}\right).$$

Comparing Proposition 1 with Proposition 6 reveals that the equilibrium number of franchises, $n^* = \frac{1}{2}\sqrt{\frac{LT}{2F}}$, does not depend on whether consumers have heterogeneous tastes. Why then is it that firms now earn strictly positive profits? The answer is that a consumer located on the extreme of a firm j franchise service area, i.e., $d_j^c = L/(2n)$, now prefers with strictly positive probability to purchase from firm j: some of these consumers have a large relative preference for firm j, and want to patronize a firm j franchise. Hence, were firm j to reduce its price at one franchise, it would now steal consumers from its other franchises. As a result, firms have lesser incentives to reduce prices, and the weakened price competition allows firms to earn strictly positive profits in equilibrium.

As before, the competitive equilibrium still features over-provision of franchises—one can show that a social planner would choose a lesser franchise concentration than what emerges in the competitive equilibrium. Again, this reflects that the social planner internalizes the competition for market share via franchise location.

The comparative statics are straightforward. As m increases, firms exploit the increased taste heterogeneity to increase profit. Profits are a decreasing function of FLT. This is because increasing FLT makes consumer demand more sensitive to endogenous spatial heterogeneity and less sensitive to the exogenous taste heterogeneity: as L or F increase, franchises are located further away, and as T increases, consumers weigh more the costs of travel. Because consumer demand is less affected by consumer tastes, any marginal price reduction at one franchise steals/cannibalizes fewer consumers from its other franchises. Hence, relative to an economy with only spatial heterogeneity [where firms earn zero profit], the costs of aggressive price cutting are reduced, implying that firm profits fall.

That firms earn positive profits when there is exogenous heterogeneity in consumer preferences between firms is not surprising. The more revealing question is: relative to an economy where firms only differ on an exogenous taste dimension, how does introducing a contestable spatial dimension affect firm profits? Recall that in our benchmark spatial setting with homogeneous firms, firms competed away all profits through franchise concentration. As a result, one might conjecture that adding consumer heterogeneity in tastes might not alter firm profits, especially since equilibrium franchise concentration is unaffected by the extent of the consumer heterogeneity in tastes. We show that this is not so—heterogeneity in tastes *intensifies* price competition. In particular, Proposition 7 shows that not only do firms compete away all profits from the spatial dimension, but they also compete away some of the rents that accrue due to the exogenous heterogeneity in consumer tastes.

Proposition 7. *Firms earn larger profits when exogenous tastes are the sole source of consumer heterogeneity than when there is also endogenous consumer spatial heterogeneity.*

To understand Proposition 7, consider the impact of introducing a slight consumer heterogeneity in tastes to the spatial model. For $m < \sqrt{2FLT}$, a marginal increase in taste heterogeneity has a differential impact on profits in the two environments of:

$$\frac{\partial \pi^*_{NC}}{\partial m} - \frac{\partial \pi^*_{C,NC}}{\partial m} = \frac{1}{2}[1 - \frac{2FLT}{(2\sqrt{2FLT} - m)^2}] > 0,$$

where π^*_{NC} is equilibrium firm profit when there is only taste heterogeneity and $\pi^*_{C,NC}$ is equilibrium firm profit when there is also a endogenous heterogeneity along a spatial dimension. In the neighborhood of $m = 0$, $\frac{\partial \pi^*_{NC}}{\partial m} - \frac{\partial \pi^*_{C,NC}}{\partial m} = \frac{1}{4}$: introducing spatial heterogeneity causes firms to compete away fully three-quarters of the potential value of an increase in taste heterogeneity, m. As m increases further, price cuts would cannibalize a greater fraction of demand that would be taken by a firm's other franchises, causing firms to compete away smaller fractions of the potential profit.

Still, the marginal profit loss remains positive, and at $m = \sqrt{2FLT}$, the total accumulated profit loss **equals** the total franchise establishment cost. In particular, for $m \geq \sqrt{2FLT}$, equilibrium pricing only reflects the exogenous taste dimension—price in the economy without the endogenous spatial dimension is the same as that when the spatial dimension is present. It follows that the firms extract no benefits from the franchises that they establish, and that all franchise establishment costs are, in some sense, wasted.

1.5 Competition between Product Lines

We now introduce multiple product lines for the two firms, allowing for both meaningful correlation in consumer preferences across product lines, and for differences in the costs of creating products across product lines. Concretely, we let a preference for Diet Coke reveal a likely preference for soda drinks over fruit juices, and sodas can be less expensive to provide, but we maintain the assumption that a preference for Diet Coke over Coke reveals nothing about preferences for Diet Pepsi vs. Pepsi. So, too, a preference for a Honda Odyssey may suggest a likely preference for vans, but a consumer's preferred Toyota could turn out to be a Camry.

To model multiple product lines, we assume that firms A and B have products associated with two spatial circles, which, for simplicity, we assume have the same length, L. Each consumer is located on one spatial circle for each firm. The cost of traveling distance d on a circle is Td, and the travel costs between a firm's circles are "high enough" that in equilibrium a consumer always purchases a product on one of the circles on which he or she is located. The unconditional probability that a consumer is located on a circle i is one-half, $i = 1, 2$. We introduce correlation in preferences over the products of the two firms by supposing that if a consumer is located on circle i of firm A, then the conditional probability that the consumer is located on circle i for firm B is ρ (and vice versa for a consumer located on circle i for firm B).[4] We introduce heterogeneity between product lines by assuming that the cost F_1 of introducing a product variety on circle 1 exceeds the cost F_2 of a product variety on circle 2, i.e., $F_1 \geq F_2$. Introducing heterogeneity along other dimensions (e.g., spatial distances) gives rise to analogous results; and relaxing the assumption that, ex ante, a citizen is equally likely to be on each circle is routine. We renormalize the measure of consumers to two.

The total profit function for firm $j = A, B$ becomes

$$\pi_j = p_{j_1}(\rho Y_{j_{11}} + (1-\rho)Y_{j_{12}}) + p_{j_2}(\rho Y_{j_{22}} + (1-\rho)Y_{j_{21}}) - (F_1 n_{j_1} + F_2 n_{j_2}),$$

where, for example, $Y_{A_{ik}}$ denotes the measure of consumers who purchase from firm A when they are located on circle i of firm A and circle k of firm B, $i, k = 1, 2$, and we omit the dependence of these measures on prices and numbers of products.

Proposition 8. *There is a unique symmetric equilibrium. In this equilibrium, both firms charge price p_i and have n_i product varieties on circle i, $i = 1, 2$. Firms earn zero profits from their products on circle 1 where product varieties are more expensive. For $\rho \in [0, 1)$ and $F_1 > F_2$, firms choose $n_2 > n_1$ and earn strictly positive profits of at least*

[4] Reisinger [2006] studies product bundling in a duopolistic multi-product environment with an ostensibly similar preference structure. His model features two products x_1 and x_2, each with their own spatial circle, both produced by two firms A and B, where firm A is located at 0 on both circles, while B is located directly opposite at 1/2. A consumer located at x_1 on circle 1 is located on $x_1 + \delta$ on circle 2, where δ is a parameter that provides a measure of how many consumers are most likely to prefer both of one firm's products. Thus, although his set up has a multi-product feature to address strategic bundling, Reisinger's model has more in common with standard spatial models than with our's: in his model, firm locations are exogenous, and given knowledge about a consumer's preference for firm A's first product, one can *exactly* determine the consumer's preference for all other products.

$(1-\rho)p_2(Y_{21} - \frac{n_1}{2n_2}) > 0$ *from their less expensive product varieties on circle 2. Firms set* $p_1 = \sqrt{\frac{LTF_1}{2}}$ *and* $p_1 > p_2$ *if and only if* $\rho > \rho^*$*, where* $\rho^* = \frac{\sqrt{F_2}\sqrt{8F_1+F_2}-3F_2}{2(F_1-F_2)} \in (0, \frac{2}{3})$*. At* ρ^**,* $\frac{n_2}{n_1} = \frac{F_1}{F_2}$*.*

Intuition for this proposition can be gleaned by considering the extreme scenarios of perfectly positive and negative correlation in consumer preferences, i.e., $\rho = 1$ and $\rho = 0$. When $\rho = 1$, a consumer on firm A's circle 1 is also on circle 1 of firm B; and when $\rho = 0$, a consumer on firm A's circle 1 is on firm B's circle 2. When $\rho = 1$, firm j's profits become

$$\sum_{k=1}^{2} p_{jk}Y_{jkk} - F_k n_{j_k}.$$

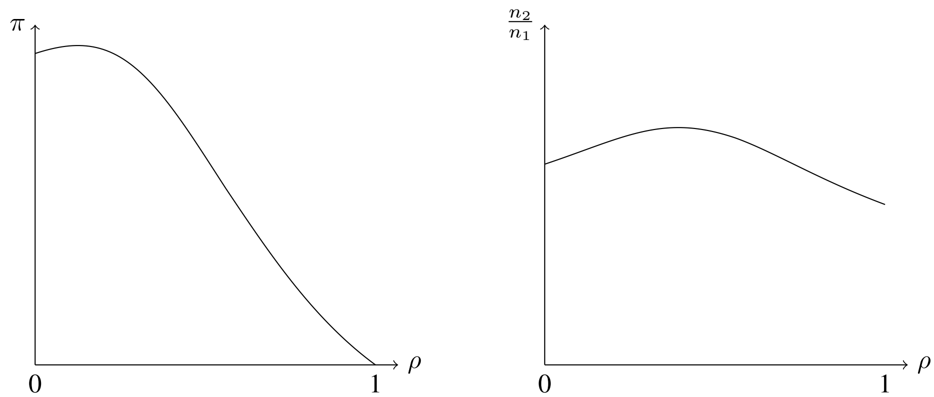
Firm j's profits from circle k only depend on the prices and product variety choices by the two firms on their k^{th} circles. The separability of the profit function across the spatial circles immediately implies that equilibrium is characterized by Proposition 1: the firms compete against each other on a circle-by-circle basis, setting prices $p_k = \sqrt{\frac{LTF_k}{2}}$, earning zero profits. Note that $F_1 > F_2$ implies that $p_1 > p_2$, and $n_1 < n_2$.

Conversely, when $\rho = 0$, a consumer who is on firm A's circle 2 is on firm B's circle 1, where products are more expensive to produce. Again profit functions are separable, with each firm having an advantaged circle 2 competing against a disadvantage circle 1, and a disadvantaged circle 1 competing against an advantaged circle 2. It immediately follows that equilibrium is characterized by Proposition 3, with firms earning zero expected profits from their disadvantaged circle 1, setting price $p_1 = \sqrt{\frac{LTF_1}{2}}$ and earning strictly positive profits from their advantaged circle 2, charging $p_2 > p_1$, as they exploit their more extensive product line. The proof shows that these qualitative results extend to intermediate levels of correlation, i.e., to $\rho \in (0, 1)$.

The general intuition is that cost differences induce firms to provide more products on their inexpensive lines than on their expensive ones. Moreover, some consumers are located on one firm's inexpensive product line, but the other firm's expensive product line. Because the inexpensive product line is more extensively stocked, i.e., $n_2 > n_1$, it competes against itself for some consumers—some consumers strictly prefer more than one product on one firm's inexpensive line to any of the other firm's products. Firms internalize this own product line competition by not expanding their inexpensive product lines to the same extent. This reduces the intensity of price competition, so that firms extract strictly positive profits.

The analysis makes clear that profits go to zero as correlation in preferences goes high, i.e., $\rho \to 1$, or as cost heterogeneity goes to zero, i.e., $F_1 \to F_2$. What is not clear is how intermediate levels of correlation affect profits when cost heterogeneity is significant. We investigate this quantitatively, exploring how $\rho \in [0, 1]$ affects equilibrium outcomes when $F_1 = 1$ and $F_2 = \frac{1}{2}$, so that products on circle 1 are twice as expensive to produce as those on circle 2 (implying that $p_1 > p_2$ if and only if $\rho > \rho^* \sim 0.562$). We normalize $\sqrt{LT} = 100$.

Figures 1.4.1 and 1.4.2 graph profits and $\frac{n_2}{n_1}$ as a function of ρ. As ρ is reduced below 1, profits initially increase



1.4.1: Firm profits as a function of ρ.

1.4.2: $\frac{n_2}{n_1}$ as a function of ρ.

Figure 1.4: Firm profits and firm franchise ratios as a function of ρ. Parameters: $F_1 = 1$, $F_2 = \frac{1}{2}$, $\sqrt{LT} = 100$.

sharply (convexly) from zero, but then the rate of increase slows down, and profits are maximized when $\rho \sim 0.124$. In particular, maximizing the probability $1 - \rho$ that an inexpensive product line competes against the expensive product line of the other firm does *not* maximize firm profits. The direct effect on profits of reducing ρ is always positive. However, at $\rho \sim 0.393$, n_1 reaches a minimum and n_2 reaches a maximum, implying that $\frac{n_1}{n_2}$ begins to rise as ρ is reduced further below 0.393. The intuition is that as ρ falls, a firm's product line 2 increasingly competes against itself for customers who are also located on the other firm's expensive product line, eventually causing a firm to reduce its n_2. Once ρ falls below 0.124, the increase in $\frac{n_1}{n_2}$ swamps the direct increase in $(1 - \rho)$, and profits begin to fall. Still, it is important to recognize that any plausible parameterization has $\rho > 0.5$, e.g., a consumer who prefers one firm's soda to its juices is more likely to prefer a soda from the other firm to its juices. This suggests that for plausible parameterizations, both $\frac{n_2}{n_1}$ and firm profits strictly increase as ρ is reduced, i.e., as the inexpensive product line is more likely to compete against the expensive one.

1.6 Conclusion

This paper endogenizes both firm pricing and franchise location within a novel spatial model in which consumers receive *firm-specific* location shocks. This renders the analysis with endogenous franchising feasible. We establish a remarkable result: when the product is homogeneous, then in the unique equilibrium, firms earn zero profits—firms over-provide franchises to such an extent that they compete away all profits. That is, even though firms face ex-post consumer heterogeneity they fail to exploit it: while ex-post consumer heterogeneity ensures positive gross firm profits, competition for market share via franchise location drives net firm profits down to zero. This qualitative result extends when firms differ in franchise costs or a firm has a better product: the "disadvantaged" firm continues to make zero net profits. We show that this competitive provision of franchises is socially excessive—a social planner internalizes the competition for market share and chooses a lesser franchise concentration. It follows that, in the competitive equilibrium, firm profits are lower and consumer surplus is higher than they would be were franchise concentration chosen by a social planner.

We then introduce an additional exogenous taste source of consumer heterogeneity. Firms profit from this exogenous taste heterogeneity. However, we find that the contestable spatial dimension enhances price competition, causing firms to compete profits from exogenous taste heterogeneity below those that would obtain were there no endogenous spatial heterogeneity.

Finally, we show that our model remains tractable even in the presence of significant heterogeneity across aspects of firms. As such, our model can be used as the foundation for the analyses of competition between networks in other settings. In particular, our framework remains tractable when firms have multiple product lines, and a preference for a good from one firm's product line contains information about likely preferences over the other firm's product lines. What is crucial for our analysis is only that information is not conveyed about relative locations on the other firm's product line. Concretely, a preference for Coke over Sprite can convey a likely preference for sodas over juices, but not for Pepsi over Diet Pepsi. When we introduce such correlation in consumer preferences, and integrate the possibility that some products are more costly to produce, we find that firms earn strictly positive profits as long as preferences are not perfectly positively correlated. In this situation, sometimes a firm's inexpensive product line competes against itself for some consumers, rather than against the other firm's expensive product line; this reduces the incentives to over-provide products, reducing the intensity of price competition, and allowing firms to earn strictly positive profits.

1.7 Chapter 1 Proofs

Calculating $y_{j_i}(d_j, S_A, S_B)$: Define

$$\underline{a}_{j_i}(S_j) = \min\{l_{j_i} - a_{j_{i-1,i}}(S_j), a_{j_{i,i+1}}(S_j) - l_{j_i}\},$$

$$\overline{a}_{j_i}(S_j) = \max\{l_{j_i} - a_{j_{i-1,i}}(S_j), a_{j_{i,i+1}}(S_j) - l_{j_i}\},$$

for $i \in N_j$, $j \in \{A, B\}$. $\underline{a}_{j_i}(S_j)$ and $\overline{a}_{j_i}(S_j)$ are the shortest and longest distances from franchise j_i's i location to the edge of their service area.

Given strategies (S_A, S_B) and location shock d_j in franchise j_i's service area, the conditional expected demand $y_{j_i}(d_j, S_A, S_B)$ is the measure of firm $-j$'s circle for which the total delivery cost of the product is lower if purchased from franchise j_i than from the lowest competing alternative. For a given d_{-j} this lowest competing alternative is identified by the partition of firm $-j$'s circle into franchise service areas. For some $-j_k$ franchises, total delivery cost by $-j_k$ to any point in its service area is lower than total delivery cost by franchise j_i to d_j. For some $-j_k$ franchises, total delivery cost by $-j_k$ to any point in its service area is higher than total delivery cost by franchise j_i to d_j. The remaining $-j_k$ franchises 'split' their service area. For distances close to l_{-j_k} total delivery cost by $-j_k$ is lower than by franchise j_i to d_j, while for distances far away,total delivery cost by $-j_k$ is higher than by franchise j_i to d_j. To reflect this, we partition N_{-j} into four sets given d_j, S_A and S_B:

$$\begin{aligned}
L_{j_i}(d_j, S_A, S_B) &= \{k \in N_{-j} : p_{-j_k} + \overline{a}_{-j_k}(S_{-j})T < p_{j_i} + |d_j - l_{j_i}|T\},\\
M_{j_i}(d_j, S_A, S_B) &= \{k \in N_{-j} : p_{-j_k} + \underline{a}_{-j_k}(S_{-j})T < p_{j_i} + |d_j - l_{j_i}|T < p_{-j_k} + \overline{a}_{-j_k}(S_{-j})T\},\\
H_{j_i}(d_j, S_A, S_B) &= \{k \in N_{-j} : p_{-j_k} < p_{j_i} + |d_j - l_{j_i}|T < p_{-j_k} + \underline{a}_{-j_k}(S_{-j})T\},\\
V_{j_i}(d_j, S_A, S_B) &= \{k \in N_{-j} : p_{j_i} + |d_j - l_{j_i}|T < p_{-j_k}\}.
\end{aligned}$$

We use this notation to calculate $y_{j_i}(d_A, S_A, S_B)$:

$$\begin{aligned}
y_{j_i}(d_j, S_A, S_B) = \Bigg[L - &\sum_{k \in H_{j_i}(d_j, S_A, S_B)} 2\left(\frac{p_{j_i} - p_{-j_k}}{T} + |d_j - l_{j_i}|\right)\\
&- \sum_{k \in M_{j_i}(d_j, S_A, S_B)} \left(\frac{p_{j_i} - p_{-j_k}}{T} + |d_j - l_{j_i}| + \underline{a}_{-j_i}(S_{-j})\right)\\
&\qquad - \sum_{k \in L_{j_i}(d_j, S_A, S_B)} \left(\underline{a}_{-j_k}(S_{-j}) + \overline{a}_{-j_k}(S_{-j})\right)\Bigg]/L.
\end{aligned}$$

Calculating $Y_{j_i}(S_A, S_B)$: By definition,

$$Y_{j_i}(S_A, S_B) = \int_{a_{j_{i-1,i}}(S_j)}^{a_{j_{i,i+1}}(S_j)} y_{j_i}(d_j, S_A, S_B) dd_j.$$

To prove some results we use a more explicit decomposition of $Y_{j_i}(S_A, S_B)$ that exploits the fact that $Y_{j_i}(S_A, S_B)$ is the sum of trapeziums. Define

$$c_{j_{i_k}} = \max\{0, (p_{-j_k} - p_{j_i})/T\} \quad \text{and} \quad c_{j_{i_{k+n_{-j}}}} = \max\{0, (p_{-j_k} - p_{j_i})/T + a_{-j_{k,k+1}} - l_{-j_k}\}.$$

$c_{j_{i_k}}$ is the distance from l_{j_i} at which a consumer who receives the location shock pair, $(d_j, d_{-j}) = (l_{j_i} + c_{j_{i_k}}, l_{-j_k})$ is indifferent to purchasing from franchise j_i or franchise $-j_k$. $c_{j_{i_{k+n_{-j}}}}$ is the distance from l_{j_i} at which a consumer who receives the location shock pair, $(d_j, d_{-j}) = (l_{j_i} + c_{j_{i_{k+n_{-j}}}}, a_{-j_{k,k+1}})$ is indifferent to purchasing from franchise j_i or franchise $-j_k$. We reorder $c_{j_{i_k}}$ so that $c_{j_{i_k}} < c_{j_{i_{k+1}}}$. Let $\overline{k}_{j_i} = |\{k : c_{j_{i_k}} < a_{j_{i,i+1}} - l_{j_i}\}|$ and $\underline{k}_{j_i} = |\{j_{i_k} :$

$c_{j_{i_k}} < l_{j_i} - a_{j_{i-1,i}}\}|$. Finally, let $c_{j_{i_0}} = 0$, $c_{\overline{k}_{j_i}+1} = a_{j_{i,i+1}} - l_{j_i}$ and $c_{\underline{k}_{j_i}+1} = l_{j_i} - a_{j_{i-1,i}}$. Hence,

$$Y_{j_i}(S_A, S_B) = \frac{1}{L}\Bigg(\sum_{k=1}^{\underline{k}_{j_i}+1} T(c_{j_{i_k}}, c_{j_{i_{k-1}}}, y_{j_i}(l_{j_i} - c_{j_{i_k}}, S_A, S_B), y_{j_i}(l_{j_i} - c_{j_{i_{k-1}}}, S_A, S_B)) + \sum_{k=1}^{\overline{k}_{j_i}+1} T(c_{j_{i_k}}, c_{j_{i_{k-1}}}, y_{j_i}(l_{j_i} + c_{j_{i_k}}, S_A, S_B), y_{j_i}(l_{j_i} + c_{j_{i_{k-1}}}, S_A, S_B)) \Bigg), \quad (1.7)$$

where $T(a, b, c, d) = (a - b)(c + d)/2$. Figure 1.5 is a graphical depiction of Y_{j_i}. Y_{j_i} is equal to the area under $y_{j_i}(d_j, S_A, S_B)$ from $a_{j_{i-1,i}}$ to $a_{j_{i,i+1}}$.

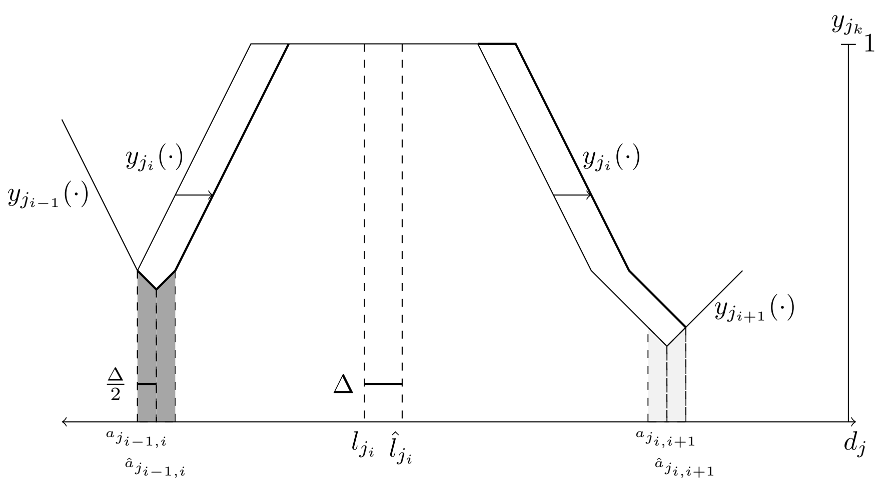


Figure 1.5: Effect of a marginal shift in the location of franchise j_i by Δ from l_{j_i} to $\hat{l}_{j_i}$. The dark gray trapeziums on the left represent the increased demand for franchises j_{i-1} and j_i. The light gray trapeziums on the right represent the decreased demand for franchises j_i and j_{i+1}.

Proof of Lemma 1: Consider franchise j_i, such that $y_{j_i}(a_{j_{i,i+1}}(S_j), S_A, S_B) = 0$ and franchise j_k such that $y_{j_k}(a_{j_{k-1,k}}(S_j), S_A, S_B) > 0$, where $i < k$. Then fixing prices and shifting $\{l_{j_m}\}_{m=i+1}^{k-1}$ marginally by the same amount counterclockwise, no franchise experiences a fall in market share (at least to a first order effect), but the market shares (sales) of franchises j_{k-1} and j_k both strictly increase. Hence, firm j's profits must increase.

Suppose instead, $y_{j_i}(a_{j_{i,i+1}}(S_j), S_A, S_B) > 0$ and $y_{j_k}(a_{j_{k-1,k}}(S_j), S_A, S_B) = 0$. Then fixing prices and shifting $\{l_{j_m}\}_{m=i+1}^{k-1}$ marginally by the same amount clockwise, no franchise experiences a fall in market share, but the market shares of franchises j_i and j_{i+1} strictly increase. Hence, firm j's profits must increase. ■

Proof of Lemma 2: Because each firm j franchise is isolated, each firm j franchise faces the same demand curve. It follows that charging the same price at each franchise, and hence capturing the same market share is a best response. To prove uniqueness, we show that $\Pi_j(S_A, S_B)$ is strictly quasi-concave in p_{j_i} $\forall i \in N_j$, implying that this best response is unique.

Note that $y_{j_i}(d_j, S_A, S_B)$ is a continuous, piecewise linear function of p_{j_i} and l_{j_i}; since $Y_{j_i}(S_A, S_B)$ is the integral of $y_{j_i}(d_j, S_A, S_B)$, $Y_{j_i}(S_A, S_B)$ is C^1 as a function of $\{p_{j_i}, l_{j_1}\}_{i=1}^{n_j}$.

The marginal profit function of firm j is differentiable with respect to p_{j_i} and l_{j_i} everywhere except at prices and locations where the partition of firm $-j$ franchises defined by $L(.)$, $M(.)$, $H(.)$ and $V(.)$ changes. At these points the number of firm $-j$ franchises against which franchise j_i competes changes discontinuously.

Because franchise j_i is isolated, $y_{j_i}(a_{j_{i-1},i}(S_j), S_A, S_B) = 0$. If $p_{j_i} \leq \min\{p_{-j_k}\}$, then $y_{j_i}(l_{j_i}, S_A, S_B) = 1$ and marginal profit decreases linearly—the coefficient on p_{j_i} is $-4/(LT)$. Hence, marginal profit is strictly decreasing over this range. If $p_{j_i} > \min\{p_{-j_k}\}$, then $y_{j_i}(l_{j_i}, S_A, S_B) < 1$ and the marginal profit function is a series of piecewise quadratic convex functions of p_{j_i} over this range—the leading term coefficient is $(2|H_{j_i}(l_{j_i}, S_A, S_B)| + |M_{j_i}(l_{j_i}, S_A, S_B)|)/(2L^2T^2)$. Within each section, the number of competing $-j$ franchises remains constant. Each section of the piecewise quadratic function has two real solutions over the domain of $\mathcal{R}^+$ (else profit can increase without bound). The larger root of the quadratic in each section is where the implied franchise j_i market share is 0. Hence, the marginal profit function has only one root associated with a maximum. ■

Proof of Lemma 3: Fix an arbitrary franchise profile, S_{-j} for the other firm, and consider a franchise profile for firm j with n_j franchises. By fixing the prices and locations of the other firm j franchises we can analyze the impact of a marginal shift in l_{j_i} and p_{j_i}. Using equation (1.7), $\frac{\partial \pi_j}{\partial l_{j_i}} = p_{j_{i-1}} \frac{\partial Y_{j_{i-1}}}{\partial l_{j_i}} + p_{j_i} \frac{\partial Y_{j_i}}{\partial l_{j_i}} + p_{j_{i+1}} \frac{\partial Y_{j_{i+1}}}{\partial l_{j_i}}$

$$
\begin{aligned}
&= \frac{p_{j_{i-1}}}{L} \frac{\partial T(c_{\overline{k}_{j_{i-1}}+1}, c_{\overline{k}_{j_{i-1}}}, y_{j_i}(l_{j_{i-1}} - c_{\overline{k}_{j_{i-1}}+1}, S_A, S_B), y_{j_{i-1}}(l_{j_{i-1}} - c_{\overline{k}_{j_{i-1}}}, S_A, S_B))}{\partial l_{j_i}} \\
&+ \frac{p_{j_i}}{L} \Bigg(\frac{\partial T(c_{\underline{k}_{j_i}+1}, c_{\underline{k}_{j_i}}, y_{j_i}(l_{j_i} - c_{\underline{k}_{j_i}+1}, S_A, S_B), y_{j_i}(l_{j_i} - c_{\underline{k}_{j_i}}, S_A, S_B))}{\partial l_{j_i}} \\
&+ \frac{\partial T(c_{\overline{k}_{j_i}+1}, c_{\overline{k}_{j_i}}, y_{j_i}(l_{j_i} - c_{\overline{k}_{j_i}+1}, S_A, S_B), y_{j_i}(l_{j_i} - c_{\overline{k}_{j_i}}, S_A, S_B))}{\partial l_{j_i}} \Bigg) \\
&+ \frac{p_{j_{i+1}}}{L} \frac{\partial T(c_{\underline{k}_{j_{i+1}}+1}, c_{\underline{k}_{j_{i+1}}}, y_{j_{i+1}}(l_{j_{i+1}} - c_{\underline{k}_{j_{i+1}}+1}, S_A, S_B), y_{j_{i+1}}(l_{j_{i+1}} - c_{\underline{k}_{j_{i+1}}}, S_A, S_B))}{\partial l_{j_i}}.
\end{aligned}
$$

As Figure 1.5 shows, this is equal to

$$
\frac{y_{j_i}(a_{j_{i-1},i}(S_j), S_A, S_B)(p_{j_i} + p_{j_{i-1}}) - y_{j_i}(a_{j_{i,i+1}}(S_j), S_A, S_B)(p_{j_i} + p_{j_{i+1}})}{2L}. \tag{1.8}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial \pi_j}{\partial l_{j_i}} &= Y_{j_i} + p_{j_{i-1}} \frac{\partial Y_{j_{i-1}}}{\partial p_{j_i}} + p_{j_i} \frac{\partial Y_{j_i}}{\partial p_{j_i}} + p_{j_{i+1}} \frac{\partial Y_{j_{i+1}}}{\partial p_{j_i}} \\
&= Y_{j_i} + \frac{p_{j_{i-1}}}{L} \frac{\partial T(c_{\overline{k}_{j_{i-1}}+1}, c_{\overline{k}_{j_{i-1}}}, y_{j_{i-1}}(l_{j_{i-1}} - c_{\overline{k}_{j_{i-1}}+1}, S_A, S_B), y_{j_{i-1}}(l_{j_{i-1}} - c_{\overline{k}_{j_{i-1}}}, S_A, S_B))}{\partial p_{j_i}} \\
&+ \frac{p_{j_i}}{L} \left(\frac{\partial T(c_{\underline{k}_{j_i}+1}, c_{\underline{k}_{j_i}}, y_{j_i}(l_{j_i} - c_{\underline{k}_{j_i}+1}, S_A, S_B), y_{j_i}(l_{j_i} - c_{\underline{k}_{j_i}}, S_A, S_B))}{\partial p_{j_i}} \right. \\
&+ \frac{\partial T(c_1, c_0, y_{j_i}(l_{j_i} - c_1, S_A, S_B), y_{j_i}(l_{j_i} - c_0, S_A, S_B))}{\partial p_{j_i}} \\
&+ \frac{\partial T(c_1, c_0, y_{j_i}(l_{j_i} - c_1, S_A, S_B), y_{j_i}(l_{j_i} - c_0, S_A, S_B))}{\partial p_{j_i}} \\
&+ \left. \frac{\partial T(c_{\overline{k}_{j_i}+1}, c_{\overline{k}_{j_i}}, y_{j_i}(l_{j_i} - c_{\overline{k}_{j_i}+1}, S_A, S_B), y_{j_i}(l_{j_i} - c_{\overline{k}_{j_i}}, S_A, S_B))}{\partial l_{j_i}} \right) \\
&+ \frac{p_{j_{i+1}}}{L} \frac{\partial T(c_{\underline{k}_{j_{i+1}}+1}, c_{\underline{k}_{j_{i+1}}}, y_{j_{i+1}}(l_{j_{i+1}} - c_{\underline{k}_{j_{i+1}}+1}, S_A, S_B), y_{j_{i+1}}(l_{j_{i+1}} - c_{\underline{k}_{j_{i+1}}}, S_A, S_B))}{\partial l_{j_{i+1}}}.
\end{aligned}
$$

As Figure 1.6 shows, this is equal to

$$
\begin{aligned}
&Y_{j_i}(S_A, S_B) - 2p_{j_i} \frac{y_{j_i}(l_{j_i}, S_A, S_B)}{LT} \\
&\qquad + \frac{y_{j_i}(a_{j_{i-1,i}}(S_j), S_A, S_B)(p_{j_{i-1}} + p_{j_i}) + y_{j_i}(a_{j_{i,i+1}}(S_j), S_A, S_B)(p_{j_{i+1}} + p_{j_i})}{2LT}. \qquad (1.9)
\end{aligned}
$$

For S_j to be a best response to S_{-j} (fixing n_j), equations (1.8) and (1.9) evaluated at (S_j, S_{-j}) must be zero. Since by assumption no firm j franchise is isolated, this gives $2n_j - 1$ equations in $2n_j - 1$ unknowns.[5] Inspection reveals that uniform franchise pricing and equal distances between franchises solves this system of equations.

An extensive numerical analysis indicates that this symmetric solution is the globally optimal best response. We compute firm j's best response of location and price given the strategy of firm $-j$ and the number of firm j franchises. Using Lemma 2, we restrict firm $-j$ strategies to those that charge a uniform price p_{-j} at each franchise, where $p_{-j} \in [0, 0.1, \ldots, 100]$. Without loss of generality we assume firm $-j$ spaces its franchises equally (expected sales of the other firm do not depend on the spacing of local monopolies).

Best responses are calculated for $n_j \in \{2, \ldots, 25\}$ and $n_{-j} \in \{2, \ldots, 25\}$ using the numerical optimization algorithm "fmincon" in Matlab. The constraints associated with the algorithm are set to ensure that firm locations are sequentially ordered, each franchise serves a non-negative measure of consumers and prices are non-negative. We normalize $\sqrt{LT}$ to 100. For each n_j, for each competing firm strategy, equidistant franchise spacing and equal franchise pricing are always the unique best response. ■

[5] l_{j_1} is normalized to 0.

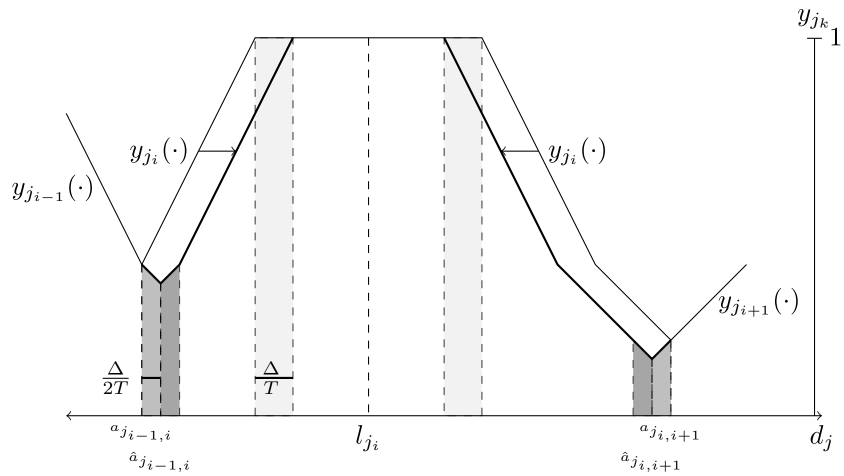


Figure 1.6: The effect of a marginal increase in the price charged by franchise j_i by Δ from p_{j_i} to $\hat{p}_{j_i}$. The light gray rectangles represent a loss in demand for franchise j_i. The darker gray trapeziums represent a gain in demand for franchise j_i. The lighter gray trapeziums represent a gain in demand for franchises j_{i-1} and j_{i+1}.

Proof of Lemma 4: First note that the marginal change in demand for firm B due to a change in p_B is the same as the marginal change in demand for firm A due to a change in p_A. To see this, let $p_d = p_A - p_B$. Then

$$\frac{\partial Y_A}{\partial p_A} = \frac{\partial Y_A}{\partial p_d} = \frac{\partial(1 - Y_B)}{\partial p_d} = -\frac{\partial Y_B}{\partial p_d} = \frac{\partial Y_B}{\partial p_B}. \tag{1.10}$$

The first-order conditions for profit maximization of each firm with respect to its price gives

$$Y_A^* = -p_A^* \frac{\partial Y_A^*}{\partial p_A^*} \text{ and } Y_B^* = -p_B^* \frac{\partial Y_B^*}{\partial p_B^*}. \tag{1.11}$$

Combining equations (1.10) and (1.11) gives

$$\frac{Y_A^*}{Y_B^*} = \frac{p_A^*}{p_B^*}.$$

Hence, $p_A^* \geq p_B^* \Leftrightarrow Y_A^* \geq Y_B^*$. It remains to show that in equilibrium firm B franchises are isolated. In contradiction to the hypothesis, suppose that firm B franchises are not isolated, i.e., $L/(2n_B^*) < (p_A^* - p_B^*)/T + L/(2n_A^*)$. Then equation (1.5) and $Y_B^* = 1 - Y_A^*$ implies

$$Y_A^* = \frac{n_A^*}{2n_B^*}\left(1 - 2n_B^* \frac{p_A^* - p_B^*}{LT}\right)^2 < \frac{n_A^*}{2n_B^*} \min\Big\{1, \frac{{n_B^*}^2}{{n_A^*}^2}\Big\} = \min\Big\{\frac{n_A^*}{2n_B^*}, \frac{n_B^*}{2n_A^*}\Big\} \leq 1/2 < Y_B^*. \qquad \blacksquare$$

Proof of Proposition 1: The first-order condition for firm A profit maximization with respect to n_A is

$$\frac{\partial \pi_A}{\partial n_A} = \frac{p_A n_B}{2n_A^2} - F = 0.$$

Hence,

$$p_A^* = \frac{2F n_A^{*^2}}{n_B^*}. \tag{1.12}$$

The first-order condition for firm B profit maximization with respect to p_B is

$$\frac{\partial \pi_B}{\partial p_B} = Y_B - \frac{2p_B n_B}{LT} = \frac{n_B}{2n_A} + 2n_B \frac{p_A - 2p_B}{LT} = 0,$$

where we substitute for Y_B using equation (1.4). Substituting for p_A^* using equation (1.12) yields

$$p_B^* = \frac{F n_A^{*^2}}{n_B^*} + \frac{LT}{8n_A^*}. \tag{1.13}$$

The first-order condition for firm A profit maximization with respect to p_A is

$$\frac{\partial \pi_A}{\partial p_A} = Y_A - \frac{2p_A n_B}{LT} = 1 - \frac{n_B}{2n_A} - 2n_B \frac{2p_A - p_B}{LT} = 0, \tag{1.14}$$

where we have substituted for Y_A using equaiton (1.3). Substituting for p_A and p_B using equations (1.12) and (1.13) into equation (1.14) then yields

$$n_B^* = 4n_A^*(LT - 6F n_A^{*^2})/LT. \tag{1.15}$$

The first-order condition for firm B profit maximization with respect to n_B is

$$\frac{\partial \pi_B}{\partial n_B} = \frac{p_B Y_B}{n_B} - F = 0. \tag{1.16}$$

Hence, equations (1.12), (1.13), (1.15) and (1.16) imply that in equilibrium

$$\frac{(LT - 8F n_A^{*^2})(144F^2 n_A^{*^4} - 32F n_A^{*^2} LT + L^2T^2)}{32 n_A^{*2}(LT - 6F n_A^{*2})} = 0.$$

Solving yields

$$n_A^* = n_B^* = \frac{1}{2}\sqrt{\frac{LT}{2F}},$$

which implies that

$$p_A^* = p_B^* = \sqrt{\frac{FLT}{2}}.$$

Hence, in the unique equilibrium $\pi_A^* = \pi_B^* = 0$.[6] The equilibrium is unique because firm profit is continuously differentiable everywhere in price and franchise concentration and the above analysis shows only one possible solution to the first-order conditions. ∎

Proof of Proposition 2: In equilibrium either $L_B/(2n_B^*) \geq L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$ or $L_B/(2n_B^*) < L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$. This implies that in equilibrium at least one firm's representative franchise is an isolated franchise (if we have equality then all franchises are isolated). As in the previous proof, the scalability of franchise concentration then immediately implies that this firm's profits must be zero. ∎

Proof of Proposition 3: As in a symmetric firm setting, letting $p_d = p_A - p_B$, we have

$$\frac{\partial Y_A}{\partial p_A} = \frac{\partial(1 - Y_B)}{\partial p_d} = -\frac{\partial Y_B}{\partial p_d} = \frac{\partial Y_B}{\partial p_B}. \tag{1.17}$$

Profit maximization implies

$$Y_A^* = -p_A^* \frac{\partial Y_A^*}{\partial p_A^*} \text{ and } Y_B^* = -(p_B^* - c_B)\frac{\partial Y_B^*}{\partial p_B^*}. \tag{1.18}$$

Combining equations (1.17) and (1.18) gives

$$\frac{Y_A^*}{Y_B^*} = \frac{p_A^*}{p_B^* - c_B}$$

Hence, $p_A^* \geq p_B^* - c_B \Leftrightarrow Y_A^* \geq Y_B^*$.

Assume that $p_A^* > p_B^* + a$ and $L_B/(2n_B^*) > L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$ (we show later that these assumptions hold in equilibrium). Firm profits are

$$\pi_A = p_A Y_A - n_A F_A = p_A \left(1 - \frac{L_A n_B}{2 L_B n_A} + 2n_B \frac{p_B - p_A + a}{L_B T}\right) - n_A F_A$$
$$\pi_B = (p_B - c_B) Y_B - n_B F_B = (p_B - c_B)\left(\frac{L_A n_B}{2 L_B n_A} - 2n_B \frac{p_B - p_A + a}{L_B T}\right) - n_B F_B.$$

The four first-order conditions are

$$\frac{\partial \pi_A}{\partial p_A} = Y_A - \frac{2 n_B p_A}{L_B T} = 0 \tag{1.19}$$
$$\frac{\partial \pi_A}{\partial n_A} = \frac{L_A n_B p_A}{2 L_B n_A^2} - F_A = 0 \tag{1.20}$$
$$\frac{\partial \pi_B}{\partial p_B} = Y_B - 2n_B \frac{p_B - c_B}{L_B T} = 0 \tag{1.21}$$
$$\frac{\partial \pi_B}{\partial n_B} = Y_B \frac{p_B - c_B}{n_B} - F_B = 0. \tag{1.22}$$

[6] The assumption that all consumers purchase the good implies that $V \geq \sqrt{\frac{9FLT}{2}}$.

The first-order condition for firm B profit maximization with respect to its franchise concentration immediately implies that firm B earns zero profits. Solving equation (1.22) for $Y_B^* = \frac{n_B^* F_B}{p_B^* - c_B}$ and substituting into equation (1.21), we solve for

$$p_B^* = \sqrt{\frac{F_B L_B T}{2}} + c_B. \tag{1.23}$$

From equation (1.20) we get

$$p_A^* = \frac{2F_A L_B n_A^{*^2}}{L_A n_B^*}. \tag{1.24}$$

Substituting $Y_A^* = 1 - Y_B^* = 1 - \frac{n_B^* F_B}{p_B^* - c_B}$ into equation (1.19), gives

$$1 - \frac{n_B^* F_B}{p_B^* - c_B} = \frac{2n_B^* p_A^*}{L_B T}.$$

Substituting for p_A^* using equation (1.24) and p_B^* using equation (1.23), we solve for

$$n_B^* = \frac{L_A L_B T - 4F_A L_B n_A^{*^2}}{L_A \sqrt{2F_B L_B T}}. \tag{1.25}$$

Substituting equations (1.23), (1.24) and (1.25) into equation (1.19), reveals that n_A is given by the solution to a cubic equation,

$$G(n_A) = -8F_A(2\alpha + 3\beta)n_A^3 + 4F_A L_A T n_A^2 + 4L_A T(\alpha + \beta)n_A - L_A^2 T^2,$$

where $\alpha = a + c_B$ and $\beta = \sqrt{2F_B L_B T}$. Because the discriminant of $G(n_A)$ is positive, $G(n_A)$ has 3 real roots. Also, since the leading term coefficient is negative and $G(0) = -L_A^2 T^2 < 0$, G has at least one negative root. To be consistent with our initial premise that $L_B/(2n_B^*) > L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$, we must have $n_A^* > L_A T/(2(\alpha + \beta))$. $G(n_A)$ evaluated at this lower bound is positive, implying that such a solution exists and n_A^* is the largest root of $G(n_A)$. Define

$$\underline{n}_A = \sqrt{\frac{L_A T(2\alpha + \beta)}{8F_A(\alpha + \beta)}} \qquad \text{and} \qquad \overline{n}_A = \sqrt{\frac{L_A T(\alpha + \beta)}{2F_A(2\alpha + 3\beta)}}.$$

Evaluating G at these points yields

$$G(\underline{n}_A) = \frac{T^2\beta(\sqrt{F_B}\sqrt{2\alpha + \beta} - \sqrt{F_A}\sqrt{\alpha + \beta})}{2\sqrt{F_A}\sqrt{(\alpha + \beta)^3}} > 0 \quad \text{and} \quad G(\overline{n}_A) = -\frac{L_A T^2 \beta}{2\alpha + 3\beta} < 0.$$

Hence, $\underline{n}_A < n_A^* < \overline{n}_A$. We now show that $p_A^* > p_B^* + a$. Using equations (1.23), (1.24) and (1.25) this is equivalent to showing

$$\frac{8F_A n_A^{*^2}(\alpha + \beta) - L_A T(2\alpha + \beta)}{2(L_A T - 4F_A n_A *^2)} > 0,$$

which holds since $\underline{n}_A < n_A^* < \overline{n}_A$.

Uniqueness is assured by showing that $p_A^* > p_B^* + a$ and $L_B/(2n_B^*) > L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$ must hold in equilibrium. To see this consider the three other possible outcomes.

Case 1: $p_A^* > p_B^* + a$**,** $L_B/(2n_B^*) \leq L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$**.** Let

$$X^* = \frac{L_B}{2n_B^*} - \left(\frac{p_A^* - p_B^* - a}{T}\right) < \min\left\{\frac{L_A}{2n_A^*}, \frac{L_B}{2n_B^*}\right\}.$$

$$Y_A^* = \frac{2n_A^* n_B^* {X^*}^2}{L_A L_B} < \frac{2n_A^* n_B^*}{L_A L_B}\left(\min\left\{\frac{L_A}{2n_A^*}, \frac{L_B}{2n_B^*}\right\}\right)^2 = \min\left\{\frac{n_B^* L_A}{2n_A^* L_B}, \frac{n_A^* L_B}{2n_B^* L_A}\right\} < 1/2 < Y_B^*.$$

but this implies $p_A^* < (p_B^* - c_B)$ which contradicts $p_A^* > p_B^* + a$.

Case 2: $p_A^* \leq p_B^* + a$**,** $L_B/(2n_B^*) \geq L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$**.** Let

$$X^* = \frac{L_A}{2n_A^*} + \left(\frac{p_A^* - p_B^* - a}{T}\right) < \min\left\{\frac{L_A}{2n_A^*}, \frac{L_B}{2n_B^*}\right\}.$$

Demand for firm B is

$$Y_B^* = \frac{2n_A^* n_B^* {X^*}^2}{L_A L_B} < \frac{2n_A^* n_B^*}{L_A L_B}\left(\min\left\{\frac{L_A}{2n_A^*}, \frac{L_B}{2n_B^*}\right\}\right)^2 = \min\left\{\frac{n_B^* L_A}{2n_A^* L_B}, \frac{n_A^* L_B}{2n_B^* L_A}\right\} < 1/2 < Y_A^*.$$

Hence, from lemma 4, $p_A^* > p_B^* - c_B$. The four first-order conditions are

$$\frac{\partial \pi_A}{\partial p_A} = 1 - Y_B - \frac{2p_A Y_B}{XT} = 0 \tag{1.26}$$

$$\frac{\partial \pi_B}{\partial p_B} = Y_B\left(1 - \frac{2(p_B - c_B)}{XT}\right) = 0 \tag{1.27}$$

$$\frac{\partial \pi_A}{\partial n_A} = \frac{p_A Y_B}{n_A}\left(\frac{L_A}{n_A X} - 1\right) - F_A = 0 \tag{1.28}$$

$$\frac{\partial \pi_B}{\partial n_B} = \frac{(p_B - c_B) Y_B}{n_B} - F_B = 0. \tag{1.29}$$

Combining equations (1.26), (1.27) and $Y_A = 1 - Y_B$ yields

$$Y_A^* = \frac{p_A^*}{p_A^* + p_B^* - c_B} \quad \text{and} \quad Y_B^* = \frac{p_B^* - c_B}{p_A^* + p_B^* - c_B}. \tag{1.30}$$

The first-order condition for firm B profit maximization with respect to its price implies

$$X^* = \frac{2(p_B^* - c_B)}{T}. \tag{1.31}$$

Substituting equation (1.31) into Y_B^* gives

$$Y_B^* = \frac{8n_A^* n_B^* (p_B^* - c_B)^2}{L_A L_B T^2}. \tag{1.32}$$

Substituting for Y_B^* using equation (1.32) into equation (1.29) we solve for

$$n_A^* = \frac{F_B L_A L_B T^2}{8(p_B^* - c_B)^3}. \tag{1.33}$$

Using the definition of X^* and equation (1.31) gives

$$L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T = \frac{2(p_B^* - c_B)}{T}.$$

Because $p_A^* \leq p_B^* + a$ this further implies that $p_B^* \geq c_B + \sqrt{\frac{F_B L_B T}{2}}$. Finally, substituting equations (1.30), (1.31) and (1.33) into equation (1.28) implies

$$8p_A^*(p_B^* - c_B)^4(4(p_B^* - c_B)^2 - F_B L_B T) - (p_A^* + p_B^* - c_B) F_A L_A F_B^2 L_B^2 T^3 = 0.$$

This equality can never be satisfied since $p_A^* > p_B^* - c_B$ and $p_B^* \geq c + \sqrt{\frac{F_B L_B T}{2}}$.

Case 3: $p_A^* \leq p_B^* + a$**,** $L_B/(2n_B^*) \leq L_A/(2n_A^*) + (p_A^* - p_B^* - a)/T$ The second assumption implies that firm A franchises are isolated, so that firm A makes zero profit. This combined with the first assumption imply that $n_A^* < n_B^*$. If $Y_A^* \geq 1/2 \geq Y_B^*$ then $p_A^* > p_B^* - c_B$, which implies firm B makes negative profit. Conversely if, instead, $Y_A^* < 1/2 < Y_B^*$, then there exists a deviation by firm A that gives it positive profit, contradicting the posited equilibrium. To see this, observe that firm A's demand is

$$Y_A^* = \frac{L_B n_A^*}{2L_A n_B^*} + 2n_A^* \frac{p_B^* + a - p_A^*}{L_A T} < 1/2 \quad \text{implying that} \quad \frac{L_B n_A^*}{L_A n_B^*} < 1. \tag{1.34}$$

Profit maximization by firm B requires that in equilibrium

$$\frac{\partial \pi_B}{\partial n_B} = (p_B - c_B)\frac{L_B n_A}{2L_A n_B^2} - F_B = 0. \tag{1.35}$$

Substituting the inequality in equation (1.34) into (1.35) reveals that $\frac{p_B^* - c_B}{2} > n_B^* F_B$. If firm A deviates and sets $p_A = p_B^* + a$ and $n_A = n_B^*$, then $Y_A = 1/2$ and its profit is strictly positive. ■

Proof of Proposition 4: Let firms 1 to $N-1$ employ symmetric strategies, $p_1 = \ldots = p_{N-1} = p$ and $n_1 = \ldots = n_{N-1} = n$. One can again show that for firm N to charge a lower markup price than the other firms in equilibrium,

N must serve a smaller share of the market. Hence, if $p_N \leq p$, then $Y_N \geq Y$. Firm N's market share is

$$Y_N = (N-1)!\frac{2^N n_N n^{N-1}}{L^N}\left(\int_0^{\frac{p-p_N}{T}} \int_0^{\frac{L}{2n}} \int_{d_{N-1}}^{\frac{L}{2n}} \ldots \int_{d_2}^{\frac{L}{2n}} 1 dd_1 \ldots dd_N \right.$$
$$\left. + \int_{\frac{p-p_N}{T}}^{\frac{L}{2n}+\frac{p-p_N}{T}} \int_{d_N-\frac{p-p_N}{T}}^{\frac{L}{2n}} \int_{d_{N-1}}^{\frac{L}{2n}} \ldots \int_{d_2}^{\frac{L}{2n}} 1 dd_1 \ldots dd_N \right) = \frac{n_N}{Nn} + \frac{2n_N(p-p_N)}{LT}.$$

Hence,

$$\pi_N = p_N\left(\frac{n_N}{Nn} + \frac{2n_N(p-p_N)}{LT}\right) - Fn_N.$$

The first-order condition for firm N with respect to n_N is

$$\frac{d\pi_N}{dn_N} = p_N\frac{Y_N}{n_N} - F = 0.$$

Hence, $\pi_N^* = p_N Y_N - Fn_N = 0$. In a symmetric equilibrium $Y_N = \frac{1}{N}$, so that

$$p_N = FNn_N. \tag{1.36}$$

The first-order condition for firm N with respect to p_N is

$$\frac{d\pi_N}{dp_N} = -p_N\frac{2n_N}{LT} + Y_N = 0. \tag{1.37}$$

Substituting $Y_N = \frac{1}{N}$ and solving equations (1.36) and (1.37) simultaneously yields $p_N^* = \sqrt{\frac{FLT}{2}}$ and $n_N^* = \frac{1}{N}\sqrt{\frac{LT}{2F}}$. ■

Proof of Proposition 5: In the second stage, firm i maximizes π_i given franchise concentrations, $n_A = n_B = n$ and the prices of the other firm j. With $n_A = n_B$, it is straightforward to show that firms choose $p_A = p_B$. But then prices drop out of the social planner's objective,

$$\begin{aligned} SS &= V - \frac{4n^2}{L^2}\int_0^{\frac{L}{2n}}\left(\int_{d_A}^{\frac{L}{2n}} Td_A dd_B + \int_0^{d_A} Td_B dd_B\right) dd_A - 2nF \\ &= V - \frac{LT}{6n} - 2nF. \end{aligned} \tag{1.38}$$

Differentiating SS with respect to n gives the social planner's first-order condition:[7]

$$\frac{\partial SS}{\partial n} = \frac{LT}{6n_{SP}^2} - 2F = 0.$$

[7] Second-order conditions are clearly satisfied.

Denoting the optimal level of franchise concentration per firm as n^*_{SP}, we solve for

$$n^*_{SP} = \sqrt{\frac{LT}{12F}} < \sqrt{\frac{LT}{8F}} = n^*,$$

$$SS^*_{SP} = V - \sqrt{\frac{4FLT}{3}} > V - \sqrt{\frac{25FLT}{18}} = SS^*. \qquad \blacksquare$$

Proof of Proposition 6: **Case 1 (small m).** We first consider the possibility that m is small enough that in equilibrium no consumer with location shocks $\{d_j, d_{-j}\} = \{\frac{L}{2n_j}, 0\}$, $j \in \{A, B\}$ purchases from firm j, i.e., $m < |p^*_A - p^*_B| + T\max\{\frac{L}{2n^*_A}, \frac{L}{2n^*_B}\}$. That is, m is small enough that, in equilibrium, any consumer who is located at the same point as one firm's franchise and at the edge of the other firm's franchise service area will patronize the former firm regardless of their taste preference. Under this assumption, firm market shares are

$$\begin{aligned} Y_A =& \frac{2n_An_B}{mL^2}\left[\int_0^{\frac{L}{2n_A}}\int_0^{\frac{L}{2n_B}}\left(\int_{(d_A-d_B)T+p_A-p_B}^{\frac{T}{2n_A}+p_A-p_B} dz - \int_m^{\frac{T}{2n_A}+p_A-p_B} dz\right) dd_B dd_A \right.\\ &+ \int_{\frac{p_B-p_A+m}{T}}^{\frac{L}{2n_A}}\int_0^{d_A+\frac{p_A-p_B-m}{T}}\int_m^{(d_A-d_B)T+p_A-p_B} dz dd_B dd_A \\ &\left. - \int_0^{\frac{L}{2n_B}+\frac{p_B-p_A-m}{T}}\int_{d_A+\frac{p_A-p_B+m}{T}}^{\frac{L}{2n_B}}\int_{(d_A-d_B)T+p_A-p_B}^{-m} dz dd_A dd_B\right] \end{aligned}$$

$$\begin{aligned} Y_B =& \frac{2n_An_B}{mL^2}\left[\int_0^{\frac{L}{2n_A}}\int_0^{\frac{L}{2n_B}}\left(\int_{-\frac{T}{2n_B}+p_B-p_A}^{(d_A-d_B)T+p_A-p_B} dz - \int_{-\frac{T}{2n_B}+p_B-p_A}^{-m} dz\right) dd_B dd_A \right.\\ &+ \int_0^{\frac{L}{2n_B}+\frac{p_B-p_A-m}{T}}\int_{d_A+\frac{p_A-p_B+m}{T}}^{\frac{L}{2n_B}}\int_{(d_A-d_B)T+p_A-p_B}^{-m} dz dd_A dd_B \\ &\left. - \int_{\frac{p_B-p_A+m}{T}}^{\frac{L}{2n_A}}\int_0^{d_A+\frac{p_A-p_B-m}{T}}\int_m^{(d_A-d_B)T+p_A-p_B} dz dd_B dd_A\right]. \end{aligned}$$

Simplifying yields

$$\begin{aligned}
Y_A =& \frac{n_A n_B}{24m}\left\{\frac{3LT(n_A+n_B)}{n_A^2 n_B^2}+\frac{12n_A(p_B-p_A+m)-6LT}{n_A^2 n_B}\right.\\
&\left.+\frac{1}{L^2T^2}\left[\left(\frac{2n_B(p_A-p_B+m)-LT}{n_B}\right)^3-\left(\frac{2n_A(p_B-p_A+m)-LT}{n_A}\right)^3\right]\right\}\\
Y_B =& \frac{n_A n_B}{24m}\left\{\frac{3LT(n_A+n_B)}{n_A^2 n_B^2}+\frac{12n_B(p_A-p_B+m)-6LT}{n_A n_B^2}\right.\\
&\left.-\frac{1}{L^2T^2}\left[\left(\frac{2n_B(p_A-p_B+m)-LT}{n_B}\right)^3-\left(\frac{2n_A(p_B-p_A+m)-LT}{n_A}\right)^3\right]\right\}.
\end{aligned}$$

Differentiating firm profit and applying symmetry yields the equilibrium outcomes:

$$p_A^* = p_B^* = p_{m_S}^* = \frac{2FLT}{\sqrt{8FLT}-m} \quad \text{and} \quad n_A^* = n_B^* = n_{m_S}^* = \frac{1}{2}\sqrt{\frac{LT}{2F}}.$$

Our initial assumption that $m < |p_A^* - p_B^*| + T\max\{\frac{L}{2n_A^*}, \frac{L}{2n_B^*}\}$ holds if and only if $m < \sqrt{2FLT}$. In equilibrium,

$$\pi_A^* = \pi_B^* = \frac{m\sqrt{2FLT}}{4(\sqrt{8FLT}-m)} > 0, \quad \text{since} \quad m < \sqrt{2FLT}. \tag{1.39}$$

The second-order conditions when m is small are

$$\left.\begin{bmatrix} \frac{\partial^2 \pi_i}{\partial p_i^2} & \frac{\partial^2 \pi_i}{\partial p_i \partial n_i} \\ \frac{\partial^2 \pi_i}{\partial p_i \partial n_i} & \frac{\partial^2 \pi_i}{\partial n_i^2} \end{bmatrix}\right|_{\substack{n_A=n_B=n_{m_S}^* \\ p_A=p_B=p_{m_S}^*}} = \begin{bmatrix} \frac{m-2\beta}{\beta^2} & \frac{2m\beta-m^2-2\beta^2}{2LT(m-2\beta)} \\ \frac{2m\beta-m^2-2\beta^2}{2LT(m-2\beta)} & \frac{8F^2}{m-2\beta} \end{bmatrix}.$$

Hence, we have a maximum since the matrix is negative definite.

Case 2 (large m). If $m > \sqrt{2FLT}$, then in equilibrium some consumers who realize location shocks $\{d_j, d_{-j}\} = \{\frac{L}{2n_j}, 0\}$, $j \in \{A, B\}$ still purchase from firm j. In this case the market shares and profits of the two firms are given by

$$
\begin{aligned}
Y_A &= \frac{2n_A n_B}{mL^2}\int_0^{\frac{L}{2n_A}}\int_0^{\frac{L}{2n_B}}\int_{T(d_A-d_B)+p_A-p_B}^{m} dz dd_B dd_A \\
&= \frac{1}{2}+\frac{p_B-p_A}{2m}+\frac{LT}{m}\left(\frac{1}{8n_B}-\frac{1}{8n_A}\right) \\
Y_B &= \frac{2n_A n_B}{mL^2}\int_0^{\frac{L}{2n_A}}\int_0^{\frac{L}{2n_B}}\int_{-m}^{T(d_A-d_B)+p_A-p_B} dz dd_B dd_A \\
&= \frac{1}{2}+\frac{p_A-p_B}{2m}+\frac{LT}{m}\left(\frac{1}{8n_A}-\frac{1}{8n_B}\right)
\end{aligned}
$$

$$
\pi_A(S_A,S_B) = \left(\frac{1}{2}+\frac{p_B-p_A}{2m}+\frac{LT}{m}\left[\frac{1}{8n_B}-\frac{1}{8n_A}\right]\right)p_A - n_A F
$$

$$
\pi_B(S_A,S_B) = \left(\frac{1}{2}+\frac{p_A-p_B}{2m}+\frac{LT}{m}\left[\frac{1}{8n_A}-\frac{1}{8n_B}\right]\right)p_B - n_B F.
$$

Differentiating firm profit and applying symmetry yields the equilibrium outcomes:

$$
p_A^* = p_B^* = p_{m_L}^* = m \quad \text{and} \quad n_A^* = n_B^* = n_{m_L}^* = \frac{1}{2}\sqrt{\frac{LT}{2F}}, \quad \text{so that}
$$

$$
\pi_A^* = \pi_B^* = \frac{1}{2}\left(m - \sqrt{\frac{FLT}{2}}\right) > 0. \tag{1.40}
$$

The second-order conditions are

$$
\left.\begin{bmatrix} \frac{\partial^2 \pi_i}{\partial p_i^2} & \frac{\partial^2 \pi_i}{\partial p_i \partial n_i} \\ \frac{\partial^2 \pi_i}{\partial p_i \partial n_i} & \frac{\partial^2 \pi_i}{\partial n_i^2} \end{bmatrix}\right|_{\substack{n_A=n_B=n_{m_L}^* \\ p_A=p_B=p_{m_L}^*}} = \begin{bmatrix} -\frac{1}{m} & \frac{F}{m} \\ \frac{F}{m} & -4\sqrt{\frac{2F^3}{LT}} \end{bmatrix}.
$$

Hence, we have a maximum since the matrix is negative definite. ■

Proof of Proposition 7: In an environment with heterogeneity only in consumer tastes, firm i captures $(m - p_i + p_j)/(2m)$ of the market. Firm i then maximizes

$$
\max_{p_i} p_i \frac{(p_j - p_i + m)}{2m}.
$$

Differentiating with respect to p_i and imposing symmetry yields $p_{NT}^* = m$ and profit $\pi_{NT}^* = m/2$. This profit exceed profits when there is both taste and spatial consumer heterogeneity, given by equation (1.40) when $m \leq \sqrt{2FLT}$, and by equation (1.39) when $m > \sqrt{2FLT}$. ■

Proof of Proposition 8: First we assume that ρ, F_1 and F_2 are such that $Y_{12} > 0$. There are four possible cases:

1. $p_1 \leq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} \leq 0$.

2. $p_1 \geq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} \leq 0$.

3. $p_1 \leq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} > 0$.

4. $p_1 \geq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} > 0$.

We will rule out the latter two possibilities (which would imply that even though $F_1 > F_2$, firms make zero profits from circle 2, but positive profits from circle 1). We present case 1 in detail; analyses of the other three cases are similar.

Case 1: In the neighborhood of an equilibrium with $p_{A_i} \geq p_{B_i}$, $i = 1, 2$, we have

$$\begin{aligned}
\pi_B = & p_{B_1}\Bigg(\rho\Bigg(1 - \frac{4n_{A_1}n_{B_1}}{L^2}\int_0^{\frac{L}{2n_{A_1}}}\int_{d_{A_1}+\frac{p_{A_1}-p_{B_1}}{T}}^{\frac{L}{2n_{B_1}}} dd_{B_1}\, dd_{A_1}\Bigg) \\
+ & \qquad (1-\rho)\Bigg(1 - \frac{4n_{A_2}n_{B_1}}{L^2}\int_0^{\frac{L}{2n_{A_2}}}\int_{d_{A_2}+\frac{p_{A_2}-p_{B_1}}{T}}^{\frac{L}{2n_{B_1}}} dd_{B_1}\, dd_{A_2}\Bigg)\Bigg) \\
& + p_{B_2}\Bigg((1-\rho)\frac{4n_{A_1}n_{B_2}}{L^2}\int_0^{\frac{L}{2n_{B_2}}}\int_{d_{B_2}+\frac{p_{B_2}-p_{A_1}}{T}}^{\frac{L}{2n_{A_1}}} dd_{A_1}\, dd_{B_2} \\
& + \rho\left(1 - \frac{4n_{A_2}n_{B_2}}{L^2}\int_0^{\frac{L}{2n_{A_2}}}\int_{d_{A_2}+\frac{p_{A_2}-p_{B_2}}{T}}^{\frac{L}{2n_{B_2}}} dd_{B_2}\, dd_{A_2}\right)\Bigg) - (F_1 n_{B_1} + F_2 n_{B_2})\,.
\end{aligned}$$

The first-order conditions for firm B are

$$\begin{aligned}
\frac{\partial \pi_B}{\partial p_{B_1}} &= \rho\left(Y_{B_{11}} - \frac{2p_{B_1}n_{B_1}}{LT}\right) + (1-\rho)\,\left(Y_{B_{12}} - \frac{2p_{B_1}n_{B_1}}{LT}\right) = 0 \\
\frac{\partial \pi_B}{\partial p_{B_2}} &= \rho\left(Y_{B_{22}} - \frac{2p_{B_2}n_{B_2}}{LT}\right) + (1-\rho)\left(Y_{B_{21}} - \frac{2p_{B_2}n_{A_1}}{LT}\right) = 0 \\
\frac{\partial \pi_B}{\partial n_{B_1}} &= \rho Y_{B_{11}}\frac{p_{B_1}}{n_{B_1}} + (1-\rho)Y_{B_{12}}\frac{p_{B_1}}{n_{B_1}} - F_1 = 0 \\
\frac{\partial \pi_B}{\partial n_{B_2}} &= \frac{(1-\rho)n_{A_1}p_{B_2}}{2n_{B_2}^2} + \rho Y_{B_{22}}\frac{p_{B_2}}{n_{B_2}} - F_2 = 0.
\end{aligned}$$

where we have substituted $\frac{n_{A_1}}{2n_{B_2}} = Y_{B_{21}} + \frac{\left(\frac{p_{B_2}-p_{A_1}}{T}\right)+\frac{L}{2n_{B_2}}-\frac{L}{2n_{A_1}}}{\frac{L}{2n_{A_1}}}$ in $\frac{\partial \pi_B}{\partial n_{B_2}}$. Imposing symmetry, substituting

$Y_{jj} = \frac{1}{2}$ and rearranging the first-order conditions yields:

$$0 = \frac{\rho}{2} + (1-\rho)Y_{12} - \frac{2p_1 n_1}{LT} \tag{1.41}$$

$$0 = \frac{\rho}{2} + (1-\rho)Y_{21} - \frac{2p_2}{LT}\left(\rho n_2 + (1-\rho)n_1\right) \tag{1.42}$$

$$0 = p_1\left(\frac{\rho}{2} + (1-\rho)Y_{12}\right) - F_1 n_1 \tag{1.43}$$

$$0 = p_2\left((1-\rho)\frac{n_1}{2n_2} + \frac{\rho}{2}\right) - F_2 n_2. \tag{1.44}$$

We see immediately from equation (1.43) that firms earn zero profits on circle 1, and from equation (1.44) that firms earn strictly positive profits on circle 2 when $(\frac{p_2 - p_1}{T}) + \frac{L}{2n_2} - \frac{L}{2n_1} < 0$ (which implies that $\frac{n_1}{2n_2} < Y_{21}$). Using equation (1.43) to substitute for

$$\frac{\rho}{2} + (1-\rho)Y_{12} = \frac{F_1 n_1}{p_1}$$

into (1.41) and solving for p_1 yields

$$p_1 = \sqrt{\frac{LTF_1}{2}}.$$

Also, n_1 and n_2 solve

$$\frac{2\sqrt{2}\sqrt{LTF_1}n_1 - LT\rho}{(1-\rho)} = \frac{n_1}{n_2}\left(LT + 4n_2\left(\frac{4F_2 n_2^2}{2[(1-\rho)n_1 + \rho n_2]} - \frac{\sqrt{LTF_1}}{\sqrt{2}}\right)\right) \tag{1.45}$$

$$LT - \sqrt{2}\sqrt{LTF_1}n_1 = 4F_2 n_2^2. \tag{1.46}$$

Solving equation (1.46) for n_1 and substituting into equation (1.45) yields

$$\frac{2LT(1-\frac{\rho}{2}) - 8F_2 n_2^2}{(1-\rho)} = \frac{\left(LT - 4F_2 n_2^2\right)}{\sqrt{2}\sqrt{(LTF_1)}n_2}\left(LT + 4n_2\left(\frac{4F_2 n_2^2}{2\rho n_2 + \frac{\sqrt{2}(1-\rho)\left(LT - 4F_2 n_2^2\right)}{\sqrt{LTF_1}}} - \frac{\sqrt{LTF_1}}{\sqrt{2}}\right)\right),$$

which can be reduced to the following fifth-degree polynomial equation:

$$64F_2^2 F_1(1-\rho)(3-\rho)n_2^5 + 16F_2\sqrt{2}\sqrt{F_1 LT}((F_2 - F_1)(2-\rho)\rho - F_2)n_2^4 - 16F_2 F_1 LT(1-\rho)(5-3\rho)n_2^3$$
$$+ 2LT\sqrt{2}\sqrt{F_1 LT}(4F_2(1-\rho)^2 + \rho F_1(4-3\rho))n_2^2 + 8L^2T^2F_1(1-\rho)^2 n_2 - L^2T^2\sqrt{2}\sqrt{F_1 LT}(1-\rho)^2 = 0.$$

Analogously, one can solve for the first-order conditions for the other three cases:

Case 2. $p_1 \geq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} \leq 0$

$$\frac{\rho}{2} + (1-\rho)Y_{12} = \frac{2p_1}{LT}\left(\rho n_1 + (1-\rho)n_2 \frac{(\frac{L}{2n_2} - \frac{p_1-p_2}{T})}{\frac{L}{2n_1}}\right) \tag{1.47}$$

$$\frac{\rho}{2} + (1-\rho)Y_{21} = \frac{2p_2 n_2}{LT}\left(\rho + (1-\rho)\frac{(\frac{L}{2n_2} - \frac{p_1-p_2}{T})}{\frac{L}{2n_1}}\right) \tag{1.48}$$

$$\frac{\rho p_1}{2} + (1-\rho)p_1 Y_{12} - F_1 n_1 = 0 \tag{1.49}$$

$$\frac{\rho p_2}{2} + (1-\rho)p_2 Y_{21} - F_2 n_2 = (1-\rho)p_2 \frac{(\frac{L}{2n_1} - (\frac{L}{2n_2} - \frac{p_1-p_2}{T}))}{\frac{L}{2n_1}}. \tag{1.50}$$

Case 3. $p_1 \leq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} \geq 0$.

$$\frac{\rho}{2} + (1-\rho)Y_{12} = \frac{2p_1 n_1}{LT}\left(\rho + (1-\rho)\frac{(\frac{L}{2n_1} - \frac{p_2-p_1}{T})}{\frac{L}{2n_2}}\right) \tag{1.51}$$

$$\frac{\rho}{2} + (1-\rho)Y_{21} = \frac{2p_2}{LT}\left(\rho n_2 + (1-\rho)n_1 \frac{(\frac{L}{2n_1} - \frac{p_2-p_1}{T})}{\frac{L}{2n_2}}\right) \tag{1.52}$$

$$\frac{\rho p_1}{2} + (1-\rho)p_1 Y_{12} - F_1 n_1 = (1-\rho)p_1 \frac{(\frac{L}{2n_2} - (\frac{L}{2n_1} - \frac{p_2-p_1}{T}t))}{\frac{L}{2n_2}} \tag{1.53}$$

$$\frac{\rho p_2}{2} + (1-\rho)p_2 Y_{21} - F_2 n_2 = 0. \tag{1.54}$$

Case 4. $p_1 \geq p_2$ and $(\frac{L}{2n_2} - \frac{p_1-p_2}{T}) - \frac{L}{2n_1} \geq 0$.

$$\frac{\rho}{2} + (1-\rho)Y_{12} = \frac{2p_1}{LT}\left(\rho n_1 + (1-\rho)n_2\right) \tag{1.55}$$

$$\frac{\rho}{2} + (1-\rho)Y_{21} = \frac{2p_2 n_2}{LT} \tag{1.56}$$

$$\frac{\rho p_1}{2} + (1-\rho)p_1 Y_{12} - F_1 n_1 = (1-\rho)p_1 \frac{\left(\left(\frac{L}{2n_2} - \frac{(p_1-p_2)}{T}\right) - \frac{L}{2n_1}\right)}{\frac{L}{2n_2}} \tag{1.57}$$

$$\frac{\rho p_2}{2} + (1-\rho)p_2 Y_{21} - F_2 n_2 = 0. \tag{1.58}$$

Cases 1 and 4 are "symmetric", as are Cases 2 and 3: relabeling p_1 as p_2, p_2 as p_1, n_1 as n_2 and n_2 as n_1 in Case 1, the sets of first-order conditions in Case 4 correspond to those in 1.

To rule out cases 3 and 4, we first characterize p_1 relative to p_2. We know that $p_1 > p_2$ at $\rho = 1$. As $p_1 = \sqrt{\frac{F_1 LT}{2}}$ for $p_1 \geq p_2$, the critical ρ that determines which price is higher sets $p_1 = p_2 = \sqrt{\frac{F_1 LT}{2}}$. At $p_1 = p_2$, the first order conditions (1.49) and (1.50) simplify to:

$$\frac{\rho p_1}{2} + (1-\rho)\frac{n_1}{2n_2}p_1 - F_1 n_1 = 0 \tag{1.59}$$

$$(1-\rho)p_1\frac{n_1}{2n_2}+\frac{\rho p_1}{2}-F_2n_2=0. \tag{1.60}$$

Therefore, at $p_1 = p_2$, we have $F_1n_1 = F_2n_2 \Leftrightarrow \frac{F_1}{F_2} = \frac{n_2}{n_1}$.

In addition, the first-order conditions (1.47) and (1.48) are,

$$\frac{\rho}{2}+(1-\rho)\frac{n_1}{2n_2}=p_1\frac{2n_1}{LT} \tag{1.61}$$

$$\frac{\rho}{2}+(1-\rho)\left(1-\frac{n_1}{2n_2}\right)=\frac{2p_1}{LT}(\rho n_2+(1-\rho)n_1), \tag{1.62}$$

where $p_1 = p_2 = \sqrt{\frac{F_1LT}{2}}$. Substituting $n_2 = n_1\frac{F_1}{F_2}$ and solving (1.61) for n_1 yields

$$n_1=\frac{\sqrt{LT}\,(\rho F_1+F_2-\rho F_2)}{2\sqrt{2}F_1^{3/2}}, \tag{1.63}$$

and solving (1.62) for n_1 yields

$$n_1=\frac{\sqrt{LT}F_2\,(2F_1-\rho F_1-F_2+\rho F_2)}{2\sqrt{2}F_1^{3/2}\,(\rho F_1+F_2-\rho F_2)}. \tag{1.64}$$

Equating these two solutions, we solve for

$$\rho^*=\frac{\sqrt{F_2}\sqrt{8F_1+F_2}-3F_2}{2\,(F_1-F_2)}. \tag{1.65}$$

Differentiation establishes that ρ^* is decreasing in F_1, and an application of L'Hospital's rule shows that $\rho^* \to \frac{2}{3}$ as $F_2 \to F_1$.

We are now in a position to rule out Cases 3 and 4. First note that if $F_1 > F_2$, then when $\rho = 0$ or $\rho = 1$, $(\frac{L}{2n_1} - \frac{p_2-p_1}{T}) - \frac{L}{2n_2} > 0$. Further, $(\frac{L}{2n_1} - \frac{p_2-p_1}{T}) - \frac{L}{2n_2}$ is continuous in ρ, so suppose there were a ρ such that $(\frac{L}{2n_1} - \frac{p_2-p_1}{T}) - \frac{L}{2n_2} = 0$. If $p_1 \leq p_2$, as in Cases 1 and 3 then $n_2 \geq n_1$ and $Y_{12} \geq Y_{21} = \frac{n_1}{2n_2}$. From the first-order equations (1.51) and (1.52) for Case 3,

$$\frac{2p_1n_1}{LT}\geq\frac{1}{2}\geq\frac{2p_2}{LT}\,(\rho n_2+(1-\rho)n_1)\geq\frac{2p_2n_1}{LT}.$$

But this implies that $p_1 = p_2$. By the assumption that $(\frac{L}{2n_1} - \frac{p_2-p_1}{T}) - \frac{L}{2n_2} = 0$, we have $n_1 = n_2$, thus $F_1 = F_2$ by $\frac{F_1}{F_2} = \frac{n_2}{n_1}$ at $p_1 = p_2$, a contradiction. The analysis for Case 4 is similar. If $p_2 \leq p_1$, as in Cases 2 and 4 then $n_1 \geq n_2$

and $Y_{21} \geq Y_{12}$. From the Case 4 first order equations (1.55) and (1.56),

$$\frac{2p_2n_2}{LT} \geq \frac{1}{2} \geq \frac{2p_1}{LT}\left(\rho n_1 + (1-\rho)n_2\right) \geq \frac{2p_1n_2}{LT}.$$

Again this implies that $p_1 = p_2$, and a contradiction obtains as above. Thus, we have that equilibrium is characterized by Cases 1 and 2.

To show that $n_1 < n_2$, we show that $n_1 \geq n_2$ implies a contradiction in Case 2. Under Case 2, $p_1 \geq p_2$ and $Y_{21} \geq Y_{12}$. If $n_1 \geq n_2$ then from equations (1.47) and (1.48)

$$\frac{2p_2n_2}{LT}(\rho + (1-\rho)\lambda) \geq \frac{1}{2} \geq \frac{2p_1}{LT}(\rho n_1 + (1-\rho)n_2\lambda) \geq \frac{2p_1n_2}{LT}(\rho + (1-\rho)\lambda),$$

where

$$\lambda = \frac{2n_1}{L}\left(\frac{L}{2n_2} - \frac{(p_1-p_2)}{T}\right).$$

Again this implies that $p_1 = p_2$. Thus, we have $n_1 > n_2$ and $F_1 > F_2$ at $p_1 = p_2$, which contradicts our finding that $\frac{F_1}{F_2} = \frac{n_2}{n_1}$ at $p_1 = p_2$.

From (1.50), firm profit in Case 2 is

$$\begin{aligned}
\pi_2 &= (1-\rho)p_2 \frac{\left(\frac{L}{2n_1} - \left(\frac{L}{2n_2} - \frac{(p_1-p_2)}{T}\right)\right)}{\frac{L}{2n_1}} \\
&= (1-\rho)p_2\left(1 - \frac{n_1}{n_2} + \frac{2n_1(p_1-p_2)}{LT}\right) \\
&= (1-\rho)p_2\left(1 - \frac{n_1}{2n_2}\left(1 - \frac{4n_2(p_1-p_2)}{LT} + \frac{4n_2^2(p_1-p_2)^2}{L^2T^2}\right) - \frac{n_1}{2n_2} + \frac{2n_1n_2(p_1-p_2)^2}{L^2T^2}\right) \\
&= (1-\rho)p_2\left(Y_{21} - \frac{n_1}{2n_2} + \frac{2n_1n_2(p_1-p_2)^2}{L^2T^2}\right) \\
&> (1-\rho)p_2\left(Y_{21} - \frac{n_1}{2n_2}\right),
\end{aligned}$$

where the third equality follows from rearranging, which allows us to write it in the form of the fourth equality, as in equation (1.6) (where 1 replaces A and 2 replaces B).

Finally, we characterize equilibrium when $Y_{12} = 0$ and $Y_{21} = 1$. This is a special case of Case 2. From the first-order conditions, one can solve explicitly for the equilibrium values:

$$n_1 = \sqrt{\frac{\rho LT}{8F_1}}, p_1 = \sqrt{\frac{F_1LT}{2\rho}}, n_2 = \sqrt{\frac{(2-\rho)LT}{8F_2}}, p_2 = \sqrt{\frac{(2-\rho)F_2LT}{2\rho^2}} \text{ and}$$

$$\pi_2 = (1-\rho)\sqrt{\frac{(2-\rho)F_2LT}{2\rho^2}}.$$

Equilibrium is characterized by $Y_{12} = 0$ if $F_1 > 8F_2$ and

$$\rho \in \left[\frac{F_1 - 2F_2 - \sqrt{F_1^2 - 8F_1F_2}}{F_1 + F_2}, \frac{F_1 - 2F_2 + \sqrt{F_1^2 - 8F_1F_2}}{F_1 + F_2}\right].$$

Note that all results of Proposition 8 hold: $n_2 > n_1$, $p_1 > p_2$ for $\rho > \rho^*$ and $\pi_2 = (1-\rho)p_2$. ■

Chapter 2

Do-Nothing Extremists and Protection against the Tyranny of an Unrepresentative Majority

2.1 Introduction

The effect of [a representative democracy is] to refine and enlarge the public views, by passing them through the medium of a chosen body of citizens, whose wisdom may best discern the true interest of the nation....

James Madison

Society's tastes for government policy vary as society itself changes. More often than not, the legislature comprised of elected representatives takes the role of ensuring that changes in government policy reflect these changes in taste. How easily this legislature can adapt policy to reflect tastes depends on the rules that govern the legislative process.

An unencumbered legislative process will ensure that policy closely reflects the preferences of the legislative body. This will benefit society if the legislators' preferences mirror those of society. However, for reasons such as party affiliation, incumbency advantage, special interest group pressure, gerrymandered districts, imperfect voter information, time between elections, and so on, an elected legislature may not well-represent society. In such instances, a degree of inertia or legislative rigidity, as encapsulated in the proportion of legislative votes required to pass a proposal, may protect society from a radical legislature's desire to implement 'bad' policy.

Examples of supermajority rules in practice are commonplace. For example, the United States Constitution establishes supermajority rules for a range of decisions such as overriding vetoes, the United States Senate requires a three-fifths majority to end a filibuster, and many states require supermajorities to raise taxes, pass spending bills, or pass legislation that restricts local communities' regulatory powers.

We develop a simple model of the legislative process to address the optimal tradeoff between flexibility and protection embedded in voting rules. Our model has three basic building blocks. First, society has quadratic preferences over policy outcomes, and society's preferred policy e may evolve from an established status quo. Second, while the preferred policy m of the median legislator may reflect society's new preferred policy *in expectation*, realizations of preferred policies may differ. Third, there is a (possibly probabilistic) rule that selects a member of the legislature, p, who gets to propose a policy. This policy is adopted if and only if it wins approval from a sufficient proportion of the legislature in a vote against a status quo alternative. The proposer could, but need not always be, the median member of the legislature. In this setting, we characterize how the primitives of our economy affect the optimal degree of inertia in the legislature, as captured by the proportion of votes necessary for a proposal to defeat the status quo.

Our model allows us to address fundamental questions. When is a simple majority optimal? How does increasing the likelihood of 'irresponsible' legislatures—legislatures with a median whose preferred policy is far from society's—affect the optimal supermajority? How will selecting a proposer whose interests are further from the median legislator's, and hence is less representative of society, affect the optimal supermajority? What is the impact of initial policy bias?

When choosing a policy, a proposer faces one of three scenarios. First, a proposer can be free from legislative constraint: he can propose his most preferred policy and win approval from enough legislators to defeat the status quo. Second, he can be completely blocked from implementing a policy change that he prefers: there may be no policy that the proposer both prefers to the status quo and would receive enough votes from legislators. In these two scenarios where a proposer is free or blocked, marginal changes in the voting rule have no effect on policy outcomes.

In the third scenario, the proposer is constrained: he can only instigate a partial movement in policy toward his bliss point. When the median legislator is conservative, i.e., when he lies close to the status quo, so do most legislators, making it difficult or impossible for a proposer to identify a policy that enough legislators prefer to the status quo: the voting rule severely constrains policy movement so that only small policy changes are possible, and the resulting policy outcome is closer to the status quo than the representative citizen typically prefers. However, when, instead, the median legislator is radical, i.e., when he lies far from the status quo, so do most legislators. As a result, the voting rule lightly constrains policy movement, both for good and for bad. We establish a simple characterization of the optimal voting rule whenever a supermajority is optimal: the optimal voting rule is such that conditional on the proposer being constrained, the expected policy outcome equals the expected bliss point of the representative/median citizen. Quite generally, neither submajority nor unanimity voting rules are ever optimal: submajorities facilitate unrepresentative shifts in policy from the status quo, leading to excessive policy movement, while unanimity eliminates all movement in policy, whereas a lesser, but still large supermajority would approve only policy changes that the representative citizen strictly prefers.

We first focus on the case where the median legislator is always the proposer. The median is never constrained by simple majority rule. Hence, simple majorities increase the representative citizen's utility if the distance between his bliss point, e and that of the median legislator m is less than that between e and the status quo. It follows directly that *if* the legislature is sufficiently representative of society *and* the median legislator proposes policy, then it is not optimal to constrain the legislature via a supermajority. In particular, if preferences in society are sufficiently variable relative to the likely representativeness of the legislature, then simple majority is optimal.

But how unrepresentative must the legislature be before it is optimal to constrain the median proposer? We consider three broad classes of distributions over bliss points of the representative citizen and the median legislator: two-point, uniform and normal. In all three classes, a supermajority becomes optimal even when legislative preferences are less volatile than society's. Moreover, when the volatility in the preferences of the median legislator around the representative citizen's preferred policy is sufficiently high vis à vis the volatility in the representative citizen's preferred policy, the optimal supermajority rises with the volatility of the legislature's preferences and falls with the volatility of society's preferences. The first-order intuition is that when the median legislator's preferences are more dispersed around the representative citizen's preferred policy, the median legislator is more likely to want to implement a "worse" policy. As a result, the social planner wants to constrain the median proposer further.

Importantly, slight supermajorities may never be optimal: for both two-point and uniform distributions, the optimal voting rule is a discontinuous function of the underlying parameters moving from majority to supermajority. This discontinuity reflects that supermajority voting rules are blunt instruments, restricting for good when the proposer is radical and for bad when he is conservative. Quite generally, when the median legislator is close to the status quo, supermajorities tend to constrain him excessively, and when he is far from the status quo, they tend to constrain him too little. Slight supermajorities always tend to restrict the less radical proposers disproportionately by more. Thus, the social planner's consideration becomes (i) should he leave the median legislator unchecked via a simple majority voting rule; or (ii) is the median legislator likely to be sufficiently unrepresentative that he should choose a large supermajority, forsaking the gains that a conservative proposer can achieve in order to restrain extreme proposers via large supermajorities? That is, slight supermajorities incur the costs of restricting conservative proposers significantly, while providing limited beneficial restraint on extreme proposers.

One's first-order intuition might suggest that greater volatility in the representative citizen's preferred policy should always reduce the optimal supermajority, as the status quo is likely to be further from his preferred policy, making change more attractive. We show constructively, using a two-point distribution of e and m, that this need not be so. A shift of e away from the status quo by one unit also shifts the distribution of m away from the status quo by one unit, so the legislature has the freedom to change policy by up to *two* units. This increases the ratio of (a) constrained proposers who move policy too far to (b) constrained proposers who move policy insufficiently; and can make greater supermajorities optimal.

The conjecture that more volatile legislative preferences, which lead to more irresponsible median legislators vis à vis society, should always increase the optimal supermajority is also false. A mean preserving spread of the distribution of the median legislator's preferences can be created that (a) reduces the measure of blocked proposers (for whom the size of the sufficient supermajority is irrelevant), and (b) increases the measure of heavily constrained proposers, who, on average, generate insufficient movement in policy, but (c) leaves unaltered the measure of most radical proposers, whom the social planner wants to constrain. Changing the measure of blocked proposers has no direct effect on the optimal voting rule. However, increasing the measure of heavily-constrained proposers, implies insufficient average movement in policy. Thus, the optimal voting rule falls.

We then consider how the selection of the policy proposer affects the optimal voting rule. To focus on the "representativeness" of a proposer, we consider proposers who are equally likely to lie distance u_p to the left or right of the median legislator. An increase in this 'polarity' distance represents a more extreme proposer. One's first-order intuition might be that with more extreme proposers, greater supermajorities should be optimal because proposers who are more extreme *within* the legislature also tend to be further from the representative citizen. Indeed, when the legislature is highly polarized, this intuition is correct. A more extreme proposer who is constrained in his change of legislative outcome indicates a median legislator and representative citizen whose bliss points lie closer to the status quo. Increasing the supermajority reduces the policy movement away from the status quo and hence typically the representative citizen.

But what is the effect of more extreme proposers on the size of the optimal voting rule when the legislature is sufficiently unrepresentative that a supermajority is optimal, but the polarity of the proposer is is only modest? When does making the proposer *less* representative of the legislature (i.e., increasing u_p) raise the optimal supermajority, and when does it reduce it?

We establish that slight polarity *reduces* the optimal supermajority whenever the median legislator's preferences tend not to mirror the representative citizen's. However, the relationship between polarity and the optimal supermajority is U-shaped. The intuition is that, on average, more extreme proposers like more extreme policies. However, *more extreme proposers are also more constrained than the median legislator by the necessity of winning approval from more moderate representatives.* As the dispersion of the median legislator around the representative citizen's bliss point rises, conditional on legislative success, it is more likely that a more extreme proposer shifts policy in the direction that society prefers. In contrast, a greater supermajority is required to protect against a rogue median proposer, who would otherwise be free to implement his preferred policy. When uncertainty is normally or uniformly distributed, we provide necessary and sufficient conditions on primitives for *greater* polarity to *reduce* the optimal voting rule.

We then study how initial policy bias affects the optimal voting rule. Allowing the initial policy to differ from the representative citizen's expected bliss point, allows us to capture aspects of legislative policy/voting dynamics. In particular, we can glean insights into how the optimal voting rule changes when past movement of policy is slow to catch up with changes in society's preferences. For example, the status quo on health care may be too conservative. A slightly unrepresentative conservative congress would be unlikely to move policy; but an unrepresentative radical congress would be able to convert the threat of bad conservative policy into equally bad (or worse) radical policy. Reducing the voting rule reduces the costs associated with the first congress, but raises the costs associated with the second. Intuition might suggest that since initial policy bias makes the status quo less representative, on average, of societal preferences, the social planner should rely more on the legislature to determine policy by reducing the size of the supermajority. When e and m are normally distributed, we prove that this is only true when the volatility in m is sufficiently high. In particular, when the dispersion of the median legislator around society's preferred policy is sufficiently small relative to the variation in society's preferred policy, then introducing slight initial policy bias *raises* the optimal voting rule. The intuition is that with slight policy bias, the proposers who are more likely to be constrained by a given supermajority are those who would move the policy in the direction *away* from what the representative citizen prefers.

Related Literature. The benefits and costs of delegating authority have been well studied. A number of studies have shown the efficacy of supermajority rules in various settings. In the earliest study of optimal voting rules, Caplin and Nalebuff [1988] show that a supermajority of 64 percent suffices to rule out Condorcet cycles when preferences are sufficiently homogeneous.

Papers with results that have flavors related to ours include Klumpp [2005], Aghion et al. [2004] and Compte and Jehiel [2010b]. Klumpp [2005] shows that in a model of indirect democracy, constituents may, in equilibrium, select a representative with preferences closer to the status quo. In his model of the legislative process a proposer is randomly selected. Electing a more conservative representative can moderate legislative outcomes, improving voter welfare. This work is complementary to ours since in our model the optimal voting rule ensures that, on average, more-moderate proposers (i.e., closer to society's ideal) determine the change in policy.

In Aghion et al. [2004], the optimal rule trades off the possibility that an unethical politician will expropriate funds, with the possibility that a minority of people will block a socially beneficial reform. They show that the optimal amount of insulation depends on the size of the aggregate improvement from reform, the aggregate and idiosyncratic uncertainties over reform outcomes, the degree of polarization of society, the individual degree of risk aversion, the availability and efficiency of fiscal transfers, and the degree of protection of property rights against expropriation. In contrast, in our model, the motivation of elected politicians is pure. Uncertainty is over the politicians' preferred policy point rather than their integrity.

Compte and Jehiel [2010b] show that in a model of collective search, unanimity is undesirable in large committees with sufficiently patient members. Generally speaking, unanimity makes it difficult to find a proposal that is acceptable to all, thereby inducing costly delay. The optimal majority rule is the one that solves best the trade-off between speeding up the decision-making process and avoiding the risk of adopting relatively inefficient proposals. In our model the optimal voting rule solves best the trade-off between a flexible legislative process that can respond to changes in society's tastes and a conservative legislative process that can guard against radical legislators.

Another strand of research examines optimal voting rules as a way to aggregate information when information is dispersed throughout the electorate. Young [1995] is a good review. This role for voting is absent in our model since all agents are endowed with the same information.

Several papers study voting rules in settings where issues of dynamic consistency can arise, and supermajorities can commit future governments to behave appropriately (see e.g., Gradstein [1999], Messner and Polborn [2004], Dal Bo [2006], Duggan and Kalandrakis [2007] and Acemoglu et al. [2008]). Our model has no dynamic consistency problems. Supermajorities arise not from concern about the damage a future government might do, but from uncertainty about what the current government might do.

Section 2.2 presents our model of the legislative process. In Section 2.3 we characterize how the nature of uncertainty over society's preferences and those of the median legislator affect optimal voting rules when the median legislator is the proposer. Section 2.4 characterizes optimal voting rules when the proposer can be less representative than the median of the legislature's preferences. Section 2.5 explores how the relative position of initial policy to society's tastes affects the optimal majority. Section 2.6 concludes. All proofs are in the Section 2.7.

2.2 The Model

Government policy, S, is defined over the real line. There is an initial government policy, S_0, normalized to zero, which we refer to as the status quo. A legislative process that we describe shortly, generates a new policy S_1. We consider a representative citizen with policy bliss point e who derives utility $U = -\mathrm{E}(S_1 - e)^2$ from the policy S_1 that is implemented. We will consider a setting that induces symmetric uncertainty over policy outcomes, in which case one can interpret the representative citizen as the median voter.[1]

To capture evolving societal tastes for government policy (e.g., due to changing economic conditions) we assume that the representative citizen's preferred policy e is distributed according to some non-trivial distribution F_e with an associated density f_e that is symmetric around zero. For example, in times of high unemployment society may prefer increased government spending, or in times of war, society may prefer reduced personal freedoms. Unfortunately for citizens, policy is determined by a legislature whose preferences may not perfectly mirror society's. This non-alignment of interests may reflect the limited information voters have about legislator preferences, the institutional design of heterogeneous districts, the impact of party affiliation, incumbency advantage, special interest group pressure, and so on. To capture this, we assume that the policy bliss point of the median legislator is given by $m = e + \mu_m$, where $\mu_m \sim F_m$ with an associated density f_m that is symmetric around e. That is, the median legislator's preferences only correspond to those of the representative citizen's *on average*. The distribution of bliss points in the legislature is described by F_l with an associated density f_l that is symmetric around m.

[1]With quadratic preferences, the ex-ante preferences of a voter whose bliss point $e + \delta$ is always δ from the median voter equal those of the median minus the constant δ^2. Hence, maximizing the median voter' ex-ante welfare also maximizes the welfare of all voters (see e.g., Bernhardt et al. [2009] and Bernhardt et al. [2011] for similar results).

Next, a representative is selected to propose a new government policy k on which the legislature will vote. The proposer's policy bliss point is given by $p = m + \mu_p$, where $\mu_p \sim F_p$. We assume symmetry, $F_p(-\mu_p) = 1 - F_p(\mu_p)$, for $\mu_p > 0$. Much of our analysis focuses on two particular distributions. We focus largely on the possibility that the median legislator is always the proposer, in which case μ_p is the degenerate random variable equal to zero. This possibility allows us to shed light on the trade-offs a social planner faces when he gives the legislature more freedom. We also consider the possibility that $\mu_p \in \{-u_p, u_p\}$. The parameter u_p, i.e., the proposer's distance from the median legislator, captures the effect of increased polarity of a proposer in a legislature in a transparent and analytically tractable way.

Finally, legislators simultaneously vote on whether or not to replace current government policy (the status quo) with the proposed policy k. A legislator is completely characterized by his policy bliss point, and so we refer to a legislator with policy bliss point b as legislator b. The objective of each legislator, including the proposer, is to minimize the distance between his bliss point and the policy that the legislature adopts. To defeat the status quo and be adopted, a proposal must garner the required proportion, α, of votes from the legislature. After the vote is taken, policy is implemented and payoffs are realized.

The social planner chooses the voting rule α (the proportion of votes required to change policy) that maximizes the representative citizen's ex-ante welfare. When deciding on the appropriate voting rule the social planner takes into consideration the incentives facing a proposer, who, in turn, must consider the incentives of his legislative colleagues.

Equilibrium. An equilibrium is a voting rule $\alpha \in [0, 1]$, a policy proposal function $k(p, m, \alpha, S_0) \to R$, and a voting rule for legislators $v(l, k, S_0, \alpha) \to \{0, 1\}$ such that:

- the voting rule α maximizes the representative citizen's ex-ante expected utility given the optimal policy choice $k(p, m, \alpha, S_0)$ by each proposer p, and legislator voting rules, $v(l, k, S_0, \alpha)$;
- for each proposer p, $k(p, m, \alpha, S_0)$ minimizes $|S_1 - p|$ given the voting rule α, the status quo S_0, the position of the median legislator m, and legislator voting rules $v(l, k, S_0, \alpha)$;
- each legislator l votes for the proposal k, $v(l, k, S_0, \alpha) = 1$, if $|k - l| \leq |l - S_0|$, and votes for the status quo alternative, $v(l, k, S_0, \alpha) = 0$, otherwise;

where the law of motion for new policy, S_1, is governed by

$$S_1 = \begin{cases} k(p, m, \alpha, S_0) & \text{if } \int_{-\infty}^{\infty} v(l, k, S_0, \alpha) dF_l(l) \geq \alpha \\ S_0 & \text{otherwise.} \end{cases}$$

Implicit in the equilibrium definition is the assumption that legislators adopt the weakly dominant strategy of voting for a policy k if and only if they weakly prefer it to the status quo. This assumption rules out uninteresting equilibria e.g., where a policy k wins because a proposer believes that everyone will vote against any other policy (even though more than measure α of legislators prefer a policy k' to k that the proposer also prefers).

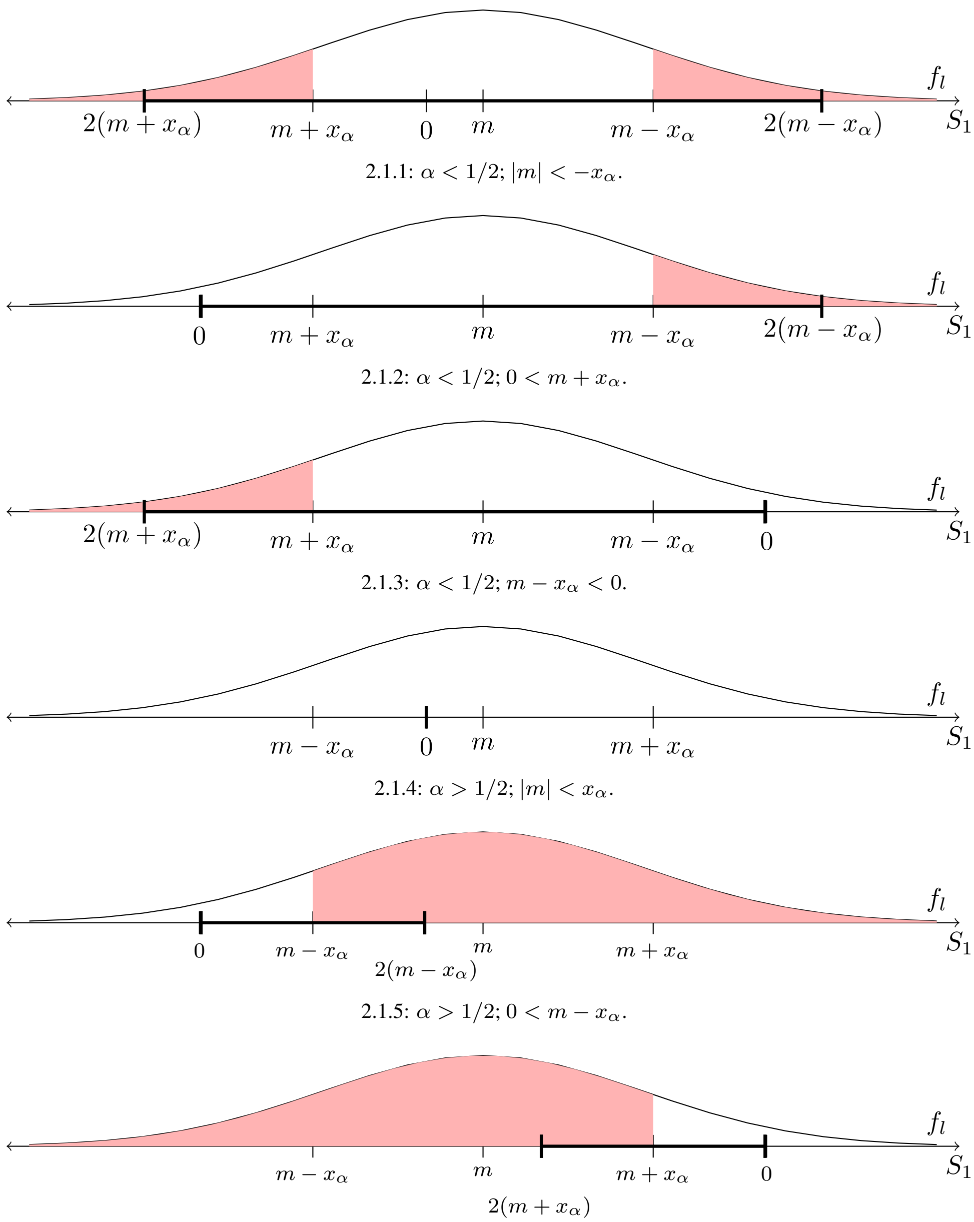
Legislator l weakly prefers policy k to the status quo of zero if and only if l lies no further from k than from the status quo. Hence, legislator $l > 0$ supports policy k if and only if $k \in [0, 2l]$. The set of policies supported by legislator $l' > l > 0$ is $[0, 2l']$, which is a superset of the set of policies supported by l. A legislator at $l < 0$ supports a policy if and only if it is in the interval $[2l, 0]$. The set of policies supported by a legislator $l' < l < 0$ is $[2l', 0]$, which is a superset of the set of policies supported by l. Finally, if two legislators l and l' support the same policy, then so do all legislators located between them.

We define $R(m, \alpha)$ to be the *feasible policy set*, i.e., the set of policies that are preferred to the status quo by the required proportion α of legislators when the median legislator is located at m. Let $x_\alpha = F_l^{-1}(\alpha) - m$. Note that if $\alpha < 1/2$, then $x_\alpha < 0$. We will show that $|x_\alpha|$ is the distance from the median legislator that identifies the legislators who are crucial in determining $R(m, \alpha)$. Following Compte and Jehiel [2010a], we refer to these legislators, located at $m - |x_\alpha|$ and $m + |x_\alpha|$, as *key legislators*. We call the set of legislators to the left of both key legislators the "*radical left*", the set of legislators to the right of both key legislators the "*radical right*", and the set of legislators between both key legislators the "*conservative middle*".

Minority Voting Rule ($\alpha < 1/2$). With a minority voting rule, a policy is feasible if and only if it is supported by at least one key legislator. For example, suppose that $|m| < -x_\alpha$ (Figure 2.1.1). The key legislator at $m + x_\alpha < 0$ supports any policy in $[2(m + x_\alpha), 0]$ against the status quo, as does the radical left. The measure of the radical left is exactly α. This group forms the required minority to change policy to any point in $[2(m + x_\alpha), 0]$. In contrast, the key legislator $m + x_\alpha < 0$ supports the status quo against any policy $k < 2(m + x_\alpha)$ as does the radical right and the conservative middle. The measure of this group is exactly $1 - \alpha$. Hence, this group forms the required majority to block any change in policy to $k < 2(m + x_\alpha)$.

The key legislator at $m - x_\alpha > 0$ supports any policy in $[0, 2(m - x_\alpha)]$ against the status quo, as does the radical right. Since the measure of the radical right is exactly α, this group forms the required minority to change policy to any point in $[0, 2(m - x_\alpha)]$. In contrast, the key legislator $m - x_\alpha > 0$ supports the status quo against any policy $k > 2(m - x_\alpha)$ as does the radical left and the conservative middle. The measure of this group is exactly $1 - \alpha$, and hence this group forms the required majority to block any change in policy to $k > 2(m - x_\alpha)$.

In general, if one key legislator supports the proposal, so too does one of the radical groups, which represents the required minority to pass the proposal. If both key legislators reject the proposal, then so too does the conservative middle and at least one radical group. These two groups form the required majority to block the proposal. Figures 2.1.2 and 2.1.3 show $R(m, \alpha)$ for the other possible values of m and x_α when $\alpha < 1/2$.



2.1.1: $\alpha < 1/2$; $|m| < -x_\alpha$.

2.1.2: $\alpha < 1/2$; $0 < m + x_\alpha$.

2.1.3: $\alpha < 1/2$; $m - x_\alpha < 0$.

2.1.4: $\alpha > 1/2$; $|m| < x_\alpha$.

2.1.5: $\alpha > 1/2$; $0 < m - x_\alpha$.

2.1.6: $\alpha > 1/2$; $m + x_\alpha < 0$.

Figure 2.1: We denote the feasible policy set, $R(m, \alpha)$, by the thick black line. Figures 2.1.1 to 2.1.3 show $R(m, \alpha)$ when $\alpha < 1/2$. Figures 2.1.4 to 2.1.6 show $R(m, \alpha)$ when $\alpha > 1/2$. The shaded areas represent the required proportion α of legislators who support a movement in policy from zero to any point in $R(m, \alpha)$.

Majority Voting Rule ($\alpha \geq 1/2$)**.** When the voting rule is a supermajority ($\alpha > 1/2$), a proposal succeeds if and only if it is supported by both key legislators, $m - x_\alpha < m$ and $m + x_\alpha > m$. For example, suppose $|m| < x_\alpha$ (Figure 2.1.4). The key legislator at $m - x_\alpha < 0$ supports the status quo against any policy $k > 0$, as does the radical left. The measure of this group of legislators is exactly $1 - \alpha$, thus forming the required minority to block a change in policy to any $k > 0$. The key legislator at $m + x_\alpha > 0$ supports the status quo against any policy $k < 0$, as does the radical right, thereby forming the required minority to block any change in policy to $k < 0$. As a result, when $|m| < x_\alpha$ the feasible policy set is just the status quo.

More generally, if both key legislators support a proposal, so too does the conservative middle and one of the radical groups. Together these two groups form the required majority α to pass a proposal. If one of the key legislators does not support a proposal, neither does one of the radical groups. This group forms the required minority $1 - \alpha$ to block the proposal. Figures 2.1.5 and 2.1.6 show $R(m, \alpha)$ for the other possible values of m and x_α when $\alpha > 1/2$.

Feasible Policy Sets. Summarizing, we have that the feasible policy sets are given by

$$\text{for } \alpha < 1/2, \quad R(m,\alpha) = \begin{cases} [0, 2(m - x_\alpha)] & \text{if } 0 < m + x_\alpha \\ [2(m + x_\alpha), 2(m - x_\alpha)] & \text{if } |m| < -x_\alpha \\ [2(m + x_\alpha), 0] & \text{if } m - x_\alpha < 0 \end{cases}, \text{ and}$$

$$\text{for } \alpha \geq 1/2, \quad R(m,\alpha) = \begin{cases} [0, 2(m - x_\alpha)] & \text{if } 0 < m - x_\alpha \\ 0 & \text{if } |m| < x_\alpha \\ [2(m + x_\alpha), 0] & \text{if } m + x_\alpha < 0. \end{cases}$$

When α is small, for any realization of m there always exist alternative policies that are more preferred by the required minority. This minority is always the radical group furthest from the status quo. When $\alpha < 1/2$ and m is sufficiently close to the status quo, then policies can be found on either side of the status quo that garner the necessary support for a policy change. As α increases, the feasible policy set $R(m, \alpha)$ shrinks. When $\alpha > 1/2$ and the median legislator is sufficiently close to the status quo then policy movement in one direction is blocked by the radical group that lies on the opposite side of the status quo. In such a case no change to the status quo is possible. For sufficiently extreme m, the interval of policies that garner sufficient support for a policy change lies to one side of the status quo.

As in Compte and Jehiel [2010a], under majority rules, the set of possible policy changes is the same as if only the key legislators were present and unanimity among key legislators were required to change policy. Under minority rules, the set of possible policy changes is the same as if only the key legislators were present and support from only one legislator was required to change policy.

Policy Outcomes. Given a voting rule α, the proposer p presents to the legislature the feasible policy that he most prefers. Thus, the implemented policy solves

$$S_1 = \min_{k \in R(m,\alpha)} |p - k|. \tag{2.1}$$

This policy k is unique since $R(m, \alpha)$ is closed and the objective function is strictly quasi-concave. Obviously, policy remains unchanged if the feasible policy set consists only of the status quo. If p lies inside the feasible policy set, the new policy is exactly p — the proposer can move policy to his bliss point. Such a proposer is *free*. When a proposer lies on the opposite side of the status quo to the feasible policy set or the feasible policy set only consists of the status quo, then he cannot make a change in policy that raises his utility. Such a proposer is *blocked*.

In all other cases, the proposer is *constrained*: the proposer can move policy, but only part of the way toward his preferred policy. When the voting rule is a minority, the policy proposal only needs to make the key legislator closest to the proposer indifferent between the proposal and the status quo. For example, suppose that $\alpha < 1/2$, $m + x_\alpha < 0$ and $p < 2(m + x_\alpha)$. Then, the feasible policy set is $[2(m + x_\alpha), 0]$ and the proposer proposes $k = 2(m + x_\alpha)$. This policy makes the key member who lies closest to p, $m + x_\alpha$, indifferent.

When the voting rule is a supermajority, the policy proposal will be preferred by the closer key legislator, but must also make the key legislator who lies closest to the status quo indifferent between the proposal and the status quo. For example, suppose that $\alpha > 1/2$, $m - x_\alpha > 0$ and $p > 2(m - x_\alpha)$. Then, the feasible policy set is $[-, 2(m - x_\alpha)]$ and he proposes $k = 2(m - x_\alpha)$, the policy that makes the key legislator closest to the status quo, $m - x_\alpha$, indifferent.

From this we derive the implemented policy S_1 as an explicit function of m, p and x_α:

$$\text{For } \alpha < 1/2, \quad S_1 = \begin{cases} 0 & \text{if } m - x_\alpha < 0 < p \text{ or } m + x_\alpha > 0 > p \\ p & \text{if } (m - x_\alpha < 0 \text{ and } 2(m + x_\alpha) < p < 0) \\ & \text{or } (m + x_\alpha > 0 \text{ and } 0 < p < 2(m - x_\alpha)) \\ & \text{or } (|m| < -x_\alpha \text{ and } |2m_t - p| < -2x_\alpha) \\ 2(m + x_\alpha) & \text{if } m + x_\alpha < 0 \text{ and } p < 2(m + x_\alpha) \\ 2(m - x_\alpha) & \text{if } 0 < m - x_\alpha \text{ and } 2(m - x_\alpha) < p. \end{cases} \tag{2.2}$$

$$\text{For } \alpha \geq 1/2, \quad S_1 = \begin{cases} 0 & \text{if } |m| < x_\alpha \text{ or } m < 0 < p \text{ or } m > 0 > p \\ p & \text{if } (m - x_\alpha > 0 \text{ and } 0 < p < 2(m - x_\alpha)) \\ & \text{or } (m + x_\alpha < 0 \text{ and } 2(m + x_\alpha) < p < 0) \\ 2(m - x_\alpha) & \text{if } m - x_\alpha > 0 \text{ and } 2(m - x_\alpha) < p \\ 2(m + x_\alpha) & \text{if } m + x_\alpha < 0 \text{ and } 2(m + x_\alpha) > p. \end{cases} \tag{2.3}$$

For very small voting rules, a proposer can typically successfully propose his own position as the new policy: there is usually a radical group that provides sufficient support. As the required vote share increases, more often he must adjust his proposal to obtain the necessary support. For large voting rules, no such adjustment exists if the status quo lies closer to the median legislator than both key legislators.

Initially we assume only that the densities f_e and f_m are *symmetric, quasi-concave, mean-zero* distributions. We later consider three classes of distributions—two-point, normal and uniform—that allow explicit characterizations of the optimal voting rule. Section 2.5 considers bias in the initial policy. Throughout, we assume there is enough dispersion in the bliss points of legislators that first-order conditions characterize the optimal majority, i.e., the support of f_l is sufficiently large.

Minority Rules Are Never Optimal. We first show that the social planner can restrict attention to majority voting rules.

Lemma 5. *A minority voting rule is never an optimal voting rule.*

Lemma 5 is intuitive. Increasing the submajority rule reduces the set of possibly successful legislative changes around the median legislator. On average, the median legislator is more representative of the populace than a non-median proposer. As a result, welfare increases. As the optimal voting rule is a majority rule, we now refer to $x_\alpha \geq 0$ as a voting rule distance.

With majority voting rules, $\alpha \geq 1/2$, the representative citizen's ex-ante expected utility is

$$
\begin{aligned}
U = &-\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{-x_\alpha}\Big[\int_{-\infty}^{2(m+x_\alpha)}(e-2(m+x_\alpha))^2 dF_p(p-m)+\int_{2(m+x_\alpha)}^{0}(e-p)^2 dF_p(p-m)\\
&+\int_{0}^{\infty}e^2 dF_p(p-m)\Big]dF_m(m-e)+\int_{-x_\alpha}^{x_\alpha}\Big[\int_{-\infty}^{\infty}e^2 dF_p(p-m)\Big]dF_m(m-e)\\
&+\int_{x_\alpha}^{\infty}\Big[\int_{-\infty}^{0}e^2 dF_p(p-m)+\int_{0}^{2(m-x_\alpha)}(e-p)^2 dF_p(p-m)\\
&+\int_{2(m-x_\alpha)}^{\infty}(e-2(m-x_\alpha))^2 dF_p(p-m)\Big]dF_m(m-e)\Big)dF_e(e). \qquad (2.4)
\end{aligned}
$$

The first three triple integrals are associated with realizations for which the median and key legislators lie to the left of the status quo. The first triple integral is associated with a key legislator who lies so close to the left of the status quo that change in policy is constrained to win the key legislator's support. The second triple integral is associated with key legislators who lie far enough to the left of the status quo that the proposer is unconstrained in proposing policy change. The third triple integral is associated with a proposer who lies to the right of the status quo and is, thus, unable to change policy. The fourth triple integral is associated with a median legislator lying on the opposite side of the status quo to a key legislator. In such a case, policy cannot be changed since the left key legislator votes against movement in policy to the right and the right key legislator votes against movement in policy to the left. The last three triple integrals have analogous interpretations, except that the median and key legislators lie to the right of the status quo.

The relevant primitive for the social planner is the voting rule distance x_α. The distribution of the legislature around the median legislator, F_l, only enters payoffs indirectly via the linear distance x_α implied by the voting rule. A social planner, in ignorance of F_l, can establish the optimal voting rule distance, x^*_α. This distance is invariant to changes in the distribution of the legislature. The optimal voting rule delivers the optimal voting distance. This implies the following proposition.

Proposition 9. *Suppose that* $F_{l_2}(\cdot|l > 0) > F_{l_1}(\cdot|l > 0)$ *for* $F_{l_1}(\cdot|l > 0) < 1$. *Then* $\alpha^*_{l_1} \geq \alpha^*_{l_2}$. *The inequality is strict if* $\alpha^*_{l_2} > 1/2$.

Unanimity Is Never Optimal. The social planner always prefers a voting rule that allows at least slight movement in policy to one that blocks all proposers. This is because as the voting rule decreases, the first proposers to be 'unblocked' are those who lie on the same side of the status quo as the representative citizen, but are further away. The marginal effect of reducing the majority rule to unblock these first proposers is always to move policy toward the representative citizen's bliss point, raising welfare. We formalize this in Proposition 10:

Proposition 10. *Let F_m have bounded support, $[-\bar{m}, \bar{m}]$, and F_e have positive variance. Voting rules that are close enough to unanimity to prevent any change in policy, can never be optimal.*

The requirements of the proposition are weak. Quasi-concavity is not required and symmetry can be dropped. Choosing a voting rule distance that slightly exceeds the maximum of the absolute value of the support of F_m does better than unanimity.

Optimal Voting Rule. Lemma 5 and Proposition 10 imply that, in general, the first-order condition that characterizes the optimal voting rule distance, x_α^*, is

$$\frac{\partial U}{\partial x_\alpha} = 8 \int_{-\infty}^{\infty} \int_{-\infty}^{-x_\alpha} (e - 2(m + x_\alpha)) F_p(m + 2x_\alpha) dF_m(m - e) dF_e(e) \leq 0, \tag{2.5}$$

with strict equality if $x_\alpha^* > 1/2$, where we use the symmetry of F_p, F_e and F_m to simplify the expression. When $x_\alpha^* > 1/2$, the social planner's optimal choice is summarized by the following maxim: choose the voting rule such that, conditional on the proposer being constrained, the expected value of new policy equals the expected bliss point of the representative citizen.

To understand the social planner's tradeoffs, recognize that a marginal increase in the voting rule affects welfare in two ways. First, raising x_α can raise the probability that a proposer is constrained: (a) there are more realizations of the legislature for which the feasible policy set is just the status quo, and hence the limit of integration over m shrinks; and (b) for those realizations of the legislature for which the feasible set is non-trivial, the probability that the proposer is constrained, $F_p(m + 2x_\alpha)$, rises. This is because with larger voting rules, a non-trivial feasible policy set requires more extreme median legislators, and hence more extreme proposers relative to the status quo, are more likely.

Second, raising x_α affects the realized policy when a proposer is constrained. Consider a proposer p, located to the left of $2(m + x_\alpha) < 0$. The proposer must win approval from the key legislator at $m + x_\alpha$, resulting in a new policy of $2(m + x_\alpha)$. A marginal increase in x_α moves policy toward the status quo by twice the increase in x_α, and the effect on utility, given e and m is $8(e - 2(m + x_\alpha))$. Therefore, if the representative citizen's bliss point lies to the right of the new policy ($e > 2(m + x_\alpha)$), then a marginal increase in x_α raises utility; but if the representative citizen's bliss point lies to the left ($e < 2(m + x_\alpha)$) then it reduces utility.

Were the dispersion in the representative citizen's preferred policy to go to zero, the social planner's optimal voting rule becomes large enough to prevent changes to the status quo. Intuitively, there is no need for a legislature if the electorate never changes its mind. At the other extreme, the optimal voting rule of a society with a perfectly representative median legislator who always proposes legislative changes is a simple majority.

These two extreme scenarios do not imply that increasing the dispersion in the median legislator's preferred policy around the representative citizen's bliss point always increases the optimal voting rule. To convey why this is so, we first characterize the optimal voting rule when the median legislator is always the proposer. This is equivalent to p being a degenerate random variable equal to zero, i.e., $F_p(z) = 0$ if z is negative and $F_p(z) = 1$ otherwise.

2.3 Median legislator proposes policy

The first-order condition that characterizes the optimal voting rule is derived from equation (2.5) where $F_p(m + 2x_\alpha)$ is zero if $m < -2x_\alpha$ and is one if $m > -2x_\alpha$. At the optimal voting rule,

$$\frac{\partial U}{\partial x_\alpha} = 8 \int_{-\infty}^{\infty} \int_{-2x_\alpha}^{-x_\alpha} (e - 2(m + x_\alpha)) dF_m(m - e) dF_e(e) \leq 0, \tag{2.6}$$

where equation (2.6) must hold as an equality if the optimal voting rule exceeds simple majority.

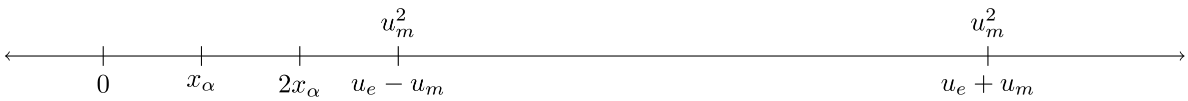
We next characterize how the optimal voting rule varies with primitives for three classes of distributions. For each class, we find that an increase in the dispersion of the median legislator around the representative citizen's bliss point increases the optimal voting rule. We then show how the simple conjecture that increased dispersion in f_m (i.e., a first-order stochastic shift in $f_m(\cdot|m \geq 0)$) *necessarily* leads to a weakly larger optimal voting rule is not well founded.

Two-point uncertainty. We first consider the possibility that the representative citizen's bliss point e and that of the median legislator m are drawn from distributions with two-point supports, characterized by u_e and u_m respectively. Here u_e measures the possible dispersion in society's bliss point and u_m measures the extent of the representativeness of society by the legislature. Thus,

$$F_m(m - e) = \begin{cases} 0 & \text{if } m - e < -u_m \\ 1/2 & \text{if } -u_m \leq m - e < u_m \\ 1 & \text{if } u_m \leq m - e. \end{cases}; \quad F_e(e) = \begin{cases} 0 & \text{if } e < -u_e \\ 1/2 & \text{if } -u_e \leq e < u_e \\ 1 & \text{if } u_e \leq e. \end{cases}. \tag{2.7}$$

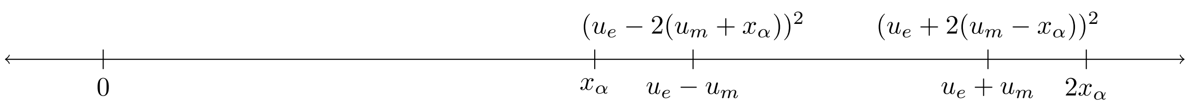
To construct the first-order condition that characterizes the optimal voting rule distance, one can substitute equation (2.7) into equation (2.6). However, it is more instructive to explicitly derive the representative citizen's utility given the two-point distribution and then derive the first-order condition. The representative citizen's utility depends on the relative values of u_e, u_m and x_α.

We focus on the case where $u_e > u_m$ and $+u_e$ is realized (analyses of the other possibilities are similar). When x_α is sufficiently small relative to both u_e and u_m, that $x_\alpha < \frac{u_e - u_m}{2}$, then the median legislator is free in his choice of new policy. As a result, the implemented policy is exactly distance u_m from u_e, so the representative citizen's utility is $-u_m^2$. As x_α is increased, eventually the left-wing median proposer, i.e., the median legislator to the left of the

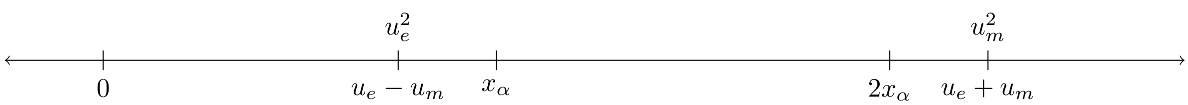


2.2.1: $x_\alpha < \frac{u_e - u_m}{2}$.

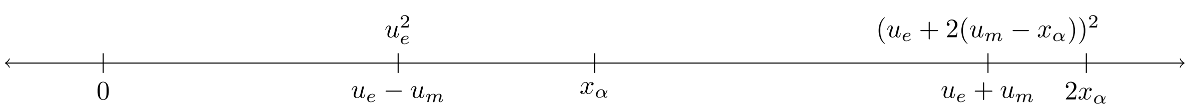
2.2.2: $\frac{u_e - u_m}{2} < x_\alpha < \min\{u_e - u_m, \frac{u_e + u_m}{2}\}$.



2.2.3: $\frac{u_e + u_m}{2} < x_\alpha < u_e - u_m$.



2.2.4: $u_e - u_m < x_\alpha < \frac{u_e + u_m}{2}$.



2.2.5: $\max\{u_e - u_m, \frac{u_e + u_m}{2}\} < x_\alpha < u_e + u_m$.

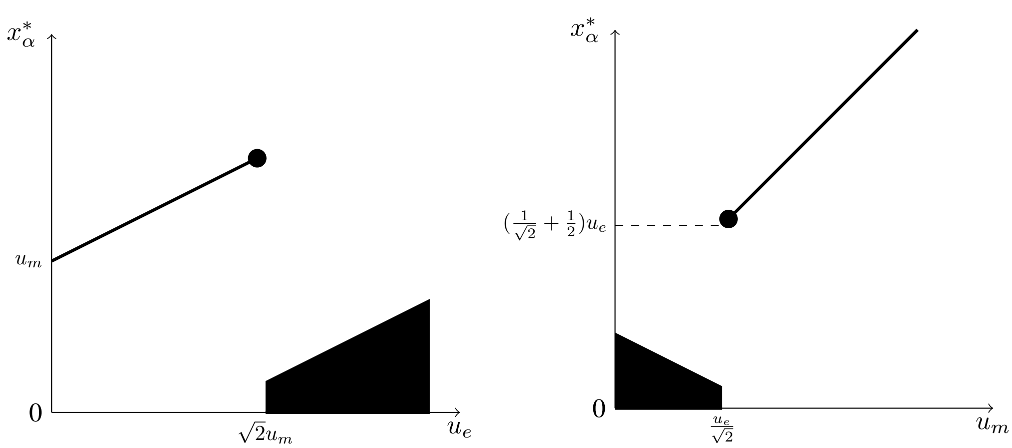
2.2.6: $u_e + u_m < x_\alpha$.

Figure 2.2: These figures show how utility changes as the required majority changes when F_e and F_m are two-point distributions characterized by the parameters u_e and u_m, when $u_e > u_m$ and $+u_e$ is drawn.

representative citizen's bliss point, becomes constrained in his choice of new policy (recall that $+u_e$ is realized), while the right-wing median legislator remains free. Increasing x_α further, the left-wing median legislator becomes blocked, while the right-wing median legislator becomes constrained; which happens first depends on the relative values of u_e and u_m. If $u_e - u_m < \frac{u_e+u_m}{2}$ the former occurs first, otherwise the latter does. Eventually, the voting rule distance becomes so large, in particular, $x_\alpha > u_e + u_m$, that both median legislators are blocked. With no change in policy, utility is $-u_e^2$.

Figure 2.2 summarizes these possible utility realizations; there are six possible cases depending on the relative values of u_e, u_m and x_α. The points on the line are labeled from below and indicate the position of the median legislator (either $u_e - u_m$ or $u_e + u_m$) relative to the status quo (0) and the important distances implied by the voting rule (x_α and $2x_\alpha$). The labels above the proposers show the disutility associated with the new policy generated under the assumed voting rule. For example, Figure 2.2.1 shows that when $2x_\alpha < u_e - u_m$, the new policy is always u_m distance from u_e. Hence, disutility is u_m^2. In contrast, Figure 2.2.6 shows that when $u_e + u_m < x_\alpha$, policy is never changed so disutility is always u_e^2.

Proposition 11. *Let $e \in \{-u_e, u_e\}$ and let $m \in \{e - u_m, e + u_m\}$, each with probability 1/2. Then, if $u_e > \sqrt{2}u_m$, i.e, if society's preferences are sufficiently more dispersed than the median legislator's, the optimal voting rule leaves the median legislator completely free to change policy, i.e., x_α^* is any element of $[0, \frac{|u_e - u_m|}{2}]$. If $u_e < \sqrt{2}u_m$, the optimal voting rule rule blocks the median legislator closest to the status quo from making any changes to policy and constrains the median legislator furthest from the status quo to enact new policy exactly equal to the representative citizen's bliss point, i.e., $x_\alpha^* = u_m + u_e/2$. When $u_e = \sqrt{2}u_m$, either policy is optimal.*



2.3.1: x_α^* as a function of u_e given u_m.

2.3.2: x_α^* as a function of u_m given u_e.

Figure 2.3: The optimal voting rule distance when F_e and F_m are both two-point distributions characterized by parameters u_e and u_m, respectively.

Figure 2.3 shows the optimal voting rule for different values of u_e and u_m. Note that as u_m increases, the optimal voting rule distance (weakly) increases. Intuitively, when the dispersion in the median legislator's bliss point around that of the representative citizen is higher, the median legislator is typically less representative of the representative citizen's preferences, as is, on average, the resulting policy outcome. As a result, the social planner prefers to rely more on the status quo than the legislative process in determining policy.

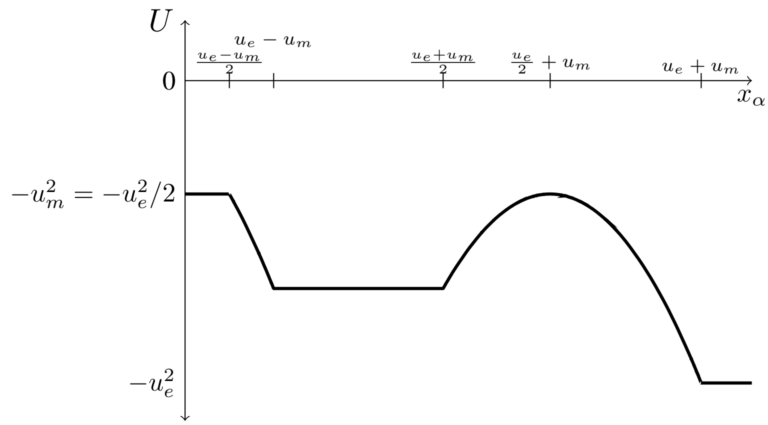


Figure 2.4: Expected utility when $u_e = \sqrt{2}u_m$.

The optimal voting rule is discontinuous in u_e and u_m going from simple majority to a large supermajority. To understand this discontinuity, one must understand how a marginal increase in x_α affects policy when x_α is small and $u_e > u_m$. Figures 2.2.1 and 2.2.2 show that when $+u_e$ is drawn, increasing x_α marginally from $x_\alpha < \frac{u_e-u_m}{2}$ to $x_\alpha > \frac{u_e-u_m}{2}$, only serves to constrain the median legislator closer to the status quo who would move policy in the direction that the representative citizen prefers, while leaving the median legislator furthest from the status quo (who can exploit the fact that most legislators are closer to his bliss point than the status quo) unconstrained. As Figure 2.4 reveals, utility falls, since policy outcomes are now constrained to lie further from the representative citizen's bliss point e. Utility continues to fall as the legislator closest to the status quo is further constrained, reaching a local minimum once he becomes blocked (Figure 2.2.4). This effect always exists whenever a proposer lies between the status quo and the representative citizen. An increase in the voting rule reduces welfare, as the new policy is constrained away from the representative citizen's bliss point.

However, once increases in x_α start to constrain the median legislator furthest from the status quo who moves policy past e, utility starts to rise as policy outcomes are constrained to lie closer to e (Figure 2.2.5). Utility reaches a local maximum once the median legislator furthest from the status quo is constrained so that he can do no better than enact exactly e. After this point, further increases in the voting rule constrain policy outcomes to lie further from e, reducing welfare.

It follows that the social planner's optimal choice of x_α is one of two alternatives: (1) simple majority, or (2) the voting rule that blocks a median legislator who is closer to the status quo than to e and exactly constrains the median legislator who is further from the status quo than the representative citizen so that the new policy that he adopts exactly equals e. As Figure 2.4 shows, when $u_e = \sqrt{2}u_m$, the social planner is indifferent between these two choices.

Uniform uncertainty. We now characterize the optimal voting rule when e and m are drawn from uniform distributions.

Proposition 12. *Let $e \sim U(-a, a)$ and $m \sim U(e - b, e + b)$. If society's preference are sufficiently more dispersed than the median legislator's, so that $a > \frac{11+4\sqrt{7}}{18}b \approx 1.2b$, then the optimal voting rule is simple majority. As society's preferences become less dispersed, x^*_α increases up to the point where $a = b$, and $x^*_\alpha = b$ for $a \leq b$.*

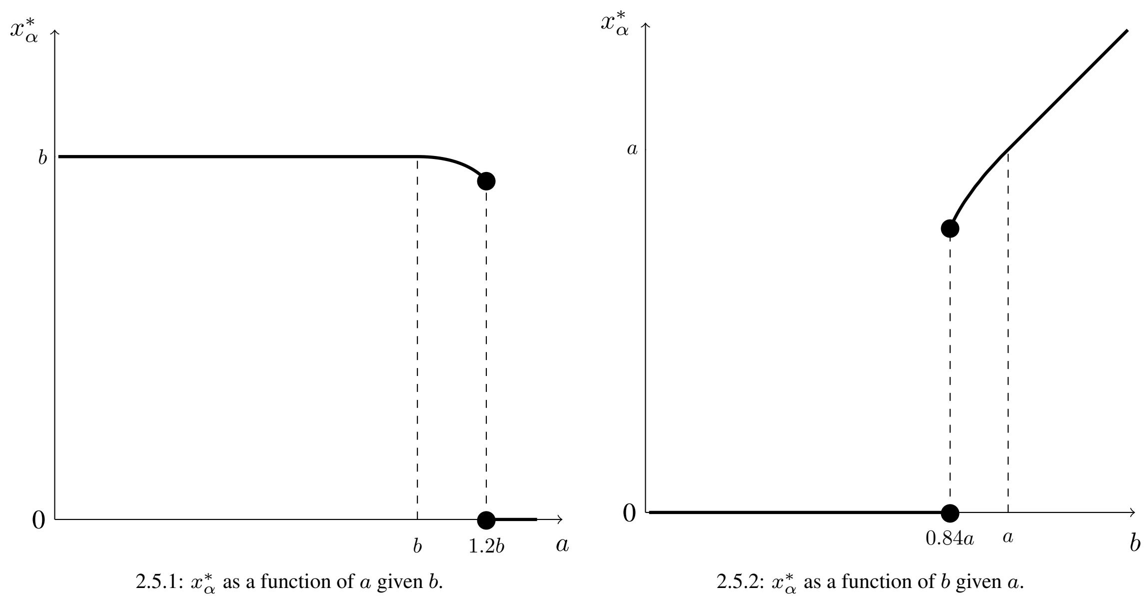


Figure 2.5: The optimal majority when $e \sim U[-a, a]$ and $m \sim U[-b, b]$.

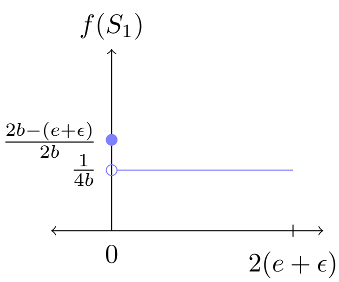
Figure 2.5 plots the optimal majority distance for different values of a and b. Once again, as dispersion in the median legislator's possible preferred policy increases, so does the optimal voting rule. The social planner prefers less flexibility in the legislative process since the median position becomes less representative of society's preferences relative to the status quo.

When $b \geq a$, i.e., when the dispersion in the median legislator's preferences around society's preferred policy exceeds the dispersion in society's preferred policy, the optimal voting rule distance is exactly b. That is, the optimal voting rule *always* constrains or blocks the median legislator—the median legislator is *never* free to propose his preferred policy. Recall that the voting rule maximizes utility when, conditional on a change in policy, the expected new policy equals the representative citizen's expected bliss point. With sufficient uniform uncertainty in the median legislator's bliss point m around e relative to that for e, the social planner's solution is particularly simple. By choosing

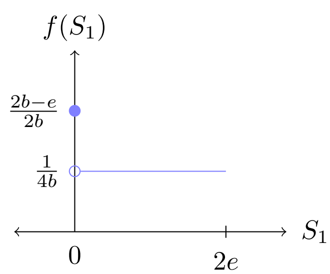
x_α equal to the maximum possible distance of the median legislator from e, i.e., $x_\alpha = b$, the social planner ensures that for any realization of e, the distribution over policy is uniformly distributed on $[0, 2e]$, implying that the expected change in new policy exactly equals e.

To convey intuition for this result, Figure 2.6 shows the distribution of new policy for three different voting rules: $x_\alpha = b - e$ (Figure 2.6.1), $x_\alpha = b$ (Figure 2.6.2) and $x_\alpha = b + e$ (Figure 2.6.3). Let $e \in [0, a]$ (analysis when $e \in [-a, 0]$ is similar) and consider first the optimal voting rule $x_\alpha = b$. The probability of a policy change is $\frac{e}{2b}$, and policy is changed when $m \in (b, e + b]$. If $m = b$, then $S_1 = 0$. If $m = e + b$, then $S_1 = 2e$. Hence, conditional on a change in policy, the new policy is distributed uniformly between zero and $2e$. For realizations where the median legislator lies close to the status quo, inertia is too high. For realizations where the median legislator lies far from the status quo, inertia is too low. However, the optimality condition that the expected new policy exactly equals the representative citizen's bliss point holds.

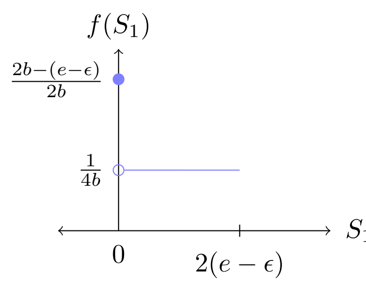
Now consider a slightly larger voting rule $x_\alpha = b + \epsilon$. For any $e \in [0, \epsilon]$, no policy change is possible. However, the effect on utility is to a third order. For any $e \in [\epsilon, a]$ the probability of a change in policy is $\frac{e-\epsilon}{2b}$. If $m = b + \epsilon$, then $S_1 = 0$. If $m = e + b$, then $S_1 = 2(e - \epsilon)$. Hence, conditional on a change in policy, the new policy is distributed uniformly between zero and $2(e - \epsilon)$. Comparing Figures 2.6.2 and 2.6.3 reveals that the net effect of a slightly larger voting rule is to reduce utility because policy outcomes marginally less than $2e$ are replaced by the status quo of zero — there is too little policy movement.



2.6.1: $x_\alpha = b - \epsilon$.



2.6.2: $x_\alpha = b$.



2.6.3: $x_\alpha = b + \epsilon$.

Figure 2.6: The distribution of new policy S_1 given $e \in [\epsilon, a]$ when $b > e$; the dot at $S_1 = 0$ is the point mass on the status quo policy outcome of zero.

The analysis for a slightly smaller voting rule, $x_\alpha = b - \epsilon$, is similar. Figure 2.6.1 shows the distribution of policy when $e \in [-\epsilon, a]$. Figures 2.6.1 and 2.6.2 reveal that the net effect is to reduce utility as policy outcomes marginally greater than $2e$ replace the status quo as the policy outcome: policy movement is too high.

To understand the discontinuity in the optimal voting rule when societal preferences are more dispersed than the median legislator's, i.e., $a > b$, recall the two-point distribution results (Proposition 11). There when $\mu_e > \mu_m$, a slight increase in the voting rule from $x_\alpha < \frac{\mu_e - \mu_m}{2}$ to $x_\alpha > \frac{\mu_e - \mu_m}{2}$ reduced utility as the higher voting rule reduced policy

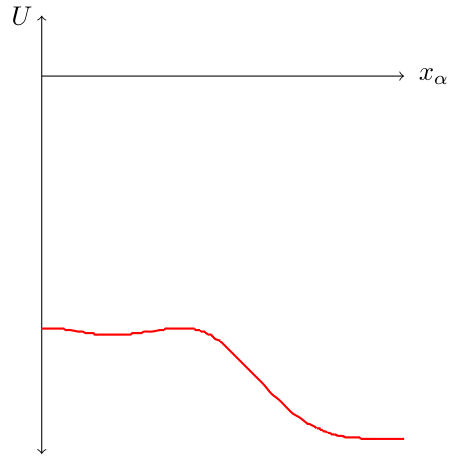


Figure 2.7: Utility for different voting rule distances when $a = \frac{11+4\sqrt{7}}{18}b$.

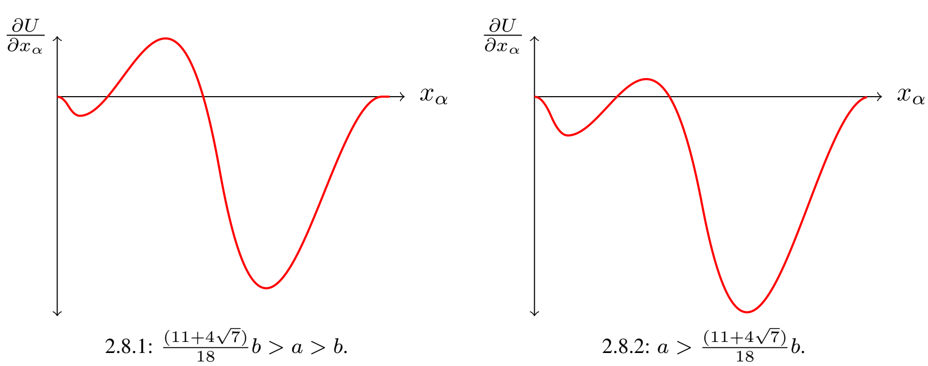


Figure 2.8: The marginal effect on expected utility of the representative citizen of an increase in the majority rule for different values of a and b.

movement toward the representative citizen. Only once the more extreme proposer was constrained did increases in x_α raise utility. When b is less than a and $x_\alpha = 0$, that same logic holds for all $e \in [-b, b]$. For all $e \notin [-b, b]$, marginally raising x_α has no effect on policy outcomes. As a result, marginally raising the voting rule when $x_\alpha = 0$ *always* reduces welfare (Figures 2.8.1 and 2.8.2).

As x_α increases, for realizations of e close to zero, the more extreme proposer becomes constrained and utility rises, while for realizations of e far from zero, only the less extreme proposer is constrained, reducing utility. As b approaches a, the measure of the former group increases relative to the latter. As Figure 2.8.1 shows, if a slightly exceeds b, there exists a supermajority for which the positive marginal effect for higher supermajorities outweighs the negative marginal effect for low supermajorities. If a substantially exceeds b, no such supermajority rule exists (Figure 2.8.2).

Normal uncertainty. We now characterize optimal voting rules when the representative citizen's bliss point and that of the median legislator are normally distributed.

Proposition 13. *Let $e \sim N(0, \sigma_e^2)$ and $m \sim N(e, \sigma_m^2)$. If $\sigma_e^2 < 2\sigma_m^2$, the optimal voting rule declines with the volatility of society's bliss point, σ_e, and rises with the volatility in the representativeness of the legislature, σ_m. Further, $x_\alpha^*(\gamma\sigma_e, \gamma\sigma_m) = \gamma x_\alpha^*(\sigma_e, \sigma_m)$ where $x_\alpha^*(\sigma_e, \sigma_m)$ is the optimal voting rule distance given σ_e and σ_m. If society's preferences are sufficiently volatile relative to the volatility in legislative preferences, $\sigma_e^2 \geq 2\sigma_m^2$, then the optimal voting rule is simple majority.*

To prove this result, we first show that the optimal voting rule distance, if positive, solves

$$\Delta(\sigma_e^2, \sigma_m^2) \equiv \frac{2\sigma_m^2 + \sigma_e^2}{2\delta^2} = \gamma \frac{[\Phi(2\gamma) - \Phi(\gamma)]}{[\phi(\gamma) - \phi(2\gamma)]}, \tag{2.8}$$

where $\delta \equiv \sqrt{\sigma_e^2 + \sigma_m^2}$ and $\gamma \equiv x_\alpha/\delta$. We then use a L'Hôpital-type rule for monotonicity due to Pinelis [2002] to show that the right-hand side of equation (2.8) is strictly increasing in γ. This implies that the optimal voting rule is unique. The comparative statics follow directly.

As with the two-point class of distributions, x_α^* increases as the median legislator becomes less representative of society, in order to constrain the median from enacting extreme policies. As the dispersion in the median legislator's bliss point around e rises, the expected bliss point of the representative citizen, conditional on the median legislator being constrained, is closer to the status quo. For a given voting rule, this would create excessive movement in policy. The optimal voting rule must therefore be higher, reducing the movement in policy, so that, in expectation, the representative citizen's bliss point and new policy are again equal.

Increased dispersion in $\mathbf{F_m}$. One might conjecture, based on consideration of Propositions 3–5, that when legislatures are less representative, i.e., when there is greater dispersion in the median legislator's location relative to the representative citizen's bliss point, then the optimal supermajority should *always* be greater in order to protect the representative citizen from proposers who are more likely to be further from his bliss point. We now show that this conjecture is false. To do this, we construct distributions F_m and G_m, where $F_m(\cdot|\mu_m \geq 0) \geq G_m(\cdot|\mu_m \geq 0)$, but $\alpha_F^* > \alpha_G^*$. The loose intuition that we detail below is that the social planner would like to constrain some proposers by more and others by less, and that some spreads of the distribution of proposers can *raise* the fraction of proposers whom the social planner wants to constrain less.

Proposition 14. *Let $e \sim U[-1, 1]$. Let $\mu_m \sim F_m$ with associated density f and $\mu_m' \sim G_m$ with associated density g, where*

$$f(z) = \begin{cases} 0.026178 & \text{if } z \in [-10, -8.9] \\ 0.052944 & \text{if } z \in (-8.9, 8.9) \\ 0.026178 & \text{if } z \in [8.9, 10] \end{cases} \text{ and } g(z) = \begin{cases} 0.026178 & \text{if } z \in [-10, -9.1] \\ 0.052356 & \text{if } z \in (-9.1, 9.1) \\ 0.026178 & \text{if } z \in [9.1, 10] \end{cases}.$$

Then $F_m(\cdot|\mu_m \geq 0) \geq G_m(\cdot|\mu_m \geq 0)$ but $x_{\alpha F}^* = 10 > 9.98934 = x_{\alpha G}^*$.

When the median legislator m is close to the status quo he is always blocked from changing policy, so increasing x_α does not affect welfare. When m is an intermediate distance from the representative citizen's bliss point, e so that the constrained policy is closer to the status quo than is e, then increasing x_α further constrains the new policy to lie further from e, reducing welfare. The counter-example replaces median legislators very close to e with those slightly further away. That is, the counter-example replaces legislators who are very likely to be always blocked with those who tend to be an intermediate distance away. As a result, conditional on a change in policy, the expected new policy is too close to the status quo and welfare is raised by marginally reducing the voting rule.

A similar effect could be realized if we replaced median legislators very far from e, who are (almost always) free, with legislators a little closer to e, who are (almost always) constrained too little. Conditional on a change in policy, expected new policy is too far from the status quo and welfare is raised by marginally increasing the voting rule.

2.4 More Extreme Proposers

We now consider proposers other than the median legislator. We are especially interested in how the extent of the proposer's extremism affects the optimal voting rule. We build this analysis up by supposing that the proposer is randomly chosen from a two-point distribution: with equal probability the proposer lies u_p either to the left or the right of the median legislator, so that $F_p(z) = 0$ if $z < -u_p$, $F_p(z) = 1/2$ if $-u_p \leq z < u_p$, and $F_p(z) = 1$ if

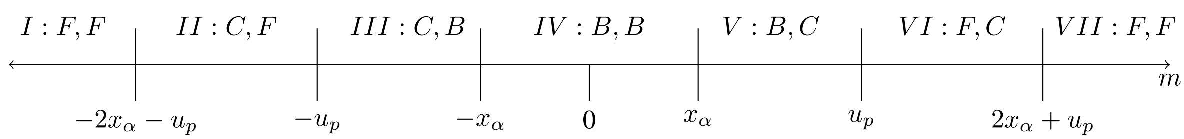
$u_p \leq z$. We refer to u_p as the proposer's *polarity*, since a higher u_p represents a more extreme proposer relative to the legislature. When u_p is negative, the proposer is to the left of the median legislator, and we call him a left-wing proposer; analogously when u_p is positive, we call him a right-wing proposer.

We next show that if u_p is large, so that the proposer is unrepresentative even of the legislature, then, as simple intuition might suggest, further increases in the proposer's polarity raise the optimal voting rule. However, this intuition does not extend when u_p is small: we identify conditions for a marginal increase in the polarity of the proposer to *reduce* the optimal voting rule.

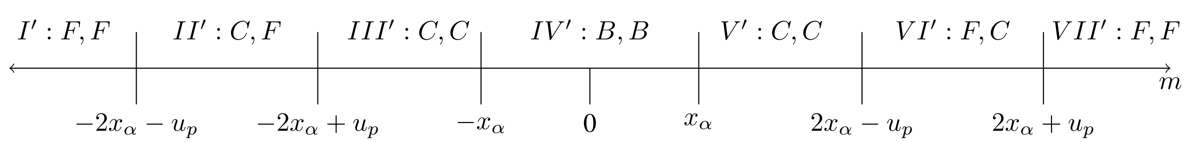
First, we analyze policy movement for different proposers. If m is sufficiently far from the status quo, i.e., $|m| > 2x_\alpha + u_p$, then the proposer is always free. For example, consider an extreme right-wing median legislator at $2x_\alpha + u_p + \epsilon$. The key legislator lies at $x_\alpha + u_p + \epsilon$. The key legislator supports any policy in $[0, 2(x_\alpha + u_p + \epsilon)]$, an interval that includes both the left-wing proposer at $2x_\alpha + \epsilon$ and the right-wing proposer at $2x_\alpha + 2u_p + \epsilon$.

As m moves toward the status quo from the right, first the right-wing and then the left wing proposer become constrained. A right-wing proposer is constrained when $x_\alpha < m < 2x_\alpha + u_p$ since $m + u_p > 2(m - x_\alpha)$, i.e., the proposer lies to the right of the feasible policy set. The right-wing proposer is blocked when $m < x_\alpha$ since the status quo becomes the sole feasible policy. A left-wing proposer is constrained when $u_p < m < 2x_\alpha - u_p$ since $m - u_p > 2(m - x_\alpha)$, and is blocked when $m < u_p$ since he then lies on the opposite side of the status quo from a key legislator. Analogously, as m moves toward the status quo from the left, first left-wing and then right-wing proposers become constrained and then blocked.

Notice that when the median legislator is to the right of the status quo, a left-wing proposer is only constrained if $u_p < m < 2x_\alpha - u_p$. This happens if and only if $x_\alpha > u_p$, i.e., if the voting rule is high relative to the proposer's polarity. Otherwise, when $x_\alpha < u_p$, the left-wing proposer is either free (when $m > u_p$) or blocked (when $0 < m < u_p$).



2.9.1: $u_p > x_\alpha$.



2.9.2: $u_p < x_\alpha$.

Figure 2.9: Realizations of proposer status (free – F, blocked – B, or constrained – C) for a given voting rule and different realizations of the median legislator.

Figure 2.9 illustrates these possibilities. For a given voting rule, we classify realizations of m into seven classes based on the movement in the status quo for right- and left-wing proposers. This classification depends on whether $x_\alpha < u_p$ (Figure 2.9.1) or $x_\alpha > u_p$ (Figure 2.9.2). The two letters above the line defining each region denote the possible policy movement facing a left- or right-wing proposer, respectively. We denote a free proposer by F, a constrained proposer by C, and a blocked proposer by B. Thus, as Figure 2.9.1 shows, when $u_p > x_\alpha$ and $x_\alpha < m < u_p$ then a left-wing proposer is blocked and a right-wing proposer is constrained.

The representative citizen's utility when $x_\alpha < u_p$ is

$$U = -\int_{-\infty}^{\infty}\left[\int_{-\infty}^{-u_p-e}(\mu_m+u_p)^2 dF_m(\mu_m) + \int_{-u_p-e}^{x_\alpha-e} e^2 dF_m(\mu_m) + \int_{x_\alpha-e}^{2x_\alpha+u_p-e}(e+2(\mu_m-x_\alpha))^2 dF_m(\mu_m) + \int_{2x_\alpha+u_p-e}^{\infty}(\mu_m+u_p)^2 dF_m(\mu_m)\right] dF_e(e).$$

If, instead, the voting rule x_α exceeds the polarity u_p of the proposer then his utility is

$$U = -\int_{-\infty}^{\infty}\left[\int_{-\infty}^{-2x_\alpha+u_p-e}(\mu_m+u_p)^2 dF_m(\mu_m) + \int_{-2x_\alpha+u_p-e}^{-x_\alpha-e}(e+2(\mu_m+x_\alpha))^2 dF_m(\mu_m) + \int_{-x_\alpha-e}^{x_\alpha-e} e^2 dF_m(\mu_m) + \int_{x_\alpha-e}^{2x_\alpha+u_p-e}(e+2(\mu_m-x_\alpha))^2 dF_m(\mu_m) + \int_{2x_\alpha+u_p-e}^{\infty}(\mu_m+u_p)^2 dF_m(\mu_m)\right] dF_e(e),$$

where we use the symmetry of the distributions to simplify the expressions.

The following proposition shows that when the proposer is sufficiently extreme, located further from the median legislator than the indifferent key legislator, then further increases in extremism of the proposer raise the optimal majority voting rule. Indexing the optimal voting rule by u_p, we have $u_p > x^*_\alpha(u_p) > 0$ implies $\frac{\partial x^*_\alpha(u_p)}{\partial u_p} \geq 0$:

Proposition 15. *If $u_p > x^*_\alpha(u_p) > 0$, a marginal increase in u_p raises the optimal supermajority.*

A marginal change in the voting rule affects utility only through its effect on constrained proposers. When the legislature is highly polarized, i.e., when u_p is large relative to x_α, *the proposer closest to the status quo is never constrained*. The optimal voting rule distance, x^*_α, ensures that conditional on the other proposer (the one furthest from the status quo) being constrained, his expected proposed policy equals the representative citizen's expected bliss point.

Consider the proposer who is *marginally free*. This proposer lies exactly on the edge of the feasible policy set. A marginal shift of this proposer away from the status quo makes the proposer constrained. For a given voting rule x_α, the marginally-free proposer $m+u_p$ is located at $2(x_\alpha+u_p)$ and the marginally-free proposer $m-u_p = -2(x_\alpha+u_p)$.

For a given voting rule, the only effect on policy outcomes of raising a proposer's polarity is when it shifts the status of the proposer from free to constrained, i.e., when the proposer is marginally free. Such proposers now enter the social planner's first-order conditions at policies $2(x_\alpha + u_p)$ and $-2(x_\alpha + u_p)$. To see that these marginally-free proposers shift policy too far, on average, observe that the median legislator is located closer to the status quo than these proposers; with the quasi-concavity of $f_e(e)$, the expected location of the representative citizen is also closer to the status quo. Hence, a further increase in the already substantial polarity u_p of the proposer makes a larger supermajority optimal.

The next proposition shows that this logic holds true more generally for distributions that place more weight on sufficiently polarized proposers.

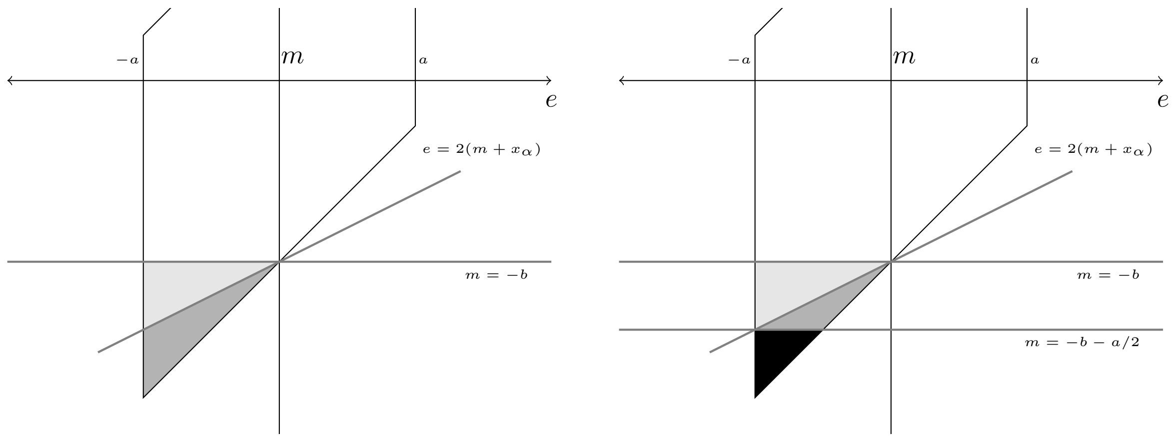
Proposition 16. *Consider distributions F_p^1 and F_p^2, where $F_p^1(z) = F_p^2(z)$ for all $-x_\alpha^{*1} \leq z \leq x_\alpha^{*1}$; but $F_p^2(z)$ has more dispersed tails so that $F_p^2(z) \leq F_p^1(z)$ for all $z > x_\alpha^{*1}$, strictly for some z and $x_\alpha^{*1} > 0$. Then $x_\alpha^{*2} > x_\alpha^{*1}$.*

The impact of increasing u_p on the optimal voting rule when *both* conservative and radical proposers can be constrained, i.e., when $x_\alpha > u_p$, is less clear. Consider conservative and radical marginally-free proposers who lie to the right of the status quo, i.e., the conservative proposer lies at $2(x_\alpha - u_p) > 0$, while the radical proposer lies further away at $2(x_\alpha + u_p)$. While both of these proposers move policy too far on average relative to the representative citizen's bliss point, the radical marginally-free proposer moves policy further, since he lies further from the status quo.

An increase in the proposer's polarity increases the measure of constrained radical proposers, and *decreases* that of constrained *conservative* proposers. Since the median legislator associated with the conservative marginally-free proposer lies closer to the status quo at $2x_\alpha - u_p$ than the median legislator associated with the radical marginally-free proposer at $2x_\alpha + u_p$, he is more likely to be realized. Therefore, the social planner places a higher probability weight on conservative marginally-free proposers than on radical ones.

We next characterize sufficient conditions for the optimal voting rule to *fall* as the polarity of the proposer rises. In general, for an increase in u_p to reduce the optimal voting rule, the dispersion of the median legislator around e must be large relative to the variation in e. When this dispersion increases, so too does x_α^*. The policy outcome induced by a conservative marginally-free proposer lies closer to the status quo, and thus is given greater weight by the social planner relative to the outcome induced by a radical marginally-free proposer.

Proposition 17. **(Uniform Uncertainty).** *Let $e \sim U[-a, a]$ and $\mu_m \sim U[-b, b]$ where $b > a$, i.e., there is more uncertainty about the median legislator's preferences than society's. Then, if the proposer's polarity is intermediate, $b - a < u_p < b - \frac{a}{2}$, the optimal supermajority is less than when the median is the proposer; abusing notation, $\alpha_{u_p}^* < \alpha_m^*$. If $u_p \leq b - a$ or $u_p \geq b$, then $\alpha_{u_p}^* = \alpha_m^*$.*



2.10.1: $u_p = 0$.

2.10.2: $u_p = b - a/2$.

Figure 2.10: These figures graphically depict the optimal voting rule first-order condition when $e \sim U[-a, a]$ and $m \sim U[e - b, e + b]$ for different polarities. The shaded area under $m = -b$ is the area of integration.

Since $b > a$, at the optimal supermajority of b, the median proposer is never free. Figure 2.10 provides a graphical representation of the proof. Figure 2.10.1 shows the first-order conditions when $a < b$, the voting rule distance is b and the proposer's polarity is zero. Figure 2.10.2 shows the first-order conditions with the same parameter values when the proposer's polarity is $u_p = b - a/2$. The light grey area in each figure represents the range of e and m for which the integrand is negative and the dark grey area represents the range of e and m for which the integrand is positive. In Figure 2.10.1 where $u_p = 0$, the light and dark grey areas exactly offset each other since the probability density of e and m over each area is constant and equal.

This is no longer the case in Figure 2.10.2 where $u_p = b - a/2$. For values of e and m in the lightly shaded areas, the density is the same as in Figure 2.10.1. The policy outcome for these values is exactly the same as when the polarity is zero. However, the density of the black lower left triangle is half that of the lightly-shaded areas. This triangle represents values of e and m for which the conservative proposer is constrained, while the radical proposer is free. The resulting distribution of new policy is too close to the status quo. Reducing the voting rule improves welfare.

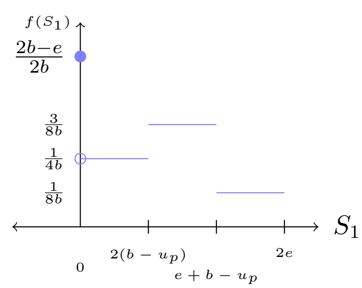


Figure 2.11: The distribution of policy given $e \in [\epsilon, a]$ when $b - u_p < e < b$ and $x_\alpha = b$; the dot at $S_1 = 0$ is the point mass on the status quo policy outcome of zero.

Figure 2.11 shows the distribution of new policy under the voting rule $x_\alpha = b$ when $b - u_p < e < b$. Comparing Figures 2.11 and 2.6.2 reveals that the distribution of S_1 close to the status quo is the same as when the median legislator proposes policy. When m is sufficiently close to the status quo, the feasible policy set is small, and both conservative and radical proposers are constrained. Shifting m one unit further away expands the feasible policy set by two units, and if e is sufficiently far from the status quo that $e - (b - u_p) > 0$, then eventually outward shifts leave the conservative polarized proposer free while the median and radical polarized proposers remain constrained. It follows that relative to when the median legislator proposes policy, the distribution of policy outcomes is shifted toward the status quo, indicating for these realizations of e that a lower supermajority, which reduces constraints on conservative polarized proposers, would raise welfare. There are two relevant possibilities at the optimal voting rule for the median proposer: either all three types of proposers are (equally) constrained, in which case they contribute in the same way to the first-order conditions; or the conservative proposer is free, but the median and radical proposers are constrained. It follows that reducing the supermajority slightly replaces a marginally-free median proposer who implements the policy $2(e + \epsilon)$ with either a marginally-free radical proposer who also implements $2(e + \epsilon)$ or a marginally-free conservative proposer who implements a policy $2(b - u_p - \epsilon)]$ that is closer to that preferred by society, raising welfare.

Proposition 18. (Normal Uncertainty). *Let $e \sim N(0, \sigma_e^2)$ and $\mu_m \sim N(0, \sigma_m^2)$. Then a marginal increase in u_p from zero reduces the optimal voting rule if and only if $\sigma_e^2/\sigma_m^2 < \beta$, where $\beta = 4\eta^2 - 2$ and η solves*

$$\frac{4\eta^2}{8\eta^2 - 2} = \eta \frac{[\Phi(2\eta) - \Phi(\eta)]}{[\phi(\eta) - \phi(2\eta)]}.$$

When u_p is zero, i.e., when the median legislator is the proposer, then the first-order effect of an increase in u_p on the optimal voting rule is zero (equation (2.16) in the Appendix). In the limit the conservative and radical marginally-free proposers realize the same policy outcome and are given the same weight by the social planner since they have the same probability of being realized.

The second-order effect is

$$\left.\frac{\partial^3 U}{\partial x_\alpha \partial u_p^2}\right|_{u_p=0} = 8 \int_{-\infty}^{\infty} 2 f_m(2x_\alpha + e) + (e + 2x_\alpha) f_m'(2x_\alpha + e) dF_e(e). \tag{2.9}$$

The first term in the integrand of equation (2.9) represents the net movement in policy away from the status quo conditional on the proposer being constrained from the additional radical constrained proposers and fewer conservative constrained proposers. This effect is always positive, and leads to higher voting rules. The second term represents the difference in probability weight that the social planner places on the conservative marginally-free proposer relative to the radical marginally-free proposer. Since the density is symmetric around $2x_\alpha$ as a function of e, this difference is strictly positive. For the sufficiently large voting rules implied in this context by the sufficiently high volatility in the legislature relative to society, $\frac{\sigma_e^2}{\sigma_m^2} < \beta$, the second effect outweighs the first.

2.5 Initial Policy Bias

We now study how initial policy bias affects the optimal voting rule: with bias the initial policy no longer equals the representative citizen's expected bliss point. This analysis sheds insights into how the optimal voting rule is affected when past policy changes by the legislature are slow to catch up with societal preferences. Intuition might suggest that initial policy bias should lead the social planner to rely more on the legislature to determine policy, and hence cause him to reduce the voting rule. We show that this conjecture is false when the dispersion in m is sufficiently small relative to the volatility in e. That is, we prove that when legislative preferences tend to be more representative of society's, then slight initial policy bias *raises* the optimal voting rule: whenever the legislature tends to be responsible, slight policy bias causes the social planner to rely *less* on the legislature! Only when legislative preferences are less likely to be representative of society's, does initial policy bias *reduce* the optimal voting rule.

To begin we must redefine the feasible policy set for a general initial policy, S_0. The feasible policy set has the same general properties as when $S_0 = 0$. In particular, for the reasons as outlined in Section 2.2, there still exist key legislators whose support is necessary and sufficient to change policy. If the median legislator is too close to the status quo, $|m - S_0| < x_\alpha$, then the feasible policy set is the singleton S_0 as one key legislator is always located on the opposite side of the status quo from the median legislator. Otherwise, the feasible policy set, $\hat{R}(m, \alpha, S_0)$, is an interval whose limit at one end is the status quo and at the other is the policy that makes the key legislator closest to the status quo indifferent between the policy change and the status quo. When $S_0 < m - x_\alpha$, the key legislator at $m - x_\alpha$ supports any policy $k \in [S_0, 2(m - x_\alpha) - S_0]$. When $m + x_\alpha < S_0$, the key legislator at $m + x_\alpha$ supports any policy $k \in [2(m + x_\alpha) - S_0, S_0]$. Summarizing, for $\alpha \geq 1/2$,

$$\hat{R}(m, \alpha, S_0) = \begin{cases} [S_0, 2(m - x_\alpha) - S_0] & \text{if } S_0 < m - x_\alpha \\ S_0 & \text{if } |m - S_0| < x_\alpha \\ [2(m + x_\alpha) - S_0, S_0] & \text{if } m + x_\alpha < S_0 \end{cases}.$$

A proposer p is free if $p \in \hat{R}(m, \alpha, S_0)$. If the closest element of the feasible policy set to the proposer is the status quo, then the proposer is blocked. This occurs if either the feasible policy is the status quo or the proposer lies on the opposite side of the status quo to the feasible policy set. Otherwise, the proposer is constrained.

From this we derive S_1 as an explicit function of m, p, S_0 and x_α. We again restrict attention to majority voting rules. For $\alpha \geq 1/2$,

$$S_1 = \begin{cases} S_0 & \text{if } |m_1 - S_0| < x_\alpha \text{ or } m_1 + x_\alpha < S_0 < p \text{ or } m_1 - x_\alpha > S_0 > p \\ p & \text{if } (m_1 + x_\alpha < S_0 \text{ and } 2(m_1 + x_\alpha) - S_0 < p < S_0) \\ & \text{or } (m_1 - x_\alpha > S_0 \text{ and } S_0 < p < 2(m_1 - x_\alpha) - S_0) \\ 2(m_1 + x_\alpha) - S_0 & \text{if } m_1 + x_\alpha < S_0 \text{ and } p < 2(m_1 + x_\alpha) - S_0 \\ 2(m_1 - x_\alpha) - S_0 & \text{if } S_0 < m_1 - x_\alpha \text{ and } 2(m_1 - x_\alpha) - S_0 < p \end{cases}.$$

As when $S_0 = 0$, if the proposer's polarity u_p exceeds x_α then the proposer who lies closest to the status quo is never constrained. He is either blocked (when the median legislator is sufficiently close to the status quo) or free (when the median legislator is sufficiently far away).

If $x_\alpha < u_p$, then the representative citizen's expected utility is

$$\begin{aligned} 2U &= -\int_{-\infty}^{\infty}\int_{-\infty}^{-u_p+S_0-e}(\mu_m+u_p)^2dF_m(\mu_m)+\int_{-u_p+S_0-e}^{x_\alpha+S_0-e}(e-S_0)^2dF_m(\mu_m) \\ &+ \int_{x_\alpha+S_0-e}^{2x_\alpha+u_p+S_0-e}(e+2(\mu_m-x_\alpha)-S_0)^2dF_m(\mu_m)+\int_{2x_\alpha+u_p+S_0-e}^{\infty}(\mu_m+u_p)^2dF_m(\mu_m) \\ &+ \int_{-\infty}^{-2x_\alpha-u_p+S_0-e}(\mu_m-u_p)^2dF_m(\mu_m)+\int_{-2x_\alpha-u_p+S_0-e}^{-x_\alpha+S_0-e}(e+2(\mu_m+x_\alpha)-S_0)^2dF_m(\mu_m) \\ &+ \int_{-x_\alpha+S_0-e}^{u_p+S_0-e}(e-S_0)^2dF_m(\mu_m)+\int_{u_p+S_0-e}^{\infty}(\mu_m-u_p)^2dF_m(\mu_m)dF_e(e), \end{aligned}$$

and if $x_\alpha > u_p$, then

$$\begin{aligned} 2U &= -\int_{-\infty}^{\infty}\int_{-\infty}^{-2x_\alpha+u_p+S_0-e}(\mu_m+u_p)^2dF_m(\mu_m)+\int_{-2x_\alpha+u_p+S_0-e}^{-x_\alpha+S_0-e}(e+2(\mu_m+x_\alpha)-S_0)^2dF_m(\mu_m) \\ &+ \int_{-x_\alpha+S_0-e}^{x_\alpha+S_0-e}(e-S_0)^2dF_m(\mu_m)+\int_{x_\alpha+S_0-e}^{2x_\alpha+u_p+S_0-e}(e+2(\mu_m-x_\alpha)-S_0)^2dF_m(\mu_m) \\ &+ \int_{2x_\alpha+u_p+S_0-e}^{\infty}(\mu_m+u_p)^2dF_m(\mu_m)+\int_{-\infty}^{-2x_\alpha-u_p+S_0-e}(\mu_m-u_p)^2dF_m(\mu_m) \\ &+ \int_{-2x_\alpha-u_p+S_0-e}^{-x_\alpha+S_0-e}(e+2(\mu_m+x_\alpha)-S_0)^2dF_m(\mu_m)+\int_{-x_\alpha+S_0-e}^{x_\alpha+S_0-e}(e-S_0)^2dF_m(\mu_m) \\ &+ \int_{x_\alpha+S_0-e}^{2x_\alpha-u_p+S_0-e}(e+2(\mu_m-x_\alpha)-S_0)^2dF_m(\mu_m)+\int_{2x_\alpha-u_p+S_0-e}^{\infty}(\mu_m-u_p)^2dF_m(\mu_m)dF_e(e). \end{aligned}$$

Because the initial policy is biased, we can no longer use symmetry to simplify the expressions.

To facilitate analysis, we suppose that F_e and F_m are normally distributed. Greater initial policy bias implies that the status quo is less representative of society's bliss point. The next two propositions provide sufficient conditions on the volatility of societal preferences, median legislator preferences and the proposer's polarity for an increase in initial policy bias from zero to either decrease or increase the optimal supermajority.

Proposition 19. *Let $e \sim N(0, \sigma_e^2)$ and $m \sim N(e, \sigma_m^2)$. Introducing a marginal bias in the initial policy away from the representative citizen's expected bliss point reduces the optimal voting rule if either (1) the median legislator is the proposer and $\sigma_e^2 < 2\sigma_m^2$, or (2) $u_p > \tilde{u}_p$ and $\lambda\sigma_e^2 < \sigma_m^2$ where $\lambda = \frac{2(\Phi(3)-\Phi(1))-\phi(1)+\phi(3)}{2(\phi(1)-\phi(3)+\Phi(1)-\Phi(3))} \approx 0.480$ and $\tilde{u}_p$ is the polarity that solves*

$$\Delta(\sigma_e^2, \sigma_m^2) = \frac{u_p(\Phi(3u_p/\delta) - \Phi(u_p/\delta))}{\delta(\phi(u_p/\delta) - \phi(3u_p/\delta))}.$$

Proposition 20. *Let $e \sim N(0, \sigma_e^2)$, $m \sim N(e, \sigma_m^2)$ and $\sigma_m^2 < \chi\sigma_e^2$ where $\chi = \frac{2(1-\Phi(1))-\phi(1)}{2(\phi(1)+\Phi(1)-1)} \approx 0.452$. There exists $\bar{u}_p$ such that for all $u_p > \bar{u}_p$, introducing a marginal bias in the initial policy away from the representative citizen's expected bliss point increases the optimal voting rule.*

The first-order effect of slight positive policy bias on the optimal policy rule is zero. Proposers to the left of the representative citizen's expected bliss point are less constrained and generate greater shifts in policy away from the status quo, but this is exactly offset by the reduced shifts away from the status quo by proposers to the right who are more constrained.

Hence, we need to understand the second-order effects. These are similar in nature to those for the impact of a marginal increase in the proposer's polarity from zero:

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2} = 4\int_{-\infty}^{\infty} [(2(x_\alpha + u_p) - e)f_m'(2x_\alpha + u_p - e) + ef_m'(x_\alpha - e)]dF_e(e). \tag{2.10}$$

There are two second-order effects of a slight positive increase in policy bias from zero: those associated with changes in the probabilities that a proposer is (a) marginally free vs. (b) marginally blocked, where a proposer is marginally blocked when the median legislator lies exactly x_α from the status quo.

With a positive marginal shift in the status quo, the marginally-free proposer to the left of the status quo is closer to the representative citizen's expected bliss point (implying insufficient policy movement), while the marginally-free proposer to the right of the status quo is further from the expected bliss point (implying excessive policy movement). Since the marginally-free proposer to the left of the status quo lies closer to the expected bliss point than the one to the right, the social planner puts greater probability weight on the marginally-free proposer to the left, suggesting a larger supermajority increases utility.

Offsetting this, however, the marginally-blocked proposer to the left of the status quo now lies closer to the representative citizen's ex-ante bliss point (implying excessive policy movement) and the marginally-blocked proposer to the right of the status quo now lies further away (implying insufficient policy movement). Since the marginally-blocked proposer to the left lies closer to the expected bliss point than the one to the right, the social planner puts greater probability weight on the marginally-blocked proposer to the left, suggesting a larger supermajority decreases utility.

Which effect dominates depends on the slopes of the density of the median legislator's bliss point at the marginally-free and marginally-blocked proposers. The optimal voting rule x_α is a bounded function of u_p, implying an upper bound to the marginally-blocked proposer effect. The weight the social planner places on the marginally-free proposer effect is determined by the optimal voting rule and u_p. When u_p is high, and σ_m^2 is small (so that x_α is sufficiently small), the weight the social planner places on the marginally-free proposer effect relative to the marginally-blocked proposer effect is sufficient to cause x_α^* to fall when slight initial policy bias is introduced.

2.6 Conclusion

Society's preferences over government policy shift as circumstances and society itself evolve. Ceteris paribus, having a legislature that can freely tailor policy to reflect society is good. However, a legislature's composition may not always reflect that of society. In particular, the views of the median legislator may sometimes be unrepresentative of society: An unchecked legislature can sometimes implement bad policy. The legislative process itself, by choosing more extreme agenda setters, may also generate less representative outcomes. We characterize how the optimal degree of inertia—the required vote share to implement a proposed policy rather than the status quo—is affected by the primitives describing the preferences of society, the median legislator, and the agenda setter.

The optimal voting rule trades off between excessively constraining "responsible" legislatures versus protecting society from irresponsible ones. With quadratic societal preferences for government policy, when the initial status quo is an unbiased representation of societal preferences, the optimal voting rule is such that, conditional on a proposer being constrained by the legislature, the expected difference between new policy implemented and society's bliss point is zero.

Neither submajority voting rules nor unanimity are ever optimal; and for some distributions, slight supermajorities may also never be optimal. Supermajority voting rules are blunt instruments, both restricting for good and for bad. For realizations of the legislature that are close to both the status quo and the representative citizen's bliss point, they are overly restrictive, while for legislatures that are far from both the status quo and the representative citizen's bliss point, they are not restrictive enough. Discontinuity in the optimal voting rule as a function of the underlying parameters from majority to supermajority can emerge as slight supermajorities tend to restrict the less radical proposers disproportionately.

When the median legislator is always the proposer, we show that for three classes of distributions—normal, uniform and two-point—increasing volatility of the legislature raises the optimal supermajority rule. The first-order intuition is that when the median legislator's preferences are more dispersed around the representative citizen's preferred policy, the median legislator is more likely to want to implement a "worse" policy. As a result, the social planner wants to further constrain him. However, more volatile legislative preferences need not always imply greater supermajorities. A mean preserving spread of the distribution of the median legislator's preferences can be created that (a) reduces the measure of blocked proposers, who are insensitive to marginal changes in the voting rule and (b) increases the measure of heavily constrained proposers, who, on average, generate insufficient movement in policy. The implication is insufficient average movement in policy. Thus, the optimal voting rule falls.

We then consider how the selection of the policy proposer affects the optimal voting rule. We consider randomly-selected proposers who lie distance u_p to the left or right of the median legislator with equal probability. When the legislature is highly polarized, more extreme proposers imply greater supermajorities since the views of proposers who are more extreme within the legislature also tend to be further from the representative citizen's, making it more important to restrain them. However, when polarity is slight, the relationship between polarity and the optimal supermajority is U-shaped. More extreme proposers, who would like to propose more extreme policies on average, are more constrained by the necessity to win the approval from moderate representatives. As the dispersion of the median legislator around the representative citizen's bliss point increases, then, it is more likely that, *conditional on legislative success*, a more extreme proposer moves policy in the direction preferred by society.

We then study how initial policy bias affects the optimal voting rule. One's intuition may be that initial policy bias should induce the social planner to rely more on the legislature to determine policy. However, we prove that when the dispersion of the median legislator around society's preferred policy is sufficiently small relative to the variation in society's preferred policy, introducing slight initial policy bias *raises* the optimal voting rule. The intuition is that with slight policy bias, the proposers who are more likely to be constrained by a given supermajority are those who would move policy in the direction away from what the representative citizen prefers.

2.7 Chapter 2 Proofs

Proof of Lemma 5: Consider $\alpha < 1/2$. From equation (2.2) and the distribution of e, m and p, utility is

$$\begin{aligned}
U = & -\int_{-\infty}^{\infty}\int_{-\infty}^{x_\alpha}\int_{-\infty}^{2(m+x_\alpha)}(e-2(m+x_\alpha))^2 dF_p(p-m) \\
& + \int_{2(m+x_\alpha)}^{0}(e-p)^2 dF_p(p-m) + \int_0^{\infty} e^2 dF_p(p-m)dF_m(m-e) \\
& + \int_{x_\alpha}^{-x_\alpha}\int_{-\infty}^{2(m+x_\alpha)}(e-2(m+x_\alpha))^2 dF_p(p-m) + \int_{2(m+x_\alpha)}^{2(m-x_\alpha)}(e-p)^2 dF_p(p-m) \\
& + \int_{2(m-x_\alpha)}^{\infty}(e-2(m-x_\alpha))^2 dF_p(p-m)dF_m(m-e) \\
& + \int_{-x_\alpha}^{\infty}\int_{-\infty}^{0} e^2 dF_p(p-m) + \int_0^{2(m-x_\alpha)}(e-p)^2 dF_p(p-m) \\
& + \int_{2(m-x_\alpha)}^{\infty}(e-2(m-x_\alpha))^2 dF_p(p-m)dF_m(m-e)dF_e(e).
\end{aligned}$$

The first-order condition is

$$\begin{aligned}
\frac{\partial U}{\partial x_\alpha} = & 4\int_{-\infty}^{\infty}\int_{-\infty}^{-x_\alpha}\int_{-\infty}^{2(m+x_\alpha)}(e-2(m+x_\alpha))dF_p(p-m)dF_m(m-e) \\
& - \int_{x_\alpha}^{\infty}\int_{2(m-x_\alpha)}^{\infty}(e-2(m-x_\alpha))dF_p(p-m)dF_m(m-e)dF_e(e) \\
= & 8\int_{-\infty}^{\infty}\int_{-\infty}^{-x_\alpha}\int_{-\infty}^{2(m+x_\alpha)}(e-2(m+x_\alpha))dF_p(p-m)dF_m(m-e)\, dF_e(e) \\
= & 8\int_{-\infty}^{\infty}\int_{-\infty}^{-x_\alpha}(e-2(m+x_\alpha))F_p(m+2x_\alpha)dF_m(m-e)dF_e(e),
\end{aligned}$$

where the second equality uses the symmetry of the distribution functions. Note that differentiation of the limits of integration cancel each other out. We prove that the optimal voting rule is not a minority by showing that the first-order condition is positive for any $\alpha < 1/2$. To do this, we divide the possible realizations of m and e into three set sequences, N_n, P_n and R_n, where $n = 1, \ldots, \infty$. For two of these sets, P_n and R_n, the value of the integrand is positive and for the third it is negative. A point-wise comparison of the elements of N_n and P_n, where the position of the median legislator is equal, will show that the value of the integrand over P_n more than offsets the value of the integrand over N_n.

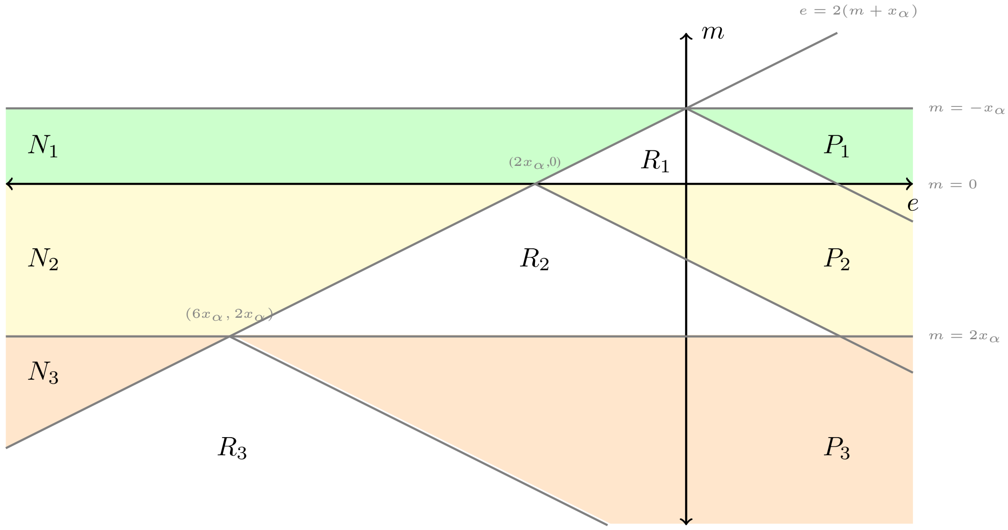


Figure 2.12: The area under $m = -x_\alpha$ is to be integrated. The shaded area to the right of $e = 2(m + x_\alpha)$, P_n, has a positively-valued integrand that more than offsets the negatively-valued integrand in the shaded area to the left of $e = 2(m + x_\alpha)$, N_n.

Define $m_0 = -x_\alpha$, $m_n = 2(m_{n-1} + x_\alpha)$ and the following sets,

$$N_n = \{(m, e)|m_n < m \leq m_{n-1}, e \leq 2(m + x_\alpha)\},$$

$$P_n = \{(m, e)|m_n < m \leq m_{n-1}, e \geq 2(2m_{n-1} + x_\alpha - m)\} \text{ and}$$

$$R_n = \{(m, e)|m_n < m \leq m_{n-1}, 2(m + x_\alpha) < e < 2(2m_{n-1} + x_\alpha - m)\}.$$

As required, the value of the integrand $(e - 2(m + x_\alpha))F_p(m + 2x_\alpha|m)f_m(m|e)f_e(e)$ is negative over the set N_n and positive over the sets P_n and R_n. Hence,

$$\begin{aligned}
\frac{\partial U}{\partial x_\alpha} = 8\sum_{n=1}^{\infty}\Bigg(&\iint_{N_n}(e - 2(m + x_\alpha))F_p(m + 2x_\alpha|m)f_m(m - e)f_e(e)dmde \\
&+ \iint_{P_n}(e - 2(m + x_\alpha))F_p(m + 2x_\alpha|m)f_m(m - e)f_e(e)dmde \\
&+ \iint_{R_n}(e - 2(m + x_\alpha))F_p(m + 2x_\alpha|m)f_m(m - e)f_e(e)dmde\Bigg) \\
> 8\sum_{n=1}^{\infty}\Bigg(&\iint_{N_n}(e - 2(m + x_\alpha))F_p(m + 2x_\alpha|m)f_m(m - e)f_e(e)dmde \\
&+ \iint_{P_n}(e - 2(m + x_\alpha))F_p(m + 2x_\alpha|m)f_m(m - e)f_e(e)dmde\Bigg).
\end{aligned}$$

To show that for each element of N_n the absolute value of the integrand is no greater than the value of the integrand of a corresponding element of P_n, let $(m, e) \in N_n$ and define $e' = 4(m_{n-1} + x_\alpha) - e$. Then $(m, e') \in P_n$, since

$e \leq 2(m+x_\alpha)$ by construction of N_n, and $4(m_{n-1}+x_\alpha)-e \geq 2(2m_{n-1}+x_\alpha-m)$. Since $2(2m_{n-1}+x_\alpha-m) < 0$, $|e| > |e'|$ which, combined with the quasi-concavity and symmetry of f_e around zero implies $f_e(e) < f_e(e')$. Also, since $m \geq m_n$, $m - e > e' - m$ which, combined with the quasi-concavity and symmetry of f_m around e implies $f_m(m|e) < f_m(m|e')$. Finally, since $m \leq m_{n-1}$, $|e - 2(m + x_\alpha)| \leq (e' - 2(m + x_\alpha))$ (Figure 2.12 provides a graphical depiction of this proof). ■

Proof of Proposition 9: Let x^*_α maximize equation (2.4). By the definition of x^*_α, $F^{-1}_{l_1}(\alpha^*_{l_1}) = F^{-1}_{l_2}(\alpha^*_{l_2})$. Since $F^{-1}_{l_2}(\alpha^*_{l_2}) \geq F^{-1}_{l_1}(\alpha^*_{l_2})$, this implies $\alpha^*_{l_1} \geq \alpha^*_{l_2}$. If $\alpha^*_{l_2} > 0$ then $F^{-1}_{l_2}(\alpha^*_{l_2}) > F^{-1}_{l_1}(\alpha^*_{l_2})$ and $\alpha^*_{l_1} > \alpha^*_{l_2}$. ■

Proof of Proposition 10: Let $x'_\alpha = \bar{m} + \epsilon$, where $\epsilon < \bar{e}$, the support of e (which could be infinite). Ex ante, the probability of policy change is strictly positive since $e + \bar{m} > \bar{m}$ for all $e > \epsilon$ and $e - \bar{m} < \bar{m}$ for all $e < -\epsilon$. For any value of μ_m, policy is either unchanged or any policy change moves policy closer to e. To see this let $e \geq 0$ (the analysis for $e < 0$ is similar). For any $m = e + \mu_m < 0$, $S_1 = 0$ since $|m| < \bar{m}$. Movement in policy is only possible if $m = e + \mu_m > \bar{m} + \epsilon$. Fix such an m. The feasible policy set, $R(m, \alpha) = [0, 2(m - \bar{m} - \epsilon)]$. For all m, $2(m - \bar{m} - \epsilon) - e = e - 2(\bar{m} + \epsilon - \mu_m) < e$ since $\mu_m < \bar{m}$. Hence, any movement in policy is toward e and utility under x'_α is greater than under a voting rule where no change in policy is possible. ■

Proof of Proposition 11: If $u_e > u_m$, then

$$U = \begin{cases} -u_m^2 & \text{if } 0 < x_\alpha < \frac{u_e - u_m}{2} \\ -\frac{(u_e - 2(u_m + x_\alpha))^2 + u_m^2}{2} & \text{if } \frac{u_e - u_m}{2} < x_\alpha < \min\{u_e - u_m, \frac{u_e + u_m}{2}\} \\ -\frac{(u_e - 2(u_m + x_\alpha))^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \frac{u_e + u_m}{2} < x_\alpha < u_e - u_m \\ -\frac{u_e^2 + u_m^2}{2} & \text{if } u_e - u_m < x_\alpha < \frac{u_e + u_m}{2} \\ -\frac{u_e^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \max\{u_e - u_m, \frac{u_e + u_m}{2}\} < x_\alpha < u_e + u_m \\ -u_e^2 & \text{if } u_e + u_m < x_\alpha \end{cases}$$

and

$$\frac{\partial U}{\partial x_\alpha} = \begin{cases} 0 & \text{if } 0 < x_\alpha < \frac{u_e - u_m}{2} \\ 2(u_e - 2(u_m + x_\alpha)) & \text{if } \frac{u_e - u_m}{2} < x_\alpha < \min\{u_e - u_m, \frac{u_e + u_m}{2}\} \\ 4(u_e + 2x_\alpha) & \text{if } \frac{u_e + u_m}{2} < x_\alpha < u_e - u_m \\ 0 & \text{if } u_e - u_m < x_\alpha < \frac{u_e + u_m}{2} \\ 2(u_e + 2(u_m - x_\alpha)) & \text{if } \max\{u_e - u_m, \frac{u_e + u_m}{2}\} < x_\alpha < u_e + u_m \\ 0 & \text{if } u_e + u_m < x_\alpha \end{cases} .$$

If $u_e < u_m$ then

$$U = \begin{cases} -u_m^2 & \text{if } 0 \leq \frac{u_m - u_e}{2} \\ -\frac{(u_e - 2(u_m - x_\alpha))^2 + u_m^2}{2} & \text{if } \frac{u_m - u_e}{2} < x_\alpha < \min\{u_m - u_e, \frac{u_m + u_e}{2}\} \\ -\frac{(u_e - 2(u_m - x_\alpha))^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \frac{u_m + u_e}{2} < x_\alpha < u_m - u_e \\ -\frac{u_e^2 + u_m^2}{2} & \text{if } u_m - u_e < x_\alpha < \frac{u_m + u_e}{2} \\ -\frac{u_e^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \max\{u_m - u_e, \frac{u_m + u_e}{2}\} < x_\alpha < u_m + u_e \\ -u_e^2 & \text{if } u_m + u_e < x_\alpha \end{cases}$$

and

$$\frac{\partial U}{\partial x_\alpha} = \begin{cases} 0 & \text{if } 0 \leq \frac{u_m - u_e}{2} \\ -2(u_e - 2(u_m - x_\alpha) & \text{if } \frac{u_m - u_e}{2} < x_\alpha < \min\{u_m - u_e, \frac{u_m + u_e}{2}\} \\ 8(u_m - x_\alpha) & \text{if } \frac{u_m + u_e}{2} < x_\alpha < u_m - u_e \\ 0 & \text{if } u_m - u_e < x_\alpha < \frac{u_m + u_e}{2} \\ 2(u_e + 2(u_m - x_\alpha)) & \text{if } \max\{u_m - u_e, \frac{u_m + u_e}{2}\} < x_\alpha < u_m + u_e \\ 0 & \text{if } u_m + u_e < x_\alpha \end{cases}.$$

The utility from a voting rule $x_\alpha \in [0, \frac{|u_e - u_m|}{2}]$ equals $-\mu_m^2$. Over each interval where utility is a function of x_α, the second-order condition for a maximum is always satisfied. It is simple to show that the solution to the first-order condition over each interval of x_α satisfies the interval constraints only when $\max\{\frac{u_e + u_m}{2}, |u_e - u_m|\} < x_\alpha < u_e + u_m$. The relevant first-order condition is $2(u_e + 2(u_m - x_\alpha)) = 0$, which implies that $x_\alpha^* = u_m + u_e/2$, yielding utility $-u_e^2/2$. This exceeds $-u_m^2$ only if $u_e < \sqrt{2}u_m$. ■

Proof of Proposition 12: The form of the first-order condition in the uniform case of the median proposer depends on the relative values of a and b.

Case 1: $a > b$

$$\frac{\partial U}{\partial x_\alpha} = \begin{cases} -\frac{2{x_\alpha}^2}{a} & \text{if } x_\alpha < \frac{a-b}{2} \\ \frac{2{x_\alpha}^2 - 4(a-b)x_\alpha + (a-b)^2}{a} & \text{if } \frac{a-b}{2} < x_\alpha < \min\{a-b, \frac{a+b}{2}\} \\ \frac{2(3{x_\alpha}^2 - 4ax_\alpha + a^2 + b^2)}{a} & \text{if } \frac{a+b}{2} < x_\alpha < a-b \\ -\frac{x_\alpha({x_\alpha}^2 - (2a-b)x_\alpha + a^2 - b^2)}{ab} & \text{if } a-b < x_\alpha < \frac{a+b}{2} \\ -\frac{(x_\alpha - b)(a+b-x_\alpha)^2}{ab} & \text{if } \max\{a-b, \frac{a+b}{2}\} < x_\alpha < a+b \\ 0 & \text{if } a+b < x_\alpha. \end{cases}$$

If $a > 5/4b$, then for all $x_\alpha \in [0, \infty)$, $\frac{\partial U}{\partial x_\alpha} < 0$. Hence, $x_\alpha^* = 0$. If $5/4b > a > b$, then there are two possible solutions to the first-order condition; $x_\alpha^* = 0$ and $x_\alpha^* = (2a - b + c)/2$, where $c = \sqrt{5b^2 - 4ab}$. Comparing utility under the two voting rules gives

$$\begin{aligned} U\,\big|_{x_\alpha=(2a-b+c)/2} &= -\frac{12a^3 + 2ab(18b - 7c) + b^2(5c - 11b) + a^2(8c - 30b)}{24a} \\ &> -b^2/3 = U\,|_{x_\alpha=0} \;\text{ iff } a < \frac{(11 + 4\sqrt{7})b}{18} \approx 1.2b. \end{aligned}$$

Case 2: $a < b$

$$\frac{\partial U}{\partial x_\alpha} = \begin{cases} \frac{4{x_\alpha}^2}{b} & \text{if } x_\alpha < \frac{b-a}{2} \\ -\frac{(b-a)((b-2x_\alpha)^2 - ab)}{ab} & \text{if } \frac{b-a}{2} < x_\alpha < \min\{b-a, \frac{a+b}{2}\} \\ 4\frac{(b-x_\alpha)^2}{b} & \text{if } \frac{a+b}{2} < x_\alpha < b-a \\ -\frac{x_\alpha({x_\alpha}^2 - (2a-b)x_\alpha + a^2 - b^2)}{ab} & \text{if } b-a < x_\alpha < \frac{a+b}{2} \\ -\frac{(x_\alpha - b)(a+b-x_\alpha)^2}{ab} & \text{if } \max\{b-a, \frac{a+b}{2}\} < x_\alpha < a+b \\ 0 & \text{if } a+b < x_\alpha. \end{cases}$$

For all values of a and b the first-order condition has a unique solution, $x_\alpha^* = b$. Hence, the optimal voting rule is characterized by

$$x_\alpha^* = \begin{cases} 0 & \text{if } a > \frac{(11+4\sqrt{7})b}{18} \\ \frac{2a-b+\sqrt{5b^2-4ab}}{2} & \text{if } \frac{(11+4\sqrt{7})b}{18} > a > b \\ b & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Proof of Proposition 13: Letting $\mu_m = m - e$ in equation (2.6) and assuming $x^*_\alpha > 0$, we have

$$\frac{\partial U}{\partial x_\alpha} = 8\int_{-\infty}^{\infty}\int_{-2x_\alpha - e}^{-x_\alpha - e}(-e - 2(\mu_m + x_\alpha))dF_m(\mu_m)dF_e(e).$$

Expanding this gives

$$\begin{aligned}\frac{\partial U}{\partial x_\alpha} &= 8\int_{-\infty}^{\infty} e(F_m(-x_\alpha - e) - F_m(-2x_\alpha - e))dF_e(e)\\ &- 2x_\alpha\int_{-\infty}^{\infty}(F_m(-x_\alpha - e) - F_m(-2x_\alpha - e))dF_e(e)\\ &- 2\int_{-\infty}^{\infty}\mu_m(F_m(-x_\alpha - \mu_m) - F_m(-2x_\alpha - \mu_m))dF_m(\mu_m),\end{aligned}$$

where the last term comes from interchanging the order of integration, μ_m and e. Converting the distributions to standard normal distributions gives

$$\begin{aligned}\frac{\partial U}{\partial x_\alpha} =& -8\int_{-\infty}^{\infty} e(\Phi(\frac{-x_\alpha - e}{\sigma_m}) - \Phi(\frac{-2x_\alpha - e}{\sigma_m}))\phi(\frac{e}{\sigma_e})de/\sigma_e\\ &- 2x_\alpha\int_{-\infty}^{\infty}(\Phi(\frac{-x_\alpha - e}{\sigma_m}) - \Phi(\frac{-2x_\alpha - e}{\sigma_m}))\phi(\frac{e}{\sigma_e})de/\sigma_e\\ &- 2\int_{-\infty}^{\infty}\mu_m(\Phi(\frac{-x_\alpha - \mu_m}{\sigma_e}) - \Phi(\frac{-2x_\alpha - \mu_m}{\sigma_e}))\phi(\frac{\mu_m}{\sigma_m})d\mu_m/\sigma_m. \end{aligned} \tag{2.11}$$

From Patel and Read [1996],

$$\begin{aligned}\int_{-\infty}^{\infty}\Phi(a+bx)\phi(x)dx &= \Phi(a/\sqrt{1+b^2}) \quad \text{and}\\ \int_{-\infty}^{\infty}x\Phi(a+bx)\phi(x)dx &= \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right).\end{aligned}$$

Defining $\delta = \sqrt{\sigma_e^2 + \sigma_m^2}$, equation (2.11) simplifies to

$$\begin{aligned}\frac{\partial U}{\partial x_\alpha} =& 8\frac{\sigma_e^2}{\delta}\left[\phi\left(\frac{-x_\alpha}{\delta}\right) - \phi\left(\frac{-2x_\alpha}{\delta}\right)\right] - 2x_\alpha\left[\Phi\left(\frac{-x_\alpha}{\delta}\right) - \Phi\left(\frac{-2x_\alpha}{\delta}\right)\right]\\ &+ \frac{2\sigma_m^2}{\delta}\left[\phi\left(\frac{-x_\alpha}{\delta}\right) - \phi\left(\frac{-2x_\alpha}{\delta}\right)\right].\end{aligned}$$

Thus, when the median legislator is always the proposer, and the optimal voting rule distance x^*_α is strictly positive, x^*_α must solve

$$8\left(\frac{2\sigma_m^2 + \sigma_e^2}{2(\sigma_m^2 + \sigma_e^2)}\right)\left[\phi\left(\gamma\right) - \phi\left(2\gamma\right)\right] - 16\gamma\left[\Phi\left(\gamma\right) - \Phi\left(\gamma\right)\right] = 0.$$

Define $\Delta(\sigma_e^2, \sigma_m^2) = \frac{2\sigma_m^2+\sigma_e^2}{2\delta^2}, \Gamma(\gamma) = \gamma\frac{[\Phi(2\gamma)-\Phi(\gamma)]}{[\phi(\gamma)-\phi(2\gamma)]}$ and $\gamma(x_\alpha) = \frac{x_\alpha}{\delta}$. Then x_α^* solves

$$\Delta(\sigma_e^2, \sigma_m^2) = \Gamma(\gamma(x_\alpha)). \tag{2.12}$$

To show that the optimal voting rule is unique it is sufficient to show that $\Gamma(\gamma)$ is strictly increasing in γ. To do this we use the following lemma due to Pinelis [2002].

Lemma 6. *Let $-\infty \leq a < b \leq \infty$ and let f and g be differentiable functions on (a, b). Assume that either $g' > 0$ everywhere on (a, b) or $g' < 0$ on (a, b). Furthermore, suppose that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$ and f'/g' is strictly increasing on (a, b). Then the ratio f/g is strictly increasing on (a, b).*

To use this lemma, let $f(\gamma) = \Phi(2\gamma) - \Phi(\gamma)$ and $g(\gamma) = [\phi(\gamma) - \phi(2\gamma)]/\gamma$. Then

$$f'(\gamma) = 2\phi(2\gamma) - \phi(\gamma) = \frac{e^{-2\gamma^2}}{\sqrt{2\pi}}(2 - e^{\frac{3\gamma^2}{2}}),$$

and

$$\begin{aligned} g'(\gamma) &= \frac{\gamma^2(4\phi(2\gamma) - \phi(\gamma)) - \phi(\gamma) + \phi(2\gamma)}{\gamma^2} \\ &= \frac{e^{-2\gamma^2}}{\gamma^2\sqrt{2\pi}}(1 + 4\gamma^2 - e^{\frac{3\gamma^2}{2}}(1 + \gamma^2)). \end{aligned}$$

Let $\bar{\gamma} \approx 0.6536$ be the unique positive root to the transcendental equation $g'(\gamma) = 0$. Then over $(0, \bar{\gamma})$, $g'(\gamma) > 0$ and over $(\bar{\gamma}, \infty)$, $g'(\gamma) < 0$. Also, $f(0) = g(0) = 0$ and $f(\infty) = g(\infty) = 0$. Therefore,

$$\frac{f'(\gamma)}{g'(\gamma)} = \frac{\gamma^2}{1 + \gamma^2 + \frac{1-2\gamma^2}{e^{\frac{3\gamma^2}{2}}-2}} \text{ and}$$

$$\left(\frac{f'(\gamma)}{g'(\gamma)}\right)' = \frac{\gamma\left(4 + 2e^{3\gamma^2} + 3e^{\frac{3\gamma^2}{2}}(-2 + \gamma^2 - 2\gamma^4)\right)}{\left(1 + 4\gamma^2 - e^{\frac{3\gamma^2}{2}}(1 + \gamma^2)\right)^2}. \tag{2.13}$$

Equation (2.13) is increasing for all $\gamma \in (0, \infty)\backslash\{\bar{\gamma}\}$ since

$$\frac{\partial}{\partial\gamma}(4 + 2e^{3\gamma^2} + 3e^{\frac{3\gamma^2}{2}}(-2 + \gamma^2 - 2\gamma^4)) = 3e^{\frac{3\gamma^2}{2}}\gamma\left(4e^{\frac{3\gamma^2}{2}} - (4 + 5\gamma^2 + 6\gamma^4)\right)$$

and

$$\frac{\partial}{\partial\gamma}(4e^{\frac{3\gamma^2}{2}} - (4 + 5\gamma^2 + 6\gamma^4)) = 2\gamma\left(-5 + 6e^{\frac{3\gamma^2}{2}} - 12\gamma^2\right),$$

which is positive for all $\gamma > 0$. Hence, by Lemma 6, Γ is strictly increasing over the intervals $(0, \bar{\gamma})$ and $(\bar{\gamma}, \infty)$. By continuity of the function, Γ is strictly increasing over $[0, \infty)$.

We now use L'Hôpital's rule twice to determine when the first-order condition characterizes x_α^*. Let $\Gamma_N(\gamma) = \gamma(\Phi(2\gamma) - \Phi(\gamma))$ and $\Gamma_D(\gamma) = \phi(\gamma) - \phi(2\gamma)$. Then

$$\lim_{\gamma \to 0} \frac{\Gamma_N'(\gamma)}{\Gamma_D'(\gamma)} = \lim_{\gamma \to 0} \frac{\gamma(2\phi(2\gamma) - \phi(\gamma)) + \Phi(2\gamma) - \Phi(\gamma)}{\gamma(4\phi(2\gamma) - \phi(\gamma))} = \frac{0}{0},$$

where we use the fact that $\phi'(\lambda\gamma) = -\lambda\gamma\phi(\lambda\gamma)$, and

$$\lim_{\gamma \to 0} \frac{\Gamma_N''(\gamma)}{\Gamma_D''(\gamma)} = \lim_{\gamma \to 0} \frac{4\phi(2\gamma) - 2\phi(\gamma) + \gamma(4\phi'(2\gamma) - \phi'(\gamma))}{4\phi(2\gamma) - \phi(\gamma) + \gamma(8\phi'(2\gamma) - \phi'(\gamma))} = \frac{2}{3} = \Gamma(0).$$

The optimal voting rule is implicitly given by the solution to equation (2.12) if $\sigma_e^2 > 2\sigma_m^2$. Otherwise, $\alpha^* = 1/2$.

Next we show that the optimal voting rule increases with σ_m^2. As σ_m^2 increases, the left-hand side of equation (2.12) increases since

$$\frac{\partial \Delta(\sigma_e^2, \sigma_m^2)}{\partial \sigma_m^2} = \frac{\sigma_e^2}{2(\sigma_e^2 + \sigma_m^2)^2} > 0.$$

An increase in σ_m^2 evaluating at $x_\alpha = x_\alpha^*$ leads to decreases in γ and $\Gamma(\gamma)$. Hence, the right-hand side of equation (2.12) falls at $x_\alpha = x_\alpha^*$. Thus, the optimal voting rule must increase with σ_m^2.

Next, we show that the optimal voting rule is decreasing in the volatility of the representative citizen's bliss point. Differentiating equation (2.12) with respect to σ_e^2 gives

$$\begin{aligned}
\left.\frac{\partial^2 U}{\partial x_\alpha \partial \sigma_e^2}\right|_{x_\alpha = x_\alpha^*} &= -\frac{\sigma_m^2\,(\phi(\gamma) - \phi(2\gamma))}{2(\sigma_e^2 + \sigma_m^2)^2} + \left(\frac{\gamma(\sigma_e^2 + 2\sigma_m^2)(4\phi(2\gamma) - \phi(\gamma))}{2(\sigma_e^2 + \sigma_m^2)}\right. \\
&- \left.(\Phi(\gamma) - \Phi(2\gamma)) - \gamma(2\phi(2\gamma) - \phi(\gamma))\right)\frac{\partial \gamma}{\partial \sigma_e^2} \\
&= \frac{\sigma_e^2(1 - \gamma^2)\phi(\gamma) - (\sigma_e^2 + 4\sigma_m^2\gamma^2)\phi(2\gamma)}{4(\sigma_e^2 + \sigma_m^2)^2} \\
&< \sigma_e^2 \frac{(1 - \gamma^2)\phi(\gamma) - (1 + 2\gamma^2)\phi(2\gamma)}{4(\sigma_e^2 + \sigma_m^2)^2}.
\end{aligned}$$

The second equality follows from using equation (2.12) to substitute for $\gamma(\Phi(2\gamma) - \Phi(\gamma))$ and substituting $\frac{\partial \gamma}{\partial \sigma_e^2}$ with $-\frac{\gamma}{2(\sigma_e^2 + \sigma_m^2)}$. The inequality follows since $2\sigma_m^2 > \sigma_e^2$. Again it is straightforward to verify that this inequality is negative for all γ.

Finally, we show that $x_\alpha^*(\gamma\sigma_e, \gamma\sigma_m) = \gamma x_\alpha^*(\sigma_e, \sigma_m)$. Let $\sigma_m' = \eta\sigma_m$ and $\sigma_e' = \eta\sigma_e$. Then, $\Delta(\sigma_e'^2, \sigma_m'^2) =$

$\Delta(\sigma_e^2, \sigma_m^2)$. Hence, $\gamma(x_\alpha'^*) = \gamma(x_\alpha^*)$, which implies

$$\frac{x_\alpha'^*}{\sqrt{\sigma_m'^2 + \sigma_e'^2}} = \frac{x_\alpha^*}{\sqrt{\sigma_m^2 + \sigma_e^2}} \Rightarrow x_\alpha'^* = \left(\frac{\sqrt{\sigma_m'^2 + \sigma_e'^2}}{\sqrt{\sigma_m^2 + \sigma_e^2}} \right) x_\alpha^* = \eta x_\alpha^*. \qquad \blacksquare$$

Proof of Proposition 14: F second-order stochastically dominates G. Since $(10 > 1 + 8.9)$, $x_\alpha = 10$ is a possible optimal voting rule under the distribution F since the first-order condition that characterizes the optimal rule when m is uniformly distributed continues to hold. It is sufficient to check that $x_\alpha = 0$ gives a lower level of utility, which it indeed does, since $U(x_\alpha = 0; F) \sim -30.0316 < U(x_\alpha = 10; F) \sim -0.32897$.

Under the distribution G, $x_\alpha = 10$ is no longer a possible optimal voting rule since the first-order condition that characterizes x_α^* when m is uniformly distributed no longer holds. Since $10 < 1 + 9.1$, there is additional density over the negative values of the integrand. Since the density of m when $m \in [e, e - 9.1]$ is twice the density of m when $m \in [e - 9.1, e - 10]$, then x_α^* must solve

$$\int_{-a}^{10-x_\alpha} \int_{e-10}^{-x_\alpha} (e - 2(m + x_\alpha)) dm de + \int_{-a}^{9.1-x_\alpha} \int_{e-9.1}^{-x_\alpha} (e - 2(m + x_\alpha)) dm de = 0.$$

Evaluating, this simplifies to

$$1069.15 - 313.415 x_\alpha + 30.65 {x_\alpha}^2 - {x_\alpha}^3 \sim 0,$$

which implies that $x_{\alpha}{}_G^* \sim 9.989$. $\blacksquare$

Proof of Proposition 15: When $u_p > x_\alpha$ then

$$\frac{\partial U}{\partial x_\alpha} = -4 \int_{-\infty}^{\infty} \int_{-2x_\alpha - u_p - e}^{-x_\alpha - e} (e + 2(\mu_m + x_\alpha)) dF_m(\mu_m) dF_e(e). \tag{2.14}$$

Differentiating equation (2.14) with respect to u_p gives

$$\frac{\partial^2 U}{\partial x_\alpha \partial u_p} = 4 \int_{-\infty}^{\infty} (e + 2x_\alpha + 2u_p) f_m(-2x_\alpha - u_p - e) dF_e(e). \tag{2.15}$$

The function $e + 2x_\alpha + 2u_p$ is symmetric around $-2x_\alpha - 2u_p$ which is less than $-2x_\alpha - u_p$. The function $f_m(-2x_\alpha - u_p - e)$ is symmetric around $-2x_\alpha - u_p$ which is less than zero, while the function $f_e(e)$ is symmetric around zero. Hence, equation (2.15), evaluated at $x_\alpha = x_\alpha^*$, is positive and the optimal voting rule must increase. $\blacksquare$

Proof of Proposition 16: Since ${x_\alpha^*}^1 > 0$ equation (2.5) holds with equality at $x_\alpha = {x_\alpha^*}^1$ under the distribution F_p^1. A point-wise comparison shows that equation (2.5) is positive at $x_\alpha = {x_\alpha^*}^1$ under the distribution F_p^2. Since $F_p^1(z) = F_p^2(z)$, for all $-{x_\alpha^*}^1 \leq z \leq {x_\alpha^*}^1$, the value of the integrand in equation (2.5) is the same under F_p^1 and F_p^2

for all m such that $-3x_\alpha^{*1} < m < -x_\alpha^{*1}$. Now we show that conditional on $m < -3x_\alpha^{*1}$, the value of the integrand over e is positive for any distribution F_p by comparing points symmetrically around $m = e$. For every $e < 2(m - x_\alpha^{*1})$ let $e' = 2m - e$. By symmetry of f_m, $f_m(m-e) = f_m(m-e')$. Also, $2(m - x_\alpha^{*1}) - e < e' - 2(m - x_\alpha^{*1})$ since $m < -3x_\alpha^{*1}$. Finally, since $m < 0$, $|e'| < |e|$, which, together with symmetry and quasi-concavity of f_e implies $f(e) < f(e')$. Since $F_p^2(z) \leq F_p^1(z)$, for all $z > x_\alpha^{*1}$, strictly for some z, then $F_p^2(m - 2x_\alpha^{*1}) \geq F_p^1(m - 2x_\alpha^{*1})$, for all $m < -3x_\alpha^{*1}$, strictly for some m. Thus, equation (2.5) is positive at $x_\alpha = x_\alpha^{*1}$ under the distribution F_p^2, which implies $x_\alpha^{*2} > x_\alpha^{*1}$. ■

Proof of Proposition 17: The first-order condition when the median legislator is the proposer is defined by equation (2.6). Since $b > a$, the optimal voting rule distance when the median legislator proposes is b. If the median legislator m is replaced as proposer by someone who with equal probability, lies distance u_p to either the left or the right of m, then the first-order condition is defined by equation (2.5), where $F_p(z) = 0$ if $z < -u_p$, $F_p(z) = 1/2$ if $-u_p \leq z < u_p$, and $F_p(z) = 1$ if $u_p \leq z$. If $b - a < u_p < b - a/2$, then $F_p(b - a/2) = 1$ and $F_p(b-a) = 1/2$. Evaluated at α_m^*, the first-order condition is negative, implying $\alpha_p^* < \alpha_m^*$. If $u_p \leq b - a$ then $F_p(b - a/2) = 1$ and $F_p(b-a) = 1$. Evaluated at α_m^*, the first-order condition is zero, implying $\alpha_p^* = \alpha_m^*$. If $u_p \geq b$ then $F_p(b) = 1/2$ and $F_p(b-a) = 1/2$. Evaluated at α_m^*, the first-order condition is zero, implying $\alpha_p^* = \alpha_m^*$. ■

Proof of Proposition 18: When $u_p < x_\alpha^*$ the effect on utility from and increase in x_α is

$$\frac{\partial U}{\partial x_\alpha} = -4\int_{-\infty}^{\infty}\int_{-2x_\alpha+u_p-e}^{-x_\alpha-e}(e + 2(\mu_m + x_\alpha))dF_m(\mu_m) + \int_{-2x_\alpha-u_p-e}^{-x_\alpha-e}(e + 2(\mu_m + x_\alpha))dF_m(\mu_m)dF_e(e).$$

We see that

$$\left.\frac{\partial^2 U}{\partial x_\alpha \partial u_p}\right|_{u_p=0} = 4\int_{-\infty}^{\infty}(e + 2x_\alpha)f_m(2x_\alpha + e) - (e + 2x_\alpha)f_m(2x_\alpha + e)dF_e(e) = 0, \tag{2.16}$$

and

$$\left.\frac{\partial^3 U}{\partial x_\alpha \partial u_p^2}\right|_{u_p=0} = 8\int_{-\infty}^{\infty}2f_m(2x_\alpha + e) + (e + 2x_\alpha)f_m'(2x_\alpha + e)dF_e(e). \tag{2.17}$$

When $e \sim N(0, \sigma_e^2)$ and $\mu_m \sim N(0, \sigma_m^2)$ equation (2.17) simplifies to

$$\begin{aligned}\left.\frac{\partial^3 U}{\partial x_\alpha \partial u_p}\right|_{u_p=0} &= 8\int_{-\infty}^{\infty}\left(2 - \frac{(e + 2x_\alpha)^2}{\sigma_m^2}\right)\frac{\phi(\frac{2x_\alpha + e}{\sigma_m})}{\sigma_m}\frac{\phi(\frac{e}{\sigma_e})}{\sigma_e}de \\ &= 8\frac{\phi(\frac{2x_\alpha}{\delta})}{\delta^{5/2}}(\sigma_e^4 + 3\sigma_e^2\sigma_m^2 + 2\sigma_m^2(\sigma_m^2 - 2x_\alpha{}^2)).\end{aligned} \tag{2.18}$$

The positive root of equation (2.18) is $\tilde{x}_\alpha^* = \frac{\sqrt{\sigma_e^4+3\sigma_e^2\sigma_m^2+2\sigma_m^4}}{2\sqrt{\sigma_m^2}}$. So if $x_\alpha^* > \tilde{x}_\alpha^*$ then a marginal increase in u_p from zero leads to a reduction in the optimal voting rule. Let $\tilde{\gamma}(\sigma_m^2, \sigma_e^2) = \frac{\sqrt{\sigma_e^2+2\sigma_m^2}}{2\sqrt{\sigma_m^2}}$. Then

$$\Delta(\sigma_e^2, \sigma_m^2) = \Gamma(\tilde{\gamma}(\sigma_m^2, \sigma_e^2)), \tag{2.19}$$

defines σ_e^2 as an implicit function of σ_m^2, $\tilde{\sigma_e}^2(\sigma_m^2)$, for which the optimal voting rule is exactly $\tilde{x}_\alpha^*$ when $u_p = 0$.

Let (σ_m^2, σ_e^2) solve equation (2.19) and let $\sigma_m' = \eta\sigma_m$ and $\sigma_e' = \eta\sigma_e$. Then, $\Delta(\sigma_e'^2, \sigma_m'^2) = \Delta(\sigma_e^2, \sigma_m^2)$ and $\tilde{\gamma}(\sigma_m'^2, \sigma_e'^2) = \tilde{\gamma}(\sigma_m^2, \sigma_e^2)$. Hence, $(\sigma_m'^2, \sigma_e'^2)$ also solves equation (2.19) and it follows that $\tilde{\sigma_e^2}(\sigma_m^2) = \beta\sigma_m^2$.

We solve for β by substituting $\sigma_e^2 = \beta\sigma_m^2$ into equation (2.19), i.e., β is the solution to

$$\frac{4\eta^2}{8\eta^2 - 2} = \eta\frac{[\Phi(2\eta) - \Phi(\eta)]}{[\phi(\eta) - \phi(2\eta)]},$$

where $\eta = \frac{\sqrt{\beta+2}}{2}$. Solving numerically gives $\beta \approx 1.33872$. Recall from Proposition 13, as σ_e^2 increases, x_α^* decreases. Hence, as σ_e^2 increases above $\tilde{\sigma}_e^2(\sigma_m^2)$, equation (2.18) becomes positive, implying that the optimal voting rule must increase, and as σ_e^2 decreases below $\tilde{\sigma}_e^2(\sigma_m^2)$, equation (2.18) becomes negative, implying that the optimal voting rule must decrease. ■

Proposition 19 preliminaries: As a preliminary step in proving Proposition 19, we give a more detailed characterization of the optimal voting rule when the initial policy is unbiased and the legislative proposers are sufficiently extreme.

Lemma 7. *Suppose $S_0 = 0$, $e \sim N(0, \sigma_e^2)$, $m \sim N(e, \sigma_m^2)$, and define $\tilde{u}_p(\sigma_e^2, \sigma_m^2)$ implicitly to be the solution to*

$$\Delta(\sigma_e^2, \sigma_m^2) = \frac{u_p(\Phi(3u_p/\delta) - \Phi(u_p/\delta))}{\delta(\phi(u_p/\delta) - \phi(3u_p/\delta))}.$$

Then if $u_p > \tilde{u}_p(\sigma_e^2, \sigma_m^2)$, the optimal voting rule increases with the volatility σ_m^2 in the representativeness of society by the legislature.

When $u_p > x_\alpha$, the proposer nearest to the status quo is never constrained. An increase in σ_m^2 implies that, on average, a constrained median proposer lies further from e. Therefore, constraining the proposer further by increasing x_α, reduces the feasible policy set, improving welfare.

The function $\tilde{u}_p(\sigma_e^2, \sigma_m^2)$ is the polarity distance, given σ_e^2 and σ_m^2 at which $x_\alpha^*(u_p) = u_p$. Characterizing this function allows us to provide sufficient parametric conditions under which $x_\alpha^*(u_p) \leq u_p$. Since $\tilde{u}_p(\sigma_e^2, \sigma_m^2)$ is homogeneous of degree one, without loss of generality, we normalize $\sigma_e^2 = 1$ and show in Figure 2.13 how $\tilde{u}_p(1, \sigma_m^2)$ varies with σ_m^2. Consistent with Lemma 7, $\tilde{u}_p(1, \sigma_m^2)$ increases in σ_m^2. As σ_m^2 increases, so does x_α^*, so that $\tilde{u}_p(1, \sigma_m^2)$ must increase to maintain equality with x_α^*.

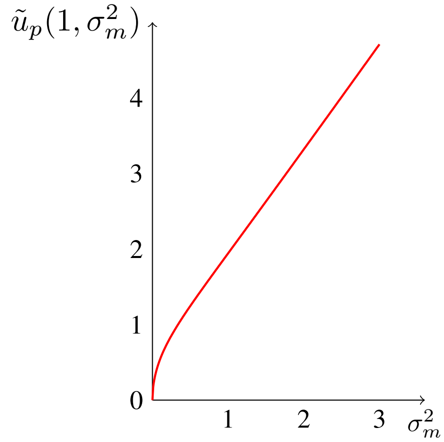


Figure 2.13: $\tilde{u}_p(1, \sigma_m^2)$ as a function of σ_m^2.

Proof of Lemma 7: When $S_0 = 0$, $e \sim N(0, \sigma_e^2)$, $m \sim N(e, \sigma_m^2)$ and $u_p > x_\alpha$ the optimal voting rule must satisfy equation (2.14). Define $\psi = \frac{u_p}{\delta}$ and $\Psi(\gamma, \psi) = \gamma \frac{(\Phi(2\gamma+\psi)-\Phi(\gamma))}{(\phi(\gamma)-\phi(2\gamma+\psi))}$. An approach similar to that used in the proof of Proposition 13 shows that at the optimal voting rule distance

$$\Delta(\sigma_e^2, \sigma_m^2) = \Psi(\gamma, \psi). \tag{2.20}$$

The function Ψ is decreasing in ψ since $\frac{\Phi(2\gamma+\psi)-\Phi(\gamma)}{\phi(\gamma)-\phi(2\gamma+\psi)}$ is the inverse of the expected value of a truncated standard normally distributed random variable between γ and $2\gamma + \psi$. As u_p increases, ψ increases. By Proposition 15, x_α^* is increasing in u_p, so γ also increases. Thus, Ψ must be increasing in γ for equation (2.20) to hold.

We now show that there exists a $\tilde{u}_p$ such that for all $u_p \geq \tilde{u}_p$, $x_\alpha^* \leq u_p$. Using the implicit function theorem, we derive how u_p affects x_α^*.

$$\frac{\partial \Psi}{\partial \psi} = \frac{\gamma(\phi(2\gamma+\psi)(\phi(\gamma)-\phi(2\gamma+\psi)) - \Phi(2\gamma+\psi) + \Phi(\gamma))}{(\phi(\gamma)-\phi(2\gamma+\psi))^2},$$

$$\begin{aligned}\frac{\partial \Psi}{\partial \gamma} =& [(\Phi(2\gamma+\psi) - \Phi(\gamma) + 2\gamma(\phi(2\gamma+\psi) - \phi(\gamma)))(\phi(\gamma) - \phi(2\gamma+\psi)) \\ &+\gamma(\Phi(2\gamma+\psi) - \Phi(\gamma))(\gamma\phi(\gamma) - 2(2\gamma+\psi)\phi(2\gamma+\psi))] / (\phi(\gamma) - \phi(2\gamma+\psi))^2,\end{aligned}$$

and

$$\frac{\partial x_\alpha^*}{\partial u_p} = -\frac{\frac{\partial \Psi}{\partial \psi}\frac{\partial \psi}{\partial u_p}}{\frac{\partial \Psi}{\partial \gamma}\frac{\partial \gamma}{\partial x_\alpha^*}} = -\frac{\frac{\partial \Psi}{\partial \psi}}{\frac{\partial \Psi}{\partial \gamma}}.$$

Hence, when $x_\alpha^* = u_p$,

$$\frac{\partial x_\alpha^*}{\partial u_p}\Big|_{x_\alpha^*=u_p} = -\frac{\psi(\phi(3\psi)(\phi(\psi)-\phi(3\psi)) - \Phi(3\psi) + \Phi(\psi))}{[(\Phi(3\psi)-\Phi(\psi)+2\psi(\phi(3\psi)-\phi(\psi)))(\phi(\psi)-\phi(3\psi)) + \psi(\Phi(3\psi)-\Phi(\psi))(\psi\phi(\psi)-6\phi(3\psi))]}. \tag{2.21}$$

At $u_p = 0$ equation (2.21) equals one, and a graphical analysis shows that the function is monotonically decreasing (see Figure 2.14). Hence, the ratio of x_α^*/u_p at $x_\alpha^* = u_p$ is decreasing. Thus, for any value of σ_m and σ_e there

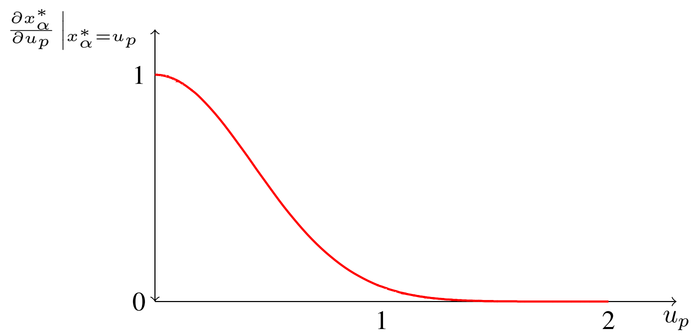


Figure 2.14: When $x_\alpha^* = u_p$, the increase in x_α^* as u_p increases is always less than 1.

is a unique u_p, $\tilde{u}_p$, at which $x_\alpha^* = \tilde{u}_p$ and for all $u_p \geq \tilde{u}_p$, $x_\alpha^* \leq u_p$. This critical polarity distance $\tilde{u}_p$ solves $\Delta(\sigma_e^2, \sigma_m^2) = \Psi(\psi, \psi)$. Note that a proportionally-equal increase in ψ and γ increases Ψ.

Now we show that if $u_p > x_\alpha^*$, then x_α^* is increasing in σ_m^2 . As shown in the proof of Proposition 13, the left-hand side of equation (2.20) is increasing in σ_m^2. As σ_m^2 increases, ψ and γ decrease by the same proportion, which implies that Ψ decreases. Hence, equation (2.20) becomes negative and x_α^* must increase to ensure the equality of equation (2.20) at the optimum. ■

Proof of Proposition 19: When the median legislator proposes policy, the change in utility from a marginal change in the voting rule distance is

$$\frac{\partial U(x_\alpha, S_0)}{\partial x_\alpha} = 4\int_{-\infty}^{\infty}\left(\int_{S_0-2x_\alpha}^{S_0-x_\alpha}\Big[e-2(m+x_\alpha)+S_0\Big]dF_m(m-e) - \int_{S_0+x_\alpha}^{S_0+2x_\alpha}\Big[e-2(m-x_\alpha)+S_0\Big]dF_m(m-e)\right)dF_e(e). \tag{2.22}$$

Differentiating equation (2.22) with respect to S_0 gives

$$\frac{\partial^2 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0} = 4\int_{-\infty}^{\infty} \Big[(e - S_0)\left(f_m(S_0 - x_\alpha - e) + f_m(S_0 + x_\alpha - e)\right) - (e - S_0 + 2x_\alpha) f_m(S_0 - 2x_\alpha - e) - (e - S_0 - 2x_\alpha) f_m(S_0 + 2x_\alpha - e) + \int_{S_0 - 2x_\alpha}^{S_0 - x_\alpha} 1 dF_m(m - e) - \int_{S_0 + x_\alpha}^{S_0 + 2x_\alpha} 1 dF_m(m - e)\Big] dF_e(e). \tag{2.23}$$

Evaluated at $S_0 = 0$, equation (2.23) is zero:

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2} = 4\int_{-\infty}^{\infty} [(e - S_0)\left(f'_m(S_0 - x_\alpha - e) + f'_m(S_0 + x_\alpha - e)\right) - (e - S_0 + 2x_\alpha) f'_m(S_0 - 2x_\alpha - e) - (e - S_0 - 2x_\alpha) f'_m(S_0 + 2x_\alpha - e) dF_e(e). \tag{2.24}$$

Evaluated at $S_0 = 0$, equation (2.24) simplifies to

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2}\Big|_{S_0=0} = 8\int_{-\infty}^{\infty} e f'_m(x_\alpha - e) - (e - 2x_\alpha) f'_m(2x_\alpha - e) dF_e(e). \tag{2.25}$$

2.25. When $e \sim N(0, \sigma_e^2)$ and $m \sim N(e, \sigma_m^2)$, equation (2.25) becomes

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2} = 8\phi\left(\frac{2x_\alpha}{\delta}\right)\frac{\sigma_m^2 \sigma_e^2}{\delta^{5/2}}\left(e^{\frac{3{x_\alpha}^2}{2\delta^2}} \sigma_e^2(\delta^2 - {x_\alpha}^2) - \sigma_e^2 \delta^2 - 4\sigma_m^2 {x_\alpha}^2\right). \tag{2.26}$$

At $x_\alpha = 0$, equation (2.26) equals zero and is decreasing in x_α for all $x_\alpha > 0$. To see this, differentiate the bracketed term in equation (2.26) with respect to x_α and define $\theta = {x_\alpha}^2/\delta^2$.

$$\begin{aligned}
\frac{\partial}{\partial x_\alpha}\left(e^{\frac{3{x_\alpha}^2}{2\delta^2}} \sigma_e^2(\delta^2 - {x_\alpha}^2) - \sigma_e^2 \delta^2 - 4\sigma_m^2 {x_\alpha}^2\right) &= -8\sigma_m^2 x_\alpha + e^{\frac{3{x_\alpha}^2}{2\delta^2}} \sigma_e^2 x_\alpha (\delta^2 - 3{x_\alpha}^2)/\delta^2 \\
&= -x_\alpha\left(8\sigma_m^2 + \sigma_e^2 e^{\frac{3}{2}\theta}(3\theta - 1)\right) \\
&< -x_\alpha \sigma_e^2\left(4 + e^{\frac{3}{2}\theta}(3\theta - 1)\right) \\
&< 0,
\end{aligned}$$

where the first inequality follows since $\sigma_e^2 < 2\sigma_m^2$.

Since equation (2.26) is negative for all x_α and $\sigma_e^2 < 2\sigma_m^2$, it must be negative for the optimal voting rule when initial policy is unbiased. Hence, the first-order condition evaluated at the optimal voting rule when initial policy is unbiased becomes negative as S_0 increases.

Next we show that a marginal movement of initial policy away from the representative citizen's expected bliss point increases the optimal voting rule if the proposer's polarity exceeds the optimal voting rule distance and $\sigma_m^2 > \lambda\sigma_e^2$. When $u_p > x_\alpha$ then

$$\frac{\partial U(x_\alpha, S_0)}{\partial x_\alpha} = 2\int_{-\infty}^{\infty}\int_{x_\alpha+S_0-e}^{2x_\alpha+u_p+S_0-e}(e + 2(\mu_m - x_\alpha) - S_0)dF_m(\mu_m) - \int_{-2x_\alpha-u_p+S_0-e}^{-x_\alpha+S_0-e}(e + 2(\mu_m + x_\alpha) - S_0)dF_m(\mu_m)dF_e(e). \tag{2.27}$$

Differentiating equation (2.27) with respect to S_0 gives

$$\begin{aligned}\frac{\partial^2 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0} = 2\int_{-\infty}^{\infty} & [F_m(-x_\alpha + S_0 - e) - F_m(-2x_\alpha - u_p + S_0 - e) \\ & -F_m(2x_\alpha + u_p + S_0 - e) + F_m(x_\alpha + S_0 - e) \\ & +(-2(x_\alpha + u_p) + S_0 - e)f_m(-2x_\alpha - u_p + S_0 - e) - (S_0 - e)f_m(-x_\alpha + S_0 - e) \\ & + (2(x_\alpha + u_p) + S_0 - e)f_m(2x_\alpha + u_p + S_0 - e) - (S_0 - e)f_m(x_\alpha + S_0 - e)]\, dF_e(e).\end{aligned} \tag{2.28}$$

Evaluated at $S_0 = 0$, equation (2.28) equals zero.

$$\begin{aligned}\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2} = 2\int_{-\infty}^{\infty} & [(2(x_\alpha + u_p) + S_0 - e)f'_m(2x_\alpha + u_p + S_0 - e) \\ & + (-2(x_\alpha + u_p) + S_0 - e)f'_m(-2x_\alpha - u_p + S_0 - e) \\ & -(S_0 - e)(f'_m(x_\alpha + S_0 - e) + f'_m(-x_\alpha + S_0 - e))]\, dF_e(e).\end{aligned} \tag{2.29}$$

Evaluated at $S_0 = 0$, equation (2.29) simplifies to (2.10). When $e \sim N(0, \sigma_e^2)$ and $m \sim N(e, \sigma_m^2)$, equation (2.10) becomes

$$\begin{aligned}\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2} = & \frac{\sqrt{8}}{\sqrt{\pi}\delta^5}\Big(e^{-\frac{{x_\alpha}^2}{2\delta^2}}\sigma_e^2\sigma_m^2(\delta^2 - {x_\alpha}^2) \\ & -e^{-\frac{(2x_\alpha+u_p)^2}{2\delta^2}}\sigma_m^3(\sigma_e^2\delta^2 + u_p^2(\delta^2 + \sigma_m^2) + 2u_p x_\alpha(\delta^2 + 2\sigma_m^2) + 4\sigma_m^2{x_\alpha}^2)\Big)\end{aligned} \tag{2.30}$$

Hence, if $x_\alpha^* > \delta$, the first-order condition evaluated at the optimal voting rule when initial policy is unbiased becomes negative as S_0 increases. The optimal voting rule decreases as S_0 increases from zero.

It remains to establish when $x_\alpha^* > \delta$ holds. Let $u_p = \delta$. Then $x_\alpha^* = \delta$ if

$$\frac{\Phi(3) - \Phi(1)}{\phi(1) - \phi(3)} = \frac{\sigma_e^2 + 2\sigma_m^2}{2(\sigma_e^2 + \sigma_m^2)}, \tag{2.31}$$

which holds when $\frac{\sigma_m^2}{\sigma_e^2} = \lambda$, where λ is defined in the proposition. By Lemma 7, as σ_m^2 increases, so does x_α^*/δ. From

Lemma 7, if $u_p > \tilde{u}_p$ then $u_p > x_\alpha^*$. From Proposition 7, if u_p increases, x_α^* increases. Hence, $x_\alpha^* > \delta$ and the optimal voting rule decreases in S_0 if $\sigma_m^2 > \lambda\sigma_e^2$ and $u_p > \tilde{u}_p$. ■

Proof of Proposition 20: In the limit as $u_p \to \infty$, $x_\alpha^* = \delta$ if

$$\frac{1-\Phi(1)}{\phi(1)} = \frac{\sigma_e^2 + 2\sigma_m^2}{2(\sigma_e^2 + \sigma_m^2)}, \tag{2.32}$$

which holds when $\frac{\sigma_m^2}{\sigma_e^2} = \chi$ where χ is defined in the proposition. By Lemma 7, as σ_m^2 increases x_α^*/δ increases. Also, from Proposition 15, as u_p increases, so does x_α^*. Hence, $x_\alpha^* < \delta$ for all u_p if $\sigma_m^2 < \chi\sigma_e^2$. By equation (2.30), as $u_p \to \infty$ the second term goes to zero, while the first term remains strictly positive. Hence, there exists a $\bar{u}_p$ such that for all $u_p > \bar{u}_p$ the optimal voting rule increases in S_0. ■

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