CENTRALIZERS IN AUTOMORPHISM GROUPS

BY

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DISSERTATION

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Abstract

In this dissertation we investigate centralizers in several automorphism groups of homogenous structures. In the first chapter, we discuss the centralizer question in ergodic theory, an open question that has served as motivation for much of the work in this dissertation. We also introduce the content of each of the subsequent chapters and describe how it relates to the centralizer question in ergodic theory.

In the second chapter, we investigate the topological complexity of the set of n-th powers in the group of isometries of Baire space. We prove that for $n > 1$, this set is not Borel.

In the third chapter, we investigate *topological similarity*, an equivalence relation on a Polish group introduced by Rosendal in [15]. We prove some results for topological similarity in general Polish groups and give some new, simplified proofs of known genericity results in the group of invertible measure-preserving transformations. We also show that a generic measure-preserving transformation is not conjugate to any of its n-th roots, for $n > 1$.

In the fourth chapter, we introduce the notion of a rank-1 homeomorphism of a zero-dimensional Polish space X, analogous in many ways to a rank-1 invertible measure-preserving transformation. We show that every rank-1 homeomorphism with a non-repeating tower representation (the class of such homeomorphisms is large) has trivial centralizer in the group of homeomorphisms of X.

To my family

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Chapter 1

Introduction

1.1 Overview

The collection of automorphisms of any mathematical structure forms a group under composition. In many instances, the structure of the underlying set naturally gives rise to a Polish topology on the automorphism group. Such a Polish topology exists for the automorphism groups that we will discuss in this dissertation, namely,

1. $\mathrm{Aut}(X, \mu)$, the group of invertible measure-preserving transformations of a standard Lebesgue space (X, μ);

2. $\mathrm{Iso}(\mathbb{N}^{\mathbb{N}})$, the group of isometries of Baire space; and

3. $\mathrm{Homeo}(X)$, the group of homeomorphisms of a Polish space X.

For a specified automorphism, one can ask many questions that relate to the size and structure of its centralizer in the automorphism group. For example, one can ask whether the automorphism commutes with only its integrals powers (i.e., has *trivial centralizer*), whether it has a square root (which would necessarily be in its centralizer), and whether it is the 1-value of a flow (a one parameter subgroup of the automorphism group).

One can also ask how the collection of the automorphisms that satisfy some particular property sits in the group of all automorphisms. For example, one can ask whether a generic automorphism has trivial centralizer (i.e., whether a comeager set of automorphisms has trivial centralizer) or inquire as to the topological complexity of the set of automorphisms that have a square root. This dissertation will answer several such questions.

We will first discuss an open question in ergodic theory that has served as motivation for much of the work in this dissertation. Then we will give an overview of the work done in each of the subsequent chapters and its relation to that question.

1.2 The Centralizer Question in Ergodic Theory

1.2.1 Introduction

Let (X, μ) be a standard Lebesgue space, e.g., the unit interval with Lebesgue measure. Let $\mathrm{Aut}(X, \mu)$ denote the group of automorphisms of (X, μ), taken modulo null sets, and equipped with the weak topology. To be more precise, let G denote the group of all bijections $f : X \to X$ such that both f and f^{-1} are measure preserving. The group $\mathrm{Aut}(X, \mu)$ is obtained from G by moding out by the equivalence relation given by $f \sim g$ iff f and g agree on a set of full measure. The weak topology can be described by saying that a sequence $\{T_n\}$ of invertible measure-preserving transformations converges to an invertible measure-preserving transformation T iff for every measurable set A, $\mu(T_n(A) \triangle T(A)) \to 0$. A subset of $\mathrm{Aut}(X, \mu)$ is then closed iff every convergent subsequence of the set converges to an element of the set. With the weak topology, $\mathrm{Aut}(X, \mu)$ is a Polish group.

With topology comes a notion of genericity. We say a generic transformation satisfies a certain property if the collection of transformations that satisfy the property is a comeager set in $\mathrm{Aut}(X, \mu)$, i.e., if its complement is a countable union of nowhere dense sets.

The general, and somewhat imprecise, question about centralizers is this: What can be said about the centralizer of a generic measure-preserving transformation?

More specific questions can also be asked. Does a generic transformation have trivial centralizer? Is the centralizer of a generic transformation abelian? The centralizer of any transformation is a subgroup of all measure-preserving transformations; what algebraic or topological properties does the centralizer of a generic transformation have?

These questions would be nicely answered if one found a Polish group G so that for generic T, the centralizer of T is isomorphic to G as a topological group. It may be the case, however, that there is no such G, i.e., that for any Polish group G, the set transformations whose centralizer is isomorphic to G is meager.

1.2.2 What is Known

Let $C(T)$ denote the centralizer of a transformation T in $\mathrm{Aut}(X, \mu)$. We want to know what $C(T)$ looks like for generic T.

A few things are immediate. Any transformation commutes with any power of itself, so for all T, $\{T^i : i \in \mathbb{Z}\} \subseteq C(T)$. Since $\mathrm{Aut}(X, \mu)$ is a topological group, we can say more: $\overline{\{T^i : i \in \mathbb{Z}\}} \subseteq C(T)$. This raises two questions. First, is $\overline{\{T^i : i \in \mathbb{Z}\}}$ strictly larger than $\{T^i : i \in \mathbb{Z}\}$ for generic T? Second, is $C(T)$ strictly larger than $\overline{\{T^i : i \in \mathbb{Z}\}}$ for generic T?

It has long been known that the answer to the first question is yes—in other terminology, that a generic transformation is rigid. In 1986, Jonathan King [12] answered the second question in the negative. His result, known as the weak closure theorem, is for rank-1 transformations (and a generic transformation is rank-1).

Theorem 1.1 (King, 1986). *If T is rank-1, then $C(T) = \overline{\{T^i : i \in \mathbb{Z}\}}$.*

It follows immediately from King's theorem that the centralizer of a generic transformation is abelian and, in some sense, as small as possible.

In another sense, however, the centralizer of a generic transformation is known to be large. In 2000, King [13] answered an old question of Halmos by showing that a generic transformation has roots of all orders. The method that King devised was later used by de Sam Lazaro and de la Rue [5] to show that $\mathbb{R}$ embeds into the centralizer of a generic T (in fact, successive roots of T can be taken in a way to produce a flow, or one parameter subgroup, whose 1-value is T). By *embedding*, we mean a continuous injective group homomorphism, but the inverse need not be continuous. King's method was also used by Ageev [1] to show that every finite abelian group embeds into the centralizer of a generic T (in essence, that the identity has abundant roots in the centralizer of T).

Stepin and Eremenko used very different methods to show in [16] that for generic T, the infinite dimensional torus $(S^1)^{\mathbb{N}}$ embeds into the centralizer of a generic transformation. This is very significant, since every compact abelian group embeds into $(S^1)^{\mathbb{N}}$.

1.3 The Complexity of the Set of n-th Powers

In the above considerations, the set of n-th powers plays an important role and one would like to understand it as completely as possible. In King's paper showing that a generic transformation has roots of all orders [13], he asked whether the set of n-th powers (for fixed $n > 1$) is Borel. If it were Borel, then a concrete description of it might yield additional insight into the size and structure of the centralizer of a generic transformation. On the other hand, it would also be of interest if the collection of n-th powers, an algebraically simple set, were not Borel, i.e., topologically complicated.

This question has also been studied in other contexts. In 1989, Humke and Laczkovich [10] showed that in the semi-group of continuous functions from an interval to itself, the set of n-th powers is not Borel. This was strengthened by Beleznay [2] who showed that the set of n-th powers in the semi-group mentioned above is complete analytic. Then in 2007, Gartside and Pejić [8] showed that the set of n-th powers is complete analytic in the group of homeomorphisms of the circle.

In both of these cases the underlying structure (the interval or the circle) carries an ordering that interacts with the functions under consideration (continuous functions or homeomorphisms). This aspect of linearity, completely absent in (X, μ), is crucial to the arguments in [10] and [8]. The main result of Chapter 2 is that the set of n-th powers is not Borel in a setting without such linearity, the group of isometries of Baire space (the metric is defined in Chapter 2).

Theorem 1.2. *For each $n > 1$, the set of n-th powers in the group of isometries of Baire space is not Borel.*

In Baire space, there is no ordering to interact with the functions under consideration. By restricting our attention to isometries, rather than considering all homeomorphisms, we also have something that resembles a measure: each basic clopen set has an associated size (the length of the finite sequence defining the basic clopen set), and this size must be preserved under all isometries.

We will prove the theorem via the following proposition, which is of interest in its own right (appropriate definitions are given in Chapter 2).

Proposition 1.3. *For each $n > 1$, the set of trees with an n-matching is not Borel in the space of trees on $\mathbb{N}$.*

1.4 Topological Similarity

We know from King's weak closure theorem that $C(T) = \overline{\{T^i : i \in \mathbb{Z}\}}$ for generic T. How can we get information about this closure? One way is through topological similarity, an equivalence relation introduced by Rosendal in [15]. We say a strictly increasing sequence $\{i_n\}$ of natural numbers sends a transformation T to the identity if $\{T^{i_n}\}$ converges to the identity. Two transformations are *topologically similar* if they are sent to the identity by the same sequences. It is not hard to show, and we will do so in Chapter 3, that knowing the sequences that send T to the identity is sufficient to determine $\overline{\{T^i : i \in \mathbb{Z}\}}$ as an abstract topological group.

When Rosendal introduced the notion of topological similarity in [15], he gave an elegant proof of an old result attributed to Rohlin: each conjugacy class in $\mathrm{Aut}(X, \mu)$ is meager. In Chapter 3 we use topological similarity to give new proofs of several other results:

1. The set of strongly mixing transformations is meager (Rohlin, [14]).

2. The centralizer of a generic transformation is not compact; in fact, if T is weakly mixing, then $\overline{\{T^i : i \in \mathbb{Z}\}}$, and hence $C(T)$, is not compact (folklore).

3. A generic T is not conjugate to T^n for $n > 1$ (del Junco and Lemańczyk, [6]).

We also prove that a generic transformation is not conjugate any of its n-th roots, for $n > 1$. King showed in [13] that a generic transformation has an n-th root for each $n > 1$, and Stepin and Eremenko showed in [16] that a generic transformations has continuum many n-th roots, for each $n > 1$.

The chapter concludes with some open questions that are related to topological similarity in the group $\mathrm{Aut}(X, \mu)$.

1.5 Rank-1 Homeomorphisms

King's weak closure theorem applies to rank-1 transformations, and we would like to understand these transformations as completely as possible. In Chapter 4, we introduce the notion of a rank-1 homeomorphism of a zero-dimensional Polish space X. A Polish space is *zero-dimensional* if the space has a countable basis of clopen sets. It is well known that every zero-dimensional Polish space is homeomorphic to a closed subset of $\mathbb{N}^{\mathbb{N}}$. We show in Theorem 4.1 that if such a rank-1 homeomorphism f satisfies some non-degeneracy condition, then the centralizer of f in the group of homeomorphisms of X is exactly $\{f^i : i \in \mathbb{Z}\}$.

In the literature, there are actually several different definitions of rank-1 transformations. These are all equivalent if we restrict our attention to transformations that are totally ergodic; a transformation T is *totally ergodic* if T^n is ergodic for each $n > 0$. A nice discussion of these several definitions and their connections can be found in the survey article [7]. The non-degeneracy condition in our theorem is analogous to restricting our attention to totally ergodic rank-1 transformations.

To motivate our definition and theorem, we now describe two ways of defining rank-1 transformations and how rank-1 measure-preserving transformations can be naturally realized as homeomorphisms of a zero-dimensional Polish spaces.

One way of defining rank-1 transformations is via symbolic systems. One can define a collection of symbolic rank-1 systems. Each system is a triple (X, μ, σ), where X is some closed, shift-invariant subset of $2^{\mathbb{Z}}$ with no isolated points, μ is a shift-invariant non-atomic probability measure supported on X, and σ is the shift. A totally ergodic rank-1 transformation is rank-1 if it is isomorphic to some symbolic rank-1 system. For a symbolic rank-1 system, the shift σ is not just a measure-preserving transformation, but also a homeomorphism of the Cantor space X. The weak closure theorem provides information about the centralizer of σ in the group $\text{Aut}(X, \mu)$, but one can also ask about the centralizer of σ in the group $\text{Homeo}(X)$. It is a consequence of our main theorem that σ has trivial centralizer in the group $\text{Homeo}(X)$.

Another way of defining rank-1 transformations is via cutting and stacking with intervals. If a transformation T is obtained from a cutting and stacking construction with intervals, one can remove from the interval $[0, 1)$ both the point 0 and each cut-point of the procedure. By doing this, one removes a countable dense subset from the interval $(0, 1)$, and thus what remains, call it Y, is homeomorphic to Baire space. The transformation T restricted to Y is still a rank-1 measure-preserving transformation, but it is also a homeomorphism of Y. Our theorem shows that as long as the original transformation T was totally ergodic, its restriction to Y has trivial centralizer in the group of homeomorphisms of Y.

Chapter 2

The Complexity of the Set of n-th Powers in the Group of Isometries of Baire Space

2.1 Introduction

2.1.1 Baire Space and its Isometries

Let $\mathbb{N}^{\mathbb{N}}$ denote the metric space whose underlying set is the collection of all sequences of natural numbers and in which the distance between two sequences, α and β, is given by $\frac{1}{1+l(\alpha,\beta)}$, where $l(\alpha, \beta)$ is the length of the longest common initial segment of α and β. We refer to $\mathbb{N}^{\mathbb{N}}$ as (the metric) Baire space. If we take $\mathbb{N}^{\mathbb{N}}$ as defined above and forget the metric, but retain the topology induced by the metric, we have the topological Baire space, which is what is usually meant by "Baire space" when that phrase is used in the literature.

Let $\mathrm{Iso}(\mathbb{N}^{\mathbb{N}})$ denote the group of isometries of $\mathbb{N}^{\mathbb{N}}$ equipped with the topology of pointwise convergence. A sub-basis for this topology is given by sets of the form

$$\{F \in \mathrm{Iso}(\mathbb{N}^{\mathbb{N}}) : \text{if } \alpha \in \mathbb{N}^{\mathbb{N}} \text{ extends } \overline{a}, \text{ then } F(\alpha) \text{ extends } \overline{b}\},$$

where $\overline{a}$, $\overline{b} \in \mathbb{N}^k$ for some $k \in \mathbb{N}$. With this topology $\mathrm{Iso}(\mathbb{N}^{\mathbb{N}})$ is a Polish group.

It immediately follows from the definition of the metric on $\mathbb{N}^{\mathbb{N}}$ that if F is an isometry and if two sequences α and β agree on their first k coordinates, then $F(\alpha)$ and $F(\beta)$ also agree on their first k coordinates. Thus we may define "level-k approximations" of an isometry F as follows. For $k \in \mathbb{N}$ define $F_k : \mathbb{N}^k \to \mathbb{N}^k$ (the level-k approximation of F) so that if $\langle a_0, a_1, \dots, a_{k-1}\rangle$ is an initial segment of α, then $F_k(\langle a_0, a_1, \dots, a_{k-1}\rangle)$ is an initial segment of $F(\alpha)$. For an isometry F and $k \in \mathbb{N}$ there is only one such $F_k : \mathbb{N}^k \to \mathbb{N}^k$ and it is a bijection.

An isometry of Baire space can be constructed by way of its level-k approximations. A sequence $(F_k : k \in \mathbb{N})$ such that each F_k is a bijection from $\mathbb{N}^k$ to $\mathbb{N}^k$ is said to be coherent provided that for all k, if $F_{k+1}(\langle a_0, a_1, \dots, a_k\rangle) = \langle b_0, b_1, \dots, b_k\rangle$, then $F_k(\langle a_0, a_1, \dots, a_{k-1}\rangle) = \langle b_0, b_1, \dots, b_{k-1}\rangle$. Such a coherent sequence gives rise to a unique $F \in \mathrm{Iso}(\mathbb{N}^{\mathbb{N}})$.

2.1.2 Trees on $\mathbb{N}$ and their Automorphisms

Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of finite sequences of natural numbers, including the empty sequence. A *tree* (on $\mathbb{N}$) is a nonempty subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under taking initial segments. The empty sequence $\emptyset$ is an element of every tree; it is called the *root*. Non-root elements of a tree are called *nodes*. A tree is naturally composed of levels; the level-k nodes are precisely those with length k. A node b is a *descendant* of $a \in T$ if b properly extends a. The node b is a *child* of a if additionally the length of b is exactly one more than the length of a. The collection of all trees is a closed subset of $2^{(\mathbb{N}^{<\mathbb{N}})}$ and so is a Polish space. It is denoted by $\mathrm{Tr}_{\mathbb{N}}$. A sub-basis for this topology is given by sets of the form $\{T \in \mathrm{Tr}_{\mathbb{N}} : a \in T \text{ and } b \notin T\}$, where a and b are elements of $\mathbb{N}^{<\mathbb{N}}$.

An *automorphism* of a tree T is a bijection $F : T \to T$ satisfying the following conditions.

1. $F(\emptyset) = \emptyset$.

2. For all $a, b \in T$, b is a child of a iff $F(b)$ is a child of $F(a)$.

An automorphism of a tree clearly preserves the levels of the tree. Each automorphism can be uniquely decomposed into its orbits. The restriction of an automorphism of a tree to a single orbit is called a *cycle*. If the orbit has cardinality n it is called an n-cycle and if the orbit is infinite it is called an infinite cycle, or an ∞-cycle. We can construct an automorphism of a tree by constructing, level by level, the cycles that make up the automorphism.

A tree of particular interest is the full tree on $\mathbb{N}$, denoted by $T_{\mathbb{N}}$, consisting of all finite sequences of natural numbers. The automorphism group of $T_{\mathbb{N}}$, denoted $\mathrm{Aut}(T_{\mathbb{N}})$, is isomorphic to the group of isometries of Baire space. Indeed, the restriction of an automorphism of $T_{\mathbb{N}}$ to the level-k nodes of $T_{\mathbb{N}}$ is precisely the level-k approximation to the corresponding isometry of $\mathbb{N}^{\mathbb{N}}$. A sub-basis for the topology on $\mathrm{Aut}(T_{\mathbb{N}})$ is given by sets of the form $\{F \in \mathrm{Aut}(T_{\mathbb{N}}) : F(\overline{a}) = \overline{b}\}$, where $\overline{a}$ and $\overline{b}$ are nodes in $T_{\mathbb{N}}$.

An *n-matching* of a tree T is an automorphism F of T such that each node of T has order exactly n under F. A tree has an n-matching iff its level-1 nodes can be partitioned into sets of size n so that two nodes in the same set have isomorphic "sets of descendants."

2.2 Results

The main result of this chapter is the following theorem.

Theorem 2.1. *For each $n > 1$, the set of n-th powers is not Borel in the group of isometries of Baire space.*

This will be proved using the following proposition.

Proposition 2.2. *For each $n > 1$, the set of trees with an n-matching is not Borel in the space of trees on $\mathbb{N}$.*

To prove the proposition, we need the following. The collection of well-founded trees (i.e., those without an infinite branch) is a complete coanalytic subset of $\mathrm{Tr}_{\mathbb{N}}$. There is a natural coanalytic rank on well-founded trees, and any collection of well-founded trees with unbounded rank (in ω_1) is not Borel and, moreover, not analytic (See Theorem 35.23 in [11]).

We first prove the theorem from the proposition and then prove the proposition.

2.2.1 Proof of the Theorem from the Proposition

Fix $n > 1$. We will describe a continuous function $\Phi : \mathrm{Tr}_{\mathbb{N}} \to \mathrm{Aut}(T_{\mathbb{N}})$ and show that $T \in \mathrm{Tr}_{\mathbb{N}}$ has an n-matching if and only if $\Phi(T)$ has an n-th root. This, together with the proposition, implies that the collection of n-th powers in $\mathrm{Aut}(T_{\mathbb{N}})$ is not Borel. Since the group of isometries of Baire space is isomorphic to the automorphism group of $T_{\mathbb{N}}$, this will prove the theorem.

First, partition $\mathbb{N}$ into infinite sets $(B_i : i \in \mathbb{N})$. Now recursively define $A_{\bar{c}}$, for $\bar{c} \in \mathbb{N}^{<\mathbb{N}}$, by $A_\emptyset = \{\emptyset\}$ and $A_{\langle c_0, c_1, \ldots, c_k \rangle} = \{\langle a_0, a_1, \ldots, a_k \rangle : \langle a_0, a_1, \ldots, a_{k-1} \rangle \in A_{\langle c_0, c_1, \ldots, c_{k-1} \rangle}$ and $a_k \in B_{c_k}\}$, for $k > 0$.

For $k > 0$, the sets $(A_{\bar{c}} : \bar{c} \in \mathbb{N}^k)$ form a partition of $\mathbb{N}^k$ and each element of that partition is infinite. We now describe the function $\Phi : \mathrm{Tr}_{\mathbb{N}} \to \mathrm{Aut}(T_{\mathbb{N}})$. This is done in such a way that each $A_{\bar{c}}$ is invariant under each $\Phi(T)$.

Define Φ in such a way that for each $T \in \mathrm{Tr}_{\mathbb{N}}$, $\Phi(T)$ is an automorphism of $T_{\mathbb{N}}$ satisfying:

1. If $\bar{c} \in T$, then $\Phi(T) \restriction A_{\bar{c}}$ consists of one n-cycle and infinitely many ∞-cycles.

2. If $\bar{c} \notin T$, then $\Phi(T) \restriction A_{\bar{c}}$ consists of infinitely many ∞-cycles.

Moreover, do this in such a way that for all $T_1, T_2 \in \mathrm{Tr}_{\mathbb{N}}$, for all $\bar{c} = \langle c_0, c_1, \ldots, c_k \rangle$

in $\mathbb{N}^{<\mathbb{N}}$, and for all $\overline{a} \in A_{\overline{c}}$,

$$\Phi(T_1)(\overline{a}) = \Phi(T_2)(\overline{a})$$

iff

$$\{\langle c_0\rangle, \langle c_0, c_1\rangle, \ldots, \langle c_0, c_1, \ldots, c_k\rangle\} \cap T_1 = \{\langle c_0\rangle, \langle c_0, c_1\rangle, \ldots, \langle c_0, c_1, \ldots, c_k\rangle\} \cap T_2.$$

To check that Φ is continuous, it suffices to show that the preimage under Φ of each set of the form $\{F \in \mathrm{Aut}(T_{\mathbb{N}}) : F(\overline{a}) = \overline{b}\}$, where $\overline{a}, \overline{b} \in \mathbb{N}^{<\mathbb{N}}$, is open in $\mathrm{Tr}_{\mathbb{N}}$. In other words, we need to check that for each $\overline{a}, \overline{b} \in \mathbb{N}^{<\mathbb{N}}$, the set $\{T \in \mathrm{Tr}_{\mathbb{N}} : \Phi(T)(\overline{a}) = \overline{b}\}$ is open in $\mathrm{Tr}_{\mathbb{N}}$. Fix $\overline{a} \in \mathbb{N}^k$ and find $\overline{c} \in \mathbb{N}^k$ so that $\overline{a} \in A_{\overline{c}}$. If $\Phi(T_0)(\overline{a}) = \overline{b}$, then for all $T \in \mathrm{Tr}_{\mathbb{N}}$, $\Phi(T)(\overline{a}) = \overline{b}$ iff

$$\{\langle c_0\rangle, \langle c_0, c_1\rangle, \ldots, \langle c_0, c_1, \ldots, c_k\rangle\} \cap T = \{\langle c_0\rangle, \langle c_0, c_1\rangle, ..., \langle c_0, c_1, \ldots, c_k\rangle\} \cap T_0.$$

This shows that $\{T \in \mathrm{Tr}_{\mathbb{N}} : \Phi(T)(\overline{a}) = \overline{b}\}$ is open in $\mathrm{Tr}_{\mathbb{N}}$.

It is easy to check that a k-cycle raised to the n-th power results in a product of $\gcd(k, n)$ disjoint $(\frac{k}{\gcd(k,n)})$-cycles (this is true when $k = \infty$ as well as when k is finite). It easily follows from this that if the n-th power of a k-cycle consists of at least one n-cycle, then $k = n^2$ and the n-th power of the k-cycle is a product of n disjoint n-cycles.

It remains to show that $T \in \mathrm{Tr}_{\mathbb{N}}$ has an n-matching if and only if $\Phi(T)$ has an n-th root.

First, let $T \in \mathrm{Tr}_{\mathbb{N}}$, let $F = \Phi(T)$, and suppose $G^n = F$. Consider an n-cycle in F. The elements of this n-cycle must be part of an n^2-cycle in G. The orbit of size n^2 of G that consists of those n^2 elements must be the union of

n orbits of size n of F. Moreover, G must permute these orbits. Since there is a bijective correspondence between the nodes of T and the n-cycles in F, G induces a permutation of all the nodes of T. It is straightforward to check that this permutation is an n-matching.

Now, let $T \in \mathrm{Tr}_{\mathbb{N}}$ and suppose T has an n-matching. By the definition of Φ, $\Phi(T)$ has only n-cycles and ∞-cycles. For $\Phi(T)$ to have an n-th root, there must be a coherent, level by level way of grouping the n-cycles into groups of size n and the ∞-cycles into groups of size n. The construction of $\Phi(T)$ was done in such a way that at each stage there are infinitely many ∞-cycles, so these can be matched into groups of n. The n-matching of T decomposes into orbits of size n, and each element of a given orbit is on the same level. Each of the nodes in an orbit corresponds to an n-cycle in $\Phi(T)$ and this provides a way of grouping the n-cycles into groups of n. It is straightforward to check that such a grouping gives rise to the desired n-th root of $\Phi(T)$.

We have shown that $T \in \mathrm{Tr}_{\mathbb{N}}$ has an n-matching if and only if $\Phi(T)$ has an n-th root. This concludes the proof of the theorem.

2.2.2 Proof of the Proposition

We will describe a continuous function $\Psi : \mathrm{Tr}_{\mathbb{N}} \to \mathrm{Tr}_{\mathbb{N}}$. The preimage of the collection of well-founded trees will be the collection of trees without an n-matching. We will show that the image under Ψ of the set of trees without an n-matching has unbounded rank. This will imply that the set of trees without an n-matching is non-Borel, since the image of any Borel set under a continuous function is analytic. This implies that the collection of trees with an n-matching is non-Borel.

We first order the elements of $\mathbb{N}^{<\mathbb{N}}$ as follows: For each $a \in \mathbb{N}^{<\mathbb{N}}$, let s_a be the sum of the entries of a plus the length of a. If s_a is less than s_b, then we say $a \prec b$.

If $s_a = s_b$, then we say $a \prec b$ if a is lexicographically less than b. It is clear that $\prec$ is an ω-ordering of $\mathbb{N}^{<\mathbb{N}}$. Let $\{r_0, r_1, r_2, \dots\}$ be the enumeration of $\mathbb{N}^{<\mathbb{N}}$ satisfying: $i < j$ iff $r_i \prec r_j$.

Given a tree $T \in \mathrm{Tr}_{\mathbb{N}}$ we produce $\Psi(T)$, a tree of attempts to produce an n-matching of T. We say that a function $f : \{r_0, r_1, \dots, r_k\} \to T$ is a *valid attempt* for T if for $i, j \leq k$:

1. If $r_i \notin T$, then $f(r_i) = \emptyset$.
2. If $r_i = \emptyset$, then $f(r_i) = \emptyset$.
3. If r_i is a node of T and a child of r_j, then $f(r_i)$ is a child of $f(r_j)$.
4. If r_i and r_j are distinct nodes of T, then $f(r_i) \neq f(r_j)$.
5. If r_i is a node of T and $r_i \in \mathrm{dom}(f^n)$, then $f^n(r_i) = r_i$.
6. If r_i is a node of T and $r_i \in \mathrm{dom}(f^m)$ with $0 < m < n$, then $f^m(r_i) \neq r_i$.

The sequence $\langle a_0, a_1, a_2, \dots, a_k \rangle$ is a node of the tree $\Psi(T)$ iff the function f defined by $f(r_i) = r_{a_i}$ is a valid attempt.

It is clear $\Psi(T)$ actually is a tree, for an initial segment of a valid attempt is a valid attempt. It is easy to see that if T has an n-matching, then $\Psi(T)$ has an infinite branch. Indeed, the initial segments of such an automorphism naturally correspond to valid attempts which in turn correspond to the nodes of an infinite branch. Also, if $\Psi(T)$ has an infinite branch, then the sequence of valid attempts corresponding to those nodes gives rise to an n-matching of T. Thus a tree T has no n-matching if and only if $\Psi(T)$ is well-founded.

It is straightforward to check that Ψ is continuous. It remains to show that the image of the set of trees without an n-matching is unbounded.

We will construct two sequences of trees: $(T_\alpha : \alpha < \omega_1)$, each of which does not have an n-matching, and $(T'_\alpha : \alpha < \omega_1)$, each of which has an n-matching. Thus T_α will not be isomorphic to T'_α. The construction will be such that the rank of $\Psi(T_\alpha)$ will be at least α. Additionally, the image under Ψ of the tree $T'_\alpha \oplus T_\alpha$ (defined below) will be well-founded and have rank greater than the rank of $\Psi(T_\alpha)$.

Let S_0 and S_1 be trees. The sequence $\langle a_0, a_1, a_2, \ldots, a_k\rangle$ is a node of $S_0 \oplus S_1$ if either $a_0 \leq (n-2)$ and $\langle a_1, a_2, \ldots, a_k\rangle$ is a node of S_0 or $a_0 = (n-1)$ and $\langle a_1, a_2, \ldots, a_k\rangle$ is a node of S_1. Thus the tree $S_0 \oplus S_1$ has exactly n level one nodes; Beneath $(n-1)$ of them is a subtree isomorphic S_0 and beneath the last of them is a subtree isomorphic to S_1. Thus $S_0 \oplus S_1$ has an n-matching iff S_0 and S_1 are isomorphic.

We now describe T_0 and T'_0. The tree T_0 consists of the root and the nodes $\langle 0\rangle, \langle 1\rangle, \ldots, \langle n-2\rangle$. The tree T'_0 consists of the root and the nodes $\langle 0\rangle, \langle 1\rangle, \ldots, \langle n-1\rangle$. It is clear T_0 does not have an n-matching and that T'_0 does have an n-matching. It is clear also that $T'_0 \oplus T_0$ does not have an n-matching and that its image under Ψ has rank greater than the rank of $\Psi(T_0)$.

We now describe how to produce $T_{\alpha+1}$ and $T'_{\alpha+1}$ from T_α and T'_α. The tree $T_{\alpha+1}$ is defined to be $T'_\alpha \oplus T_\alpha$ and the tree $T'_{\alpha+1}$ is defined to be $T'_\alpha \oplus T'_\alpha$. By assumption, the rank of $\Psi(T'_\alpha \oplus T_\alpha)$ is at least $\alpha + 1$. We need further to show that the rank of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ is greater than the rank of $\Psi(T_{\alpha+1})$. Consider the structure of the two trees $T_{\alpha+1}$ and $T'_{\alpha+1} \oplus T_{\alpha+1}$.

The structure of the tree $T_{\alpha+1}$ is this: There are exactly n level-1 nodes, $\langle 0\rangle, \langle 1\rangle, \ldots, \langle n-1\rangle$. Beneath the first $n-1$ of these is a subtree isomorphic to T'_α and beneath the last is subtree isomorphic to T_α.

The structure of the tree $(T'_{\alpha+1} \oplus T_{\alpha+1})$ is this: There are exactly n level-1 nodes: $\langle 0\rangle, \langle 1\rangle, \ldots, \langle n-1\rangle$. There are exactly n^2 level-2 nodes: all $\langle i, j\rangle$ with i and

j less than n. Beneath the node $\langle n-1, n-1\rangle$ there is a subtree isomorphic to T_α, beneath each of the other level-2 nodes there is a subtree isomorphic to T'_α.

Notice that in each of these two cases, the entire tree doesn't have an n-matching because T'_α and T_α are not isomorphic.

We will define an injective tree homomorphism (a map between trees which preserves the descendence relation) $H : \Psi(T_{\alpha+1}) \to \Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$. The homomorphism H will be such that each level-2 node in $\Psi(T_{\alpha+1})$ will get sent to a node of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ on a level greater than 2. It is easy to see that $\Psi(T_{\alpha+1})$ has only finitely many level-2 nodes; it follows that the rank of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ is greater than the rank of $\Psi(T_{\alpha+1})$.

We will first given an informal description of H and then give a formal description of H and show that it is a tree homomorphism. There will be similar functions used later in the paper, but only the informal versions of these later functions will be given.

The informal description of H is this: H takes as an input a valid attempt for $T_{\alpha+1}$. This corresponds to an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-1\rangle$ in $T_{\alpha+1}$ are pairwise isomorphic. This attempt translates directly to an attempt to show that the sets of descendants beneath the nodes $\langle 0, n-1\rangle$, $\langle 1, n-1\rangle$, ..., $\langle n-1, n-1\rangle$ in $T'_{\alpha+1} \oplus T_{\alpha+1}$ are pairwise isomorphic (the collections of "sets of descendants" are the same in the two situations).

It is easy to translate this into an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-1\rangle$ in $T'_{\alpha+1} \oplus T_{\alpha+1}$ are pairwise isomorphic, because for each $i \neq n-1$, the sets of descendants beneath the nodes $\langle 0, i\rangle$, $\langle 1, i\rangle$, ..., $\langle n-1, i\rangle$ in $T'_{\alpha+1} \oplus T_{\alpha+1}$ are pairwise isomorphic. This in turn corresponds to a valid attempt for $T'_{\alpha+1} \oplus T_{\alpha+1}$. This is the output of our function H.

We will now give the formal description of H. Consider $\langle a_0, a_1, \ldots, a_k \rangle$, any node of $\Psi(T_{\alpha+1})$. This corresponds a valid attempt $f : \{r_0, r_1, \ldots, r_k\} \to \mathbb{N}^{<\mathbb{N}}$ satisfying $f(r_i) = r_{a_i}$. For each nonempty $z = \langle z_0, z_1, \ldots, z_m \rangle \in \mathbb{N}^{<\mathbb{N}}$, define $\overline{z}$ to be $\langle z_0, n, z_1, \ldots, z_m \rangle$ (in the case $m = 0$, $\overline{z} = \langle z_0, n \rangle$). Let $\overline{\emptyset} = \emptyset$.

We now define a function g as follows. For $r_i \in dom(f)$, define $g(\overline{r_i})$ to be $\overline{f(r_i)}$. This function g serves as a skeleton for a full function $g' : \{r_0, r_1, \ldots, \overline{r_k}\} \to \mathbb{N}^{<\mathbb{N}}$. We extend g to g' in the following way. Let $r_j \in \{r_0, r_1, \ldots, \overline{r_k}\} \setminus dom(g)$. If $r_j \notin T$, then define $g'(r_j) = \emptyset$. If, on the other hand, $r_j = \langle c_0, c_1, \ldots, c_l \rangle \in T$ (in this case $c_1 \neq n$) define $g'(r_j) = \langle d, c_1, c_2, \ldots, c_l \rangle$, where d is the unique entry of $f(c_0)$.

It is easy to check that g' is a valid attempt for $T'_{\alpha+1} \oplus T_{\alpha+1}$. The node of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ that corresponds to g' is the definition of $H(\langle a_0, a_1, \ldots, a_k \rangle)$.

It is easy to see that if a and b are nodes in $\Psi(T_{\alpha+1})$ with b a descendant of a, then $F(b)$ is a descendant of $F(a)$ in $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$. It is also straightforward to check that every level-2 node in $\Psi(T_{\alpha+1})$ gets sent to a node in $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ that is on a level greater than 2. Since there are only finitely many level-2 nodes in $\Psi(T_{\alpha+1})$, the rank of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ is strictly greater than the rank of $\Psi(T_{\alpha+1})$.

We will next describe how to produce T_β and T'_β when β is a limit ordinal. First choose an increasing sequence of ordinals $(\alpha_i : i \in \mathbb{N})$ whose supremum is β.

We now define the auxiliary tree S_j for each $j \in \mathbb{N}$. The sequence $\langle a_0, a_1, \ldots, a_n \rangle$ is a node of S_0 if and only if: for some $r \in \mathbb{N}$, $a_r = 1$, $a_i = 0$ if $i < r$, and $\langle a_{r+1}, a_{r+2}, \ldots, a_n \rangle \in T'_{\alpha_r}$. For $j > 0$, the sequence $\langle a_0, a_1, a_2, \ldots, a_n \rangle$ is a node of S_j if and only if either $a_j = 1$, $a_i = 0$ for $i < j$, and $\langle a_{j+1}, a_{j+2}, \ldots, a_n \rangle \in T_{\alpha_j}$ or for some $r \neq j$, $a_r = 1$, $a_i = 0$ if $i < r$, and $\langle a_{r+1}, a_{r+2}, \ldots, a_n \rangle \in T'_{\alpha_r}$. Since T_{α_i} is not isomorphic to T'_{α_i}, S_0 is not isomorphic to any S_j with $j > 0$.

We will now describe T_β and T'_β. For each $i \in \mathbb{N}$, $\langle i \rangle$ is a level-1 node of T_β. Beneath each of the level-1 nodes $\langle 0 \rangle, \langle 1 \rangle, \ldots, \langle n-2 \rangle$ there should be a subtree isomorphic to S_0. Beneath each of the other level-1 nodes should be a subtree isomorphic to some S_j for some $j > 0$. Furthermore, for each $j > 0$, there should be infinitely many level-1 nodes such that the nodes beneath them are isomorphic to S_j. The tree T'_β is defined in the same way as T_β, except that the nodes beneath the node $\langle n-1 \rangle$ should also be isomorphic to S_0. It is clear that T'_β has an n-matching and that T_β does not have an n-matching.

It remains to verify two things. First that the rank of $\Psi(T_\beta)$ is at least β. Second, that the rank of $\Psi(T'_\beta \oplus T_\beta)$ is greater than the rank of $\Psi(T_\beta)$.

Claim 1. The rank of $\Psi(T_\beta)$ is at least β.

Proof. It suffices to show that the rank of $\Psi(T_\beta)$ is greater than or equal to the rank of $\Psi(T'_{\alpha_r} \oplus T_{\alpha_r})$ for each $r > 0$. The structure of the tree $T'_{\alpha_r} \oplus T_{\alpha_r}$ is this: There are exactly n level-1 nodes, $\langle 0 \rangle, \langle 1 \rangle, \ldots, \langle n-1 \rangle$. Beneath the first $n-1$ of these is a subtree isomorphic to T'_{α_r} and beneath the last is a subtree isomorphic to T_{α_r}.

The structure of the tree T_β is more complicated and we will describe in detail only a part of it. Beneath each of the nodes $\langle 0 \rangle, \langle 1 \rangle, \ldots, \langle n-2 \rangle$ is a subtree isomorphic with S_0. We can also choose some $m > n-2$ so that beneath the node $\langle m \rangle$ is a subtree isomorphic to S_r. So for each $i \leq n-2$ there is a subtree isomorphic to T'_{α_r} beneath the node $\langle i, 0, 0, \ldots, 0, 1 \rangle$, where there are r-many zeros written. Also, there is a subtree isomorphic to T_{α_r} beneath the node $\langle m, 0, 0, \ldots, 0, 1 \rangle$, where again there are r-many zeros written.

We will give an informal description of a tree homomorphism $G_r : \Psi(T'_{\alpha_r} \oplus T_{\alpha_r}) \to \Psi(T_\beta)$ that is similar to the function H described previously. We leave it to the reader to completely formalize G_r and show that it is a tree homomorphism, but it is similar to what was done for H previously. This will imply that the rank of $\Psi(T_\beta)$ is greater than or equal to the rank of $\Psi(T'_{\alpha_r} \oplus T_{\alpha_r})$.

Here is the informal description of G_r: G_i takes as an input a valid attempt for $T'_{\alpha_r} \oplus T_{\alpha_r}$, which corresponds to an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-1\rangle$ in $T'_{\alpha_r} \oplus T_{\alpha_r}$ are pairwise isomorphic. This translates directly to an attempt to show that the sets of descendants beneath the nodes $\langle 0,0,0,\dots,0,1\rangle$, $\langle 1,0,0,\dots,0,1\rangle$, ..., $\langle n-2,0,0,\dots,0,1\rangle$ and $\langle m,0,0,\dots,0,1\rangle$ in T_β are pairwise isomorphic (the collections of "sets of descendants" are the same in the two situations). This then translates into an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-2\rangle$ and $\langle m\rangle$ in T_β are pairwise isomorphic, which corresponds to a valid attempt for T_β. This is the output of the function G_r. □

Claim 2. The rank of $\Psi(T'_\beta \oplus T_\beta)$ is greater than the rank of $\Psi(T_\beta)$.

Proof. We will give an informal description of a function $F : \Psi(T_\beta) \to \Psi(T'_\beta \oplus T_\beta)$ that is similar the function H described previously. We leave it to the reader to completely formalize G_r and show that it is a tree homomorphism.

The function F takes as an input a valid attempt for T_β, which corresponds to an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-2\rangle$ and $\langle m\rangle$ (some $m \geq n-1$) in T_β are pairwise isomorphic. This translates to an attempt to show that the sets of descendants beneath the nodes $\langle 0, n-1\rangle$, $\langle 1, n-1\rangle$, ..., $\langle n-2, n-1\rangle$ and $\langle n-1, m\rangle$ in $T'_\beta \oplus T_\beta$ are pairwise isomorphic (the collections of "sets of descendants" are the same in the two situations). This then translates into an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-2\rangle$ and $\langle n-1\rangle$ in $T'_\beta \oplus T_\beta$ are pairwise isomorphic, which corresponds to a valid attempt for $T'_\beta \oplus T_\beta$. This is the output of F.

It is easy to see that every level-2 node in $\Psi(T_\beta)$ is sent to a node in $\Psi(T'_\beta \oplus T_\beta)$ on a level greater than 2. Since $\Psi(T'_\beta \oplus T_\beta)$ has only finitely many level-2 nodes, this implies that the rank of $\Psi(T'_\beta \oplus T_\beta)$ is greater than the rank of T_β. □

Chapter 3

Topological Similarity

3.1 In General Polish Groups

Let g be an element of a Polish group G. If $\{i_n\}$ is a strictly increasing sequence of natural numbers and $g^{i_n} \to h \in G$, we say that the sequence $\{i_n\}$ sends g to h and that g converges along the sequence $\{i_n\}$. Let $\Omega(g)$ denote the set of strictly increasing sequences of natural numbers along which g converges and $\Omega_e(g)$ denote the set of such sequence that send g to the identity. We would like to know what information we can recover about g if we know something about $\Omega(g)$ or $\Omega_e(g)$. The following definition essentially appears in [15].

Definition. Two elements of a Polish group G are *topologically similar* if they are sent to the identity by the same sequences.

It is clear that topological similarity is an equivalence relation that is at least as coarse as the conjugacy equivalence relation. Also, the definition of topological similarity is independent of the ambient group G in the following sense: if F is a group homomorphism that homeomorphically maps G into a larger group G', then two elements are topologically similar in G iff their images under F are topologically similar in G'. We will say that $g \in G$ and $h \in H$ are topologically similar if $\Omega_e(g) = \Omega_e(h)$. This is true iff (g, e_H) and (e_G, h) are topologically similar in the group $G \times H$.

We say that an element g of a Polish group is *rigid* if $\Omega_e(g)$ is non-empty.

The terminology comes from dynamical systems, where one is interested in the asymptotic behavior of a structure-preserving bijection $T : X \to X$ (e.g., when X is the unit interval equipped with Lebesgue measure). The system is rigid if there exists a strictly increasing sequence $\{i_n\}$ where $T^{i_n}(X)$ looks more and more like the original state of the system, $T^0(X) = X$.

It is worth noting that the only element of a Polish group that is sent to the identity by the sequence $\{1, 2, 3, ...\}$ is the identity. Indeed, if g^n converges to the identity, then g^{n+1} both converges to the identity and to g (since $g^{n+1} = gg^n$), and thus, g is the identity. It immediately follows from this that if g is sent to the identity by the sequence $\{k, 2k, 3k, ...\}$ then g^k is the identity.

For an element g in a Polish group G, $\langle g \rangle$ denotes the subgroup of G generated by g. The next two propositions show a strong connection between $\Omega(g)$, $\Omega_e(g)$, and $\overline{\langle g \rangle}$ as an abstract topological group.

Proposition 3.1. *Let G and H be Polish groups with $g \in G$ and $h \in H$. The following are equivalent.*

1. $\Omega(g) \subseteq \Omega(h)$.

2. $\Omega_e(g) \subseteq \Omega_e(h)$.

3. *There is a continuous group homomorphism $F : \overline{\langle g \rangle} \to \overline{\langle h \rangle}$ such that $F(g) = h$.*

Proof. We first show that (1) implies (2). Suppose that $\Omega(g) \subseteq \Omega(h)$, but that $\Omega_e(g) \nsubseteq \Omega_e(h)$. Then there is some increasing sequence of natural numbers $\{i_n\}$ so that $\{g^{i_n}\}$ converges to the identity while $\{h^{i_n}\}$ converges to some h' not equal to the identity. Let $\{i_{n_k}\}$ be a subsequence of $\{i_n\}$ so that $\{i_{n_k} - i_k\}$ is increasing. Intertwine this with $\{i_n\}$ to get $\{j_n\}$ (both $\{i_n\}$ and $\{i_{n_k} - i_k\}$ should occur as subsequences of $\{j_n\}$, $\{j_n\}$ should be increasing, and every element of $\{j_n\}$ should be an element of $\{i_n\}$ or $\{i_{n_k} - i_k\}$). Then $\{g^{j_n}\}$ converges to the identity while $\{h^{i_n}\}$ does not converge, for it has one subsequence converging to the identity and another subsequence converging to h', which is not the identity. Thus $\Omega(g) \nsubseteq \Omega(h)$, a contradiction.

We now show that (2) implies (3). Suppose $\Omega_e(g) \subseteq \Omega_e(h)$ and consider the map $F' : \langle g \rangle \to \langle h \rangle$ sending g^i to h^i. This is clearly a group homomorphism. To show that F' is continuous, it suffices to check continuity at the identity. If F' is not continuous at the identity, then there is some open $V \subseteq \langle h \rangle$ containing the identity such that for each open $U \subseteq \langle g \rangle$ containing the identity, $F'(U) \nsubseteq V$. Choose $\{U_n\}$, a decreasing neighborhood basis for the identity in $\langle g \rangle$ and for each $n \in \mathbb{N}$, choose $j_n \in \mathbb{Z}$ so that $g^{j_n} \in U_n$ and so that $h^{j_n} \notin V$. First note that if $\{j_n\}$ has a subsequence $\{j_{n_k}\}$ that is constantly equal to $j \in \mathbb{Z}$, then since $\{U_n\}$ is a neighborhood basis for the identity in $\langle g \rangle$, g^j is the identity, which contradicts the fact that $g^j \notin V$. Thus, the sequence $\{j_n\}$ is either unbounded from above or unbounded from below. If unbounded from above, let $\{i_n\}$ be a strictly increasing subsequence of $\{j_n\}$ such that $i_0 \geq 0$. If not unbounded from above, let $\{i_n\}$ be such that $\{-i_n\}$ is a strictly decreasing subsequence of $\{j_n\}$ with $-i_0 \leq 0$. In either case, $\{i_n\}$ is a strictly increasing sequence of natural numbers that sends g to the identity. Since $\Omega_e(g) \subseteq \Omega_e(h)$, $\{i_n\}$ also sends h to the identity. But this is impossible, since each $h^{i_n} \notin V$ and V is an open set that contains the identity.

We have F' a continuous group homomorphism from a subgroup of a Polish group G to subgroup of a Polish group H. It is well known (see, for example, Proposition 5 of Section 3.3 of Chapter 5 of [4]) that such a map extends uniquely to a continuous group homomorphism $F : \overline{\langle g \rangle} \to \overline{\langle h \rangle}$.

We now prove that (3) implies (1). Suppose $F : \overline{\langle g \rangle} \to \overline{\langle h \rangle}$ is a continuous group homomorphism sending g to h. Since F is a group homomorphism, $F(g^i) = h^i$, for each $i \in \mathbb{Z}$. Since F is continuous, if $g^{i_n} \to a$ for some $a \in G$, then $F(g^{i_n}) \to F(a) \in H$. But $F(g^{i_n}) = h^{i_n}$, for each $n \in \mathbb{N}$. Thus, $\Omega(g) \subseteq \Omega(h)$. □

Note there is at most one continuous group homomorphism from $\overline{\langle g \rangle}$ to $\overline{\langle h \rangle}$ that sends g to h, since $\langle g \rangle$ is dense in $\overline{\langle g \rangle}$.

Proposition 3.2. *Let G and H be Polish groups with $g \in G$ and $h \in H$. The following are equivalent.*

1. $\Omega(g) = \Omega(h)$.

2. $\Omega_e(g) = \Omega_e(h)$.

3. *There is a topological group isomorphism $F : \overline{\langle g \rangle} \to \overline{\langle h \rangle}$ such that $F(g) = h$.*

Proof. By the previous proposition it suffices to show that if there exists a continuous group homomorphism $F_0 : \overline{\langle g \rangle} \to \overline{\langle h \rangle}$ sending g to h and there exists a continuous group homomorphism $F_1 : \overline{\langle h \rangle} \to \overline{\langle g \rangle}$ sending $h = g$, then F_0 and F_1 are inverses.

Consider $F_1 \circ F_0 : \overline{\langle g \rangle} \to \overline{\langle g \rangle}$, a continuous group homomorphism which clearly sends g^i to g^i for each $i \in \mathbb{Z}$. Let $a \in \overline{\langle g \rangle} \setminus \langle g \rangle$ and consider $a' = F_1 \circ F_0(a)$. Find $\{i_n\}$ so that $g^{i_n} \to a$, which implies $F_1 \circ F_0(g^{i_n}) \to a'$. But since $F_1 \circ F_0(g^{i_n}) = g^{i_n}$, $a = a'$. Thus $F_1 \circ F_0$ is the identity map on $\overline{\langle g \rangle}$. Similarly, $F_0 \circ F_1$ is the identity map on $\overline{\langle h \rangle}$. □

We can explicitly characterize compactness of the group $\overline{\langle g \rangle}$ in terms of $\Omega(g)$.

Proposition 3.3. *Let G be a Polish group and $g \in G$. The group $\overline{\langle g \rangle}$ is compact iff every strictly increasing sequence $\{i_n\}$ of natural numbers has a subsequence along which g converges.*

Proof. Let G be a Polish group and let $g \in G$. If $\overline{\langle g \rangle}$ is compact, then every sequence $\{g^{i_n}\}$ has a convergent subsequence.

Suppose now that $\overline{\langle g \rangle}$ is not compact. Then there is a sequence $\{a_n\}$ in $\overline{\langle g \rangle}$ that has no convergent subsequences. We want to show that there is an increasing sequence $\{i_n\}$ of natural numbers so that $\{g^{i_n}\}$ has no convergent subsequences.

If g is not rigid, then any strictly increasing sequence of natural numbers will do. Suppose, then, that g is rigid. It is easy to see that in this case $\{g^i : i \in \mathbb{N}\}$ is dense in $\overline{\langle g \rangle}$. Indeed, let j_n be such that $\{g^{j_n}\}$ converges to the identity and notice that for each $k \in \mathbb{N}$, $\{g^{j_n - k}\}$converges to g^{-k}. This implies that $\{g^i : i \in \mathbb{N}\}$ is dense in $\langle g \rangle$ and thus is dense in $\overline{\langle g \rangle}$.

For each n, find a strictly increasing sequence $\{i_{n,k}\}$ of natural numbers that sends g to a_n. By passing to a subsequence if necessary, we can assume that for all k, $d(g^{i_{n,k}}, a_n) < 2^{-n}$. Now choose a strictly increasing sequence $\{i_{n,k_n}\}$ and consider the sequence $\{g^{i_{n,k_n}}\}$ If any subsequence of $\{g^{i_{n,k_n}}\}$ converges to some $a_\infty \in G$, then since $d(g^{i_{n,k_n}}, a_n) < 2^{-n}$, a subsequence of $\{a_n\}$ would also converge to a_∞, which is impossible. Thus $\{g^{i_{n,k_n}}\}$ has no convergent subsequences. □

Corollary 3.4. *Let G and H be Polish groups with $g \in G$ and $h \in H$. If $\Omega(g) \subseteq \Omega(h)$ and $\overline{\langle g \rangle}$ is compact, then $\overline{\langle h \rangle}$ is also compact.*

In a similar vein we have the proposition below for elements of the circle, $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$. An element x of a Polish group is *periodic* if there is some natural number $p > 0$ so that $x^p = 1$. The *period* of a periodic element is the smallest natural number $p > 0$ so that $x^p = 1$.

Proposition 3.5. *Let $x \in \mathbb{T}$ and let g be an element of a Polish group G so that $\Omega_e(x) \subseteq \Omega_e(g)$. Then there is some $k \in \mathbb{Z}$ so that $\Omega_e(x^k) = \Omega_e(g)$.*

Proof. We may assume that g is not the identity, for otherwise $k = 0$ works.

Suppose $x \in \mathbb{T}$ is periodic with period p. Then $\{p, 2p, 3p, \dots\} \in \Omega_e(x) \subseteq \Omega_e(g)$. Now g^p must be the identity in G and thus g is also periodic with period q dividing p. Clearly $x^{p/q}$ is periodic with period q and thus $\Omega_e(x^{p/q}) = \Omega_e(g)$.

Now suppose $x \in \mathbb{T}$ is not periodic and that $\Omega_e(x) \subseteq \Omega_e(g)$. Then there is a continuous group homomorphism $F : \mathbb{T} \to \overline{\langle g \rangle}$ so that $F(x) = g$. Since $\mathbb{T}$ is compact, the image of F is compact in $\overline{\langle g \rangle}$. This clearly implies that the image of F equals $\overline{\langle g \rangle}$. Consider the kernel of F. It must be finite, for it is a closed subgroup of $\mathbb{T}$ that does not contain x (we know $F(x) = g$ which is not the identity in G). Moreover, it must be generated by a periodic element of $\mathbb{T}$. This implies that $\mathbb{T}/\ker(F)$ is isomorphic to $\mathbb{T}$. Let $G : \mathbb{T}/\ker(F) \to \mathbb{T}$ be such an isomorphism.

The continuous group homomorphism $F : \mathbb{T} \to \overline{\langle g \rangle}$ factors through the group $\mathbb{T}/\ker(F)$. Let $F_0 : \mathbb{T} \to \mathbb{T}/\ker(F)$ and $F_1 : \mathbb{T}/\ker(F) \to \overline{\langle g \rangle}$ be the factor maps. We will first show that for some $k \in \mathbb{Z}$, $G \circ F_1(x) = x^k$. Then we will show that $\Omega_e(x^k) = \Omega_e(g)$ (recall that while the first isomorphism theorem for groups guarantees that F_2 is a group isomorphism, there is no "first isomorphism theorem for topological groups" and F_2 need not, a priori, be a homeomorphism).

First, since $G \circ F_1$ is a continuous group homomorphism from $\mathbb{T}$ to $\mathbb{T}$, there is some $k \in \mathbb{Z}$ so that for all $a \in \mathbb{T}$, $G \circ F_1(a) = a^k$. In particular, $G \circ F_1(x) = x^k$.

Next, consider the continuous group isomorphism $F_2 \circ G^{-1} : \mathbb{T} \to \overline{\langle g \rangle}$. We know that x^k is sent to g and thus that $\Omega_e(x^k) \subseteq \Omega_e(g)$. Suppose that $\Omega_e(g) \not\subseteq \Omega_e(x^k)$. Let $\{i_n\}$ be a strictly increasing sequence of natural numbers that sends g to the identity but does not send x^k to 1. By passing to a subsequence, if necessary, we can assume that $\{(x^k)^{i_n}\}$ is bounded away from 1. By passing to a subsequence, if necessary, we can assume that $\{(x^k)^{i_n}\}$ converges to some $y \in \mathbb{T}$ ($y \neq 1$). But then $F_2 \circ G^{-1}(y)$ is the identity in G, which contradicts the fact that $F_2 \circ G^{-1}$ is a group isomorphism.

Therefore, $\Omega_e(x^k) = \Omega_e(g)$. □

3.2 In the Group Aut(X, μ)

3.2.1 Meager Topological Similarity Classes

When Rosendal introduced the notion of topological similarity in [15] he gave an elegant proof of the fact, attributed to Rohlin, that each conjugacy class in Aut(X, μ) is meager. In fact he proved more, that each topological similarity class is meager in Aut(X, μ). This implies that even if one allows conjugation by elements of $U(L^2(X, \mu))$, each conjugacy class is meager in Aut(X, μ).

Theorem 3.6. *In* Aut(X, μ)*, each topological similarity class is meager. Moreover, if I is an infinite set of positive integers, then the set*

$$\{T : \text{ for some increasing } \{i_n\} \subseteq I, T^{i_n} \to E_\Omega\}$$

is a dense G_δ.

With a little more effort, we can strengthen the previous result.

Proposition 3.7. *Let $T \in$ Aut(X, μ) be rigid and not the identity. Then* $\{S \in \text{Aut}(X,\, u) : \Omega_e(S) \subseteq \Omega_e(T) \text{ or } \Omega_e(T) \subseteq \Omega_e(S)\}$ *is meager.*

Proof. Find an increasing sequence $\{i_n\}$ that sends T to the identity. The set of transformations T for which there is a subsequence of $\{i_n + 1\}$ sending T to the identity is comeager. But if a non-identity $S \in$ Aut(X, μ) is sent to the identity by a subsequence $\{i_{n_k} + 1\}$ of $\{i_n + 1\}$, then

1. The sequence $\{i_{n_k}\}$ sends T to the identity and S to S^{-1}, which is not the identity.

2. The sequence $\{i_{n_k} + 1\}$ sends S to the identity and T to T, which is not the identity.

Thus $\Omega_e(S) \nsubseteq \Omega_e(T)$ and $\Omega_e(T) \nsubseteq \Omega_e(S)$. □

3.2.2 Weak Closures and Centralizers in Aut(X, μ)

We say that $T \in \mathrm{Aut}(X, \mu)$ has the *weak closure property* if $\overline{\langle T \rangle} = C(T)$. King's theorem is that a generic transformation has the weak closure property.

Proposition 3.8. *Let T have the weak closure property. If $S^k = T$, then S has the weak closure property.*

Proof. Suppose $S^k = T$. It is clear that $T \in \overline{\langle S \rangle}$. This clearly implies that $\overline{\langle T \rangle} \subseteq \overline{\langle S \rangle}$, but it also implies that $C(S) \subseteq C(T)$. Since T has the weak closure property, we also have $S \in \overline{\langle T \rangle}$. This implies that $\overline{\langle S \rangle} \subseteq \overline{\langle T \rangle}$ and that $C(T) \subseteq C(S)$. Together we have $\overline{\langle T \rangle} = \overline{\langle S \rangle}$ and $C(S) = C(T)$. Since T has the weak closure property, so does S. □

The following theorem immediately follows from Corollary 1 in [17].

Theorem 3.9. *Fix $k > 0$ and let $F : \mathrm{Aut}(X, \mu) \to \mathrm{Aut}(X, \mu)$ be given by $F(T) = T^k$. The preimage under F of any comeager set is comeager.*

Corollary 3.10. *If a generic T satisfies a certain property, then for generic T, T^k satisfies the property for each $k > 0$.*

We can now prove the following proposition.

Proposition 3.11. *For $k > 1$, a generic transformation is not topologically similar (and hence not conjugate) to its k-th power or any of its k-th roots.*

Proof. Fix $k > 1$ and let T have the weak closure property and be such that T commutes with a transformation R of order k (a generic transformation satisfies these properties by King's weak closure theorem and Ageev's theorem).

To see that T and T^k are not topologically similar, find a sequence $\{i_n\}$ that sends T to R and notice that $\{i_n\}$ sends T^k to the identity. Since T and T^k are not sent to the identity by the same sequences, they are not topologically similar (and hence are not conjugate).

Now suppose $S^k = T$. By Proposition 3.8, S also has the weak closure property and so we can find a sequence $\{i_n\}$ that sends S to R. Notice that $\{i_n\}$ sends T to the identity. Hence S and T are not topologically similar (and hence are not conjugate). □

King proved in [13] that a generic transformation has roots of all orders and Stepin and Eremenko extended this result in [16] by proving that a generic transformation has continuum many k-th roots for each $k > 1$. Thus, the second part of the proposition is not vacuously true. The first part of the proposition is also a consequence of Theorem 1 in [6].

It is worth noting that while the weak closure property is conjugacy invariant, it is not invariant under topological similarity. That is, there exist two measure preserving transformations, exactly one of which has the weak closure property, that are topologically similar.

Indeed, it is known that irrational rotations are rank-1 and so by King's theorem they have the weak closure property. On the other hand, let T be a transformation that has the weak closure property and a non-compact centralizer into which $\mathbb{T}$ embeds (in fact a generic transformation satisfies these properties). Let S be the image, under an embedding of $\mathbb{T}$ into $C(T)$, of some irrational α. By Proposition 3.5, S is similar to an irrational rotation. Now $T \in C(S)$. We claim that T is not in the weak closure of S. If T were in the closure of S, then S and T would have the same closure (since S is in the closure of T). But this is impossible, because the closure of S is compact (isomorphic to $\mathbb{T}$), while the weak closure of T is not.

3.2.3 Mixing and Rigidity

Mixing and rigidity are, in some sense, in conflict with each other. This is because mixing implies a sort of asymptotic independence, while rigidity requires that along a strictly increasing sequence of times $\{i_n\}$, T^{i_n} should look more and more like the identity map.

Proposition 3.12. *If T is strongly mixing, then no strictly increasing sequence of positive integers sends T to the identity.*

Proof. If $\{i_n\}$ is an increasing sequence of positive integers that sends T to the identity, then for each measurable A,

$$\lim_{n\to\infty} \mu(T^{i_n} A \cap A) = \mu(A).$$

But if T is strongly mixing, then it also must be the case that

$$\lim_{n\to\infty} \mu(T^{i_n} A \cap A) = \mu(A)\mu(A).$$

This is a contradiction for any A satisfying $0 < \mu(A) < 1$. □

The following corollary was first proved by Rohlin in [14].

Corollary 3.13. *A generic measure-preserving transformation is not strongly mixing.*

Proof. All strongly mixing transformations are not rigid, and thus are in the same topological similarity class. Each topological similarity class is meager. □

The following proposition can also be seen as showing the incompatibility of asymptotic independence and rigidity.

Proposition 3.14. *If T is weakly mixing, then $\overline{\langle T\rangle}$ is not compact.*

Proof. It suffices to find a sequence $\{i_n\}$ so that no subsequence of $\{T^{i_n}\}$ converges. We may assume (X, μ) is the unit interval with Lebesgue measure.

Let $\{A_i\}$ be an enumeration all dyadic intervals (sets of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$). Since T is weakly mixing, for any i and j there exists $I(i,j) \subseteq \mathbb{N}$ of density 1 so that

$$\lim_{k \in I(i,j)} \mu(T^k A_i \cap A_j) = \mu(A_i)\mu(A_j).$$

We will now construct an increasing sequence of positive integers $\{i_n\}$ so that for each $j \in \mathbb{N}$, $\mu(T^{i_n}[0,1/2] \cap A_j) \to \mu([0,1/2])\mu(A_j) = \frac{1}{2}\mu(A_j)$. Choose i_0 to be any element of $I(1,0)$ and for each n choose $i_{n+1} > i_n$ so that $i_{n+1} \in I(1,j)$ for each $0 \leq j \leq n+1$. We can choose these elements because the finite intersection of density 1 subsets of $\mathbb{N}$ has density 1 and so is unbounded.

We now show that T does not converge along any subsequence of $\{i_n\}$. Suppose some subsequence of $\{i_n\}$ sends T to S. Then $\mu(S([0,1/2]) \cap A_j) = \frac{1}{2}\mu(A_j)$ for all j. But no measurable set, e.g. $S([0,1/2])$, can intersect every dyadic rational in this fashion (this is implied, for example, by the Lebesgue density theorem). □

Since a generic transformation is weakly mixing [9], we have the following corollary.

Corollary 3.15. *For generic T, $\overline{\langle T\rangle}$, and hence $C(T)$, is not compact.*

3.2.4 Questions

Corollary 3.4 and Proposition 3.14 imply that that if $x \in \mathbb{T}$ and if T is weakly mixing, then $\Omega_e(x) \nsubseteq \Omega_e(T)$. The following questions are closely connected to considerations in [3].

Question 1. For which $x \in \mathbb{T}$ does there exist a sequence $\{i_n\} \in \Omega_e(x)$ so that $\{i_n\} \notin \Omega_e(T)$ for every weakly mixing transformation T?

If $x \in \mathbb{T}$ is periodic, there is such a sequence, the multiples of the period of x. It is not known whether there is any non-periodic $x \in \mathbb{T}$ for which there is such a sequence.

On the other hand we can ask the following question.

Question 2. For which weakly mixing T does there exist a non-identity $x \in \mathbb{T}$ so that $\Omega_e(T) \subseteq \Omega_e(x)$?

For any non-rigid, weakly mixing transformations (e.g., any strongly mixing transformation), any non-identity $x \in \mathbb{T}$ will do. For any transformation T, the existence of such an x is equivalent to the existence of a non-trivial character of $\overline{\langle T \rangle}$ (i.e. a continuous group homomorphism from $\overline{\langle T \rangle}$ to $\mathbb{T}$ that does not send T to the identity). It is an open question whether a monothetic group with no non-trivial characters must be extremely amenable. It would be interesting know whether, for generic T, $\overline{\langle T \rangle}$ has no non-trivial characters and or is extremely amenable. Either of these would, in some sense, attest to the largeness of $C(T)$, for generic T.

To show that T is such that $\Omega_e(T) \nsubseteq \Omega_e(x)$ for all non-identity $x \in \mathbb{T}$, it would suffice to find a sequence $\{i_n\}$ that sends T to the identity but not any non-identity $x \in \mathbb{T}$. It is unknown whether there is any weakly mixing T for which such a sequence exists.

Chapter 4

Rank-1 Homeomorphisms of Zero-Dimensional Polish Spaces

4.1 Preliminaries

While the definition given below is for any Polish space X, it should be noted that the existence of a rank-1 homeomorphism of X implies, by condition (5), that X has a basis of clopen sets, i.e., it is zero-dimensional.

Definition. A homeomorphism f of a Polish space X is *rank-1* if there exists a strictly decreasing sequence $\{B_n\}$ of clopen sets and a strictly increasing sequence $\{h_n\}$ of natural numbers so that

1. the sets B_n, $f(B_n)$, ..., $f^{h_n-1}(B_n)$ are pairwise disjoint;
2. $\bigcup_{0\leq i<h_n} f^i(B_n) \subseteq \bigcup_{0\leq j<h_{n+1}} f^j(B_{n+1})$;
3. If $f^j(B_{n+1}) \cap f^i(B_n) \neq \emptyset$ and $0 \leq j < h_{n+1}$ and $0 \leq i < h_n$, then $f^j(B_{n+1}) \subseteq f^i(B_n)$.
4. $f^{h_{n+1}-1}(B_{n+1}) \subseteq f^{h_n-1}(B_n)$; and
5. the orbits of the sets B_n under f and orbits of the sets $L_n := X \setminus \bigcup_{0\leq i<h_n} f^i(B_n)$ under f form a basis for the topology of X.

We will first establish some terminology for rank-1 homeomorphisms. If f is a homeomorphism of a Polish space X, and the sequences $\{B_n\}$ and $\{h_n\}$ witness that f is a rank-1 homeomorphism, then we call the pair $(\{B_n\}, \{h_n\})$ a *tower representation* of f. Every rank-1 homeomorphism has multiple tower representations. For example, if $(\{B_n\}, \{h_n\})$ is a tower representation of f, then by removing a single entry B_k from the sequence $\{B_n\}$ and the corresponding entry h_k from the sequence $\{h_n\}$, one obtains another tower representation of f.

Let f be a rank-1 homeomorphism of a Polish space with a fixed tower representation $(\{B_n\}, \{h_n\})$. For $n \in \mathbb{N}$ and $0 \leq i < h_n$, the set $f^i(B_n)$ is called the i-th level of the stage-n tower. We also call B_n the *base* of the stage-n tower and call $f^{h_n - 1}(B_n)$ the *top* of the stage-n tower. We say that h_n is the *height* of the stage-n tower and call L_n the *leftover piece* of the stage-n tower.

The definition of a rank-1 homeomorphism requires that each level of each tower be either a subset of some level of the stage-0 tower or a subset of L_0. Borrowing terminology from the measure-preserving situation, we sometimes refer to levels of towers that are contained in L_0 as *spacers*. Define $W_n \colon h_n \to \{0, 1\}$ so that $W_n(i) = 0$ iff $f^i(B_n)$ is contained in some level of the stage-0 tower. Since the sequences $\{B_n\}$ and $\{f^{h_n - 1}(B_n)\}$ are each decreasing, each W_n begins and ends with 0. Since $B_{n+1} \subseteq B_n$, W_n is a proper initial segment of W_{n+1}. Thus, we can define $W_\infty \colon \mathbb{N} \to \{0, 1\}$ so that for all $n \in \mathbb{N}$, $W_\infty(i) = A_n(i)$ whenever $W_n(i)$ is defined. There are three possibilities:

1. The sequence W_∞ is periodic. In this case, we say that the tower representation of f is repeating.

   Odometers form an important class of measure-preserving transformations, and each can be realized as a rank-1 homeomorphism of a Cantor space with a repeating tower representation (in fact, with $L_0 = \emptyset$).

2. The sequence W_∞ is not periodic, but there is a bound on the number of consecutive 1s in W_∞. In this case, we say that the tower representation of f is

non-repeating and furthermore that it has bounded sequences of consecutive spacers.

The symbolic version of Chacon's (measure-preserving) transformation is a homeomorphism of a Cantor space. The natural choice for a tower representation of this homeomorphism witnesses that it is rank-1 and has bounded sequences of consecutive spacers.

3. There are arbitrarily long sequences of consecutive 1s in W_∞. In this case, we say the tower representation of f is non-repeating and furthermore that it has arbitrarily long sequences of consecutive spacers.

It is clear that a tower representation of a rank-1 homeomorphism satisfies exactly one of the above conditions, but it is unknown whether a rank-1 homeomorphism can have different tower representations that satisfy different conditions above.

Our main result is the following theorem.

Theorem 4.1. *Let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation. If g is a homeomorphism of X that commutes with f, then there is some $i \in \mathbb{Z}$ so that $f^i = g$.*

Before we proceed with the general analysis that will lead us to the proof of the theorem, we prove a technical proposition about the words W_n that come from a non-repeating tower representation of a rank-1 homeomorphism f. Let $(\{B_n\}, \{h_n\})$ be a non-repeating tower representation of a rank-1 homeomorphism f. Let $m > n$ and consider B_m, $f(B_m)$, $\ldots$, $f^{h_m-1}(B_m)$, the levels of the stage-m tower.

Condition (3) of the definition of rank-1 implies that each level of the stage-m tower is either a subset of a level of the stage-n tower or is contained in L_n. Since

$B_m \subseteq B_n$, level i of the stage-m tower is contained in level i of the stage-n tower, for $0 \leq i < h_n$. Thus, W_m begins with an occurrence of W_n.

It is easy to see that either $f^{h_n}(B_m) \subseteq B_n$ or $f^{h_n}(B_m) \subseteq L_n$. Indeed, if $f^{h_n}(B_n) \subseteq f^i(B_n)$ with $0 < i < h_n$, then $f^{h_n - i}(B_m) \subseteq B_n$, which contradicts the fact that B_n, $f(B_n)$, ..., $f^{h_n-1}(B_n)$ are pairwise disjoint. By similar reasoning, the smallest $j \geq h_n$ for which $f^j(B_m)$ is contained in some level of the stage-n tower satisfies $f^j(B_m) \subseteq B_n$. For this j, if $h_n \leq k < j$, then $f^k(B_m) \subseteq L_n$. Thus, the initial W_n in W_m is followed by $(j - h_n)$-many 1s and then another occurrence of W_n. This pattern continues and W_m can be viewed as a disjoint collection of occurrences of W_n interspersed with 1s. This is described more concretely below.

Let $E_{m,n} = \{i \in [0, h_m) : f^i(B_m) \subseteq B_n\}$. It is clear that if $i \in E_{m,n}$, then W_m has an occurrence of W_n beginning at position i. Such an occurrence of W_n in W_m is called *expected*. The arguments in the preceding paragraph show that each 0 in W_m is a part of exactly one expected occurrence of W_n. In particular, if i and j are consecutive elements of $E_{m,n}$ and $i + h_n \leq k < j$, then $W_m(k) = 1$.

It is possible for W_m to have unexpected occurrences of W_n, but the following proposition guarantees that knowing a sufficiently large subword of W_m that begins with a specified occurrence of W_n is enough to determine whether that specified occurrence of W_n is expected (and "large enough" is independent of m).

Proposition 4.2. *For any $n \in \mathbb{N}$, there is some $l(n) \in \mathbb{N}$ so that for any $m > n$, if s is a subword of length $l(n)$ of W_m that begins with an expected occurrence of W_n, then every occurrence of s in W_m begins with an expected occurrence of W_n.*

To prove this proposition, we need the following lemma.

Lemma 4.3. *Suppose W_m has an expected occurrence of W_n that begins at i and is followed by r 1s and then another expected occurrence of W_n. Suppose further* *that $i < j < i + h_n$ and that W_m also has a (necessarily unexpected) occurrence of W_n that begins at j that is followed by s 1s and then another occurrence of W_n. Then $r = s$.*

Proof. There are four known occurrences of W_n, beginning at i, $i + h_n + r$, j, and $j + h_n + s$. Let α be the finite word that begins at j and ends at $i + h_n - 1$; α has length between 1 and $h_n - 1$, inclusive. Let a be the number of 0s in α (thus, $a > 0$). Let β be the finite word that begins at $i + h_n + r$ and ends at $j + h_n - 1$; β has length between 1 and $h_n - 1 - r$, inclusive. Let b be the number of 0s in β (thus, $b > 0$). Notice that the occurrence of W_n that begins at j consists exactly of $\alpha 1^r \beta$, so there are exactly $a + b$ 0s in W_n. Notice also that the occurrence of W_n that begins at i ends with α and so the word W_n must end with α. Notice also that the occurrence of W_n that begins at $i + h_n + r$ begins with β and so the word W_n must begin with β. By counting the number of 0s in W_n, we see that W_n can also be expressed as $\beta 1^r \alpha$. In particular, the W_n that begins at $i + h_n + r$ has the form $\beta 1^r \alpha$ and also the form $\beta 1^s \alpha'$, where α' is an initial segment of the occurrence of W_n that begins at $j + h_n + s$. Since $\beta 1^r \alpha = \beta 1^s \alpha'$ and both α and α' begin with 0, $r = s$. □

We now give the proof of Proposition 4.2.

Proof. Suppose $n \in \mathbb{N}$ is such that for each $l \in \mathbb{N}$ there is some $m \in \mathbb{N}$ and a subword s of W_m of length l that begins with an expected occurrence of W_n, so that there is also an occurrence of s in W_m that does not begin with an expected occurrence of W_n.

Since W_∞ is not periodic, there is some $k > n$ so that the number of 1s that separate expected occurrences of W_n in W_k is not constant. Let $l = 2h_k + h_n$. Find $m \in \mathbb{N}$ and s a subword of length l of W_m that begins with an expected occurrence of W_n, and so that W_m has an occurrence of s that does not begin with an expected occurrence of W_n.

First consider the occurrence of s in W_m that begins with an expected occurrence of W_n. Because of the regularity with which expected occurrences of W_n appear in W_m, s can be written as $W_n 1^{r_1} W_n 1^{r_2} \dots W_n 1^{r_t} A$, where A is a proper initial segment (possibly empty) of W_n.

Now consider the occurrence of s in W_m that does not begin with an expected occurrence of W_n (say it begins at position i in W_m). Then there are occurrences of W_n that begin at positions i, $i+h_n+r_1$, $i+2h_n+r_1+r_2$, $\dots$, $i+(t-1)h_n+(r_1+\dots+r_{t-1})$. It is easy to check that since W_n begins and ends with 0, none of these occurrences of W_n are expected in W_m and, moreover, that each of them intersects exactly two expected occurrences of W_n in W_m. Repeated application of Lemma 4.3 shows that $r_1 = r_2 = \dots = r_{t-1}$ and, moreover, that if two occurrences of W_n are completely contained in s and separated only by 1s, then they are separated by exactly r_1-many 1s.

Also, since there is an expected occurrence of W_n that contains the 0 at position $i + (t)h_n + (r_1 + \dots + r_{t-1}) - 1$ and that expected occurrence of W_n ends with 0, $r_t < h_n$. This clearly implies that in s there is no consecutive sequence of 1s with length h_n.

Any occurrence of s in W_m begins with a 0, which is a part of some expected occurrence of W_k. The length of s is large enough to guarantee that this occurrence of s will completely contain the next expected occurrence of W_k in W_m. This implies that each pair of consecutive expected occurrences of W_n in W_k are separated by exactly r_1-many 1s. This contradicts the choice of k. □

4.2 Initial Analysis

Let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation $(\{B_n\}, \{h_n\})$.

Proposition 4.4. *Each level of the stage-n tower is a disjoint union of at least two levels of the stage-$(n+1)$ tower.*

Proof. Let $0 \leq i < h_n$ and consider $f^i(B_n)$, level i of the stage-n tower. If $x \in f^i(B_n)$, then for some $0 \leq j < h_{n+1}$, $x \in f^j(B_{n+1})$, which implies that $f^j(B_{n+1}) \subseteq f^i(B_n)$. Since the levels of the stage-$(n+1)$ tower are pairwise disjoint, $f^i(B_n)$ is a disjoint union of some levels of the stage-$(n+1)$ tower. To see that $f^i(B_n)$ contains at least two levels of the stage-$(n+1)$ tower, notice that $B_{n+1} \subsetneq B_n$, which guarantees that $f^i(B_{n+1}) \subsetneq f^i(B_n)$. $\square$

Proposition 4.5. *Every nonempty open set contains a level of a tower.*

Proof. It suffices to show that for each $n \in \mathbb{N}$ and each $i \in \mathbb{Z}$, each of the sets $f^i(B_n)$ and $f^i(L_n)$ contains a level of some tower.

The image, under f, of a non-top level of the stage-m tower is a level of the stage-m tower. The top of the stage-m tower contains at least two levels of the stage-$(m+1)$ tower, and only one of these can be the top of the stage-$(m+1)$. Thus, the image, under f, of the top of the stage-m tower contains a level of the stage-$(m+1)$ tower. Similarly, the pre-image under f of any level of the stage-m tower contains a level of the stage-$(m+1)$ tower. We thus have the following: if A contains a level of a tower, then so does each of $f(A)$ and $f^{-1}(A)$. It now suffices to show that each L_n contains a level of some tower.

If $m > n$, then each level of the stage-m tower is either contained in L_n or is disjoint from L_n. If for every $m > n$, every level of the stage-m tower is disjoint from L_n, then $W_\infty = W_n W_n W_n \dots$, which contradicts the fact that $(\{B_n\}, \{h_n\})$ is a non-repeating tower representation of f. Therefore, L_n contains some level of some tower. □

Let $B_\infty = \bigcap B_n$, $T_\infty = \bigcap f^{h_n - 1}(B_n)$, and $L_\infty = \bigcap L_n$. We say that a point $x \in X$ is an *interior point* (with respect to f) if no point in the orbit of x is in B_∞, T_∞, or L_∞. Interior points are relatively simple to deal with and will play a crucial role in the proof of the main theorem.

Proposition 4.6. *The set of interior points is comeager in X.*

Proof. It suffices to show that each of the sets L_∞, B_∞, and T_∞ are nowhere dense. That L_∞ is nowhere dense is an immediate consequence of the previous proposition. We now prove that B_∞ is nowhere dense. (The proof that T_∞ is nowhere dense essentially the same.)

Suppose U is a non-empty open set. We need to show that there is some non-empty open $V \subseteq U$ that does not intersect B_∞. By the previous proposition, U contains a level of a tower. If it is a level that is not the base, then we can set V equal to that level. If, on the other hand, there is a base of a tower, say B_n, contained in U, then we know that B_n is the disjoint union of at least two levels of the stage-$(n+1)$ tower. As only one of those can be the base of the stage-$(n+1)$ tower, another of them can be chosen for V. In any case, we have V, a non-base level of a tower. The clopen set V is disjoint from B_∞. □

Proposition 4.7. *Let x be an interior point. Then for each $k \in \mathbb{N}$ there is some $N \in \mathbb{N}$ so that for all $n > N$, there is some i satisfying $k \leq i < h_n - k$ and $x \in f^i(B_n)$.*

Proof. Suppose x is an interior point. Since $x \notin L_\infty$, $x \in f^{j_M}(B_M)$ for some $M \in \mathbb{N}$ and $0 \leq j_M < h_M$. By condition (2) of the definition of rank-1 homeomorphisms, there is, for each $m \geq M$, a unique j_m satisfying $0 \leq j_m < h_m$ so that $x \in f^{j_m}(B_m)$. We first claim the sequence $(j_m : m \geq M)$ and the sequence $(h_m - j_m : m \geq M)$ are each increasing.

To see that $j_{m+1} \geq j_m$ for each $m \geq M$, recall that $B_{m+1} \subseteq B_m$. Suppose that $j_{m+1} < j_m$. Then $x \in f^{j_{m+1}}(B_{m+1})$, and hence $f^{-j_{m+1}}(x) \in B_{m+1} \subseteq B_m$. This implies that $x \in f^{j_{m+1}}(B_m)$. But $0 \leq j_{m+1} < j_m < h_m$, hence $f^{j_{m+1}}(B_m)$ and $f^{j_m}(B_m)$ are disjoint, which is a contradiction, since x belongs to both. So, $j_{m+1} \geq j_m$.

To see that $h_{m+1} - j_{m+1} \geq h_m - j_m$ for each $m \geq M$, recall that $f^{h_{m+1}-1}(B_{m+1}) \subseteq f^{h_m-1}(B_m)$. Suppose that $h_{m+1} - j_{m+1} < h_m - j_m$. Then $x \in f^{j_{m+1}}(B_{m+1})$ and hence $f^{h_{m+1}-j_{m+1}-1}(x) \in f^{h_{m+1}-1}(B_{m+1}) \subseteq f^{h_m-1}(B_m)$. This implies that

$$x \in f^{(h_m-1)-(h_{m+1}-j_{m+1}-1)}(B_m) = f^{(h_m)-(h_{m+1}-j_{m+1})}(B_m).$$

But $j_m < h_m - (h_{m+1} - j_{m+1}) < h_m$, and hence $f^{(h_m)-(h_{m+1}-j_{m+1})}(B_m)$ and $f^{j_m}(B_m)$ are disjoint, which is a contradiction, since x belongs to both. So, $h_{m+1} - j_{m+1} \geq h_m - j_m$.

We now have that the sequences $(j_m : m \geq M)$ and $(h_m - j_m : m \geq M)$ are both increasing. To prove the proposition above, it suffices to show that neither of these sequences is eventually constant. If $\{j_m\}$ is eventually constantly c, then $f^{-c}(x)$ is in B_m for all sufficiently large m, which implies that $f^{-c}(x)$ is in B_m for all m, i.e., $x \in B_\infty$, and this contradicts the fact that x is an interior point. Similarly, if $\{h_m - j_m\}$ is eventually constantly c, then $f^{c-1}(x)$ is in T_∞, and this contradicts the fact that x is an interior point. □

Corollary 4.8. *Let x be an interior point. Then for each i, there is some n so that $f^i(x)$ is in a level of the stage-n tower that is neither the base nor the top.*

Proposition 4.9. *If x and y are distinct interior points, then there is some level of some tower that contains exactly one of x and y.*

Proof. Suppose that x and y are interior points that are contained in the same levels of the same towers. We must show that $x = y$. We do this by showing that x and y are in the same basic clopen sets.

Suppose that for some $n \in \mathbb{N}$ and some $k \in \mathbb{Z}$, either $f^k(B_n)$ or $f^k(L_n)$ contains exactly one of x and y. By Proposition 4.7 we can find $m > n$ so that there are i, j satisfying $|k| \leq i, j < h_m - |k|$ so that $x \in f^i(B_m)$ and $y \in f^j(B_m)$. But, since x and y are in the same levels of the stage-m tower, $i = j$. Now $f^{-k}(x)$ and $f^{-k}(y)$ are both in the same level of the stage-m tower. By repeated application of condition (3) of the definition of a rank-1 homeomorphism, either there is some level of the stage-n tower that contains both $f^{-k}(x)$ and $f^{-k}(y)$, or $f^{-k}(x)$ and $f^{-k}(y)$ are both in L_n. Thus, neither $f^k(B_n)$ nor $f^k(L_n)$ contains exactly one of x and y.

Therefore, x and y are in exactly the same basic clopen sets, and so $x = y$. □

Proposition 4.10. *If U is a nonempty open set and $x \notin \bigcup_{i \in \mathbb{Z}} f^i(U)$, then x is the unique fixed point of f.*

Proof. Let U be nonempty and open. By Proposition 4.5, U contains some level of some tower. So, $\bigcup_{i \in \mathbb{Z}} f^i(U)$ contains some B_n, which contains B_m for every $m > n$. It follows that $\bigcup_{i \in \mathbb{Z}} f^i(U)$ contains every level of every tower and thus contains everything that is not in L_∞.

Suppose that $x \notin \bigcup_{i \in \mathbb{Z}} f^i(U)$. It is clear that for all $k \in \mathbb{Z}$, $f^k(x) \in L_\infty$. Thus for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$, both x and $f(x)$ are elements of $f^k(L_n)$ and not elements of $f^k(B_n)$. So, x and $f(x)$ are in the same basic clopen sets and thus $x = f(x)$.

Now suppose that y is a fixed point of f. Since no element of any level of any tower is a fixed point, $y \in L_\infty$. Thus, for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$, $y \in f^k(L_n)$ and $y \notin f^k(B_n)$. Thus, x and y are in the same basic clopen sets, and so $x = y$. □

4.3 Further Analysis

4.3.1 Simplifying Assumptions

As before, let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation $(\{B_n\}, \{h_n\})$. Let g be a homeomorphism of X that commutes with f. We will work towards proving that there is some $k \in \mathbb{Z}$ so that $f^k = g$. For the proof of Lemma 4.17 below, we will distinguish between two cases: either the tower representation $(\{B_n\}, \{h_n\})$ has arbitrarily long sequences of consecutive spacers, or it has bounded sequences of consecutive spacers. If the latter holds, then let a_{max} denote the length of the longest sequence of consecutive spacers in W_∞. Equivalently, a_{max} is the largest natural number for which there exist $n, m \in \mathbb{N}$ so that $0 \leq m \leq m + a_{max} < h_n$ and so that for each j satisfying $m < j \leq m + a_{max}$, $f^j(B_n) \subseteq L_0$.

Before proceeding further, we make some simplifying assumptions, which we can do without loss of generality. If the tower representation $(\{B_n\}, \{h_n\})$ has bounded sequences of consecutive spacers, we can assume that $h_0 > a_{max}$. Indeed, we can modify the witnessing sequences $\{B_n\}$ and $\{h_n\}$ by deleting the initial entry of each sequence and still have a tower representation for f that is non-repeating and has absolutely bounded sequences of consecutive spacers. Doing this repeatedly will guarantee that the initial element of the height sequence will be larger than a_{max}.

For the next two simplifying assumptions, consider $g^{-1}(B_0)$. Since g is a homeomorphism, this nonempty set is open and thus must contain $f^m(B_n)$, for some $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Notice that $gf^m(B_n) \subseteq B_0$ and that gf^m is a homeomorphism of X that commutes with f. If gf^m is an integral power of f, then so is g. Thus we may assume that $m = 0$ and thus that $g(B_n) \subseteq B_0$. Also, we may assume that $n = 1$, for otherwise we can delete the entries in the sequences $\{B_n\}$ and $\{h_n\}$ that are indexed by 1 through $(n-1)$, inclusive. With these two simplifying assumptions we now have the following. If $x \in B_1$, then $g(x) \in B_0$.

4.3.2 The Set Z(x)

For any $x \in X$, let

$$Z(x) = \{i \in \mathbb{Z} : f^i(x) \in B_1\}.$$

The crucial fact is that if x is an interior point, then the set $Z(x)$ contains enough information to recover x.

Lemma 4.11. *If x and y are interior points with $Z(x) = Z(y)$, then $x = y$.*

Proof. Suppose that x and y are distinct interior points and that $Z(x) = Z(y)$. By Proposition 4.9, some level of some tower contains exactly one of x and y. It is easy to see that since x and y are interior points, there is some $n \in \mathbb{N}$ so that x and y are in different levels of the stage-n tower. Let $0 \leq i, j < h_n$ so that $x \in f^i(B_n)$ and $y \in f^j(B_n)$. Without loss of generality, assume $i < j$. By Proposition 4.7 there is some N so that neither x nor y are in any of the top $l(n)$ levels of the stage-N tower (i.e., for all $0 \leq k < l(n)$, neither $f^k(x)$ nor $f^k(y)$ is in $f^{h_N - 1}(B_N)$,the top of the stage-N tower). Recall that $l(n) \in \mathbb{N}$ is such that if s is subword of W_N of length $l(n)$ that begins with an expected occurrence of W_n, then every occurrence of s in W_N begins with an expected occurrence of W_n (see Proposition 4.2).

Since $Z(x) = Z(y)$, we know that for all $k \in \mathbb{Z}$, $f^k(x) \in B_1$ iff $f^k(y) \in B_1$.

It immediately follows from this that if $0 \le m < h_1$, then for all $k \in \mathbb{Z}$, $f^k(x) \in f^m(B_1)$ iff $f^k(y) \in f^m(B_1)$. In other words, we know that for all k, $f^k(x)$ is in level m of the stage-1 tower iff $f^k(y)$ is in level m of the stage-1 tower. But each level of the stage-1 tower is either completely contained in a level of the stage-0 tower or is disjoint from all levels of the stage-0 tower. Thus for all $k \in \mathbb{Z}$,

$$f^k(x) \in \bigcup_{0 \le m < h_0} f^m(B_0) \quad \text{iff} \quad f^k(y) \in \bigcup_{0 \le m < h_0} f^m(B_0). \tag{4.1}$$

Now consider the points $f^{-i}(x)$, $f^{-i+1}(x)$, $\ldots$, $f^{-i+l(n)-1}(x)$. Since x is not in any of the top $l(n)$ levels of the stage-N tower, these points correspond to a subword of length $l(n)$ in W_N that begins with an expected occurrence of W_n. Call this subword s.

But now consider the points $f^{-i}(y)$, $f^{-i+1}(y)$, $\ldots$, $f^{-i+l(n)-1}(y)$. Since y is not in any of the top $l(n)$ levels of the stage-N tower, these points correspond to a subword of length $l(n)$ in W_N. In fact, by (4.1) above, this subword is exactly s. However, since y is on level j of the stage-n tower and $i < j$, the occurrence of s in W_N that corresponds to the points $f^{-i}(y)$, $f^{-i+1}(y)$, $\ldots$, $f^{-i+l(N)-1}(y)$ does not begin with an expected occurrence of W_n. This is a contradiction. □

We will analyze the relationship between $Z(x)$ and $Z(g(x))$, but first we mention two facts and prove a proposition. It is easy to see that distinct elements of $Z(x)$ cannot be two close to each other. Indeed, if $f^i(x) \in B_1$, i.e., if $f^i(x)$ is in the base of the stage-1 tower, then $f^{i+1}(x)$ must be in the first level of the stage-1 tower, $f^{i+2}(x)$ must be in the second level of the stage-1 tower, etc. If $f^{i+k}(x)$ is again in the base of the stage-1 tower, with $k > 0$, then it must be the case that $k \ge h_1$. We thus have the following fact.

Fact 4.12. *For any $x \in X$, if $i \ne i'$ are both in $Z(x)$, then $|i - i'| \ge h_1$.*

More generally, and for similar reasons, we have:

Fact 4.13. *For any $x \in X$, if $i \neq i'$ and both $f^i(x)$ and $f^{i'}(x)$ are in the same level of the stage-n tower, then $|i - i'| \geq h_n$.*

Proposition 4.14. *If x is an interior point, then $Z(x)$ is neither bounded above nor from below.*

Proof. It is clear from the definition of interior point that x is an interior point iff every element of the orbit of x under f is an interior point. It thus suffices to show that for each interior point x, $Z(x)$ contains both a positive and a negative element.

By proposition 4.7 there is some n and some i satisfying $h_1 \leq i < h_n - h_1$ and so that $x \in f^i(B_n)$. Now $f^{-i}(x) \in B_n \subseteq B_1$, so $-i$ is a negative element of $Z(x)$. But also, $f^{h_n - i - 1}(x)$ is in the top of the stage-n tower and thus is in the top of the stage-1 tower. So $f^{h_n - i - h_1}(x) \in B_1$ and hence $h_n - i - h_1$ is a positive element of $Z(x)$. □

4.3.3 The Function ϕ_x

We now work towards showing a very rigid connection between $Z(x)$ and $Z(g(x))$, as long as x and $g(x)$ are interior points.

For any $x \in X$, there is a natural way to define a function from $Z(x)$ to $Z(g(x))$. If $i \in Z(x)$, then $f^i(x) \in B_1$. This implies that $gf^i(x) \in B_0$. So there is a unique m with $0 \leq m < h_1$ so that $gf^i(x)$ is in level m of the stage-1 tower, i.e., in $f^m(B_1)$. Now $f^{i-m}g(x) \in B_1$, so $i - m \in Z(g(x))$. We thus have a function $\phi_x : Z(x) \to Z(g(x))$, given by $\phi_x(i) = i - m$. It is clear that for each $i \in Z(x)$,

$$i - h_1 < \phi_x(i) \leq i.$$

Lemma 4.15. *For any x, the function ϕ_x is an order preserving injection.*

Proof. Suppose i and i' are distinct elements of $Z(x)$ with $i > i'$. By Fact 1 above, we have $i - i' \geq h_1$ and so $i - h_1 \geq i'$. But we also have $\phi_x(i) > i - h_1$ and $i' \geq \phi_x(i')$. Together these give $\phi_x(i) > \phi_x(i')$. □

In the next Lemma, the levels of the stage-n tower that are subsets of B_1 will be important. For $n > 0$, let r_n denote the number of such levels. Since the definition of rank-1 ensures that the top of the stage-n tower is a subset of the top of the stage-1 tower, the highest level of the stage-n tower that is contained in B_1 is level $h_n - h_1$ (for $n > 0$).

Lemma 4.16. *If x is an interior point, then the function ϕ_x is surjective.*

Proof. Let x be an interior point and suppose $j \in Z(g(x)) \setminus \mathrm{rng}(\phi_x)$. Consider the point $f^{j+h_1-1}(x)$. Since x is interior, we can find some $n \in \mathbb{N}$ and $0 \leq m < h_n$ so that $f^{j+h_1-1}(x) \in f^m(B_n)$. We can assume that $n > 1$. Then $f^{j+h_1-1-m}(x)$ is in the base of the stage-n tower.

Now consider the interval $I = [j + h_1 - 1 - m, j + h_n - 1 - m]$. The set $\{f^i(x) : i \in I\}$ contains one element from each of the bottom $h_n - h_1 + 1$ levels of the stage-n tower. This includes all of the levels that are subsets of B_1. So $|I \cap Z(x)| = r_n$.

Now consider the interval $J = [j - m, j + h_n - 1 - m]$. We claim that $|J \cap Z(g(x))| > r_n$. First, if $i \in I \cap Z(x)$, then, since $i - h_1 < \phi_x(i) \leq i$, $\phi_x(i) \in J \cap Z(g(x))$. But j is also in $J \cap Z(g(x))$ and, by assumption, $j \notin \mathrm{rng}(\phi_x)$. Therefore, $|J \cap Z(g(x))| > r_n$.

This implies the existence of distinct $i, i' \in J$ so that both $f^i(x)$ and $f^{i'}(x)$ are in the same level of the stage-n tower. By fact 2 above, $|i - i'| \geq h_n$. This is impossible, since $J = [j - m, j + h_n - 1 - m]$. □

4.3.4 The Function Ψ_x

For x an interior point, we define a function $\Psi_x : Z(x) \to \mathbb{N}$ as follows. Let $i \in Z(x)$ and find $j > i$ as small as possible so that $j \in Z(x)$. Let $\Psi_x(i) = j - i$.

Lemma 4.17. *Suppose x and $g(x)$ are interior points. Then for each $i \in Z(x)$, $\Psi_x(i) = \Psi_{g(x)}(\phi_x(i))$.*

The proof of this Lemma 4.17 will be done differently for the two cases. We first give the proof in the case that the tower representation $(\{B_n\}, \{h_n\})$ has bounded sequences of consecutive spacers.

Proof. Let x be such that x and $g(x)$ are interior points and let $i \in Z(x)$. We want to show that $\Psi_x(i) = \Psi_{g(x)(\phi_x(i))}$. Let j be the smallest element of $Z(x)$ that is greater than i. We have $\Psi_x(i) = j - i$. Also, since we are in the case with absolutely bounded sequences of consecutive spacers, we have:

$$0 \leq \Psi_x(i) - h_1 \leq a_{max} \tag{4.2}$$

Since $\phi_x : Z(x) \to Z(g(x))$ is an order preserving bijection, we have that $\phi_x(j)$ is the largest element of $Z(g(x))$ that is greater than $\phi_x(i)$. So we have $\Psi_{g(x)}(\phi_x(i)) = \phi_x(j) - \phi_x(i)$. And, as before, we have:

$$0 \leq \Psi_{g(x)}(\phi_x(i)) - h_1 \leq a_{max} \tag{4.3}$$

Equations (4.2) and (4.3) clearly imply:

$$|\Psi_x(i) - \Psi_{g(x)}\phi_x(i)| \leq a_{max} \tag{4.4}$$

We will show that in fact $\Psi_x(i) = \Psi_{g(x)}(\phi_x(i))$.

First, consider the point $f^i g(x)$. Since $i \in Z(x)$, $f^i g(x) \in B_0$. We claim that also $f^{i+\Psi_{g(x)}(\phi_x(i))} g(x) \in B_0$. Indeed, since $f^i g(x) \in B_0$, the point $f^i g(x)$ must be in some level of the stage-1 tower. Let $0 \leq m < h_1$ be such that $f^i g(x) \in f^m(B_1)$. Then $f^{i-m} g(x) \in B_1$ and $\phi_x(i) = i - m$. Now $f^{i-m+\Psi_{g(x)}(\phi_x(i))} g(x) \in B_1$, and so $f^{i+\Psi_{g(x)}(\phi_x(i))} g(x) \in f^m(B_1)$. Since we know that $f^m(B_1)$ intersects B_0, it must be contained in B_0. Thus $f^{i+\Psi_{g(x)}(\phi_x(i))} g(x) \in B_0$.

Next, consider the point $f^j g(x)$. Since $j \in Z(x)$, $f^j g(x) \in B_0$. But $j = i + \Psi_x(i)$, so $f^{i+\Psi_x(i)} g(x) \in B_0$.

Suppose that $\Psi_{g(x)}(\phi_x(i)) \neq \Psi_x(i)$. Then $i + \Psi_{g(x)}(\phi_x(i)) \neq i + \Psi_x(i)$. Fact 2 then implies that

$$|(i + \Psi_{g(x)}(\phi_x(i))) - (i + \Psi_x(i))| \geq h_0.$$

Since $h_0 > a_{max}$, we have

$$|\Psi_{g(x)}(\phi_x(i)) - \Psi_x(i)| > a_{max},$$

which contradicts equation (4.4) above. □

The proof of Lemma 4.17 is more involved in the case that the tower representation $(\{B_n\}, \{h_n\})$ has arbitrarily long sequences of spacers. In this case, we need to show that Ψ_x and $\Psi_{g(x)}$ exhibit an almost periodic behavior. Since the elements of $Z(x)$ are not equally spaced, it will be easier to describe this type of periodicity if we first identify $\mathbb{Z}$ with $Z(x)$. This can be done because, by Proposition 4.14, $Z(x)$ is neither bounded from above nor from below.

Fix some $i_0 \in Z(x)$ and let this correspond to $0 \in \mathbb{Z}$. This extends uniquely to an order preserving correspondence of $\mathbb{Z}$ with $Z(x)$. Let i_k denote the element of $Z(x)$ that corresponds to $k \in \mathbb{Z}$. This correspondence between $\mathbb{Z}$ and $Z(x)$ extends through the order preserving bijection $\phi : Z(x) \to Z(g(x))$ to a correspondence between $\mathbb{Z}$ and $Z(g(x))$. For $k \in \mathbb{Z}$, let j_k denote $\phi(i_k) \in Z(g(x))$. In the ensuing discussion of Ψ_x and $\Psi_{g(x)}$ we will take the domain of each function to be $\mathbb{Z}$; that is, we will write $\Psi_x(k)$ in place of $\Psi_x(i_k)$ and we will write $\Psi_{g(x)}(k)$ in place of $\Psi_{g(x)}(j_k)$.

Recall that for $n > 0$, r_n is the number of levels in the stage-n tower that are contained in B_1.

Claim 3. Let x be an interior point.

1. For each $n > 1$, Ψ_x is constant each congruence class mod r_n except one. On this last congruence class mod r_n, Ψ_x is unbounded.

2. For each $k \in \mathbb{Z}$, there is an $n > 1$ so that Ψ_x is constant on the congruence class of k mod r_n.

Proof. For each $n > 0$ we will define R_n, a sequence of natural numbers of length $r_n - 1$. Let z be any point in the base of the stage-n tower. Let $\{i_0, i_1, ..., i_{r_n-1}\}$ enumerate, from smallest to largest, the elements of the set $\{i \in [0, h_n - 1] : f^i(x) \in B_1\}$. We now define R_n to be the sequence $(i_1 - i_0, i_2 - i_1, ..., i_{r_n-1} - i_{r_n-2})$.

It is clear that the definition of R_n is independent of which point $z \in B_n$ is chosen. So whenever $i_k \in Z(x)$ is such that $f^{i_k}(x)$ is in the base of the stage-n tower, there is an occurrence of the word R_n that begins at position k in Ψ_x. But since x is an interior point, if k is such that $f^{i_k}(x) \in B_n$, then both $f^{i_{k+r_n}}(x)$ and $f^{i_{k-r_n}}(x)$ are in the base of the stage-n tower. So for such a k the function Ψ_x is constant on the congruence class of $k + m$ mod r_n, for each $0 \leq m < r_n - 1$. But since we are in the case that the tower representation has unbounded sequences of consecutive spacers, the last congruence class mod r_n must be unbounded. This proves part 1.

Let $k \in \mathbb{Z}$. Since $f^{i_k}(x)$ is in the base of the stage-1 tower, $f^{i_k+h_1-1}(x)$ is in the top of the stage-1 tower. Since x is an interior point, there is some n so that $f^{i_k+h_1-1}(x)$ is not in the top of the stage-n tower. For this n, Ψ_x is constant on the congruence class of k mod r_n. □

We now give the proof of Lemma 4.17 in the case that the tower representation has arbitrarily long sequences of spacers.

Proof. Let x and $g(x)$ be interior points. We want to show that $\Psi_x = \Psi_{g(x)}$. Suppose that $\Psi_x(i) \neq \Psi_{g(x)}(i)$. Choose n so that $\Psi_{g(x)}$ is constant on the set $\{i+k(r_n) : k \in \mathbb{Z}\}$ and let j be such that $\Psi_{g(x))}$ is unbounded on the set $\{j+k(r_n) : k \in \mathbb{Z}\}$. Clearly i and j are in different congruence classes mod r_n. If Ψ_x is unbounded on the set $\{j + k(r_n) : k \in \mathbb{Z}\}$, then Ψ_x agrees with $\Psi_{g(x)}$ at each position except perhaps those in $\{j + r(z_n) : r \in \mathbb{Z}\}$ (which contradicts the fact that Ψ_x and $\Psi_{g(x)}$ differ at i). Thus Ψ_x is constant on the set $\{j + k(r_n) : k \in \mathbb{Z}\}$.

Now find $k \in \mathbb{Z}$ so that $\Psi_{g(x)}(k) - \Psi_x(k) \geq h_1$. We have that $i_{k+1} - i_k = \Psi_x(k)$ and also that $j_{k+1} - j_k = \Psi_{g(x)}(k)$. It follows that

$$\Psi_{g(x)}(k) - \Psi_x(k) = (j_{k+1} - i_{k+1}) + (i_k - j_k).$$

But we know that $j_{k+1} - i_{k+1} \leq 0$ and that $i_k - j_k < h_1$. Thus, $\Psi_{g(x)}(k) - \Psi_x(k) < h_1$, a contradiction. □

Lemma 4.18. *If x and $g(x)$ are interior points, then for some $k \in \mathbb{Z}$, $f^k(x) = g(x)$.*

Proof. Suppose x and $g(x)$ are interior points. We know that for each $i \in Z(x)$, $i - h_1 < \phi_x(i) \leq i$.

We claim that $i - \phi_x(i)$ is independent of i. Indeed, if $i - \phi_x(i)$ is not independent of i, then we can find consecutive i and j in $Z(x)$ so that $i - \phi_x(i) \neq j - \phi_x(j)$. By saying that i and j are consecutive elements of $Z(x)$, we formally mean that

$j \in Z(x)$ is as small as possible satisfying $j > i$. So we have

$$\phi_x(j) - \phi_x(i) \neq j - i.$$

Since i and j are consecutive elements of $Z(x)$, $j - i = \Psi_x(i)$. Since $\phi_x : Z(x) \to Z(g(x))$ is an order preserving bijection we also have that $\phi_x(i)$ and $\phi_x(j)$ are consecutive elements of $Z(g(x))$ and so $\phi_x(j) - \phi_x(i) = \Psi_{g(x)}(\phi_x(i))$. But then

$$\Psi_{g(x)}(\phi_x(i)) \neq \Psi_x(i)$$

and this contradicts Lemma 4.17.

Thus $i - \phi_x(i)$ is independent of i. Let k be such that for all $i \in Z(x)$, $i - \phi_x(i) = k$. Since $\phi_x : Z(x) \to Z(g(x))$ is a bijection we have that $i \in Z(x)$ iff $i-k \in Z(g(x))$. But clearly $i \in Z(x)$ iff $i-k \in Z(f^k(x))$. Thus $Z(f^k(x)) = Z(g(x))$ and, by Proposition 4.11, $f^k(x) = g(x)$. $\square$

4.4 Proof of the Main Theorem

We now prove Theorem 4.1.

Proof. Let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation and let g be a homeomorphism of X that commutes with f. By Proposition 4.6, the set of interior points is comeager in X. So the set $\{x \in X : x \text{ and } g(x) \text{ are interior points}\}$ is also comeager in X.

Since both g and f are homeomorphisms of X, each $A_i = \{x \in X : g(x) = f^i(x)\}$ is closed. By Lemma 4.18, each element of the comeager set $\{x \in X : x \text{ and } g(x) \text{ are interior points}\}$ is in some A_i. Therefore, some A_i is dense in some nonempty open set U. Since A_i is closed, it contains U. Since g and f commute,

A_i is invariant under T. By Proposition 4.10, $\bigcup_{i \in \mathbb{Z}} f^i(U)$ is either all of X or all of X except the unique fixed point of f. But the unique fixed point of f must be a fixed point of g (since g and f commute) and thus must be in A_i.

We now have that A_i is all of X. Thus $g = f^i$. $\square$

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