The Determination of Spot and Future Prices with Storable Commodities

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Abstract

This paper analyzes the effects of futures trading in a market for a storable commodity, in which producers and speculators are assumed to be risk averse and specifications of the aggregate supply and inventory demand functions are derived from explicit optimization. A critical aspect is how the parameters of these functions change with the introduction of the future market as it is through these induced parameter changes that the futures market exerts its influence on the spot price. The effects of the futures market on both the long-run average spot price and its variance are analyzed. While we are unable to draw any definitive conclusions on this issue, we find that in all cases considered the futures market stabilizes the spot price, as well as lowering its long-run mean.
1. INTRODUCTION

For many years now, the question of the effects of futures trading on the stability of spot prices has been extensively debated. This topic has been discussed at various levels. For example, farmers and agricultural interest groups have claimed that futures trading destabilizes spot prices, thereby imposing welfare losses on the economy. In the United States, Congress has decided that futures trading can cause price destabilization and has subjected futures trading to the regulation of the Commodity Futures Trading Commission. At a more analytical level, economists have been investigating the issue, both empirically, and more recently, theoretically as well.

The empirical work extends as far back as Emery (1896). The typical approach is to consider two periods, one with, and the other without, futures trading and to compare the variances of the spot prices in the two cases. This has been carried out for a large number of specific commodities markets and the results are not uniform. Several authors find that futures markets definitely reduce price fluctuations: see, e.g., Hieronymus (1960), Working (1960), Gray (1963), and Cox (1976) for the onions markets; Hooker (1901) and Tomek (1971) for the wheat market; Emery (1896) for the cotton market; Powers (1970), Taylor and Leuthold (1974) and Cox (1976) for the cattle market. Other authors find that there is no essential gain in stability: see, e.g., Johnson (1973) for the onions market; Naik (1970) for the hessian market. These are just a small sample of what has become a vast literature.

Parallel to this, several economists have tried to provide a theoretical answer to the issue; see, e.g., Peck (1976), Turnovsky (1979) and Sarris (1980). ¹ The approach adopted by these authors is to compare the stability of the spot price, as measured by its asymptotic variance, with and without a futures market, for given (arbitrarily specified) behavioral relationships for the agents in the market. The general conclusions of these theoretical analyses is that under their respective assumptions, the futures market almost certainly stabilizes the spot price.
As has been recently pointed out by Kawai (1981a) and Turnovsky (1981) these theoretical studies suffer from a fundamental deficiency. Specifically, they assume that the coefficients of the relevant supply and inventory demand functions remain unchanged with the introduction of the futures market. But except in the polar case where all individuals are risk neutral, this assumption is invalid. If producers and speculators are risk averse, the slopes of the relevant supply and inventory demand functions depend upon the degree of price stability and this in turn varies with the introduction of the futures market. To incorporate these important aspects adequately, it is necessary to derive the behavioral relationships for firms and speculators from underlying optimizing considerations, similar to those employed by Danthine (1978), Holthausen (1979), Feder, Just and Schmitz (1980), Newbery and Stiglitz (1981) and others. Indeed, unless one casts the analysis in such a framework, one has no firm theoretical basis for drawing any inference as to the effects of futures market trading on the distribution of the spot price.

Unlike the earlier studies we have noted, the Kawai and Turnovsky analyses are restricted to non-storable commodities. While futures markets for some such commodities exist, it is important to extend this approach to the more general case where commodities are assumed to be storable, thereby permitting private individuals to hold inventories. This is undertaken in the present paper. In doing so, we employ a rational expectations framework, in which all agents have access to identical information. We restrict our attention to the question of stabilization in the face of final demand and aggregate supply disturbances, which generally are presumed to be the underlying sources of the random fluctuations in the spot price.

The remainder of the paper proceeds as follows. Section 2 discusses the determination of the market equilibrium price in the absence of the futures market. Section 3 introduces the futures market and derives the optimal behavior of risk
averse firms and speculators. Based on this behavior, the solutions for the short-run market clearing spot and futures prices are derived. The next section presents general expressions for comparing the long-run mean spot price and its variance with and without the futures market. While a general unambiguous comparison is not possible, several channels through which the introduction of the futures market exerts its influence can be identified. Section 5 undertakes these comparisons for a number of important special cases. These include: (i) risk neutral speculators; (ii) risk neutral producers; (iii) risk neutral speculators and producers; (iv) pure production (i.e., no inventories); (v) pure inventory holdings (i.e., no production); (vi) constant marginal inventory costs. Taken together, these results suggest that for the classes of disturbances considered by the analysis, namely disturbances in final demand and/or supply, the introduction of a futures market almost certainly provides a stabilizing influence. The main conclusions and some further comments are given in Section 6.

2. PRICE DETERMINATION IN THE ABSENCE OF A FUTURES MARKET

There are three groups of participants in the market: (a) firms, (b) speculators, (c) consumers. We assume for convenience, but without any essential loss of generality, that private storage is the specialized activity of speculators. We begin by considering the behavioral relationships of representative individuals from each group.

A. Derivation of Underlying Microeconomic Behavioral Relationships

The representative firm is assumed to be perfectly competitive and to produce its output subject to a quadratic cost function. Its profit function in period $t$ is therefore described by

$$\pi^f_t = p_t y_t - \frac{1}{2} c y^2_t$$

where

$y_t =$ planned output, chosen by the representative firm, upon which costs are incurred,
\( y_t = \) actual output of the firm,

\( P_t = \) spot price of output at time \( t \) taken as parameterically given to the firm.

We shall assume that the firm makes its production decision for time \( t \) at time \( t-1 \), before the actual spot price \( P_t \) is known. Because of random fluctuations in production conditions, assumed to be beyond the control of the firm, actual and planned outputs are related by

\[
y_t = \bar{y}_t + v'_t
\]

where \( v'_t \) is an additive random variable, having zero mean and finite variance.

Combining equations (1) and (2) yields

\[
\pi_t = P_t(\bar{y}_t + v'_t) - \frac{1}{2} c y_t^{-2}.
\]

To preserve the linearity of the model, we shall assume that the firm maximizes the following one-period function of expected profit and its variance

\[
V^f_t \equiv \pi^*(t,t-1) - \frac{1}{2} a \sigma^2(t,t-1)
\]

where

\[
\pi^*(t,t-1) \equiv E_{t-1}(\pi_t) = \text{conditional expectation of profit for time} \ t, \ \text{formed at time} \ t-1,
\]

\[
\sigma^2(t,t-1) \equiv E_{t-1}[\pi_t - E_{t-1}(\pi_t)]^2 = \text{conditional variance of profit for time} \ t, \ \text{formed at time} \ t-1.
\]

The criterion (3) may be justified as being consistent with expected utility maximization in the case where the underlying random variables are multivariate-normal and the utility function is characterized by constant absolute risk aversion, with \( a \) being the coefficient of absolute risk aversion. For a risk averse producer \( a > 0 \), while \( a = 0 \) corresponds to risk neutrality.

By direct calculation, we obtain from (1) and (2)

\[
\pi^*(t,t-1) = P^*_{t,t-1} \bar{y}_t + E_{t-1}(P_t v'_t) - \frac{1}{2} c y_t^{-2}
\]

\[
\sigma^2(t,t-1) = \sigma^2_p(t,t-1) \bar{y}_t^{-2} + 2 \text{cov}_{t-1}(P_t, P_t v'_t) \bar{y}_t + \text{var}(P_t v'_t)
\]

where \( \text{cov}_{t-1}, \ \text{var}_{t-1} \) denote the conditional covariance and variance respectively, and \( P^*_{t,t-1}, \ \sigma^2_p(t,t-1) \) denote the one period conditional mean and variance of the
price for time \( t \), formed at time \( t-1 \). In order to simplify the subsequent analysis, it is useful to introduce an approximation based on the following argument. We shall assume that there are \( n \) identical firms, each of which contributes equally to the aggregate supply disturbance \( v_t \), so that for the representative firm \( v_t' = v_t/n \). We shall assume that the price responds proportionally to the aggregate supply disturbance \( v_t \) so that the conditional cross moments formed at time \( t-1 \) between \( P_t \) and \( v_t \) are finite and of order 1; i.e., \( E_{t-1}(P_t v_t) = O(1) \), where \( O(\cdot) \) denotes order notation. Thus we have

\[
\sigma_p^2(t,t-1) = \text{Var}_{t-1}(P_t) = O(1) \\
E_{t-1}(P_t v_t') = \frac{1}{n} E_{t-1}(P_t v_t) = O(\frac{1}{n}) \\
\text{cov}_{t-1}(P_t P_t v_t') = \frac{1}{n} \text{cov}_{t-1}(P_t P_t v_t) = O(\frac{1}{n}) \\
\text{var}_{t-1}(P_t v_t') = \frac{1}{n} \text{var}_{t-1}(P_t v_t) = O(\frac{1}{n}).
\]

Assuming that the number of firms is sufficiently large, the expressions \( E_{t-1}(P_t v_t') \), \( \text{var}_{t-1}(P_t v_t') \), \( \text{cov}(P_t P_t v_t') \) are all at least of an order smaller than \( \sigma_p^2(t,t-1) \). Thus to the first order, we may approximate the one period mean and variance of profit by

\[
\pi^*(t,t-1) = P^*_{t,t-1} \overline{y}_t - \frac{1}{2} \sigma_{P_t}^2 \overline{y}_t^2 \\
\sigma_{\pi}^2(t,t-1) = \sigma_p^2(t,t-1) \overline{y}_t^2.
\]

Substituting these two expressions into (3) yields

\[
V_t^f \equiv P^*_{t,t-1} \overline{y}_t - \frac{1}{2} \sigma_{P_t}^2 \overline{y}_t^2 - \frac{1}{2} a_0^2(t,t-1) \overline{y}_t^2
\]

and maximizing \( V_t^f \) with respect to \( \overline{y}_t \), we derive the following expression for the optimal planned output

\[
\overline{y}_t = \frac{P^*_{t,t-1}}{c + a_0^2(t,t-1)}.
\]

Thus the planned output of the representative firm varies positively with the expected spot price, and inversely with its one-period variance.

We turn now to the second group, speculators. In the present context they are assumed to trade in inventories of the commodity in anticipation of price changes.
We shall let $i_{t-1}$ denote the net position in the commodity by the speculator entered at time $t-1$. If $i_{t-1} > 0$, the speculator holds positive stocks of the commodity, while $i_{t-1} < 0$ denotes that he is holding the commodity short. The profit of the representative speculator over the period $(t-1, t)$ is postulated to be

$$\pi^s_t = i_{t-1}(P_t - P_{t-1}) - \frac{1}{2}d i_{t-1}^2$$

where the quadratic term denotes the costs associated with trading in inventories. These consist of storage costs (if $i_{t-1} > 0$) together with transactions and interest costs. While the assumption of quadratic costs is a gross simplification, introduced to preserve linearity, the requirement $d > 0$ is necessary for a well-defined inventory demand function to exist, once a futures market is introduced; see section 3 below.

Analogous to firms, the objective function of the speculator is to maximize

$$V^s_t = i_{t-1}(P^*_{t,t-1} - P_{t-1}) - \frac{1}{2}d i_{t-1}^2 - \frac{1}{2} \beta \sigma^2_{P}(t,t-1) i_{t-1}^2 .$$

The parameter $\beta$ measures the degree of risk aversion and need bear no particular relation to $\alpha$. Maximizing (6) with respect to $i_{t-1}$, yields the following inventory demand function

$$i_{t-1} = \frac{P^*_{t,t-1} - P_{t-1}}{d + \beta \sigma^2_{P}(t,t-1)} .$$

This specification has been familiar since the time of Muth (1961). It asserts that risk averse speculators take a long or short position in inventories, depending upon whether they expect the spot price to rise or fall over the period.

It is not necessary to derive the demand functions for the third group of market participants, consumers, from underlying utility maximization. Instead, it suffices to simply postulate some convenient aggregate function.

B. Aggregate Market Relationships

We assume that aggregate demand for the commodity, $D_t$, is specified by the linear relationship

$$D_t = A - aP_t + u_t \quad A, a > 0 .$$
Summing over the representation firms leads to the aggregate supply function

\[ S_t = bP_{t,t-1}^* + v_t \]  

(8)

where

\[ b \equiv \frac{1}{c + \alpha \sigma_p^2(t,t-1)} . \]  

(9)

Strictly speaking, the coefficient in the aggregate supply function should be multiplied by the number of representative firms, but without essential loss of generality this factor can be set to unity. We shall assume that the probability distribution of \( P_t \) is stationary, so that the one period variance \( \sigma_p^2(t,t-1) \) is independent of \( t \) and shall be denoted simply by \( \sigma_p^2(1) \).

Now that even though the representative firm may make its production plans by ignoring the correlation between \( P_t \) and \( v_t' \), at the aggregate level, the stochastic component of \( S_t \) must be taken into account. The disturbances \( u_t \) and \( v_t \) are assumed to be additive, independently distributed over time, and to have zero means and finite variances

\[ E(u_t) = E(v_t) = 0 \]

\[ E(u_t^2) = \sigma_u^2; E(v_t^2) = \sigma_v^2 \]

\[ E(u_t v_t) = \sigma_{uv} . \]

Likewise, the aggregate demand for inventories, \( I_{t-1} \), is postulated to be

\[ I_{t-1} = \omega(P_{t,t-1}^* - P_{t-1}) \]  

(10)

where

\[ \omega \equiv \frac{1}{d + \beta \sigma_p^2(1)} . \]  

(11)

The final component of the aggregate model is the market clearing condition

\[ D_t + I_t = S_t + I_{t-1} . \]  

(12)

C. Solution for Spot Price

Substituting (7), (8) and (10) into (12) yields the following difference equation in \( P_t, P_{t+1}^*, P_{t,t-1}^* \).
\[ A - \alpha P_t + u_t + \omega(P^*_{t+1} - P_t) \]
\[ = \omega(P^*_{t,t-1} - P_{t-1}) + bP^*_{t,t-1} + v_t. \]  

(13)

We define the long-run average price attained when expectations are realized \((P^*_{t+1} = P^*_{t,t-1} = P_t)\) by \(\overline{P}\), it follows from (13) that
\[ \overline{P} = \frac{A}{\alpha + b}. \]  

(14)

Next, we define the following variables in deviation form, \(p_t \equiv P_t - \overline{P}\),
\[ p^*_{t+1,t} \equiv P^*_{t+1,t} - \overline{P} \] and let
\[ e_t \equiv u_t - v_t \]
equation (13) becomes
\[ -\alpha p_t + \omega(p^*_{t+1,t} - P_t) = \omega(p^*_{t,t-1} - P_{t-1}) + \omega(p^*_{t,t-1} - P_{t-1}) - e_t \] 

(15)

where
\[ E(e_t) = 0, \quad E(e_t^2) = \sigma_e^2 = \sigma_u^2 + \sigma_v^2 + 2\sigma_{uv}. \]

Equation (15) describes the behavior of current market clearing prices in terms of their conditional variances. It is observed to be identical to Muth (1961, eq. (4.12) and Turnovsky (1979, eq. (10)). Thus following the solution procedure of these authors (which employs standard rational expectations methodology), the stable solution for \(P_t\) is given by
\[ p_t = rP_{t-1} + \frac{e_t}{\alpha + \omega(1-r)} \] 

(16)

where \(r\) is the smaller root (which is real and lies in the range \(0 < r < 1\)) of the quadratic equation
\[ \omega(1-r)^2 = (a+b)r. \]  

(17)

For convenience the formal argument is repeated in Appendix A. Thus we deduce that the one-period variance \(\sigma^2_p(1)\) and asymptotic variance \(\sigma^2_p\) of spot prices are given by
\[ \sigma^2_p(1) = \frac{\sigma_e^2}{\left(a + \omega(1-r)\right)^2} \] 

(18)
\[ \sigma^2_p = \frac{\sigma^2_p(1)}{\left(1-r^2\right)^2} = \frac{\sigma_e^2}{\left(1-r^2\right)^2\left[a + \omega(1-r)\right]^2} \]  

(19)

where \(r\) is the smaller root of (17), and \(b, \omega\) are given by (9) and (11) respectively.
Formally, the expressions for $\overline{P}$ and $\sigma_{P}^{2}$ given in (14), (19) are identical to the corresponding expressions obtained by Turnovsky (1979) for rational expectations and private storage. There is however, one fundamental difference. From the expressions (9) and (11) the parameters $b$, $c_{t-1}$ are seen to be functions of the one-period variance $\sigma_{P}^{2}(1)$, and therefore of $\sigma_{P}^{2}$ and $r$. As a consequence, equations (17) and (19) define a pair of highly non-linear relationships between $\sigma_{P}^{2}$ and $r$, or equivalently $\sigma_{P}^{2}(1)$ and $r$. As pointed out in a slightly simplified version of this model by McCafferty and Driskill (1980), this non-linearity is the cause of possible non-uniqueness and non-existence problems concerning rational expectations equilibria. That is, when we take the definitions of $b$ and $c_{t-1}$ given in (9) and (11) into account, it is possible that there is in fact no value of $r$ lying in the range $0 < r < 1$, or alternatively there may be a multiplicity of such roots. We shall note below an example where such non-existence problems may arise.

3. FUTURES MARKET

A. Underlying Microeconomic Behavior

We now introduce a futures market and assume that at time $t-1$, when production decisions must be made, the firm can enter into a contract to deliver a fixed specified quantity of output at an agreed contract price at time $t$. Profit at time $t$ is therefore given by

$$\pi_{t}^{f} = P_{t}(y_{t} - z_{t-1}) + p_{t-1}^{f}z_{t-1} - \frac{1}{2}c_{y_{t}}^{2}$$

where $y_{t}$, $\overline{y}_{t}$, $P_{t}$ are as defined previously and

- $p_{t-1}^{f}$ = one period futures price prevailing at time $t-1$ for delivery at time $t$,
- $z_{t-1}$ = firm's net position in futures contracts entered into at time $t-1$.

If $z_{t-1} > 0$, the firm is selling futures contracts, while $z_{t-1} < 0$ denotes a purchase of futures contracts.

Repeating the analysis of Section 2, the firm's objective (3) with the addition of a futures market, becomes
The firm's decision problem is to choose: (i) its planned level of output $\bar{y}_t$, (ii) its net position in the futures market $z_{t-1}$, to maximize (20). The first order optimality condition are

$$P_{t,t-1}^* - \frac{c}{2} y_t - \alpha \sigma^2_p(1) (\bar{y}_t - z_{t-1}) = 0$$

$$P_{t-1}^f - P_{t,t-1}^* + \alpha \sigma^2_p(1) (\bar{y}_t - z_{t-1}) = 0$$

from which we obtain the explicit solutions

$$\bar{y}_t = \frac{P_{t-1}^f}{c}$$

$$z_{t-1} = \frac{P_{t-1}^f}{c} + \frac{1}{\alpha \sigma^2_p(1)} (P_{t-1}^f - P_{t,t-1}^*).$$

The implications of these two equations for producers have been discussed by Turnovsky (1981). There are two important aspects worth highlighting. First, in (21) the futures price replaces the expected spot price as the relevant price governing total production plans; secondly the sensitivity of supply to price changes is increased. Risk and the degree of risk aversion play no role in determining the total production decision; rather these determine the hedging or speculative behavior undertaken by the firm and are described by the second term in (22). Four possible cases can be identified and have been discussed; see also Feder, Just, and Schmitz (1980).

With the existence of a futures market, we assume that speculators may now speculate in the purchase and sales of futures contracts, as well as in inventories. We shall let $x_{t-1}$ denote the net position of the representative speculator in the futures market entered at time $t-1$, with $x_{t-1} > 0$ denoting sales and $x_{t-1} < 0$ denoting the purchases of futures contracts, respectively. The profit of the speculator over the period $(t-1,t)$ now becomes

$$\pi_t^s = i_{t-1}(P_t - P_{t-1}) + x_{t-1}(P_{t-1}^f - P_t) - \frac{1}{2} dt_{t-1}^2 - \frac{1}{2} dt_{t-1}^2$$

while the objective is to choose $i_{t-1}$ and $x_{t-1}$ to maximize.
\[ v^*_t = i_{t-1}(P^*_t, t-1 - P_{t-1}) + x_{t-1}(P^f_{t-1} - P^*_t, t-1) \]
\[ - \frac{1}{2} \sigma^2_{t-1} - \frac{1}{2} \beta \sigma^2_p (1)(i_{t-1} - \bar{x}_{t-1})^2. \]  

(23)

The first order optimality conditions are

\[ P^*_t, t-1 - P_{t-1} - d\bar{x}_{t-1} - \beta \sigma^2_p (1)(i_{t-1} - \bar{x}_{t-1}) = 0 \]
\[ P^f_{t-1} - P^*_t, t-1 + \beta \sigma^2_p (1)(i_{t-1} - \bar{x}_{t-1}) = 0 \]

which may be solved to yield

\[ i_{t-1} = \frac{1}{d} (P^f_{t-1} - P_{t-1}) \]  

(24)
\[ x_{t-1} = \frac{1}{d} (P^f_{t-1} - P_{t-1}) + \frac{1}{\beta \sigma^2_p (1)} (P^f_{t-1} - P^*_t, t-1) \]  

(25)

These equations are parallel to (24) and (25) derived above for the firm.

In the first place, the total inventory decision, \( i_{t-1} \) is separated from the hedging decision. Total inventory demand depends upon the difference between the current spot price and the current one-period futures price. Moreover, the coefficient in this demand function is now increased to \( d \) and is independent of risk or the coefficient of risk aversion. Indeed, (24) highlights the significance of assuming \( d > 0 \). If instead we have \( d = 0 \), \( P^f_{t-1} = P_{t-1} \), and the current spot and futures prices are always equal. The extent to which inventory holders hedge their holdings of inventories, or alternatively speculate on the futures market, is reflected in the second term of (25) which depends upon the difference between the current futures prices and the expected spot price for next period. In this case, four possibilities again exist. First, if \( P^f_{t-1} = P^*_t, t-1 \) then \( x_{t-1} = i_{t-1} \) and the inventory holders hedge their entire holdings of inventories. Secondly if

\[ \frac{\beta \sigma^2_p (1)P_{t-1} + dP^*_t, t-1}{d + \beta \sigma^2_p (1)} < P^f_{t-1} < P^*_t, t-1 \]

then \( 0 < x_{t-1} < i_{t-1} \) and a fraction of inventory holdings will be hedged. Thirdly, if

\[ P^f_{t-1} < \frac{\beta \sigma^2_p (1)P_{t-1} + dP^*_t, t-1}{d + \beta \sigma^2_p (1)} \]
\( x_{t-1} < 0 \) and the inventory holders will speculate by buying futures contracts. Finally, if \( p^f_{t-1} > P^*_{t,t-1} \), they will speculate by selling futures contracts in excess of their total inventory holdings.

In addition, there may be pure speculators, who deal only in futures contracts. But these do not add anything and can be ignored without any loss of generality.

B. Aggregate Market Relationships

The aggregate relationships describing the goods and futures markets may now be specified as follows

**Goods Market**

\[
D_t = A - aP_t + u_t \tag{26}
\]

\[
S_t = B + \frac{1}{c} P^f_{t-1} + v_t \tag{27}
\]

\[
I_{t-1} = \frac{1}{d}(P^f_{t-1} - P_{t-1}) \tag{28}
\]

\[
D_t + I_t = S_t + I_{t-1} \tag{29}
\]

**Futures Market**

\[
Z_{t-1} = \frac{1}{c} P^f_{t-1} + \frac{1}{\alpha^2 p (1)} (P^f_{t-1} - P^*_{t,t-1}) \tag{30}
\]

\[
X_{t-1} = \frac{1}{d} (P^f_{t-1} - P_{t-1}) + \frac{1}{\beta^2 p (1)} (P^f_{t-1} - P^*_{t-1}) \tag{31}
\]

\[
Z_{t-1} + X_{t-1} = 0 \tag{32}
\]

where \( Z_{t-1}, X_{t-1} \) represent aggregates over the representative individuals. Again the coefficients on these aggregate equations should be multiplied by the number of individuals in the aggregate, but as before this is absorbed in the other parameters.

The spot market is basically as before and requires no further comment.

For equilibrium in the futures market to prevail, the excess demand for futures contracts by both producers and speculators must sum to zero. Substituting (30) and (31) into (32), yields
Thus with risk averse behavior, the current futures price is a weighted average (with weights summing to less than unity) of the current spot price and the expected future spot price. Only if either producers or speculators are risk neutral so that $\alpha + 0$ or $\beta + 0$ does $P^f_{t-1} = P^*_{t-1}$ and the futures price become an unbiased predictor of the future spot price. Otherwise, the futures price is a biased predictor, with the direction of the bias depending upon the magnitudes of the cost parameters $c$ and $d$.

Although it is possible for $P^f_{t-1}$ to overpredict $P^*_{t-1}$, on average one would expect it to underpredict the future spot price. To see this, we may note that $P_t$ will be shown below to follow an equation analogous to (16), (see (41) below) so that the expected price is given by

$$P^*_{t-1} - \bar{P} = r_1 (P_{t-1} - \bar{P})$$

(34)

where $0 < r_1 < 1$, is determined below. If $P_{t-1}$ happens to equal its equilibrium value $\bar{P}$, it is evident from (33) and (34) that $P^f_{t-1} < P^*_{t-1} = \bar{P}$. Only for values of $P_{t-1}$ sufficiently greater than the long-run mean $\bar{P}$ will the current futures price overpredict the future spot price.

The implied behavior for firms and speculators can be obtained by substituting (33) and (34) into their respective optimality conditions. Again taking the case where $P_{t-1} = \bar{P}$, we can show that at that point

$$0 < z_{t-1} < \bar{y}$$

$$x_{t-1} < i_{t-1} < 0$$

That is, firms will sell futures to partially hedge their planned output, while speculators will sell their inventories short and purchase futures contracts in excess of the short position they have taken.
C. Solution for Spot and Futures Prices

To solve the system, we first substitute the relevant demand and supply functions into the goods market and futures market equilibrium conditions. These yield the following pair of equations in the spot and futures prices

\[ A - aP_t + \frac{1}{d} (P^f_t - P_t) = \frac{1}{c} P^f_{t-1} + \frac{1}{d} (P^f_{t-1} - P_{t-1}) - e_t \]

\[ \frac{1}{c} P^f_{t-1} + \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \frac{1}{\sigma^2_p(1)} (P^f_{t-1} - P^*_{t-1}) + \frac{1}{d} (P^f_{t-1} - P_{t-1}) = 0. \]

The long-run average spot and futures prices \( \bar{P}, \bar{P}^f \) respectively, are given by the expressions

\[ \bar{P} = \frac{A}{a_1 + b_1} \]  

\[ \bar{P}^f = \left\{ \frac{1}{\sigma^2_p(1)} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) + \frac{1}{d} \right\} \bar{P} < \bar{P} \]

where for notational convenience we let

\[ a_1 \equiv a + \frac{1}{\sigma^2_p(1)} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) + \frac{1}{c} + \frac{1}{d} \]  

\[ b_1 \equiv \frac{1}{\sigma^2_p(1)} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \]

Written in this way we see that the expression for the long-run equilibrium spot price \( \bar{P} \) is of the same form as (14) with \( a, b \) replaced by \( a_1, b_1 \) respectively. The other point to observe is that the long-run average futures price is in general below the long-run average spot price. This is necessary in order to induce producers to hedge permanently part of their output. The long-run bias will disappear if either \( \alpha \to 0, \beta \to 0, c \to \infty, \) or \( d \to 0. \)

Substituting for \( P^f_{t-1} \) from the futures market equilibrium condition, the following difference equation is obtained for the spot price, expressed in deviation form
\[-a_1 p_t + \omega_1 (p^*_{t+1}, t - p_t) = b_1 p^*_{t, t-1} + \omega_1 (p^*_{t, t-1} - p_{t-1}) - e_t \quad (39)\]

where \( a_1, b_1 \) are defined in (37) and (38), and

\[
\omega_1 = \frac{1}{d \sigma^2_p (1)} \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] \quad (40)
\]

The crucial thing to observe about this equation is that it is of identical form to (15) applicable in the absence of a futures market, with \( a, b, \omega \), as defined previously, being replaced by \( a_1, b_1, \omega_1 \), respectively, the latter being defined in (37), (38) and (40). The effect of the futures market on the behavior of the spot price is therefore equivalent to inducing changes in the price sensitivity of:

(a) the demand function; (b) the supply function; (c) the inventory demand function.

Of course only the latter two actually change, since by assumption \( a \) remains constant.

The solution for the spot price (in deviation form) is therefore

\[
p_t = r_1 p_{t-1} + \frac{e_t}{a_1 + \omega_1 (1-r_1)} \quad (41)
\]

where \( r_1 \) is the smaller root \( (0 < r_1 < 1) \) of the quadratic equation

\[
\omega_1 (1-r_1)^2 = (a_1 + b_1) r_1 \quad (42)
\]

The one-period variance and the asymptotic variance of the spot price are given by

\[
\sigma^2_p (1) = \frac{\sigma^2_e}{[a_1 + \omega_1 (1-r_1)]^2} \quad (43)
\]

\[
\sigma^2_p = \frac{\sigma^2_e}{(1-r_1^2)[a_1 + \omega_1 (1-r_1)]^2} \quad (44)
\]

where \( r_1 \) is the smaller root of (42) and now \( a_1, b_1, \omega_1 \) are given by (37), (38), and (40).

Using the steady-state relationship (36) to write (33) and (34) in derivation form, the current futures price can be expressed in terms of the current spot price as
Thus given (41), the evolution of $p_{t-1}^f$ is determined. From this it follows that the asymptotic variance of the futures price, $\sigma_f^2$, is

$$\sigma_f^2 = n^2 \sigma_p^2 < \sigma_p^2$$

and a similar relationship holds between the corresponding one-period variances. Consequently both in the short run and in the long run the futures price is more stable than the spot price.

4. COMPARISON OF LONG RUN EQUILIBRIUM DISTRIBUTIONS FOR SPOT PRICES: GENERAL CASE

We now attempt to compare the long-run means and variances of the spot price with, and without, the futures market. The solutions for the determination of the spot price in these two circumstances have been derived in Sections 2 and 3 above and for convenience they are summarized below.

No Futures Market

$$\overline{P}(N) = \frac{A}{a+b} \quad (45)$$

$$\sigma_p^2(N) = \frac{\sigma_p^2(1;N)}{1-r^2} \quad (46)$$

$$\sigma_p^2(1;N) = \frac{\sigma_e^2}{[a + \omega(1-r)]^2} \quad (47)$$

$$\omega(1-r)^2 = (a + b)r \quad (48)$$

$$b = \frac{1}{c + a\sigma_p^2(1;N)} \quad (49)$$

$$\omega = \frac{1}{d + \beta\sigma_p^2(1;N)} \quad (50)$$

where the index $N$ in $\overline{P}(N)$, $\sigma_p^2(N)$, is used to designate these variables in the absence of a futures market and $\sigma_p^2(1;N)$ denotes the one-period variance. All other parameters remain as defined previously.
Futures Market

\[
\bar{P}(F) = \frac{A}{a_1 + b_1} \quad (51)
\]

\[
\sigma_p^2(F) = \frac{\sigma^2_{P(1;F)}}{1-r_1} \quad (52)
\]

\[
\sigma_p^2(1;F) = \frac{\sigma^2_e}{[a_1 + \omega_1(1-r_1)]^2} \quad (53)
\]

\[
\omega_1(1-r_1)^2 = (a_1 + b_1)r_1 \quad (54)
\]

\[
a_1 = a + \frac{1}{\sigma_p^2(1;F)} \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] + \frac{1}{c} + \frac{1}{d} \quad (55)
\]

\[
b_1 = \frac{1}{\sigma_p^2(1;F)} \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] - \frac{1}{c \sigma_p^2(1;F)} \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] \quad (56)
\]

\[
\omega_1 = \frac{1}{d \sigma_p^2(1;F)} \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] \quad (57)
\]

where \( F \) is used to identify the futures market.

Turning first to the set of equations relevant in the absence of a futures market, it is seen that the four variables \( \sigma_p^2(1,N) \), \( r \), \( b \), and \( \omega \) are jointly determined by equations (47)-(50). We have already commented on the non-linear nature of these equations and how this may create problems of non-existence and non-uniqueness of solutions. Having determined these four quantities (assuming that a solution exists), the long run mean \( \bar{P}(N) \) and asymptotic variance \( \sigma_p^2(N) \) of the spot price are then determined from (45) and (46) respectively.

The second set of equations, applicable in the presence of a futures market, has the same general structure. Indeed, it will be observed that equations (51)-(54) are identical to the corresponding equations (45)-(48) in the
first set, with \( a_1, b_1 \) and \( \omega_1 \) replacing \( a, b, \) and \( \omega, \) respectively. The differences are in the determination of the parameters \( a, b, \omega \) so that the effects of the futures market can be analyzed in terms of its impact on these latter parameters. 

These two sets of equations form the basis for the analysis of a futures market on the distribution of the spot price. It is possible by substitution to reduce equations (45)-(50) to a pair of equations in \( \sigma_p^2(N) \) and \( r, \) and likewise for the second set, (51)-(57). In either case, the relationships turn out to be highly complex and to be quite intractable for analytical purposes, especially in the light of the possible non-existence and non-uniqueness problems we have noted might arise. Indeed, it is possible for a feasible equilibrium value of \( \sigma_p^2(N) \) say, but not \( \sigma_p^2(F) \) to exist, and vice versa.

But without explicit comparison of these two sets of equations in the general case, a good deal of insight can be obtained by exploiting the similarity of the solution structure in terms of the parameters \( a, b, \) and \( \omega. \) Specifically, we may consider (45) as defining \( \overline{P}(N) \) in terms of \( a, b, \) while (46)-(48) determine \( \sigma_p^2(1;N), \sigma_p^2(N), r \) in terms of \( a, b, \) and \( \omega \) and in particular, we may write

\[
\overline{P}(N) = G(a, b) \\
\sigma_p^2(N) = H(a, b, \omega)
\]

Moreover, the corresponding equations (51) and (52)-(54) determine \( \overline{P}(F) \) and \( \sigma_p^2(1,F), \sigma_p^2(F), \) \( r \) as identical functions of \( a_1, b_1, \omega_1 \) and thus we have

\[
\overline{P}(F) = G(a_1, b_1) \\
\sigma_p^2(F) = H(a_1, b_1, \omega_1).
\]

For these functions \( G(\cdot) \) and \( H(\cdot) \) we can establish (suppressing the indices \( N, F) \)

\[
\frac{\partial \overline{P}}{\partial a} < 0; \quad \frac{\partial \overline{P}}{\partial b} < 0 \\
\frac{\partial \sigma_p^2}{\partial a} < 0; \quad \frac{\partial \sigma_p^2}{\partial b} < 0;\quad \frac{\partial \sigma_p^2}{\partial \omega} < 0.
\]

Inequalities (58) are obtained by direct partial differentiation of (45) or (51) while (59) are somewhat more complicated and are derived in Appendix B. Given that
b, \( \omega \) (and \( a_1 \), \( b_1 \), and \( \omega_1 \)) are endogenous, these partial derivatives require careful interpretation. They describe *ceteris paribus* changes in the particular coefficients, which in turn arise from changes in underlying exogenous parameters. For example, in order for \( b \) to increase while \( \omega \) remains constant, a decrease in \( c \) say (which generates the increase in \( b \)) must be offset by an appropriate change in some other parameter \( d \) say, which will exactly offset the change in \( \omega \) which would otherwise occur through the induced changes in \( r \) and \( \sigma_p^2(1) \). Noting these comments, we see that an increase in the price responsiveness of demand and supply will tend to lower the equilibrium mean price and the asymptotic variance of the spot price. An increase in the price responsiveness of storage will tend to stabilize the spot price. These statements hold whether or not a futures market exists. Given that the functions \( G(\cdot) \) and \( H(\cdot) \) are identical in the two cases, it follows from (58) and (59) that if the futures market were to lead to an increase in the parameters \( a, b, \omega \), then \( P(F) < \bar{P}(N), \sigma_p^2(F) < \sigma_p^2(N) \); the futures market will reduce the mean of the spot price as well as reducing its variance. But we are unable to demonstrate this in the general case.

Some insight into the role of the futures market is obtained by focusing on each parameter separately. Turning first to (55), we see that \( a_1 > a \), so that one effect of the futures market is equivalent to increasing the price sensitivity of demand and this has an unambiguously stabilizing effect on the spot price. By contrast, the effects of the futures market on the price sensitivity of the supply and inventory demand functions are not unambiguous. From the definitions of \( b_1, b, \omega_1, \omega \), we can establish the following effects on the relative magnitudes of the relevant slope coefficients

\[
\text{sgn}(b_1-b) = \text{sgn}[(ad-\beta c)\sigma_p^2(1;N) + \beta(c+d)(\sigma_p^2(1;N) - \sigma_p^2(1;F))\]
\[
\text{sgn}(\omega_1-\omega) = \text{sgn}[(\beta c-ad)\sigma_p^2(1;N) + \alpha(c+d)(\sigma_p^2(1;N) - \sigma_p^2(1;F))\]

Because of the fact that the one-period variances are endogenous, nothing definite can be inferred. Nevertheless, a comparison of the first coefficients in these
two expressions suggests that some offsetting effects on the supply and inventory price responsiveness are likely to operate. Clearly if \( \sigma^2_p(1;N) = \sigma^2_p(1;F) \) the introduction of the futures market will affect the slopes of the supply and inventory demand functions differently; one will tend to be stabilizing, but the other destabilizing. It is this conflict in effects which makes the general comparison of \( \sigma^2_p(N) \) with \( \sigma^2_p(F) \) so difficult.

One can carry out this kind of general qualitative analysis on \( \overline{P}(N) \) and \( \overline{P}(F) \). While the general comparison is again intractable (since it involves the asymptotic variances), one can establish that a sufficient condition, but certainly not necessary condition, for \( \overline{P}(F) < \overline{P}(N) \) is that

\[
\frac{\sigma^2_p(1;F)}{\sigma^2_p(1;N)} < \frac{\alpha + \beta}{\beta} .
\]

(60)

5. COMPARISON OF LONG-RUN EQUILIBRIUM DISTRIBUTIONS FOR SPOT PRICES: SPECIAL CASES

Despite the fact that it seems impossible to make general comparisons regarding the stabilizing properties of a futures market, it is possible to draw firm conclusions in a number of important special cases. In some instances this can be done more conveniently by considering the expressions summarized in (45)–(50) and (51)–(57); in other cases it is simpler to compare the expressions for the coefficients \((a, b, \omega)\) with \((a_1, b_1, \omega_1)\) and to use (58) and (59).

A. Risk Neutral Speculators

This is obtained by setting \( \beta = 0 \). Thus we deduce from equations (49), (50), (55), (56) and (57) that \( a_1 = a, b_1 = 1/c > b, \) and \( \omega_1 = 1/d = \omega \). In this case, the only effect of the futures market is to increase the price responsiveness of the supply curve, leaving the slopes of the demand and inventory demand curves unchanged. Hence it follows from (58) and (59) that

\[
\sigma^2_p(F) < \sigma^2_p(N) \\
\overline{P}(F) < \overline{P}(N) .
\]
Thus the presence of risk neutral speculators (in conjunction with risk averse producers) ensures that the futures market has a stabilizing effect on the spot price, while also reducing its long-run mean.

B. Risk Neutral Producers

This case is parallel and is characterized by $\alpha = 0$. We deduce from equations (49), (50), (55), (56) and (57) that $a_1 = a$, $b_1 = 1/c = b$, $\omega_1 = 1/d > \omega$. The net effect of the future market is therefore to increase the price responsiveness of the inventory demand function, leaving the slopes of the other functions unchanged. Thus again (59) implies the futures market stabilizes the spot price, although its long-run mean remains unchanged.

C. Risk Neutral Speculators and Producers

If both producers and speculators are risk neutral, $\alpha = \beta = 0$. In this case we find $a_1 = a$, $b_1 = b$, $\omega_1 = \omega$. The slopes of the demand function, supply function, and inventory demand function all remain unchanged and the introduction of the futures market leaves the long-run mean and variance of the spot price unaffected.

Taken together, the results in 5.A-5.C make good intuitive sense. As long as one group in the market is risk averse, the certainty offered by the futures market will increase the price responsiveness of the behavioral relationship pertaining to the risk averse group, thereby providing some stabilizing influence to the spot price.

D. Pure Production

This case arises when $d \to \infty$, so that inventories are infinitely costly to store. No inventories are therefore held and the model reduces to the no-storage case analyzed in the previous papers by Kawai (1981) and Turnovsky (1981). Letting $d \to \infty$ in (50) and (57), we find $\omega = \omega_1 = 0$ and hence (47) and (54) imply $r = r_1 = 0$. It then immediately follows that
\[ \sigma_p^2(N) = \sigma_p^2(F) = \frac{\sigma^2}{a^2} \]

so that the futures market leaves the variance of the spot price unchanged. This result is identical to that obtained in these earlier studies which found that in the absence of storage, gains in price stability arise only if the additive disturbances are autocorrelated. There are no gains, if they are independently distributed over time, as is being assumed here.

E. Pure Inventory Holding Case

In this case production costs are infinitely large \((c \to \infty)\). No production occurs, and the model reduces to one of pure inventory holdings. Letting \(c \to \infty\), (49) and (56) imply \(b = b_1 = 0\). Equations (46)-(48) and (52)-(54) together yield

\[ \sigma_p^2(N) = \frac{\sigma^2}{a^2} \frac{(1-r)}{1+r}, \quad \sigma_p^2(F) = \frac{\sigma^2}{a^2} \frac{(1-r_1)}{1+r_1} \]

Inserting these expressions into (46) and (52), and hence into (50) and (52) respectively, and using (47) and (54), we can show that \(r_1 > r\). Hence we deduce that \(\sigma_p^2(N) > \sigma_p^2(F)\) so that the long-run variance of the spot price is once again decreased. Furthermore, \(\sigma_p^2(1;N) > \sigma_p^2(1;F)\), implying that the mean spot price is reduced as well; see (60).

F. Constant Marginal Inventory Costs

To illustrate the possible problems of non-existence of equilibrium, we shall consider the case where \(d = 0\), so that the marginal costs of holding inventories are constant. Setting \(d = 0\) in (55)-(57), the solution in the presence of a futures market simplifies to

\[ \sigma_p^2(F) = \sigma_p^2 \frac{(1-r_1)}{1+r_1} \frac{1}{(a+\frac{1}{c})^2} \]

\[ \frac{(1-r_1)(\frac{1}{a} + \frac{1}{b})}{\sigma_p^2(F)(1+r_1)} = (a+\frac{1}{c})r_1. \]

These equations may be solved explicitly for \(r_1\) and \(\sigma_p^2(F)\).
\[-23-\]

\[ r_1 = \frac{(a+\frac{1}{c})(\frac{1}{a}+\frac{1}{\beta})}{\sigma_e^2} \]

\[ \sigma_p^2(F) = \frac{\sigma_e^2 [\sigma_e^2 - (\frac{1}{a}+\frac{1}{\beta})(a+\frac{1}{c})]}{[\sigma_e^2 + (\frac{1}{a}+\frac{1}{\beta})(a+\frac{1}{c})][a+\frac{1}{c}]^2} \]

It is evident from this last pair of equations that for a consistent solution to exist (i.e., the variance \( \sigma_p^2(F) \geq 0 \)), we require

\[ \sigma_e^2 \geq (\frac{1}{a}+\frac{1}{\beta})(a+\frac{1}{c}) \]

If this inequality is reversed—as it might well be if, for example, the degree of risk aversion is sufficiently low—then no consistent solution will exist in the presence of a futures market.

While similar problems of non-existence arise in the absence of a futures market, the conditions are in general different. Suppose that inventory holders are risk neutral. Setting \( \beta = 0 \), the feasibility condition is clearly violated. On the other hand, setting \( \beta = 0 \), together with \( d = 0 \) in (50) implies \( r = 1 \).

Eliminating \( \omega(l-r) \) between (46) and (47) the asymptotic variance for price may be written as:

\[ \sigma_p^2(N) = \frac{\sigma_e^2(1-r)}{(1+r)(a+br)^2} \]

and setting \( r = 1 \) we find \( \sigma_p^2(N) = 0 \). That is, with risk neutral speculators and constant inventory costs, a perfectly feasible solution exists in the absence of a futures market.

While each of these examples is special, the range of cases they cover suggests that they should provide a reasonable indication as to the effects of futures trading on spot price in the general case. One can identify among them polar pairs for appropriate types of behavior or parameter values. Take for example, cases A and B, which represent one such polar pair. These two examples show that if only one group is risk averse (producers in the former case, speculators in the latter), then the opening of a futures market will have both a stabilizing effect
on the spot price, while also reducing its long-run mean. Provided that the
effects of introducing risk averse behavior to these two sets of agents simult-
aneously are approximately additive, it seems reasonable to expect that the
general case, in which both producers and speculators are risk averse, should
not respond too differently from the two extremes.

Cases D and E represent another polar pair. In the pure production model
(no inventories) it is shown that the introduction of the futures market has no
effect on the distribution of the spot price, at least under the present assump-
tions regarding stochastic specification. In the other extreme of pure inventory
holding (no production) the futures market has the usual stabilizing effect.
Again, assuming that the effects of introducing production and inventorying behavior
simultaneously are approximately additive, one would expect that the general case
to be some kind of average of these two cases, so that the futures market should
again have a stabilizing effect on the spot price.

A third polar pair is to contrast risk neutral speculators ($\beta = 0$) with
infinitely risk averse speculators ($\beta \to \infty$). While the latter case is of no par-
cular interest (except as a polar case) and is therefore not reported, it can be
shown that like case B it implies that futures trading is stabilizing. The general
case, which in this case corresponds to finite risk averse speculation ($0 < \beta < \infty$),
should once again be some kind of average of these two extremes, suggesting again
that futures trading will have a stabilizing effect.

6. CONCLUSIONS AND SOME FURTHER ISSUES

In this paper we have analyzed the effects of futures trading in a market
for a storable commodity, in which producers and speculators are assumed to be
risk averse and their respective behavior is derived from explicit optimization.
From these underlying micro considerations, the specifications of the aggregate
supply and inventory demand functions are derived. A critical aspect is how the
parameters of these functions change with the introduction of the futures market and it is through these induced parameter changes that the futures market exerts its effect on the spot price. We have focused on two main issues.

First, we have discussed some of the properties of the underlying behavioral functions and shown how the introduction of the futures market introduces a "separation" into both production and inventory decisions. In both cases, the total decision depends upon the futures price and is independent of risk and risk attitudes. These influence the degree of hedging activity. We have also considered in some detail the relationship between the futures price and the expected future spot price, an issue which has been widely discussed in the literature. We have shown how the current futures price is a weighted average (with weights summing to less than unity) of the current spot price and the expected future spot price.

Secondly, we have analyzed the effects of the futures market on both the long-run average spot price and its variance. In general, it seems impossible to draw any definitive conclusions on this issue. This is partly because of non-existence problems which may arise due to the non-linearity of the underlying model and partly due to the intractability of therelevant expressions. We are, however, able to break down the impact of the futures market into effects which operate through the slopes of the demand, supply, and inventory demand, functions, and these turn out to provide insight as to how the introduction of the futures markets impinges on the distribution of the spot price. A number of special cases were considered and apart from one case where there is zero effect, we find that the futures market both stabilizes the long-run spot rate as well as lowering its long run mean. The range of extremes covered by these examples suggests that the response of the long run distribution for the spot price in the general case, which in a sense is interior to them all, should not be too different.

To conclude, two further points should be made. First, as noted at the outset, our analysis is restricted to final demand and supply disturbances, our
treatment of the latter involving some reasonable approximations. Kawai (1981b) has pointed out that the introduction of a futures market will most likely provide a destabilizing influence for disturbances originating with inventory demand. With a little reflection this result is not surprising. Clearly if inventory behavior is subject to randomness, then any institutional change, such as the introduction of a futures market, which tends to encourage this activity, will thereby introduce more randomness into the system.

Secondly, one can easily extend the analysis to consider questions relating to spot and futures market intervention by some stabilization authority. Some work along these lines has been done, but is not reported here. Again it turns out to be difficult to draw definitive conclusions as to the stabilizing properties of these policies. The analysis does suggest that, at least in many important instances, spot market intervention is likely to be the more effective means of stabilizing both the spot price and the futures price. But further investigation of this issue is better left for some other occasion.
A. Rational Expectations Solution to (15)

The condition for market clearing is given by

\[ -a p_t + \omega (p^*_{t+1} - P_t) = b p^*_{t}, t-1 + \omega (p^*_{t}, t-1 - P_{t-1}) - e_t \]  (A.1)

Taking conditional expectations of this equation at time \( t-1 \) for an arbitrary period \( j = 0, 1, 2, \ldots, t+j \), yields

\[ \omega p^*_{t+j+1, t-1} - (2\omega + a + b)p^*_{t+j, t-1} + \omega p^*_{t+j-1, t-1} = 0 \]  (A.2)

where in deriving (A.2) we have used the well-known property of conditional expectations

\[ E_{t-1}[E_t(p_{t+j})] = E_{t-1}p_{t+j} \]  (A.3)

and have defined

\[ p^*_{t-1, t-1} \equiv p_{t-1} \]

This last assumption is simply a statement that the price at time \( t-1 \) is known at that time, so that the conditional expectation is simply the true observed value.

Equation (A.2) is a second order difference equation in the predictions \( p^*_{t+j, t-1} \), the solution to which is

\[ p^*_{t+j-1, t-1} = B_1(t-1)r_1^j + B_2(t-1)r_2^j \]  (A.4)

where \( r_1 \) and \( r_2 \) are the roots to the quadratic equation

\[ \omega (1-r)^2 = (a+b)r \]  (A.5)

It is immediately seen that \( r_1 \) and \( r_2 \) are both real and positive, lying on opposite sides of unity, with say \( 0 < r_1 < 1, r_2 > 1 \). Unless \( B_2(t-1) = 0 \), the forecast of prices expected to prevail in the infinite future will diverge. In order to rule this out, we set \( B_2(t-1) = 0 \). Moreover taking \( j = 0 \) and recalling our definition of \( p^*_{t-1, t-1} \), we see \( B_1(t-1) = p_{t-1} \). Hence the solution (A.4) simplifies to

\[ p^*_{t+j-1, t-1} = r_1^j p_{t-1} \quad j = 0, 1, 2, \ldots \]  (A.6)

Equation (A.6) gives the rational expectations held at time \( t-1 \) over varying future time horizons. Setting \( j = 1 \) gives the one period rational forecast
Substituting (A.7) into (A.1), the spot price follows the first order stochastic difference equation

\[-ap_t + \omega(r_1-1)p_t = br_1p_{t-1} + \omega(r_1-1)p_{t-1} - e_t\]

which we may rewrite as

\[p_t = rp_{t-1} + \frac{e_t}{a+\omega(1-r)}\]

where for convenience we drop the subscript 1 from r. This last equation is the solution (16) given in the text.

B. Derivation of Inequalities (59)

Consider the pair of equations

\[\sigma_p^2 = \frac{\sigma^2}{(1-r^2)[a+\omega(1-r)]^2}\]  
(A.8)

\[\omega(1-r)^2 = (a+b)r\]  
(A.9)

Eliminating r between them yields the function

\[\sigma_p^2 = H(a, b, \omega)\]

defined in the text. To determine the signs of the partial derivatives it is convenient to define

\[\phi \equiv (1-r)^2[a+\omega(1-r)]^2\]  
(A.10)

It then follows immediately that to establish the results in the text it is necessary and sufficient to show

\[\frac{\partial \phi}{\partial i} > 0 \quad i = a, b, \omega\]  
(A.11)

(i) Differentiating the pair of equations (A.8) and (A.9) with respect to a, we obtain

\[\frac{\partial \phi}{\partial a} = 2[a+\omega(1-r)](1-r)^2\frac{\partial \phi}{\partial a} - r[a+\omega(1-r)]\frac{\partial \phi}{\partial a}\]  
(A.12)

\[\frac{\partial \phi}{\partial a} = \frac{-r}{a+b+2\omega(1-r)} < 0\]  
(A.13)

Since 0 < r < 1, and the coefficient on \(\partial \phi/\partial a\) in (A.12) is negative, it immediately follows from these two equations that \(\partial \phi/\partial a > 0\); i.e., \(\partial \sigma_p^2/\partial a < 0\).
(ii) Differentiating (A.8) and (A.9) with respect to \( b \), yields

\[
\frac{\partial \phi}{\partial b} = 2[a+\omega(l-r)]\{- (1-r)^2 \omega \frac{\partial r}{\partial b} - r [a+\omega(l-r)] \frac{\partial r}{\partial b}\} \tag{A.14}
\]

\[
\frac{\partial r}{\partial b} = \frac{-r}{a+b+2\omega(l-r)} < 0 \tag{A.15}
\]

which as in (i) combine to imply \( \partial \phi/\partial b > 0 \).

(iii) Differentiating (A.8) and (A.9) with respect to \( \omega \), yields

\[
\frac{\partial \phi}{\partial \omega} = 2[a+\omega(l-r)]\{(1-r)^2(1-r-\omega \frac{\partial r}{\partial \omega}) - r [a+\omega(l-r)] \frac{\partial r}{\partial \omega}\} \tag{A.16}
\]

\[
\frac{\partial r}{\partial \omega} = \frac{(1-r)^2}{a+b+2\omega(l-r)} > 0 \tag{A.17}
\]

Substituting (A.17) into (A.16) we obtain

\[
\frac{\partial \phi}{\partial \omega} = \frac{2[a+\omega(l-r)](1-r)^2[a+b+br+\omega(l-r)]}{a+b+2\omega(l-r)} > 0 \tag{A.18}
\]

implying \( \partial^2 \phi/\partial \omega < 0 \).
FOOTNOTES

*The helpful comments of three referees and the Co-editor are gratefully acknowledged.

1. Early contributions to the theory of futures price determination and the potential role of futures market for price stabilization, see, e.g., Working (1958), Stein (1961) and McKinnon (1967).

2. The idea that speculators may have superior information is considered by Danthine (1978).

3. This assumption can easily be relaxed without any significant alteration to the analysis.

4. The assumption that costs depend upon planned rather than actual output can be justified if costs are incurred on non-stochastically determined inputs, chosen at the time the production decision is made. The random fluctuations in output appearing in revenue are due to stochastic disturbances in production conditions, which occur after the inputs have been purchased.

5. The additive form of the disturbance term appearing in (2) corresponds to a production function $f(m_t)$ of the form $y_t = f(m_t) + v'_t$, where $m_t$ is a vector of inputs. The result we shall obtain below, that production and futures trading decisions may be "separated," depends on this assumption and would not hold for a more general specification of the disturbance term $y_t = f(m_t,v'_t)$; see Danthine (1978).

6. The general expression for the optimal planned output is given by

$$ y^*_t = \frac{P^*_{t,t-1} - \alpha \, \text{cov}_{t-1}(P_t, P_{t-1}v'_t)}{c + \alpha \sigma_p^2(t,t-1)} $$

To derive the approximation used in the text it is sufficient to observe that the covariance term $\text{cov}_{t-1}(P_t, P_{t-1}v'_t)$ is $0(1/n)$. The advantage in reporting all the approximations is that the expressions for $V^e_t$ given in (4) and (20) can be simplified at the outset.

7. The futures contracts we shall consider are for single period which coincides the production period. A brief discussion of multiperiod futures contracts is contained in Turnovsky (1979).

8. The firm may (i) hedge its total planned output; (ii) hedge a fraction of the planned output; (iii) speculate by buying futures contracts; (iv) speculate by selling futures contracts in excess of its planned production. These are parallel to the possibilities for inventory holders we note below.

9. Taking the one-period conditional expectations of (41), yields

$$ P^*_{t,t-1} = r_t P_{t-1} $$

which in non-deviation form is precisely (34).

10. Writing

$$ b = \frac{1}{\alpha \sigma_p^2(1;N)} + \frac{1}{c} $$

where

$$ \frac{1}{\alpha \sigma_p^2(1;N)} $$

is the solution of

$$ \frac{1}{\alpha \sigma_p^2(1;N)} = -\frac{1}{\alpha \sigma_p^2(1;N)} + \frac{1}{c} $$
\[
\omega = \frac{1}{d \beta \sigma^2 (1; N)} \left( \frac{1}{\beta \sigma^2 (1; N)} + \frac{1}{d} \right)
\]

the analogies between \( b_1 \) and \( b \) and between \( \omega_1 \) and \( \omega \) become readily apparent.
REFERENCES


