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Average and Truncated Average Functions

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Stability in Games Induced by the Average and Truncated Average Functions

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Abstract

In a number of competitive sports the score of each participant is decided by averaging the grades given by each judge in a panel of judges. We characterize the sets of Nash and strong Nash equilibria of the n-person noncooperative games induced by the average and the truncated average functions.

Key words: Nash equilibrium, Strong Nash equilibrium, Committee voting, Average function

1. Introduction

In a number of competitive sports the score of each participant is decided by a panel of judges. Every competitor's performance is assigned a numerical value by each judge and these grades are incorporated by a prespecified rule into a single number which serves as the participant's score.

Two functions which are used extensively to aggregate the grades reported by the judges into a single score, are the average function and the truncated average function (the average of all the numbers without the maximum and minimum values). E.g., the truncated average function is used in judging the competition in gymnastics (see [3]), and diving (see[5]), while the average function is used in grading the competitors in synchronized swimming when the number of judges is three (see[6]).

It is widely recognized that the average function is open to manipulation by "sophisticated" judges. Namely, by misrepresenting his true opinion, a judge may increase or reduce the score of a given competitor. Thus, the decision process induces an n-person game among the judges.

Apparently, the truncated function was adopted by some organizations as an aggregation rule, in an attempt to curtail strategic behavior and force judges to vote truthfully.

Here we consider a general case in which a committee has to assign a numerical value to a given alternative. Each committee member submits a single numerical value, and the submitted numbers are incorporated by a prespecified rule into a single number. This number serves as the value the committee assigns to the alternative.

In this article we investigate properties of the n-person games induced when the weighted average function and the truncated average function are used as the aggregation rules, and committee members derive (directly or indirectly) satisfaction from the committee's choice.

It should be noted that we do not discuss here the purpose of the number assigned to an alternative and how it may be used. Thus, we consider only the strategic behavior arising from the choice process itself, and we do not consider strategic behavior across alternatives.

We first investigate stability properties of the games as represented by the Nash equilibrium when the committee members are assumed to behave noncooperatively, and by the strong Nash equilibrium when members may collude and coordinate their choices. Then we examine whether these two functions elicit truthful behavior by the players.

The results presented here are of interest both because they constitute an interesting example of games with strong Nash equilibria, and because of their practical implications.

2. <u>Definitions</u>

Let N be a set of n players. We will denote an n-person noncooperative game as a 2n+1 tuple $(S_1,\ldots,S_n;F;v_1,\ldots,v_n)$ where S_i is the strategy set of player i, F is the function or mechanism which decides the outcome of the game, defined on the n-dimensional strategy space, and v_i is the i-th player utility or value function defined on the outcomes of the game. A strategy profile X is an n-dimensional vector in which the i-th component, x_i, is the strategy of player i. If profile Y is such that for every $i \neq j$, $y_i=x_i$ and $y_j=y$, we denote it as $Y=(X|y)_j$. A strategy profile X is said to be a <u>Nash equilibrium</u> of the game, if for every player i and every strategy y in S_i, $v_i(F(X)) \leq v_i(F((X|y)_i))$, (see Nash[4]).(In this article we assume that each player desires to minimize his value function, and we also consider only pure strategy equilibria.) <u>A coalition</u>, is a subset of the player set. A strategy profile X is said to be a <u>strong Nash equilibrium</u>, if for no coalition C there exists a strategy Y such that: for every player i not in C, $y_i=x_i$, for every player j in C, $v_j(F(Y)) \leq v_j(F(X))$ and for at least one player k in C, $v_k(F(Y)) < v_k(F(X))$, (see Aumann [1]).

In the games discussed here each strategy set is an interval I in R, and all are assumed to be identical. Namely, each player chooses and reports a number from the same interval. Four different types of intervals are considered: $I=(-\infty,\infty)$, $I=[L,U]$, $I=[L,\infty)$, and $I=(-\infty,U]$.

In addition, let t_i be the number that player i wants the committee to assign to the alternative. We assume that the value function v_i, of each player i, is a strictly increasing function of the absolute value of the difference between the game's outcome t and t_i. For simplicity we assume:

$v_i(t)=|t-t_i|$.

Thus, each t_i completely describes the corresponding value function. (Although the validity of these value functions is not discussed here, it should be noted that the above class of value functions represents a class of the well known single peaked preferences (see Black[2].).

Without loss of generality we may assume that $t_1 \leq \ldots \leq t_n$, with $t_1 < t_n$.

Let A(X) be the <u>weighted average function</u>,

$A(X)=\sum_{i=1}^{n} w_i x_i$, where $n \geq 2$, $w_i > 0$, and $\Sigma w_i = 1$.

Let TA(X) be the <u>truncated average function</u>,

$TA(X)=(\sum_{i=1}^{n} x_i - Mx - mx)/(n-2)$, where $n \geq 3$, $Mx=\max\{x_i | i \in N\}$ and $mx=\min\{x_i | i \in N\}$. Namely, TA(X) is the average of the strategies x_i without (one of) the largest and smallest reported numbers.

Let G_1 and G_2 be the games in which the outcome function F is A and TA respectively.

3. Stability

The attraction of the Nash and strong Nash equilibria as solutions for games stems from the stability they introduce into the game. If no equilibrium exists, then time and resources may be wasted by the players in attempting to outwit their rivals, and to create coalitions that will benefit them. Once it is known to a player that the rest of the players will follow their Nash strategies, this player has the incentive to play his Nash strategy. If in addition, the solution is also a strong Nash, then there is no incentive to create coalitions. Thus, time and resources are saved.

We first investigate the existence of equilibria for the game G_1. The following proposition demonstrates that not every game G_1 has an equilibrium.

Proposition 1.

If $I=(-\infty,\infty)$ then no Nash equilibrium exists for the game G_1 .

Proof:

Suppose there exists an equilibrium point X for G_1. Then without loss of generality there exists a player j such that $t_j<A(X)$. Let $\delta=A(X)-t_j$ and let $y=x_j-\delta/w_j$, then $A((X|y)_j)=t_j$. Since $v_j(t_j)<v_j(A(X))$, a contradiction is reached. QED

The nature of the results changes once I is restricted. Not only do Nash equilibrium points exist, but each one of them is also a strong Nash. In the following discussion we may assume, without loss of generality, that $L\leq t_1\leq\ldots\leq t_n\leq U$, and that L=0, U=1 (when they exist). Let Y_j be the profile in which $x_i=0$ for i=1,...,j and $x_i=1$ for i=j+1,...,n. Let $a_j=A(Y_j)$, and let $m=\max\{i\varepsilon N|ti\leq ai\}$.

Proposition 2.

a. If the strategy set I is either $[L,\infty)$, $(-\infty,U]$, or

[L,U] then there exists a Nash equilibrium for the game G_1.

b. Every Nash equilibrium of G_1 is also a strong Nash equilibrium.

Proof:

a. Consider the following cases.

i. I=[0,∞). Let $X=(o,\dots,o,t_n/w_n)$, then $A(X)=t_n$. Notice that for every player j, $t_j \le t_n$. Clearly, no player j for which $t_j=t_n$, can improve the value of the game's outcome for himself. Let player k be such that $t_k<t_n$. If player k replaces his strategy in X with strategy y, then y>0 and therefore $A((X|y)_k)>t_n$ and $v_k(A(X|y)_k))>v_k(t_n)$. Hence X is an equilibrium.

ii. I=(-∞,1]. Let $X=((t_1-\sum_{i=2}^{n} w_i)/w_1,1,\dots,1)$, then $A(X)=t_1$. The proof that X is an equilibrium is similar to the proof of part i.

iii. I=[0,1], and $t_{m+1} \ge a_m$. Let $X=Y_m$, then $A(X)=a_m$. It is easy to prove that X is a Nash equilibrium.

iv. I=[0,1], and $t_{m+1}<a_m$. Let $X=(Y_m|x)_{m+1}$ where x is such that $A(X)=t_{m+1}$. Notice that such an x exists since $a_{m+1}<t_{m+1}<a_m$, and A is a continuous function of the strategy of player m+1. Again, it is simple to check that X is a Nash equilibrium.

b. To prove that any Nash equilibrium in G_1 is also a strong Nash equilibrium, let assume that X is a Nash equilibrium and that there exists a coalition C of players and a profile Z such that: for every $i \notin C$, $z_i=x_i$, for every

$i \varepsilon C$, $v_i(A(Z)) \leq v_i(A(X))$, and there exists at least one player k in C such that $v_k(A(Z)) < v_k(A(X))$. Without loss of generality, let assume that $A(Z) > A(X)$. This indicates that for every member i in C, $t_i > A(X)$, and that there exists at least one member l in C such that $z_l > x_l$. Then $A((X|z)_l) > A(X)$, and because of the continuity of A, there exists some y_l, $x_l < y_l \leq z_l$, such that $v_l(A(X|y)_l) < v_l(A(X))$, contradicting the assumption that X is a Nash equilibrium. QED

The difference between games G_1 and G_2 is illuminated immediately by the following proposition.

<u>Proposition 3.</u>

For every game G_2, there exists at least one Nash equilibrium .

<u>Proof:</u>

Consider any profile X such that for every player i, $x_i = y$ for some y in I. Clearly X is a Nash equilibrium, since for any player i and any strategy z, $TA((X|z)_i) = TA(X)$. QED

Thus, G_2 always has many equilibria. In the following proposition we identify all the strong Nash equilibria of G_2.

<u>Proposition 4.</u>

A strategy profile X is a strong Nash equilibrium of G_2 if and only if:

I. $I=(-\infty,1]$, and $X=(x_1,\ldots,x_k,1,\ldots,Mx)$ such that $x_1=mx$, $t1\le TA(X)=t2=\ldots=tk$, or

II. $I=[0,\infty)$ and $X=(mx,0,\ldots,0,x_{k+1},\ldots,x_n)$ such that $x_n=Mx$, $TA(X)=t_{k+1}=\ldots=t_{n-1}\le t_n$, or

III. $I=[0,1]$ and $X=(mx,0,\ldots,0,x_{k+1},\ldots,x_{k+1},1,\ldots,1,Mx)$ such that $TA(X)=t_{k+1}=\ldots=t_{k+1}$.

Proof:

a. We first prove that a profile X satisfying any one of the above defined conditions is a strong Nash equilibrium.

I. Suppose $I=(-\infty,1]$ and X is as defined in case (I). It is easy to verify that X is a Nash equilibrium. If X is not a strong Nash equilibrium, then there exists a coalition C and a profile Z such that for every player i not in C, $z_i=x_i$, and such that for every player i in C, $v_i(TA(Z))\le v_i(TA(X))$ with at least one player j for which $v_j(TA(Z))<v_j(TA(X))$. Clearly none of players 2 to k is in C. If $TA(Z)>TA(X)$, then player 1 is not in C and for at least one player j among players k+1 to n, $z_j>x_j=1$, which is impossible. Therefore $TA(Z)<TA(X)$ and only player 1 is in C, but then X is not a Nash equilibrium, a contradiction.

The proofs for cases II and III follow similar lines and are omitted.

b. We now prove that if X is a strong Nash equilibrium of G_2 it must satisfy the above definition.

I. Let $I=(-\infty,1]$ and let X be a strong Nash. Clearly $TA(X)=t_2$, and $x_1=mx$. This, since players 1 and 2 can collude and force it. If only $t_n>t_2$, then any $x_n=Mx$ is an equilibrium strategy for player n. Otherwise, for every player i such that $t_i > t_2$ it must be $x_i = 1$, which completes the proof.

The proofs of cases II and III follow similar lines and are omitted. QED

How the sets of equilibrium points of the games G_1 and G_2 relate to each other is still an open question. Clearly some of the equilibria of the G_1 game are not equilibrium points of the corresponding G_2 game. For example, if $X=(0,\ldots,0,t_n/w_n)$ is an equilibrium of G_1 and $t_{n-1}>0$, then X is not an equilibrium of G_2.

4. Truthful Choice

The above results indicate that the profile $(t_1,\ldots,t_n)$ is rarely a Nash equilibrium of either G_1 or G_2. Thus, participants in such games usually will not report their true choice values. Are there rules, besides the trivial dictatorial rules, which will cause $(t_1,\ldots,t_n)$ to be an equilibrium in the induced game? (Notice that here we impose only the nondictatorial requirement on the choice function. For a detailed analysis of requirements and properties of choice functions for committees see Peleg[7] for example.)

In the following discussion, for simplicity's sake, let $n=2k+1$ for some integer $k \geq 0$. Let M be the median of the set $\{x_i\}$ (namely, there are k reported numbers in the profile X which are not as large as M, and k reported numbers which are at least as large as M). Define the median function M(X) as

$M(X)=M$, and let G_3 be the game induced by $M(X)$. Then:

Proposition 5.

The profile $T=(t_1,...,t_n)$ is a Strong Nash equilibrium of the game G_3.

Proof:

Suppose there exists a coalition C and a profile X such that for every player i not in C, $x_i=t_i$, for every player j in C, $v_j(M(X)) \leq v_j(M(T))$, and for at least one player k in C $v_k(M(X)) < v_k(M(T))$.

Without loss of generality assume, $M(X) > t_{k+1}$, (notice that $t_{k+1}=M$). This indicates that for every player i in C, $t_i \geq t_{k+1}$, and $x_i \geq t_{k+1}$. This implies $M(X)=M(T)$, and a contradiction is reached. QED

Actually, T is a dominant strategy Nash equilibrium. Namely, t_i is the best strategy for player i to follow, independently of the actions of the other players. Thus, we may conclude that the median function elicits truthful voting from the players.

5. Conclusions

In this article we identified and characterized all the strong Nash equilibria of the games induced by the average and the truncated average functions. As a result we may conclude that rarely is it in the best interest of participants in these games to report their true choices. Thus, if indeed the objective in using the truncated average function rather then the regular average function, is to elicit truthful voting from committee members, then this objective is not achieved.

We also demonstrated that the median function is an aggregation rule which elicits truthful voting by the participants.

The implications of the above results will be reported in future work.

References

1. Aumann, R. J., "Acceptable Points in General Cooperative n-person Games," AnMS, 40 (1959),287-324.

2. Black, D., "The Decision of a Committee Using a Special Majority," Econometrica, 16 (1948).

3. Code of Points, International Gymnastics Federation (1979)

4. Nash, J. F. Jr., "Equilibrium Points in N-Person Games," Proceedings of The National Academy of Science, U.S.A., 36 (1950),48-49.

5. Official Diving Rules And Regulations of United State Diving, Inc., United States Diving, Inc. (1985).

6. Official 1978 AAU Synchronized Swimming Handbook, C. Tackett (Ed.), Amateur Athletic Union of The United State (1978).

7. Peleg, B., Game Theoretic Analysis of Voting in Committees, Cambridge University Press (1984).

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