BEBR

FACULTY WORKING
PAPER NO. 833

Ascent Ray Theorems and Some Applications

Charles Blair

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

BEBR

FACULTY WORKING PAPER NO. 833

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

December 1981

Ascent Ray Theorems and Some Applications

Charles Blair, Associate Professor
Department of Business Administration

Abstract

We show how a consistency condition for semi-finite systems of linear inequalities can be applied in various contests. We conclude with some speculations regarding systems with an infinite number of variables.

Ascent Ray Theorems and Some Applications

1. Introduction In Spring 1973 I took a course on Convex Analysis taught by Bob Jeroslow. One of the exercises assigned was "prove the best result you can about when a semi-infinite system of linear inequalities $a_i x \geq b_i$ has a solution." The natural conjecture was

Theorem 1.1: Let $a_i \in R^n$, $b_i \in R$ for all $i \in I$. If there is no x such that $a_i x \geq b_i$, $i \in I$ then there is a $w \in R^n$ such that, for every N, there is a finite $F \in I$ such that $a_i x \geq b_i$, $i \in F$ implies $wx \geq N$.

After a week or so I found a proof. Bob's reaction was "you should write that up". With his unselfish assistance I did. That was my first paper [2].

The proof of Theorem 1 in [2] was based on the Kuhn-Fourier theorem [12], which deals with the case in which I is finite. I also made use of the fact, that, given a fixed basis for a vector space, there is a bound on the size of coefficients used to represent any vector of unit length in terms of the basis. Today, I would not dismiss this result as "...a trivial exercise in linear algebra...."* Otherwise, I would use the same proof.

I regard theorem 1.1 as the foundation of semi-infinite programming, possibly for sentimental reasons. In this paper, I wish to show how other results may be derived from it.

The vector w in theorem 1.1 was described as an "ascent ray" by Ken Kortanek. Shortly after I showed him the result, he found the strengthening

---

*It is essentially the result that any two norms are equivalent for a finite dimensional space.

Corollary 1.2: If there are no x such that $a_i x \geq b_i$, then the conclusion of Theorem 1 holds for any w in the relative interior of the cone generated by a_i.

Proof: Let w be the ascent ray given by Theorem 1*, and w' be in the relative interior of $C(a_i)$. Then for some finite $G \subset I$, $\beta > 0$, $\alpha_i \geq 0$ $w' = \beta w + \sum_{i \in G} \alpha_i a_i$. For any N, there is a finite $F \subset I$ such that $a_i x \geq b_i$, $i \in F$ implies $wx \geq \frac{1}{\beta}(N - \sum_{i \in G} \alpha_i b_i)$. Then if $a_i x \geq b_i$, $i \in F \cup G$

$$w'x = \beta(wx) + \sum_{i \in G} \alpha_i (a_i x) \geq \beta(wx) + \sum_{i \in G} \alpha_i b_i \geq N.$$ Q.E.D.

Kortanek also found a proof of Theorem 1.1 based on a theorem involving systems of inequalities with infinite elements [7, 10]. I returned the compliment by extending the infinite element theorem to cases involving strict inequalities [3]. That was my second paper.

---

*The proof of theorem 1.1 in [2] uses for w the sum of any set of basis vectors for a_i.

2. Application to Semi-Infinite Programming

Theorem 2.1: Suppose there are x such that $a_i x \geq b_i$, $i \in I$ and that every such x satisfies $cx \geq d$ (i.e., the semi-infinite program has value $\geq d$). Then there is a w and sequences of finite sets $F_j \subset I$ and numbers N_j, M_j such that

(i) If $a_i x \geq b_i$, $i \in F_j$ then $(w + N_j c)x \geq M_j$ for $j = 1,2,\ldots$

(ii) $\lim_{j\to\infty} N_j = \infty$

(iii) $\underline{\lim}_{j\to\infty} \frac{M_j}{N_j} \geq d$

Proof: Choose w to be a fixed vector in the relative interior of $\{a_i\} \cup \{c\}$. For each j, there is no x such that $a_i x \geq b_i$ and $-cx \geq -d + \frac{1}{j}$. By Corollary 1.2, there is a finite $F_j \subset I$ such that $a_i x \geq b_i$ $i \in F_j$ $-cx \geq -d + \frac{1}{j}$ implies $wx \geq j$. By the duality theorem of linear programming there are $\alpha_i \geq 0$ and $N_j \geq 0$ such that $\sum_{i\in F_j} \alpha_i a_i - N_j C = w$, and $\sum \alpha_i a_i + N_j(-d + \frac{1}{j}) \geq j$. If we take $M_j = N_j(d - \frac{1}{j}) + j$ then (i) and (iii) are clearly satisfied. (ii) follows because otherwise there would be no x satisfying $a_i x \geq b_i$, $i \in I$. Q.E.D.

The proof of Theorem 2.1 in [2 , corollary 2] was cumbersome, partly because I obtained the result from Theorem 1.1 rather than corollary 1.2.

The following strengthening is due to Jeroslow [8].

Corollary 2.2: Under the same assumptions as Theorem 2.1, there exists $w' \in R^n$ and w_0 such that, for every $0 < \theta < 1$, there is a finite $F_\theta \subset I$ such that, if $a_i x \geq b_i$ $i \in F_\theta$, then $(\theta w + c)x \geq d - \theta w_0$.

Proof: First we apply Theorem 2.1 as above to obtain w, N_j, M_j, F_j. We will show that the conclusion of Corollary 2.2 holds if we take $w' = \frac{1}{N_1} w$,

$w_0 > d - \frac{M_1}{N_1}$. Let $0 < \theta < 1$. Let α_j be such that $\alpha_j + (1 - \alpha_j)(\frac{N_1}{N_j}) = \theta$.
By (ii) $\text{Lim}\ \alpha_j = \theta$. By (i) if $a_i x \geq b_i$, $i \in F_1 \cup F_j$ then

$$(\alpha_j(w' + c) + (1 - \alpha_j)(\frac{N_1}{N_j}w' + c))x = (\theta w' + c)x \geq \alpha_j \frac{M_1}{N_1} + (1 - \alpha_j)\frac{M_j}{N_j}.$$

By (ii) and (iii) we may choose j so that the right-hand side is $\geq d - \theta w_0$.

Q.E.D.

3. Intersections of Convex Sets

The conditions under which an infinite family of convex sets has non-empty intersection have been studied by many people [1, 4, 9]. The case of a family of closed convex sets can be treated immediately using Theorem 1.1.

Corollary 3.1: Let K_i, $i \varepsilon I$ be closed convex sets. If $\cap K_i = \emptyset$, then there is a w such that, for every N, there is a finite $F \subset I$, and closed half spaces $H_i \supset K_i$ such that $wx \geq N$ for every $x \varepsilon \bigcap_{i \varepsilon F} H_i$.

Proof: Each K_i may be represented as the solution set of a system of linear inequalities corresponding to supporting hyperplanes. Since the intersection of the K_i is empty, the system formed by taking all the representing systems is inconsistent, and we may apply Theorem 1.1. The existence of suitable half spaces follows from the duality theorem for linear programming. Q.E.D.

To study families of open convex sets we make use of representations as solutions of systems of strict inequalities. Hence we need to extend Theorem 1.1 to systems including strict inequalities.

Lemma 3.2: If there is no x such that $a_i x \geq b_i$ $i \varepsilon I$ and $a_i x > b_i$ $i \varepsilon J$, then for some $j \varepsilon J$ every x such that $a_i x \geq b_i$ $i \varepsilon I \cup (J-j)$ satisfies $-a_j x \geq -b_j$. (hence Theorem 2.1 and Corollary 2.2 apply).

Proof: We will show that, if there is an x_j satisfying $a_j x_j \geq b_i$ $i \varepsilon I \cup (J-j)$ and $a_j x_j > b_j$ for every j, then there is an x with $a_i x \geq b_i$ $i \varepsilon I$ and $a_i x > b_i$ $i \varepsilon J$. Let y_1, y_2, ... be a countable of the x_j such that every x_j is a limit point of a subsequence. This implies $a_j y_i \geq b_j$ for every i and $a_j y_i > b_j$ for at least one i. Choose $\alpha_i > 0$ so that $\Sigma\alpha_i = 1$ and $\Sigma\alpha_i ||y_i|| \leq 1 + \inf ||y_i||$. Then $x = \Sigma\alpha_i y_i$ is defined and satisfies the desired system. Q.E.D.

<u>Theorem 3.3</u>: Let $V_i \subset R^n$, $i \varepsilon I$, be open convex sets. If $\cap V_i = \emptyset$ then either (i) the closures of the V_i have empty intersection (and corollary 3.1 applies) or (ii) there is a $j \varepsilon I$, a halfspace $\{x | cx<d\} \supset V_j$, $w \varepsilon R^n$, and $w_0 \varepsilon R$ such that, for every $0 < \theta < 1$, there is a finite $F \subset I$ and closed halfspaces $H_i \supset V_i$ such that $(\theta w+c)x \geq d-\theta w_0$ for every $x \varepsilon \bigcap_{i \varepsilon F} H_i$.

The intuitive content of Theorem 3.3 is that there are three ways in which the V_i may have empty intersection. The least interesting possibility is that some finite subfamily has empty intersection (in this case Helly's Theorem implies there are n+1 with empty intersection). The second possibility is that the V_j "march off to infinity" in the manner described in Corollary 3.1. Finally case (ii) describes a situation in which the intersection gets squashed to nothingness. A typical example illustrating case (ii) for $n = 2$ would be the family
$\{(x,y) | y<0\}$ $\{(x,y) | \frac{1}{k}x+y>- \frac{1}{k}\}$ $k = 1,2, \ldots$

<u>Proof of Theorem 3.3</u>: The assertion $\cap V_i = \emptyset$ implies that a system of strict inequalities $a_j x > b_j$, $j \varepsilon J$ has no solution, where each inequality is a supporting hyperplane of some V_i. Lemma 3.2 implies that one of the inequalities is negated by all the others. If case (i) does not hold then the system consisting of the others has a solution and we may apply Corollary 2.2. As in Corollary 3.1 the appropriate halfspaces may be obtained by the duality theorem of linear programming. Q.E.D.

Corollary 3.1 and Theorem 3.3 give necessary and sufficient conditions for families of closed or open convex sets to have empty intersection. We conclude with an example illustrating how Theorem 3.3 may be used to obtain Corollary 3.1 of [4].

Corollary 3.4: Let g: $R^n \times T \to R \cup \{\infty\}$ (T compact) be such that

(α) For every $t \in T$ $\{x | g(x,t) < 0\}$ is an open convex set.

(β) For every $t_1, \ldots t_{n+1} \in T$ there is an x such that $g(x,t_i) < 0$ for all i. (Helly's Theorem then implies that there is a suitable x for every finite subset of T).

(ν) If t_i is a sequence, $\lim t_i = t$, $H_i \supset \{x | g(x,t_i) < 0\}$ are closed halfspaces, and $H \subset \bigcup_{i=1}^{\infty} \bigcap_{k=1}^{\infty}$ is a closed halfspace, then $\{x | g(x,t) < 0\} \subset H$ (this is equivalent to a semi-continuity assumption about g).

Then there is an x for which $g(x,t) < 0$, $t \in T$.

Proof: We apply Theorem 3.3 with $V_t = \{x | g(x,t) < 0\}$. If there is no suitable x, then either (i) or (ii) of Theorem 3.3 occurs.

In case (i) we apply Corollary 3.1 to obtain, for each N, closed halfspaces $H_{iN} \supset \{x | g(x,t_{iN}) < 0\}$ $1 \leq i \leq n$ such that $wx \geq N$ for all x in $\bigcap_1^n H_{iN}$ (the fact that at most n halfspaces are needed for each N follows from a standard result of Caratheodory [12, 2.2.11]). If t_i is a limit point of t_{iN}, $1 \leq i \leq n$ then (ν) implies the intersection of the n V_{t_i} is empty, which contradicts (β).

If case (ii) occurs there is a set with $cx < d$ for $x \in V_s$ and, for each N, closed halfspaces H_{iN} as above such that $x \in \cap H_{iN}$ implies $V_s \cap V_{t_i} = \phi$, contradicting (β). Q.E.D.

4. Convex Optimization Application

In this section we use the results of Section One to analyze optimization problems involving closed convex functions.

Theorem 4.1: Let $f = \sup \{\alpha_i x+\beta_i | i\varepsilon I\}$; $g_j = \sup \{\gamma_{ij}x+\delta_{ij} | i\varepsilon I_j\}$, $j\varepsilon J$ (here α, x, $\gamma\varepsilon R^n$). Assume there is x for which $g_j(x) \leq 0$ and that $f(x) \geq L$ for all such x (i.e., the convex optimization program is feasible and has value $\geq L$). Then there is $w\varepsilon R^n$ such that for all $\varepsilon,N>0$ there are affine functions $h_j \leq g_j$, $h \leq f$; $\theta \geq 0$; $F \subset J$ finite; $\lambda_j \geq 0$ such that

$$(4.1) \qquad h(x) + \sum_{j\varepsilon F} \lambda_j h_j(x) + \theta wx \geq L - \varepsilon + N\theta$$

for all x.

Proof: The hypothesis implies that there are no x satisfying the semi-infinite system of inequalities $\alpha_i x \leq - \beta_i + L - \varepsilon$, $\gamma_{ij}x \leq - \delta_{ij}$. Theorem 1.1 gives a w such that, for every N, there is a finite subsystem which implies $wx \geq N$. The duality theorem of linear programming implies that the inequality $wx \geq N$ may be obtained as a non-negative linear combination of the members of the finite subsystem. This means there are finite $D \subset I$, $F \subset J$, $F_j \subset I_j$ and r_i, $s_{ij} \geq 0$ such that

$$(4.2) \qquad \sum_{i\varepsilon D} r_i\alpha_i + \sum_{j\varepsilon F} \sum_{i\varepsilon F_j} s_{ij}\gamma_{ij} = -w$$

$$(4.3) \qquad \Sigma r_i(-\beta_i+L-\varepsilon) + \Sigma \Sigma s_{ij}(-\delta_{ij}) \leq -N$$

Since there is an x for which $g_j(x) \leq 0$ we may assume $\Sigma r_i = R > 0$. Let $\theta = 1/R$, $\lambda_j = \theta(\Sigma s_{ij})$, $h_j(x) = (\theta/\lambda_j)\Sigma s_{ij}(\gamma_{ij}x+\delta_{ij})$, $h(x) = \theta\Sigma r_i(\alpha_i x+\beta_i)$. Note that h_j and h are convex combinations of affine supporting functions, hence are also affine supporting functions. (4.1) may be obtained by multiplying (4.2) by $-\theta x$ and (4.3) by θ and adding. Q.E.D.

Theorem 4.1 in turn implies a "limiting Lagrangian" result of Jeroslow [8].

<u>Corollary 4.2</u>: Assume that, in addition to the assumptions of Theorem 4.1, there is an $\underline{x}$ with $g_j(\underline{x}) \leq 0$, $f(\underline{x})$ finite. Then for θ arbitrarily close to zero there are λ_j with

(4.4) $$h(x) + \Sigma\lambda_j h_j(x) + \theta wx \geq L - \varepsilon.$$

<u>Proof</u>: It suffices to show that θ approaches zero as N goes to infinity in (4.1). Clearly we have $\theta \leq (N-w\underline{x})^{-1}(f(\underline{x})-L+\varepsilon)$. Given two values of θ, λ_j suitable for any intermediate value of θ may be obtained as convex combinations of the λ_j for the extreme values. Q.E.D.

We can also obtain the Slater point theorem from Theorem 4.1.

<u>Corollary 4.3</u>: Suppose J is finite and there is an $\underline{x}$ with $g_j(x) < 0$, $f(\underline{x})$ finite. Then there are λ_j with

(4.5) $$f(x) + \Sigma\lambda_j g_j(x) \geq L$$

for all x.

<u>Proof</u>: For $N > w\underline{x}$ (4.1) implies $\lambda_j \leq (-1/g_j(\underline{x}))(f(\underline{x}) - L + \varepsilon)$. As ε approaches zero, compactness implies the have a limit point which satisfies (4.5) Q.E.D.

Uzawa, and more recently, Duffin [5], have shown that in the case in which some of the constraints are affine the Slater point need only be strict for the nonaffine constraints. We establish a finite-dimensional version of this result.

Corollary 4.4: Suppose J is finite and that $||\alpha_i||$, $|\beta_i|$, $||\gamma_{ij}||$, $|\delta_{ij}| \leq M$ for all i, j (this is equivalent to assuming the functions are continuous). If there are $\underline{x}$, β for which $g_j(\underline{x}) \leq 0$ and $\gamma_{ij}\underline{x} + \delta_{ij} \leq \beta < 0$ for all but finitely many (i,j) then there are λ_i for which (4.4) holds.

Proof: We repeat the proof of Theorem 4.1, showing that for large N and small ε the values of the λ_j stay bounded. Then a compactness argument as in Corollary 4.3 completes the proof.

From (4.1) to (4.2) we obtain

$$(4.6) \qquad (L-F)-\dot{h}(x) + \theta\Sigma\Sigma - s_{ij}(\gamma_{ij}x+\delta_{ij}) \leq \theta(wx-N)$$

for all x. When $N > w\underline{x}$ this implies $\theta \dot{s}_{ij} \leq \frac{1}{\beta}(L-f(\underline{x})-\varepsilon)$ for all but finitely many s_{ij}. Further, only n s_{ij} can be nonzero by Caratheodory's lemma. To establish bounds on the remaining θs_{ij} we use the facts that θ approaches zero for N large (see corollary 4.2) and that the optimal solution to a linear program may be assumed to be in a bounded region when the right hand side varies through a bounded region. These results establish bounds on λ_j. Q.E.D.

5. What About Systems With Infinitely Many Variables?

I consider semi-infinite programming a nearly complete theory, but there are interesting questions lurking on the boundary between semi-infinite programming and infinite systems.

One way to consider such problems is via functional analysis. A representative result is a corollary of a theorem of Hahn [6 , p. 86].

Theorem 5.1: If X is a reflexive Banach space and a_i are continuous linear functionals, the system of inequalities $a_i x \geq b_i$, $i \varepsilon I$ has no solution if and only if, for every N, there is a finite $F \subset I$ and a functional w_N with $||w_N|| \leq 1$ such that $w_N x \geq N$ for all x such that $a_i x \geq b_i$, $i \varepsilon F$.

When X is ℓ^2, Theorem 5.1 gives a condition under which a system with countably many variables has a solution with the sum of squares converging. Early work of this type is reported in Chapter 3 of the still interesting [11].

It is easy to see that Theorem 1.1 does not generalize to the context of Theorem 5.1. We cannot have w_N the same for all N. A simple example is to take a_i an orthonormal basis for ℓ^2 and b_i go to infinity.

If one is interested in solutions to infinite systems which are not in ℓ^2 some restriction is necessary on the type of inequalities allowed. A natural restriction is that each of the inequalities involve only finitely many variables. Morley has announced a result of this type. It claims that, if a system is inconsistent, then the system is equivalent to a canonical inconsistent system, in the sense that each inequality of the canonical system is implied by finitely many inequalities of the original system. In Morley's result the canonical system has uncountably many inequities.

We outline a possible analysis based on countable systems. Let (S) be an inconsistent system. An FV subsystem is a subsystem of (S) involving only finitely many variables. Define an ordinal-valued rank for (S) as follows: (S) has rank zero iff (S) contains an inconsistent FV subsystem (Theorem 1.1 characterizes such subsystems). (S) has rank α if (S) has no smaller rank and there is a sequence of inequalities $a_i x \geq b_i$ of rank zero such that for any i and any $\varepsilon > 0$ the system $(S) \cup \{-a_i x \geq -b_i + \varepsilon\}$ is an inconsistent system of rank $< \alpha$.

I conjecture that for every countable ordinal α, there are systems of rank α and (less confidently) that every inconsistent (S) has a countable ordinal rank.

Finally, we wish to give an example indicating the difficulties that confront any attempt to allow infinitely many variables in each inequality. Consider a system involving the variables x_S for every $S \subset [0,1]$ as follows: each variable is between zero and one, $x_{[0,1]} = 1$, $x_{\{r\}} = 0$ for all r, and if the sets S_i are disjoint and $T = \cup S_i$ then $-x_T + \Sigma x_{S_i} = 0$. This system has a solution if and only if there is a countably additive measure defined on all subsets of the reals. Banach and Kuratowski proved this system is inconsistent assuming the continuum hypothesis. Later work indicates that some special axiom is needed to prove this system has no solutions.

REFERENCES

1. A. Ben-Tal, E. Rosinger and A. Ben-Israel, "A Helly-Type Theorem and Semi-infinite Programming," in Constructive Approaches to Mathematical Models, edited by Coffman and Fix, Academic Press, 1979.

2. C. Blair, "A Note on Infinite Systems of Linear Inequalities in R^n," Journal of Mathematical Analysis and its Applications 48, pp. 150-154.

3. C. Blair, "An Extension of a Theorem of Jeroslow and Kortanek," Israel Journal of Mathematics 17, pp. 111-115.

4. J. Borwein, "Direct Theorems in Semi-Infinite Convex Programming," Mathematical Programming 21, pp. 301-318.

5. R. Duffin, "Convex Programs Having Some Linear Constraints," Proceedings National Academy of Science 74, pp. 26-28.

6. N. Dunford and J. Schwartz, Linear Operators vol. I, Interscience 1958.

7. R. Jeroslow and K. Kortanek, "On Semi-Infinite Systems of Linear Inequalities," Israel Journal of Mathematics 10, pp. 252-258.

8. R. Jeroslow, "A Limiting Lagrangian for Infinitely Constrained Convex Optimization in R^n," Journal of Optimization Theory and Applications 33, pp. 479-495.

9. V. Klee, "The Critical Set of a Convex Body," American Journal of Mathematics 75, pp. 178-188.

10. K. Kortanek, "Constructing a Perfect Duality in Infinite Programming," Applied Mathematics and Optimization 3, pp. 357-372.

11. F. Riesz, Les Systemes d'Equations Lineaires à une Infitè d'Inconnues, Gauthier-Villars 1913.

12. J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, Springer-Verlag 1970.