URBAN GENERAL EQUILIBRIUM MODELS WITH NON-CENTRAL PRODUCTION

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Abstract

The paper develops general equilibrium conditions for urban models with non-central production of a local "retail" commodity. The commodity is produced by workers who commute circumferentially to work and is purchased by consumers who make circumferential shopping trips.
Urban General Equilibrium
Models with Non-Central Production
by Jan Brueckner*

The standard microeconomic model of an urban area assumes that city residents commute to the central business district, where they produce some good. Housing is the only commodity produced outside the center of the city, and consumer locational equilibrium conditions require that the unit rental price of housing decline with distance from the CBD. Locational equilibrium for housing producers generates declining land rent, which, in conjunction with falling housing prices, means that population density falls off as distance from the CBD increases. Consumers in this simple model acquire a non-housing consumer good at a fixed price.¹

The purpose of the models presented in this paper is to increase the realism of the standard model by adding more structure to the non-housing consumer good production activity. The modification introduced

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¹See Mills (1972) and Muth (1969) for detailed analysis of the standard model.
was suggested by the apparent fact that consumer shopping trips are short compared to commuting trips, with consumers travelling close to home to acquire daily necessities such as food. Of course, shopping travel for infrequent purchases of items such as consumer durables may be more extensive, but this is ignored in the model. The models constructed are characterized by location of consumer good producers at all distances from the CED. Consumers shop "locally" by making costless circumferential shopping trips to nearby producers. The production process we envision is a "retailing" process; the output is goods sold in a particular spot, while the inputs include goods produced by other industries. Thus, shipping of the produced good by producers does not occur.

This paper presents two general equilibrium models of an urban area with non-central production of the "retail" commodity, which is referred to as the "local" good. The first model has two classes of labor, one of which is the sole input to the production of the local good, while the second model has only one labor type but adds produced goods and building space to the inputs required for local goods production. The models impose Cobb-Douglas utility and production functions so that analytical solutions are achievable for functions of interest.

The purpose of the analysis is the derivation of general equilibrium conditions for urban economies with the special features we have discussed. A comparative-static analysis is possible because of the complexity of the models. However, since the goal was to illustrate how urban economies with non-central production "fit together", neither generality nor
detailed results were desired. For the development and testing of a related empirically-oriented model, see Brueckner (1976a).

The next section of the paper contains preliminary analysis, while subsequent sections develop the general equilibrium models.

I.

The basic assumptions used are the following:

A1) Production which requires labor input occurs in the CBD, and CBD commuters live at every distance in the city from the CBD.

A2) Housing, h, and the local good, g, are the only arguments of utility functions.

A3) The money cost of radial travel is exogenous and is increasing and concave in radial distance, while circumferential travel is costless.

A4) Local goods production requires labor input.

A5) Perfect competition prevails in all markets.

A6) Consumers make the same number of commuting and shopping trips per period.

A7) Consumers acquire the retail good at the distance at which they live.

While some of these assumptions are in the spirit of the standard model, A3, A6, and A7 require discussion. We ignore the time cost of travel, because the principal results are unchanged when it is included. The money cost of radial travel is $T(k)$, where $k$ is radial distance travelled, and $T' > 0$, $T'' < 0$. Exogeneity of the function $T$ requires
zero congestion at all traffic levels, which we assume, and requires that the transportation system uses no resources whose prices are endogenous, a requirement that our system, which uses no resources whatever, satisfies. The zero-cost assumption for circumferential travel is artificial, but in conjunction with A7, it permits construction of a model where consumers make "short" shopping trips with zero cost. In the real world, shopping trips appear to be short compared to CBD commuting trips, but while shopping costs may be small, they are not zero. Thus, consumers value accessibility to local goods producers, and a realistic model must be multicentered, with each local good producer a different center. A3 and A7 generate short shopping trips while allowing us to avoid the intractable multicentered problem that arises with small positive shopping costs. Since shopping costs are zero, the need for A6 is not immediately obvious, but it will become apparent below in the demonstration of the consistency of all the other assumptions. A6 seems to be a natural assumption, although shopping behavior could be endogenous in a more detailed model.

We now explore some implications of the assumptions. If $\gamma$ is the number of commuter trips per period, $2\gamma T(k)$ is commuting transport cost per period from a residence at distance $k$. A1 and A2 imply that housing is produced at all distances from the CBD, and A1, A2, and A7 imply that local good production occurs at every distance from the CBD. A1, A2, and A4 mean that labor is locally employed at every $k$.

In a model with one class of labor, we may demonstrate that these workers must reside at the distance of their place of employment,
travelling circumferentially to work and incurring no commuting cost, as follows. The disposable income per period of a CBD commuter living at distance $k$ is $y - t(k)$, where $y$ is the endogenous CBD wage per period. If the local wage rate at $k$ exceeded $y - t(k)$, all CBD commuters at $k$ would switch to local employment, violating Al. Hence $w(k) \leq y - t(k)$, where $w(k)$ is the local wage at $k$. Competition among firms bids up the local wage until it equals $y - t(k)$, and hence $w(k) = y - t(k)$. Since $t' > 0$, the wage of locally employed workers decreases with $k$.

We may ask if a worker will work outside the CBD and commute radially to his place of work. Outward commuting yields a lower wage and extra costs, and it will never occur. A worker will never commute inward from $k_0$ to $k_1$, $0 < k_1 < k_0$, when

$$w(k_1) - t(k_0 - k_1) < w(k_0)$$

which requires $t(k_0 - k_1) > t(k_0) - t(k_1)$. This inequality holds when $t' > 0$ and $t'' < 0$ with the exception of the case $t(k) \equiv \beta k$, in which case equality holds.\(^2\) When transport costs have the latter form, consumers are indifferent between zero radial commuting and commuting any distance inward. Otherwise, inward commuting lowers disposable income and will never occur.

Demonstrating that non-CBD radial commuting does not occur in the two-class model requires using some of the results from the actual

\(^2\)Consider the function $x(k) = t(k + \alpha) - t(k)$ where $\alpha > 0$. Now $x'(k) = t'(k + \alpha) - t'(k) \leq 0$ since $t'' < 0$. Let $\alpha = k - k_1$. Substituting $k = 0$ and $k = k_1$ in $x(k)$ and noting $x'' \leq 0$, we have $t(k_0) - t(k_1) \leq t(k_0 - k_1) - t(0) \leq t(k_0 - k_1)$. The only way equality can hold all the way through this relation is when $t'' = 0$ and $t(0) = 0$, that is if $t(k) \equiv \beta k$, $\beta > 0$. Otherwise, $t(k_0) - t(k_1) < t(k_0 - k_1)$.
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solution of the model. It turns out, as shown below, that the CBD commuters with the low CBD wage live in an inner annulus, while high-wage CBD commuters live in the outer annulus. The lower paid CBD worker type is employed at every distance since the local good is produced at every distance. In the inner annulus, the local wage is $y_2 - t(k)$ where $y_2$ is the low CBD wage, and the above argument shows that no non-CBD inward commuting occurs from residences in the inner annulus. The local wage in the outer annulus is proportional to $y_1 - t(k)$, where $y_1$ is the high CBD wage, and the above argument also shows that no inward commuting occurs with origin and destination in the outer annulus. For inward commuting with origin in the outer annulus and destination is the inner annulus, it must be true that $y_2 - t(k_0) - t(k_1 - k_0) < w_1(k_1)$, where $k_0$ and $k_1$ are in the inner and outer annuli respectively and $w_1$ is the local wage function in the outer annulus. Since $y_2 - t(k_1) < w_1(k_1)$ by equation (8) below, the inequality holds if we use the above argument. Thus non-CBD inward commuting of any kind never occurs in the two-class model. Since the local wage is monotonically decreasing, outward commuting also never occurs.

The salient features of the models developed below are thus:
1) Producers of g and h locate at all distances; 2) CBD workers and locally employed workers, who travel circumferentially to work at zero cost, reside at all distances; 3) Both types of workers make circumferential shopping trips at zero cost; 4) perfect competition prevails in all markets.

One question which might occur to the reader is: Does the spatial equilibrium in the models developed have the property that people have
no incentive the deviate from the seemingly arbitrary behavior postulated in A7? The answer is affirmative: the structure of the models is such that the behavior postulated is A7 is optimal, in other words, that people have no incentive to deviate from the circumferential shopping pattern. This is shown as follows for the one-class model. This model is characterized by declining local good and housing price functions. Hence, consumers will never travel radially inward to purchase g because extra transport costs are incurred and g is more expensive closer to the center. We can show also that outward shopping travel also reduces utility. Let \( V \) be the utility level of a worker living at \( k_0 \) and shopping at \( k_1 > k_0 \). His shopping costs are given by \( t(k_1 - k_0) \), since by A6 the number of shopping trips per period equals \( \gamma \), the number of commute trips per period. His disposable income is \( w(k_0) - t(k_1 - k_0) \), which, from above, is less than \( w(k_1) \), the disposable income of a worker living and shopping at \( k_1 \). Since the latter worker faces the same local good price, a lower housing price, and has higher disposable income than the worker who travels radially to shop, his utility level, \( U(k_1) \), exceeds \( V \). But locational equilibrium in the model with circumferential shopping requires \( U(k_1) = U(k_0) \), where \( U(k_0) \) is the utility level of a worker living and shopping at \( k_0 \). Hence \( U(k_0) > V \) and the radial shopper is better off shopping circumferentially at \( k_0 \). A similar argument may be made for the two class model. Some other set of assumptions may generate a spatial equilibrium which validates those assumptions. The attractiveness of our assumptions lies in their apparent realism.
and in the simplicity of the models they generate.

II.

This section develops the two-class general equilibrium model. There are two kinds of labor in the city, both of which are required for production of the CBD commodity \( x \) according to the firm production function \( f(L_1, L_2) \), where \( L_1 \) is type-one labor and \( L_2 \) is type-two labor. We require that \( f_1 \) exceed \( f_2 \) over most of the domain of the production function, guaranteeing that the value of the marginal product and hence the wage is higher for type-one than for type-two CBD workers. We can imagine that type-one workers are trained to be managers while type-two workers are clerical or perhaps unskilled workers. For simplicity, the CBD is assumed to be a point, reflecting the assumption that CBD production uses no non-labor inputs.

Local goods production requires only type-two labor. We can imagine that local production is a simple process which does not require highly-trained overseers. The firm production function is \( \alpha L_2^\tau, \alpha > 0, 0 < \tau < 1 \). Excluding non-labor inputs leads to a stylized model which is easy to manipulate and no less suggestive than a more general one. Housing producers use the Cobb-Douglas function \( \rho_1 L_2^\rho_2 \), where \( N \) and \( \ell \) are non-land capital and land inputs respectively.
The city exports \( x \) in return for \( N \), which is distributed to housing producers all over the city at the same delivered price \( n \). For simplicity we assume that the cost of transporting \( x \) and \( N \) is zero. Consumers in the city consume the local good \( g \) and housing \( h \), and both types of workers have the utility function \( g^m h^n \). In this model, \( x \), the CBD commodity, is purely an export commodity. The number of radians available for settlement at distance \( k \) is \( F(k) \), \( 0 \leq F(k) \leq 2\pi \). Hence the residential area in a ring of inner radius \( k \) and width \( dk \) is \( (2\pi - F(k))dk \equiv \xi(k)dk \). (Recall that local goods production is not land-using.)

Our first task is to derive the three price functions for the city, \( p(k) \), the housing unit rental price, \( r(k) \), land rent, and \( s(k) \), the unit price of the locally-produced good. We require that a CBD commuter live at every distance \( k \) from the center. Since local production occurs at every distance, local type-two workers must live at every distance. Our approach will be to generate price functions which make CBD commuters of each type locationally indifferent in the areas where they live. Since the CBD incomes of the workers are different, two different sets of price functions are required. We then generate a local wage function for type-two workers which (a) makes them locationally indifferent when locally employed, and (b) gives them the same utility level as type-two CBD commuters. An equilibrium with both CBD commuting and ubiquitous local employment of type-two workers cannot exist unless both (a) and (b) hold.

The CBD commuter's Lagrangean is
\[ g_1 h^2 = \lambda (p(k)h + s(k)g - (y_1 - t(k))) \quad i=1,2. \]

The Lagrangean reflects the assumption that people acquire the local good \( g \) at the distance at which they reside in that the relevant local good price is \( s(k) \) and no shopping transportation costs are incurred.

The first-order conditions yield demand functions

\[ g = \frac{m_1}{m_1 + m_2} \frac{y_i - t(k)}{s(k)} \]
\[ h = \frac{m_2}{m_1 + m_2} \frac{y_i - t(k)}{p(k)} \quad i=1,2 \quad (2) \]

and the locational equilibrium condition

\[ p'(k)h + s'(k)g = -t'(k), \]

which after substituting (2) becomes

\[ m_2 \frac{p'(k)}{p(k)} + m_1 \frac{s'(k)}{s(k)} = - \frac{(m_1 + m_2)t'(k)}{y_i - t(k)} \quad i=1,2. \quad (3) \]

Assuming (3) holds at all \( k \) allows us to integrate it, yielding

\[ m_2 \log p(k) + m_1 \log s(k) = (m_1 + m_2) \log (y_i - t(k)) + A_i \]

\[ p(k)^{m_2} s(k)^{m_1} = a_i (y_i - t(k))^{m_1 + m_2} \quad i=1,2 \quad (4) \]

where \( A_i, i=1,2 \), are integration constants and \( a_i = e^{A_i} \), \( i=1,2 \). Now (4)
says that there are different price functions \( p \) and \( s \) which make the two types of CBD commuters locationally indifferent. Rewriting (4)

\[
p_i(k)^{m_2} s_i(k)^{m_1} = a_i(y_i - t(k))^{m_1 + m_2} \quad i = 1, 2
\]

emphasizes this. The indirect utility functions of the workers when they face the price functions which make them locationally indifferent are

\[
\frac{m_1}{m_1 + m_2} (m_1 + m_2) - \frac{(m_1 + m_2)}{a_i} \quad (5)
\]

We can identify the areas in which the CBD commuters live as follows. It must be the case that the utility level of a type-one worker facing the prices in the type-two area (prices which make type-two workers locationally indifferent) is lower than the constant level in the type-one area, given by (5) above with \( i = 1 \). If this were not the case, he would reside in what we have assumed is the type-two area, contrary to assumption. The utility level of a type-one worker facing type-two prices is

\[
\frac{m_1}{m_1 + m_2} (m_1 + m_2) - \frac{(m_1 + m_2)}{a_2} \left( \frac{y_i}{y_2} - t(k) \right)^{m_1 + m_2} \quad (6)
\]

which is found by substituting (2) with \( i = 1 \) into the utility function and then using (4) with \( i = 2 \). If \( k^* \) is in the type-two area, then (6) evaluated at \( k^* \) must be less than or equal to (5) by the above argument. This requires
Similarly, if $k^{**}$ is in the type-one area it must be true that the utility level of type-two workers at $k^{**}$ is less than or equal to (5) with $i=2$. This requires
\[
\frac{y_1 - t(k^{**})}{y_2 - t(k^{**})} \leq \frac{a_2}{a_1}
\]  

(7)

Differentiating the LHS of (7) with respect to $k^*$, we get
\[
(m_1 + m_2) \left( \frac{y_1 - t(k^*)}{y_2 - t(k^*)} \right)^{m_1 + m_2 - 1} \frac{t'(k^*)(y_1 - y_2)}{(y_2 - t(k^*))^2} > 0
\]

since $y_1 > y_2$. Thus the LHS of (7) is monotonically increasing in $k^*$, and the $k^*$ that satisfy (7) belong to the closed interval $[0, \hat{k}]$, for some $\hat{k} > 0$. The same argument shows that the $k^{**}$ which satisfy (8) satisfy $k^{**} \geq \hat{k}$. Since the city is bounded at $\bar{k}$, $k^{**} \in [\hat{k}, \bar{k}]$. Thus the type-two CBD commuters live in an inner annular area and the richer type-one commuters live in an outer annulus. At the boundary $\hat{k}$ we have
\[
a_1(y_1 - t(\hat{k}))^{m_1 + m_2} = a_2(y_2 - t(\hat{k}))^{m_1 + m_2}.
\]  

(9)

We can now derive the wage function for local type-two workers. In the inner annulus requirements (a) and (b) above are satisfied when
\[
w_2(k) = y_2 - t(k).
\]  

(10)
The subscript 2 on the local wage indicates that it is the local wage function for the inner annulus, where type-two commuters live. Satisfaction of requirements (a) and (b) in the outer annulus occurs when

\[\frac{m_1}{m_2} \left(\frac{m_1 + m_2}{m_2} \right) - \left(\frac{m_1 + m_2}{m_1 + 2m_2} \right) - \frac{m_1 + m_2}{a_1} \left(\frac{\frac{m_1}{m_2} - t(k)}{v_1 - t(k)}\right) \]

\[\equiv \frac{m_1}{m_2} \left(\frac{m_1 + m_2}{m_1 + 2m_2} \right) - \left(\frac{m_1 + m_2}{m_1 + 2m_2} \right) - \frac{m_1 + m_2}{a_2} \]

The RHS of (11) is the utility level of all type-two workers in the inner annulus and the LHS is the utility level of workers receiving a local wage \(w_1(k)\) and facing the type-one price functions in the outer annulus. This implies

\[w_1(k) \equiv \frac{a_1}{a_2} \left(\frac{1/(m_1 + m_2)}{v_1 - t(k)}\right). \]

Now a necessary requirement for equilibrium is that \(a_1 < a_2\), or that, from (5), the utility level of the type-one workers exceeds that of type-two workers. If this were not the case, type-one workers would switch to the menial jobs, violating our assumptions on employment patterns. Thus \(a_1/a_2 < 1\) and \(w_1(k) < y_1 - t(k)\). From (9), (10) and (12), we get \(\hat{w}_1(k) = \hat{w}_2(k)\); the local wage function is continuous at the boundary, as it should be.

Other locational equilibrium conditions must be satisfied as well. The first-order conditions for the perfectly competitive local goods producers are
$s(k) \alpha L_2^{-1} = w(k)$

$s'(k) \alpha L_2^{-1} = w'(k)$,  \hspace{1cm} (13)

which yield

$$\frac{s'(k)}{s(k)} = \tau \frac{w'(k)}{w(k)}, \hspace{1cm} (14)$$

where

$$w(k) = \begin{cases} 
    w_2(k) & 0 \leq k \leq \hat{k} \\
    w_1(k) & \hat{k} \leq k \leq \bar{k} .
\end{cases} \hspace{1cm} (15)$$

Integrating (14) gives

$$s(k) = bw(k)^\tau = \begin{cases} 
    \left[ \frac{b(y_2 - t(k))^\tau}{\frac{\tau}{m_1+m_2}} \right] & 0 \leq k \leq \hat{k} \\
    \frac{a_1}{a_2}^{\frac{\tau}{m_1+m_2}} (y_1 - t(k))^\tau & \hat{k} \leq k \leq \bar{k} .
\end{cases} \hspace{1cm} (16)$$

Substituting (16) in (4) we get

$$p(k) = \begin{cases} 
    \frac{1/m_2 - x_1/m_2}{a_2} (y_2 - t(k))^{\frac{(1-\tau)m_1+m_2}{m_2}} & 0 \leq k \leq \hat{k} \\
    \frac{-m_2}{a_2} (y_1 - t(k))^{\frac{(1-\tau)m_1+m_2}{m_2}} & \hat{k} \leq k \leq \bar{k} .
\end{cases} \hspace{1cm} (17)$$

It is easy to check that $p$ and $s$ are continuous at $\hat{k}$.

The housing producers maximize $p(k)N^{\rho_1 \rho_2} - r(k)k - nN$ yielding
\[ p(k) \rho_1^{\rho_1 - 1} \rho_2 = n \]
\[ p(k) \rho_2^{\rho_2 - 1} = r(k) \]  \hspace{1cm} (18)
\[ p'(k) \rho_1^{\rho_1 - 1} \rho_2^{\rho_2 - 1} = r'(k), \]

which yield

\[ \frac{r'(k)}{r(k)} = \frac{1}{\rho_2} \frac{p'(k)}{p(k)}. \]  \hspace{1cm} (19)

or, integrating;

\[ r(k) = c \rho_2^{1/\rho_2}. \]

We may summarize by saying that locational equilibrium conditions generate prices \( w_1, w_2, p_1, p_2, s_1, s_2, r_1, r_2 \) with the following functional dependencies:

\[ v_2(k, y_2); w_1(k, a_1, a_2, y_1) \]
\[ p_2(k, a_2, b, y_2); p_1(k, a_1, a_2, b, y_1) \]
\[ s_2(k, b, y_2); s_1(k, a_1, a_2, b, y_1) \]
\[ r_2(k, a_2, b, c, y_2); r_1(k, a_1, a_2, b, c, y_1). \]

All of the partial derivatives of these functions with respect to distance are negative. Again, the subscript refers to the annulus to which the function pertains (1-outer, 2-inner). Substituting prices into the demand functions (2) we get
\[ g_2 = \frac{m_1}{m_1 + m_2} \frac{y_2 - t(k)}{s_2(k, b, y_2)} = \frac{m_1}{m_1 + m_2} b^{-1} (y_2 - t(k))^{1-\tau} = g_2(k, b, y_2) \]

\[ h_2 = \frac{m_2}{m_1 + m_2} \frac{y_2 - t(k)}{p_2(k, a_2, b, y_2)} = \frac{m_2}{m_1 + m_2} a_2^{-1/m_2} b_{1/m_2}^{1/m_2} (y_2 - t(k))^{(r-1)m_1/m_2} \]

\[ = h_2(k, a_2, b, y_2) \]  

(22)

as the demand functions for residents of the inner annulus. Similarly, we get \( g_{11}(k, a_1, a_2, b, y_1) \) and \( h_{11}(k, a_1, a_2, b, y_1) \) as demand functions for the type-one residents of the outer annulus. Substituting prices and \( w_1 \) in place of \( \dot{y}_1 - t(k) \) in (2), we get \( g_{12}(k, a_1, a_2, b, y_1) \) and \( h_{12}(k, a_1, a_2, b, y_1) \) as demand functions of local workers living in the outer annulus. Housing output per unit land from (18) is

\[ r_i(k)/\rho_2 p_i(k) \] in the \( i \)th annulus, and per capita land consumption is the product of this expression and per capita housing consumption, or

\[ \frac{m_2 \rho_2}{m_1 + m_2} \frac{y_2 - t(k)}{r_i(k)} = \frac{c m_2 \rho_2}{m_2} a_2^{-1/m_2} b_{1/m_2}^{1/m_2} (y_2 - t(k))^{(m_2 (1-\rho_2)-\rho_2 (1-\tau)m_1)/m_2} \]

\[ = \bar{\ell}_2(k, a_2, b, c, y_2). \]

Similarly, the per capita land consumption of type-one and type-two workers in the outer annulus is

\[ \bar{\ell}_{11}(k, a_1, a_2, b, c, y_1) \]

\[ \bar{\ell}_{12}(k, a_1, a_2, b, c, y_1). \]
From (18) the ratio of non-land capital to land in the hour industry in the inner annulus is

$$\frac{1}{\rho_2} \frac{r_2(k, a_2, b, c, y_2)}{n} = \gamma_2(k, a_2, b, c, y_2).$$

Similarly, this ratio in the outer sector is

$$\frac{1}{\rho_1} \frac{r_1(k, a_1, a_2, b, c, y_1)}{n} = \gamma_1(k, a_1, a_2, b, c, y_1).$$

We now proceed to calculate patterns of local employment which equate supply and demand for local goods in every ring. In the ring with inner radius $k$ for $0 < k < k$, the demand for the local good is

$$\xi(k) \frac{E(k, b, y_2)}{z_2(k, a_2, b, c, y_2)} dk,$$

which is the residential land area in the ring, $z(k)dk$, divided by per capita land consumption $L_2$, which equals population in the ring, times per capita demand for $g$. This demand is insensitive to the mix of local and non-workers, since in the inner annulus their disposable income equals. The number of firms producing local goods at distance $k$ in the inner annulus is $n_2(k)$, and for $0 < k < k$, this function must satisfy

$$n_2(k) aL_2 dk = \frac{\xi g_2}{z_2} dk,$$  \hspace{1cm} (23)
where \( L_2 \) is the optimal labor input for a local goods firm. Equation (23) says that the supply of the local good in the ring must equal demand for it. Now the volume of local employment in the ring which generates an output such that demand is satisfied is \( n_2(k)L_2dk \). The proportion of ring residents locally employed which leads to the clearing of the local goods market is thus

\[
\varepsilon = \frac{\frac{n_2(k)L_2dk}{(\xi_2)}(\xi_2)}{L_2},
\](24)

since the denominator equals the ring population. Substituting for the \( n_2(k) \) solution from (23) above, we get

\[
\varepsilon = \frac{s_2}{\alpha L_2^{\tau-1}}.
\](25)

Substituting for \( \alpha L_2^{\tau-1} \) from (13) we have

\[
\varepsilon = \varepsilon \frac{s_2 s_2}{w_2} = \frac{\tau^{m_1}}{m_1+n_2} < 1,
\](26)

since budget shares are constant with a Cobb-Douglas utility function from (2). Thus, a constant fraction \( \varepsilon \) of the residents in any ring in the inner annulus is engaged in local employment, and \( \varepsilon \xi(k)dk \) of the land is occupied by the residences of local workers.

Solving for the distribution of population between local employees and CBD commuters in the outer annulus is more complicated because the population mix does affect demand for local goods as a
result of the unequal incomes of the two types of workers. Let $\theta$ be the proportion of residential land occupied by local workers in a ring in the outer annulus. Then the demand for $g$ in the ring is

$$\frac{\theta \xi(k) g_{12}(k,a_1,a_2,b,y_1)}{\xi_{12}(k,a_1,a_2,b,c,y_1)} + \frac{(1-\theta) \xi(k) g_{11}(k,a_1,a_2,b,y_1)}{\xi_{11}(k,a_1,a_2,b,c,y_1)} \, dk. \hspace{1cm} (27)$$

Now all the type-two labor in the ring must be used by local firms. The number of local firms is thus equal to total type-two labor divided by the optimal labor input per firm, or

$$n_1(k) = \frac{\theta \xi(k) dk}{L_2}. \hspace{1cm} (28)$$

Since $n_1(k)\alpha L_2^T$ must equal (27) we have, using (28) and (13),

$$\alpha L_2^{\tau-1} \frac{\theta \xi(k)}{L_{12}} = \frac{\theta \xi(k) g_{12}}{L_{12}} + \frac{(1-\theta) \xi(k) g_{11}}{L_{11}} \hspace{1cm} (29)$$

$$\theta = \frac{g_{11}}{g_{12} + \frac{1}{L_{12}} \left( w_1 - g_{12} \right)}.$$ 

Now since $\tau < 1$ and $s_1 g_{12}/w_1 = m_1/m_1 + m_2$ from (2), the expression in parentheses in (29) is positive and $0 < \theta < 1$. Noting the functional dependencies of the variables on the RHS of (29), we have

$$\theta = \theta(k, a_1, a_2, b, c, y_1). \hspace{1cm} (30)$$
Unlike in the inner annulus, the fraction of residential land occupied by locally employed workers does vary over distance. It should be noted that the above analysis ignores the fact that $n_1$ and $n_2$ must be integer-valued.

What kind of market forces bring about an allocation of labor to local goods production which clears the local goods market at every distance? In each ring in the inner annulus, there is a perfectly elastic supply of local labor to the market for local workers at the wage rate $w_2$ since CBD workers are indifferent between local and CBD employment. Suppose there were an excess demand for local goods in the ring, that is, not enough labor locally employed. There would be upward pressure on the price of the local good in the ring resulting in entry of firms. The bidding of firms for a total amount of labor inadequate to supply each firm with its optimal labor input would put upward pressure on the local wage rate, causing reallocation of labor from CBD to local employment. This process would continue until the local goods market cleared, relieving upward pressure on the local good price and the local wage.

Inadequate local employment in a ring in the outer annulus would cause upward pressure on the local good price, entry of firms, and upward pressure on the local wage rate. Workers must be attracted from some other ring to restore equilibrium, and these workers must bid away housing from type-one workers in the ring in order to establish residences. An infinitesimal temporary increase in the local wage may allow this.
We may now discuss general equilibrium in this model. The exogenous variables in the model are the populations of the two groups in the city $L_1$, $L_2$, the agricultural rent $r_A$, the unit prices of $x$ and $N$, $v$ and $n$ respectively (the assumption that $x$ is imported in exchange for $N$ means these prices must be set in external markets, and the city is assumed to be a price taken in these markets), and the profit levels $\pi_G$ and $\pi_H$ in the local goods and housing industries. (A more satisfactory profit constraint would be zero profits, but the homogeneous production functions employed above for their analytical usefulness rule out this assumption.) The question of the expenditure of profits by the firm owners fortunately does not arise with a zero profit constraint. To eliminate the complication of how positive profits get spent in this model, we assume firm owners live outside the city. In addition, we assume that the landowners who receive the urban land rent also live outside the city. A model incorporating expenditures on housing and local goods in the city by rentiers and capitalists could be developed without difficulty, although the gains from this additional complexity would be marginal.

From (13) and (18), the profit levels of local goods and housing producers are respectively

$$\pi_G(b) = (1-\tau)(ab^T)^{1/1-\tau}$$

$$\pi_H(c) = (1 - \rho_1 - \rho_2) (\rho_1/n)^{\rho_1/1-\rho_1-\rho_2} (\rho_2/c)^{\rho_2/1-\rho_1-\rho_2}$$

We may now develop general equilibrium conditions for the model. Let $J$ be the number of $x$-producing firms in the CBD and let $L_1$ and $L_2$
be the inputs of the two labor types in each firm. Equilibrium conditions for the \( x \) sector are

\[
\begin{align*}
  vf_1(L_1, L_2) &= y_1 \\
  vf_2(L_1, L_2) &= y_2 \\
  Jvf(L_1, L_2) &= n \int_0^{\hat{k}} \xi(k)\gamma_2(k,a_2,b,c,y_2)dk + n \int_{\hat{k}}^{\bar{k}} \xi(k)\gamma_1(k,a_1,a_2,b,c,y_1)\\
  L_1 &= \frac{1}{J} \bar{L}_1 \\
  L_2 &= \frac{(1-\varepsilon)}{J} \int_0^{\hat{k}} \frac{\xi(k)}{\lambda_2(k,a_2,b,c,y_2)} dk.
\end{align*}
\]

The first two equations state that firm labor inputs are profit maximizing given the wage rates the firm faces, which are themselves endogenous. The third equation is a trade balance equation which states that the value of \( x \) production equals the value of \( N \) consumed by the housing industry. The integrands on the RHS of (35) are residential land times non-land capital per unit land, or non-land capital used in a ring of radius \( k \). Equation (36) states that each firm employs the same amount of type-one labor, and that total employment exhausts the supply \( \bar{L}_1 \). The last equation states that each firm employs the same amount of type-two labor, and that total CBD employment of type-two labor equals the number of type-two CBD commuters (remember that \( 1-\varepsilon \) of the population in each ring in the inner annulus commutes to the CBD, and hence that \( 1-\varepsilon \) of the total inner annulus population, which is the
integral in (37), commutes to the CBD). There is no zero- or constant-profit condition for x producers because such a condition is already built into the external solution for the price v.

Profit conditions for local goods and housing producers are

\[ \pi_G(b) = \bar{\pi}_G \]  
\[ \pi_H(c) = \bar{\pi}_H. \]  

(38)  
(39)

Conditions which guarantee that the city is able to house each group of workers are

\[ \int_{\hat{k}}^{k} \frac{\xi(k)(1 - \theta(k,a_1,a_2,b,c,y_1))}{z_{11}(k,a_1,a_2,b,c,y_1)} \, dk = \bar{L}_1 \]  
\[ \int_{\hat{k}}^{k} \frac{\xi(k)}{z_{2}(k,a_2,b,c,y_2)} \, dk + \int_{\hat{k}}^{k} \frac{\xi(k)\theta(k,a_1,a_2,b,c,y_1)}{z_{12}(k,a_1,a_2,b,c,y_1)} \, dk = \bar{L}_2. \]  

(40)  
(41)

where \( \bar{L}_1 \) and \( \bar{L}_2 \) are type-one and type-two populations. Boundary conditions are

\[ r_1(\bar{k},a_1,a_2,b,c,y_1) = r_A, \]  

(42)

which says that urban land rent must just equal agricultural rent, \( r_A \), at the periphery of the urban area, and

\[ a_1(y_1 - t(k))^{m_1+m_2} = a_2(y_2 - t(k))^{m_1+m_2}, \]  

(43)

which was developed above.
The unknowns in the model are the integration constants $a_1$, $a_2$, $b$, $c$; the income levels $y_1$, $y_2$; the number of CBD firms and their labor inputs $J$, $L_1$, $L_2$; and the boundary distances $\hat{k}$ and $\bar{k}$. There are eleven variables and we have eleven equations above to solve for them. We can solve directly for $b$ and $c$ in (38) and (39), reducing the system to a fully simultaneous nine-equation system with nine unknowns.

It should be noted that hidden requirements for the consistency of the above system are $y_1 > y_2$, which guarantees that the model is not constructed backwards, $a_1 < a_2$, which guarantees that type-one workers cannot gain by switching to type-two jobs, and $y_1 - t(\bar{k}) > 0$, which guarantees that no one in the city spends an amount greater than or equal to his income on commuting expenses ($y_1 - t(\bar{k}) > 0$ implies that $y_1 - t(k) > 0$ in the outer annulus, which by (43) guarantees $y_2 - t(k) > 0$ in the inner annulus). We also require positive values for all the endogenous variables.

The model implicitly assumes that firms are free to migrate between cities through the constant profit conditions for producers. However, allowing $a_1$ and $a_2$ to be solved for in the city means that utility levels are unrelated to those prevailing outside the urban area. In other words, the above model implicitly allows firm migration without allowing worker migration. An easy modification which frees workers to migrate is to set $a_1 = \tilde{a}_1$ and $a_2 = \tilde{a}_2$ and make $\bar{L}_1$ and $\bar{L}_2$ endogenous. Then population will adjust until the utility levels of both groups equal prevailing utility levels in the economy at large.

While comparative-static analysis using this model is possible in principle, pencil-and-paper results are out of the question in a
practical sense because of the large number of equations. The only ways of investigating the sensitivity of the model solution to variations in the exogenous variables would be to conduct computer solutions using specific numerical values for parameters and exogenous variables. Interesting exercises would be an examination of the effect of a change in the population of one group on the wages and utility levels of both groups and an examination of how a change in the exogenous utility level of one group in the mobile-worker model effects population and wages for both groups.

Since the purpose of this model has been to illustrate the structure of an urban economic model with local goods, the lack of comparative static results is defensible. We have shown how incomes, utility levels, boundary configurations, and patterns of labor allocation are all endogenously determined in a complete model of an urban area. The lesson to be drawn from this kind of model is that partial equilibrium urban economic analysis may ignore complex and important general equilibrium effects.

III.

The model developed in this section is similar to the one explored above, but its emphasis is different. The local goods sector now uses local labor, a produced input Q, which can be thought of as "wholesale goods," and commercial real estate R, or "store space." Like housing producers, the producers of commercial real estate use
land and non-land capital as inputs, but they produce a distinctly different product. Local goods production uses land in this model since land is embodied in its R input. We dispense with the complications introduced by multiple income groups by assuming all workers are identical. While the clearing of the local goods market in the previous model required the proper division of workers between local and CBD employment in a given ring, it turns out that the proper division of land between local goods and housing producers guarantees market clearing in the model developed below.

We assume that the produced input Q is manufactured in the CBD. Output must satisfy the city's demand for Q and pay for imports of N used by housing and commercial real estate producers.

The utility and housing production functions are the same as those above, while the production functions for commercial real estate and local goods are

\[ R = N^{\mu_1} L^{\mu_2} \]
\[ G = Q^{g_1} L^{g_2} R^{g_3}. \]  

(44)

Utility maximization again results in the demand equations (2) and the locational equilibrium condition

\[ P(k)^{m_2} q(k)^{m_1} = a(y - t(k))^{m_1 + m_2} = a(w(k))^{m_1 + m_2}. \]  

(45)

Housing producer equilibrium again results in
\[ p(k) = cr(k)^{p_2}, \quad (46) \]

and commercial real estate producer equilibrium yields the analogous condition

\[ z(k) = dr(k)^{r_2}, \quad (47) \]

where \( z \) is the unit rental price of \( R \) and \( d \) is an integration constant. Local goods producers solve

\[
\begin{align*}
sg_1 \frac{G}{Q} - q &= 0 \\
sg_2 \frac{G}{L} - w &= 0 \\
sg_3 \frac{G}{R} - z &= 0 \\
sl'G - w'L - z'R &= 0,
\end{align*}
\quad (48)
\]

where \( q \) is the unit price of \( Q \), which is invariant over space by assumption. Dividing the last equation in (48) by \( R \) and substituting from the first three equations, we have

\[ \frac{s'(k)}{s(k)} - g_2 \frac{w'(k)}{w(k)} - g_3 \frac{z'(k)}{z(k)} = 0, \]

which after integration yields

\[ s(k) = bw(k)^{g_2} z(k)^{g_3}, \quad (49) \]
where $b$ is an integration constant. Now (45), (46), (47), and (49) form a non-linear equation system in the five prices, which may be solved for all the prices in terms of $w$. The solution is

$$
\begin{align*}
    r &= r_0(a,b,c,d)w(k)^c_0 \\
    z &= z_0(a,b,c,d)w(k)^\mu_zc_0 \\
    p &= p_0(a,b,c,d)w(k)^\rho_pc_0 \\
    s &= s_0(a,b,c,d)w(k)^\frac{\rho_3\mu_2c_0}{2},
\end{align*}
$$

(50)

where

$$
\begin{align*}
    r_0 &= \begin{bmatrix} -m_1 & -m_2 & -g_3m_1 \end{bmatrix}^\kappa \\
    z_0 &= \begin{bmatrix} \mu_2 & -\mu_zm_1 & -\mu_zm_2 & \rho_2m_2 \end{bmatrix}^\kappa \\
    p_0 &= \begin{bmatrix} \rho_2 & -\rho_2m_1 & g_3\mu_2m_1 & -g_3\rho_2m_1 \end{bmatrix}^\kappa \\
    s_0 &= \begin{bmatrix} g_3\mu_2 & \rho_2m_2 & -g_3\mu_2m_2 & g_3\rho_2m_2 \end{bmatrix}^\kappa \\
    \kappa &= (\rho_2m_2 + g_3\mu_2m_1)^{-1} \\
    c_0 &= \kappa(m_1(1-g_2) + m_2).
\end{align*}
$$

Substituting (50) into the demand equations (2) gives us demand functions
\[ g(k, a, b, c, d, y) \]

\[ h(k, a, b, c, d, y). \]

As above, per capita land consumption is \( h \) times the inverse of housing output per unit land from (18), or

\[ \zeta(k, a, b, c, d, y). \]

The ratios of non-land capital to land in housing and commercial real estate are, respectively,

\[
\frac{\rho_1^r}{\rho_2^n} = \gamma_H(k, a, b, c, d, y)
\]

\[
\frac{\mu_1^r}{\mu_2^n} = \gamma_R(k, a, b, c, d, y). \tag{52}
\]

The ratio of commercial real estate output per unit land input from a system analogous to (18) is

\[
\frac{r}{\mu_2^p} = \phi(k, a, b, c, d, y).
\]

The ratios of \( Q \) input, \( G \) output, and \( L \) input to land used in the \( G \) industry are, respectively,

\[
\left( \frac{Q}{G} \right)_R = \left( \frac{Q}{R} \right)\phi = \frac{g_1^2}{g_3 q} \phi = \delta_1(k, a, b, c, d, y)
\]
\[
\left( \frac{G}{G} \right) = \left( \frac{G}{R} \right) \phi = \frac{z}{s_3} \phi = \delta_2(k,a,b,c,d,y)
\]

\[
\left( \frac{L}{L} \right) = \left( \frac{L}{R} \right) \phi = \frac{z_2}{s_3} \phi = \delta_3(k,a,b,c,d,y),
\]

(53)

where the expressions following the second set of equality signs come from (48).

Let \( \lambda \) be the fraction of available land used to produce local goods, that is, the fraction used by producers of commercial real estate. In order that the market for local goods clear in a ring at distance \( k \), it must be true that

\[
\lambda \xi(k) \delta_2(b,a,b,c,d,y) dk = \frac{(1-\lambda) \xi(k) g(k,a,b,c,d,y)}{\xi(k,a,b,c,d,y)} dk.
\]

(54)

The LHS of (54) is land used for local goods production times local goods output per unit land input, or total local goods output in the ring. The RHS is residential land divided by per capita land consumption, which equals ring population times per capita demand for the local good. The equality requires that output of local goods equal demand for them. Hence

\[
\lambda \equiv \frac{\xi}{\xi + \delta_2} = \lambda(k,a,b,c,d,y)
\]

(55)

is the fraction of land devoted to local goods production which guarantees market equilibrium at every distance. As before, a constant fraction of the ring labor force works locally. The fraction is
which, using (55), (53), and (2), reduces to \( \varepsilon = g_2 w_1 / m_1 + m_2 \).

Ring equilibrium comes about as in the previous model. Excess demand for local goods bids up the price of commercial real estate as local goods producers enter in response to an increased unit price for their output. This induces increased output of commercial real estate as new firms enter. These firms must bid land away from housing producers. Labor flows to the expanding local goods sector as local wages threaten to rise.

Profits for housing producers are \( \pi_H(c) \) as in (32) while profits for commercial real estate producers are analogously \( \pi_R(d) \). Now local goods producers' profits are

\[
(1-g_1-g_2-g_3) \left[ s \left( \frac{q_1}{q} \right) \frac{g_1}{w} \frac{g_2}{z} \frac{g_3}{1-g_1-g_2-g_3} \right]
\]

\[
= (1-g_1-g_2-g_3) \left[ \frac{g_1}{q} \frac{g_2}{g_3} \frac{g_2}{g_3} \frac{g_3}{1-g_1-g_2-g_3} \right] = \pi_G(b).
\]

We may now characterize general equilibrium. The conditions on CBD production are as follows:

\[
qf'(L) = y \tag{56}
\]

\[
Jqf(L) = \int_0^k \{ \lambda(q_1(k,a,b,c,d,y) + n_{Y_R}(k,a,b,c,d,y)) + (1-\lambda)n_{Y_H}(k,a,b,c,d,y) \} \xi(k)dk \tag{57}
\]
The first condition says that the competitive Q producers, whose production function is \( f \), maximize profits in their choice of labor input \( L \).

The second condition says that the value of Q production equals the value of the Q used in the city, corresponding to the first term in the integral, plus the value of N imported from the outside, which corresponds to the last two terms in the integral. Again we assume transport costs for N and Q are zero. The third condition states that each firm employs \( \frac{1}{J} \) of the labor commuting to the CBD, which is just equal to \( 1 - \varepsilon \) of the population of the urban area. Above, we have omitted the arguments of \( \lambda \).

We also have the requirement that the city house its population,

\[
\int_{0}^{k} \frac{(1 - \lambda) \xi(k)}{\xi(k,a,b,c,d,y)} \, dk = \bar{L},
\]

as well as the boundary condition

\[
x_0(a,b,c,d)w(k)^c_0 = r_A.
\]

We also have the three profit conditions

\[
\pi_G(b) = \bar{\pi}_G,
\]

\[
\pi_H(c) = \bar{\pi}_H,
\]

\[
\pi_R(d) = \bar{\pi}_R.
\]
Solving (58) and (61) - (63) and substituting the solutions for b, c, d, and L into the rest of the system reduces it to a four-equation system in the five unknowns, a, y, J, k. Again, comparative statics results are impracticable, but computer solution of the model may provide useful insights.

The model illustrates that a more complicated sectoral breakdown of the urban economy is possible and interesting. Indeed, a model which incorporates a land-using local goods sector as well as different types of labor could be constructed, escalating the level of realism even further. The local goods market equilibrium requirements for such a model would involve simultaneously determining land and labor allocation to the local goods sector in rings where different types of labor cohabit. The separate models, however, amply illustrate the problem of land and labor allocation to local goods producers.
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