Faculty Working Papers

EXPECTED-UTILITY-MAXIMIZING PRICE SEARCH WITH LEARNING[1]

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Summary:

We consider a model of search when the distribution of prices (wages) is unknown. The effect of changing the objective function from minimizing expected cost to maximizing expected utility is examined.

1. Introduction

A considerable literature[2/] analyzes the behavior of an individual who is to buy one unit of a good in a market with differing prices for the good. The consumer is assumed to sample, at a fixed cost, from a known probability distribution over a finite set of potential prices, and to minimize the expected sum of price plus search costs. The optimal strategy is characterized by a reservation price, with search ceasing as soon as a price quotation no higher than the reservation price is elicited.

Rothschild [2] is rightly critical of the assumption that the searcher behaves as if he knew the distribution of prices, and looks instead at an environment where a searcher with prior beliefs about the unknown price distribution "learns" through Bayesian updating of beliefs as prices are observed.

For this environment, the characterization which would be analogous to a reservation price is the reservation-price property: the existence of a threshold price function which determines a reservation price for each history of observed price offers. Rothschild establishes the reservation-price property for an expected-cost minimizing search strategy, in the case where the searcher's prior beliefs have a Dirichlet distribution (cf below). Bayesian updating with a Dirichlet prior has the crucial neutrality property; observing price p_i does not affect the relative perceived likelihood of prices p_j and p_k, $j, k \neq i$.

This paper considers the extension to an expected-utility-maximizing searcher with the option of purchasing different quantities of the commodity. In particular, we consider a searcher who derives utility from leisure and a consumption good. A price quotation for the good can be obtained by foregoing a fixed amount of leisure. After any price is observed, the searcher may exchange any (or all) of his (or her) remaining leisure for the consumption good at this price. The quantity ultimately purchased will in general depend upon the prices he has observed.

In the case where the commodity in question is a durable good, and the quantity purchased is invariant with respect to the price, we might expect that the quality of the good is not invariant. If so, the quality may be treated as a proxy for quantity purchased. Thus even for durable consumer goods expected-cost minimization may yield different results from expected-utility maximization.

We show that for expected-utility maximization the reservation-price property may not hold. This is demonstrated by means of a simple, non-pathological example where prior beliefs have a Dirichlet distribution, as in Rothschild.

We also show that if we assume a searcher's indirect utility function globally satisfies a particular inequality, then his expected-utility-maximizing stopping rule will exhibit the reservation-price property if he has a Dirichlet prior. The inequality which guarantees the reservation-price property is satisfied for all direct utility functions which are homogeneous.

2. Preferences and Beliefs

A searcher faces one market where the numeraire good, leisure, is exchanged for a consumption good. A price quotation can be elicited by foregoing c units of leisure.

Suppose the individual has decided to cease searching upon observing the (relative) price p. The quantity to purchase is then determined by the standard (full-information) consumer problem,[3/]

$$u(p,I) = \max_{x,L} w(x,L) \text{ subject to } I - L - px \leq 0 \qquad (1)$$

where $w(\cdot)$: $R_+^2 \to R^1$ is the direct utility function, (x,L) the vector of consumption of the purchased good and leisure, and I the "current income" or the finite leisure endowment $\bar{I}$ less leisure foregone for search. w is assumed to be concave and continuous. As properties of (1) have been well-studied, x, L, and $w(\cdot)$ will be submersed, and the analysis concerned with the indirect utility function u(p,I) and the decision whether to search again with current income, and quoted price, given updated beliefs.

The environment is the formal extension of Rothschild's model to expected-utility maximization, altered only to maintain consistency with that objective. Search is without recall, eliciting each time an element of the finite set

$$P = \{p_1, p_2, \ldots, p_n\}$$

labelled ascendingly, so $p_i < p_{i+1}$.[4/]

The actual (but unknown) probability distribution over P, a multinomial, is characterized by $\Pi = (\pi_1, \ldots, \pi_n) \in \Delta$, where Δ is the simplex in R_+^n, π_i the probability that any given price quotation will be p_i. The searcher updates by Bayes' rule a subjective prior distribution over Δ as price quotations are obtained. If this experience is summarized by $N = (N_1, \ldots, N_n)$, the number of times each price has been observed, let $\rho = {}^1/\Sigma N_i$ and $\mu_i = \rho N_i$, i = 1, ..., n. Then

$$(\mu,\rho) \in \Gamma = \Delta \times [0,1]$$

completely characterizes experience. With an observation of p_i, (μ,ρ) becomes

$$h_i(\mu,\rho) = \left(\frac{\mu_1}{1+\rho}, \ldots, \frac{\mu_{i-1}}{1+\rho}, \frac{\mu_i+\rho}{1+\rho}, \frac{\mu_{i+1}}{1+\rho}, \ldots, \frac{\mu_n}{1+\rho}; \frac{\rho}{1+\rho}\right) . \qquad (2)$$

Let $\lambda(\mu,\rho)$ be the current prior over P for a searcher with history (μ,ρ). Note that

$$\lim_{\rho \to 0} \lambda_i(\mu,\rho) = \mu_i, \ i = 1, \ldots, n . \qquad (3)$$

We maintain the assumption that the searcher's prior is a Dirichlet distribution (in which case updated beliefs are also Dirichlet), which has been completely characterized.

Proposition 1 (Rothschild [2]): A searcher has a Dirichlet prior if and only if it is possible to parametrize his experience so that

$$\lambda_i(\mu,\rho) = \mu_i \text{ for all } (\mu,\rho) \in \Gamma .$$

By (3) above, then, all searchers come with enough experience to behave as if their priors were Dirichlet.

3. The Optimal Strategy

Let

$$V_1(\mu,\rho,I) = \sum_i \lambda_i(\mu,\rho)\, u(p_i,I) . \qquad (4)$$

$V_1(\mu,\rho,I)$ is the expected (indirect) utility for a consumer allowed to search only once, assuming history (μ,ρ), and holding current income I (after costs of this one allowed search have been paid). Designate

$$V_T(\mu,\rho,I) = \sum_i \lambda_i(\mu,\rho) \max \{u(p_i,I),\ V_{T-1}[h_i(\mu,\rho),\ I-c]\} \qquad (5)$$

$V_T(\mu,\rho,I)$ is then the expected attainable utility for an individual who can search at most T times, has prior experience (μ,ρ), and income I after costs of the first search have been paid. Clearly

$$V_{T-1}(\mu,\rho,I) \leq V_T(\mu,\rho,I) \leq u(p_1,I)$$

for all T, so the $V_T(\mu,\rho,I)$ converge.

Defining

$$V(\mu,\rho,I) = \lim_{T\to\infty} V_T(\mu,\rho,I) , \qquad (6)$$

it is clear that

$$V(\mu,\rho,I) = \sum_i \lambda_i(\mu,\rho) \max\{u(p_i,I),\ V[h_i(\mu,\rho),I-c]\} , \qquad (7)$$

so the optimal stopping policy is to cease search, upon eliciting price p_i with experience (μ,ρ) and income I, if $u(p_i,I) \geq V[h_i(\mu,\rho),I-c]$, otherwise to search again.

Rothschild [2] proves that the optimal strategy for an expected-cost-minimizing searcher implies a finite number of searches. His argument extends to this model, although finite income implies finite search trivially.

4. Non-Reservation-Price Examples

In general, searchers who learn about the unknown distribution of prices do not exhibit the reservation-price property. Rothschild's example imagines that $P = \{1, 2, 3\}$, $c = .01$ $(\overline{I} \geq .5)$, and prior beliefs admit only two possible price distributions: a) prob (3) = 1, or b) prob (1) = .99, prob (2) = .01. Then an initial quote of 3 will end search, but an initial price of 2 will cause further search, as a much better price is expected.

In his example, a lower price offer has a direct effect, higher attaintable utility if search ceases $\{u(2,I) > u(3,I)\}$. It also has an indirect effect, measured in utility terms as $V[h_2(\mu,\rho),I-c] - V[h_3(\mu,\rho),I-c]$. In this case, the indirect effect of a lower price quote swamps the direct effect. When that cannot happen, the reservation-price property obtains.

Proposition 2. If for all $(\mu,\rho) \varepsilon \Delta$, all $I > 0$,

$$|u(p_k,I) - u(p_i,I)| \geq |V[h_k(\mu,\rho),I-c] - V[h_i(\mu,\rho),I-c]| \ , \tag{8}$$

then

$$V[h_i(\mu,\rho),I-c] \leq u(p_i,I) \leq u(p_k,I) \tag{9}$$

implies

$$V[h_k(\mu,\rho),I-c] \leq u(p_k,I) \ . \tag{10}$$

That (9) implies (10) is the reservation-price property. We adopt and maintain the notational convention $p_i > p_k$.

Proof: Trivial if $V[h_k(\mu,\rho),I-c] \leq V[h_i(\mu,\rho),I-c]$. If not, by (8),

$$u(p_k,I) - u(p_i,I) \geq V[h_k(\mu,\rho),I-c] - V[h_i(\mu,\rho),I-c] \ ,$$

so

$$u(p_k,I) - V[h_k(\mu,\rho),I-c] \geq u(p_i,I) - V[h_i(\mu,\rho),I-c] \geq 0 \ .$$

When the searcher's objective is to minimize expected cost, the indirect effect of a lower price quote is solely the informational value of that price (the expected-cost evaluation of the effect upon updated beliefs). In the case of a Dirichlet prior, the informational value of a lower price cannot swamp its direct effect, and Rothschild is able to prove that expected-cost-minimizing searchers with Dirichlet priors exhibit the reservation-price property.

However, the Dirichlet assumption is not sufficient to establish the reservation-price property for expected-utility-maximizing search, as now computing the indirect effect of observing a lower price must take into account the expected change in the marginal utility of leisure. That is, while the nominal cost of search is fixed, the expected cost in foregone utility will depend on price observations.

Consider a simple illustration. Let

$$w(x,L) = \begin{cases} x + 40\,, & \text{if } x \geq 2/3\,, \\ 122x - 40\ 2/3\,, & \text{if } 0 \leq x \leq 2/3\,, \end{cases}$$

$$P = \{{}^{1}/320,\ {}^{1}/4,\ 3\}, \text{ and}$$

$$(\mu,\rho) = \left({}^{1}/8,\ {}^{1}/8,\ {}^{3}/4;\ {}^{1}/4\right)$$

is the searcher's experience. Search cost, c, is 1, and income after the first search is 2.

It is obvious that one or two searches will be optimal. If the first search elicits p = 3, the posterior is $(\mu,\rho) = \left({}^{1}/10,\ {}^{1}/10,\ {}^{8}/10;\ {}^{1}/5\right)$. A second search would yield expected utility

$$EU = {}^{1}/10\ w(320,0) + {}^{1}/10\ w(4,0) + {}^{8}/10\ w({}^{1}/3,0) = 40.4\ .$$

Without search $w(2/3,0) = 40\ 2/3$ can be achieved, so search ceases.

Suppose instead the first search observed $p = {}^1/4$. The posterior is then $(\mu,\rho) = ({}^1/10,\ {}^3/10,\ {}^3/5;\ {}^1/5)$. A second search would yield expected utility

$$EU = {}^1/10\ w(320,0) + {}^3/10\ w(4,0) + {}^3/5\ w({}^1/3,0) = 49.2\ ,$$

while stopping after the first search can only attain $w(8,0) = 48$.

So, in this example, if the first search elicits the highest price, 3, search stops and 2 units are purchased. But if the first price quoted is lower, ${}^1/4$, expected utility is maximized by searching again. Note that allowing for partial purchase after the first search would not alter the example.

Also, as the example relies only upon the value of the indirect utility function at a finite number of points, a smooth utility function approximating the w function above would not change the conclusion. Similarly any sufficiently small perturbation of this utility function will give rise to the same search behavior. In particular, the preferences over x and L could have been strictly monotonic and convex without altering the conclusion.

5. A Reservation-Price Theorem

The following property is useful for reservation-price behavior:

$$|u(p_k,I-c) - u(p_i,I-c)| \leq |u(p_k,I) - u(p_i,I)|,\ \text{any}\ p_i,p_k \in P,\ \text{and}\ I > c\ .^{5/}\ (*)$$

Proposition 3: If $w(x,L)$, the direct utility function, is homogeneous of degree h, property (*) obtains, for all $h \geq 0$.

Proof:

$$w(\alpha x, \alpha L) = \alpha^h w(x,L) \text{ for all } \alpha > 0$$

implies $$u(p_i, \alpha I) = \alpha^h u(p_i, I) \text{ for all } \alpha > 0 .$$

Thus,

$$u(p_k, I-c) - u(p_i, I-c) = \left(\frac{I-c}{I}\right)^h [u(p_k, I) - u(p_i, I)] \leq u(p_k, I) - u(p_i, I) .$$

Theorem. Assume property (*), and that prior beliefs are Dirichlet. Then the stopping rule for an expected-utility-maximizing searcher exhibits the reservation-price property.

The proof, which is nearly parallel to Rothschild's argument, is presented in the Appendix.

6. Concluding Remarks

From the theorem we see that the reservation-price property obtains if the searcher has a (direct) utility function which is homogeneous of degree h, any $h \geq 0$. Thus, marginal utility of income can diminish (at arbitrarily high rates) without destroying the reservation-price property so long as it diminishes "regularly." In the example, the marginal utility of income is constant for low income, then diminishes, then is once again constant. This suggests that third-order considerations are related to the question of reservation price behavior.

Appendix

Theorem II. Assume property (*), and that prior beliefs are Dirichlet. Then the stopping rule for an expected-utility-maximizing searcher exhibits the reservation-price property.

Proof: Given Propositions 1 and 2 above, it is sufficient to prove that

$$\lambda_i(\mu,\rho) = \mu_i \qquad (11)$$

implies, for all positive integers, s:

$$|u(p_k,I) - u(p_i,I)| \geq |V[h_k^s(\mu,\rho),I-c] - V[h_i^s(\mu,\rho),I-c]| \ , \qquad (12)$$

where

$$h_i^s(\mu,\rho) = \left(\frac{\mu_1}{1+s\rho}\ , \ \ldots, \ \frac{\mu_i+s\rho}{1+s\rho}\ , \ \ldots, \ \frac{\mu_n}{1+s\rho}\ , \ \frac{\rho}{1+s\rho}\right) \ ,$$

a second-order approximation to the updating of experience which would result from observing price p_i s times.

It is sufficient to prove that, for all positive integers, t:

$$|u(p_k,I) - u(p_i,I)| \geq |V_t[h_k^s(\mu,\rho),I-c] - V_t[h_i^s(\mu,\rho),I-c]| \qquad (13)$$

which is done by induction. By (11),

$$V_1[h_i^s(\mu,\rho),I-c] = \frac{1}{1+s\rho} \sum_j [\mu_j u(p_j,I-c)] + \frac{s\rho}{1+s\rho} u(p_i,I-c) \ .$$

Thus,

$$V_1[h_k^s(\mu,\rho),I-c] - V_1[h_i^s(\mu,\rho),I-c]$$

$$= \left(\frac{s\rho}{1+s\rho}\right)[u(p_k,I-c) - u(p_i,I-c)] \leq u(p_k,I-c) - u(p_i,I-c)$$

$$\leq u(p_k,I) - u(p_i,I) ,$$

with the last inequality resulting from property (*). So (13) holds for t=1. The following lemmas show that if (13) is satisfied for t=T-1, it must be true for t=T.

Lemma 1. $u(p_k,I) \geq u(p_i,I)$ implies

$$V_T[h_k^s(\mu,\rho),I-c] \geq V_T[h_i^s(\mu,\rho),I-c] .$$

Proof: As shown above,

$$V_1[h_k^s(\mu,\rho),I-c] - V_1[h_i^s(\mu,\rho),I-c]$$

$$= \left(\frac{s\rho}{1+s\rho}\right)[u(p_k,I-c) - u(p_i,I-c)] \geq 0 .$$

Presume the lemma holds for t=T-1, and adopt

$$h_k h_i^s(\mu,\rho) = h_k[h_i^s(\mu,\rho)] = h_i^s h_k(\mu,\rho) .$$

Then,

$$V_T[h_k^s(\mu,\rho),I-c] - V_T[h_i^s(\mu,\rho),I-c]$$

$$= \left(\frac{1}{1+s\rho}\right) \sum_j \mu_j[\max\{u(p_j,I-c),\ V_{T-1}[h_k^s h_j(\mu,\rho),\ I-2c]\}$$

$$- \max\{u(p_j,I-c),\ V_{T-1}[h_i^s h_j(\mu,\rho),I-2c]\}]$$

$$+ \left(\frac{s\rho}{1+s\rho}\right)[\max\{u(p_k,I-c),\ V_{T-1}[h_k^{s+1}(\mu,\rho),I-2c]\}$$

$$- \max\{u(p_i,I-c),\ V_{T-1}[h_i^{s+1}(\mu,\rho),I-2c]\}] . \quad (14)$$

In the right-hand side of (14), the terms in brackets within Σ_j are of the form $J(A,B,C) = \max(A,B) - \max(A,C)$, with $A = u(p_j, I-c)$, $B = V_{T-1}[h_k^s h_j(\mu,\rho), I-2c]$, and $C = V_{T-1}[h_i^s h_j(\mu,\rho), I-2c]$. By the induction hypothesis, $B \geq C$, so all these terms are nonnegative.

The last term in brackets is of the form $J(A,B,C,D) = \max(A,B) - \max(C,D)$ with $A = u(p_k, I)$, $B = V_{T-1}[h_k^{s+1}(\mu,\rho), I-2c]$, $C = u(p_i, I)$, and $D = V_{T-1}[h_i^{s+1}(\mu,\rho), I-2c]$. By convention, $p_i \geq p_k$, so $A \geq C$, $B \geq D$ by the induction hypothesis, so this term is nonnegative. (14), a weighted average of nonnegative quantities, is nonnegative.

Lemma 2. If (13) holds for $t=T-1$, then

$$\max\{u(p_j, I-c),\ V_{T-1}[h_k^s h_j(\mu,\rho), I-2c]\}$$

$$- \max\{u(p_j, I-c),\ V_{T-1}[h_i^s h_j(\mu,\rho), I-2c]\}$$

$$\leq u(p_k, I) - u(p_i, I) \qquad (15)$$

Proof: The left-hand side of (15) is of the form $J(A,B,C) = \max(A,B) - \max(A,C)$, with $A = u(p_j, I-c)$, $B = V_{T-1}[h_k^s h_j(\mu,\rho), I-2c]$, and $C = V_{T-1}[h_i^s h_j(\mu,\rho), I-2c]$.

$$0 \leq B - C \leq u(p_k, I-c) - u(p_i, I-c) \leq u(p_k, I) - u(p_i, I)\ . \qquad (16)$$

The first inequality comes from Lemma 1, the second from the induction hypothesis, the third from property (*). If $C \geq A$, then $B \geq A$, and $J = B - C$. If $C < A$, either $B \leq A$ and $J = A - A = 0 \leq B - C$, or $B > A$ and $J = B - A \leq B - C$. So $J \leq B - C$, and (16) provides the desired conclusion.

Lemma 3: If (13) holds for $t = T-1$, then

$$\max\{u(p_k, I-c),\ V_{T-1}[h_k^{s+1}(\mu,\rho), I-2c]\}$$

$$- \max\{u(p_i, I-c),\ V_{T-1}[h_i^{s+1}(\mu,\rho), I-2c]\}$$

$$\leq u(p_k, I) - u(p_i, I)\ . \qquad (17)$$

Proof: The left-hand side of (17) is of the form

$J(A,B,C,D) = \max(A,B) - \max(C,D)$, where $A = u(p_k, I-c)$,

$B = V_{T-1}[h_k^{s+1}(\mu,\rho), I-2c]$, $C = u(p_i, I-c)$, and $D = V_{T-1}[h_j^{s+1}(\mu,\rho), I-2c]$.

By Lemma 1, $B \geq D$.

$$0 \leq A - C \leq u(p_k, I) - u(p_i, I) \tag{18}$$

The first inequality is by convention, the second is property (*). To show $J \leq A - C$, four cases must be considered:

1. $A \geq B$, $C \geq D$, so $J = A - C$
2. $A \geq B$, $C < D$, so $J = A - D \leq A - C$
3. $A < B$, $C < D$, so $J = B - D \leq A - C$ by the induction hypothesis.
4. $A < B$, $C \geq D$. This implies $B - D > A - D \geq A - C$, which would contradict the induction hypothesis, so this case may be discarded.

Thus, $J \leq A - C$, and (18) provides the conclusion.

Lemma 4: If (13) holds for $t = T-1$, it holds for $t=T$.

Proof: Use Lemmas 2 and 3 to calculate:

$$V_T[h_k^s(\mu,\rho), I-c] - V_T[h_i^s(\mu,\rho), I-c]$$

$$= \left(\frac{1}{1+s\rho}\right) \sum_j \mu_j[\max\{u(p_j, I-c), V_{T-1}[h_k^s h_j(\mu,\rho), I-2c]\}$$

$$- \max\{u(p_j, I-c), V_{T-1}[h_i^s h_j(\mu,\rho), I-2c]\}]$$

$$+ \left(\frac{s\rho}{1+s\rho}\right)[\max\{u(p_k, I-c), V_{T-1}[h_k^{s+1}(\mu,\rho), I-2c]\}$$

$$- \max\{u(p_i, I-c), V_{T-1}[h_i^{s+1}(\mu,\rho), I-2c]\}]$$

$$\leq \left(\frac{1}{1+s\rho}\right) \sum_j \mu_j[u(p_k, I) - u(p_i, I)]$$

References

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