Relevant and Irrelevant Internal Rates of Return

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ABSTRACT

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One problem with internal rate of return is the possibility that given cash flows may result in more than one internal rate of return. The purpose of this paper is to develop a method for determining relevance of an internal rate of return. An internal rate of return is considered relevant if its derivatives with respect to each of the cash flows are positive. It is determined that the necessary and sufficient conditions for relevance are: (1) the net future value function has negative derivative at a relevant root; and (2) a relevant internal rate of return must be greater than -1.
One problem often associated with the use of the internal rate of return is the possibility that given cash flows may result in more than one internal rate of return. Within three years of Lorie and Savage's 1955 article [12] with its famous pump example, at least three solutions to the problem of multiple internal rates of return were proposed.

The first of these solutions was proposed by Solomon [19] and involves the use of an assumed reinvestment rate. Teichroew, et. al., [20] later extended this train of thought by assuming a project financing rate (cost of capital) that could be different from the reinvestment rate.

The second solution was proposed by Hirshleifer [7, p. 352]. He suggested not using the internal rate of return for longer than two-period comparisons. This approach was subsequently extended by Bailey [1].

The third solution, and the one rationalized and qualified in this paper, was initially suggested by Lorie and Savage [12, p. 237] and involves a criterion for distinguishing between relevant and irrelevant rates. The term relevant rate of return comes from a footnote in the Lorie and Savage article [12, p. 237].

Lorie and Savage asserted that a relevant rate of return is found where the net present value is a decreasing function of the discount rate. Although they did not say so, we suspect that their rationale might have been to identify an internal rate of return that when compared to the required rate of return may yield a decision consistent with the positive net present value (NPV) criterion. Solomon [19, pp. 128 & 129] was the first to hint at the rationale that is used in this paper;
relevance relates to the investor's view of what should happen to a rate of return given various changes in cash flows. It would be possible to go through the same analysis, but with different results, to define a relevant rate for a borrower.

Although the third solution was subsequently used by McLean [14, p. 67], Massé [13, pp. 20-23], and Wright [23], it has been dismissed by others. Quirin stated that, "Mathematically, there are no grounds for distinguishing one rate from another, as all are roots of a single polynomial equation" [16, p. 54]. Jean has said that, "There seems to be no particular justification for this selection method for the situation of multiple rates of return ..." [9, p. 28]. In a more general criticism, Ramsey stated that the search for a method to differentiate roots and internal rates of return, "was mistaken in its aims" [17, p. 1022].

Like Lorie and Savage, neither McLean nor Massé provided a rationale for the criterion of declining net present value at a relevant rate and Wright's attempt to do so essentially resulted in a restatement of the criterion. Further, no basis was provided for choosing among more than one internal rate of return which meets the criterion of declining net present value. Massé attempted this by providing a second criterion, stating that the relevant rate is the rate closest to the prevailing rate of interest [13, p. 22]. Massé did not provide a rationale for his second criterion.

The overall purpose of this paper is to develop a method for determining the relevance of an internal rate of return. There are four specific objectives: (1) to provide a rationale for determining
the economic relevance of an internal rate of return (i.e., essentially to define relevance); (2) to show that a condition similar to the Lorie and Savage criterion (Masse's first criterion) is a necessary condition for a relevant internal rate of return; (3) to identify a second condition (different from Masse's second criterion) which together with the first criterion provide the necessary and sufficient conditions for a relevant internal rate of return; and, (4) to develop the implications of certain patterns of cash flow for the relevance of internal rates of return.

I. THE CONDITIONS FOR RELEVANCE

An internal rate of return of an investment over n years is a real number i* satisfying

$$\sum_{j=0}^{n} \frac{I_j}{(1-i^*)^j} = 0 \quad (1)$$

where $I_j$ is the net income at the end of the jth period. For any $j = 0, 1, \ldots, n$, $I_j$ may be positive, negative or 0. The internal rate of return is not necessarily unique, but certain values of $i^*$ can be rejected as being not relevant, and the others accepted as being relevant. A value of $i^*$ will be referred to as relevant if it admits to sensible economic interpretation in the context of changing cash flows. For example, if the cash flow for any one period is increased, a relevant internal rate of return will increase.

By letting $x^* = 1 + i^*$ and multiplying (1) through by $(x^*)^n$, one sees that (1) is equivalent to

$$I_0(x^*)^n + I_1(x^*)^{n-1} + \ldots + I_{n-1}x^* + I_n = 0. \quad (2)$$
For different rates of discount $i$, $I_0x^n + I_1x^{n-1} + \ldots + I_n$ is the net future value (NFV) at time $n$ of the cash flows at the rate of discount $i = x - 1$. So, in particular, when $i = i^*$, NFV = 0.

To help visualize what follows, consider the case $n = 2$, where $I_0 < 0$. The graph of $NFV = I_0x^2 + I_1x + I_2$ may appear as in Figure 1. There are two roots, $x^*_1$ and $x^*_2$. If $I_2$ increases to $I_2'$ with $I_0$ and $I_1$ remaining fixed, we get a new graph (dashed line) and $x^*_2$ has moved to the new position $x^*_2'$ as shown in Figure 2. The increase in $I_2$ represents a larger income to the investor and so should be reflected in an increase in the internal rate of return, which is the case with $x^*_2$. The roots of (2) that increase when any $I_j$ increases, the other $I_k$'s remaining constant, are relevant roots. All other roots are irrelevant. For example, the negative root $x^*_1$ in Figure 2 is irrelevant because it decreases when $I_2$ increases.

**Definition.** A root $x^*$ of (2) is called relevant if: 1) whenever $I_j$ increases (decreases), for any $j$, all other $I_k$'s being held constant, then $x^*$ also increases (decreases) and 2) simultaneous "small" changes in $I_0$, $\ldots$, $I_n$ result in a polynomial with just one root "near" $x^*$.

The internal rate of return $i^* = x^* - 1$ is relevant if $x^*$ is relevant.

Although we are using the words "small" and "near" in an intuitive sense, the concepts here have a rigorous foundation. We are concerned that the conditions of the implicit function theorem can be satisfied. The interested reader is referred to Bartle [2, p. 261].

With the definition of relevant internal rate of return, one may give a simple condition for deciding on the relevance of $i^*$ provided that $x^*$ is a simple root of (2). The condition actually provides an
Figure 1
Figure 2
algorithm with wide application. Unfortunately, the mathematical machinery breaks down when \( x^* \) is a repeated root of (2). This breakdown is reflected in the economic interpretation of relevance.

**Theorem:** Let \( I_0, I_1, \ldots, I_n \) be the cash flow associated with a particular investment and let \( i^* \) be an internal rate of return for the investment, i.e.,

\[
\sum_{j=0}^{n} \frac{I_j}{(1+i^*)^j} = 0.
\]

Further, assume that \( x^* = 1 + i^* \) is a simple root of (2), \( I_0(x^*)^n + I_1(x^*)^{n-1} + \ldots + I_n = 0 \) (that is, \( (x-x^*)^2 \) is not a factor of the polynomial and therefore \( x^* \) is not a repeated root). Let

\[
NFV(x) = I_0x^n + I_1x^{n-1} + \ldots + I_n.
\]

Then \( i^* \) is a relevant internal rate of return if and only if \( i^* > -1 \) and \( NFV'(x^*) < 0 \).

The proof of this theorem is divided into two main parts. Part A below shows that if the slope of the NFV function is negative, internal rates of return greater than negative one are relevant. Part B shows that: (1) internal rates of return of less than negative one are irrelevant; (2) internal rates of return equal to negative one are irrelevant, and; (3) internal rates of return must be irrelevant if the slope of the NFV function is positive.

**Proof:** A. Assume \( i^* > -1 \) or \( x^* > 0 \) and \( NFV'(x^*) < 0 \). The graph of NFV against \( x \) (for a fixed set of values \( I_0, I_1, \ldots, I_n \)) has negative slope at \( x^* \) (see Figure 1) and \( x^* > 0 \).

Now consider the expression \( G(I_0, I_1, \ldots, I_n, x^*) = I_0(x^*)^n + I_1(x^*)^{n-1} + \ldots + I_n = 0 \), where for different values of...
$I_0, I_1, \ldots, I_n$, we can solve for different values of $x^*$. For the particular values of $I_0, \ldots, I_n$ associated with the investment, one of the roots of $C(I_0, I_1, \ldots, I_n, x^*) = 0$ would be the $x^*$ in the statement of the theorem.

By differentiating $C(I_0, I_1, \ldots, I_n, x^*) = 0$ implicitly (see Bartle [2, p. 261] or Purcell [15, p. 729]), and then evaluating at the given values of $I_0, \ldots, I_n, x^*$, we find that

$$\frac{\partial x^*}{\partial I_j} = -\frac{\partial C}{\partial I_j} \text{ } \frac{\partial C}{\partial x^*},$$

however,

$$\frac{\partial C}{\partial x^*} = NFV'(x^*) < 0, \text{ and } \frac{\partial C}{\partial I_j} = (x^*)^{n-j}, \text{ so}$$

$$\frac{\partial x^*}{\partial I_j} = -\frac{(x^*)^{n-j}}{NFV'(x^*)}.$$  \hspace{1cm} (3)

For $x^* > 0$, since $NFV'(x^*) < 0$, clearly $\frac{\partial x^*}{\partial I_j} > 0$ so $x^*$ increases (decreases) with increasing (decreasing) $I_j$ (all j) and so $x^*$ is a relevant root, i.e., $i^*$ is a relevant internal rate of return.

F. Conversely, suppose $i^*$ is a relevant internal rate of return. Then $x^*$ is a relevant root of (2). Since we have assumed that $x^*$ is not a repeated root of (2), it follows that $NFV'(x^*) \neq 0$ and so

$$\frac{\partial C}{\partial x^*} = 0. \text{ Thus, we may again use the implicit function theorem }$$

and equation (3).

In case $x^* < 0$, as $n - j$ varies from 0 to $n$, (3) implies that $\frac{\partial x^*}{\partial I_j}$ alternates signs. Thus, $x^*$ decreases with increasing $I_j$ for alternate j's regardless of the sign of $NFV'(x^*)$. Thus, $x^*$ is not relevant.
In case \( x^* = 0 \), since \( x^* \) satisfies \( NFV(x^*) = 0 \), we have
\[
I_0(x^*)^n + \ldots + I_{n-1}x^* + I_n = 0, \text{ so } I_n = 0.
\]
Since \( I_n = 0 \), a change in any \( I_j \), \( j = 0 \) to \( n - 1 \), results in a polynomial in which \( x^* = 0 \) is still a root so that \( x^* \) is irrelevant. Thus, the relevance of \( x^* \) implies \( x^* > 0 \).

Given \( x^* > 0 \), it is possible to show the necessity that \( NFV'(x^*) \) be less than zero for relevance. Suppose that \( NFV'(x^*) > 0 \). Then using (3), \( \frac{2x^*}{I_j} \) would be negative for all \( j \). This would imply that \( x^* \) decreases with respect to increasing \( I_j \), contradicting that \( x^* \) is relevant. Thus, we must have \( NFV'(x^*) < 0 \). QED

Unfortunately we do not have straight-forward conditions when \( x^* \) is a repeated root of (2). The following examples illustrate the difficulties of the repeated root case.

First, consider the case \( n = 2 \), where \( NFV(I_0, I_1, I_2, x^*) = -(x^*)^2 + 2x^* - 1 = 0 \), i.e., \( I_0 = -1, I_1 = 2 \) and \( I_2 = -1 \). We can think of this as being a cash flow where \( $1 \) is invested, income is a positive \( $2 \) at the end of period 1, and income is a negative \( $1 \) at the end of period 2. The graph of \( NFV = -x^2 + 2x - 1 \) is shown in Figure 3. Note that if \( I_2 \) increases ever so slightly, with \( I_0 \) and \( I_1 \) remaining fixed, then the graph becomes as shown in Figure 4. We observe that the repeated root \( x^* \) has been replaced by two roots, one greater than and one less than \( x^* \). Because of this difficulty, we cannot consider \( x^* = 1 \) as a relevant root in this case.

The case where \( NFV \) has a graph such as shown in Figure 5 is less apparent. This case is especially important because it meets Lorie and Savage's
Figure 3
Figure 4
criterion for relevance. Assume \( NFV = -(x-1)^3 = -x^3 + 3x^2 - 3x + 1 \) and \( x^* = 1 \) is a repeated root of \( NFV \). Here \( I_0 = -1, I_1 = 3, I_2 = -3, \) and \( I_3 = 1 \). We note that \( \frac{\partial G}{\partial x^*} = 0 \) for these values at \( x^* = 1 \).

Now if we change two of the values ever so slightly, we run into a problem similar to that in the quadratic example. Consider the polynomial:

\[
NFV = -(x-1)(x-1+\epsilon)(x-1-\epsilon)
\]

\[
= -x^3 + 3x^2 - (3-\epsilon^2)x + (1-\epsilon^2).
\]

This results in increasing \( I_2 \) by \( \epsilon^2 \) and decreasing \( I_3 \) by \( \epsilon^2 \) simultaneously. The graph of this polynomial is shown in Figure 6.

In the conditions of the theorem, since \( NFV'(x^*) \neq 0 \), it also holds that \( \frac{\partial G}{\partial x^*} \neq 0 \) at the given values of \( I_0, \ldots, I_n \) and the value \( x^* \) such that \( G(I_0, \ldots, I_n, x^*) = 0 \). Thus, by the implicit function theorem, a number of small changes in \( I_0, \ldots, I_n \) yield just one new value for \( x^* \). However, in the example under discussion, the two small changes in \( I_2 \) and \( I_3 \) (\( I_2 \) changing from \(-3\) to \(-3+\epsilon^2\) and \( I_3 \) from \(1\) to \(1-\epsilon^2\)) result in three roots of the new polynomial, namely \(1-\epsilon, 1, \) and \(1+\epsilon\). Since in practice we would want to make simultaneous changes in the cash flow, we see that the case where \( x^* \) is a repeated root leads to ambiguity.

II. SIMPLE CASES

In the cases where \( n = 1 \) or \( 2 \), application of the conditions for relevance is simple since the existence of uncomplicated algebraic solutions and Descartes' rule of signs [21, p. 125] provide for easy analysis. With higher degrees, Descartes' rule of signs involves more
cases to analyze \([2^{n+1}] \) cases, where \(n\) is the degree of \(NFV(x)\). For \(n = 3, 4\), while algebraic formulas for the roots of an \(n\)-degree polynomial exist, they are complicated. In the cases where \(n \geq 5\), the results of Galois theory [8, p. 302] show that there cannot exist algebraic solutions for the roots of a general \(n\)th degree polynomial. [By an algebraic solution for a root of a general \(n\)th degree polynomial, we mean a formula that would express the roots as functions of the coefficients where the function involves the usual arithmetic operations (i.e., addition, subtraction, multiplication, and division) and the taking of roots of various orders.]

When \(n = 1\), \(NFV(x) = I_0 x + I_1\). Since we require that \(NFV'(x^*) < 0\), we immediately see that we require \(I_0 < 0\). Since we also require that \(x^* > 0\), we also need \(I_1 > 0\). The graph of \(NFV\) under these conditions is as shown in Figure 7. It is clear that if \(I_1\) increases, \(x^*\) increases. Similarly, if \(I_0\) increases (since \(I_0 < 0\), this means \(|I_0|\) decreases), \(-I_1/I_0\) increases and again \(x^*\) increases.

In the other three subcases of the linear case, \(x^*\) is not a relevant root even though there is only one root, which is easily seen either analytically or graphically. A simple summary for the

<table>
<thead>
<tr>
<th>(I_0)</th>
<th>(I_1)</th>
<th>(x^*) Relevant</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>+</td>
<td>Yes</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>No</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>No</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>No</td>
</tr>
</tbody>
</table>
Figure 7
linear case is presented in Table I. Table I shows that \( x^* \) is a relevant root, when \( n = 1 \), if and only if \( I_0 < 0 \) and \( I_1 > 0 \).

It is not clear from Table I what the situation is if \( I_0 \) or \( I_1 = 0 \). Should \( I_0 = 0 \), then \( NFV = I_1 \) and there are no roots of \( NFV = 0 \). Therefore, there is no internal rate of return when there is no initial investment, either positive or negative. Should \( I_1 = 0 \), then \( NFV = I_0 x^* \) and \( NFV = 0 \) only if \( x = 0 \) (not a relevant root) or \( I_0 = 0 \), for which it has just been shown that there are no roots.

A somewhat more complicated analysis is required in the case of \( n = 2 \). Our conditions are quite useful in determining whether a root is relevant or irrelevant based on the signs of \( I_0 \), \( I_1 \), \( I_2 \).

Since we must rule out the case of a repeated root of \( NFV = 0 \), we note that \( I_0(x^*)^2 + I_1 x^* + I_2 = 0 \) has a repeated root if and only if \( I_1^2 - 4 I_0 I_2 = 0 \). Also, there are no real roots if \( I_1^2 - 4 I_0 I_2 < 0 \) since the roots of \( I_0(x^*)^2 + I_1 x^* + I_2 = 0 \) are \((-I_1 \pm \sqrt{I_1^2 - 4 I_0 I_2})/2I_0 \). Thus, we henceforth assume \( I_1^2 - 4 I_0 I_2 > 0 \).

Since for \( n = 1 \) the graph of \( NFV(x) \) is a parabola, we see that for \( I_0 > 0 \), the slope at the left most root \( x_L^* \) of \( I_0(x^*)^2 + I_1 x^* + I_2 = 0 \) is negative, while the slope at the right most root \( x_R^* \) is positive. Thus, in this case we can only have a relevant root at \( x_L^* \).

By our Theorem in Section II, we also require \( x_L^* \) to be positive if it is to be relevant. Thus, we require that \( I_0(x^*)^2 + I_1 x^* + I_2 = 0 \) have two positive roots \((x_L^* > 0 \text{ implies } x_R^* > 0)\). By Descartes rule of signs, there are two (or zero) positive roots if there are two changes in sign. Since we have assumed \( I_0 > 0 \), this now happens only if \( I_1 < 0 \) and
From our underlying assumption that $I_1^2 - 4I_0 I_2 > 0$, we have exactly two positive roots and so $x_L^*$ is relevant.

When $I_0 < 0$, then the slope of $NFV(x)$ is positive at $x_L^*$ and negative at $x_R^*$, so only $x_R^*$ can qualify as a relevant root. To be so, $x_R^*$ must be positive. Again using Descartes rule of signs, we see that the $NFV(x) = 0$ has a positive root when there is just one change of sign or, under the assumption $I_1^2 - 4I_0 I_2 > 0$, when there are two changes of sign. Using this fact, we can now construct Table II below.

Table II

<table>
<thead>
<tr>
<th>$I_0$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$x_L^*$ Relevant</th>
<th>$x_R^*$ Relevant</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>No</td>
<td>No</td>
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<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>No</td>
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<td>+</td>
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<td>Yes</td>
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<td>-</td>
<td>-</td>
<td>No</td>
<td>No</td>
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</tbody>
</table>

provided $I_1^2 - 4I_0 I_2 > 0$

Table II shows that $x_L^*$ is a relevant root, when $I_0 > 0$, if and only if $I_1 < 0$ and $I_2 > 0$: $x_R^*$ is never a relevant root when $I_0 > 0$.

Table II also shows that $x_L^*$ is never relevant when $I_0 < 0$ and
that $x^*_R$ is relevant when $I_0 < 0$ except when both $I_1$ and $I_2$ are less than zero. It should be noted that there are other analytic approaches that would lead to the same table.

One can also analyze the case $n = 2$ where some of the $I_j$'s are zero. Working with the conditions that $NFV'(x^*) < 0$ and $x^* > 0$, one can determine that relevant roots exist as follows:

1) $I_0 = 0, I_1 < 0, I_2 > 0$
2) $I_0 < 0, I_1 = 0, I_2 > 0$
3) $I_0 < 0, I_1 > 0, I_2 = 0$

We leave the details of this to the reader and also leave to the reader that there is not a relevant root in case two of the $I_j$'s are 0.

III. APPLICATIONS

In practice, our conditions can be applied for any number of periods $(n)$ to determine whether a root $x^*$ (and hence, the internal rate of return, $i^* = x^* - 1$) is relevant. From the cash flow $I_0, I_1, ..., I_n$, determine

$$NFV(x) = I_0 x^n + I_1 x^{n-1} + ... + I_n.$$ 

By any standard method (e.g., Newton's method, successive bisection, etc.), determine the roots $x^*$ of $NFV(x) = 0$. If $n$ clearly distinct values are obtained, all roots are simple roots and one need only check the sign of $NFV'(x^*)$ for each $x^*$ greater than zero. Care should be exercised to make sure that rounding errors are not affecting the
signs of \( x^* \) and \( NFV'(x^*) \). As long as \( x^* \) and \( NFV'(x^*) \) are not too close to 0, this will, in general, not be a problem.

In case \( n \) distinct roots of \( NFV(x) = 0 \) are not clearly obtained, there could be some difficulty, especially since a root could be a repeated root. This would require a more sophisticated approach to the numerical analysis than we pursue here. However, if one does do the required numerical analysis, our conditions would be applicable once a root is determined to be a simple root.

As indicated previously, the algebraic solution for the roots of the case \( n = 3 \) is too complicated for a brief analysis. However, a numerical example for \( n = 3 \) is presented to illustrate the application of the conditions for relevance. Assume that \( I_0 = 1 \), \( I_1 = -6 \), \( I_2 = 10.8 \), and \( I_3 = -5.7 \). A summary of the pertinent data is presented in Table III.

<table>
<thead>
<tr>
<th>( x^* )</th>
<th>( i^* )</th>
<th>( NFV'(x^*) )</th>
<th>Relevant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.949</td>
<td>-0.051</td>
<td>2.114</td>
<td>No</td>
</tr>
<tr>
<td>1.915</td>
<td>0.916</td>
<td>-1.179</td>
<td>Yes</td>
</tr>
<tr>
<td>3.133</td>
<td>2.133</td>
<td>2.665</td>
<td>No</td>
</tr>
</tbody>
</table>

It can be seen from Table III that there are three roots. Applying the conditions for relevance reveals that only one root is relevant. Therefore, it is possible to select one relevant root even when there are more than two roots. However, if the signs for each \( I_j \) in this example were reversed, a graph of \( NFV(x) \) would appear as in Figure 6.
Two roots would satisfy the conditions for relevance and there would be no basis for choosing between them. This limitation is discussed in the following section.

IV. A LIMITATION

In a case where more than one relevant root is found, our analysis cannot be extended to choose among the relevant roots. That is, a single truly relevant root cannot be chosen from among two or more roots that meet the conditions for relevance.

It can be shown for the example illustrated in Figure 6 that two of the roots I-ε and I+ε meet the conditions for relevance. It is possible to add a marginal cash flow that will cause the relevant roots of the base cash flow reflected in Figure 6 to move in opposite directions. When this is done, in some cases both roots move in sensible directions (i.e., toward a relevant root of the marginal cash flow) and in some other cases the one root that moves in a sensible direction depends on the characteristics of the marginal cash flow (i.e., change that character and the root that moves in a sensible direction may change).

To illustrate a case in which the relevant roots move in opposite, but sensible directions, assume the marginal cash flow is such that

\[ \frac{d}{dx} \text{NPV} = \begin{cases} \text{positive and small} & \text{if } x < 1 \\ \text{positive and small} & \text{if } x > 1 \end{cases} \]

The net future value function of the marginal cash flow shown as the dashed line in Figure 7 has one relevant root at \( x^* = 1 \). Adding the marginal cash flow to the base cash flow illustrated in Figure 7 causes the left-most relevant root of the base cash flow to increase and the right-most one to decrease. Both sensibly move towards the relevant root of the marginal cash flow as indicated by the arrows in Figure 8.
To illustrate a case in which only one root moves in a sensible direction, assume the marginal cash flow is such that $\delta$ is added to $I_0$, $(\delta + \delta \gamma)$ is subtracted from $I_1$ and $\delta \gamma$ is added to $I_2$ of the base cash flow, where $0 < \gamma < 1 - \epsilon$. The net future value of this marginal cash flow is a convex, quadratic function of $x$ as shown by the dashed line in Figure 9. The marginal cash flow has two roots, a relevant root at $\gamma$ and an irrelevant root at $1$. Adding this cash flow to the base cash flow illustrated in Figure 6 causes the left most root of the base cash flow to decrease (i.e., move toward the relevant root of the marginal cash flow). The right most root increases (i.e., moves away from the relevant root of the marginal cash flow). This might suggest that the left most root is the one truly relevant root. However, counter examples can be found.

Consider a marginal cash flow such that $\delta$ is subtracted from $I_0$, $\delta (2 + \epsilon + \gamma)$ is added to $I_1$ and $\delta (1 + \epsilon + \gamma)$ is subtracted from $I_2$ of the base cash flow. Net future value of this marginal cash flow is a concave, quadratic function of $x$ as shown by the dashed line in Figure 10. It has two roots, a relevant root at $1 + \epsilon + \gamma$ and an irrelevant root at $1$. Adding this marginal cash flow to the base cash flow again causes the left most root of the base cash flow to decrease, but this time it moves away from the relevant root of the marginal cash flow. The right most root again increases, but this time it moves toward the relevant root of the marginal cash flow, suggesting the right most root is the one truly relevant root. Therefore, neither relevant root of the base cash flow may be chosen over the other.
V. CONCLUSIONS

It is possible to determine the economic relevance of an internal rate of return for a cash flow sequence which yields a polynomial with no repeated roots. There are two conditions, each of which is necessary for relevance. One necessary condition is that the net future value function have a negative derivative at a relevant root. The second necessary condition is that a relevant internal rate of return must be greater than -1. Together, these two conditions are sufficient for relevance. The rationale for these conditions is that any marginal increase (decrease) in an income will result in an increase (decrease) in a relevant internal rate of return.
References


