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Preferential Factors in Order of Operations

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UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

University of Illinois Committee on School Mathematics

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Abstract

An experiment was designed to uncover (1) preferences in order of operations which 7th-grade children posess before instruction and (2) the extent to which variable factors in expressions like '5 + 3 × 8' affect their performance. Fifteen of 38 subjects exhibited 1 of 2 stable "rules" in the first 10 of 40 trials; 32 exhibited 1 of 5 in the last 10. Two "rules", multiplication first and a left-to-right order, predominated. Anova revealed that spacing within the expression interacted strongly with the tendency to work from left to right. A byproduct finding: confusion of multiplication signs with addition signs occurred in about 1 trial in 10, but the frequency is reduced about 50% when the dot is used as a multiplication sign rather than the cross.

Introduction

It is not unusual for teachers to find college freshmen who seem to be unaware that a convention exists for order of operations in expressions such as (*) in which multiplication appears with addition. The surprise which these

$$5 + 3 \times 8 \qquad (*)$$

students show when the "ambiguity" of such an expression is demonstrated suggests an interesting possibility. In order for an expression to appear ambiguous, more than one simplification must be evident. If only one solution is evident to a student, it is possible that he is either acting under an idiosyncratic set of rules of order of operation (which may or may not coincide with the conventional "Multiplication before Addition Order"), or he is under the impression that associativity holds for all operations. In the first case no ambiguity would result because the idiosyncratic rule used would permit only one method of solution, as does the standard rule of order of operations. In the second case the student would admit that more than one method of solution was possible but would assume that all of these produced the same result.

The experiment reported in this paper was designed to detect preferences for order of operations and to determine the factors involved in the rules for order of operations which are used by individuals but which are not based on instruction in the usual order of operations convention. Typically, entering seventh graders have seen indicated sums and indicated products written in horizontal form but have not mastered the convention for order of operations for expressions such as (*) (although some of them have encountered it). "Average" seventh graders were selected as subjects in an experiment conducted during the first month of school, before they had received any instruction on order of operations.

Experimental Design

The experiment reported in this paper was designed to detect untaught preferences for order of operations and to determine factors involved in the formation of such rules. Display (*) is an example of such an expression. Four different factors were varied to determine possible effects on preference in order of operations: (1) order of presentation, i.e., the multiplication sign preceded the addition sign or vice versa, (2) relative spacing, i.e., the distance between the centers of the 1st and 2nd numerals and the distance between the centers of the 2nd and 3rd numerals were in the ratio 7 : 3, 3 : 2, 1 : 1, 2 : 3, or 3 : 7, (3) form of multiplication sign used, either "×" or "•", and (4) relative difficulty of the multiplication and the addition. With respect to the latter factor an operation of either kind was defined as relatively "easy" when the sum of operands was 10 or less and "difficult" when the sum of operands exceeded 10. Exactly one of the two operations was always in the difficult category. Thus in expression (*) the addition, 5 + 3, would be considered "easy" and the multiplication, 3 × 8, "difficult". It should be noted that this definition applies only to that operation which the student performs first.

When an expression such as (*) is considered as a whole, the use of the conventional order will insure that the smaller of the two answers will be found. The assumption was made that, of the computations the student must perform in simplifying an expression, only the difficulty of the first would influence his choice of the order in which to compute. If this assumption is true, the larger answer and possibly greater difficulty encountered by adding first would not influence the student's preference.

Since 40 combinations of the factors mentioned are possible, this was the minimum number of trials necessary for a complete factorial experimental design. Accordingly, 40 problems of the forms shown in Fig. 1 were written and reproduced on 3" × 5" cards. (The complete set is reproduced in Appendix A.)

$$A \times B + C \qquad A \cdot B + C$$

$$A + B \times C \qquad A + B \cdot C$$

Fig. 1

Problem Forms

In the expressions presented to the students 'A', 'B', and 'C' were replaced by different numerals ranging from '2' through '9'. Ten problems were made up in each of the four forms shown. The ten problems showed each of the five spacing ratios, mentioned on the previous page (and exhibited in Appendix A) with either $A + B > 10$ and $B + C \leq 10$ or with $A + B \leq 10$ and $B + C > 10$. Subjects were shown a card with one sample problem which used a cross multiplication sign and one problem which used a dot multiplication sign. Each student was asked if he or she was familiar with both the dot and cross signs. Less than 25% had never used the dot before. These students were told, "This sign means exactly the same thing as the cross multiplication sign." All students were able to solve the two sample problems without further aid. None of the students who stated that they had never used the dot multiplication sign before showed any difficulty in understanding the function of the dot either on the sample or on subsequent problems. The 40 problems in Appendix A were shown in the indicated order to each of the students. It should be noted that for individual problems of this type any <u>rule</u> followed by the student can lead to one of only two possible orders of operation.

Answers obtained by <u>either</u> of the two possible orders of operation for a given problem were accepted, provided that no error was made in any of the computations. Where errors were due to incorrect multiplication or addition or to misreading of operation signs, the error type was determined (by questioning the student where necessary) and recorded. The subject was then asked to "try again" or "look at the problem a little more carefully" until one of the two acceptable answers was given.

Results

One advantage in noting all errors was a demonstration of the confusion resulting from use of the cross (×) multiplication sign. In the total of 1520 problems solved by the 38 subjects there were 273 errors due either to incorrect multiplication and addition or to misreading the problem. About 53% (145) of all the errors were due to errors in multiplication or addition while about 47% (128) of them were due to one of three types of reading errors. Table 1 shows these reading errors and the relative number of errors when a dot or a cross multiplication sign was used. The difference is significant in favor of the dot ($\chi^2 = 7.61$, df = 2, $P < .025$).

Error	Dot	Cross	Total
Multiplication sign read as addition sign	13	23	36
Addition sign read as multiplication sign	22	30	52
Multiplication and addition signs transposed	14	26	40
	49	79	128

Table 1

Frequencies of Reading Errors

Source	df	MS	F	P
M	1	1.27	7.65	>.25
S	4	.09	.54	>.25
O	1	86.21	510.	<.05
D	1	.38	2.28	>.25
M × S	4	.33	1.99	>.25
M × O	1	.18	1.08	>.25
M × D	1	.13	.78	>.25
S × O	4	4.13	24.8	<.005
S × D	4	.13	.78	>.25
O × D	1	.04	.24	>.25
M × S × O	4	.18	1.08	>.25
M × S × D	4	.12	.72	>.25
S × O × D	4	.02	.12	>.25
O × D × M	1	.31	1.81	>.25
M × S × O × D	4	.22	1.32	>.25
within	1480	.167		
Total	1519			

Table 2

Results of Analysis of Variance

An analysis of variance was performed on final responses. The results of this analysis are given in Table 2, where factor M is the type of multiplication sign used, S is the spacing ratio, O is the order in which the addition and multiplication signs were placed, and D is relative difficulty. The measure used for the spacing ratio was chosen so as to retain information on both the magnitude and the direction of the space, i.e., to allow discrimination between the effects of the larger space being to the left as opposed to the right, of the middle numeral. Magnitude, *per se*, of the ratio is not related to this measure since the factorial design balances out magnitude effects except in interactions.

The order of placement of signs, the O factor, was found to be significant. Examination of the data indicates that this may be interpreted to mean that the order of placement of signs was ignored by a significant portion of the group; that is, some students worked the problems by consistently performing the leftmost or rightmost operation first without regard for whether it was an addition or a multiplication. The S × O interaction was highly significant. Examination of the data in this case indicates that the tendency of the group to perform operations in a set direction (left-to-right or right-to-left) was differentially affected by the relative size of the spaces between numerals. That is, there was a greater tendency for the group as a whole to work from left to right when the space on the left was the smaller than to work from right to left when the space on the right was the smaller.

In order to examine the effect of the magnitude of the spacing ratio alone it is necessary to examine responses to individual items. If we predict that the operation between the two closest numerals will be performed first, we can enumerate the number of student responses which meet or fail to meet this prediction. Even in the case of equal (1 : 1) spacing more ''empty'' space is left around a dot than around a plus sign and, in turn, more around a plus sign than around a multiplication cross of the size used in this experiment. This variation in ''empty'' space allows us to make predictions even for the 1 : 1 cases. Table 3 shows the results of the testing of these predictions with the data from this study. Wide and moderate spacing, in both cases easily discerned as being unequal, lead to highly significant effects, and even equal spacing with variations in amounts of open space between operation symbols and numerals has a moderately significant effect.

Spacing ratio	% in Predicted Direction	χ^2(1 d.f.)	P
7 : 3	64	47.5	< .001
3 : 2	57	11.6	< .001
1 : 1	56	4.8	< .05

Table 3

Effect of Spacing on Order of Operations

Students almost always worked the problems in a straightforward fashion without pauses such as would have been expected had they been considering the difficulty of the second computation in choosing which to do first. Moreover, an analysis of the second computations required showed no evidence that students favored the order which would make the second operation easier. (The response data are presented in Appendix B.)

If rules possibly used by individuals are noted for the first and last ten problems, it is found that five different rules can account for the behavior of 32 of the 38 subjects in the experiment. Table 4 shows the number of subjects using the various rules stated in terms of the operation performed first.

Operation performed first	Used on first ten problems	Used on last ten problems
Multiplication	8	12
Leftmost operation	7	12
Closest spaced	0	5
Addition	0	2
Rightmost operation	0	1

Table 4

Order of Operation Rules Assumed Used

The criterion for assumption that a given rule was being used was the production of the solution predicted by that rule for nine or ten of the ten problems examined. The binomial probability of such an occurrence by chance alone is .012. The rules ''multiplication first'' and ''leftmost operation first'' were the most popular for the problems. Since some of these students had been exposed to the convention, the data given in Table 4 should not be regarded as indicative of the preferences to be expected from completely naive subjects. What should be noted is the relative dominance of these two rules and the tendency for the subjects to follow consistent rules of order at the end of a large block of problems even when they did not do so at the outset. In addition, subjects appeared to attack problems more confidently as they became practiced. Consistent use of rules could be interpreted as being the result of learning about the general task which allowed use of ''shortcuts.'' Such ''shortcuts'' should not be confused with effects of practice on a basic skill. They are learned patterns of organizing previously known skills and, as such, enable a subject to do a task for which a pattern exists in a more efficient manner than if he possessed only the individual skills necessary.

Conclusions

Evidence was obtained in favor of using the dot multiplication sign rather than the cross sign. The number of errors of interpretation was significantly higher for the cross multiplication sign.

It was determined that there is a significant tendency for subjects to perform arithmetic operations in a left-to-right order. While large spacing differences produced significant changes in operation orders, smaller differences and moderate term-by-term differences in difficulty had little or no effect on order-of-operations behavior. When large spacing differences exist, a significant interaction between the left-to-right or right-to-left rule and spacing is found.

APPENDIX A

A complete list of expressions used in this experiment. Item numbers correspond to the rows in APPENDIX B.

1. $3 \cdot 7 + 9$
2. $9 + 4\times2$
3. $3+7 \times 8$
4. $6 \times 8+2$
5. $4\cdot6 + 7$
6. $7+4 \times 3$
7. $4 + 3\times9$
8. $7 \times 6+3$
9. $2 \cdot 8+5$
10. $5 + 6\cdot2$
11. $6 + 4 \cdot 9$
12. $5\cdot3 + 9$
13. $8+6 \times 4$
14. $2 + 4\times7$
15. $9\cdot3 + 4$
16. $8\times4 + 6$
17. $4\times5 + 7$
18. $8+7 \cdot 2$
19. $2+7 \cdot 8$
20. $9 \cdot 2 + 5$
21. $2\times3 + 8$
22. $9 + 6 \cdot 3$
23. $4+5 \cdot 7$
24. $3 \times 6 + 8$
25. $8 + 5\times2$
26. $2 + 5 \times 9$
27. $9 \times 7 + 3$
28. $2 \times 5+9$
29. $8+4 \cdot 5$
30. $7 \cdot 5+4$
31. $2 + 6\cdot9$
32. $8 \cdot 3+5$
33. $2 \times 6+9$
34. $5 + 7 \times 3$
35. $2 + 8 \cdot 9$
36. $3 \cdot 4+7$
37. $9\cdot6 + 2$
38. $6\times7 + 2$
39. $9 + 4\cdot2$
40. $3+6 \times 5$

APPENDIX B

Data on order of operations

Multiplication first is scored 1 and addition first is scored 0.

ITEM NO.	SUBJECTS 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	1	1	0	1	0	0	1	1	0	1	1	1	1	1	1	1	1
3	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0
4	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	0
5	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	0
6	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
7	1	0	1	1	1	0	1	1	0	1	1	0	0	1	1	1	1	1	1	1
8	1	1	0	1	1	1	0	1	1	1	1	1	1	0	1	1	0	1	1	0
9	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	0	0	1	1	0
10	1	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	1	0
11	1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
13	1	0	0	1	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0
14	1	0	1	1	1	0	1	1	0	1	0	0	0	0	1	1	0	0	0	0
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
18	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
19	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
20	1	1	0	0	1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	0
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
22	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
23	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
24	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
25	1	0	1	1	1	1	1	1	0	1	0	0	0	0	1	1	0	1	0	0
26	1	0	1	0	1	0	1	1	0	1	0	0	0	0	1	1	0	1	0	0
27	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
28	1	1	0	0	1	1	0	0	1	1	1	1	1	1	1	0	1	0	1	0
29	1	0	0	1	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0
30	1	1	0	1	0	0	0	0	1	1	1	1	1	1	1	0	1	0	1	0
31	1	0	1	1	1	0	1	1	0	1	0	0	0	0	1	1	0	1	0	0
32	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	0	1	0	1	0
33	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	0	1	0	0	0
34	1	0	0	1	0	0	0	1	0	1	0	1	0	0	1	1	0	1	0	0
35	1	0	1	0	1	0	1	1	0	1	0	0	0	0	1	1	0	1	0	0
36	1	1	0	1	0	0	0	1	1	1	1	1	1	1	1	0	1	0	0	0
37	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	1	0	0	0
38	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
39	1	0	1	1	1	1	1	1	0	1	0	0	0	0	1	1	0	1	0	0
40	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0

APPENDIX B. Data on order of operations (concluded)

ITEM NO.	SUBJECTS																		
	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	Σx
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	38
2	1	1	1	0	1	0	1	0	1	0	1	1	0	0	1	0	1	1	26
3	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	10
4	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	0	1	32
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	36
6	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	9
7	1	1	1	0	1	0	1	0	0	0	1	0	0	0	1	0	1	1	24
8	1	1	1	1	0	1	1	1	1	1	1	0	1	1	0	1	0	1	29
9	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	1	1	1	31
10	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	12
11	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	9
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	37
13	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	10
14	1	1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	16
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	37
16	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	1	35
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	37
18	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	1	1	11
19	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	10
20	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	1	1	1	31
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	35
22	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	9
23	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	9
24	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	36
25	1	1	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	1	21
26	1	0	1	0	1	0	1	0	0	0	1	0	0	0	1	0	0	1	16
27	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	36
28	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	29
29	1	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	1	14
30	1	0	1	1	0	1	1	1	0	1	1	0	1	1	0	0	0	1	23
31	1	1	1	0	1	0	1	0	0	0	1	0	0	0	1	0	0	1	18
32	1	0	1	1	0	1	1	1	1	1	1	0	1	1	0	1	1	1	29
33	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	30
34	1	1	1	1	1	0	1	0	1	0	1	0	0	0	1	0	0	1	18
35	1	1	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	1	19
36	1	1	1	1	0	1	1	1	0	1	1	0	1	1	1	1	1	1	27
37	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	32
38	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	36
39	1	1	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	1	21
40	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	1	1	12