Faculty Working Papers

URBAN LAND VALUE FUNCTIONS AND THE PRICE ELASTICITY OF DEMAND FOR HOUSING

James B. Kau and C. F. Sirmans

#371

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign
URBAN LAND VALUE FUNCTIONS AND THE PRICE ELASTICITY OF DEMAND FOR HOUSING

James B. Kau and C. F. Sirmans

#371
URBAN LAND VALUE FUNCTIONS AND THE PRICE ELASTICITY OF DEMAND FOR HOUSING

(LAND VALUE FUNCTIONS)

James B. Kau
Visiting Associate Professor
University of Illinois
and
Associate Professor
University of Georgia

C. F. Sirmans*
Assistant Professor
Department of Finance
University of Illinois
320 David Kinley Hall
Urbana, Illinois 61801

*To whom correspondence should be addressed
ABSTRACT

The structural characteristics of an urban area are determined to a great extent by the spatial pattern of land values. It is thus important to develop urban land value functions which accurately describe the spatial structure of cities. The purpose of this paper is to provide some empirical evidence on the functional form of the relationship between land values and distance from the center of the city. Using historical data for Chicago, the Box and Cox transformation technique is employed to examine the land value function. The functional form parameter can be used to determine the price elasticity of demand for housing.
1. Introduction

The spatial pattern of land values determines to a great extent the structural characteristics of an urban area. Changes in exogenous factors such as income, transportation cost and migration affect land values which in turn alters production factor ratios and the corresponding urban structural patterns. Hence, it is important to develop urban land value functions which accurately describe changes in spatial structure. Muth [10] and Mills [8], using Cobb-Douglas demand and supply functions for housing, derived the conditions necessary for the existence of an exponential function between land values and distance. One of the conditions necessary for an exponential function is the existence of a price elasticity of demand for housing equal to minus one.

The purpose of this paper is to provide some empirical evidence on the functional form of the relationship between land values and distance from the center of the city and to re-evaluate the results obtained by Mills [7] in his study of urban land values. Statistical transformation procedures developed by Box and Cox [1] and techniques developed by Kau and Lee [5, 6] for determining the price elasticity of demand for housing services are used. The Box and Cox transformation technique introduces a functional form parameter to generalize the land value function.² This same parameter can be used to
test whether the negative exponential function provides an accurate description of land value patterns. The functional form parameter can also be used to determine price elasticity of demand for housing.

This study is divided into five sections. In the second section, Mills' derivation of the negative exponential function will be reviewed. The third section provides a discussion of the general relationship between urban land values and distance. In the fourth section, data similar to that used by Mills [7] for the Chicago area over the time period 1830 to 1930 is used to test both the negative exponential and generalized functional relationships discussed in the third section. Finally the fifth section will summarize the results.

2. The Exponential Land Value Function

Following Mills [8], the output of housing services at distance $u$, $X_s(u)$ is defined as

$$X_s(u) = AL(u)\alpha K(u)^{1-\alpha}$$

(1)

where $K(u)$ and $L(u)$ represent inputs of capital and land in the production of housing services $u$ miles from the center; and $A$ and $\alpha$ are scale and distribution parameters, respectively, for a Cobb-Douglas production function.

In deriving a negative exponential function to describe the relationship between the land rent $R(u)$ and the distance $u$, Mills has also defined a demand function for housing services at $u$, $X_d(u)$, as
\[ X_d(u) = Bw^{\Theta_1}p(u)^{\Theta_2} \]  

(2)

where:

- \( B \) = a scale parameter and depends upon the units which housing services are measured;
- \( w \) = income for workers;
- \( p(u) \) = price of housing services at distance \( u \);
- \( \Theta_1 \) = income elasticity; and
- \( \Theta_2 \) = price elasticity.

Using the first order conditions of equation (1), the aggregate demand derived from equation (2) and other equilibrium conditions, Mills [8, pp. 83-84] has derived the following relationship between land rents \( R(u) \) and distance.

\[ R(u) = \left( R^\beta + \beta t E(\bar{u}-u) \right)^{\frac{1}{\beta}} \]  

if \( \beta \neq 0 \)  

(3)

and

\[ R(u) = R e^{t E(\bar{u}-u)} \]  

\( \beta = 0 \)  

(4)

where:

- \( R(u) \) = land rents at distance \( u \);
- \( \beta = \alpha(1+\Theta_2) \)  

(5)

\[ E^{-1} = \alpha \beta w^{\Theta_2} \left[ A \alpha(1-\alpha)^{1-\alpha} \right] - (1+\Theta_2) \]  

- \( r \) = rental rate for housing capital;
- \( t \) = the commuting cost per mile;
- \( \bar{u} \) = the distance from the center to the edge of the urban area;
- \( \bar{R} \) = rent on non-urban uses of land; and
- \( R(\bar{u}) = \bar{R} \).

Equation (4) demonstrates that rents decline exponentially with distance if and only if \( \beta \) is equal to zero. This result also implies that the population density declines exponentially
with distance if and only if the price elasticity, $\Theta_2$, is minus one.

3. The Generalized Rent Function

Following the procedure developed by Kau and Lee [5, 6] and after some rearrangements, equation (3) can be rewritten as

$$\frac{R^\beta(u)-1}{\beta} = \frac{R^\beta-1}{\beta} + tE(u-u) \quad (6)$$

It can be shown that equation (6) will become equation (4) when $\beta$ approaches zero. Since there exists only two observable variables, $R(u)$ and $u$ in equation (6), it can be written as

$$\frac{R^\lambda(u)-1}{\lambda} = R_0 - \gamma u \quad (7)$$

where:

$$\gamma = \beta, \quad R_0 = \frac{R^\beta-1}{\beta} + \gamma \bar{u} \quad \text{and} \quad \gamma = tE$$

Equation (7) belongs to one of the cases that Box and Cox [1] have derived for determining the true functional form. If $\lambda$ approaches zero, then equation (7) will become

$$\log R(u) = R_0 - \gamma u \quad (8)$$

This implies that the negative exponential function is only a special case of equation (7); therefore, equation (7) can be regarded as a generalized functional form to be used to determine the true relationship between rent and distance in an urban area. An additive stochastic term can be introduced into equation (7) and the relationship defined as,

$$\frac{R^\lambda(u)-1}{\lambda} = R_0 - \gamma u + \epsilon_i \quad (9)$$
where $\epsilon_i$ is normally distributed with zero mean and variance $\sigma^2$ and $R_c$ and $\gamma$ are regression parameters with $\lambda$ a functional form parameter.

An alternative functional form, with less theoretical foundation than equation (7), can be used to incorporate the double-log function use by Mills [7]. The generalized functional form is

$$\frac{R(u)^\lambda - 1}{\lambda} = \gamma_0' - \gamma_1' \frac{(u_i^\lambda - 1)}{\lambda'} + \epsilon_i'$$  \hspace{1cm} (10)

where $\gamma_0'$, $\gamma_1'$ are regression parameters, $\lambda'$ a functional form parameter and the disturbance term $\epsilon_i'$ is normally distributed with zero mean and variance $\sigma^2$, respectively.

For equation (9), if $\lambda = 1$, linear rent is regressed on distance; if $\lambda$ approaches zero the dependent variable is the natural logarithm of rent. Similarly, equation (10) will reduce to the linear form when $\lambda'$ is equal to one and to a double logarithmic form when $\lambda'$ approaches zero.

Under the assumptions of normality, the probability rent function for $\epsilon_i$ in equation (9) is

$$f(\epsilon_i) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2}(\epsilon_i^2/\sigma^2)\right)$$  \hspace{1cm} (11)

If the $\epsilon_i$'s are identically and independently distributed, the log likelihood function for (11) is written as

$$\log L = -\frac{N}{2} \log 2\pi\sigma^2 + (\lambda-1) \sum_{i=1}^{N} \log R(u) - \frac{1}{2\sigma^2} \sum_{i=1}^{N} \frac{R(u_i^\lambda - 1)^2}{\lambda'} - \gamma_0 + \gamma_1 u_i$$  \hspace{1cm} (12)
The logarithmic likelihood is maximized with respect to \( c^2, \gamma_0 \) and \( \gamma_1 \), given \( \lambda \). The maximum likelihood estimates if \( \sigma^2 \) for the given \( \lambda \), \( \hat{\sigma}^2 (\lambda) \) is then the estimated variance of the disturbances of regressing \((R_i^{\lambda-1})/\lambda \) on \( u_i \).

Replacing \( \sigma^2 \) by \( \hat{\sigma}^2 (\lambda) \), the maximum log likelihood \([L_{\text{MAX}}(\lambda)]\) for equation (8) is

\[
L_{\text{MAX}}(\lambda) = -\frac{n}{2} \log 2\pi \hat{\sigma}^2 (\lambda) - \frac{n}{2} + (\lambda-1) \sum_{i=1}^{n} \log R_i (u) \quad (13)
\]

Box and Cox (1, p. 216) indicate that an approximate 95 percent confidence region for \( \lambda \) is obtained from

\[
L_{\text{MAX}}(\hat{\lambda}) - L_{\text{MAX}}(\lambda) < 1/2\chi^2 (.05) = 1.92 \quad (14)
\]

Equations (13) and (14) with \( \lambda' \) can also be derived for equation (10) by a similar procedure. Equation (14) is used to test whether the functional form parameters, \( \lambda \) and \( \lambda' \) are significantly different from zero and/or one. Note that estimates of the optimum point estimate, \( \hat{\lambda} \) provides an estimate of \( \beta \). From the relationship between \( \beta \) and \( \theta_2 \) defined in equation (5), the assumption of unitary price elasticity can also be tested statistically.

4. **Empirical Results**

Initially three regressions were estimated for each year; each representing the specific functional forms of linear, exponential and double logged with land values as the dependent variable and distance as the independent variable.

The data for this analysis are land values for the Chicago metropolitan area for the time periods 1836, 1857, 1973,
1892, 1910 and 1928 from a study by Homer Hoyt [4]. This sample was selected so that comparison could be made with the Mills study and because there is no other comparable historical series that allows comparing land value patterns over long periods of time. Land values are dollars per acre and distance is airline miles from the intersection of State and Madison Streets. This initial set of regressions is provided to exemplify the flexibility of the functional form approach. The coefficients are presented in Table 1 and in most cases are similar with Mills' results.

The functional form parameter is determined by transforming \( R(u) \) and \( u_i \) in accordance with equation (7) and (10) using \( \lambda \)'s between -0.50 and 1.50 at intervals of 0.1. Twenty-one different regressions are run for each time period. The \( L_{\text{MAX}}(\lambda) \)'s of the six time periods are calculated by equation (13).

The maximum likelihood estimates of \( \lambda \) for the six time periods for Chicago are listed in Table 2. In general, \( L_{\text{MAX}}(\lambda) \) is not symmetrically distributed with respect to its maximum value; therefore, the lower bound and upper bound are calculated respectively. Using equation (14), the point estimate and the 95 percent confidence region for \( \lambda \) are calculated and listed in Table 2.

The results of the functional form analysis as presented in Table 2 indicate that for 4 out of the 6 time periods the estimated \( \lambda \) is significantly different from zero at the 5 percent level. Since \( \lambda = \beta \) and because \( \beta = \alpha(1 + \theta_2) \), it can be concluded that the price elasticity of demand for
<table>
<thead>
<tr>
<th>Year</th>
<th>Regression</th>
<th>Constant</th>
<th>Distance</th>
<th>$R^2$</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>361.591</td>
<td>-36.665</td>
<td>.187</td>
<td>208</td>
</tr>
<tr>
<td></td>
<td>(8.120)</td>
<td>(6.982)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1836</td>
<td>Exponential</td>
<td>5.632</td>
<td>-.403</td>
<td>.781</td>
<td>208</td>
</tr>
<tr>
<td></td>
<td>(44.832)</td>
<td>(27.168)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log-Log</td>
<td>7.294</td>
<td>-2.472</td>
<td>.896</td>
<td>208</td>
</tr>
<tr>
<td></td>
<td>(61.900)</td>
<td>(42.305)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>5874.752</td>
<td>-576.87</td>
<td>.182</td>
<td>211</td>
</tr>
<tr>
<td></td>
<td>(7.931)</td>
<td>(6.919)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1857</td>
<td>Exponential</td>
<td>8.748</td>
<td>-.513</td>
<td>.858</td>
<td>211</td>
</tr>
<tr>
<td></td>
<td>(70.886)</td>
<td>(35.724)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log-Log</td>
<td>10.533</td>
<td>-3.023</td>
<td>.876</td>
<td>211</td>
</tr>
<tr>
<td></td>
<td>(66.811)</td>
<td>(38.625)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>24173.00</td>
<td>-2361.37</td>
<td>.177</td>
<td>206</td>
</tr>
<tr>
<td></td>
<td>(8.090)</td>
<td>(6.721)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1873</td>
<td>Exponential</td>
<td>9.980</td>
<td>-.344</td>
<td>.682</td>
<td>206</td>
</tr>
<tr>
<td></td>
<td>(71.655)</td>
<td>(21.011)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log-Log</td>
<td>11.263</td>
<td>-2.075</td>
<td>.751</td>
<td>206</td>
</tr>
<tr>
<td></td>
<td>(67.632)</td>
<td>(24.897)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>31569.6</td>
<td>-2865.35</td>
<td>.253</td>
<td>173</td>
</tr>
<tr>
<td></td>
<td>(9.800)</td>
<td>(7.701)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1892</td>
<td>Exponential</td>
<td>10.043</td>
<td>-.246</td>
<td>.418</td>
<td>173</td>
</tr>
<tr>
<td></td>
<td>(52.588)</td>
<td>(11.169)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log-Log</td>
<td>11.519</td>
<td>-1.758</td>
<td>.540</td>
<td>173</td>
</tr>
<tr>
<td></td>
<td>(46.011)</td>
<td>(14.254)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>32316.7</td>
<td>-3100.61</td>
<td>.301</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>(9.487)</td>
<td>(7.332)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1910</td>
<td>Exponential</td>
<td>10.564</td>
<td>-.319</td>
<td>.566</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>(52.018)</td>
<td>(12.678)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log-Log</td>
<td>11.879</td>
<td>-1.968</td>
<td>.624</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>(44.120)</td>
<td>(14.265)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>34411.90</td>
<td>-7055.42</td>
<td>.206</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>(9.393)</td>
<td>(6.061)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1928</td>
<td>Exponential</td>
<td>11.736</td>
<td>-.2203</td>
<td>.497</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>(72.380)</td>
<td>(11.735)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Log-Log</td>
<td>12.456</td>
<td>-1.267</td>
<td>.425</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>(43.407)</td>
<td>(10.161)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*a t-values are in parentheses.*
<table>
<thead>
<tr>
<th>Year</th>
<th>( eta )-Value(^a )</th>
<th>Price Elasticity of Demand(^b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1836</td>
<td>-.25 (-.30 - -.20)</td>
<td>Elastic (-2.25 (-1.67 - -3.00))</td>
</tr>
<tr>
<td>1857</td>
<td>-.07 (-.08 - -.05)</td>
<td>Elastic (-1.35 (-1.17 - -1.53))</td>
</tr>
<tr>
<td>1873</td>
<td>-.10 (-.16 - -.05)</td>
<td>Elastic (-1.50 (-1.17 - -2.07))</td>
</tr>
<tr>
<td>1892</td>
<td>-.09 (-.15 - -.02)</td>
<td>Elastic (-1.40 (-1.07 - -2.00))</td>
</tr>
<tr>
<td>1910</td>
<td>0.00 (-.15 + .05)</td>
<td>Unitary (-1.00 (-.83 - -2.00))</td>
</tr>
<tr>
<td>1928</td>
<td>0.00 (-.12 + .05)</td>
<td>Unitary (-1.00 (-.83 - -1.80))</td>
</tr>
<tr>
<td>1970(^c )</td>
<td>0.00 (-1.22 + .24)</td>
<td>Unitary (-1.00)</td>
</tr>
</tbody>
</table>

\(^a\) 95% confidence interval in parentheses

\(^b\) The point estimate of the elasticity of demand for housing is based on \( \alpha = .20 \). The earliest estimate of \( \alpha \) from Hoyt [4] indicated \( \alpha = .27 \). The elasticities in parentheses are based on estimates of \( \alpha \) of .30 and .15, using the largest and smallest estimates of \( \beta \), respectively.

\(^c\) This result was taken from the study of density gradients by Kau and Lee [5].
housing services is significantly different from minus one when the $\beta$ is significantly different from zero. Given downward sloping demand curves for housing, then $\alpha > \beta$ when $\beta > 0$. Thus when $\beta > 0$ the price elasticity of demand is inelastic. When $\beta < 0$ and since $\alpha > 0$, the demand curve for housing services is elastic.

In all significant cases for equation (7), $\beta$ is less than zero implying an elastic demand for housing services in the earlier years. For 1910 and 1928, $\beta$ was not significantly different from zero thus implying unitary elasticity. Kau and Lee's study [5] of density gradient functional forms indicates a minus one price elasticity for housing in 1970 for Chicago. Their study also indicated that for the total 50 cities analyzed no cases resulted in a significant elastic demand for housing and approximately 50% of the cities had inelastic demands. Thus over time, there appears to be a shift away from elastic to unitary and inelastic demands for housing services. This is demonstrated in Figure I where the $I_{\text{MAX}}(\lambda)$ for the three time periods 1836, 1892 and 1928 are plotted and reveal a movement towards a $\lambda$ value of zero. Note that the cases provided are significantly different from each other.

The results for equation (10) indicate that the $\lambda$ values are significantly different from zero in three of the cases. Thus the double-log form is inappropriate. Given the previous result with equation (7), this is not unexpected but confirms the necessity to use a more generalized functional form when analyzing urban spatial structure.
The generalized rent gradient is derived by rewriting equation (7) as

$$R(u) = [\lambda R_0 - 1 - \lambda \gamma u]^{1/\lambda}$$

(15)

and taking the derivative of equation (15) with respect to $u$, and rearranging terms, the generalized rent gradient is,

$$-\gamma = \frac{\partial R}{\partial u} \frac{R^\lambda}{R}$$

(16)

The elasticity of rent with respect to distance is

$$\eta_\gamma = \frac{\partial R}{\partial u} \frac{u}{R} = -\frac{\gamma u}{R^\lambda}$$

(17)

If $\lambda = 0$, then $\eta_\gamma = -\gamma u$ which is the elasticity for the exponential rent gradient. The generalized rent gradient and its elasticity are a function not only of distance ($u$) but also of $R(u)$. 
The effect of this additional term, \( R(u) \) is to reduce or increase the elasticity relative to the exponential gradient as \( \lambda \) is greater or less than zero, respectively. Thus the conclusion for Chicago is that during the 1830 to 1930 time period the elasticity of the rent gradient has been decreasing over time. If it is accepted that past historical rental patterns affect current rents, then equation (17) allows measurement of the impact of past development on current rental gradients.

5. Summary and Conclusions

The purpose of this paper was to provide some empirical evidence on the functional form of the relationship between land values and distance from the center of the city. Using historical data for Chicago, the Box and Cox transformation technique was used to test alternative forms of the land value gradient. These estimates help to clarify the previous work by Mills [7].

The results indicated that the price elasticity for housing tended to be shifting from elastic to unitary elasticity of demand. The negative exponential form derived in the theoretical model by Muth [10] and Mills [8] proved to be the correct form in only two of the six time periods tested. The generalized rent gradient derived in this paper indicates that the elasticity of rent with respect to distance is a function not only of distance but also of rent. This result has important implications for analyzing the importance of the impact of historical rental patterns on current rates.
FOOTNOTES

1 This technique has been used to test the functional form of the population density gradient (Kau and Lee [5, 6]), the money demand function (Zarembka [11]), and the earning-schooling relationship (Heckman and Polachek [3]).

2 Following Mills [7], the data are on urban land values. It is assumed that the capitalization rate, at any given time, differs little from one place to another in an urban area. Hence values and rents will be a constant ratio.

3 It should be noted that Mills [7] did not include the land value data for 1892 in his study.

4 These results are available from the authors.

5 The slight variations in the estimated parameters as compared to Mills [7] is probably due to inexact methods of measuring distance to the CBD.

6 The estimates indicated that the double-log form was inappropriate for the three earlier time periods.

BIBLIOGRAPHY


