Faculty Working Papers

UNEMPLOYMENT, INFLATION, AND INTEREST IN MULTI-SECTOR NEOCLASSICAL GROWTH

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By Hans Breen

58-word Summary:

The purpose of the present paper is to see if a multi-sector neoclassical model of steady-state imbalanced growth has room for familiar Keynesian and monetarist ideas on unemployment and inflation. It has. It can be shown to have infinitely many solutions, one for each value of the employment fraction. Solutions differ with respect to rate of inflation, nominal but not real interest rate, and rates of growth of money and national money income. Switching from one solution to another requires a monetary policy permitting both interest rates to deviate temporarily from their steady-state equilibrium solutions.
I. PURPOSE

The purpose of the present paper is to see if a multi-sector neoclassical model of steady-state imbalanced growth has room for familiar Keynesian and monetarist ideas on unemployment and inflation. It has. It can be shown to have infinitely many solutions, one for each value of the employment fraction. Solutions differ with respect to rate of inflation, nominal but not real interest rate, and rates of growth of money and national money income. Switching from one solution to another requires a monetary policy permitting both interest rates to deviate temporarily from their steady-state equilibrium solutions.
Our neoclassical growth model will have two goods in it, each of which serves interchangeably as a consumers' or a producers' good. Three gains result. First, the model can allow for substitution both in consumption and production, hence let the price mechanism come into play. Second, the model can allow for realistic steady-state but imbalanced growth. Third, having heterogeneous physical capital stock, the model will be immune to Cambridge, England criticism. Despite the resulting complication the model is solvable—and solved. It will use the following notation.

II. NOTATION

Variables

C ≡ physical consumption

D ≡ demand for money
\[ g_v \equiv \text{proportionate rate of growth of variable } v \in \{C, D, I, K, L, M, P, r, p, S, w, X, Y\} \]

\[ I \equiv \text{physical investment} \]

\[ k \equiv \text{present gross worth of another physical unit of capital stock} \]

\[ \kappa \equiv \text{physical marginal productivity of capital stock} \]

\[ L \equiv \text{labor employed} \]

\[ \lambda \equiv \text{proportion employed of available labor force} \]

\[ M \equiv \text{supply of money} \]

\[ N \equiv \text{present net worth of entire physical capital stock} \]

\[ n \equiv \text{present net worth of another physical unit of capital stock} \]

\[ P \equiv \text{price of good} \]

\[ r \equiv \text{nominal rate of interest} \]

\[ p \equiv \text{real rate of interest} \]

\[ S \equiv \text{physical capital stock} \]

\[ U \equiv \text{utility} \]

\[ w \equiv \text{money wage rate} \]

\[ X \equiv \text{physical output} \]

\[ Y \equiv \text{money income} \]
Parameters

\(a\) \(\equiv\) multiplicative factor of production function
\(\alpha, \beta\) \(\equiv\) exponents of production function
\(c\) \(\equiv\) propensity to consume money income
\(F\) \(\equiv\) available labor force
\(g_p\) \(\equiv\) proportionate rate of growth of parameter \(p\) \(\equiv a\) and \(F\)
\(m\) \(\equiv\) multiplicative factor of demand for money function
\(\mu\) \(\equiv\) exponent of demand for money function
\(p\) \(\equiv\) multiplicative factor of Phillips function
\(\pi\) \(\equiv\) exponent of Phillips function
\(u\) \(\equiv\) multiplicative factor of utility function
\(\upsilon\) \(\equiv\) exponent of utility function

All parameters are stationary except \(a\) and \(F\) whose growth rates are. Time coordinates are \(t\) for general time and \(T\) for specific time. Euler's number \(e\) is the base of natural logarithms. \(G, H, J,\) and \(K\) stand for agglomerations to be defined as we go along.
III. THE MODEL

1. Definitions

Define the proportionate rate of growth

\( g_v = \frac{dv}{dt} \frac{1}{v} \)  

Define investment as the derivative of capital stock with respect to time:

\( I_{ij} = \frac{dS_{ij}}{dt} \)
2. Production

For its production each good needs capital stock of both goods: The output $X_j$ of the $j$th good is produced from labor $L_j$ and two immortal capital stocks $S_{ij}$, where $i$ is the sector of origin and $j$ the sector of installation. As a result, there are four distinct physical capital stocks $S_{ij}$ in the model. Let every entrepreneur have access to a Cobb-Douglas production function

$$X_1 = a_1^{L_1} S_{11}^{B_{11}} S_{21}^{B_{21}}$$

$$X_2 = a_2^{L_2} S_{12}^{B_{12}} S_{22}^{B_{22}}$$

where $0 < a_j < 1; 0 < B_{ij} < 1; \alpha_1 + B_{11} + B_{21} = 1; \alpha_2 + B_{12} + B_{22} = 1; \text{ and } a_j > 0$. Assume a fairly strong interindustry dependence: Let each industry be at least as dependent upon the capital stock supplied by the other as upon that supplied by itself, then $B_{11} \leq B_{21}, B_{22} \leq B_{12}$.
In each industry let profit maximization under pure competition equalize real wage rate and physical marginal productivity of labor:

\[
\frac{w}{P_i} = \frac{\partial x_i}{\partial L_i} = a_i \frac{x_i}{L_i}
\]

(5)

Physical marginal productivities of capital stock are

\[
\kappa_{ij} = \frac{\partial x_i}{\partial S_{ij}} = \beta_{ij} \frac{x_j}{S_{ij}}
\]

(6)

3. Investment Demand

Let \( N_j \) be the present net worth of new capital stock \( S_{ij} \) installed by an entrepreneur in the \( j \)th industry. Let his de-
To find desired capital stock, proceed as follows. Let entrepreneurs be purely competitive ones, hence price of output $P_j$ is beyond their control. At time $t$, therefore, value marginal productivity of another physical unit of capital stock is $k_{ij}(t)P_j(t)$. As seen from the present time $t$, value marginal productivity at time $t$ is $k_{ij}(t)P_j(t)e^{-r(t - \tau)}$, where $r$ is the stationary nominal rate of interest used as a discount rate. Define present gross worth of another physical unit of capital stock as the present worth of all its future value marginal productivities over its entire useful life

\begin{equation}
(9) \quad k_{ij}(\tau) = \int_{\tau}^{\infty} k_{ij}(t)P_j(t)e^{-r(t - \tau)} \, dt
\end{equation}

Let entrepreneurs expect physical marginal productivity of capital stock to be growing at the stationary rate $g_{Kij}$.
\[ K_i^j(t) = K_i^j(\tau) e^{g_{Kij}(t - \tau)} \]

and price of output to be growing at the stationary rate \( g_{Pj} \):

\[ P_j(t) = P_j(\tau) e^{g_{Pj}(t - \tau)} \]

Insert (10) and (11) into (9), define

\[ \rho_{ij} = r - (g_{Kij} + g_{Pj}) \]

and write the integral (9) as

\[ k_{ij}(\tau) = \int_{\tau}^{\infty} k_{ij}(\tau) P_j(\tau) e^{-\rho_{ij}(t - \tau)} dt \]

Neither \( k_{ij}(\tau) \) nor \( P_j(\tau) \) are functions of \( t \), hence may be taken outside the integral sign. Our \( g_{Kij} \), \( g_{Pj} \), and \( r \) were all said to be stationary, hence the coefficient \( \rho_{ij} \) of \( t \) is
stationary, too. Assume $\rho_{ij} > 0$. As a result, find the integral to be

$$k_{ij} = \kappa_{ij} P_j / \rho_{ij}$$

Find present net worth of another physical unit of capital stock as its gross worth minus its price:

$$n_{ij} = k_{ij} - P_i = \kappa_{ij} P_j / \rho_{ij} - P_i$$

Our appendix proves that (13) satisfies the second-order conditions for a maximum $N_j$. Applying the first-order condition (7) to our result (13) find equilibrium physical marginal productivity of capital stock

$$\kappa_{ij} = \rho_{ij} P_i / P_j$$

Take (14) and (6) together and find desired capital stock
(15) \[ S_{ij} = \beta_{ij} p_j x_j / (\rho_{ij} p_i) \]

Apply the definitions (1) and (2) to (15) and find desired investment as the derivative of desired capital stock with respect to time:

(16) \[ I_{ij} \equiv g_{sij} S_{ij} = g_{sij} \beta_{ij} p_j x_j / (\rho_{ij} p_i) \]

(15) and (16) are capital stock and investment desired by an individual entrepreneur in the jth industry. Except \( x_j \) everything on the right-hand sides of (15) and (16) is common to all entrepreneurs of the industry. Factor out all common factors, sum over all entrepreneurs of the industry, then \( x_j \) becomes industry output, and (15) and (16) become capital stock and investment desired by the industry.

So investment in the ith good by the industry producing the jth good is in direct proportion to, first, the rate of
growth $g_{Sij}$ of desired capital stock; second, the elasticity $\beta_{ij}$ of the output of the jth good with respect to the stock of the ith good; third, the relative price $P_j/P_i$ of the jth and the ith good; and fourth, the output $X_j$ of the jth good. Investment is in inverse proportion to what will turn out to be the real rate of interest $\rho_{ij}$.

4. Consumption Demand

Let every consumer have the utility function

$$U = uC_1^{v_1}C_2^{v_2}$$

where $0 < v_i < 1$, and $u > 0$. Let every consumer spend the fraction $c$, where $0 < c < 1$, of his money income $Y$. Then his budget constraint is
Maximize the consumer's utility subject to his budget constraint and find his two demand functions

\[ C_i = \frac{c_i Y}{P_i} \]

where \( c_i = \frac{c_i U_i}{(u_1 + u_2)} \). (17) is consumption desired by an individual consumer. Except \( Y \) everything on the right-hand side of (17) is common to all consumers. Factor out all common factors, sum over all consumers, then \( Y \) becomes national money income, and (17) becomes national desired consumption. But with immortal capital stock, the entire value of national output represents value added, i.e., national money income

\[ Y = \sum_{i=1}^{2} (P_i X_i) \]

Insert (18) into (17) and write national desired consump-
5. Goods-Market Equilibrium

Goods-market equilibrium requires output to equal the sum of consumption and investment demand for it:

\[(21) \quad X_i = C_i + \sum_{j=1}^{2} I_{ij} \]

6. Employment and the Phillips Function

Let labor employed be the proportion \( \lambda \) of available labor force, where \( 0 < \lambda < 1 \) and \( \lambda \) is not a function of time:
Within their province let labor unions seek a relative and temporary gain by raising the money wage rate $w$. Knowing that the gain will be temporary will not keep them from seeking it; on the contrary, in anticipating inflation they are compelled to contribute to it. But let their compulsion be tempered by unemployment. Subtract employment (22) from available labor force $F$ and find unemployment $(1 - \lambda)F$, and express a Phillips curve relationship

$$g_w = p(1 - \lambda)^\pi$$

where $\pi < 0$, and $p > 0$.

7. Money

Let the demand for money be a function of national money income.
and the nominal rate of interest:

(24) \[ D = mYr^\mu \]

where \( \mu < 0 \), and \( m > 0 \).

Money-market equilibrium requires the supply of money to equal the demand for it:

(25) \[ M = D \]

Let us now solve our model for growth rates as well as for levels at an instant of time.
IV. STEADY-STATE EQUILIBRIUM GROWTH-RATE SOLUTIONS

In our derivation of investment demand in Sec. III, 3 above, entrepreneurs were using a stationary nominal rate of interest \( r \) as a discount rate and expecting price and the physical marginal productivity of capital stock to be growing at stationary rates \( g_{pj} \) and \( g_{kij} \). Are such expectations self-fulfilling? In other words, may the system display steady-state equilibrium growth? It may. By taking derivatives with respect to time of all equations involving the variables \( C, D, I, K, L, M, P, r, \rho, S, w, X, \) and \( Y \) the reader may convince himself that the system (1) through (25) is satisfied by the following steady-state equilibrium growth-rate solutions:

\[
(26) \quad g_{Ci} = g_{Xi}
\]
\[(27)\] 
\[g_D = g_M\]

\[(28)\] 
\[g_{ij} = g_{Xi}\]

\[(29)\] 
\[g_{kij} = g_{Xj} - g_{Sij}\]

\[(30)\] 
\[g_{Li} = g_F\]

\[(31)\] 
\[g_M = g_Y\]

\[(32)\] 
\[g_{P1} = g_w - \frac{(1 - \beta_{22})g_{a1} + \beta_{21}g_{a2}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}\]

\[(33)\] 
\[g_{P2} = g_w - \frac{(1 - \beta_{11})g_{a2} + \beta_{12}g_{a1}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}\]

\[(34)\] 
\[g_r = 0\]
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(35) $g_{pij} = 0$

(36) $g_{sij} = g_{xi}$

(37) $g_w = p(1 - \lambda)q$

(38) $g_{x1} = \frac{(1 - \beta_{12})g_{a1} + \beta_{21}g_{a2}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}} + g_r$

(39) $g_{x2} = \frac{(1 - \beta_{11})g_{a2} + \beta_{12}g_{a1}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}} + g_r$

(40) $g_y = g_r + g_w$

Are (26) through (40) meaningful? Use our assumptions $\alpha_1 + \beta_{11} + \beta_{21} = 1$ and $\alpha_2 + \beta_{12} + \beta_{22} = 1$ to show that the denominator of (32), (33), (38), and (39) equals $\alpha_1\alpha_2 + \alpha_1\beta_{12} + \alpha_2\beta_{21}$. The denominator of (37) is $1 - \lambda$. Neither can be zero, hence all solutions are meaningful.

Is growth balanced? Use our assumptions $\alpha_1 + \beta_{11} + \beta_{21}$
\[ \alpha_2 + \beta_{12} + \beta_{22} = 1 \] upon (38) and (39) to see that

\[ g_{x_1} > g_{x_2} \text{ if } \frac{g_{a_1}}{g_{a_2}} > \frac{\alpha_1}{\alpha_2} \]

Growth may be balanced, then, but only by an odd piece of luck. And now let us solve for levels at an instant of time.

V. STEADY-STATE EQUILIBRIUM LEVEL SOLUTIONS

1. Rates of Interest

Into desired investment (16) insert the solution (36). Then insert desired investment (16) and desired consumption (19) and (20) into the good-market equilibrium condition (21), multiply the condition for the first good by \( p_1 \) and that for
the second by $P_2$, rearrange and find

\[(42) \quad (1 - c_1 - \beta_{11} g_{X1}/\rho_{11}) P_1 X_1 = (c_1 + \beta_{12} g_{X1}/\rho_{12}) P_2 X_2 \]

\[(43) \quad (c_2 + \beta_{21} g_{X2}/\rho_{21}) P_1 X_1 = (1 - c_2 - \beta_{22} g_{X2}/\rho_{22}) P_2 X_2 \]

Into the definition (12) insert (29) and (36). Use (32), (33), (38), and (39) to realize that $g_{P1} + g_{X1} = g_{P2} + g_{X2}$. Use that result to find

\[(44) \quad \rho_{ij} = r - g_{pi} \]

hence $\rho_{11} = \rho_{12} = r - g_{P1}$, and $\rho_{21} = \rho_{22} = r - g_{P2}$. So in (42) and (43) replace $\rho_{12}$ by $\rho_{11}$ and $\rho_{21}$ by $\rho_{22}$, respectively. Divide (42) by (43), getting rid of $P_i X_j$, and find the nonlinear equation

\[(45) \quad \rho_{11} \rho_{22} - J_1 \rho_{11} - J_2 \rho_{22} + \kappa = 0 \]
where

\[ J_1 \equiv [\beta_{22}(1 - c_1) + \beta_{21}c_1]g_{X_2}/(1 - c) \]

\[ J_2 \equiv [\beta_{11}(1 - c_2) + \beta_{12}c_2]g_{X_1}/(1 - c) \]

\[ K \equiv (\beta_{11}\beta_{22} - \beta_{12}\beta_{21})g_{X_1}g_{X_2}/(1 - c) \]

and where, according to (17), \( c \equiv c_1 + c_2 \). Use (44) to write

\begin{equation}
\rho_{22} = \rho_{11} + g_{P_1} - g_{P_2}
\end{equation}

Insert (46) into (45) and solve the latter for \( \rho_{11} \) and \( \rho_{22} \):
\[(47)\]
\[
\rho_{11} = \frac{J_1 + J_2 - (g_{p1} - g_{p2})}{2}
\]
\[
\pm \sqrt{\left(\frac{J_1 + J_2 - (g_{p1} - g_{p2})}{2}\right)^2 + J_2 (g_{p1} - g_{p2}) - \kappa}
\]

\[(48)\]
\[
\rho_{22} = \frac{J_1 + J_2 + g_{p1} - g_{p2}}{2}
\]
\[
\pm \sqrt{\left(\frac{J_1 + J_2 + g_{p1} - g_{p2}}{2}\right)^2 - J_1 (g_{p1} - g_{p2}) - \kappa}
\]

where \(g_{p1}\) and \(g_{p2}\) stand for the solutions (32) and (33). The roots of a quadratic equation may be real or complex and may be positive and/or nonpositive. In the cases of (47) and (48) which will it be? Constraints upon the parameters \(\beta_{ij}\) and c
already imposed guarantee that $J_1 > 0$ and $K \leq 0$. Consider four possibilities.

First, if $g_{P1} < g_{P2}$ then in (48) $- J_1(g_{P1} - g_{P2}) - K > 0$. Consequently the brace of (48) will be positive, and the absolute value of the square root in (48) will be greater than $(J_1 + J_2 + g_{P1} - g_{P2})/2$ regardless of the sign of the latter. As a result, (48) will have one positive and one negative root, and both will be real.

Second, if $g_{P1} = g_{P2}$ and $K < 0$ the same will be true.

Third, if $g_{P1} = g_{P2}$ and $K = 0$ then (47) and (48) will be identical and will have the positive root $J_1 + J_2$ and the root zero.

Fourth, if $g_{P1} > g_{P2}$ then in (47) $J_2(g_{P1} - g_{P2}) - K > 0$. Consequently the brace of (47) will be positive, and the absolute value of the square root in (47) will be greater than $[J_1 + J_2 - (g_{P1} - g_{P2})]/2$ regardless of the sign of the latter. As a result, (47) will have one positive and one negative root, and both will be real.
All roots (47) and (48) are meaningful: We are dividing by 2 and 1 - c only, and neither can be zero. Our four possibilities exhaust the universe. Each generates a nonpositive root to be rejected, because it violates the constraint $\rho_{ij} > 0$ under which the integral (9) was taken. But if one side of (46) is single-valued, the other must be. Consequently, as constrained, the system has one and only one positive root for every $\rho_{ij}$. That root is real and meaningful. Once that root has been found, (44) determines the nominal rate of interest $r$.

Derivation with respect to time of (44), (47), and (48) will show that they are stationary, as (34) and (35) say.

2. Employment

Write (42) and (43) as

\[
\frac{p_1 x_1}{p_2 x_2} = H = \frac{c_1 + \beta_{12} g_{x1} / \rho_{12}}{1 - c_1 - \beta_{11} g_{x1} / \rho_{11}} = \frac{1 - c_2 - \beta_{22} g_{x2} / \rho_{22}}{c_2 + \beta_{21} g_{x2} / \rho_{21}}.
\]
where \( g_{Xi} \) stands for (38) and (39) and \( \rho_{ij} \) for (47) and (48). Use (5) and (49) to write \( L_1/L_2 = H\alpha_1/\alpha_2 \), insert that into (22), and find the solutions for employment:

\[
(50) \quad L_1 = H\alpha_1 \lambda F/(H\alpha_1 + \alpha_2)
\]

\[
(51) \quad L_2 = \alpha_2 \lambda F/(H\alpha_1 + \alpha_2)
\]

Derivation with respect to time of (50) and (51) will show that the latter are indeed growing at the proportionate rate (30).

3. Output

Insert (49) into desired capital stock (15) and find

\[
(15a) \quad S_{11} = \beta_{11} X_1/\rho_{11}
\]
which are indeed growing the the proportionate rate (36). Insert (15a) through (15d) into the production functions (3) and (4) and find the solutions for output

\[
X_1 = (G_1 G_2) \left( 1 - \beta_{21} \right)^{1 \left[ (1 - \beta_{11})(1 - \beta_{22}) - \beta_{12} \beta_{21} \right]}
\]

\[
X_2 = (G_1 G_2) \left( \beta_{12} \right) \left( 1 - \beta_{11} \right)^{1 \left[ (1 - \beta_{11})(1 - \beta_{22}) - \beta_{12} \beta_{21} \right]}
\]

where
and where $H$ stands for (49), $L_i$ stands for (50) and (51), and $\rho_{ij}$ for (47) and (48). Derivation with respect to time of (52) and (53) will show that the latter are indeed growing at the proportionate rates (38) and (39), respectively.

4. Relative Prices

Write (49) as a solution for relative price

\[(54) \quad \frac{P_1}{P_2} = \frac{HX_2}{X_1}\]

where $X_i$ stands for (52) and (53). Derivation with respect to
time of (54) will show that the latter is consistent with (32), (33), (38), and (39).

5. Real Wage Rate

Equation (5) will be a solution for the real wage rate \( w/P_i \) if \( L_i \) stands for (50) and (51) and \( X_i \) for (52) and (53). Derivation with respect to time of (5) will show that the latter is consistent with (30), (32), (33), (38), and (39).

6. National Money Income

Write (5) as \( P_i X_i = wL_i/\alpha_i \), insert (50) and (51), insert result into (18), and find national money income

\[
(55) \quad Y = w\lambda F(1 + H)/(H\alpha_1 + \alpha_2)
\]

Derivation with respect to time of (55) will show that it is growing at the proportionate rate (40).
7. Required Money Supply

To the monetary authorities the money supply is a decision variable. Solving for it means finding the value of it required to uphold steady-state equilibrium growth at a given money wage rate \( w \) and a given employment fraction \( \lambda \). Insert (24) into (25) and find required money supply

\[
M = mYr^\mu
\]

where \( r \) stands for (44) and \( Y \) for (55). Derivation with respect to time of (56) will show that it is growing at the proportionate rate (31).

Equation (56) concludes our solving. We have solved our model for growth rates and levels alike. We are now ready to study its properties.
VI. PROPERTIES OF SOLUTIONS

1. Underemployment Steady-State Equilibrium Growth—a Dynamic Analogy to Keynes

Since no right-hand side of our growth-rate solutions (26) through (40) is a function of time, our solutions are steady-state growth solutions. Since goods-market and money-market equilibrium conditions (21) and (25) are satisfied, our solutions (26) through (56) are equilibrium solutions.

But our steady-state equilibrium growth solutions are underemployment solutions, found under the assumption that $0 < \lambda < 1$, and $\lambda$ was not a function of time. We have not solved for $\lambda$ and cannot: The system has infinitely many solutions, one for each value of $\lambda$. In other words, nothing keeps employment from being less than full and staying so. The expect-
iterations of entrepreneurs and consumers are still self-fulfilling. We have a neoclassical dynamic analogy to the celebrated Keynesian static underemployment equilibrium.

2. What Difference Does the Employment Fraction $\lambda$ Make?

The employment fraction $\lambda$ makes a difference, both for certain levels and certain growth rates.

Beginning with levels, we find five physical quantities being neatly in direct proportion to $\lambda$. First, no $\lambda$ enters into the definition (49) of $H$, consequently our solutions (50) and (51) for employment $L_i$ are in direct proportion to $\lambda$. Second, insert those solutions as well as our assumptions $\alpha_1 + \beta_{11} + \beta_{21} = 1$ and $\alpha_2 + \beta_{12} + \beta_{22} = 1$ into our solutions (52) and (53) for output and find the employment fraction $\lambda$ entering as a factor raised to the power one. Consequently physical output $X_i$ is in direct proportion to $\lambda$. 
If employment $L_i$ and physical output $X_i$ are both in direct proportion to $\lambda$, then in our solutions (5) for the real wage rate $w/P_i$ and (54) for relative price $P_1/P_2$ $\lambda$ cancels in numerator and denominator. According to (32) and (33) $g_w$ and with it $\lambda$ cancels in the difference $g_{P_1} - g_{P_2}$ entering our solutions (47) and (48) for the real rate of interest $\rho_{ij}$. Real wage rate, relative price, and the real rate of interest are independent of $\lambda$! But then, third, fourth, and fifth, according to (15), (16), (19), and (20), desired physical capital stock $S_{ij}$, investment $I_{ij}$, and consumption $C_i$ are in direct proportion to $X_i$, hence to $\lambda$.

Turning to growth rates we find $g_w$ and with it $\lambda$ to be absent from the growth-rate solutions for the five physical quantities $C_i$, $I_{ij}$, $L_i$, $S_{ij}$, and $X_i$. They are also absent from the growth rate $g_w - g_{P_i}$ of the real wage rate $w/P_i$ and from the growth rate $g_{P_1} - g_{P_2}$ of relative price $P_1/P_2$: Our solutions (32) and (33) can express those differences in terms containing no $\lambda$. But our growth-rate solutions for money supply $M$, prices $P_i$, and national money income $Y$ do include $g_w$ which, according to
(37), is the higher, the higher is \( \lambda \). Then according to (44), (47), and (48) the nominal but not the real rate of interest is the higher, the higher is \( \lambda \).

VII. SWITCHING FROM ONE STEADY-STATE TRACK TO ANOTHER

1. Nonsteady-State Nonequilibrium Growth

Once settled on a steady-state equilibrium growth track, the economy will stay on it. Could it be switched from a low-\( \lambda \) steady-state equilibrium growth track to a high-\( \lambda \) one? Let the monetary authorities try to switch it as follows. Fig. 1 shows the time curves of the nominal and the real rate of interest \( r \) and \( \rho_{ij} \). In accordance with (44) the vertical distance between the two curves is the rate of growth \( g_{pi} \) of price. Before Time 1 and after Time 2 let there be steady-state
Fig. 1. Time Paths of Nominal and Real Rates of Interest
equilibrium growth. Before Time 1 λ is low, after time 2 λ is higher, hence \( g_{p_i} \) is higher, and the gap between \( r \) and \( p_{ij} \) higher. At Time 1 let the monetary authorities raise the money supply \( M \) beyond its equilibrium level (56). At a so far unchanged inflationary expectation \( g_{p_i} \), the nominal and real rate of interest \( r \) and \( p_{ij} \) will then fall below their equilibrium levels (44), (47), and (48).

On the resulting nonsteady-state nonequilibrium growth our model is silent and must be: Our integral (9) can be taken only if entrepreneurs are using a stationary nominal rate of interest \( r \) as a discount rate and expecting price and physical marginal productivity of capital stock to be growing at stationary rates \( g_{p_j} \) and \( g_{x_ij} \).

Little may be said, then, about what will happen once the economy has been derailed, as it were. But five things may be said about the forces released by the derailment.
2. Excess Demand Will Emerge

The first thing we may say is that reducing $\rho_{ij}$ will generate excess demand: In (16) we found an entrepreneur's desired investment $I_{ij}$ to be in inverse proportion to $\rho_{ij}$. As a result, in the equilibrium condition (21) the right-hand side, demand, is up while the left-hand side, supply, remains the same: There is excess demand.

3. Excess Demand Will Stimulate Output and Employment

Excess demand is a signal to expand output, and entrepreneurs would like to heed the signal. Can they? Write (5) as $L_i = \alpha_i P_i X_i / w$. The second thing we may say is that expanding physical output $X_i$ will expand employment $L_i$ in direct proportion——which is feasible because of unemployment.
4. Expanding Output Will Not Eliminate Excess Demand

Let entrepreneurs heed the signal. According to (6), (19), and (20) desired physical investment $I_{ij}$ and consumption $C_i$ are in direct proportion to physical output $X_i$. The third thing we may say, then, is that in the equilibrium condition (21) the right-hand side, demand, and the left-hand side, supply, are expanding in the same proportion, hence their difference, excess demand, is expanding in the same proportion.

5. Price Adjustments Will Not Eliminate Excess Demand

But could not appropriate price changes eliminate excess demand? Take a closer look at (16), (19), and (20) and write them out as follows:

\[(16a) \quad I_{11} = g_{S11} \beta_{11} X_1 / \rho_{11}\]
\begin{equation}
I_{12} = g_{s12} b_{12} P_2 X_2 / (\rho_{12} P_1)
\end{equation}

\begin{equation}
I_{21} = g_{s21} b_{21} P_1 X_1 / (\rho_{21} P_2)
\end{equation}

\begin{equation}
I_{22} = g_{s22} b_{22} X_2 / \rho_{22}
\end{equation}

\begin{equation}
C_1 = c_1 (X_1 + P_2 X_2 / P_1)
\end{equation}

\begin{equation}
C_2 = c_2 (P_1 X_1 / P_2 + X_2)
\end{equation}

No price change could pare down $I_{11}$ and $I_{22}$, for no prices appear in them. The only price change which could pare down $I_{12}$ and $C_1$ would be a reduction of the ratio $P_2 / P_1$. But that ratio cannot be reduced without increasing its reciprocal $P_1 / P_2$, and such an increase would expand $I_{21}$ and $C_2$. The fourth thing we may say is that no change in relative prices will eliminate excess demand. If absolute prices changed in the same proportion, relative prices would remain unchanged. The fifth thing
we may say is that no change in absolute prices will eliminate excess demand.

6. Restoring Steady-State Growth Equilibrium?

The one thing which could eliminate excess demand would be the restoration of the equilibrium levels (47) and (48) of the real rate of interest $p_{ij}$. At Time 2 in Fig. 1 let the monetary authorities restore them. Won't the economy then relapse into its old low-$\lambda$ track? Doesn't its capital stock then look too large? It doesn't. The higher desired capital coefficient $S_{ij}/X_j$—encouraged according to (15) by the low $p_{ij}$—hasn't materialized! Every individual entrepreneur wanted to raise his capital coefficient but, under excess demand, was able to do so only at the expense of others.

The economy may find its way, then, into a high-$\lambda$ steady-state equilibrium growth track. Will it? Rigorously we don't know: Our model is silent on anything else than steady-state equilibrium growth.
We have built a multi-sector neoclassical model of steady-state imbalanced growth equilibria. Our model has infinitely many solutions, one for each value of the employment fraction $\lambda$.

Our model is Wicksellian—but only in the sense that it determines an interest rate at which saving equals investment. Unlike a Wicksellian model it allows for unemployment and distinguishes between a nominal and a real rate of interest. Its infinitely many solutions, one for each value of the employment fraction $\lambda$, all have the same real but a different nominal rate of interest.

Our model is Keynesian—but only in the sense that it may generate underemployment equilibria. Unlike the static Keynesian model it traces the time paths of such equilibria.

Our model is monetarist—but only in the sense that it distinguishes between a nominal and a real rate of interest. Its infinitely many solutions, one for each value of the employment fraction $\lambda$, all have the same real but a different money wage rate. Indeed, all have the same level $w/P_i$ of the real wage rate as well as
the same rate of growth \( g_w - g_{pi} \) of it. In that sense any value of the employment fraction \( \lambda \) is a Friedmanian "natural rate of unemployment". But if so, the model has scope for employment policy. Raising employment might require a monetary policy permitting both interest rates to deviate temporarily from their steady-state equilibrium solutions.
APPENDIX
SECOND-ORDER CONDITIONS FOR A MAXIMUM $n_j$ ARE SATISFIED

Insert (6) into (13), take the derivatives of $n_{ij}$ defined by (13) as ordered by the Hessian (8), and write the latter as

\[
\begin{vmatrix}
\beta_{1j} (\beta_{1j} - 1) \frac{P_j X_j}{\rho_{1j} s_{1j}^2} & \beta_{1j} \beta_{2j} \frac{P_j X_j}{\rho_{1j} s_{1j} s_{2j}} \\
\beta_{1j} \beta_{2j} \frac{P_j X_j}{\rho_{2j} s_{1j} s_{2j}} & \beta_{2j} (\beta_{2j} - 1) \frac{P_j X_j}{\rho_{2j} s_{2j}^2}
\end{vmatrix}
\]

\[
= \beta_{1j} \beta_{2j} \frac{P_j x_{1j}^2}{\rho_{1j} \rho_{2j} s_{1j}^2 s_{2j}^2}
\]

\[
\begin{vmatrix}
\beta_{1j} - 1 & \beta_{2j} \\
\beta_{1j} & \beta_{2j} - 1
\end{vmatrix}
\]
Use our assumptions $\alpha_j + \beta_{1j} + \beta_{2j} = 1$ to see that the value of the last determinant is $\alpha_j$. So the Hessian is positive. Since $\beta_{1j} - 1 < 0$, the principal minor of the Hessian is negative.
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In his 1967 presidential address, delivered at the eightieth annual meeting of the American Economic Association, Friedman defined a "natural rate of unemployment" as one at which "real wage rates are tending on the average to rise at a 'normal' secular rate, i.e., at a rate that can be indefinitely maintained so long as capital formation, technological improvements, etc., remain on their long-run trends."

In our steady-state equilibrium growth model the real wage rate is always rising like that. But it can rise like that for any value of the employment fraction λ: We have found λ cancelling in the level \( w/P_i \) of the real wage rate as well as in its rate of growth \( g_w - g_{P_i} \). Any value of λ, then, is a Friedmanian "natural rate of unemployment". Friedman's rate is not unique!

Including equations, diagrams, spaces the length of the paper is the equivalent of 6,400 words.