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OPEN LOOP NASH EQUILIBRIUM OF N-PERSON NON-ZERO SUM LINEAR-QUADRATIC DIFFERENTIAL GAMES WITH MAGNITUDE RESTRAINTS

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by

Ronald J. Stern*

Abstract

We prove the existence of an open loop Nash equilibrium of an N-person non-zero sum linear-quadratic differential game with bounded controllers. Due to the method employed, the computational method of [5], section 5, can be used to uniformly approximate the equilibrium.

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OPEN LOOP NASH EQUILIBRIUM OF N-PERSON NON-ZERO SUM

LINEAR-QUADRATIC DIFFERENTIAL GAMES WITH MAGNITUDE RESTRAINTS

1. Introduction.

The dynamics of the differential game to be considered are given by the following differential system in R^m:

$$(1.1) \qquad \dot{x} = A(t)x + \sum_{i=1}^{n} B_i(t)u_i \qquad (t_o \leqq t \leqq T_o)$$

An initial condition is given by

$$(1.2) \qquad x(t_o) = x_o$$

$A(t)$ is a continuous $(m \times m)$ matrix on $[t_o, T_o]$, and the $B_i(t)$ are continuous $(m \times q_i)$ matrices on $[t_o, T_o]$. Admissible controls u_i are Lebesque-measurable functions which almost everywhere on $[t_o, T_o]$ are valued in $\mathcal{U}_i$, the unit ball in the Euclidean space R^i. We now define N cost functionals of "quadratic type":

$$(1.3) \qquad J_i(u_1, u_2, \ldots, u_N) = \left\langle x(T_o) - \xi_i, W_i \left[x(T_o) - \xi_i\right] \right\rangle + \int_{t_o}^{T_o} \left\langle z_i(t) - C_i(t)x(t), Q_i(t) \left[z_i(t) - C_i(t)x(t)\right] \right\rangle dt + \int_{t_o}^{T_o} |u_i(t)|^2 \, dt$$

Here $\langle \; , \; \rangle$ denotes inner product, and $| \; |$ is the appropriate Euclidean norm, depending on i. The ξ_i are vectors in R^i; $z_i(t)$ are continuous functions on $[t_o, T_o]$; all matrices in (1.3) are continuous on $[t_o, T_o]$, while W_i is constant; and we also assume W_i and $Q_i(t)$ are symmetric.

In this paper we will prove the existence of an open loop Nash equilibrium point for the above game called G; that is, admissible controls $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$ such that

$$(1.4) \qquad J_i(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N) \leq J_i(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, \bar{u}_N)$$

for any admissible u_i, $(1 \leq i \leq N)$. We also will give a method for approximating the equilibrium costs $J_i(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$, $(1 \leq i \leq N)$. We will use a certain penalty function method which was introduces in [5] for approaching this problem; for a positive integer k, define N cost functionals by

$$(1.5) \qquad J_i^k(u_1, u_2, \ldots, u_N) = J_i\ (u_1, u_2, \ldots, u_N) + \int_{t_o}^{T_o} |u_i(t)|^{2k}\, dt$$

Denote the differential game with costs given by (1.5) and where admissible controls u_i need only satisfy $\int_{t_o}^{T_o} |u_i(t)|^{2k}\, dt < \infty$ by G_k. The proof that G_k has an open loop Nash equilibrium (which yields a method for approximating the equilibrium costs of G_k) is deferred until section 3. In section 2, the results of section 3 are used to prove the existence of an open loop Nash equilibrium for G, and it is proven that for large k, the equilibrium costs of G_k approximate those of G. The results of sections 2 and 3 require a bound on $T_o - t_o$.

2. Main results. Proof of the following result, as well as a computational method associated with its proof, will be dealt with in the next section.

Theorem 2.1. *If $T_o - t_o$ is sufficiently small, then G^k has a unique open loop Nash equilibrium.*

We will denote the bound required on $T_o - t_o$ by M, and the equilibrium of G^k by $(u_1^k, u_2^k, \ldots, u_N^k)$. The following maps of R^i into R are now defined:

$$\phi_i^k(v) = \begin{cases} 0 \text{ if } |v| \leqq k^{-\frac{1}{2}k} \\ |v|^{2k} \quad \text{otherwise} \end{cases}$$

The following payoff functionals are also defined for i = 1, 2, ..., N:

$$J_i^{\phi_i^k}(u_1, u_2, \ldots, u_N) = J_i(u_1, u_2, \ldots, u_N) + \int_{t_o}^{T_o} \phi_i^k\left(u_i(t)\right) dt.$$

It is not difficult to see that

$$(2.2) \qquad \left|J_i^k(u_1, u_2, \ldots, u_N) - J_i^{\phi_i^k}(u_1, u_2, \ldots, u_N)\right| \leqq \frac{T_o - t_o}{k}$$

for any vector of controls $(u_1, \cdots, u_N)$ which is feasible for G_k.

We will require the following lemma:

Lemma 2.1. *If $T_o - t_o \leqq M$, then* $\max_i \sup_k \int_{t_o}^{T_o} |u_i^k(t)|^{2k} dt \leqq Q < \infty.$

The proof of Lemma 2.1 is a computation employing Holder's inequality. Details will be omitted.

Let $\mathcal{U}_i^k$ be the set of vectors in R^i with Euclidean length $\leqq 1 - k^{-\frac{1}{2}k}$, and let $\rho_{i,k}(.)$ denote the Euclidean distance in R^i from $\mathcal{U}_i^k$. Lemma 2.1 implies

$$(2.3) \qquad \int_{t_o}^{T_o} \rho_{i,k}^2\left(u_i^k(t)\right) dt \leqq \frac{Q}{k^{\frac{1}{2}}}$$

From Theorem 2.1 and (2.2) we have for any u_i feasible in G_k

$$(2.4)\qquad J_i(u_1^k, u_2^k, \ldots, u_N^k) \leq J_i^{\phi_i^k}(u_i^k, u_2^k, \ldots, u_{i-1}^k, u_i, u_{i+1}^k, \ldots, u_N^k) + \frac{T_o - t_o}{k}$$

for $1 \leq i \leq N$ and each positive integer k.

Let $\bar{G}^k$ denote the game with the same costs as G, namely (1.3), where admissible controls are Lebesque measurable functions u_i which almost everywhere on $[t_o, T_o]$ take values in $\mathcal{U}_i^k$, the ball of radius $k^{-\frac{1}{2}k}$ in R^i.

We now prove

Lemma 2.2. Let $T_o - t_o \leq M$. Then there exists a constant $D > 0$ such that the following is true: For each positive integer k there is a vector of controls $(\hat{u}_1^k, \hat{u}_2^k, \ldots, \hat{u}_N^k)$ feasible for $\bar{G}_k$ such that

$$(2.5)\qquad J_i(\hat{u}_1^k, \hat{u}_2^k, \ldots, \hat{u}_N^k) \leq J_i(\hat{u}_1^k, \hat{u}_2^k, \ldots, \hat{u}_{i-1}^k, u_i, \hat{u}_{i+1}^k, \ldots \hat{u}_N^k) + D(k^{-\frac{1}{2}} + 1 - k^{-\frac{1}{2}k}) + \frac{(T_o - t_o)}{k}$$

for any u_i feasible for $\bar{G}_k$, $i = 1, 2, \ldots, N$.

Proof. Let v be an arbitrary vector in R^i, and let $\mathcal{U}_i^k(v)$ denote the subset of $\mathcal{U}_i^k$ such that

$$\rho_{i,k}(v) = \rho_{i,k}\left(v, \mathcal{U}_i^k(v)\right), \text{ the distance of } v \text{ to } \mathcal{U}_i^k(v).$$

A lexicographic ordering on R^i is defined as follows: We say

$$(v_1, v_2, \ldots, v_i) << (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_i)$$

if either $v_1 < \bar{v}_1$ or $v_j = \bar{v}_j$ for $j \leqq s$ and $v_s \leqq \bar{v}_s$ for some $s = 2, 3, \ldots, i$. Thus, given $v \in R^i$, there is a unique $\hat{v} \in \mathcal{U}_i^k(v)$ such that $\hat{v} << \bar{v}$ for all $\bar{v} \in \mathcal{U}_i^k(v)$. Thus we have a single valued mapping "^" of R^i onto $\mathcal{U}_i^k(v)$.

Now let $v(t)$ be any Lebesque-measurable function valued in R^i. (This is equivalent to the components of $v(t)$ being real valued Lebesque-measurable functions.) By a result due to Fillipov (see [1] or [2]) we have that $\hat{v}(t)$ is also Lebesque-measurable.

By a routine calculation (additional details are in [5]), we now have that (2.5) holds; (2.3) is made use of here.

We now can prove the following result:

<u>Theorem 2.2</u>. <u>If</u> $T_o - t_o \leqq M$, <u>then</u> G <u>has an open loop Nash equilibrium</u>.

<u>Proof</u>. In view of the fact that $\{u_i^k\}$ is a bounded sequence in the space of square-integrable controls, $i = 1, 2, \ldots, N$, there is a sequence k' such that for each i

$$(2.6) \qquad \hat{u}_i^{k'} \xrightarrow{w} \bar{u}_i \qquad \text{(weak convergence)}$$

for a measurable control $\bar{u}_i$ valued almost everywhere in $\mathcal{U}$; this last by the proof of Lemma 2.4.1 in [2]. Let $\hat{x}^{k'}$ denote the trajectory corresponding to $(\hat{u}_1^{k'}, \hat{u}_2^{k'}, \ldots, \hat{u}_N^{k'})$. By results in section 2.4 in

[2] we have $\hat{x}^{k'} \to \bar{x}$ uniformly, where $\bar{x}$ is the trajectory corresponding to $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$.

By [4] p. 209, we have for i = 1, 2, ..., N

(2.7) $$\liminf_{\kappa'} \int_{t_o}^{T_o} |\hat{u}_i^{k'}(t)|^2 \, dt \geqq \int_{t_o}^{T_o} |\bar{u}_i(t)|^2 dt.$$

Using (2.5), the form of the J_i, the convergence of $\{\hat{x}^{k^1}\}$ and (2.7), the proof of the theorem is completed.

3. Proof of Theorem 2.1. By [2], section 8.5 and also [4], we have that the second Gateaux differentials of the J_i are positive when $T_o - t_o$ is sufficiently small. The condition of stationarity (that is all the J_i having zero first Gateaux derivatives) is given by the following system of integral equations:

(3.1) $$2u_i(s) + 2k \, |u_i(s)|^{2k-2} u_i(s)$$

$$= - B_i^*(s) \, \Phi^*(T_o, s) \, W_i[x(T_o) - \xi_i] + \int_{t_o}^{T_o} B_i^*(s) \, \Phi^*(t, s) \, C_i^*(t) \, Q_i(t)$$

$$[z_i(t) - x(t)] \, dt$$

for i = 1, 2, ..., N. Here * denotes the matrix transpose and Φ is the fundamental solution of $\dot{x} = A(t)x$. The symmetry of the W_i and $Q_i(t)$ were used in the derivation of (3.1). A solution to (3.1) is an open loop Nash equilibrium for G_k. We rewrite the system (3.1) as follows:

$$(3.2)\qquad [M_1^k\left(u_1(t)\right), M_2^k\left(u_2(t)\right), \ldots, M_N^k\left(u_N(t)\right)]$$

$$= T_k\,[M_1^k\left(u_1(t)\right), M_2^k\left(u_2(t)\right), \ldots, M_N^k\left(u_N(t)\right)].$$

Here the M_i^k are maps of R^i into itself given by

$$M_i^k(v) = 2v + 2k|v|^{2k-2}v.$$

The M_i are invertible, and the inverses are given by

$$M_i^{-1}(w) \quad \frac{w}{2+2k[r_k(|w|)]^{2k-2}}$$

where $r_k(|w|)$ is the unique real root of $2kx^{2k-1} + 2x - |w|$. Also, T_k is an operator taking $C^{0,1}[t_o, T_o] \times C^{0,2}[t_o, T_o] \times \ldots \times C^{0,N}[t_o, T_o]$ into itself. Notice that the invertibility of the M_i imply that (3.1) can be expressed in the form (3.2).

We now claim that each operator T_k is a contraction when $T_o - t_o \leqq M$. This is an analog of Theorem 4.2 in [5], and details are omitted here. A computational procedure for uniformly approximating the solution of (3.1), which avoids the difficulty that r_k has no explicit form for $k > 2$, is given in section 5 of [5] for N=2, but easily generalizes to the present case.

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