A Note On Nondictational Conditions and the Relations Between Choice Mechanisms and Social Welfare Functions

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Abstract

It is demonstrated that weakening the strong nondictatorship requirement stated by Gibbard and by Satterthwaite enables the construction of strategy-proof choice mechanisms by shifting the difficulties from the phase of choice mechanism construction to the phase of agenda formation.

Ever since Arrow (1963), Gibbard (1973) and Satterthwaite (1975) proved their famous impossibility results, the relation between Arrow's required axioms for social welfare functions and Gibbard's and Satterthwaite's stated axioms for strategy-proof choice mechanisms has attracted a great deal of attention. Satterthwaite (1975) used his (and Gibbard's (1973)) results on the impossibility of constructing strategy-proof, nondictatorial choice mechanisms to re-prove Arrow's (1963) impossibility theorem on constructing nondictatorial social welfare functions. Blin and Satterthwaite (1978) proved that when preferences are unrestricted the existence of a social welfare function that satisfies the independence of irrelevant alternatives and monotonicity conditions implies the existence of strategy-proof choice mechanisms. Maskin (1976) correctly observed that to prove complete equivalence between the two problems one must show equivalence on any given domain of restricted preferences. He gives an example (see also Muller and Satterthwaite (1983)) of a restricted domain of preferences that permits the construction of a nondictatorial strategy-proof choice mechanism but not the construction of an Arrow type social welfare function, thus refuting complete equivalence. He goes on to prove, as
did Kalai and Muller (1977), the equivalence over restricted domains of preferences of the class of Arrow type social welfare functions with the class of rational strategy-proof choice mechanisms. For this purpose they used a weaker notion of nondictatorship as a requirement for choice mechanisms than the nondictatorship condition used by Gibbard and by Satterthwaite. Blair and Muller (1983) replaced the nondictatorial requirement used by Maskin and by Kalai and Muller with the stronger requirement of essentiality, and they proved the equivalence over restricted domains of preferences of the class of essential Arrow type social welfare functions with the class of rational, essential strategy-proof choice mechanisms. This body of research was summarized and discussed in detail in Muller and Satterthwaite (1983) and is an excellent reference for the interested reader.

The purpose of this note is to demonstrate that if the nondictatorial condition postulated by Gibbard (1973) and by Satterthwaite (1975) is replaced with the weaker nondictatorial condition required by Maskin (1976), Kalai and Muller (1977), Ritz (1983) and others, then it is possible to construct choice mechanisms which possess all the required properties. These mechanisms depend heavily on the agenda at issue. Thus the difficulties with strategy-proof choice mechanisms shift from the choice mechanism construction phase to the agenda formation phase.

Notations and Definitions

The notations and definitions used here follow those in Kalai and Muller (1977) and Muller and Satterthwaite (1983). Let $N = \{1, \ldots, n\}$ be a set (society) of $n$ individuals or agents ($|n| > 2$). Let $A$ denote a
finite set of feasible alternatives ($|A| > 2$), and let $\Sigma$ denote the set of all transitive antisymmetric complete binary relations on $A$. (For the sake of brevity the discussion is restricted to the case of strict preferences only.) An element of $\Sigma$ is called a preference relation. Let $\Omega$ be a nonempty subset of $\Sigma$; $\Omega$ is the set of all the admissible preference relations in the society and referred to as the restricted domain of preferences. $\Omega^n$ represents the set of all $n$-tuples of preferences from $\Omega$, and element of $\Omega^n$, $P = (p_1, \ldots, p_n)$ is called an $n$-person profile. An $n$-person social welfare function (SWF) or $\Omega$ is a function $f: \Omega^n \to \Sigma$. The following are a number of properties a SWF $f$ may possess.

**Unanimity:** For every $P$ in $\Omega^n$ and $x, y$ in $A$, if $x p_i y$ for $i = 1, \ldots, n$ then $x f(P) y$.

**Independence of Irrelevant Alternatives (IIA):** For any $x, y$ in $A$ and $P, Q$ in $\Omega^n$ if $[x p_i y$ if and only if $x q_i y$ for $i = 1, \ldots, n]$ then $[x f(P) y$ if and only if $x f(Q) y$].

**Weak Nondictatorship:** There is no individual $i$ in $N$ such that $f(P) = p_i$ for every $P$ in $\Omega^n$.

**Essentiality:** For every individual $i$ in $N$ there exists a $P$ in $\Omega^n$ and $q_i$ in $\Omega$ and a pair of alternatives $x, y$ in $A$ such that $x f(P) y$ and $y f(P/q_i x$ ($P/q_i = (p_1, \ldots, p_{i-1}, q_i, p_{i-1}, \ldots, p_n)$).

Let $\Pi$ be the set of all nonempty subsets of $A$. $\Pi$ is the set of all possible agendas. An $n$-person choice mechanism is a function $F: \Omega^n \times \Pi \to A$.

The following is a list of properties a choice mechanism may possess.

**Feasibility:** For every $B$ in $\Pi$ and every $P$ in $\Omega^n$, $F(P, B)$ is in $B$.  

Unanimity (Pareto Efficiency): For every $P$ in $\Omega^n$ and every $B$ in $\Pi$, if $x, y \in B$ and $x^i \neq y$ for $i = 1, 2, \ldots, n$ then $y \neq F(P, B)$.

Independence of Nonoptimal Alternatives (INOA): For every $P$ in $\Omega^n$ and every $B$ in $\Pi$, if $D \subseteq B$ and $F(P, B)$ in $D$ then $F(P, D) = F(P, B)$.

Manipulability: There exists a $B$ in $\Pi$, $P, Q$ profiles in $\Omega^n$ such that for some individual $i$, $p_i \neq q_i$, for every individual $k \neq i$, $p_k = q_k$ and $F(Q, B)p_i F(P, B)$. $F$ is said to be strategy-proof (or nonmanipulable) if it is not manipulable.

Rationality: There exists $f$, a social welfare function such that for every $B$ in $\Pi$ and $P$ in $\Omega^n$, $F(P, B) = \max_B f(P)$.

Monotonicity: For every feasible set $B$ in $\Pi$ and any element $x$ in $B$, whenever (i) two profiles $P, Q$ in $\Omega^n$ agree on the set $B - \{x\}$, and (ii) $x^i \neq y$ implies $x^j \neq y$ for all $y$ in $B - \{x\}$, then $F(P, B) = x$ implies $F(Q, B) = x$.

Weak Nondictatorship: No individual $i$ in $N$ exists such that, for all feasible sets $B$ in $\Pi$, $F(P, B) = \max_B (p_i)$ for all $P$ in $\Omega^n$ (max$_B (p_i)$ is the alternative in $B$ most preferred in $p_i$).

Full Agenda Nondictatorship: No individual $i$ in $N$ exists such that $F(P, A) = \max_A (p_i)$ for all $P$ in $\Omega^n$.

Strong Nondictatorship: No individual $i$ in $N$ exists such that, for at least one feasible set $B$ in $\Pi$, $\mid B \mid > 2$ $F(P, B) = \max_B (p_i)$ for all $P$ in $\Omega^n$.

Essentiality: For every individual $i$ in $N$ there exists a feasible set $B$ in $\Pi$, a profile $P$ in $\Omega^n$ and a preference $q_i$ in $\Omega$ such that $F(P, B) \neq F(P/q_i, B)$.

Thus, a choice mechanism is weakly nondictatorial if no individual has decisive powers over all possible agendas, it is full agenda non-
dictatorial if there exists no individual who always possesses decisive powers when the largest possible agenda is considered, and it is strong nondictatorial if no individual exists that has decisive powers on even one possible agenda.

In the following representation we emphasize the different notions of nondictatorship used by Arrow (1963) and by Gibbard and Satterthwaite (1975).

**Result 1 (Arrow (1963))**

If \(|A| > 3\) and preferences are unrestricted \((\Omega = \Sigma)\), then a social welfare function \(f\) cannot simultaneously satisfy unanimity, independence of irrelevant alternatives and weak nondictatorship.

**Result 2 (Gibbard (1973), Satterthwaite (1975))**

If \(|A| > 3\) and preferences are unrestricted \((\Omega = \Sigma)\), then a choice mechanism cannot simultaneously be strategy-proof and satisfy unanimity and strong nondictatorship.

Maskin (1976) and Kalai and Muller (1977) in their discussions of the equivalence between classes of strategy-proof choice mechanisms and Arrow type SWFs over restricted domains of preferences, relaxed the original requirement of strong nondictatorship postulated by Gibbard and Satterthwaite, and used instead the weak nondictatorship requirement. The same assumption is used in Ritz (1983) for the case analyzed there. (In the following representation of the results derived by Kalai-Muller and Maskin, we omit the part that deals with the characterization of the restricted domains of preferences.)
Result 3 (Kalai and Muller (1977), Maskin (1976))

For \( n > 2 \) the following two statements are equivalent for every \( \Omega \).

1. \( \Omega \) permits the construction of an \( n \)-person weakly nondictatorial, rational choice mechanism that is strategy-proof and also satisfies unanimity.

2. \( \Omega \) permits the construction of an \( n \)-person weakly nondictatorial social welfare function that satisfies unanimity and independence of irrelevant alternatives.

The difficulty in implementing stronger nondictatorship requirements for choice mechanisms is demonstrated by the following trivial example.

Let \( A = \{a, b, c\} \), \( N = \{1, 2, 3\} \) and \( \Omega = \{p_1, p_2\} \) such that \( a \prec p_1 b \prec c \prec p_1 b \). Let's define a social welfare function \( f \) as follows: for every \( x, y \) in \( A \) and \( P \) in \( \Omega^3 \), \(xf(P)y\) if and only if the majority in \( N \) prefers \( x \) to \( y \). Clearly \( f \) satisfies unanimity, independence of irrelevant alternatives and is weakly nondictatorial. On the other hand, any choice mechanism \( F \) that satisfies unanimity will choose for any \( P \) in \( \Omega^3 \), \( F(P, A) = a \), thus every agent in \( N \) is a full agenda dictator. This demonstrates that the problem of how choice mechanisms satisfying the original requirements of Gibbard and of Satterthwaite relate to nondictatorial SWFs is still an open one. Blair and Muller (1983) replaced the weak nondictatorship requirement with the requirement that every agent will have some decision powers—the essentiality requirement, and they proved:

Result 4 (Blair and Muller (1983))

The following two statements are equivalent.
a. The domain \( \Omega \) permits the construction of \( n \)-person essential, monotonic Arrow type social welfare functions.

b. The domain \( \Omega \) permits the construction of \( n \)-person rational, essential and strategy-proof choice mechanisms.

They also demonstrated that the largest restricted domain of preferences that admits essential, rational, strategy-proof choice mechanisms is the domain with "inseparable pair" defined in Kalai and Ritz (1979), which is of size \(|A|!/2 + (|A|-1)!\). Kim and Roush (1981) demonstrated that this is also the largest restricted domain that permits the construction of strategy-proof, rational, weakly nondictatorial choice mechanisms which also satisfy unanimity. Muller and Satterthwaite (1983) observe that the question of the maximal size of a domain that admits weakly nondictatorial, strategy-proof choice mechanisms which are not necessarily rational, is still an open question.

In Kalai and Muller (1977), Maskin (1976), Blair and Muller (1983), Ritz (1983) and other works, the relaxation of the strong nondictatorship condition to the weak nondictatorship condition was considered an acceptable modification. Here we will demonstrate that if the weak nondictatorship property is indeed considered to be a proper replacement for the strong nondictatorship property among the original properties defined by Gibbard and by Satterthwaite, and no additional requirements are specified, then it is possible to construct such choice mechanisms.
Theorem 1

For every society $N$ of at least two agents and every set of alternatives $A$ of at least two alternatives, when preferences are unrestricted there exist strategy-proof choice mechanisms that satisfy unanimity and weak nondictatorship.

Indeed, it is even possible to replace the weak nondictatorship requirement with the essential requirement and still prove that:

Theorem 2

For any society $N$ of at least two agents and every alternative set $A$ of at least two alternatives when preferences are unrestricted, there exist essential, strategy-proof mechanisms that also satisfy unanimity.

The following trivial corollary answers the question posed in Muller and Satterthwaite (1983).

Corollary 1

The largest domain admitting essential (therefore also weak nondictatorial) strategy-proof choice mechanisms is of size $|A|!$

It is enough to prove theorem 2.

Proof (of Theorem 2)

Let us define a choice mechanism $F$ as follows. Choose two alternatives $\hat{a}, \hat{b}$ in $A$. Then for any $P$ in $\Omega^n$ and $B$ in $\Pi$: if $B \neq \{\hat{a}, \hat{b}\}$, then $F(P, B) = \max_B(p_1)$; if $B = \{\hat{a}, \hat{b}\}$ and (i) $n$ is odd, then $F(P, B)$ is the alternative preferred by the majority of all the agents in $N$; (ii) $n$ is even but $n > 2$, then $F(P, B)$ is the alternative preferred by the
majority among agents 2, 3, ..., n; (iii) \( n = 2 \) then \( F(P, B) \) is the alternative preferred by the majority among agents 1, 2 and the preference \( p_1 \), where \( \sim p_1 \).

The proof that \( F \) is well defined, is essential and strategy-proof, and also satisfies unanimity can be derived easily and is omitted.

Q.E.D.

It is easy to verify that \( F \) is not a rational choice mechanism, which explains its sensitivity to changes in the agenda at issue.

The above results do not eliminate the existence difficulties with choice mechanisms. Rather these results demonstrate that by a proper selection of requirements for strategy-proof choice mechanisms, the difficulties can be kept either internal to the choice mechanism or external to it. By requiring either strong nondictatorship, full agenda nondictatorship, or weak nondictatorship and rationality—the difficulties are in the possibility that agents may take advantage of the specified choice mechanism and will misrepresent their preferences. This is a difficulty internal to the choice mechanism. On the other hand, if only weak nondictatorship is required, the difficulties are in how to decide on the agenda, prior to the choice process. Clearly, different agendas may give the decisive powers to different agents in society. This can be considered as a difficulty external to the choice mechanism. The difficulties in establishing an agenda are widespread and well known. For example, the president of the United States has the powers to decide on limited military operations, while he needs the approval of Congress to wage war. The difficulty is then, as was
demonstrated on a number of occasions, to decide whether an operation is of limited scale or an act of war.

The results derived in this note suggest an interesting direction of research in the investigation of mechanisms that both select agendas and choose outcomes out of these agendas.
REFERENCES


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