Systematic Risk and Market Imperfections

K. C. Chen

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois. Urbana-Champaign
Systematic Risk and Market Imperfections

K. C. Chen, Assistant Professor
Department of Finance

I am grateful for the helpful comments of Andrew H. Chen and R. Stephen Sears. Special thanks are due to S. Ghon Rhee for his critical, insightful comments and encouragement. All errors, of course, remain mine.
Abstract

This paper has demonstrated how personal taxes, default risk, leverage-related costs, and limited liability of securityholders affect the systematic risks of equity and debt respectively. When personal taxes are nonexistent and default risk is remote and leverage-related costs are negligible and the limited liability of securityholders is not recognized, the results derived in this paper have been shown to collapse into Hamada and Rubinstein's traditional formulation. Since more than one of the aforementioned market imperfections exist in the real world, the results derived here are thus more general in specification.
II. Market Values of Different Claims

Let \( \tilde{X} \) denote the firm's end-of-period cash flows which are assumed to be jointly normally distributed with the return on the market portfolio so that \( \tilde{X} \sim N(\bar{X}, \sigma_X^2) \) for any given assessment of \( \bar{R}_m \) and \( \sigma_m^2 \). The after-tax cash flows to the owners of the unlevered firm are

\[
\tilde{Y}_u = (1 - \tau_{PS}) \left\{ \begin{array}{ll}
\tilde{X}(1-\tau) & \text{if } \tilde{X} > 0 \\
0 & \text{if } \tilde{X} \leq 0
\end{array} \right.
\]

(2)

where \( \tau_{PS} \) is the constant marginal personal tax rate on equity income; and \( \tau \) is the proportional corporate income tax rate. Therefore, the market value of the unlevered firm, according to the CAPM, is given by

\[
V_u = (1 - \tau_{PS})(1-\tau)[E_0(\tilde{X}) - \lambda \text{Cov}_0(\tilde{X}, \bar{R}_m)](R)^{-1},
\]

(3)

where

\[
E_0(\tilde{X}) = \int_0^{\infty} \tilde{X} \overline{f(\tilde{X})} d\tilde{X};
\]

\( \lambda = \) the market price of risk;

\[
\text{Cov}_0(\tilde{X}, \bar{R}_m) = E[(\tilde{X} - E_0(\tilde{X}))(\bar{R}_m - E(\bar{R}_m))]
\]

\( = \) the partial covariance between \( \tilde{X} \) (truncated above 0) and \( \bar{R}_m; \)

\( R = 1 + R_f, \) wherein \( R_f \) is the risk-free interest rate.

For simplicity, assume that the firm issues only common equity and a single-period balloon-payment bond, both of which have limited liability. The total promised payment to bondholders, \( D \), is tax deductible. Because of the use of leverage, the firm is subject to the type of agency problems pointed out by Jensen and Meckling [11] who
argue that the presence of risky debt may induce the firm to undertake suboptimal investments. Since the incentive for suboptimal investments increases with the amount of risky debt outstanding, the ex-ante agency costs of debt is assumed (for simplicity) to be a percent of the total promised payment to bondholders. That is, \( \alpha D \) reflects the change in cash flows due to both the change in investment policy and the monitoring and bonding costs. Bankruptcy is defined as the state in which the firm's end-of-period cash flows are less than the ex-ante agency costs and the total promised payment to bondholders, \( \widetilde{X} < (1+\alpha)D \). At the end of the period, shareholders receive the after-tax residual value of the firm if it remains solvent, or they receive nothing if the firm goes bankrupt. On the other hand, bondholders receive their contractual claims of \( D \) at the end of the period if the firm is solvent. Otherwise, the end-of-period cash flows will be transferred to the bondholders who have to incur the costly bankruptcy penalties, \( K \). Under such a setting, the respective end-of-period cash flows to shareholders and bondholders are

\[
\begin{align*}
\tilde{Y}_S &= (1-\tau_{PS}) \left\{ \begin{array}{ll}
(1-\tau)(\tilde{X}-\alpha D-D) & \text{if } \tilde{X} > (1+\alpha)D \equiv B \\
0 & \text{if } \tilde{X} \leq B,
\end{array} \right. \\
\tilde{Y}_D &= (1-\tau_{PD}) \left\{ \begin{array}{ll}
D & \text{if } \tilde{X} > B \\
\tilde{X}-\alpha D-K & \text{if } H \equiv \alpha D+K < \tilde{X} < B \\
0 & \text{if } \tilde{X} \leq H,
\end{array} \right.
\end{align*}
\]

where \( \tau_{PD} \) is the constant marginal personal tax rate on debt income. The respective market values of equity and debt can be expressed as follows:

\[
\begin{align*}
\bar{Y}_S &= (1-\tau_{PS}) \left\{ \begin{array}{ll}
(1-\tau)(\tilde{X}-\alpha D-D) & \text{if } \tilde{X} > (1+\alpha)D \equiv B \\
0 & \text{if } \tilde{X} \leq B,
\end{array} \right.
\]

\[
\bar{Y}_D = (1-\tau_{PD}) \left\{ \begin{array}{ll}
D & \text{if } \tilde{X} > B \\
\tilde{X}-\alpha D-K & \text{if } H \equiv \alpha D+K < \tilde{X} < B \\
0 & \text{if } \tilde{X} \leq H,
\end{array} \right.
\]

where \( \tau_{PD} \) is the constant marginal personal tax rate on debt income. The respective market values of equity and debt can be expressed as follows:
\[ V_S = (1-\tau_{PS})(1-\tau)[E_B(\tilde{X}) - \lambda\text{Cov}_B(\tilde{X}, \tilde{R}_m) - (1+\alpha)D[1-F(B)]](R)^{-1} \] (6)

\[ V_D = (1-\tau_{PD})[D[1-F(B)] + E_H(\tilde{X}) - \lambda\text{Cov}_H(\tilde{X}, \tilde{R}_m) - (\alpha D + K)[F(B)-F(H)]](R)^{-1} \] (7)

where \( F(B) = \int_{-\infty}^{\infty} f(\tilde{X})d\tilde{X} \), default probability; and \( F(H) = \int_{-\infty}^{\infty} f(\tilde{X})d\tilde{X} \).

By adding (6) and (7) and then substituting \( V_u \) in (3) and \( V_D \) in (7), the market value of the firm can be expressed as

\[ V_L = V_S + V_D = V_u + \left[ 1 - \frac{(1-\tau)(1-\tau_{PS})}{1-\tau_{PD}} \right] V_D - V_K \] (8)

where \( V_K = (1-\tau_{PS})(1-\tau)[E_H(\tilde{X}) - \lambda\text{Cov}_H(\tilde{X}, \tilde{R}_m) + \alpha D[1-F(H)] + K[F(B)-F(H)]](R)^{-1} \)

= the present value of "ex ante" leverage-related costs.

Equation (8) shows that the market value of the levered firm is the sum of the market value of the unlevered firm plus the tax subsidy on debt and minus the present value of "ex ante" leverage-related costs.

III. Systematic Risk and Leverage

From the CAPM, the systematic risk of equity is defined as

\[ \beta^S = \frac{\text{Cov}(\tilde{Y}_S, \tilde{R}_m)}{V_S \sigma_m^2}, \] (9)

where \( \sigma_m^2 \) is the variance of market portfolio's returns. Using

\[ \text{Cov}(\tilde{Y}_S, \tilde{R}_m) = (1-\tau_{PS})(1-\tau)\text{Cov}(\tilde{X}, \tilde{R}_m)[1-F(B)], \] (9) can be rewritten as

\[ \beta^S = \frac{(1-\tau_{PS})(1-\tau)\text{Cov}(\tilde{X}, \tilde{R}_m)}{V_S \sigma_m^2} \cdot [1-F(B)]. \] (10)
Likewise, substituting $\text{Cov}(\tilde{Y}_u, \tilde{R}_m) = (1-\tau_{PS})(1-\tau)\text{Cov}(\tilde{X}, \tilde{R}_m)[1-F(0)]$ into the definition of the unlevered firm's systematic risk of the unlevered firm yields

$$\beta^u = \frac{\text{Cov}(\tilde{Y}_u, \tilde{R}_m)}{\nu \sigma_m^2} = \frac{(1-\tau_{PS})(1-\tau)\text{Cov}(\tilde{X}, \tilde{R}_m)}{\nu \sigma_m^2} \cdot [1-F(0)] \quad (11)$$

where $[1-F(0)]$ reflects the limited liability of the unlevered shareholders.

By combining (10) and (11), the relationship between $\beta^S$ and $\beta^u$ can be obtained as shown below;

$$\beta^S = \beta^u \left(\frac{V_u}{V_S}\right) [\frac{1-F(B)}{1-F(0)}]. \quad (12)$$

This result shows that the systematic risk of equity is equal to the systematic risk of the unlevered firm adjusted for the difference in equity value of the two firms and the survival probability (the bracketed expression in (12)). Substituting $V_u$ from rearranging (8) into (12) yields

$$\beta^S = \beta^u \left[1+\left(\frac{1-\tau_{PS}}{1-\tau_{PD}}\right) \frac{V_D}{V_S} + \frac{V_K}{V_S} \frac{1-F(B)}{1-F(0)} \right]. \quad (13)$$

Equation (13) indicates that the systematic risk of equity is a function of the systematic risk of the unlevered firm $\beta^u$, the corporate income tax rate $\tau$, the personal tax rates $\tau_{PS}$ and $\tau_{PD}$, the leverage ratio ($V_D/V_E$), the present value of "ex ante" leverage-related costs $V_K$ (mainly agency costs of debt), the level of total promised payment $D$, the variance rate on the cash flows of the firm $\sigma^2_X$, protection of limited liability (1 - $F(0)$), and the probability of default $F(B)$.

Therefore, the presence of default risk in the model implies that even
if the systematic risk of the unlevered firm is stationary, the systematic risk of equity will be nonstationary. Furthermore, the systematic risk of equity is a nonlinear function of leverage due to the presence of default risk and limited liability. When personal taxes are nonexistent and default risk is remote and the present value of "ex ante" leverage-related costs is negligible and the limited liability of securityholders is not recognized, equation (13) simply reduces to Hamada and Rubinstein's traditional formulation. Thus, the systematic risk of equity as defined in (13) is more general in its specification than the traditional formulation. To illustrate this, two special cases of (13) are discussed below.

Case I: assuming $\tau_{PS} = \tau_{PD} = 0$, riskless debt, and no limited liability, then

$$\beta^S = \beta^U \left[ 1 + (1 - \tau) \frac{V_D}{V_S} \right],$$

(13-1)

where

$$\beta^U = (1 - \tau) \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{(V_u \sigma^2_m)},$$

$$V_u = (1 - \tau) \left[ E(\tilde{X}) - \lambda \text{Cov}(\tilde{X}, \tilde{R}_m) \right] (R)^{-1},$$

$$V_D = D(R)^{-1},$$

and

$$V_S = (1 - \tau) \left[ E(\tilde{X}) - \lambda \text{Cov}(\tilde{X}, \tilde{R}_m) - D \right] (R)^{-1}.$$  

When debt is riskless, equation (13) simply collapses into Hamada and Rubinstein's traditional formulation as shown in (13-1).

Case II: assuming $\tau_{PS} = \tau_{PD} = 0$, risky debt but zero leverage-related costs ($\alpha = \kappa = 0$), and limited liability, then
\[ \beta^S = \beta^u \cdot \frac{V_D}{V_S} \frac{1-F(D)}{1-F(0)}, \quad (13-2) \]

where \( \beta^u = \frac{(1-\tau) \text{Cov}(\tilde{X}, \tilde{R}_m)}{V_u \sigma_m^2} \cdot [1-F(0)], \)

\[ V_u = (1-\tau) \left[ E_0(\tilde{X}) - \lambda \text{Cov}_0(\tilde{X}, \tilde{R}_m) \right](R)^{-1}, \]

\[ V_D = \left[ D[1-F(D)] + E_D(\tilde{X}) - \lambda \text{Cov}_D(\tilde{X}, \tilde{R}_m) \right](R)^{-1}, \]

\[ V_S = (1-\tau) \left[ E_D(\tilde{X}) - \lambda \text{Cov}_D(\tilde{X}, \tilde{R}_m) - D[1-F(D)] \right](R)^{-1}. \]

This version of beta, (13-2), is derived under the assumptions of risky debt and limited liability of corporate securityholders. Even this model is more general than Hamada and Rubinstein's traditional formulation as shown in (13-1) because of the presence of survival probability and protection of limited liability.

In similar fashion, the systematic risk of debt can be defined as

\[ \beta^D = \frac{(1-\tau_{PD}) \text{Cov}(\tilde{X}, \tilde{R}_m)}{V_D \sigma_m^2} \cdot [F(B)-F(H)]. \quad (14) \]

By solving (11) and (14) for \( \text{Cov}(\tilde{X}, \tilde{R}_m)/\sigma_m^2 \) and substituting \( V_u \) from re-arranging (8), produces

\[ \beta^D = \beta^u \left[ (1-\tau) + \frac{V_S}{V_D} + \frac{V_K}{V_D} \frac{1-\tau_{PD}}{(1-\tau_{PS})(1-\tau)} \frac{F(B)-F(H)}{1-F(0)} \right], \quad (15) \]

where \( F(B)-F(H) \) measures the probability that bondholders are in default after their limited liability is taken into account.

We can further demonstrate the linkage between the systematic risk of equity, the systematic risk of the unlevered firm, and the systematic risk of debt as follows.
\[ \beta^S = \beta^u \left[ 1 + \left( \frac{1 - \tau_{PS}}{1 - \tau_{PD}} \right) \frac{V_D}{V_S} \right] + \frac{V_K}{V_S} \]

\[ - \beta^D \left( \frac{V_D}{V_S} \right) \left[ \left( 1 - \tau_{PS} \right) \left( 1 - \tau_{PD} \right) \right] \left( F(B) - F(0) \right). \]

Equation (16) simply reduces to

\[ \beta^S = \beta^u \left[ 1 + \left( \frac{V_D}{V_S} \right) \left( 1 - \tau \right) \right] - \beta^D \left[ (1 - \tau) \frac{V_D}{V_S} \right]. \] (17)

This result is consistent with Conine (1980) in the presence of risky debt. Therefore, the model derived in (16) is more general. When debt becomes riskless, \( \beta^D = 0 \), equation (17) is identical to Hamada and Rubinstein's traditional formulation as shown in (13-1).

V. Conclusion

The paper has demonstrated how personal taxes, default risk, leverage-related costs, and limited liability of securityholders affect the systematic risks of equity and debt respectively. When personal taxes are nonexistent and default risk is remote and leverage-related costs are negligible and the limited liability of securityholders is not recognized, the results derived in this paper have been shown to collapse into Hamada and Rubinstein's traditional formulation. Since more than one of the aforementioned market imperfections exist in the real world, the results derived here are thus more general in specification.
Footnotes

1 Recently Yagill [27] attempts this but his valuation model with bankruptcy costs is arbitrary without an explicit specification of the bankruptcy cost function.

2 For discussion of truncation, refer to Lintner [15] and Chen [3, 4]. For closed form expressions of the partial mean and partial covariances, see Gonzales, Litzenberger, and Rolfo [8] and Rhee [23].

3 A single-period discount bond as assumed by Merton [17] and Galai and Masulis [7] could also be used.

4 Gonzales, Litzenberger, and Rolfo [8] discuss the issue of the absurdity of the mean-variance model for optimal capital structure decisions. Specifically, they point out that the market value of a levered firm is not a monotonically increasing function of its financial leverage. Recently, Rhee [23] has correctly proved that the "reductio ad absurdum" argument does not necessarily hold when the restriction on corporate borrowing is explicitly considered in light of shareholder limited liability.

5 This includes accounting and legal expenses incurred, loss of sales and increased costs due to disruption of operations during periods of financial distress including bankruptcy. For an excellent discussion of bankruptcy costs, see Kim [12].

6 In a recent excellent synthesis by Kim [13], the ex-ante agency costs of debt are assumed to be random and an increasing function of the total promised payment to bondholders. Without loss of generality, it is assumed in this paper that the ex-ante agency costs of debt are proportionate to the total promised payment to bondholders.

7 Miller [18], in his horse-and-rabbit stew criticism, argues that bankruptcy costs should be economically insignificant. However, several post-Miller papers (see [1], [6], [13], and [26]) suggest that leverage-related costs (bankruptcy costs and agency costs) are still important factors to derive the unique interior optimal leverage even in the presence of personal taxes.

8 See Lintner [15], p. 19.

9 The proof is available from the author upon request.
References


