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The Equilibrium APT and Optimal Portfolio Decisions

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The Equilibrium APT and Control Portfolio Decisions

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Abstract

The optimal portfolio decision problems have generally been ignored in the arbitrage pricing theories, mainly due to the derivation methods assuming the existence of a security market with infinite assets and investors holding well-diversified portfolios. This paper develops the equilibrium APT using the utility maximization approach, taking into account the portfolio decision explicitly and compares it with the Ross's APT and the CAPM of Merton and Long. In particular, it is demonstrated that the Ross's APT is a special case of the equilibrium APT only under the assumption of perfect-diversification in the infinite-assets market. However, if this unrealistic assumption is relaxed, the equilibrium APT becomes either Long's or Merton's CAPM and the market portfolio plays a prominent role in pricing assets.
The Equilibrium APT and Optimal Portfolio Decisions

As an alternative to the capital asset pricing model (CAPM), the arbitrage pricing theory (APT) has been proposed by Ross [10,11] and extended by Huberman [5], Chen and Ingersoll [2] and others. One of the reasons that the APT is generally accepted as a plausible alternative to the CAPM is that the market portfolio does not play special role in the APT whereas the identification of the true market portfolio is critical for testing the validity of the CAPM (see Roll [8]). However, the optimal portfolio decision problem of risk averse investors has never been explicitly discussed in the APT world. The reason may be that the derivation methods of the APT exploits the concept of a large security market only from the perspective of an investor who holds a well-diversified portfolio. This paper demonstrates that the APT of Ross can be derived using an optimization technique and thus the optimal portfolio decision can be explicitly taken into account in the process. In addition, the APT developed in this paper (we will call it "equilibrium APT") is related to the CAPM in the market of both infinite assets and finite assets.

The purpose of this paper is multifold: First, this paper develops the general equilibrium APT using the utility maximization approach. Second, it is shown that the Ross's model is a special case of the equilibrium APT when a risk-averse investor is assumed to hold a well-diversified portfolio of many assets. Third, this paper demonstrates that the market portfolio plays an important role in the APT if the assumption of perfect diversification is relaxed. Fourth, relatedly,
the equilibrium APT developed in this paper is shown to be in the same spirit as the intertemporal CAPM of Merton [7] or the multifactor model of Long [6]. In addition, the fund separation theorems (k+1, and k+2 funds separation theorems) are developed. The equilibrium APT in this paper is a general and linear factor model without any approximation as mentioned by Ross [10,11] or the assumption of a risk-averse agent holding a well-diversified portfolio (see Chen and Ingersoll [2]).

Section I derives the equilibrium APT using the utility maximization and compares it with the Ross's model. Section II discusses the portfolio decision problem and develops the k+1 fund separation theorem in the APT world. Section III discusses the equilibrium APT in the finite-assets market and develops the k+2 fund separation theorem. In particular, it is shown that the market portfolio plays a prominent role in pricing capital assets when the assumptions of perfect diversification is relaxed. A brief conclusion is contained in the last section.

I. Utility Maximization and the Equilibrium APT

We assume that the capital markets are perfectly competitive and frictionless. Individuals are assumed to believe homogeneously that random returns on the underlying assets are generated by a linear K-factor model of the form:

\[ \tilde{R}_{it} = E_{i} + b_{i1}f_{it} + \ldots + b_{iK}f_{kt} + \tilde{e}_{it}, \]  

(1)

where \( \tilde{R}_{it} \) = the random return on security i during period t, 
\( E_{i} \) = the expected return on security i, 
\( b_{ij} \) = the factor loading of security i on factor j,
\( f_{jt} \) = the factor score on systematic factor \( j \) during period \( t \),

\( e_{it} \) = the random error term.

The factor scores have mean zero and are assumed to be mutually independent and independent of random error terms. The random error terms are assumed to have mean zero and to be mutually independent. We also assume that all investors are risk-averse, single period maximizers of expected utility, and returns on common factors and on securities follow a multivariate normal distribution. Given the \( K \)-factor return generating process for \( n \) risky assets as shown in equation (1) and a risk-free asset if it exists (referred to 0-th asset), the return for any portfolio of the investor \( p \) with a composition of \( \mathbf{X}^p \) with \( (\mathbf{X}^p)'\mathbf{1} = 1 \) can be given by

\[
\mathbf{r}^p = (\mathbf{X}^p)'\mathbf{f} + (\mathbf{X}^p)'\mathbf{e} 
\]

where \( \mathbf{B} \) is a \( n \times k \) matrix of \( b_{ij} \). Further, we assume that portfolio cash flows for the investor \( p \) are generated at the end of the period. Under the above assumptions, the investor's utility is only a function of the mean and variance of his end-of-period cash flows, the covariance of his end-of-period cash flows with returns on common factors, and his current-period consumption. His utility function can be written as follows:

\[
U^p \{ E(\mathbf{w}^p), \text{Var}(\mathbf{w}^p), \text{Cov}(\mathbf{w}^p, \mathbf{e}), c^p \}
\]

We will drop the superscript \( p \) throughout this paper, except where required for clarity. Let us assume that the portfolio choosen by the
investor is well diversified, that is, $X'e = 0$. Then, $\tilde{W}$, $E(\tilde{W})$, $\text{Var}(\tilde{W})$, and $\text{Cov}(\tilde{W}, \tilde{f})$ can be defined by

$$\tilde{W} = W(1 + \tilde{R}^P)$$

$$E(\tilde{W}) = W(1 + E(\tilde{R}^P)) = W(1 + X'E)$$

$$\text{Var}(\tilde{W}) = (W)^2 \text{Var}(\tilde{R}^P) = (W)^2(X'BDB'X)$$

$$\text{Cov}(\tilde{W}, \tilde{f}) = WX'B$$

where $D$ = a $KxK$ diagonal variance-covariance matrix of returns on common factors, $\tilde{f}$,

$X$ = a $NxL$ vector of assets weights held by the investor,

$W$ = the wealth allocated to all assets for the investor,

$\tilde{W}$ = the end of wealth for the investor.

Let us denote $W_T$ as the total initial wealth for the investor. Now, the problem for the investor is to find the set of weights, $X$, and consumption, $c$, which maximize his expected utility subject to his budget constraint. That is,

$$\text{Max} \quad E[U\{E(\tilde{W}), \text{Var}(\tilde{W}), \text{Cov}(\tilde{W}, \tilde{f}), c\}]$$

$$\text{subject to } W + c = W_T \text{ and } X'1 = 1.$$  

Now, we can solve the individual's constrained optimization by forming the Lagrangian function, $L$, from (3) as follows:

$$L = E[U\{E(\tilde{W}), \text{Var}(\tilde{W}), \text{Cov}(\tilde{W}, \tilde{f}), c\}] + q_1(W_T - W - c)$$

$$+ q_2(X'1 - 1)$$
where $q_1$ and $q_2$ are Lagrangian multipliers. Taking partial derivatives with respect to $\mathbf{X}$ and $c$, and equalling them to zero, we have

\[
W_1 \mathbf{E} + 2W^2 \mathbf{B}^T \mathbf{D}^T \mathbf{B} \mathbf{X} + \mathbf{W} \mathbf{E} \mathbf{D} \mathbf{U} = q_2 = 0
\]  
\[
\partial \mathbf{E} / \partial c - q_1 = 0
\]

In equation (5), $U_1 = \partial \mathbf{E} / \partial \mathbf{E}(\mathbf{W})$, $U_2 = \partial \mathbf{E} / \partial \mathbf{var}(\mathbf{W})$, and $U_\mathbf{\bar{r}} = \partial \mathbf{E} / \partial \mathbf{cov}(\mathbf{\bar{W}}, \mathbf{\bar{r}})$. The Lagrangian multiplier $q_1$ in (4) is defined as the rate of change of the maximum value of $\mathbf{E} \mathbf{U}$ in (3) with respect to a small increase in the invested wealth $\mathbf{W}$, that is, $q_1 = \partial \mathbf{E} / \partial \mathbf{W}$. Inserting this definition into (6), we get $\partial \mathbf{E} / \partial c = \partial \mathbf{E} / \partial \mathbf{W}$. This is the envelope condition to equate the marginal utility of current consumption to the indirect marginal utility of wealth for future consumption.\(^7\)

Assuming that the expected return on the zero-beta portfolio with weights of $\mathbf{X}^0$, or the risk-free asset (if it exists), is $\mathbf{E}_0$, equation (5) pre-multiplied by $(\mathbf{X}^0)'$, and setting $(\mathbf{X}^0)' \mathbf{B} = 0'$ and $(\mathbf{X}^0)' \mathbf{I} = 1$, we obtain $q_2 = W_1 \mathbf{E}_0$. Substituting this result back into (5) and rearranging, we have

\[
T^P (\mathbf{E} - \mathbf{E}_0 \mathbf{I}) = \mathbf{B} \mathbf{D}^T \mathbf{X}^0 \mathbf{W}^P - \mathbf{B} \mathbf{D} \mathbf{H}^P
\]

where $T^P = -U^P / 2U^P$, a measure of absolute risk tolerance, and $H^P = -U^P / 2U^P$. Summing up (7) for all investors in the market and rearranging,

\[
\mathbf{E} - \mathbf{E}_0 \mathbf{I} = \mathbf{B} \mathbf{D}^T \mathbf{X}^m \mathbf{I} / \mathbf{T}^m - \mathbf{B} \mathbf{D} \mathbf{H}^m / \mathbf{T}^m
\]

\[
= \mathbf{B} (\mathbf{D}^T \mathbf{X}^m - \mathbf{D} \mathbf{H}^m / \mathbf{I})
\]
where $X^m$ is the weight of the market portfolio, $M$ is the total market value of all assets, $T^m = \Sigma^P$, $R^m = M/T^m$, and $H^m = \Sigma^P$. Here, we can consider $T^m$ and $R^m$ as the market absolute risk tolerance and the market relative risk aversion, respectively.

The portfolio whose return is perfectly correlated with common factor $j$, with weights vector $X^j$, has the following properties:

1. $(X^j)'B$, a row vector, which has zero elements, except for the $j$-th column which is unity and
2. $(X^j)'F = E^j$. We call these portfolios the fundamental portfolios (or fundamental funds). Then, premultiplying (8) by $(X^j)'$, we have

$$E^f - E_0 = DB'X^mR^m - DH^mR^m/M = 
\begin{bmatrix}
-b_1^2 - \sigma_1^2 h_{1}/M \\
\vdots \\
-b_j^2 - \sigma_j^2 h_{j}/M \\
\vdots \\
-b_k^2 - \sigma_k^2 h_{k}/M
\end{bmatrix}$$

where $E^f = [E^1, \ldots, E^k]$; $\sigma_j^2$ is the local variance of the return on factor $j$; $b^m = B'X^m$; $b_{jm}$, the $j$-th element of $b^m$, is the factor loading of the market portfolio on factor $j$; and $h_j^m$ is the $j$-th row of $H^m$. Let us denote the market risk premium for factor $j$, $\lambda_j = (E^j - E_0)$. Then equation (9) can be written as

$$\lambda_j = E^j - E_0 = [b_{jm} - h_{j}/M]R^m \sigma_j^2$$
in which the market risk premium for factor $j$ is proportional to its variance, and the market relative risk aversion. If the investor's utility is not a function of common factor $j$, and if $b_{jm}$ is normalized to one, the market premium for factor $j$ is only the product of its own variance and the relative market risk aversion.

Substituting (9) into (8) for the individual security,

$$E_i - E_0 = b_{i1}(E^1 - E_0) + ... + b_{ik}(E^k - E_0)$$

$$= b_{ik} \lambda_1 + ... + b_{ik} \lambda_k$$  \hspace{1cm} (11)

It can be seen that equation (11) is the same as the APT derived by Ross [10,11]. However, through the approach adopted in this paper, we can take into account the optimal portfolio decision problems explicitly and investigate the factor risk premiums from (10) simultaneously.

II. Optimal Portfolio Decisions and the Fund-Separation Theorem

To investigate the portfolio decision for the investor $p$, we need to find optimal weight of each asset in his portfolio from (7).

Because the rank of the $n \times n$ matrix $BB'$ is $k$, which is less than $n$, the inversion of $BB'$ does not exist. However, we can form the $k$ independent and simultaneous equation system from (7) by matrix inversion, as follows:

$$B'X^P = D^{-1}(B'B)^{-1}B'(E - E_0)(TP/WP) - H^P/WP$$  \hspace{1cm} (12)

Because there are $n$ unknown variables for $X^P$ but only $k$ independent equations in (12) and $k < n$, the solutions of this equation system are
infinite. However, we can get a unique solution based upon the k fundamental portfolios as defined before. Define $\mathbf{X}_f = [\mathbf{X}^1, \ldots, \mathbf{X}^k]$. Then, $\overline{\mathbf{X}}^P$ can be expressed by the following linear combination of $\mathbf{X}_f$ by post-multiplying the $k \times 1$ unique transformation vector $\mathbf{Q}^P$ with $(\mathbf{Q}^P)'_1=1$:

$$
\overline{\mathbf{X}}^P = \mathbf{X}_f \mathbf{Q}^P
$$

(13)

Below can be seen that $\mathbf{Q}^P$ must be a unique vector for the investor $p$. By substituting (13) into (12) and rearranging, we obtain the unique solution of $\mathbf{Q}^P$ based on $\mathbf{X}_f$ as follows:

$$
\mathbf{Q}^P = (B'\mathbf{X}_f)^{-1}D^{-1}(B'B)^{-1}B'(E - E_01)(T^P/W^P)
- (B'\mathbf{X}_f)^{-1}(H^P/W^P)
$$

(14)

which is a function of his wealth and preference.

**THEOREM** ("k+1 Fund" Theorem)

Given $n$ risky assets whose returns are generated by a $k$-factor linear model of equation (1) and a risk-free asset (if it exists, assign it to the 0-th asset), and if the assumption of a well diversified condition holds for every investor, then there are $k+1$ portfolios (mutual funds) such that: All risk-averse investors, who are expected utility maximizers, will be indifferent between choosing portfolios from among the original assets or from these $k+1$ funds; the proportions of each fund invested in the individual assets depends only upon the parameters $[E,F_0,B,D,E^f]$ and not on the investors' preferences; the investor's demand for the funds depends neither on
the knowledge of the investment opportunity set of individual assets nor the information on asset proportions held by the funds.

**Proof**

From (12) to (14), we know that the investor is indifferent about choosing among the original assets or the k fundamental portfolios plus the risk-free asset (or a zero-beta portfolio). Part 1 of the theorem is proved.

Obviously, the fundamental portfolios are independent of investor's preference, so we prove part 2 and 3 of the theorem. However, from (14), the investor allocates his wealth to the k+1 funds depending upon his utility function and his wealth.

O.E.D.

This theorem applies to each investor. The corollary below shows that, in equilibrium, each investor is indifferent between holding the original assets and the k+1 funds.

**Corollary**

In equilibrium, every investor behaves as if he allocates his wealth to

1. the risk-free asset (Fund 0), and
2. the k hedging funds (Funds 1, ..., k).

**Proof**

From (12)-(14), any investor's optimal portfolio can be replaced by a nonsingular linear combination of the k+1 funds. The market portfolio is simply an aggregation of all investors' optimal portfolios. Starting from (8), we can prove that there exists a unique k vector
to transform the market portfolio of risk assets into the k fundamental portfolios by using the similar procedures of (14) as follows:

\[ Q_m = (B'X_f)^{-1} D^{-1} (B'B)^{-1} (E - E_0) R^m - (B'X_f)^{-1} H^m/M. \]  

(15)

Therefore, the market portfolio of the risky assets is a linear combination of the k hedging funds.  

O.E.D.

From the theorem and its Corollary, we can see that the market portfolio plays no important role in the portfolio selection, and that the market portfolio may not necessarily be one of the k fundamental funds. This result further proves the arguments pointed out by Roll and Ross [9], and Ross [10,11] about the function of the market portfolio in the APT.

III. The Equilibrium APT in the Finite-Assets Market

The previous section has demonstrated that the market portfolio plays no prominent role in asset-pricing if investors hold well-diversified portfolios in the infinite-assets market. However, if the portfolio chosen by the investor with weights of \( X \) is not well diversified, which is a more general case, then \( \text{Cov}(W) = (W)^2 X' V_{aa} X \), where \( V_{aa} \) is a nxn variance-covariance matrix of returns on all risky assets.

Under this general situation, following the above procedures, the equations corresponding to those of (8), (9), and (11) can be written as follows:

\[ E - E_0 = V_{aa} X^m R^m - BDH^m R^m / M \]

\[ = a_{am} R^m - BDH^m R^m / M \]  

(16)
\[
\begin{bmatrix}
E^m - E_0 \\
E^f - E_0^1
\end{bmatrix} = \begin{bmatrix}
\sigma_m^2 & \sigma_{m1} & \cdots & \sigma_{mk} \\
\sigma_{1m} & \sigma_{11}^2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\sigma_{km} & 0 & \cdots & \sigma_k^2
\end{bmatrix} \begin{bmatrix}
R^m \\
-R^m h_1^m/M \\
\vdots \\
-R^m h_k^m/M
\end{bmatrix}
\]

\[
= \Sigma \begin{bmatrix}
R^m \\
-H^m R^m / M
\end{bmatrix}
\]

(17)

\[
E - E_0^1 = (V_{aa}, BD) E^{-1} \begin{bmatrix}
E^m - E_0 \\
E^f - E_0^1
\end{bmatrix}
\]

\[
= \beta_{a, mf} \begin{bmatrix}
E^m - E_0 \\
E^f - E_0^1
\end{bmatrix}
\]

(18)

where \(E^m\) is the expected market return, \(\sigma_m^2\) is the variance of the market portfolio, and \(\beta_{a, mf}\) is a \(n \times (k+1)\) matrix of multiple regression betas for all assets on the market and on the assets perfectly correlated with the changes of the common factors. Since the market portfolio is in general correlated with the common factors, \(\beta_{a, mf} \neq (\beta^m_{a}, B)\), where \(\beta^m_{a}\) are the beta coefficients in the CAPM. If and only if the
market portfolio is independent of all common factors, the equality holds.

It is important to note that the risk premium for common factor j of (17) is somehow different from (9). The market risk premium is a function of the variance of the market portfolio, the market relative risk aversion, the variances of the common factors, and the correlations between the market portfolio and the common factors. More importantly, we can see that the equilibrium APT of (18) holds even in the world of finite assets. However, the market portfolio will play an important role in the market equilibrium APT world, if the assumption of perfect diversification does not hold.

In addition, given the assumption that the inversion of \( V_{aa} \) exists in (16), the "k+2" fund instead of the "k+1" fund separation theorem is developed since any portfolio \( X^p \) is just a linear combination of the market portfolio, the risk-free asset, and the k fundamental funds. In this case, the market portfolio can no longer be expressed by a linear combination of the k fundamental funds, so one of the k+2 funds must be the market portfolio.

In this sense, the equilibrium APT can be regarded as being equivalent to Long's CAPM, so that the former is subject to the same criticism of the latter: the requirement of identification of the common factors and the market portfolio. Nevertheless, the empirical evidence that the market portfolio plays a major role in a linear factor model indicates at least that the equilibrium APT is more appropriate than the Ross APT in describing the return-risk relation of capital assets (see Wei [12]).
IV. Conclusion

The Ross's APT and its extensions have in general ignored the optimal portfolio decision problems of risk-averse investors, mainly because the derivation methods of the APT exploits the concept of a large security market only from the perspective of an investor holding a well-diversified portfolio. This paper has developed the equilibrium APT using the utility maximization approach, taking into account explicitly the portfolio decision problem. The equilibrium APT is a general and linear factor model without any approximation of the Ross's model or the assumption of a risk-averse agent holding a well-diversified portfolio. This paper has shown that the Ross's model is a special case of the equilibrium APT when investors are assumed to hold well-diversified portfolios. Of particular interest is that the market portfolio plays an important role in asset-pricing if the assumption of perfect diversification is relaxed. In addition, the equilibrium APT developed in this paper is demonstrated to be in the same spirit as either Long's or Merton's CAPM. Finally, the fund separation theorems have been derived from the equilibrium APT.

For the given empirical evidence that the market portfolio plays a prominent role in a linear factor model, we argue that the equilibrium APT is more general and appropriate than the Ross's APT in pricing capital assets.
Footnotes

1 See Dybvig and Ross [4] for a support of the APT and a complete bibliography of articles on the APT. See also Shanken [12] for a criticism of the APT.

2 The APT of Ross [10,11] was originally appealing since it did not rely on constrained optimization technique. However, for the same reason, it has a disadvantage of omitting the explicit consideration of individuals' portfolio selection problems.

3 Connor [3] independently derives a competitive APT, in which the market portfolio is also included.

4 All of the results in the paper have also been derived in a continuous-time framework. They will be available upon request to the authors.

5 For simplicity, we assume here for a moment that there does not exist a risk-free asset. It is easy to generalize from the result in the absence of a risk-free asset to the case that a risk-free asset exists simply assuming that the weight of the portfolio includes the risk-free asset.

6 In section III, this assumption will be released.

7 This is similar to the result derived from the intertemporal CAPM of Merton [7].

8 Breeden [1] has shown that it is not required to have a perfect correlation between the portfolio return on fund j and the change in state variable j. It is sufficient to derive the model with the maximum correlation between the return on fund j and the change in state variable j. Even though the above argument is directly applied to the intertemporal CAPM, it is also applied to the linear factor model in this paper.

9 Here we assume $X^p$ is the portfolio of only risky assets and the weight of the risk-free asset can be derived from $1 - X^p$. However, the constraint of equation (3) is replaced by $X^p + X^0 = 1$.

10 In a continuous-time framework, this model can be regarded as Merton's intertemporal CAPM.
References


