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by

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ABSTRACT

A fundamental problem of the theory of oligopoly is the indeterminacy of equilibria. Even limiting consideration to reaction function models, different conjectural variations can result in a plethora of equilibria. The purpose of this article is to investigate the basic premise that the various oligopoly model result in an unacceptable divergence in market equilibria. In the first half of the paper a theoretical model of a static conjectural variation duopoly with symmetric costs is presented. Using this model, we develop and quantify the concept of the indeterminacy of equilibria. In particular we define the coefficient of dispersion of equilibria and investigate its behavior. It turns out that the coefficient of dispersion is solely a function of the basic conditions of the market—costs and demand. This result is of empirical significance because it is possible to determine ex ante based on easily observable information, the extent to which conjectural variation (market conduct) will influence market equilibrium. To focus more closely on the empirical importance of this result, we examine the Western U.S. coal market and compute all of the well-known duopoly equilibria in this market (Cournot, Bertrand, Stackelberg, consistent conjectures).
I. INTRODUCTION

A fundamental problem of the theory of oligopoly is the indeterminacy of equilibria. By changing the assumptions made about the actions, interactions and strategies of firms, numerous market equilibria can be generated, ranging from pure monopoly to marginal cost pricing. There is a long history of attempts to narrow the set of plausible equilibria, with two significant recent developments. Bresnahan (1981) and Laitner (1980) have proposed new and supposedly superior models where each firm acts with full information about the strategies and actions of opponents. Bresnahan has shown that with modest restrictions on marginal cost and demand functions, such behavioral assumptions result in a unique static equilibrium. Another development has been in the applied economics literature where several authors (Iwata, 1974; Appelbaum, 1978; Just and Chern, 1980) have sought to econometrically determine, for specific markets, the beliefs held by firms about their opponents in those markets.

The purpose of this article is to investigate the basic premise that the various oligopoly models result in an unacceptable divergence in market equilibria. The question we address here is how the divergence of equilibria is affected by (a) the basic conditions of the market (costs and demand); and (b) the conduct of the market (the conjectural variations each firm holds with respect to its opponents). The general conclusion of our analysis is that conjectural variations are of secondary importance in determining market equilibria. The empirical significance of this is that a great deal can be understood
about a market through costs and demand, without information on conjectural variations. We also show that, by appropriate selection of a decision variable, the importance of knowing the conjectural variation can be further reduced.

We first examine the indeterminancy of equilibria from a theoretical perspective, using a static conjectural variation duopoly model assuming symmetric costs for the two firms. Using this model we define a measure of the indeterminacy, the coefficient of dispersion, as the ratio of the dispersion of the noncooperative conjectural variation duopoly equilibria relative to all plausible equilibria in the two-firm case. The coefficient of dispersion has several desirable properties, including a sole dependence on the relative slope of the demand and marginal cost functions in the market. Other properties of the coefficient of dispersion and the duopoly model in general are also developed.

The second part of the paper takes a closer look at the divergence of equilibria through an empirical analysis of a major natural resource market. We examine the market for Western U.S. coal where two producing regions (Montana and Wyoming) dominate production and appear to exercise market power. We find that in this market, the various duopoly equilibria are quite clustered. We also discover some empirically appealing properties of the Bresnahan/Laitner oligopoly equilibrium.

In the empirical analysis we introduce a new decision variable, intermediate between quantity and price, which is a more relevant control variable to our duopolists. The decision variable we consider is the markup over marginal cost.\(^3\) For most duopoly markets this decision
variable has appealing characteristics which should give more realistic results than output quantities (the traditional decision variables) or price (the problematic decision variable suggested by Bertrand). We calculate the standard conjectural variation equilibria with this decision variable to both expand our menu of duopoly solutions and also to examine the empirical properties of this new decision variable. In doing so, we are able to characterize the differences in market performance when quantities and markups are used as the decision variable and thus indicate when one may be a better choice than the other.

II. BACKGROUND

In contrast to the case of perfect competition, oligopoly theory is plagued by a plethora of theories for explaining the equilibria that result when a few firms interact (see Friedman, 1977, 1983). The oldest and probably the most common class of oligopoly models consists of the conjectural variation models. This class consists of the true dynamic models in which firms dynamically interact, reacting to each other's moves. Each firm's reaction function defines the way the firm reacts to its rivals. Such models are appropriately called reaction function models but the same term is also frequently used for the static conjectural variation models, such as originally developed by Cournot. With no dynamic interaction, it is a little difficult to describe the static models in terms of reacting to opponents' actions. However, this does not detract from the substance of the static models. In the same way that much can be learned from static general equilibrium models, static conjectural variation models are useful, to a large extent because of their simplicity.
A. Static vs. Dynamic Models

In both the static and dynamic models, each firm (i) has a quasi-concave profit function, \( \pi_i(x) \), which depends on the actions of all firms (x). Firm i maximizes profits by setting his action, \( x_i \), to satisfy the first-order condition

\[
\nabla \pi_i \cdot \frac{\partial x_i}{\partial x_i} = 0
\]

(1)

assuming an interior solution. It is in the vector \( \frac{\partial x}{\partial x_i} \) that the first static-dynamic ambiguity arises. For \( j \neq i \), the firm hypothesizes the value of \( \frac{\partial x_j}{\partial x_i} \). This is the firm's conjectural variation with respect to its opponents. Firm i conjectures how firm j will change its decision variable (e.g., output or price) in response to a small change in firm i's decision variable. Let firm i's conjecture about \( \frac{\partial x_j}{\partial x_i} \) be \( r_{ij} \). If \( r_{ij} \) is viewed as a response, then a multi-period interaction among the firms is implied. Viewing \( r_{ij} \) as i's conjecture of how firm j decides upon \( x_j \) is consistent with a static model. Substituting \( r_{ij} \) into equation (1) yields

\[
\frac{\partial \pi_i}{\partial x_i} + \sum_{j \neq i} r_{ij} \frac{\partial \pi_i}{\partial x_j} = 0
\]

(2)

A Nash equilibrium is the vector \( x \) (and there may be more than one) which simultaneously satisfies eqn. (2) for all oligopolists. In the static conjectural variation model, as in the static competitive equilibrium model, we are not concerned with the process for moving to an equilibrium, we are only concerned with the equilibrium.

A second static-dynamic ambiguity arises when eqn. (2) is rewritten
\[ x_i = f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, r_i) \]  

(3)

for the case of \( n \) oligopolists, where \( r_i \) is the vector of firm \( i \)'s conjectural variations. Clearly a Nash equilibrium is also a solution to the \( n \) equations of the form of eqn. (3). However, eqn. (3) is usually termed firm \( i \)'s reaction or best-reply function, indicating firm \( i \)'s best reply to the action of all the other firms. However, in a true static model firm \( i \) never sees the actions of its opponents until after it has made its move. The notion of reacting to opponent's actions is purely a dynamic phenomenon. Writing eqn. (2) in the form of eqn. (3) in a static model is perfectly correct; it is only in terming eqn. (3) a reaction function that the ambiguity arises.

In this paper we will follow the extremely common but misleading practice of dealing with static conjectural variation models in terms of reaction functions of the form of equation (3). As we have noted above, provided one does not incorrectly interpret the word "reaction function," this approach is entirely correct.

B. Static Conjectural Variation Models

Having defined static conjectural variation models, we now turn to the specific static models of Cournot, Stackelberg, Bertrand and Bresnahan (or Laitner). The only difference among these oligopoly models is the presumed conjectural variation.

On the continuum of possible assumptions about conjectural variations, extreme behavior is associated with the naïve models of Cournot and Bertrand. Without loss of generality, let quantity be the decision variable and let us now restrict ourselves to duopoly. In the Cournot
model, a firm presumes opponents will not respond to changes in the firm's output (zero conjectural variation). The original Bertrand model is based on a zero conjectural variation with price as the decision variable. In the case where goods are perfect substitutes and marginal costs are constant, each firm assumes any attempt to raise prices will result in a total abandonment of the market to the other firm. This description can be translated to the case where quantity is the decision variable. Each firm assumes that any attempt to exercise market power by reducing output will be totally foiled by an equivalent increase in output by its opponent (conjectural variation of -1). This assumption implies that at a Nash equilibrium, producers will price at marginal cost if the goods are perfect substitutes. Both the Bertrand and Cournot models assume very simple behavior of opponents.

The recently proposed consistent (or rational) conjectures equilibrium (Bresnahan, 1981; Laitner, 1979) involves the other extreme of availability of information about opponents. Each firm has rational expectations (perfect foresight) regarding conjectural variations and thus acts with full knowledge of the actual reaction functions of opponents. Thus, in equilibrium, the assumed conjectural variations are in fact the slopes of opponents' reaction functions. The consistent conjectures equilibrium (CCE) has some desirable properties relative to the other models. The other models fall prey to the criticism that although each firm is on its best-reply function, this best-reply output is "right for the wrong reasons" (Fellner 1949). Each firm's best-reply is based on an incorrect perception of its opponent's behavior.
For a CCE the assumption one makes about opponent behavior is the same as actual opponent behavior in equilibrium—one is "right for the right reasons." However, because of its use of a reaction function, the CCE is stretching the limits of the static model. To be correct, dynamics should really be explicitly introduced (see Cyert and DeGroot, 1970).

Intermediate between the two informational extremes of the CCE and the Cournot or Bertrand models is the Stackelberg model which posits an information asymmetry among the duopolists. One oligopolist acts naively as a Cournot oligopolist. The other duopolist, a leader, acts with full knowledge of the reaction function of its opponent.

All of these models involve conjectural variations (with quantity as the decision variable) between 0 and -1. Obviously, a continuum of oligopoly models can be generated by presuming different conjectural variations between 0 and -1. Each, of course, will generally result in a different market equilibrium.

The markup decision variable introduced in this paper avoids some of the ambiguity regarding conjectural variation. In many markets, including the Western U.S. coal market examined here, the assumption of zero conjectural variation in markup (tax rate) space appears to be very realistic. Markups tend to be relatively constant over time (as opposed to quantities or prices) and, in the standard oligopoly situation, are difficult if not impossible to observe for one's opponents. To observe markup, it is necessary to know one's opponents' cost structure. Further, a zero conjectural variation in terms of markup is equivalent to some nonzero conjectural variation in terms of quantities, suggesting behavior intermediate between the Cournot and Bertrand
models. Thus, by selecting a decision variable that shows little empirical temporal variation, such as per cent markup over marginal cost, an assumption of zero conjectural variation for these variables is a very useful way of partially avoiding Fellner's criticism of the Cournot duopoly equilibria.

III. THE DIVERGENCE OF DUOPOLY EQUILIBRIA

The basic question of this paper is how divergent are the equilibria from the various conjectural variation duopoly models. We answer this question in this section by examining the range of output associated with noncooperative duopoly relative to all possible economically meaningful output levels. We focus on a simple static conjectural variation model of duopoly where both firms have identical costs and produce identical products. However, to assess the effect of conjectural variation on equilibria, we allow the firms to have different conjectural variations.

Before considering the potential output levels associated with noncooperative duopoly, consider the potential output levels in a two-firm market, for any market conduct assumption. The economically meaningful set of output levels are those where each firm prices at or above marginal cost (even then, profits may be negative but we will ignore this). Furthermore, output levels below the contract curve (the curve associated with cooperative behavior) can be excluded since movement from such levels to the contract curve makes both producers and consumers better off. Consequently the region of meaningful output levels lies between the contract curve and the output curve associated with marginal cost pricing. Within this region lie the levels of output associated
with the usual models of noncooperative duopoly. The important question to ask is what is the potential dispersion of output levels from various noncooperative duopoly models relative to the overall possible dispersion of output levels.

We first define the set of models of noncooperative static duopoly. We argue that it is reasonable to define this set as the set of conjectural variation equilibria where each firm's conjectural variation lies between 0 and -1. Extremes of industry output are associated with the Cournot and "Bertrand" duopoly (conjectural variations of 0 and -1, respectively). As mentioned before, with identical goods, the Bertrand duopoly is equivalent to marginal cost pricing and thus is associated with maximum output. A conjectural variation below -1 implies pricing below marginal cost. Thus -1 is a natural lower bound on the conjectural variation. Lower output is associated with a conjectural variation of zero. As Bresnahan (1981) argues, a conjectural variation greater than zero implies a form of collusion. A firm conjectures that as it lowers its output, its opponent will do likewise. Thus zero is a plausible upper limit on conjectural variation. All of the other duopoly models (including the consistent conjectures model) involve conjectural variations between these extremes.

We examine a market for a homogeneous good with an inverse demand function\( P(q_1 + q_2): \)

\[
P(q_1 + q_2) = a + b(q_1 + q_2). \quad (3)
\]

Assume \( b < 0 \). There are two producing firms in this market, each with identical production costs, given by
\[ C(q_i) = C_0 + C_1 q_i + \frac{C_2}{2} q_i^2 \] (4)

with \( C_1, C_2 > 0 \) and \( a > C_1 \).

Profits for the \( i \)th firm are thus given by

\[ \pi_i = \left[ a + b(q_i + q_{<i>}) \right] q_i - C_0 - C_1 q_i - \frac{C_2}{2} q_i^2 \] (5)

where \( <i> \) denotes the other firm. First-order profit maximizing conditions are

\[ u_i = a + b(q_i + q_{<i>}) + bq_i(1 + r_i) - C_1 - C_2 q_i \leq 0 \] (6)

\[ q_i \geq 0, u_i q_i = 0, \]

where \( r_i \) is the firm \( i \)'s conjectural variation: \( \partial q_{<i>}/\partial q_i \), conjectured. Since \( C_2 > 0 \), eqn. (6) can be rewritten as

\[ w_i = \beta + \left[ a(2 + r_i) - 1 \right] q_i + a q_{<i>} \leq 0, q_i \geq 0, w_i q_i = 0 \] (7)

where \( \alpha = \frac{b}{C_2} \) and \( \beta = \frac{a - C_1}{C_2} \).

Note that \( \alpha \) is negative and \( \beta \) is positive. Assuming an interior solution to equation (7) for both firms, the Nash equilibrium can be easily computed:

\[ q_i^* = \frac{\beta [1 - \alpha (1 + r_{<i>})]}{[\alpha (2 + r_i) - 1][\alpha (2 + r_{<i>}) - 1] - \alpha^2}. \] (8)

It is now possible to derive the earlier statements that maximum output is associated with the Bertrand equilibrium and minimum output with the Cournot equilibrium. This result is embodied in the following proposition:
Prop. 1: Assuming a quadratic duopoly model with identical cost structures for the two firms, the equilibrium output for each firm is monotone with respect to each of the conjectural variations:

\[
\frac{\partial q_i^*}{\partial r_i} < 0, \quad \frac{\partial q_i^*}{\partial r_{<i>}^*} > 0
\]  

(9)

Proof: The proof follows directly from differentiating equation (8):

\[
\frac{\partial q_i^*}{\partial r_i} = -\beta [1 - \alpha (1 + r_{<i>})] \frac{\alpha \alpha (2 + r_{<i>}) - 1}{\alpha (2 + r_{<i>}) - 1 - \alpha^2}^2
\]

(10)

Since by assumption \( \alpha < 0, \beta > 0 \) and \(-1 \leq r \leq 0\), the above expression is negative. Similarly the sign of \( \frac{\partial q_i^*}{\partial r_{<i>}} \) can be shown to be positive.

Q.E.D.

Thus output for each firm is greater under a Bertrand model than under a Cournot model, holding the other firm's conjectural variation constant. However, the proposition implies that maximum output for firm \( i \) is associated with \( r_i = -1, r_{<i>} = 0 \) and similarly minimum output is associated with \( r_i = 0, r_{<i>} = -1 \).

The result is shown in Figure 1 for the case of \( \alpha = -1 \). The scatter of points in the figure is the set of Nash equilibria associated with 1000 randomly selected pairs of conjectural variations \((r_1, r_2)\), each element of which is chosen from an independent uniform distribution over \([-1,0]\). As we have argued, the region represents the set of possible non-cooperative equilibria. By comparing the size of the region with the size of the region of nonnegative output between ABCDE (maximum output) and GH (the contract curve) a measure of the extent of cluster of the non-cooperative equilibria can be developed.
**Defn:** Define the coefficient of dispersion, $\sigma$, as the ratio of the area (in $q_1 - q_2$ space) of the set of noncooperative equilibria to the area of the set of economically feasible equilibria, bounded by the contract curve and the marginal cost pricing curve.

For Figure 1, with $\alpha = -1$, the coefficient of dispersion, $\sigma$, is equal to approximately 17%. This means that of the total area between the contract curve and the maximum output curve, only 17% corresponds to noncooperative equilibria.

The coefficient of dispersion, $\sigma$, has two interesting properties. One is that it is independent of $\beta$. Only the relative slopes of the marginal cost and demand function ($\alpha$) determine $\sigma$. Further, $\sigma$ is monotone in $\alpha$, increasing as $\alpha$ becomes larger in absolute value, approaching a limiting value of $4/9$:

**Prop 2:** Assuming a quadratic duopoly model with identical cost structures for the two firms and conjectural variations between $-1$ and 0, then the coefficient of dispersion, $\sigma$, is monotone increasing in the absolute value of $\alpha$ (the relative slope of the demand and marginal cost functions, $b/C_2$) approaching a limiting value of $4/9$ as $\alpha$ approaches minus infinity.

**Proof:** See appendix.

Figure 2 indicates the behavior of $\sigma$ as a function of $\alpha$, the relative slope of the marginal cost and demand functions. Two cases would lead to a maximum value of $\sigma$: steeply sloped demand or a shallowly
sloped marginal cost function. On the other hand, the dispersion will be small if demand is shallowly sloped (highly price sensitive) or marginal cost is steeply sloped (rapidly rising marginal costs).

The significance of this result is that one can tell the importance of the conjectural variation in determining a duopoly equilibrium solely on the basis of observable characteristics of the market; it is not necessary to observe the conjectural variation. Thus this result has empirical utility, since costs and demand relations are much easier to observe than market conduct.

IV. AN EMPIRICAL ANALYSIS OF DUOPOLY

The principal purpose of this section is to examine the clustering of duopoly equilibria in an empirical setting. There are several reasons for this. In an empirical setting, we can examine a larger variety of duopoly models than is readily possible in the theoretical analysis. We can explore criteria other than total output (e.g., unit profit) as measures of the clustering of equilibria. We can also quantify, for an actual market, the extent to which the equilibria associated with the various models actually diverge. Finally, we can explore the performance of markup over marginal cost as a decision variable.

The market we examine is the Western U.S. coal market. In a previous paper (Kolstad and Wolak, 1983), we developed an empirical model of this market. That model will serve as the basis for this analysis. The market is dominated by two states, Wyoming and Montana. Each of these states possesses substantial market power because of large reserves of desirable (low sulfur) and inexpensively extractable coal. This coal
is by far the cheapest in the U.S. to produce (excluding low quality lignite) and even after significant transport costs are added, competes very favorably in distant midwestern and southern U.S. markets. The popularity of coal from these two states has caused a rapid increase in its production over the last decade and the imposition of substantial severance taxes.6

We assume that these two states act as duopolists even though there may be many producers in each state. Each state is in a position to intercede between producers and consumers, taxing at any level it wishes, coal which is sold. The two states have the market power and are thus duopolists. The duopolists interact with a competitive fringe of producing states and thus face a residual demand curve. The two states are assumed to act to maximize tax revenues where the tax is expressed a fractional markup over marginal production cost. We examine the conventional duopoly models of Cournot, Stackelberg, and the CCE. In order to expand this menu of models somewhat, we examine these models from the perspective of tax rate as the decision variable as well as output quantity as the decision variable. Of course a particular conjectural variation where tax rate is the decision variable can be translated into a conjectural variation where quantity is the decision variable.

Tax rate is an obvious decision variable in the case of severance taxation. However, if the tax rate is viewed as a per cent markup over cost, its applicability becomes more general. In many cases, oligopolistic firms do seem conscious of the percentage markup over cost of their product and in many cases utilize that value as the operable
variable in their maximization decision. Also, for most oligopolistic environments, institutional and political factors make the optimal setting of quantity very difficult. In these cases, optimally setting markup and satisfying demand at that markup level is far more tractable. Further, it would appear that the level of the markup remains relatively constant over time, particularly for the Western U.S. coal market suggesting the appropriateness of a zero conjectural variation in markup.

In the next section of the paper we develop the various duopoly models in a general framework and discuss the equilibria associated with the several models. We then present the results of the analysis and discuss the clustering of the various equilibria.

A. Duopoly Equilibria

The market we examine consists of two states that produce a homogeneous good and are separated from a single aggregate consumer. Each individual producer falls under a single political jurisdiction and therefore will be aggregated into one of the producing states. For the \( i \)th producing state, we denote \( q_i, t_i, \tau_i \) and \( S_i(q_i) \) as, respectively, the output, tax rate (as a fraction, to be applied to marginal cost), per unit transport cost from producers to consumers and the marginal cost function for production of the good. Let the inverse demand function be denoted by \( P(\sum_j q_j) \).

1. Quantity as the Decision Variable. We first consider the classic quantity-based duopoly models. In such cases, the state restricts output through the sale of export licenses or through some other device in order to maximize the monopoly rents accruing to the state. The objective function for each state is:
Max \( q_i \left[ P(\sum_j q_j) - S_i(q_i) - \tau_i \right] \), \( \forall i \)  \hspace{1cm} (11)

The first order conditions for this maximization (assuming positive output) are

\[ P(\sum_j q_j) - S_i(q_i) - \tau_i + q_i \left[ P'(\sum_j q_j)(1 + r_i) - S'_i(q_i) \right] = 0, \forall i \]  \hspace{1cm} (12)

The term \( r_i = \frac{\partial q_i}{\partial q_i} \) represents the \( i^{th} \) state's conjectural variation with respect to the other state \( (i') \). Note that because of the form of the objective function, a conjectural variation of -1 does not result in marginal cost pricing unless marginal costs are constant.

We can solve these first order conditions to determine reaction functions which give each \( q_i \) in terms of the other state's output level, \( q_i \), and the \( i^{th} \) state's conjectural variation, \( r_i \):

\[ q_i = R_i(q_i, r_i) \]

Given values of \( r_i \) for each state, we then have two equations in two unknown quantities. These reaction functions can be solved to obtain the duopoly equilibrium.

It is precisely the value of \( r_i \) assumed that determines the different equilibria. For the case of the Cournot equilibrium the conjectural variations are assumed to be zero (i.e., \( r_i = 0, \forall i \)). The Stackelberg equilibrium is a bit more difficult to characterize in this framework. Assume one state \( (i) \) is the leader and the other state \( (i') \) is the follower. We first determine the Cournot reaction function for the follower by assuming \( r_{i'} = 0 \). Given this reaction function we then calculate its derivative with respect to the leader's output, \( q_i \). This
derivative is set equal to the leader's conjectural variation, $r_i$. The reaction function for the leader state can then be calculated which in turn permits the computation of a Stackelberg equilibrium.

The other equilibrium concept we will consider is the consistent conjectures equilibrium (CCE). In this case each firm's (i) conjectural variation ($r_i$) is equal to the derivative of the other firm's ($<i>$) reaction function with respect to $q_i$. This condition allows us to completely determine all $r_i$ and thus solve for a CCE. It should be pointed out that in the case of the CCE, we assume a constant conjectural variation, independent of output levels (as does Bresnahan, 1981). Thus, we make a linear approximation to a duopolist's conjecture about its opponent's reaction function.

2. **Markup (Tax rate) as the Decision Variable.** Using the same basic framework, we can calculate oligopoly equilibria based on fractional markup over marginal cost, or in our specific case, a tax rate:

$$t_i = \frac{P(\sum q_j) - S_i(q_i) - \tau_i}{S_i(q_i)}.$$ (13)

The market equilibrium for a given set of taxes is determined by

$$(1 + t_i)S_i(q_i) + \tau_i = P(\sum q_j), \quad \forall i$$ (14)

Because there are many producers within a state, the market price net of tax will be marginal cost, $S_i(q_i)$. Thus, Eq. (14) states that the producer prices (including tax) plus transport cost must equal the price to consumers.
For the purposes of tax equilibrium, we assume that each producing state $i$ seeks to maximize tax revenues given the boundary equilibrium conditions (14). The simultaneous solution of the first order conditions for this maximization and the equilibrium conditions results in a set of tax reaction functions giving a state's tax rate ($t_i$) as a function of the other state's tax rate ($t_{<i>$}) and state $i$'s conjectural variation:

$$t_i = T_i(t_{<i>}, r_i).$$

(15)

This approach can be taken for each state yielding a tax reaction function such as Eq. (15) for each jurisdiction. This results in a set of equations in the same number of unknowns as tax rates, which can then be solved for a set of equilibrium tax rates as a function of the conjectural variations. Computation of the simple Nash (zero conjectural variation), Stackelberg and consistent conjectures equilibria follow the discussion of the previous section where quantity was the decision variable.

B. The Empirical Model

We are now ready to proceed to the actual empirical analysis of a duopoly. As mentioned earlier, we will treat two states, Montana and Wyoming, as duopolists. For our purposes we will treat these states as if they sell coal at a single, distant demand center (e.g., the "midwest").

In previous work (Kolstad and Wolak, 1983) we presented an econometric model of this market. To summarize, a large and detailed spatial equilibrium model of the U.S. coal market was used to generate
"pseudo-data" which were then used to estimate a simple model of the market, as described in the previous section. The model consists of five linear\(^{11}\) equations: a single demand function, two marginal cost (net of tax) functions for coal produced in the two states and two transport cost functions indicating the cost of moving coal from each of the states to the demand center. Transport costs (per unit) are assumed to be a function of the mine-mouth price of the coal (including tax), principally to capture the effect that as coal price goes up, a producer's market area shrinks, thus reducing average transport costs. Thus the model is quite similar in structure to that discussed in the previous section. For further information on the model, the reader is referred to Kolstad and Wolak (1983).\(^ {12}\)

V. RESULTS

Given the estimated model of the Western U.S. coal market, we can proceed to compute the various duopoly equilibria. For each model, and for each state, we examine equilibrium output, the tax rate necessary to support the equilibrium and resulting severance tax revenues. For reference, we have also computed these variables for the zero tax rate situation as well as collusion in tax setting. The collusive situation results in a variety of outcomes (the contract curve), which depend on the weight given to revenue from each of the states.

Table I presents the output, tax rates and revenue associated with the various equilibria. These data are presented graphically in Figs. 3 and 4. Figure 3 shows the feasible region for output levels for the two states. Shown in the figure is the line of maximum output, assuming taxes cannot be negative. In Fig. 3, the band between the origin and
the maximum output line represents the region of nonnegative outputs. This region can be further restricted by assuming that output levels in the region between the origin and the contract curve are unlikely since the contract curve represents maximum producer power and producers are assumed to behave rationally. Consumers and producers could be made better off by moving to the contract curve from its interior. Also shown in this figure is output associated with the various duopoly models, keyed to the case number in Table I. Furthermore, the broken line in the Figure represents the boundary of the set of non-cooperative equilibria where both conjectural variations lie between -1 and 0. Figure 4 shows the feasible region for tax rates for the two states. Although any tax rate can be set, by assuming profit maximization the region between this line and the origin represents plausible values for tax rates, by the same arguments as were used for Fig. 3. Tax rates associated with the various duopoly equilibria are also shown.

The same pattern occurs in the two figures. For a given decision variable the equilibria are very close together and regardless of the decision variable, the equilibria are still quite clustered when viewed relative to the set of possible or plausible outcomes. This clustering can be quantified by looking at the entire set of equilibria as the conjectural variation for each state ranges through the values between -1 and 0. The boundaries of the set of duopoly equilibria with quantities as the decision variable are shown by the broken line in Figures 3-4. In both figures the set of duopoly equilibria is a significant but modest subset of possible tax rates or output quantities. As was mentioned earlier, due to the form of the tax objective (eq. 11), a
conjectural variation of -1 for both states does not quite correspond to marginal cost pricing. This is clear from Figure 3.

Consider Figure 3, which is the empirical analog of Figure 1. Plausible output levels lie between the contract curve and the maximum output frontier. The region of duopoly equilibria is enclosed by the broken line. The area of this region is such to give a coefficient of dispersion of \( \sigma = 15\% \). Of the overall plausible region of equilibria, only 15\% is associated with noncooperative duopoly. And as can be seen, a significantly smaller region corresponds to the specific duopoly models considered in this paper. How does a \( \sigma \) of 15\% correspond to the predictions of the theory developed earlier in the paper? Unfortunately, the empirical model differs somewhat from the theoretical model, principally in terms of differing (rather than identical) cost structures and slightly different objectives (e.g., eq. 5 vs. eq. 11) for the two duopolists. The ratios of the slope of the demand and marginal cost functions \( (\alpha) \) are 1.76 for Wyoming and 12.3 for Montana. From Figure 2, this would imply a \( \sigma \) between .2 and .4, a somewhat higher figure that in actuality. In general, a high \( \alpha \) for Montana would lead one to expect a larger variation in output levels for the state than for Wyoming. This is in fact confirmed by inspection of Figure 3. The equilibrium output levels for Montana cover a range of about 300 million tons per year compared to a similar value of about 200 million tons per year for Wyoming.

Turn now to Figure 4 where the equilibrium tax rates are shown. Note that when Montana uses a conjectural variation of -1, its tax rate drops significantly to less than 10\%. Although it cannot be totally
inferred from the figure, this drop-off occurs very rapidly. For \( r_m = -0.7 \), the Montana tax rate is 23% with \( r_w = -1 \) and 32% for \( r_w = 0 \). As one moves only slightly away from the -1 conjectural variation, equilibrium tax rates rise appreciably.

Now let us examine the specific duopoly models enumerated in Table I. By closer inspection of the diagrams one can see that in all cases but the CCE, the tax-based duopoly equilibria are further from the contract curve than the quantity based-solutions. This simply means that the duopolists in the tax-based case are more conscious of the effect of their own behavior on their rival's behavior. This recognition effectively reduces producer power. Thus, the markup or tax-based equilibrium represents an attractive intermediate structure between the unrealistically competitive marginal cost pricing and the highly non-competitive Cournot model.

Note that the consistent conjectures equilibrium is largely invariant to the decision variable used. On reflection, it is easy to see that both of the solutions should be the same, because both require full knowledge of the opponent's behavior. Consequently, we would expect that the same amount of information about one's opponent, regardless of the decision variable used, would yield the same market outcome.

However, it is interesting to note that for the quantity based CCE more information is known about one's opponent than in the markup based CCE. For the quantity based equilibrium, the CCE is global because the slope of the opponent's reaction function is everywhere equal to the other player's conjectural variation. For the tax-based equilibrium this is true only in a small neighborhood of the CCE. Thus, for the
tax-based solution each player has knowledge only of the first order properties of the other's reaction function, whereas for the quantity based equilibrium, each has complete knowledge of the other's reaction function. We can see that the additional information associated with the quantity based CCE results in a slightly more competitive solution than the tax-based case. In this instance we can see that more correct information about one's opponent leads to a more competitive outcome.

This invariance of equilibria with respect to the instrument used does not carry over to the other types of duopoly equilibria. Nevertheless, certain characteristics of the cluster of the oligopoly solutions is invariant to the instrument chosen. Both sets of equilibria display the characteristic that when one state is a leader its market position relative to its opponents increases when compared to the Cournot and CCE equilibria.

Another interesting feature of the consistent conjectures solution is that it is an outlier for equilibria under both sets of decision variables. In the case of the quantity based decision variable, the CCE is the solution furthest from the contract curve, so in a sense, an outer bound for that set of equilibria. For the tax-based equilibria, the consistent conjectures solution is the closest to the contract curve, and therefore is an inner bound for this set of solutions.

This leads to another point about the set of oligopoly equilibria. In the case of the markup decision variable, we can see that as more information is utilized about one's opponent's reaction functions, duopoly equilibria become closer to the contract curve. Arranged in order of increasing information utilized, are the Nash-Cournot,
Stackelberg, and then the CCE. The justification for this ordering is that the Nash-Cournot model assumes nothing about one's opponent's costs or behavior. The Stackelberg model assumes that the leader knows the reaction function of the follower, but that the follower know nothing about the leader's strategy. Finally, it is easy to see that the CCE requires the most information.

VI. POLICY IMPLICATIONS

For the quantity based equilibria more information has the opposite impact. In this case, we see that increases in information lead to more competitive market outcomes (i.e., further from the contract curve). We can conclude that regardless of the decision variable, as more and more information about an opponent's strategy is utilized all of these noncooperative equilibria approach the consistent conjectures solution.

What interpretation should be given to these results in terms of actual and potential behavior of Montana and Wyoming in levying severance taxes on coal? First of all, it would appear that even though both states may produce approximately the same amount of coal, Wyoming has more market power. Since both states face the same demand curve, that difference must be due to different cost structures in the two states.

Also, the range of possible duopoly equilibria is significant but moderate. Tax rates for Wyoming are generally between 30% and 65% whereas Montana's rates are between 10% and 65%. The consistent conjectures equilibrium seems to be an approximate median for the specific
duopoly models considered, yielding tax rates and output levels intermediate between the extremes of the two Nash-Cournot models. The consistent conjecture equilibria with output as the decision variable yields taxes of $\left( t_m, t_w \right) = (0.35, 0.44)$ which is very close to actual rates for Montana (33%) but somewhat high for Wyoming (compare to 17%). Thus tax rates may go higher than at present, based on this model of the tax-setting process. See Kolstad and Wolak (1984) for a different view on this issue.

VII. CONCLUSIONS

The goal of this paper is to investigate the variation in equilibria that results from different assumptions about market conduct within a conjectural variation duopoly model. We have derived two basic sets of results.

At the theoretical level we have developed a measure of the spread of duopoly equilibria, the coefficient of dispersion, $\sigma$. We have shown that this coefficient is dependent only on the relative slopes of the demand and marginal cost curves. Further, $\sigma$ is monotonically increasing in this relative slope. No dispersion ($\sigma = 0$) is associated with perfectly elastic demand or perfectly inelastic marginal costs. Maximal dispersion ($\sigma = 4/9$) is associated with perfectly inelastic demand or constant marginal costs.

The results regarding the coefficient of dispersion are particularly important for empirical work. The fact that market conduct is so difficult to observe has lead to significant problems in understanding duopolistic markets. However, as was shown, it is observable characteristics of the market (costs and demands) which determine the importance
of the conjectural variation in the realized equilibria. If the market is such that $\sigma$ is low, then it is relatively unimportant to try to measure the conjectural variation in that market.

These results we confirmed in our empirical analysis of the western U.S. coal market. While we argued that the states of Montana and Wyoming are effectively duopolists in this market, we were unable to argue for a particular market conduct model. However, in generating the equilibria associated with a variety of market conduct assumptions, we were able to significantly narrow the range of possible equilibria. Using the coefficient of dispersion measure developed in the first part of the paper, possible noncooperative equilibria constitute only 15% of all possible equilibria. And if the extreme cases of Montana's conjectural variation being near $-1$ are rejected, then the spread of possible equilibria shrinks even further. For the cases of the eight specific duopoly models considered in the paper, there appeared to be a tendency for the equilibria to cluster about the consistent conjecture equilibria, suggesting that the consistent conjectures equilibrium is a good choice when there is no a priori information about market conduct.

Our results also reveal several characteristics of the markup based oligopoly equilibrium. We can see from the results that this solution concept with zero conjectural variation is close to the CCE. In cases where there may not be a unique CCE solution or a CCE is not well defined (Robson, 1983), this type solution represents a very attractive (and computationally tractable) alternative. This solution concept is
preferred for oligopolistic markets where competitors recognize the effect their decision has on their opponents but have very little information about the reaction function of their opponents. The traditional quantity-based analysis is better suited for cases when competitors are not nearly as conscious of each other's effect on one another.
FOOTNOTES

1 University of Illinois and Harvard University, respectively. Work was supported in part by the U.S. Department of Energy through the Economics Group of the Los Alamos National Laboratory in New Mexico. The assistance of Bill Beyer and Myron Stein is gratefully acknowledged.

2 A number of other authors have recently proposed or discussed similar models [Ulph (1982); Kamien and Schwartz (1981); Capozza and van Order (1979)].

3 This decision variable has a long history in the theory of imperfect competition. To quote Joan Robinson (1969, p. vii): "Prices are formed by setting a gross margin, in terms of a percentage on prime costs."

4 Laitner (1980) discusses the limited "rationality" of the consistent conjectures equilibrium in the static context.

5 In the rest of this paper, we use the term Bertrand equilibrium to refer to the case where quantity is the decision variable and the conjectural variation is -1. This may not always be the same as the original Bertrand model.

6 Tax rates (including property taxes) in both states are in terms of a per cent of the price (net of tax), FOB mine. The 1982 rate in Montana for surface coal was 33%; the rate in Wyoming was 17% (Blackstone 1982).

7 In retail establishments, for example, prices are often set through a fixed percent markup over costs (Scherer 1980).

8 Of course, just because a zero conjectural variation seems reasonable does not mean that the slope of opponents' reaction functions will be zero which would be required for consistency.

9 Bresnahan (1981) has shown that in a linear model such as this with quantity as the decision variable, then a constant conjectural variation involves no approximation at all.

10 More details on this derivation may be found in our earlier paper (Kolstad and Wolak, 1983).

11 The assumption of linearity is principally to lessen the already sizeable computational burden to calculate equilibria in the markup case, although the data appear to fit a linear model reasonably well.
The markup model only was developed in Kolstad and Wolak (1983). However, the quantity model can be easily developed using the coefficients presented in our earlier paper and the model of eqs. 11-13 in this paper. This results in the following expressions to be maximized (eq. 11):

\[ \pi_m = q_m [10.426 - .01781q_m - .01647q_w] \]

\[ \pi_w = q_w [10.569 - .01835q_w - .01171q_m] \]
REFERENCES


David Ulph, "Rational Conjectures in the Theory of Oligopoly," mimeo, Department of Political Economy, University of College London (October 1982).
APPENDIX

Proof of Prop 2:

Define the areas under the maximum output curve and contract curves as \( M \) and \( C \), respectively. Define the area of the region of duopoly equilibria as \( D \). Clearly \( \sigma = D/(M - C) \). The areas of \( D \) and \( C \) can be easily computed using the coordinates of the various vertices as shown in Fig. 1. These coordinates result from a simple application of eq. (8) and Prop. 1:

\[
\text{Area of } D = \frac{\alpha^2 \beta^2}{(\alpha^2 - 3\alpha + 1)(1 - 3\alpha)} \quad (A-1)
\]

\[
\text{Area of } M = \frac{\beta^2}{(1 - 2\alpha)(1 - \alpha)} \quad (A-2)
\]

The contract curve is slightly more complicated to derive. It is defined maximizing producer \( i \)'s profits subject to producer \( \langle i \rangle \) being at a constant profit level

\[
\max_{q_i, q_{\langle i \rangle}} \left[ a + b(q_i + q_{\langle i \rangle}) \right]q_i - C_0 - C_1q_i - \frac{C_2}{2}q_i^2 \quad (A-3a)
\]

s.t. \[
\left[ a + b(q_i + q_{\langle i \rangle}) \right]q_{\langle i \rangle} - C_0 - C_1q_{\langle i \rangle} - \frac{C_2}{2}q_{\langle i \rangle}^2 = \pi_0 \quad (A-3b)
\]

\[
q_i, q_{\langle i \rangle} \geq 0 \quad (A-3c)
\]

First order conditions for this maximization, assuming both producers are at positive output levels are

\[
a + b(q_i + q_{\langle i \rangle}) + bq_i - C_1 - C_2q_i + \lambda bq_{\langle i \rangle} = 0 \quad (A-4a)
\]
\[ bq_i + \lambda [a + b(q_i + q_{<i>}) + bq_{<i>} - C_1 - C_2q_{<i>}] = 0 \quad (A-4b) \]

These two equations can be reduced to
\[ [(2\alpha - 1)q_i + \alpha q_{<i>} + \beta] [(2\alpha - 1)q_i + \alpha q_{<i>} + \beta] - \alpha^2 q_i q_{<i>} = 0 \quad (A-5) \]

which can be solved for \( q_{<i>} \) as a function of \( q_i \) and integrated from 0 to \( q_i = \beta/(2\alpha - 1) \):

Area of \( C: \quad \frac{\beta^2}{(1-2\alpha)^2(1-4\alpha)^2} \left\{ (1-3\alpha)(1-4\alpha) - \alpha^2 \sqrt{1-4\alpha} \left[ \sinh\left(\frac{3\alpha-1}{2\alpha}\right) + \sinh\left(\frac{1}{2\alpha}\right) \right] \right\} \]  

(A-6)

The coefficient of dispersion can then be written as
\[ \sigma(\alpha) = \frac{(1-2\alpha)(1-4\alpha)^2(1-\alpha)^2}{(1-3\alpha)(\alpha^2-3\alpha+1)(\alpha(1-4\alpha)(5\alpha-2)+\alpha^2(1-\alpha)\sqrt{1-4\alpha} [\sinh\left(\frac{3\alpha-1}{2\alpha}\right) + \sinh\left(\frac{1}{2\alpha}\right) ]} \]  

(A-7)

Note that \( \beta \) has disappeared from the above equation. In proving that \( \sigma(\alpha) \) is monotone increasing, we need only show that \( \sigma'(\alpha) < 0 \) for \( \alpha < 0 \). In applying the quotient rule to finding \( \sigma'(\alpha) \) from eq. (A-7), we need only be concerned with the sign of the numerator of the resulting expression since the denominator will be the square of the denominator of eq. (A-7). It is easy to evaluate the sign of the numerator of the resulting derivative with the exception of a term
\[ \sinh\left(\frac{3\alpha-1}{2\alpha}\right) + \sinh\left(\frac{1}{2\alpha}\right) + \frac{\sqrt{1-4\alpha}}{\alpha}, \]  

(A-8)

whose sign at first appears to be ambiguous. The arguments of the inverse hyperbolic sines are positive, thus they too are positive. The last term is negative since \( \alpha < 0 \). However, eq. (A-8) turns out
to be negative since the limit of eq. (A-8) as \( \alpha \) goes to \( -\infty \) is zero, and the derivative of eq. (A-8) is negative for \( \alpha < 0 \). This permits the deduction that \( \sigma'(\alpha) < 0 \) for \( \alpha < 0 \). Further,

\[
\lim_{\alpha \to -\infty} \sigma(\alpha) = \frac{4}{9}. \tag{A-9}
\]

This is obtained by applying l'Hopital's rule several times to parts of the reciprocal of eq. (A-7). 

Q.E.D.
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(\frac{1}{1 - \alpha}, 0)
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(\frac{1}{1 - \alpha}, 0)
\frac{1}{2a} / \beta a
\frac{1}{2a} / \beta a
\text{Contract Curve}
\text{Max Output}
\text{Noncooperative Equilibria}
\text{Max Output}
\text{Fig. 1}
Figure 3: Output for the duopolists.
Figure 4: Tax-rates for the duopolists.