Nonlinear Corporate Taxes and Asset Pricing

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Abstract

This paper addresses the implications of non linear taxes for standard single period asset pricing models. In particular, the paper discusses problems in estimating the market model and in testing the CAPM, that may arise under conditions of non linear taxes and changing tax shields. Some possible implications are discussed for event studies that rely on the cumulative residual or similar techniques. Finally, a correction factor is developed for the problems due to changing tax shields under nonlinear tax system.
Firms may shield part of their income from taxation. Interest on debt, depreciation and tax credits are convenient examples of such tax shields. Such shields may be transferred in full, or in part, to other tax years under the carry back/carry forward provisions. In the absence of full and costless offsets, the effect of tax shields is to produce a non linear tax function with "option like" qualities. Such characteristics have been addressed recently in a number of studies. Majd and Myers [9] examine the effects of tax non-linearity on the value of risky investment opportunities facing the firm, and Green and Talmor [8] address similar issues. Galai [6] and Pitts and Franks [11] examine the value of the Government's tax "option" and its implications for financing decisions. Smith and Stultz [12] show the effects of non linear taxes on corporate hedging policies.

The purpose of the present paper is to address the implications of non linear taxes for standard single period mean-variance asset pricing models. We show that non linear taxes produce an estimation bias in the market model under conditions of changing tax shields. Second, the paper outlines problems that may arise in conventional tests of CAPM. In particular, we show that the use of portfolio procedures to address the "errors-in-variables" problem associated with the estimated betas may not be effective if the error is due to changing tax shields. Third, we discuss some possible implications for event studies that use the cumulative average residual or similar techniques. In addition, a correction factor is developed for the beta estimates produced by the market model.
I. ASSUMPTIONS AND NOTATION

Since our purpose is to show the effects of tax non-linearities on the standard mean-variance asset pricing model, we will simplify the tax structure by ignoring the partial offsets allowed under the carry over provisions. This simplification corresponds with most of the other papers cited earlier. All earnings above the value of the firm's tax shield are taxed at the constant marginal rate (e.g., 46%). We adopt one of the alternative assumptions associated with CAPM that the earnings of the firm before tax are approximately normally distributed.

Under proportional taxes, a proportional claim on the firm's earnings is equivalent to a proportional claim on its value. This feature greatly simplifies the single period model. But with non linear taxes we must address the sequence of taxable cash flows in order to derive a capitalized value. To keep within the spirit of the single period model which we wish to evaluate, we will assume a "stationary" world in which the distributions of relevant values, except tax shields, are held constant. Thus, the distributions of pre tax earnings and the return on the market portfolio are stationary. The riskless interest rate is also held constant. Investors anticipate this stationarity and expectations currently held on these values are not revised. However, tax shields may change as may expectations concerning future tax shields. These features permit isolation of the effects of tax shields on the single period model.

Notation

\[ Y = \text{pre tax corporate earnings} \]
\[ Y_T = \text{after tax earnings} \]
\[ V = \text{value of the firm's pre tax earnings} \]
\[ V_T = \text{after tax value of the firm} \]
\[ T = \text{corporate marginal tax rate} \]
\[ F(\cdot) = \text{cumulative normal density evaluated at (\cdot)} \]
\[ f(\cdot) = \text{normal density evaluated at (\cdot)} \]
\[ E(\cdot) = \text{expectations operator} \]
\[ \text{Cov}(\cdot, \cdot) = \text{covariance} \]
\[ E_A^B(\cdot) = \text{expectation of (\cdot) having been truncated from below at} \]
\[ \text{A and above at B} \]
\[ R_M = \text{return on market portfolio} \]
\[ R_f = \text{risk free rate} \]
\[ S = \text{tax shield, earnings below S subject to zero tax; earnings} \]
\[ \text{above S taxed at the statutory marginal rate T} \]
\[ \beta = \text{before tax beta} \]
\[ \beta_T = \text{after tax beta} \]

II. **PROPORTIONAL TAXES**

In a simple proportionate tax world, the after tax value of the firm will simply be the before tax value times one minus the tax rate. This is true since all cash flows are subject to the same tax rate, i.e.,

\[ V_T = (1-T)V. \quad (1) \]

The before tax rates of return that would accrue to investors could they purchase non taxed earnings would be

\[ E(R) = \frac{E(Y)}{V} - 1 \quad (2) \]
However, investors purchase after tax earnings for an after-tax value. Consequently the after-tax return is

\[ E(R_T) = \frac{E(Y_T)}{V_T} - 1 = \frac{E(Y)(1-T)}{V(1-T)} - 1 = E(R) \] (3)

which implies equality of the before and after tax betas

\[ \beta_T = \beta. \] (4)

For some purposes it may be useful to measure after-tax returns, denominated by the before-tax value, e.g., for evaluating the "fair" return on a pre tax equity investment in a regulated industry. In such circumstances, the post tax return would be \((1-T)\) times the pre tax return. But our purpose is to examine the valuation of post tax earnings. In this context, we note the equality of pre tax and post tax earnings and the equality of the pre tax and post tax betas as shown by equations (3) and (4).

III. NON LINEAR TAXES

A simple non linear tax structure can be expressed implicitly by the after tax value of earnings \(Y_T\)

\[ Y_T = \begin{cases} Y & \text{if } Y \leq S \\ Y(1-T) & \text{if } Y > S \end{cases} \] (5)

Pre tax earnings, \(Y\), are assumed normally distributed and stationary. From the properties of the normal distribution, the truncated after tax earnings expectation and its covariance with the return on the market portfolio can be expressed as (see Winkler et al [14]).
\[ E(Y_t) = E(Y) - T \sigma_s(Y) \]

\[ = E(Y)[1-T(1-F(S))] - T\sigma^2(Y)f(S) \quad (6) \]

and

\[ \text{Cov}(Y_t, R_M) = \text{Cov}(Y, R_M)[1-T(1-F(S))] \quad (7) \]

These expressions will be used to provide the after-tax value of the firm.

From the single period CAPM, the firm's value at time \( t \) is

\[ V_{T,t} = \frac{E_t(Y_{T,t+1})-\lambda_t \text{Cov}_t(Y_{T,t+1}, R_{M,t+1})}{1+R_{f,t}} + \frac{E_t(V_{T,t+1})-\lambda_t \text{Cov}_t(V_{T,t+1}, R_{M,t+1})}{1+R_{f,t}} \quad (8) \]

and

\[ V_{T,t+1} = \frac{E_{t+1}(Y_{T,t+2})-\lambda_{t+1} \text{Cov}_{t+1}(Y_{T,t+2}, R_{M,t+2})}{1+R_{f,t+1}} + \frac{E_{t+1}(V_{T,t+2})-\lambda_{t+1} \text{Cov}_{t+1}(V_{T,t+2}, R_{M,t+2})}{1+R_{f,t+1}} \quad (9) \]

and so on. Using the stationarity assumptions, and dropping time subscripts, successive substitution yields the current after tax value \( V_T \)

\[ V_T = \frac{E(Y) - \lambda \text{Cov}(Y, R_M)}{R_f} \quad (10) \]

Substituting from (6) and (7) gives

\[ V_T = \frac{E(Y) - \lambda \text{Cov}(Y, R_M)}{R_f} [1-T(1-F(S))] - T[\sigma^2(Y)f(S)] \]

\[ = V[1-T(1-F(S))] - T[\sigma^2(Y)f(S)] \quad (10) \]

It can be noticed that the after tax value is related to the before tax value \( V \), the tax rate \( T \) and the probability of paying tax \( (1-F(S)) \). However, the second term in (10) also relates the tax
liability to the variance of earnings. This is not surprising since the taxman’s claim is a simple call option, the value of which is related to the variance of the underlying cash flow. Thus variance enters, not because total risk is priced, but because it serves to determine the value of the firm’s tax liability.

Equations (6) and (10) now are used to determine the expected after tax rate of return $E(R_T)$.

$$E(R_T) = \frac{E(Y_T)}{V_T} - 1 = \frac{E(Y)M-N}{VM-N/R_f} - 1$$

where $M \equiv 1-T(1-F(S))$

$$N \equiv T\sigma^2(Y)f(S)$$

Tedious rearrangement gives a linear relationship between the after tax return $E(R_T)$ and the before tax return $E(R)$

$$E(R_T) = E(R)\left[\frac{MVR_f}{MVR_f-N}\right] + \left[\frac{N(1-R_f)}{MVR_f-N}\right]$$

$$= E(R)f_1(V,T,\sigma^2(Y),R_f,S) + f_2(V,T,\sigma^2(Y),R_f,S)$$

where $f_1$ and $f_2$ describe the functions defined by the respective square brackets.

We now describe a comparable relationship between the before tax and after tax betas, $\beta$ and $\beta_T$.

$$\beta_T \equiv \frac{\text{Cov}(R_T,R_M)}{\sigma^2_M} = \frac{\text{Cov}(\frac{Y_T}{V_T} - 1, R_M)}{\sigma^2_M}$$

$$= \frac{1}{V_T} \text{Cov}(Y_T,R_M)/\sigma^2_M$$

Substitution from (7) and (10) yields,
\[ \beta_T = \left( \frac{\text{VMR}_f}{\text{VMR}_f - N} \right) \beta \]

\[ = f_1 \beta \quad \text{(14)} \]

Thus, collecting together the relationships between the before and after tax returns to the firm, and the before and after tax betas,

\[ E(R_T) = f_1 E(R) + f_2 \quad \text{(12)} \]

\[ \beta_T = f_1 \beta \quad \text{(14)} \]

Now suppose that the single period CAPM holds on the after corporate tax earnings delivered by the firm to its securityholders

\[ E(R_T) = R_f + \beta_T [E(R_M) - R_f] \quad \text{(15)} \]

Then the comparable before tax relationship is established by substituting (12) and (14) into (15)

\[ E(R) = \frac{R_f}{f_1} - \frac{f_2}{f_1} + \beta [E(R_M) - R_f] \quad \text{(16)} \]

The main results under the proportional and non proportional tax structures are now summarized. The expression for firm value (equation 10), expected after tax return (equation 12) and beta (equation 14) all degenerate to the proportional tax values (equations (1), (3) and (4) respectively) if the tax shield is set at minus infinity. Such an extreme tax shield is equivalent to a proportional tax. In general, however, tax shields imply a higher after tax value, a higher expected return and a higher beta than would prevail in a proportional tax environment.
The results of this section imply that certain problems may arise in the standard proportional tax analysis of CAPM. Such problems arise, for the most part, with changes in tax shields. In particular, beta estimation problems will arise if tax shields and investors' expectations concerning future tax shields are changing over the sampling period. The next section hinges on this problem and develops a correction factor.

IV. PROBLEMS WITH NONLINEAR CORPORATE TAXES

A. Problems in the Market Model

A number of studies have raised questions on the validity of the market model in terms of estimation of the systematic risk and the variance structure (i.e., the existence of unconditional and conditional heteroscedasticities in the market model) (see Bey and Pinches [2], Brown, Lockwood and Lummer [4], Giaccotto and Ali [7], and Bera, Bubnys and Park [1]). Our analysis shows that the changes in tax shields have a direct bearing on this issue.

Consider the market model in its anticipated form

\[ \bar{R}_T = \bar{\alpha}_T + \bar{\beta}_T R_M + \varepsilon. \]  

(17)

where \( E(\varepsilon) = 0 \) and \( \sigma^2(\varepsilon) \) is constant. The term \( \bar{R}_T \) reflects current expectations of after-tax return including current expectations concerning the tax shield \( \bar{S} \). \( \bar{\beta}_T \) is the anticipated beta arising from current expectations. Furthermore, the values \( \bar{f}_1 \) and \( \bar{f}_2 \) may be defined on tax shields currently expected by investors. Estimating the model on past data involves historic returns \( \bar{R}_T \), based upon prior
values of the tax shields. The value of tax shields over the sampling period is denoted $S$ and values $f_1$ and $f_2$ may be similarly defined on $S$. Now consider the implications of changing expectations concerning tax shields. From equation (12),

$$R_T = f_1R + f_2 \quad \text{and} \quad \bar{R}_T = \bar{f}_1R + \bar{f}_2$$

Combining the above two equations,

$$\bar{R}_T = (R_T + (f_1/\bar{f}_1)\bar{f}_2 - f_2)(\bar{f}_1/f_1)$$

(18)

Substitution of eq. (18) into eq. (17) yields,

$$R_T = \bar{a}_T(f_1/\bar{f}_1) + \bar{b}_T(f_1/\bar{f}_1)R_M + \bar{f}_2 + (f_1/\bar{f}_1)(\varepsilon - \bar{f}_2)$$

(19)

Changes in tax shields will give rise to changing values of $f_1$ and $f_2$. Consequently, ordinary least squares regressions using historical data will lead to biased estimators of anticipated betas unless adjustment is made for the changes in tax shields. It is also apparent that the complex error term in equation (19) does not necessarily follow the convenient properties of the error, $\varepsilon$ and may be subject to heteroscedasticity.

B. Problems in the Testing CAPM

The two-step regression methodology for testing CAPM was introduced by Black, Jensen and Scholes [3] and has been used extensively since. The technique consists of a second-pass cross-sectional regression to estimate the market price of risk where the estimated betas for this regression are provided from first-pass time series regressions in the preceding period. The second-pass regression is usually conducted on
portfolios derived from the ranking of security betas estimated in the first-pass. The formation of portfolios is undertaken because of the "errors-in-variables" problem. The estimated security betas from the first-pass, $\hat{\beta}_i$, may be stated as the true beta, $\beta_i$ plus an error $e_i$

$$\hat{\beta}_i = \beta_i + e_i$$

If the error terms are well behaved then their weighted average within the portfolio should exhibit an expected value of zero and a variance which asymptotically approaches zero as the size of the portfolio increases.

The analysis of the market model above reveals a potential "errors-in-variables" issue arising from changes in tax shields. We will show that this form of error is not well behaved even if the changes in tax shields are randomly distributed across firms. As a result, the second pass regression will produce biased and inconsistent estimators.

The betas required for the second-pass test reflect investors expectations on the value of tax shields at the time denoted for the second-pass, i.e., $\overline{B}_{iT}$. However, the first-pass regressions produce estimates of security betas, $\hat{\beta}_{iT}$, based upon tax shield expectations held during the earlier sampling period. For example, using five years data for the first pass, $\hat{\beta}_{iT}$ will reflect tax shields in place up to five years before the time of the second pass cross sectional run. Thus we define

$$\overline{B}_{iT} = \overline{\beta}_{iT} + \mu_i \quad (20)$$

From equation (19) it follows that
\[ u_i = [\left(\frac{f_{li}}{\bar{f}_{li}}\right) - 1]B_{iT} \]  

(21)

Now consider the implications of forming a "n" asset equally weighted portfolio, subscripted p. The estimation error for the portfolio beta, \( \mu_p \), is:

\[ \mu_p = \frac{1}{n} \Sigma (f_{li}/\bar{f}_{li})B_{iT} - \frac{1}{n} \Sigma B_{iT} \]  

(22)

To produce acceptable estimators in the second pass regression, the error term \( \mu_p \) should satisfy \( E(\mu_p) = 0; \sigma(\mu_p) \to 0 \) as \( n \to \infty \). The first of these conditions \( E(\mu_p) = 0 \) will only be satisfied if the changes in tax shields follow a stochastic process satisfying;

\[ \bar{f}_{li} = f_{li}(\frac{\bar{B}_{iT}}{\bar{B}_{iT} + \delta_i}) \]  

(23)

where \( \delta_i \) is a random variable having \( E(\delta_i) = 0 \) and

\[ \bar{f}_{li} = \frac{[1-T(1-F(S_i))]V_iR_f}{[1-T(1-F(S_i))]V_iR_f - \tau^2(Y_i)f(S_i)} \]

\[ f_{li} = \frac{[1-T(1-F(S_i))]V_iR_f}{[1-T(1-F(S_i))]V_iR_f - \tau^2(Y_i)f(S_i)} \]

Notice that under this process the changes in tax shield are related to estimated betas. We cannot offer an economic process which would satisfy equation (23). Moreover, if \( \mu_p \) is to satisfy \( \sigma(\mu_p) \to 0 \) as \( n \to \infty \), then we must add the further restriction on the term \( \delta_i \).

\[ \text{Cov}(\delta_i, \delta_j) = 0 \]
But changes in tax shields across firms are likely to be responsive to changes in the tax environment which simultaneously affects all firms. Such systematic changes in tax shields would preclude the convergence of $\mu_p$ on zero. Thus, in general, we find no reason to suppose that the "error-in-variable" term, $\mu_p$, will be removed with the formation of portfolios if this error is due to changing tax shields.

C. Event Studies

The predominant methodology in event-studies is to specify a predetermined time interval on both sides of an event and the betas for each security are estimated through the market model, using the data prior to the predetermined time interval. These betas are in turn used to estimate average abnormal rate of return for the time interval (see Fama, Fisher, Jensen and Roll [5] for a typical example).

In light of the above analysis, problems may arise with residual analysis if systematic changes in tax shield occur in event time. Of particular concern is the prospect that some events chosen for residual analysis might themselves be associated with a change in the value of tax shields. For example, a merger between two firms may be associated with a reduction in earnings variance which would in turn reduce the value of the "taxman's" call option on the firms (see Pitts and Franks [11]). But this effect would also operate through $f_1$ to change the after-tax beta in event time. Divestitures may cause similar problems. Such possibilities underscore the importance of testing for structural change in the regression parameters surrounding the event date (see Brown, Lockwood and Lummer [4] for a recent discussion). If such changes may be reasonably associated with changes in tax shields (as
opposed to more general conditions such as informational uncertainty surrounding the event), then empirical results may be misinterpreted.

D. A Correction to the Market Model

These problems can be addressed by correcting the beta estimates produced by the market model. We know that the beta estimated by the market model will reflect the expectations of the values of tax shields held by investors during the historical period from which the sample was taken, S. The anticipated market model is based upon current expectations of tax shields, $\overline{S}$. Thus, we may define betas before and after adjustment of changes in tax shields as

$$\beta_T = \frac{1}{f_1} \beta$$

and

$$\overline{\beta}_T = \frac{1}{f_1} \beta = (\frac{1}{f_1} \frac{f}{f_1}) \beta_T$$

(24)

Since we require an estimate of $\overline{\beta}_T$, but obtain an estimate of $\beta_T$, the multiplicative correction factor $\frac{f_1}{f_1}$ shown above should solve the problem.

From eq. (12)

$$\frac{f_1}{f_1} = \frac{[\text{MVR}_f/(\overline{\text{MVR}}_f-N)]}{[\overline{\text{MVR}_f}/\overline{\text{MVR}}_f-N]}$$

(25)

with M and N correspondingly defined on $S$ and $\overline{S}$.

Is this correction factor operational? Expected tax shields are not observable. However, it may be reasonable to use actual average tax shields in place over the data period to estimate $S$ and the current tax shield to estimate $\overline{S}$. A further problem arises since the before
tax value \( V \) also is not observable. But we can observe \( V_T \) both currently and during the data period. Substitution of \( V \) by \( V_T \) from eq. (10) leads to a more convenient form for the correction factor

\[
\frac{\bar{f}_1}{\bar{f}_2} = \left[ \frac{(\bar{V}_T R_f + \bar{N})}{(\bar{V}_T R_f + \bar{N})} \right] \cdot \left( \frac{V_T}{\bar{V}_T} \right)
\]  

(26)

The effect of this correction factor on the observable form of the market model may be seen by substitution into equation (19).

\[
(\frac{\bar{f}_1}{\bar{f}_2}) R_T = \bar{\alpha}_T + \bar{\beta}_T R_M + \bar{\epsilon}_2 (\frac{\bar{f}_1}{\bar{f}_2}) + \epsilon - \bar{f}_2
\]  

(27)

But

\[
R_T = \frac{f_1}{f_2} R + \bar{f}_2 \text{ and } \bar{R}_T = \frac{f_1}{f_2} R + \bar{f}_2
\]

yielding

\[
\bar{R}_T = (\frac{\bar{f}_1}{\bar{f}_2}) \bar{R}_T - \frac{f_2}{f_1} (\frac{\bar{f}_1}{\bar{f}_2}) + \bar{f}_2.
\]

Substituting into (27) yields the following well behaved observable form for the market model

\[
(\frac{\bar{f}_1}{\bar{f}_2}) R_T = \bar{\alpha}_T + \bar{\beta}_T R_M + \epsilon
\]  

(28)

V. CONCLUSION

The combination of non linear corporate taxes and changing tax shields will give rise to estimation bias in the market model. This bias is unlikely to be eliminated with the portfolio procedures usually adopted for testing asset pricing models. Moreover, further problems may arise in event studies particularly where the event itself affects the value of tax shields. A correction for this estimation bias is developed. The correction factor may be used either to transform return data to estimate the market model or equivalently correct the biased estimators produced from untransformed data.
Footnotes

1 Using the assumption of lognormal terminal values, a similar valuation expression to equation (10) can be derived in an option pricing framework. Treating the taxman's claim as T times a call option on the before-tax value of the firm with the exercise price equal to the tax-shield.

\[ V_T = V - TC(V,S) \]

Using the Black-Scholes option pricing model,

\[ V_T = V - T[VN(h) - S'N(h - \sigma\sqrt{T})] \]
\[ = V[1 - TN(h)] + TS'N(h - \sigma\sqrt{T}), \]

where \( S' = S e^{-R_f T} \)

\[ h = \ln(V/S')/\sigma\sqrt{T} + 1/2\sigma\sqrt{T} \]

\[ N(h) = \int_{-\infty}^{h} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \]

The parallels with the valuation derived under normality are clear

\[ V_T \equiv V[1 - T(1-F(S))] - Ta^2 f(S)/R_f. \quad \text{(eq. 10)} \]
References


