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L162
The Iterative Step in the Linear Programming Algorithm of N. Karmarkar

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The Iterative Step in the Linear Programming Algorithm of N. Karmarkar

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We simplify and strengthen the analysis of the improvement obtained in one step of Karmarkar's algorithm.

The recently published [1] algorithm of N. Karmarkar uses the following step:

Suppose \( x = (a_1, \ldots, a_n) > 0 \) is a feasible solution to the LP:

\[
\begin{align*}
\text{minimize} & \quad cx \\
\text{subject to} & \quad Ax = 0, \ x \geq 0, \ \Sigma x_i = 1
\end{align*}
\]

(1)

We will assume the optimal solution to (1) has objective function value \( \leq 0 \), and that \( ca > 0 \). We refer to [1, section 6] for proofs that a method of solving this type of problem yields a method of solving any LP.

Let \( x = (z_1, \ldots, z_n) \) be the optimal solution to

\[
\begin{align*}
\min & \quad c(a_1 x_1, \ldots, a_n x_n) \\
\text{subject to} & \quad A(a_1 x_1, \ldots, a_n x_n) = 0, \ \Sigma x_i = 1, \ x \cdot (\frac{1}{n}, \ldots, \frac{1}{n}) \leq a(n(n-1))^{-\frac{1}{2}}
\end{align*}
\]

(2)

where \( a < 1 \) is a parameter to be specified. [1, Theorem 5] shows that (2), which is a minimization of a linear objective function on a sphere, can be carried out using \( O(n^3) \) operations.

The next feasible solution to (1) generated by the algorithm is

\( w = \gamma(a_1 z_1, \ldots, a_n z_n) \) where the scalar \( \gamma \) is chosen so that \( \Sigma w_i = 1 \).
Let \( f(x) = (cx)^n / \pi x \) (this is the same as the potential function \( f \) in [1], except we do not use logarithms). To show that the new solution \( w \) is "better" than the previous solution, [1, Theorem 2] shows

**Theorem 1:** For some \( k < 1 \) (dependent on \( \alpha \)) \( f(w) \leq kf(a) \).

Since \( \sum x_i = 1 \) and \( x_i \geq 0 \) implies \( \prod x_i \leq n^{-n} \), Theorem 1 implies that, if the optimal solution to (1) has objective function value zero and \( v \) is obtained from \( a \) after \( t \) iterations \( (cv)^n \leq k^n f(a) \). As indicated in [1], this property yields a polynomial-time algorithm.

In this note, we give a new proof of Theorem 1, which gives a slightly better value of \( k \) and is more elementary in that logarithms are not used.

**Lemma 2:** \( \sum \lambda_i^2 \leq n^{-1} (1-\alpha/(n-1)) \sum \lambda_i a_i z_i \).

**Proof:** Since the optimal solution to (1) is assumed to have value \( \leq 0 \), there is a \( u \geq 0 \) satisfying \( A(a_1 u_1, \ldots, a_n u_n) = 0 \), \( \sum u_i = 1 \), and \( \sum \lambda_i a_i u_i \leq 0 \). Since \( \| u - (\frac{1}{n}, \ldots, \frac{1}{n}) \|_2^2 \leq (1 - \frac{1}{n})^2 + (n-1)n^{-2} = (n-1)n^{-1} \), and \( z \) is the optimal solution to (2), \( \sum \lambda_i a_i z_i \) must be

\[ \leq (1-\lambda)(\sum \lambda_i a_i n^{-1}) + \lambda (\sum \lambda_i a_i u_i) \leq (1-\lambda)n^{-1} (\sum \lambda_i a_i), \]

where

\[ \lambda = (\alpha(n(n-1))^{-\frac{1}{2}})/( (n-1)n^{-1})^\frac{1}{2} = \alpha/n-1. \]

Q.E.D.

*This is the same as [1, Theorem 3].
Lemma 3: If $Q > R > S > 0$, then there exist $\varepsilon, \delta > 0$ such that (i) \[(Q-\varepsilon)^2 + (R+\varepsilon+\delta)^2 + (S-\delta)^2 = Q^2 + R^2 + S^2 \text{ and (ii) } (Q-\varepsilon)(R+\varepsilon+\delta)(S-\delta) < QRS.\]

Proof: For $\delta$ close to zero, there exists an $\varepsilon$ close to zero such that (i) holds. Since $\frac{\varepsilon}{\delta} \to (R-S)/(Q-R)$ as $\delta \to 0$, \[\lim_{\delta \to 0} \frac{1}{\delta} (QRS - (Q-\varepsilon)(R+\varepsilon+\delta)(S-\delta)) = (QR-QS) + (S^2-RS) = (R-S)(Q-S) > 0.\] Q.E.D.

Lemma 4: If $Q > R > 0$, then there exist $\varepsilon, \delta > 0$ such that (i) \[(Q-\varepsilon)^2 + (R+\varepsilon+\delta)^2 + (R-\delta)^2 = Q^2 + 2R^2 \text{ and (ii) } (Q-\varepsilon)(R+\varepsilon+\delta)(R-\delta) < QR^2.\]

Proof: For $\delta$ close to zero, there exists $\varepsilon$ close to zero such that (i) holds. Since $\lim_{\varepsilon \to 0} \varepsilon^2 = (Q-R)^{-1}$, \[\lim_{\delta \to 0} \delta^{-2} (QR^2 - (Q-\varepsilon)(R+\varepsilon+\delta)(R-\delta)) = Q - R > 0.\] Q.E.D.

Lemma 5: If $\|x-(\frac{1}{n}, \ldots, \frac{1}{n})\| = \alpha(n(n-1))^{-\frac{1}{2}}$ and $\sum x_i = 1$, then $\prod x_i \geq n^{-n}(1+\alpha/(n-1))^{n-1}(1-\alpha)$.

Proof: By continuity, there is an $x^*$ which minimizes $\prod x_i$ among those $x$ which satisfy the assumptions. $\alpha < 1$ implies $x_i > 0$ for all $i$, since $(n-1)(n^{-1}-(n-1)^{-1})^2 + n^{-2} = (n(n-1))^{-1}$. By Lemma 3, we cannot have $x_i^* > x_j^* > x_k^*$. (Note that $\sum x_i = 1$ and $\sum x_i^2 = \sum (x_i^*)^2$ imply $\|x-(\frac{1}{n}, \ldots, \frac{1}{n})\| = x^* - (\frac{1}{n}, \ldots, \frac{1}{n})$. Thus the components of $x^*$ have two different values. By Lemma 4, there cannot be more than one component of $x^*$ having the smaller value. Thus $x^*$ consists of $n-1$ components with a larger value, and one component with a smaller value. This occurs only if $n-1$ components of $x^*$ are $n^{-1}(1+\alpha/(n-1))$ and one component is $n^{-1}(1-\alpha)$. Q.E.D.
Theorem 6: If \( a \) is a feasible solution and \( w \) the next solution given by the algorithm

\[
f(w) \leq (1-\alpha/n-1)^n(1+\alpha/n-1)^{1-n}(1-\alpha)^{-1}f(a)
\]  

(3)

Proof: Recall that \( w = \gamma(a_1z_1, \ldots, a_nz_n) \), hence

\[
f(w) = f(a_1z_1, \ldots, a_nz_n). \quad \text{By Lemma 2,} \quad (\Sigma c_ia_iz_i)^n \leq n^{-n}(1-\alpha/(n-1))^{n}(\Sigma c_ia_i)^n.
\]

Since \((1+\alpha/n-1)^{n-1}(1-\alpha)\) is monotone decreasing as a function of \( \alpha \),

Lemma 5 and \( \|z-(\frac{1}{n}, \ldots, \frac{1}{n})\| \leq \alpha(n(n-1))^{-\frac{1}{2}} \) imply

\[
\prod a_iz_i = \prod a_i \prod z_i \geq (\prod a_i)^{-n}(1+\alpha/n-1)^{n-1}(1-\alpha). \quad \text{Thus}
\]

\[
f(w) \leq (1-\alpha/(n-1))^n(\Sigma c_ia_i)^n/(\prod a_i)(1+\alpha/n-1)^{n-1}(1-\alpha). \quad \text{Q.E.D.}
\]

For comparison, [1, Theorem 4] shows that, for \( \alpha = \frac{1}{4} \) and \( n \) large,

\[
f(w) \leq \exp(-13/96)f(a). \quad \text{Theorem 6 yields} \quad f(w) \leq \frac{4}{3} \exp(-\frac{1}{2})f(a).
\]

The right-hand-side of (3) is minimized when \( \alpha = (n-1)/(2n-3) \).

This may be the best single choice of \( \alpha \), if it is to be kept constant through all iterations.

Reference
