COSIMPLICIAL INVARIANTS AND CALCULUS OF HOMOTOPY FUNCTORS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

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Abstract

The origin of these investigations was the successful attempt by myself and coauthors to generalize rational equivalences of two constructions which suggest possible definitions of deRham cohomology of "brave new" rings, one of Rezk (using a cosimplicial resolution) and the other by Waldhausen (using a variant of Goodwillie's Taylor tower). The proof of agreement of these constructions relies heavily on the fact that the functors involved take values in Spectra.

Goodwillie conjectured the extension of these results to include the case of functors taking value in *Spaces*. The main result of my thesis is a proof of this conjecture (Theorem 7.1.2), using significantly different methods than in the stable setting of the joint work. This makes strong use of the intermediate constructions $T_n F$ in Goodwillie's Calculus of homotopy functors. I give a new model which naturally gives rise to a new family of towers filtering the Taylor Tower of a functor. I also establish a surprising equivalence between the homotopy inverse limits of these towers and the homotopy inverse limits of certain cosimplicial resolutions. This equivalence gives a greatly simplified construction for the homotopy inverse limit of the Taylor tower of a functor F under general assumptions.

To my grandfather, who would have been proud.

Acknowledgements

Many people have given generously of their time and support towards the culmination of this dissertation. Perhaps the biggest lesson I have learned in the process has been how much can change within just a year.

I would like to thank my advisor, Randy McCarthy, for his encouragement and time, passing on a wealth of knowledge in his philosophy of "Sit long, talk much". Thanks to the other members of my committee for their support, Matthew Ando, Charles Rezk, and Bert Guillou, who all at various times fielded my questions and made very detailed editorial comments. There are several people who have born my miscellaneous, sometimes deadline-driven, questions (mathematical and otherwise) with good humor and patience: Tyler Lawson, Inna Zakharevich and Marcy Robertson.

The results belonging to the main chapter of this dissertation really began to coalesce at the May 2010 Georgia Topology conference, and I would like to thank Michael Ching for organizing such a great event. Many thanks go to Tom Goodwillie for several conversations about cartesian cubes and inverse limits, for giving generously and patiently of his time, and being willing to explain everything down to the most particular nuts and bolts and up to the greatest breadth. These discussions that led to the conjectural form of the main results of this dissertation. I would like to thank Bill Dwyer as well for many conversations which helped greatly with the original proof the the results of Chapter 5 as well as in the development of the proofs of the main chapter, including lending insight into cofinality arguments. Thanks also go to Victor Turchin, who pointed out an error in an earlier statement of Theorem 7.3.1.

One of the most important things I was told when I started this program was to remember to spend at least *some* time on something besides Math. For coffee runs, Sunday dinners, and reminders to take time for oneself, I thank Jane Butterfield, Tracy Grauman, Joe Gezo, Chris Bonnell, Hannah DeBerg, Richard Hislop, Qai Goss, and Melih Sener.

And thanks to my family for always knowing that I could do it.

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Chapter 1

Introduction

The motivation behind my research has been to gain a better understanding of the structure of cosimplicial objects. One of the most historic and prominent examples of cosimplicial objects in use by topologists is the collection of cosimplicial abelian groups, the category of which is equivalent to cochain complexes associated to spaces which are concentrated in nonnegative degrees. Simplicial sets are Quillen equivalent to spaces, and simplicial abelian groups to chain complexes[1]. The dualization to cochain complexes and cohomology yields new, rich structure, e.g. there are cohomology operations that lead to differentials in certain spectral sequences. We are then poised for an expectation of new, rich structure when dualizing from arbitrary simplicial *objects* to cosimplicial objects in the same category.

Our methods have not caught up to the rich structure of arbitrary cosimplicial objects. A glaring example of this can be seen in the process of setting up the first page of the Bousfield-Kan spectral sequence of a cosimplicial (based) space, which is a second-quadrant sequence that we force to converge by *throwing away* the information in the lower half of that quadrant.

Goodwillie Calculus of homotopy functors is a powerful tool which we have used to hunt for new structure in cosimplicial objects. To apply this tool, we replaced our spaces X with cosimplicial resolutions $p \mapsto \Delta^p * X$, which are then replaced with weakly equivalent cubical constructions, and refined our study from $\{\mathrm{sk}_k \triangle^\bullet * X\}_k$ to $\{F(\mathrm{sk}_k \triangle^\bullet * X)\}_{k,F}$ for F a (homotopy) functor from spaces to spaces. We then work in the environment of Figure 1.1, where each $\mathbb{T}\mathrm{ot}_n$ is a cubical diagram whose homotopy limit is Tot_n.

Figure 1.1 bears strong resemblance E_0 page of the Bousfield-Kan Spectral Sequence of a cosimplicial space [BK72]. In that case, however, our first step would be to replace with '0' everything below the codiagonal (the line of $\mathrm{Tot}_k \mathrm{sk}_k$). My results fall below that line, in what would be analogous to 'negative homotopy data'.

The degree n approximation to a functor F is analogous to the nth Taylor polynomial of a function, the sum of its first n terms, although only for some functors F do their $P_n F$ split as the sum (or product) of their layers (terms). T_n is an endofunctor of homotopy functors F, which for us will have domain and codomain

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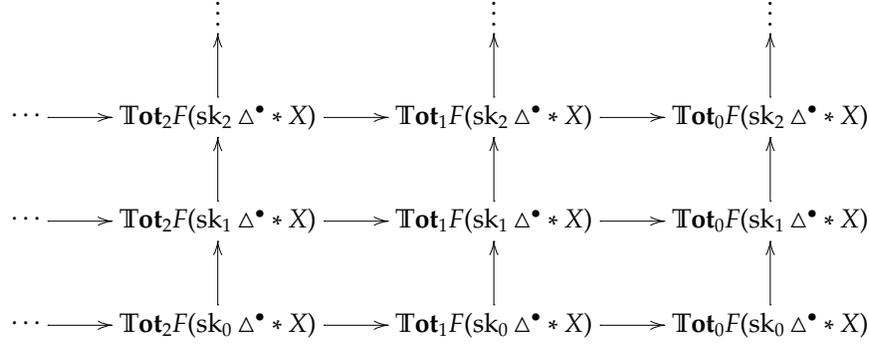[1]These are likewise concentrated in nonnegative degrees.

$$\cdots \longrightarrow \mathbb{T}\mathbf{ot}_2 F(\mathrm{sk}_2 \triangle^\bullet * X) \longrightarrow \mathbb{T}\mathbf{ot}_1 F(\mathrm{sk}_2 \triangle^\bullet * X) \longrightarrow \mathbb{T}\mathbf{ot}_0 F(\mathrm{sk}_2 \triangle^\bullet * X)$$

$$\cdots \longrightarrow \mathbb{T}\mathbf{ot}_2 F(\mathrm{sk}_1 \triangle^\bullet * X) \longrightarrow \mathbb{T}\mathbf{ot}_1 F(\mathrm{sk}_1 \triangle^\bullet * X) \longrightarrow \mathbb{T}\mathbf{ot}_0 F(\mathrm{sk}_1 \triangle^\bullet * X)$$

$$\cdots \longrightarrow \mathbb{T}\mathbf{ot}_2 F(\mathrm{sk}_0 \triangle^\bullet * X) \longrightarrow \mathbb{T}\mathbf{ot}_1 F(\mathrm{sk}_0 \triangle^\bullet * X) \longrightarrow \mathbb{T}\mathbf{ot}_0 F(\mathrm{sk}_0 \triangle^\bullet * X)$$

Figure 1.1: $\mathbb{T}\mathbf{ot}_n \, \mathrm{sk}_k$ diagram

Spaces. It is equipped with a natural map $1 \to \mathrm{T}_n$ giving the maps $\mathrm{T}_n^k \to \mathrm{T}_n^{k+1}$ with the property that, if $\mathrm{P}_n F$ is the degree n approximation to F, then $\mathrm{P}_n F(X) := \mathrm{hocolim}_k \, \mathrm{T}_n^k F(X)$. One of our main results is a computation of T_n^k, Theorem 7.3.1, which is utilized in the proof of Theorem 7.1.2. It provides a computationally easier and more geometric definition for the T_n's, and under fairly general assumptions, gives us simplified definitions of $\mathrm{P}_\infty F$.

1.0.1 Initial Cosimplicial Structure Results

McCarthy observed[McC10] that work of Segal[Seg74] could be reinterpreted as existence of a lift at one location of the sk_1-row of Figure 1.1. A lift is a map of 'slope 1' commuting with the maps in the diagram, and this was specifically in the case of F a linear functor. Inspired by existence of this one lift we provide constructions of explicit geometric lifts in Figure 1.1 of slope 1, for arbitrary functors F, not just linear, and at every point in the diagram.

This immediately gives information about the horizontal maps, that they essentially ignore (up to homotopy) the information differentiating Tot_n from Tot_{n-1}.

1.0.2 Cubical Results for structure of Cosimplicial objects

Preliminary results led me to posit that for $k > 0$, there are no 'slope 2' maps in Figure 1.1. As this picture bears a strong resemblance to the E_0 page of the spectral sequence used to calculate homotopy groups of a cosimplicial space, the slope 1 maps look like data used to calculate the 1st negative homotopy group and this obstruction then looks like the failure in general of the 2nd negative homotopy group to be constructible.

Seeking to classify obstruction to existence of slope 2 maps for general homotopy functors F relied on knowing the form of all n-cubes which are cartesian with cartesianness preserved by all homotopy functors.

This is because for a map of n-cubes $A \to B$, we have $\mathrm{holim}_0(A \to B) \simeq A$ when B itself is cartesian. To build a geometric lift, we needed it prior to application of a functor F (i.e. to also work for the identity functor), so we needed the cubes to be cartesian and remain cartesian.

Definition 1.0.1 (E.)**.** *A n-cube is **absolutely cartesian** if its cartesianness is preserved by all homotopy functors. For us, this will mean from Spaces to Spaces.*

Functorial 1-cubes are maps which are homotopy equivalences. We conjectured the form of all functorial squares and proved it (Theorem 1.0.2) for squares diagrams in Spaces or Spectra[2].

Theorem 1.0.2 (E.)**.** *A cartesian 2-cube of spaces is absolutely cartesian if and only if it is a map of functorial 1 cubes. That is, of the following form:*

$$
\begin{array}{ccc}
A & \xrightarrow{\simeq} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\simeq} & D
\end{array}
$$

1.0.3 Cubical Results generalizing linearity

Following Theorem 1.0.7 it is natural to ask if there are alternate conditions on cubes for degree n, since it implies that $\mathrm{P}_n F$ of certain general cocartesian (n+1)-cubes are cartesian, where the source cube is not *strongly* co-cartesian, as required in the definition of n-excisive [Goo91][Goo03].

Definition 1.0.3 (E.)**.** *Let an n-cube be **k-coCartesian** if every sub k-cube is coCartesian. For example, 2-coCartesian is the same as strongly coCartesian.*

Let $\mathcal{G}_n$ be the functor $S \mapsto [\#S - 1]$ (and inclusion maps to their equivalent induced maps) from $\mathscr{P}([n])$ (viewed as a category) to the full subcategory of Δ, denoted $\Delta_{\leq n}$, whose objects are $[k]$ such that $k \leq n$. Then our transition to a cube whose homotopy limit is tot is given as $\mathrm{Tot}(X^\bullet) := \mathrm{holim}_{\Delta_{\leq n}} X^\bullet \sim \mathrm{holim}_{\mathscr{P}([n])} X^\bullet \circ \mathcal{G}_n$. We then define:

Definition 1.0.4. *The cube $\mathbb{T}ot_n \, \mathrm{sk}_k \triangle^\bullet$ is the $\mathscr{P}([n])$-diagram*

$$
(\mathbb{T}ot_n \, \mathrm{sk}_k \triangle^\bullet)(U) = \begin{cases} \mathrm{Tot}_n \, \mathrm{sk}_k \triangle^\bullet & U = \emptyset \\ \mathrm{sk}_k \triangle^{\mathcal{G}_n(U)} & o.w. \end{cases}
$$

That is, $\mathbb{T}ot_n \, \mathrm{sk}_k \triangle^\bullet$ has the property $\mathrm{holim}_0 \, \mathbb{T}ot_n \, \mathrm{sk}_k \triangle^\bullet \simeq \mathrm{holim}_{\Delta_{\leq n}} \, \mathrm{sk}_k \triangle^\bullet$

---

[2]Several correspondences with Bill Dwyer were key in setting up the successful attack of this proof.

Proposition 1.0.5 (E.)**.** *The cube* $\mathbb{T}ot_n \, \mathrm{sk}_k \, \triangle^\bullet$ *is a* $(k+1)$*-coCartesian* $(n+1)$ *cube for all* k, n.

This, combined with Theorem 1.0.7 led to a conjecture that an equivalent condition for F being degree k would be for F to take strongly k-cocartesian n-cubes to cartesian n-cubes. However,

Theorem 1.0.6 (E.)**.** *Let F be a homotopy functor. If F takes every k-coCartesian n-cube to a Cartesian n-cube, for some $k > 2$ and $k \leq n$, then F is 1-excisive.*

1.0.4   Stages of the Taylor tower and partial Tot in the stable setting

The following was the main contribution I made to [BEJM11] which allowed for the pro/finite stage equivalence to be realized:

Theorem 1.0.7 (E.)**.** *Given F a homotopy functor commuting with filtered colimits, from Spaces X under A to Spectra, the following maps exist and are equivalences:*

$$\mathrm{P}_n F(X) \to \mathrm{Tot}_{(k+1)n} \, \mathrm{P}_n F(\mathrm{sk}_k \, \Delta^*_* * X)$$

This result gives the following equivalences

1.0.5   Results for Functors to Spaces, Theorems 7.1.2 and 7.3.1

The main result is the following, which was conjectured by Tom Goodwillie [Goo10]:

Theorem 7.1.2: *Let F be a homotopy functor from Spaces to Spaces. Then the following are weakly equivalent:*

$$\underset{\leftarrow_n}{\mathrm{holim}} \, \mathrm{T}_n^{k+1} F(X) \sim \underset{\leftarrow_n}{\mathrm{holim}} \, \mathrm{Tot}_n \, F(\mathrm{sk}_k \, \triangle^\bullet * X)$$

4

Whose proof requires the ancillary theorem:

Theorem 1.0.8. *There is a weak equivalence*

$$\mathrm{T}_n^k F(X) \sim \mathrm{holim}_{\Delta \leq nk} \operatorname{diag}(\operatorname{cosk}^{\vec{n}} F((\underbrace{\operatorname{sk}_0 \Delta^\bullet \ast \cdots \ast \operatorname{sk}_0 \Delta^\bullet}_{k \text{ copies}}) \ast X))$$

In particular, as $n \to \infty$, we have as an immediate consequence the following equivalence:

$$\mathrm{holim}_n \mathrm{T}_n^k F(X) \sim \mathrm{holim}_\Delta \operatorname{diag} F((\underbrace{\operatorname{sk}_0 \Delta^\bullet \ast \cdots \ast \operatorname{sk}_0 \Delta^\bullet}_{k \text{ copies}}) \ast X)).$$

1.0.6 Consequences of Theorems 7.1.2 and 7.3.1

Theorem 7.3.1 suggests the incorporation of cosimplicial methods (e.g. associated spectral sequences) as strong tools that can be brought to bear in calculations of Goodwillie calculus.

Here are several immediate consequences of Theorem 7.1.2, assuming F to be a homotopy functor from Spaces to Spaces:

Corollary 1.0.9 (Strengthened BEJM result). *Let F be a homotopy functor. Then, for any k , since $T_n P_j \simeq P_j$ for $n \geq j$, we have*

$$\mathop{\mathrm{holim}}_{\overleftarrow{n}} \mathrm{T}_n^{k+1} F(X) \simeq \mathop{\mathrm{holim}}_{\overleftarrow{n}} \mathrm{T}_n^{k+1} \mathrm{P}_j F(X) \simeq \mathrm{P}_j F(X)$$

and combined with Theorem 7.1.2, we extend a theorem of BEJM to spaces:

$$\mathrm{P}_j F(X) \simeq \mathop{\mathrm{holim}}_{\overleftarrow{n}} \mathrm{Tot}_n \, \mathrm{P}_j F(\operatorname{sk}_k \Delta^\bullet \ast X)$$

Corollary 1.0.10. *By Prop 1.0.13 and Theorem 7.1.2, $\mathrm{P}_\infty F(X) \sim \mathop{\mathrm{holim}}_{\overleftarrow{n}} \mathrm{T}_n^{\rho+1} F(X)$ when F is ρ-analytic.*

Remark 1.0.11. *Since the identity is 1-analytic, it follows from Corollary 7.1.5 that $\mathrm{P}_\infty Id(A) \sim \mathop{\mathrm{holim}}_{\overleftarrow{n}} \mathrm{T}_n^2 Id(A)$.*

Remark 1.0.12. *By a similar proof to that showing that the identity is 1-analytic, it follows that if a functor F commutes with realizations and preserves filtered colimits, then it is 1-analytic. Corollary 7.1.5 implies that for such an F, we have $\mathrm{P}_\infty F(X) \sim \mathop{\mathrm{holim}}_{\overleftarrow{n}} \mathrm{T}_n^2 F(X)$.*

Several of the above rely on the following

Proposition 1.0.13. *If F is ρ-analytic, then* $\mathrm{sk}_\rho \, \Delta^\bullet * X$ *is in the radius of convergence of F.*

Proof. Assume F is ρ-analytic. Note that for all n, $\mathrm{sk}_\rho \, \Delta^n * X$ is at least ρ-connected. Thus, $F(\mathrm{sk}_\rho \, \Delta^n * X) \simeq P_\infty F(\mathrm{sk}_\rho \, \Delta^n * X)$. Levelwise equivalences of cosimplicial objects yield an equivalence on the total spaces, yielding

$$
\begin{aligned}
\mathrm{Tot}_\infty \, F(\mathrm{sk}_\rho \, \Delta^\bullet * X) \quad &\simeq \quad \mathrm{Tot}_\infty \, P_\infty F(\mathrm{sk}_\rho \, \Delta^\bullet * X) \\
&\simeq \quad \mathrm{holim}_n \, \mathrm{Tot}_\infty \, P_n F(\mathrm{sk}_\rho \, \Delta * X) \quad \text{by commuting holims} \\
&\simeq \quad \mathrm{holim}_n \, P_n F(X) \quad \text{by Corollary 7.1.4} \\
&=: \quad P_\infty F(X) \quad \text{by definition}
\end{aligned}
$$

$\square$

1.0.7 Organization

Chapter 2 will provide greater background in the Goodwillie calculus. Chapter 7 states and provides proofs for the main theorem, as well as necessary support theorems and its corollaries. Chapter 3 contains a lot of the motivation for the ensuing chapters, and provides the lifts mentioned in §1.0.1. Chapters 5 and 6 present results closer in nature to approaching cosimplicial invariants, proving the results stated in sections 1.0.2 and 1.0.3. The result of section 4 motivates the work that led to the main result of Chapter 5.

Chapter 2

Background on Goodwillie Calculus

For this section, we restrict our attention to functors F from spaces to spaces (not necessarily based) which commute with filtered colimits. F is a *homotopy functor* if it preserves weak equivalences.

2.1   Excisive functors

In [Goo90], Goodwillie establishes the following condition for a functor, which is in analogy with a function being polynomial of degree 1:

Definition 2.1.1. *A functor is 1-excisive if F takes homotopy pushout (called cocartesian) squares to homotopy pullback (called cartesian) squares.*

This may not be the most familiar statement of excision, compared to its usual statement as one of the axioms of a (generalized) homology theory (as in [ES52]) . There is a nice discussion in [MO02, p.22] of how to get from excision as usually stated in the Eilenberg-Steenrod axioms to this definition.

2.2   Excisive approximation

The following is a homotopy pushout of finite sets. It is also a diagrammatic representation of the category $\mathscr{P}([1])$.

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & \{0\} \\
\downarrow & & \downarrow \\
\{1\} & \longrightarrow & \{0,1\}
\end{array}
$$

We make the following definition:

$$\mathrm{T}_1 F(X) := \mathrm{holim}_{U \in \mathscr{P}_0([1])} F(U * X)$$

Then we have our natural transformation $t_1 : F(X) \to T_1 F(X)$, induced by the natural map:

$$F(X) = F(\emptyset * X) \to \operatorname*{holim}_{U \in \mathscr{P}_0([1])} (U \mapsto F(U * X)).$$

That is, the map from the initial object of the square, $F(X)$ to the homotopy pullback of the rest, $T_1 F(X)$. We can take T_1 of $T_1 F$, and also have the same natural transformation from initial to homotopy pullback, now $t_1 : T_1 F(X) \to T_1(T_1 F(X)) =: T_1^2 F(X)$. See Figure 2.1.

We define the 1-excisive approximation to F, $P_1 F$, as the following homotopy colimit:

$$P_1 F(X) := \operatorname{hocolim}(T_1 F(X) \xrightarrow{t_1} T_1^2 F(X) \xrightarrow{t_1} \cdots)$$

$$T_1^2 F(X) \quad := \quad \operatorname{holim} \left(\begin{array}{ccc} & & T_1 F(\{0\} * X) \\ & & \downarrow \\ T_1 F(\{1\} * X) & \longrightarrow & T_1 F(\{0,1\} * X) \end{array} \right)$$

$$\simeq \quad \operatorname{holim} \left(\begin{array}{ccc} \left(\begin{array}{ccc} & & F(\{1\} * \{0\} * X) \\ & & \downarrow \\ F(\{1\} * \{1\} * X) & \longrightarrow & F(\{1\} * \{0,1\} * X) \end{array} \right) & \rightarrow & \left(\begin{array}{ccc} & & F(\{0,1\} * \{0\} * X) \\ & & \downarrow \\ F(\{0,1\} * \{1\} * X) & \longrightarrow & F(\{0,1\} * \{0,1\} * X) \end{array} \right) \end{array} \right)$$

(with upper-right block)
$$\left(\begin{array}{ccc} & & F(\{0\} * \{0\} * X) \\ & & \downarrow \\ F(\{0\} * \{1\} * X) & \longrightarrow & F(\{0\} * \{0,1\} * X) \end{array} \right)$$

Figure 2.1: $T_1^2 F(X)$

2.3 Higher Degree Functors

As for the 1-excisive case, we begin with a diagramatical representation of the powerset category, now $\mathscr{P}([n])$, which is an $(n + 1)$-cube indexed by subsets of $[n]$.

Definition 2.3.1. *We say that a $\mathscr{P}([n])$-indexed diagram (i.e. an $(n + 1)$-cube) $\mathscr{X}$ is <u>strongly co-cartesian</u> if every square (i.e. 2-dimensional) subface is cocartesian.*

Definition 2.3.2. *We say that a functor F is <u>n-excisive</u> if it takes strongly co-cartesian $(n + 1)$-cubes to cartesian $(n + 1)$ cubes.*

As in the 1-excisive case, we make the following definition:

$$T_n F(X) := \text{holim}_{U \in \mathscr{P}_0([n])} F(U * X)$$

This allows us to express $t_n : F(X) \to T_n F(X)$ as the natural map:

$$F(X) = F(\emptyset * X) \to \underset{U \in \mathscr{P}_0([n])}{\text{holim}} (U \mapsto F(U * X)).$$

And as before, we define the degree n polynomial approximation to F, $P_n F$, as the following homotopy colimit:

$$P_n F(X) := \text{hocolim}(T_n F(X) \xrightarrow{t_n} T_n^2 F(X) \xrightarrow{t_n} \cdots)$$

2.4   Taylor Tower

The collection of polynomial approximations to a functor F, $\{P_n F\}_{n \geq 0}$, come with natural maps between them. Using Goodwillie's [Goo03] definition

$$(T_n^i F)(X) := \underset{(U_1, \dots, U_i) \in \mathscr{P}_0([n+1])^i}{\text{holim}} F(X * (U_1 * \cdots * U_i))$$

we then have a natural map $T_n^i F \xrightarrow{q_{n,i}} T_{n-1}^i F$ induced by the inclusion of categories, $\mathscr{P}_0([n])^i \hookrightarrow \mathscr{P}_0([n+1])^i$. Taking the colimit along i gives us the induced map $P_n F \xrightarrow{q_n} P_{n-1} F$. With these maps we form a tower, the Goodwillie Taylor Tower of $F(X)$:

$$\cdots \to P_n F(X) \xrightarrow{q_n} P_{n-1} F(X) \xrightarrow{q_{n-1}} \cdots \longrightarrow P_1 F(X) \xrightarrow{q_1} P_0 F(X)$$

We denote by $P_\infty F(X)$ the homotopy inverse limit of the tower.

It is worth noting now that this defines $P_\infty F(X)$ as the homotopy inverse limit of a collection of constructions which are themselves homotopy colimits (of finite homotopy inverse limits). That is, it is not expected that this construction will commonly commute with either colimits or limits (there are several special cases set out in [Goo03]).

2.5 Analyticity

Let ρ be an integer greater than or equal to zero. We say that a functor F is ρ-*analytic* if its failure to be n-excisive is controlled by ρ as n increases.

A key feature of a ρ-analytic functor, F, is that for ρ-connected spaces, X, the following natural map is an equivalence $F(X) \xrightarrow{\simeq} P_\infty F(X)$.

Note: Higher values of ρ mean that X is 'closer' to 0 (i.e. $*$), since increasing connectivity means that X has more vanishing homotopy groups. A lower value of ρ means a larger "radius of convergence" of the functor F.

2.6 Join Lemma

This section is needed for fuller generality of the results contained in this dissertation. We would like to be able to work with functors either in the traditional setting that Goodwillie establishes, the calculus involving spaces *over* a fixed space, or using the unbased or under calculus of [BJME10]. Let $\mathscr{S}$ be either spaces or spectra, $f : A \to B$ a morphism in $\mathscr{S}$, and $\mathscr{S}_f$ be the category of factorizations of f, i.e. objects (X, α_X, β_X) such that

$$
\begin{array}{ccc}
 & X & \\
\alpha_X \nearrow & & \searrow \beta_X \\
A & \xrightarrow{\quad f \quad} & B
\end{array}
$$

Let $\emptyset$ denote the initial object of $\mathscr{S}$ and $\bullet$ denote its final object (which are the same when $\mathscr{S}$ is Spectra). Then the notions of over and under category correspond to letting f be either $\emptyset \to B$ or $A \to \bullet$.

$$
\begin{array}{ccc}
 & X & \\
\nearrow & & \searrow \beta_X \\
\emptyset & \xrightarrow{\quad\quad} & B
\end{array}
\qquad
\begin{array}{ccc}
 & X & \\
\alpha_X \nearrow & & \searrow \\
A & \xrightarrow{\quad\quad} & \bullet
\end{array}
$$

$$(\mathscr{S} \downarrow B) \qquad\qquad (A \downarrow \mathscr{S})$$

We need to discuss taking the join in the categories $\mathscr{S}_f$, and note that all of the arguments hold for $\mathscr{S}$ replaced by a simplicial model category C which has arbitrary coproducts and a functorial cofibrant replacement construction.

We will denote by $X \otimes_A -$ the a functor from *Sets* to $\mathscr{S}_f$ which is the left adjoint of $\mathrm{Hom}_{\mathscr{S}_f}(X, -)$. The following homotopy pushout in $\mathscr{S}_f$ we will take as definition of the *join*:

$$
X *_f U = \mathrm{hocolim} \left(\begin{array}{ccc} X \otimes_A U & \xrightarrow{\beta} & B \otimes_A U \\ \downarrow{\scriptstyle +} & & \\ X & & \end{array} \right)
$$

This may be prolonged to give the join of an object $X \in \mathscr{S}$ with a simplicial set U by defining, where $\|Y.\|$ denotes geometric realization,

$$
X *_f U := \|[n] \to X *_f U_n\|.
$$

For the method of proof of our theorems from Chapter 7 to be independent of the join, we need the following lemma about iterating the join construction.

Lemma 2.6.1 (Join Lemma). *Given U, V simplicial sets and $(X, \alpha_X, \beta_X) \in \mathscr{S}_f$ with α_X and β_X cofibrations, there is a natural isomorphism*[1]

$$
(X *_f U) *_f V \cong X *_f (U * V).
$$

Proof of Lemma 2.6.1[2] One property of $\otimes_A$ is that there is an isomorphism $(X \otimes_A U) \otimes_A V \cong X \otimes_A (U \otimes_\emptyset V)$. Since α_X and β_X are cofibrations, $X \otimes_A \bullet$ and $B \otimes_A \bullet$ commute with homotopy colimit. Using this and our definitions, we have the following isomorphisms:

$$
\begin{aligned}
X *_f (U * V) \;=\; & X *_f \mathrm{hocolim} \left(\begin{array}{ccc} U \times V & \longrightarrow & U \\ \downarrow & & \\ V & & \end{array} \right) \\[2em]
=\; & \mathrm{hocolim} \left(\begin{array}{c} \left(\begin{array}{ccc} X \otimes_A (U \times V) & \longrightarrow & X \otimes_A U \\ \downarrow & & \\ X \otimes_A V & & \end{array} \right) \Rightarrow \left(\begin{array}{ccc} B \otimes_A (U \times V) & \longrightarrow & B \otimes_A U \\ \downarrow & & \\ B \otimes_A V & & \end{array} \right) \\[2em] \Downarrow \\[1em] \left(\begin{array}{ccc} X & = & X \\ \| & & \\ X & & \end{array} \right) \end{array} \right)
\end{aligned}
$$

We can use a natural map $X \otimes_A V \to X$ to alter our diagram, as it induces an isomorphism on homotopy

---

[1] For $(\mathscr{S} \downarrow B)$, this statement may be found in [Goo03, p.8, §1], where $*_f$ is called the *fiberwise join*, and denoted by $*_B$.
[2] We thank Randy McCarthy for explaining this lemma.

colimits as follows:

$$\mathrm{hocolim}\left(\begin{array}{ccc} X \otimes_A V & \longrightarrow & X \\ \| & & \\ X \otimes_A V & & \end{array}\right) \xrightarrow{\cong} \mathrm{hocolim}\left(\begin{array}{ccc} X & = & X \\ \| & & \\ X & & \end{array}\right)$$

We now substitute this in and then by reversing the order of homotopy colimits, this becomes isomorphic to the iterated homotopy colimit of

$$\left(\begin{array}{c} \left(\begin{array}{ccc} X \otimes_A U \otimes_A V & \longrightarrow & B \otimes_A U \otimes V \\ \downarrow & & \\ X \otimes_A V & & \end{array}\right) \Rightarrow \left(\begin{array}{ccc} B \otimes_A (U \times V) & \longrightarrow & B \otimes_A U \\ \downarrow & & \\ X & & \end{array}\right) \\ \Downarrow \\ \left(\begin{array}{ccc} X \otimes_A V & \longrightarrow & B \otimes_A V \\ \| & & \\ X \otimes_A V & & \end{array}\right) \end{array}\right)$$

Then using that $A \to X \otimes_A U$ and $A \to B \otimes_A U$ are cofibrations, this is equivalent to

$$\mathrm{hocolim}\left(\begin{array}{ccc} X *_f U \otimes_A V & \longrightarrow & X *_f U \\ \downarrow & & \\ B \otimes_A V & & \end{array}\right) \cong (X *_f U) *_f V$$

Chapter 3

Differential-like Lifts

Let $\mathscr{C}_{f:A \rightarrowtail B}$ be the category of spaces or spectra factoring a fixed cofibration. Our functors will be from $\mathscr{C}_{f:A \rightarrowtail B}$ to spaces or spectra. We aim to prove everything in full generality, i.e in the unbased setting of $\mathrm{Top}_{f:A \rightarrowtail B}$, from which the based case can be recovered by allowing the map f to be the inclusion $X \rightarrowtail CX$. To simplify things, we will represent Figure 1.1 from the introduction as Figure 3.1. That is, at position (n, k) is $\mathbb{T}\mathrm{ot}_n F(\mathrm{sk}_k \triangle^\bullet *_f B)$.

McCarthy observed[McC10] that work of Segal[Seg74] could be reinterpreted as existence of a lift for F a linear functor from $\mathrm{Tot}_2 F(\mathrm{sk}_1 \triangle)$ to $\mathrm{Tot}_3 F(\mathrm{sk}_2 \triangle)$. By 'lift', we mean a map from (n, k) to $(n + j, k + 1)$ commuting with the maps in the diagram of Figure 3.1. We provide constructions of explicit geometric lifts in Figure 3.1 of slope 1, for arbitrary functors F, not just linear, and at every point in the diagram below the $n = k$ line; see Figure 3.2. That is, from all positions (n, k) to $(n + 1, k + 1)$, where $n > k \geq 0$. We state these results as Proposition 3.0.2.

Proposition 3.0.2. *For an arbitrary homotopy functor F, we have maps $\mathrm{Tot}_n F(\mathrm{sk}_k \triangle^\bullet *_f B) \to \mathrm{Tot}_{n+j} F(\mathrm{sk}_{k+1} \triangle^\bullet *_f B)$ for $j = 1, n > k \geq 0$ and for arbitrary $j > 1$ when $k = 0, n > k$.*

Additionally, there is more interesting behavior along the $k = (n - 1)$ line, stated as Proposition 3.0.3. By definition, $\mathrm{T}_n F(A) \simeq \mathrm{holim}_{\mathscr{P}_0([n])} F(\mathrm{sk}_0 \triangle^\bullet *_f B)$, and using cofinality of the functor $\mathscr{P}_0([n]) \to \triangle_{\leq n}$[1], we have that $\mathrm{T}_n F(A) \simeq \mathrm{Tot}_n F(\mathrm{sk}_0 \triangle^\bullet *_f B)$.

Proposition 3.0.3. *There are equivalences*

$$\mathrm{Tot}_n F(\mathrm{sk}_{n-1} \triangle *_f B) \simeq \mathrm{T}_1^n F(A)$$

Additionally, the maps $\mathcal{L}_1$ of Prop 3.0.2 along the $k = (n-1)$ line are up to homotopy the maps $t_1 : \mathrm{T}_1^n F(A) \to \mathrm{T}_1^{n+1} F(A)$. That is, the following square commutes:

---

[1] This is the base case of Theorem 7.3.1

$$\begin{CD}
\text{Tot}_n\, F(\text{sk}_{n-1}\, \Delta^\bullet *_f B) @>{\mathcal{L}_1}>> \text{Tot}_{n+1}\, F(\text{sk}_n\, \Delta^\bullet *_f B) \\
@AA{\simeq}A @AA{\simeq}A \\
\text{T}_1^n F(A) @>{t_1}>> \text{T}_1^{n+1} F(A)
\end{CD}$$

$k = 4$

$k = 3$

$k = 2$

$k = 1$

$k = 0$

$\leftarrow$ varying k in sk_k $\uparrow$

$\ldots$ $n = 4$ $n = 3$ $n = 2$ $n = 1$

$\leftarrow$ varying n in Tot_n $\rightarrow$
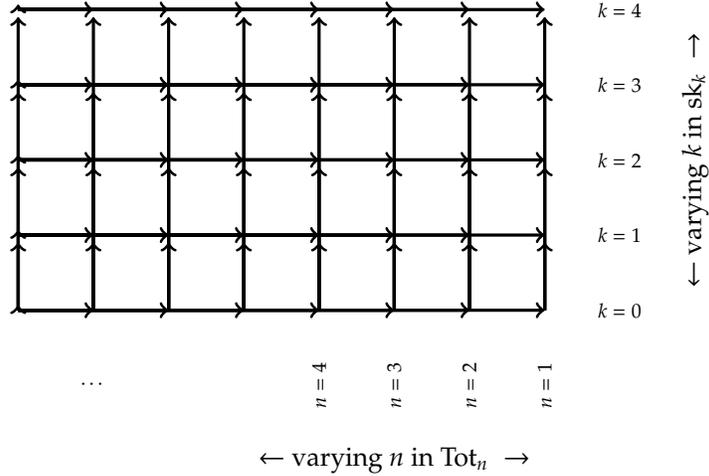
Figure 3.1: $\text{Tot}_n\, F(\text{sk}_k\, \Delta^\bullet *_f B)$ grid

$k = 4$

$k = 3$

$k = 2$

$k = 1$

$k = 0$

$\leftarrow$ varying k in sk_k $\uparrow$

$\ldots$ $n = 4$ $n = 3$ $n = 2$ $n = 1$

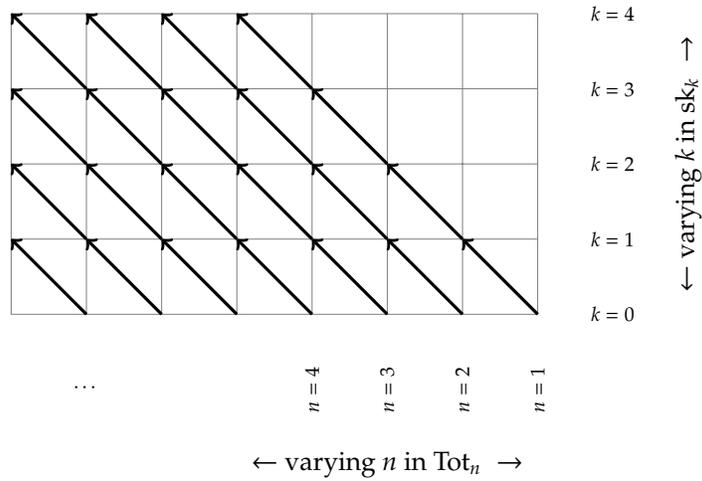$\leftarrow$ varying n in Tot_n $\rightarrow$

Figure 3.2: Slope 1 maps

This gives us not only the base case of the proof of Theorem 4.0.2, but also

Corollary 3.0.4. *We have a diagonal tower in the diagram of Figure 3.1 whose colimit is* $\text{P}_1 F(A)$, *and whose maps are all equivalences when F is 1-excisive:*

$$P_1 F(A) \simeq \text{hocolim}_n(\text{Tot}_1\, F(\text{sk}_0 \triangle^\bullet *_f B) \to \text{Tot}_2\, F(\text{sk}_1 \triangle^\bullet *_f B) \to \cdots)$$

Note that this is the rightmost line of slope 1 arrows in Figure 3.2.

In the introduction, we mentioned the similarity of the above diagram to the E_0 page of the spectral sequence of a cosimplicial space. This suggests an interpretation of the lifts of Prop 3.0.2 as being analogous to differentials (even though they do not satisfy $d \circ d = 0$), where modding out by their image would be replaced by applying $P_n F$ for geometric lifts of slope n. This viewpoint yields Conjecture 3.0.5.

Conjecture 3.0.5. *For functors F from spaces to spectra,*

(i) *We have equivalences $\text{T}_n^{k+1} F(A) \simeq \text{Tot}_{n(k+1)}\, F(\text{sk}_k \triangle^\bullet *_f B)$*

(ii) *There are maps $\text{Tot}_{n(k+1)} F(\text{sk}_k \triangle^\bullet *_f B) \to \text{Tot}_{n(k+2)} F(\text{sk}_{k+1} \triangle^\bullet *_f B)$*

(iii) *The maps in (ii) are up to homotopy the maps $t_n : \text{T}_n^k F(A) \to \text{T}_n^{k+1} F(A)$.*

(iv) *The maps of (iii) give us a directed system whose colimit is $P_n F$:*

$$P_n F(A) \simeq \text{hocolim}(\text{Tot}_n\, F(\text{sk}_0 \triangle^\bullet *_f B) \to \text{Tot}_{2n}\, F(\text{sk}_1 \triangle^\bullet *_f B) \to \cdots) \to \text{Tot}_{n(k+1)}\, F(\text{sk}_k \triangle^\bullet *_f B) \to \cdots)$$

Our answer so far to this conjecture is Theorem 4.0.2, which says that when F is degree n, the collection $\{\text{Tot}_{n(k+1)}\, F(\text{sk}_k \triangle^\bullet *_f B)\}$ are all equivalent. However, they are zig-zag equivalences, not realized by a direct map. See Figure 3.3. The nodes are labeled both by color and the excisiveness of the functor that causes them to be equivalent. That is, the red nodes with 2^k all receive maps from $F(A)$ (and we have some expectation of the Tot-cube at that location looking like T_2^k), and those maps are equivalences when F is 2-excisive. The 1-excisive line is shown with arrows expressing the maps of Prop 3.0.2 between, and unlabeled black nodes. We reserve the proof of Theorem 4.0.2 for the next chapter.

Pursuit of Conjecture 3.0.5 also lead to the main theorem of this dissertation, Theorem 7.1.2 of Chapter 7. which further ties together the T_n^k and $\text{Tot}_n\, \text{sk}_k \triangle^\bullet$-towers.

If we have indexing categories C, D with $C \hookrightarrow D$, then we get a map of homotopy inverse limits the other direction, $\text{holim}_D F \to \text{holim}_C F \circ (inc)$ for a D-diagram F. That is, it is more 'natural' to map from larger to smaller cubical diagrams. Our lifts are maps from smaller to larger cubical diagrams.

To make a lift, then, we need to take a cartesian n-cube $\mathscr{X}$ and produce an $(n + k)$ cube $\widetilde{\mathscr{X}}$ such that $\text{holim}_0 \mathscr{X} \simeq \text{holim}_0 \widetilde{\mathscr{X}}$. We use the following lemma:
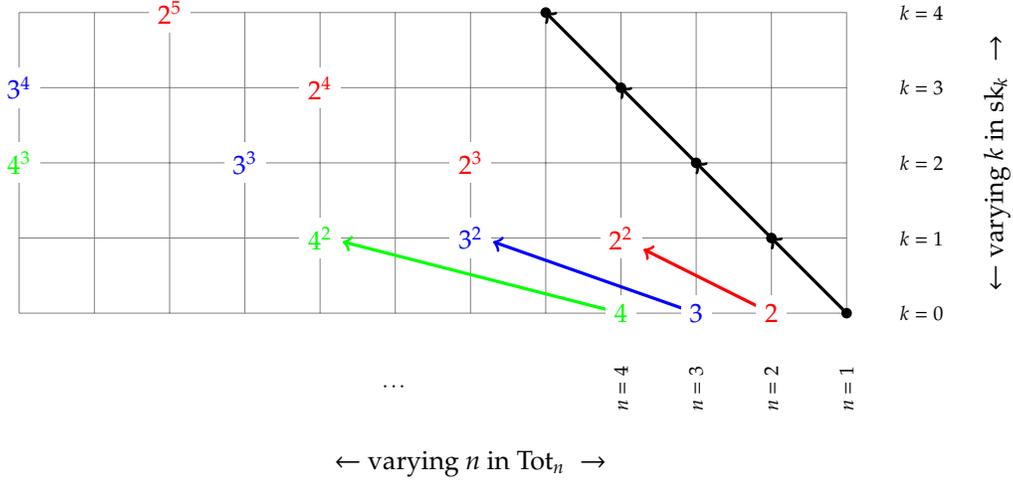
Figure 3.3: Representation of Theorem 4.0.2

Lemma 3.0.6. *If A is an n-cube, and B is a cartesian n-cube, that $\mathrm{holim}(A \to B) \simeq \mathrm{holim}\, A$, and their punctured homotopy limits are also the same.*

Proof. By the Covering Lemma, stated as Lemma 3.2.1. Let $\mathscr{X} = A \to B$. The Covering Lemma gives us that $\mathscr{X}$ is cartesian iff the following square is cartesian:

$$
\begin{array}{ccc}
\mathscr{X}(\emptyset) = A(\emptyset) & \longrightarrow & \mathrm{holim}_0\, \mathscr{X}_{Top} = \mathrm{holim}_0\, A \\
\downarrow & & \downarrow \\
\mathscr{X}_{Bottom}(\emptyset) = B(\emptyset) & \longrightarrow & \mathrm{holim}_0\, \mathscr{X}_{Bottom} = \mathrm{holim}_0\, B
\end{array}
$$

We are starting with A and B themselves cartesian, which means the top and bottom (i.e. the two right-pointing) arrows are equivalences, so the square is cartesian and $\mathscr{X}$ is itself cartesian. That is, $\mathscr{X}(\emptyset) = \mathrm{holim}_0\, \mathscr{X}$, and we know that $A(\emptyset) = \mathscr{X}(\emptyset)$, so the (punctured) homotopy limit of the whole $(n + 1)$-cube determined by $A \to B$ is the (punctured) homtopy limit of A. $\qquad\square$

Given an n-cube $\mathscr{X}$, our lift to an $(n + 1)$ cube will be determined by providing another cartesian n-cube which $\mathscr{X}$ maps to, such that the $(n + 1)$-cube maps to the target. Our starting n-cube is $\mathbb{T}\mathbf{ot}_{n-1}F(\mathrm{sk}_k \triangle^\bullet *_f B)$, and we aim to provide an n-cube $\mathscr{Y}$ such that there is a map of $(n + 1)$-cubes $(\mathbb{T}\mathbf{ot}_{n-1}F(\mathrm{sk}_k \triangle^\bullet *_f B) \to \mathscr{Y}) \Rightarrow \mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{k+1} \triangle^\bullet *_f B)$.

Since we want this for all homotopy functors, we want it for the identity first. We also want our cartesian n-cube $\mathscr{Y}$ to remain cartesian under application of any homotopy functor. We therefore need a class of n-cubes which are cartesian and remain cartesian under application of any homotopy functor. We call such

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cubes **absolutely cartesian**. Working with these constructions led to the following conjecture[2]:

Conjecture 3.0.7 (E.)**.** *An n-cube is absolutely cartesian if and only if it can be written as a map of two absolutely cartesian $(n-1)$-cubes.*

The base case of this, for $n = 2$, is the main theorem, 5.0.2, of Chapter 5. Such a square has at least one pair of parallel arrows both equivalences, given that a cartesian 1-cube is an equivalence, and therefore immediately preserved by all homotopy functors. It should be clear that building up an n-cube inductively as maps of these absolutely cartesian squares will yield an absolutely cartesian n-cube. It is not yet certain if the other direction of the conjecture is true.

The cubes constructed for our lifts are all of this inductive form.

3.1 Proof of Prop 3.0.2 , lifts

In this section, we will prove Prop 3.0.2:

Proposition 3.0.2 *For an arbitrary homotopy functor F, we have maps* $\mathrm{Tot}_n F(\mathrm{sk}_k \triangle^\bullet *_f B) \to \mathrm{Tot}_{n+j} F(\mathrm{sk}_{k+1} \triangle^\bullet *_f B)$ *for $j = 1, n > k \geq 0$ and for arbitrary $j > 1$ when $k = 0, n > k$.*

We will first show that the slope 1 maps exist everywhere. We will then give the lifts of (somewhat) arbitrary slope from the 0th to 1st rows.

As mentioned before, our lifts will be made with absolutely cartesian n-cubes, and we'll start with lifts for the identity. Given the absolute cartesianness, it suffices to construct lifts for the $\mathbb{T}\mathbf{ot}_n \, \mathrm{sk}_k \triangle^\bullet$ diagrams, as these will be preserved by applying $- *_f B$ and then any arbitrary homotopy functor F.

3.1.1 Slope 1 lifts

As was mentioned in the last section, to make our slope 1 lifts, we need to start with an n-cube $\mathbb{T}\mathbf{ot}_{n-1}F(\mathrm{sk}_k \triangle^\bullet)$ and provide an n-cube $\mathscr{Y}$ which it maps to, such that there is a map of $(n + 1)$-cubes $(\mathbb{T}\mathbf{ot}_{n-1}F(\mathrm{sk}_k \triangle^\bullet) \to \mathscr{Y}) \Rightarrow \mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{k+1} \triangle^\bullet)$.

Since we would like this lift for arbitrary functors, we provide $\mathscr{Y}$ for the identity, and assert that $\mathscr{Y}$ for subsequent homotopy functors F is given by F applied to this $\mathscr{Y}$.

Notice that $C\triangle^n = \triangle^{n+1}$, i.e. coning takes simplices up to one-higher-dimensional simplices. We also observe that we have inclusions

---

[2]Thanks go to Kristine Bauer for suggesting a more clear statement of this.

$$\mathrm{sk}_k \Delta^n \hookrightarrow C(\mathrm{sk}_k \Delta^n) \hookrightarrow \mathrm{sk}_{k+1} \Delta^{n+1}$$

The cube $\mathbb{T}\mathbf{ot}_{n-1} \mathrm{sk}_k \Delta^\bullet$, has at each U-indexed spot a copy of $\mathrm{sk}_k \Delta^{\#U-1}$, and maps are induced by inclusions that increase dimension by one, that is between $\mathrm{sk}_k \Delta^{\#U-1}$ and $\mathrm{sk}_k \Delta^{\#U}$. With the above observed inclusions, we can build our $(n+1)$-cube with homotopy limit equivalent to that of $\mathbb{T}\mathbf{ot}_{n-1} \mathrm{sk}_k \Delta^\bullet$. This is $\mathbb{T}\mathbf{ot}_n \mathrm{sk}_k \Delta^\bullet \Rightarrow C(\mathbb{T}\mathbf{ot}_n \mathrm{sk}_k \Delta^\bullet)$. For $n = 1$, see Figure 3.1.1.

We acknowledge that this is 'uninteresting' for the identity (since all of the holims are $\emptyset$, but become interesting after applying our functors F, and working at this level gives us our constructions for all F).
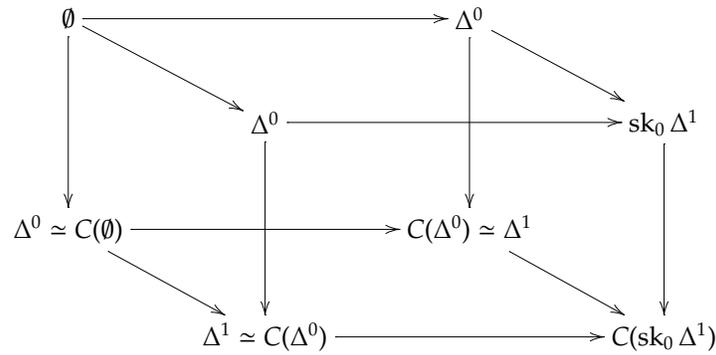

Figure 3.4: An example lift cube, for $\mathrm{Tot}_1 \mathrm{sk}_0 \Delta^\bullet$ mapping to $\mathrm{Tot}_2 \mathrm{sk}_1 \Delta^\bullet$

Notice that the following diagram of inclusions commutes:

Given that, our proposed map of $(n+1)$-cubes is well-defined.

We also need to show that they commute with the maps induced by maps $\mathrm{Tot}_n X \to \mathrm{Tot}_{n-1} X$ and $\mathrm{sk}_k \Delta^\bullet \hookrightarrow \mathrm{sk}_{k+1} \Delta^\bullet$, i.e. those of Figure 3.1. Using the indexing of that Figure, we'll represent our situation as follows, with the squiggly arrow representing our lift:

We first show that the upper triangle commutes up to homotopy. The lift map takes a tuple in the holim,

say $\vec{\gamma}$, and sends it to itself, plus a bunch of trivial stuff (since the holim of the lift/larger cube is the same as the smaller). That is, it lands in the local copy of $\mathrm{Tot}_n \, \mathrm{sk}_{k+1} \, \Delta^\bullet$. Therefore, lifting and projecting down is (up to homotopy) the same as just including from (n, k) to $(n, k+1)$.

Up to homotopy commutativity of the lower triangle follows from the fact that the inclusion of $(\mathrm{sk}_k \, \Delta^\bullet)_{1 \leq \bullet \leq n}$ in $(\mathrm{sk}_{k+1} \, \Delta^\bullet)_{1 \leq \bullet \leq n}$ is nulhomotopic. That is, the image of that map is effective only the information from the $\mathbb{T}\mathbf{ot}_n$ portion, which is what we get via our lift map.

3.1.2 Lifts of arbitrary slope, sk_0-row to sk_1-row

We establish lifts $\mathrm{Tot}_n \, \mathrm{sk}_0 \, \Delta \xrightarrow{\varrho^k} \mathrm{Tot}_{n+k} \, \mathrm{sk}_1 \, \Delta$ for all k such that $1 \leq k \leq n+1$. Note that the upper bound on k is equal to $\#(\mathrm{sk}_0 \, \Delta^n)$.

1. The *first* lift is constructed by coning the $(n+1)$−cube $\mathrm{Tot}_n \, \mathrm{sk}_0$, labeling the cone point as $n+2$.

    Example.

    We can express this more compactly if we allow commas to denote disjoint unions, and words are simplices spanning the vertices they contain. That is, 2 would be a point, while $2, 3$ would be the disjoint union of two points and 23 would be a copy of Δ^1 with vertices $2, 3$. We can give this example lift then in more compact notation as:

    $$\begin{bmatrix} \emptyset & 2 \\ 1 & 1,2 \end{bmatrix} \xrightarrow{C_1^3} \begin{bmatrix} 3 & 2,3 \\ 13 & 13,2 \end{bmatrix}$$

2. Each *subsequent* lift $(\varrho^2, \ldots, \varrho^{n+1})$ is produced from the previous by adding one new cone point (so one new $(n+1)$−cube) and pushing out over it.

Example.

$$
\overbrace{\phantom{\begin{bmatrix} \emptyset & 2 \\ 1 & 1,2 \end{bmatrix}}}^{\mathrm{Tot}_1\,\mathrm{sk}_0}
$$

$$
\left.\left.\begin{array}{ccc}
\begin{bmatrix} \emptyset & 2 \\ 1 & 1,2 \end{bmatrix} & \xrightarrow{\;C_1^3\;} & \begin{bmatrix} 3 & 2,3 \\ 13 & 2,13 \end{bmatrix} \\[4ex]
\Big\downarrow{\scriptstyle C_2^4} & & \Big\downarrow \\[2ex]
\begin{bmatrix} 4 & 24 \\ 1,4 & 1,24 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 3,4 & 24,3 \\ 13,4 & 24,13 \end{bmatrix}
\end{array}\right\}\mathfrak{L}^1\right\}\mathfrak{L}^2
$$

Notice how each coning happens to add a new point as well as exhaust one of the points we started with. At this point we can't cone over any new points from $\mathrm{sk}_0\,\Delta^1$, so we're out of lifts.

3. *Pushing out gives valid lifts.* Since each coning adds a new point and cones a different point, pushing out will never attach the segments, so pushing out over the previous cubes is equivalent to disjoint unioning. At this level, parity determines equivalence, disjoint union effectively preserves parity, done. That is, all added cubes are functorially cartesian, and include well into $\mathrm{sk}_1\,\Delta^{\bullet}$.

$\square$

3.2 Proof of Prop 3.0.3

3.2.1 Proof of Proposition3.0.3, Tot and T_n^k equivalences

In this section, we prove

Proposition 3.0.3 *There are equivalences*

$$
\mathrm{Tot}_n\, F(\mathrm{sk}_{n-1}\,\Delta *_f B) \simeq T_1^n F(A)
$$

Additionally, the maps $\mathcal{L}_1$ of Prop 3.0.2 along the $k = (n-1)$ line are up to homotopy the maps $t_1 : T_1^n F(A) \to T_1^{n+1} F(A)$. That is, the following square commutes:

$$\text{Tot}_n\, F(\text{sk}_{n-1}\,\Delta^\bullet *_f B) \xrightarrow{\;\mathcal{L}_1\;} \text{Tot}_{n+1}\, F(\text{sk}_n\,\Delta^\bullet *_f B)$$

$$\text{T}_1^n F(A) \xrightarrow{\quad t_1 \quad} \text{T}_1^{n+1} F(A)$$

We showed earlier in this chapter that we have equivalences (by cofinality of the natural functor $\mathscr{P}_0([n]) \to \Delta_{\leq n}$.

$$\mathbb{T}_1^n F(A) \simeq \mathbb{T}\mathbf{ot}_n F(\text{sk}_0\,\Delta^\bullet *_f B) \tag{3.1}$$

To prove Prop 3.0.3, we first use the Covering Lemma (stated as Lemma 3.2.1) to reduce the $\mathbb{T}\mathbf{ot}_n\, \text{sk}_k\,\Delta^\bullet$-cube to a cartesian square. We then show that this square is $\mathbb{T}_1(\text{Tot}_{n-1}\,\text{sk}_{n-2}\,\Delta^\bullet *_f B)$. Combined with 3.1, we get that $\text{T}_1(\text{Tot}_{n-1}\, F(\text{sk}_{n-2}\,\Delta^\bullet *_f B) \simeq \text{T}_1^n F(A)$. E.g.

$$\text{Tot}_1\, F(\text{sk}_0\,\Delta^\bullet *_f B) \longrightarrow \text{Tot}_2\, F(\text{sk}_1\,\Delta^\bullet *_f B) \longrightarrow \cdots$$

$$\text{T}_1 F(A) \qquad \text{T}_1(\text{Tot}_1\, F(\text{sk}_0\,\Delta^\bullet *_f B) \simeq \text{T}_1(\text{T}_1 F(A))$$

Lemma 3.2.1 (Covering Lemma). *Let $\mathscr{X}$ be an $n+1$-cube. We'll define $\mathscr{X}_{Top}$ to be the sub-n-cube indexed by the elements of $\mathscr{P}([n-1])$ and $\mathscr{X}_{Bottom}$ is the sub-n-cube indexed by elements of $\mathscr{P}([n-1]) \cup \{n\}$ that is, each element $S \in \mathscr{P}([n-1])$ with $\{n\}$ adjoined. Then $\mathscr{X} : \mathscr{X}_{Top} \to \mathscr{X}_{Bottom}$ is cartesian $\iff$ the following is*

$$\mathscr{X}(\emptyset) \longrightarrow \text{holim}_0\, \mathscr{X}_{Top}$$

$$\mathscr{X}_{Bottom}(\emptyset) \longrightarrow \text{holim}_0\, \mathscr{X}_{Bottom}$$

(By $\mathscr{X}_{Bottom}(\emptyset)$, we mean the initial object of that sub-n-cube .)

Recall that

$$\text{T}_1^n F(A) \simeq \text{holim}_0 \left(\begin{array}{cc} & \text{T}_1^{n-1} F(\{0\} *_f B) \\ & \downarrow \\ \text{T}_1^{n-1} F(\{1\} *_f B) \longrightarrow & \text{T}_1^{n-1} F(\{0,1\} *_f B) \end{array} \right)$$

To show that $\text{Tot}_n\, F(\text{sk}_{n-1}\,\Delta^\bullet *_f B) \simeq \text{T}_1(\text{Tot}_{n-1}\, F(\text{sk}_{n-2}\,\Delta^\bullet *_f B)$, it suffices to show

(1) $(\mathbb{T}\mathbf{ot}_n F(\text{sk}_{n-1}\,\Delta^\bullet *_f B))(\{n\}) \simeq \text{Tot}_{n-1}\, F(\Delta^0 *_f B)$

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(2) $\mathrm{holim}_0(\mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{n-1}\,\Delta^\bullet *_f B)_{\mathrm{top}} \simeq \mathrm{Tot}_{n-1} F(\Delta^0 *_f B)$

(3) $\mathrm{holim}_0(\mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{n-1}\,\Delta^\bullet *_f B)_{\mathrm{bottom}} \simeq \mathrm{Tot}_{n-1} F(\mathrm{sk}_{n-2}\,\Delta^\bullet * S^0 *_f B)$

(1) is immediate, since $(\mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{n-1}\,\Delta^\bullet *_f B))(U) \simeq F(\mathrm{sk}_{n-1}\,\Delta^{\#U} *_f B)$, which for $U = \{n\}$ is $F(\Delta^0 *_f B)$.

(2) is true because $\mathrm{holim}_0(\mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{n-1}\,\Delta^\bullet *_f B)_{\mathrm{top}} \simeq \mathrm{Tot}_{n-1} F(\mathrm{sk}_{n-1}\,\Delta^\bullet *_f B) \simeq F(\Delta^0 *_f B)$, being a constant diagram (up to homotopy) of F applied to contractible things.

(3) follows from showing that $(\mathbb{T}\mathbf{ot}_n F(\mathrm{sk}_{n-1}\,\Delta^\bullet)_{\mathrm{bottom}} \simeq S(\mathbb{T}\mathbf{ot}_{n-1}(\mathrm{sk}_{n-2}\,\Delta^\bullet))$. Notice that $(\mathbb{T}\mathbf{ot}_{n-1}\,\mathrm{sk}_{n-2}\,\Delta^\bullet)([n-1]) \simeq S^{n-2}$, so first off we have that

$$(\mathbb{T}\mathbf{ot}_n\,\mathrm{sk}_{n-1}\,\Delta^\bullet)([n]) \simeq S^{n-1} \simeq S(S^{n-1}) \simeq S\big((\mathbb{T}\mathbf{ot}_{n-1}\,\mathrm{sk}_{n-2}\,\Delta^\bullet)([n-1])\big)$$

Also, $(\mathbb{T}\mathbf{ot}_{n-1}(\mathrm{sk}_{n-2}\,\Delta^\bullet))(U) \simeq *$ for each U indexing the bottom of the cube besides the final point. Suspension of a contractible thing is still contractible. We have therefore shown that $(\mathbb{T}\mathbf{ot}_n(\mathrm{sk}_{n-1}\,\Delta^\bullet)_{\mathrm{bottom}} \simeq S(\mathbb{T}\mathbf{ot}_{n-1}(\mathrm{sk}_{n-2}\,\Delta^\bullet))$ and (3) follows.

Therefore, $\mathrm{Tot}_n F(\mathrm{sk}_{n-1}\,\Delta^\bullet *_f B) \simeq \mathrm{T}_1(\mathrm{Tot}_{n-1} F(\mathrm{sk}_{n-2}\,\Delta^\bullet *_f B))$. □

Chapter 4

Finite stage collapse

In this section, we will prove the following theorem:

Theorem 4.0.2. *Given F a homotopy functor commuting with colimits, from $\mathrm{Top}_{f:A \rightarrowtail B}$ to Spectra, the following maps exist and when F is degree n they are equivalences:*

$$F(A) \rightarrow \mathrm{Tot}_{(k+1)n} F(\mathrm{sk}_k \Delta^*_* *_f B)$$

The significance of this result is, for our purposes, predictive. It further suggests the existence of the differential-like maps pursued in Chapter 3. The results of that chapter were established first and were used to predict this theorem as well as offer the base case of its proof. These are Propositions 3.0.2 and 3.0.3, whose immediate corollary is one proof of the base case of Theorem 4.0.2 as it provides the requisite maps and it is clear that when F is 1-excisive, they are equivalences.

4.1 Existence of the maps from $F(A)$

The maps $F(A)$ to $\mathrm{Tot}_{n(k+1)} F(\mathrm{sk}_k \Delta^\bullet *_f B)$ are induced by the inclusion of the empty set into $\mathrm{sk}_k \Delta^\bullet$. We are working in the category $\mathrm{Top}_{f:A \rightarrowtail B}$. Recall that $\otimes_A$ is the tensor inherited by viewing Top as a simplicial model category, and which induces a simplicial tensor in $\mathrm{Top}_{f:A \rightarrowtail B}$ respecting the factorization.

In this context, we have $\emptyset *_f B \simeq A$, regarded as the constant cosimplicial object A. $F(A)$ may then be rewritten as $\mathrm{Tot}_j F(\emptyset *_f B)$ for all j. The inclusion map $\emptyset \hookrightarrow \mathrm{sk}_k \Delta$ induces maps $\emptyset *_f B \rightarrow \mathrm{sk}_k \Delta *_f B$, and also maps $F(A) = \mathrm{Tot}_j F(\emptyset *_f B) \rightarrow \mathrm{Tot}_j F(\mathrm{sk}_k \Delta *_f B) \ \forall j$.

4.2 Proof of Theorem 4.0.2 by Induction

Recall that the base case of our induction follows as a corollary of Proposition 3.0.3.

Now we want to show that

$$D_nF(A) \xrightarrow{\simeq} \mathrm{Tot}_{n(k+1)} D_nF(\mathrm{sk}_k \Delta *_f B)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\widetilde{cr}_nF(A)_{h\Sigma_n} \longrightarrow \mathrm{Tot}_{n(k+1)} \widetilde{cr}_nF(A)_{h\Sigma_n}$$

Note that by Goodwillie's classification theorem, $D_nF(A) \simeq \widetilde{cr}_nF(A)_{h\Sigma_n}$. Considering the following diagram, where the bottom equivalence is due the fact that we are landing in Spectra and can therefore commute finite limits/colimits (i.e. can worry about $(\ )_{h\Sigma_n}$ last)

$$\widetilde{cr}_nF(A)_{h\Sigma_n}$$

$$\downarrow \qquad \searrow^{(-)_{h\Sigma_n}}$$

$$\mathrm{Tot}_{n(k+1)} \widetilde{cr}_nF(\mathrm{sk}_k \Delta *_f B)_{h\Sigma_n} \xrightarrow{\simeq} [\mathrm{Tot}_{n(k+1)} \widetilde{cr}_nF(\mathrm{sk}_k \Delta *_f B)]_{h\Sigma_n}$$

Then, since $X \mapsto (X)_{h\Sigma_n}$ preserves equivalences, we can check for equivalence *before* taking homotopy orbits.

(1) Recall that we have the following important collection of fibrations

$$D_nF \to P_nF \to P_{n-1}F$$

(2) $D_nF \simeq \widetilde{cr}_nF(-)_{h\Sigma_n}$ is a symmetric diagonal of an n-multilinear functor. That is, of the form $G(-, \ldots, -)$ such that $G(x_1, \ldots, x_{n-1}, -)$ is linear, and given a $\sigma \in \Sigma$, $G(x_1, \ldots, x_n) \simeq G(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. .

(3) Because the initial result was for 1-excisive functors, we have

$$P_1F(A) \xrightarrow{\simeq} \mathrm{Tot}_{1(k+1)}(P_1F(\mathrm{sk}_k \Delta *_f B).$$

Our **Inductive Hypothesis** is that $P_{n-1}F(A) \xrightarrow{\simeq} \mathrm{Tot}_{(n-1)(k+1)} P_{n-1}F(\mathrm{sk}_k \Delta *_f B)$.

We have the following diagram:

$$D_nF(A) \longrightarrow P_nF(A) \longrightarrow P_{n-1}F(A)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \text{by hyp} \ \sim$$

$$\mathrm{Tot}_{n(k+1)} D_nF(\mathrm{sk}_k \Delta *_f B) \longrightarrow \mathrm{Tot}_{n(k+1)} P_nF(\mathrm{sk}_k \Delta *_f B) \longrightarrow \mathrm{Tot}_{n(k+1)} P_{n-1}F(\mathrm{sk}_k \Delta *_f B)$$

24

Now proving $D_n F(A) \xrightarrow{\simeq} \mathrm{Tot}_{(n-1)(k+1)} D_n F(\mathrm{sk}_k \Delta *_f B)$ suffices to prove for P_n, because for a diagram of this form of spectra, if the outer two maps are equivalences, the middle one is. In spaces, we would have to be concerned with the righthand horizontal maps being surjective on π_0.

Lemma 4.2.1. $D_n F(A) \xrightarrow{\simeq} \mathrm{Tot}_{(n-1)(k+1)} D_n F(\mathrm{sk}_k \Delta *_f B)$

Proof. This is achieved by combining the inductive hypothesis with the stabilization lemma. Recall that $D_n F(-)$ is the symmetric diagonal of a multilinear functor, G. Let diag_k denote the diagonal in some k slots. Note that G composed with the diagonal (of a given space) in all but one slot is a linear functor, i.e. $G \circ \mathrm{diag}_{n-1}$ and

$$(G \circ diag_{n-1})(\mathrm{sk}_k \Delta *_f B) \simeq \mathrm{cosk}_{1(k+1)}(G \circ \mathrm{diag}_{n-1})(\mathrm{sk}_k \Delta *_f B)$$

for each choice of $n - 1$ slots. Then the n-cosimplicial spectrum given by $(G \circ \Delta)(\mathrm{sk}_k \Delta *_f B)$ has the following property (by the extended stabilization lemma):

$$(G \circ \Delta)(A) \simeq \mathrm{cosk}_{\sum_{i=1}^{n}(k+1)}(G \circ \mathrm{diag}_{n-1})(\mathrm{sk}_k \Delta *_f B)$$

That is, $\mathrm{Tot}[(G \circ \Delta)(A)] \simeq \mathrm{Tot}_{n(k+1)}(G \circ \Delta)(\mathrm{sk}_k \Delta *_f B)$. This then implies the result for $D_n F$. □

This finishes the proof using the previous discussion, that

$$P_n F(A) \simeq \mathrm{Tot}_{n(k+1)} P_n F(\mathrm{sk}_k \Delta *_f B)$$

□

Chapter 5

Absolutely cartesian squares

We'll call a diagram $\mathscr{D}$ **absolutely cartesian** if $F(\mathscr{D})$ is cartesian for all homotopy functors F. This is a sensible notion for diagrams in spaces or spectra and functors F that go from spaces/spectra to spaces/spectra. We prove the following classification theorem for absolutely cocartesian squares:

Theorem 5.0.2 (E.)**.** *A square*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

is absolutely cartesian $\iff$ *it is a map of two absolutely cartesan 1-cubes. That is, of one of the following forms:*

$$
\begin{array}{ccccccc}
A & \overset{\sim}{\longrightarrow} & B & \quad & A & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\wr} \\
C & \overset{\sim}{\longrightarrow} & D & & C & \longrightarrow & D
\end{array}
$$

If a homotopy functor $F : \mathrm{Top} \to \mathrm{Top}$ takes cartesian diagrams to cartesian diagrams, it follows that such a cube will be taken to cartesian diagrams in Spectra. A pushout (respectively pullback) of spectra is levelwise a pushout (respectively pullback) of its constituent spaces. To show the result for functors to/from spectra or from Top to Spectra, it suffices to show for functors Top to Top.

Theorem 5.0.2 is the base case of the following conjecture:

Conjecture 5.0.3 (E.)**.** *An n-cube is absolutely cartesian if and only if it can be written as a map of two absolutely cartesian $(n-1)$-cubes.*

It should be clear that building up an n-cube inductively as maps of these absolutely cartesian squares will yield an absolutely cartesian n-cube. That is, the $\Leftarrow$ direction of the if and only if. It is not yet certain if the other direction is true. We may observe that the absolutely cartesian squares are also absolutely cocartesian. Thus, there is an additional conjecture (also unanswered) that

Conjecture 5.0.4 (E.)**.** *An n-cube is absolutely cartesian if and only if it is absolutely cocartesian.*

Proof of Theorem 5.0.2. [1]Note: this relies on switching briefly to the setting of spectra and using this to deduce properties of the original diagram of spaces.

Consider an absolutely cartesian square:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

Now apply the functor $\Sigma^\infty \mathrm{Map}(D, -)$ to our square:

$$
\begin{array}{ccc}
\Sigma^\infty \mathrm{Map}(D, A) & \longrightarrow & \Sigma^\infty \mathrm{Map}(D, B) \\
\downarrow & & \downarrow \\
\Sigma^\infty \mathrm{Map}(D, C) & \longrightarrow & \Sigma^\infty \mathrm{Map}(D, D)
\end{array}
$$

By assumption, this resultant square is still cartesian. Since the square is in spectra, we know that it is also cocartesian. Recall that Σ^∞ commutes with colimits.

We then have the following chain of equivalences:

$$
\pi_0 \Sigma^\infty \mathrm{hocolim}(\mathrm{Map}(D, B) \leftarrow \mathrm{Map}(D, A) \rightarrow \mathrm{Map}(D, C)) \quad \simeq \quad \pi_0 \Sigma^\infty \mathrm{Map}(D, D).
$$
$$
\|
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|
$$
$$
H_0(\mathrm{hocolim}(\mathrm{Map}(D, B) \leftarrow \mathrm{Map}(D, A) \rightarrow \mathrm{Map}(D, C))) \qquad H_0(\Sigma^\infty \mathrm{Map}(D, D))
$$
$$
\|
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|
$$
$$
\mathbb{Z}[\pi_0(\mathrm{hocolim}(\mathrm{Map}(D, B) \leftarrow \mathrm{Map}(D, A) \rightarrow \mathrm{Map}(D, C)))] \qquad \mathbb{Z}[\pi_0 \mathrm{Map}(D, D)]
$$

We can interpret this as telling us that $(\pi_0 \mathrm{Map}(D, B) \cup \pi_0 \mathrm{Map}(D, C))/\sim$ surject onto $\pi_0 \mathrm{Map}(D, D)$. Consider $id \in \mathrm{Map}(D, D)$. This has a preimage (up to homotopy) in $\mathrm{Map}(D, B)$ and/or $\mathrm{Map}(D, C)$. Assume $\mathrm{Map}(D, B)$. This gives a section $D \to B$. We can then rewrite our original diagram as in Figure 5, with our new map in the preimage of the identity.

We can add the homotopy pullback of $(A \to B \leftarrow D)$ to the diagram. Then the whole diagram is a pullback, being a composition of pullback squares. This lets us pull back the identity map, as in Figure 5

This yields that the whole diagram is itself absolutely cartesian, which gives that the top square is also absolutely cartesian.

Now that the top square is absolutely cartesian, we can proceed in the same way with B and obtain a

---

[1]The current form (and brevity) of this proof is influenced heavily by Tom Goodwillie. We had chatted for a few days at a conference about redoing this proof in a way that would lend a clearer route towards attacking the more general conjecture.
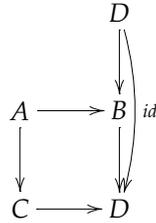
$$
\begin{array}{ccc}
& & D \\
& & \downarrow \\
A & \longrightarrow & B \quad id \\
\downarrow & & \downarrow\downarrow \\
C & \longrightarrow & D
\end{array}
$$

Figure 5.1: New information included in diagram

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
\downarrow\downarrow & & \downarrow\downarrow \\
id \quad A & \longrightarrow & B \quad id \\
\downarrow\downarrow & & \downarrow\downarrow \\
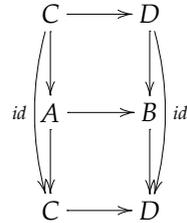C & \longrightarrow & D
\end{array}
$$

Figure 5.2: Adding the pullback of the top punctured square and pulling back the identity

section to D or A. If the section is to D, we're done.

Else, we work in the other direction. We add our section $B \to A$ to our diagram, in Figure 5

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
\downarrow & & \downarrow \\
B \longrightarrow A & \longrightarrow & B
\end{array}
$$

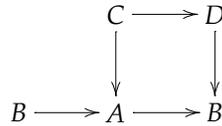Figure 5.3: Original diagram with section $B \to A$

Then we pullback the left square and observe that this gives the whole diagram (the square of squares) is cartesian. This again allows us to pull back our identity, this time to the 'top', so that we know $B' = D$ (see Figure 5).

Then since the right square and the whole square is absolutely cartesian, so is the lefthand square. Then we return to A. We get a section $A \to C$ or $A \to B$. If $A \to B$ is a section, we're done – the one-sided inverse (our original section) has an inverse on the other side and $A \xrightarrow{\simeq} B$. Then since we have the original 'top' square cartesian, as in Figure 5, we pull back the equivalence between A and B to conclude that $C \xrightarrow{\simeq} D$ as well.

If instead $A \to C$ is a section, same thing because the map $C \to A$ was a section obtained earlier. We conclude $A \xrightarrow{\simeq} C$. Recall the 'upper left' square, shown in Figure 5. Since it's also cartesian, we can pullback the equivalence and conclude that $D \xrightarrow{\simeq} B$. Since this map was a section (one-sided inverse) to $B \to D$, we
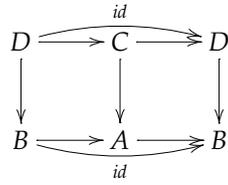
28

$$D \underset{}{\overset{id}{\rightrightarrows}} C \rightrightarrows D$$

Figure 5.4: Pulled back identity to top of diagram

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ A & \underset{\sim}{\longrightarrow} & B \end{array}$$

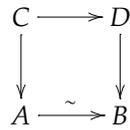Figure 5.5: Two splittings giving our equivalence

can conclude that our original map $B \xrightarrow{\simeq} D$ was an equivalence.

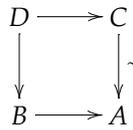$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow {\sim} \\ B & \longrightarrow & A \end{array}$$

Figure 5.6: Upper Left cartesian square

$\square$

Chapter 6

Strongly k-cocartesian n-cubes: Generalized linearity

In this section, we will prove that a reasonable potential generalization of the notion of degree n for a functor actually ends up implying that the functor is degree 1, and in the initial case, we can get the stronger condition of $\mathfrak{F}$ linear. Recall that a functor $\mathfrak{F}$ is degree 1 if it takes squares of the following form to cartesian squares:

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & A \amalg B
\end{array}
$$

Theorem 1.0.7 states that a degree n functor F has the property that $F(\mathbb{T}\mathrm{ot}_{n(k+1)}(\mathrm{sk}_k \Delta * X))$ is cartesian $\forall k \geq 0$. A reasonable conjecture to then make is the converse, that we can use these diagrams to generate an alternate definitions of degree n.

Definition 6.0.5. *We will refer to an n-cube as **strongly k-coCartesian** if every k-face (every sub-cube which is k-dimensional) is coCartesian.*

Lemma 6.0.6. $\mathbb{T}\mathrm{ot}_n(\mathrm{sk}_k \Delta^\bullet * X)$ *is a* $(k + 1)$-*coCartesian* $(n + 1)$-*cube for all* k, n.

Given Lemma 6.0.6, a reasonable conjecture would be

Conjecture 6.0.7. *If F takes strongly k-cocartesian* $(n + 1)$-*cubes to cartesian cubes for all k and fixed n, F is degree n.*

Surprisingly, this turns out to not be the case. Instead, we have Theorem 6.0.8.

Theorem 6.0.8. *Fix a k and n, with* $k \leq n$, $k \geq 3$. *Let F be a homotopy functor that takes k-coCartesian n-cubes to Cartesian n-cubes. Then F is degree 1.*

We'll prove Theorem 6.0.8 in pieces. Section 6.1 is the proof of Lemma 6.0.6. Section 6.2 is the case $k = n$. Section 6.3 is the proof for general n, k. Section 6.4 is to justify omission of considering diagonal sub k-cubes in the arguments of the sections preceeding it.

6.1 Proof of Lemma 6.0.6

We can observe that $\mathbb{T}\text{ot}_n \, \text{sk}_0 \, \triangle^\bullet$ are strongly coCartesian n-cubes. We will show that $\mathbb{T}\text{ot}_n \, \text{sk}_k \, \triangle^\bullet$ can be obtained from $\mathbb{T}\text{ot}_n \, \text{sk}_0 \, \triangle^\bullet$ by application of a functor that loosens the cocartesianness from strongly coCartesian to strongly k-cococartesian. Applying $- * X$ preserves cocartesianness. This yields Lemma 6.0.6.

We will call our functor $*^n(-)$. Let $*^n(X)$ be defined on sets as

$$X \mapsto \operatorname*{hocolim}_{U \subset \binom{X}{n}}(*U)$$

where $*U$ means to join all of the points in U.

Examples:

(1) $*^2(S^0) = \triangle^1$

(2) $*^2(S^0 \vee S^0) = \triangle^2$

Observation 6.1.1. $*^n(\text{sk}_0 \, \triangle^k) = \text{sk}_n \, \triangle^k$ *and therefore* $\mathbb{T}\text{ot}_n \, \text{sk}_k \, \triangle^\bullet = *^k(\mathbb{T}\text{ot}_n \, \text{sk}_0 \, \triangle^\bullet)$.

Recall the following definition:

Definition 6.1.2. *We call a functor $\mathfrak{F}$ **rank n** if for every strongly co-cartesian $(n + 1)$-cube $\mathfrak{D}$, we have that $\mathfrak{F}(\mathfrak{D})$ is co-cartesian.*

Lemma 6.1.3. $*^n$ *is a rank n functor.*

Proof. From [McC] we know that that the following functor, "choose n" is rank n

$$\binom{-}{n} : X \mapsto \left(\frac{\prod\limits^{n} X}{\mathbb{D}} \right) / \Sigma_n$$

where $\mathbb{D} = \{ (x_1, \ldots, x_n) \mid \exists i, j \text{ s.t. } x_i = x_j \}$, the fat diagonal.

$*^n$ is equalent to $\binom{-}{n}$ followed by join followed by a hocolim. We know that $*^n$ is a homotopy functor since the functors it is a composition of are all homotopy functors. Applying $\binom{-}{n}$ to a strongly co-cartesian $(n + 1)$ cube yields a cocartesian $(n + 1)$-cube, whose cocartesianness is preserved by join and hocolim since both join and hocolim are hocolim constructions. Therefore, $*^n$ is rank n. □

Note that for $n \leq k$, the $\mathbb{T}\text{ot}_n \, \text{sk}_k \, \triangle^\bullet$ cubes are constant. This means that they are trivially $(k + 1)$-coCartesian and trivially coCartesian of all degrees. Therefore, we need only proceed with our argument assuming $n > k$.

Since $\mathbb{T}\mathbf{ot}_n \, \mathrm{sk}_0 \, \Delta$ is strongly co-cartesian $(n+1)$-cube and $*^k$ is rank k by Lemma 6.1.3, then $\mathbb{T}\mathbf{ot}_n \, \mathrm{sk}_k \, \Delta^\bullet$ is $(k+1)$-coCartesian. As mentioned above, this implies that $\mathbb{T}\mathbf{ot}_n(\mathrm{sk}_k \, \Delta^\bullet * X)$ is $(k+1)$-coCartesian. $\qquad\square$

6.2 Case $k = n$: Pushouts to pullbacks implies F 1-excisive

In this section, we'll address a motivating case of Theorem 6.0.8, $k = n$. That is, if a homotopy functor $\mathfrak{F}$ has the property that for some or all k, $\mathfrak{F}$ takes co-cartesian k-cubes to cartesian k-cubes, then $\mathfrak{F}$ is 1-excisive.

Let's start with the case $k = 3$. Given any co-cartesian square

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_{12}
\end{array}
$$

We can build a cocartesian 3-cube by mapping to

$$
\begin{array}{ccc}
* & \longrightarrow & * \\
\downarrow & & \downarrow \\
* & \longrightarrow & *
\end{array}
$$

as a map of two squares is a 3-cube. Then applying $\mathfrak{F}$ yields a cartesian 3-cube. Since the target/bottom square is trivial, this 3-cube's homotopy limit will be the same as the homotopy limit of the top. Therefore, $\mathfrak{F}$ takes cocartesian squares to cartesian squares, i.e. is 1-excisive.

This proof generalizes immediately to all k.

6.3 General n, k

Method: Recall that the degree 1 condition is based off of a pushout square, $\mathfrak{C}$.

Step 1 Define a coCartesian 3-cube $\mathfrak{D}$ with $\mathrm{holim}_0 \, \mathfrak{D} = \mathrm{holim}_0 \, \mathfrak{C}$.

Step 2 Build a functorially Cartesian & coCartesian 3 cube $\mathscr{E}$ such that $\widetilde{\mathfrak{D}} := \mathfrak{D} \to \mathscr{E}$ is a 3-coCartesian 4-cube.
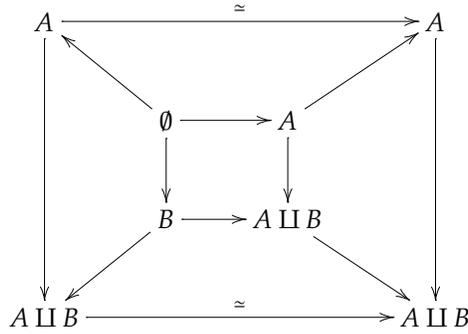
Step 3 Then the following is a 3- (and hence 4- and 5-)coCartesian 5 cube:

$$
\begin{array}{ccc}
\mathfrak{D} & \longrightarrow & \mathscr{E} \\
\downarrow & & \Downarrow \\
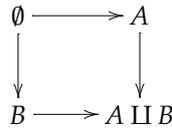\mathscr{E} & \longrightarrow & \mathscr{E}
\end{array}
$$

Step 4 Iterate.

Step 5 Each cube has $F(\emptyset) = \mathrm{holim}_0 F(\text{cube}) = \mathrm{holim}_0 F(\mathfrak{D})$. Therefore, F is degree 1.
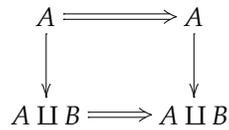
[Step 1] Consider the following square:

This will be our $\mathfrak{D}$. Its 'top' is just the square used for the degree 1 condition

and its 'bottom' is a functorially coCartesian & Cartesian square

Therefore, we immediately gain that this blowup is a coCartesian 3-cube, and for any F, $F(\text{bottom})$ is cartesian, which gives that $\mathrm{holim}_0(F(\mathfrak{C}) \to F(\text{bottom})) \simeq \mathrm{holim}_0 F(\mathfrak{C})$.

[Step 2] Let $\mathscr{E}$ be the following 3-cube of figure 6.1.

This gives us our 3-coCartesian 4-cube $\widetilde{\mathfrak{D}}$, in figure 6.2. Proof that $\widetilde{D}$ is a3-coCartesian 4-cube is postponed to section 6.3.1.

[Step 3 & 4] By extension, we also have a 3-(and 4-)coCartesian 5-cube as in figure 6.3.

And again, given a functor F taking 3-(or 4-)coCartesian 5 cubes to Cartesian 5-cubes, $\mathrm{holim}_0 F$ of the diagram will be $\mathrm{holim}_0 F(\mathfrak{D}) \simeq F(\emptyset)$, i.e. F is degree 1.
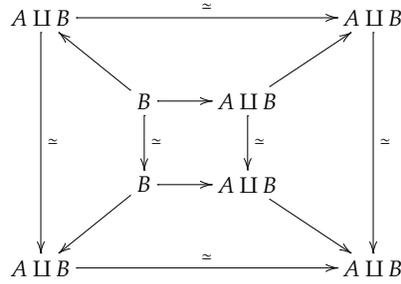
33
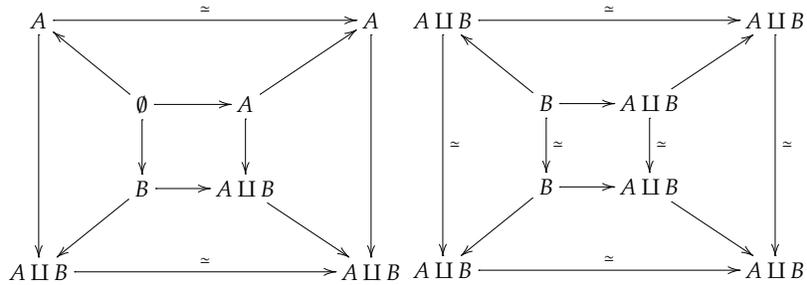
Figure 6.1: $\mathscr{E}$

Figure 6.2: $\widetilde{\mathfrak{D}}$

[Step 5]   Adding on more and more copies of $\mathscr{E}$ gives larger and larger $3-, 4-, 5-, \cdots$ coCartesian n cubes. Given a k–coCartesian such n-cube (which is going to be a 3-coCartesian n-cube, where 3 implies any k) and a functor F taking such to Cartesian n cubes, we get that $F(\emptyset) = \mathrm{holim}_0 F(\mathrm{cube}) = \mathrm{holim}_0 F(\mathfrak{C})$.

Therefore, F has degree 1.

6.3.1   Proof that $\widetilde{\mathfrak{D}}$ is a 3-co-cartesian 4-cube

Recall that the lefthand cube is $\mathfrak{D}$, the righthand $\mathscr{E}$ and the whole 4-cube is $\widetilde{D}$. Given our numbering, we can specify 3-faces by pairs of faces which map to each other.

$$\widetilde{\mathfrak{D}} \longrightarrow \mathscr{E}$$
$$\downarrow \qquad \Downarrow$$
$$\mathscr{E} \Longrightarrow \mathscr{E}$$
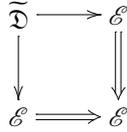
Figure 6.3: Larger k-cocartesian cubes
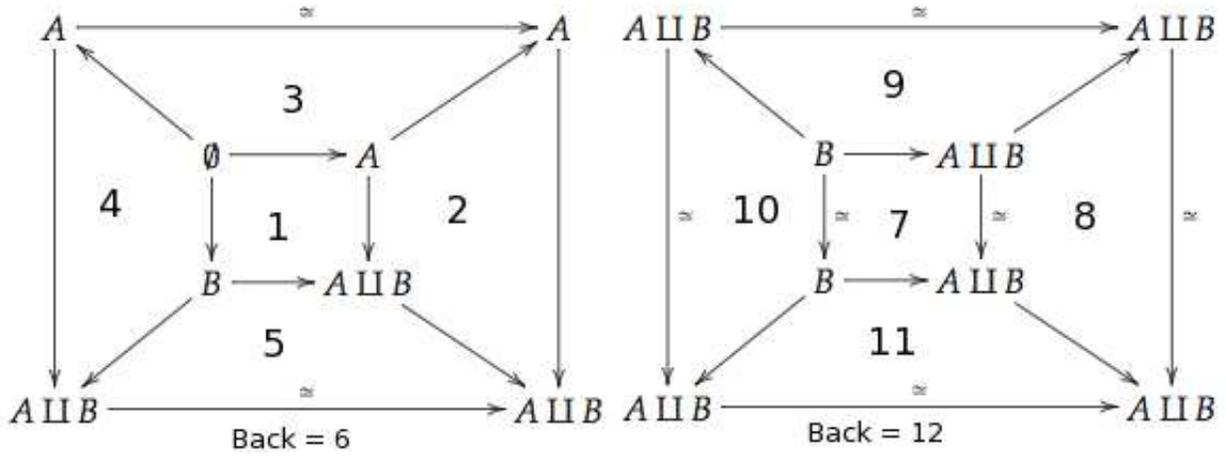
Figure 6.4: The 4-cube $\widetilde{\mathfrak{D}}$

$1 \to 7$	is a co-cartesian 3-cube because 1 and 7 are both pushout squares
$2 \to 8$	because its sides are pushout squares, see figure 6.3.1
$3 \to 9$	is a co-cartesian 3-cube because 1 and 7 are both pushout squares
$4 \to 10$	is a co-cartesian 3-cube because 4 and 10 are both pushout squares
$5 \to 10$	is a co-cartesian 3-cube because the vertical maps are all equivalences
$6 \to 12$	is a co-cartesian 3-cube because 6 and 12 are both pushout squares
$\mathscr{D}$	is co-cartesian because 1 and 6 are pushout squares
$\mathscr{E}$	is co-cartesian because 7 and 12 are pushout squares
$\therefore$	$\widetilde{\mathfrak{D}}$ is a strongly 3-co-cartesian 4-cube

Figure 6.5: The 3-cube $2 \to 8$

6.4 Diagonal sub k-Cube Lemma

Any argument involving sub-k-cubes of an n-cube can be proven by looking only at non-diagonal sub-cubes due to the following lemma.

Lemma 6.4.1. *Any diagonal sub-k-cube of an an n-cube is decomposable as a composition of non-diagonal sub$-k-$cubes.*
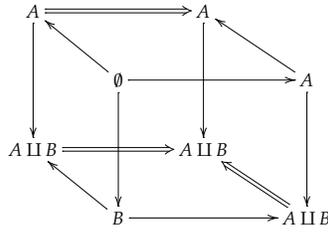
Remark 6.4.2. *It suffices to show for $k = 2$. Decomposing diagonal 3-cubes' faces generates a non-diagonal decomposition of 3-cube. In general, decomposing diagonal k-cubes into non-diagonal follows from decomposing faces. So, start at the "bottom" and percolate up.*

Proof. We reduce to the faces of the 2-cube. Given a $2-$cube

$$
\mathscr{X} = \begin{array}{ccc} X_{U_0} & \longrightarrow & X_{U_1} \\ \downarrow & & \downarrow \\ X_{U_2} & \longrightarrow & X_{U_{12}} \end{array}
$$

where the U_i are finite indexing sets. Then $U_{12} - U_0$ is a collection of elements, let's call them $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_j\}$. Being a commutative n-cube, either map from X_{U_0} to $X_{U_{12}}$ is equivalent to the other. We aim to factor each map as a composition of maps between X_{U_i} such that $U_{i+1} - U_i = 1$.

We can build

$$
\mathscr{X} = \begin{array}{ccc} X_{U_0} & \longrightarrow & X_{U_0 + \epsilon_1} \\ \downarrow & & \downarrow \\ X_{U_0 + \epsilon_2} & \longrightarrow & X_{U_0 + \epsilon_1 + \epsilon_2} \end{array}
$$

Adding on ϵ_i at every stage.

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$$\mathcal{X} = \begin{array}{ccccccccc}
X_{U_0} & \longrightarrow & X_{U_0+\epsilon_1} & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_3} & \longrightarrow & \cdots & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_3+\cdots+\epsilon_j} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
X_{U_0+\epsilon_2} & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_2} & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_2+\epsilon_3} & \longrightarrow & \cdots & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_2+\epsilon_3+\cdots+\epsilon_j} = X_{U_{12}}
\end{array}$$

Where $U_1 = U_0 + S$ for $S \in \mathscr{P}(U_{12} - U_0)$ and $U_2 = U_0 + R$ for $R \in \mathscr{P}(U_{12} - U_0)$. $\qquad\square$

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$$\mathcal{X} = \begin{array}{ccccccccc}
X_{U_0} & \longrightarrow & X_{U_0+\epsilon_1} & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_3} & \longrightarrow & \cdots & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_3+\cdots+\epsilon_j} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
X_{U_0+\epsilon_2} & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_2} & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_2+\epsilon_3} & \longrightarrow & \cdots & \longrightarrow & X_{U_0+\epsilon_1+\epsilon_2+\epsilon_3+\cdots+\epsilon_j} = X_{U_{12}}
\end{array}$$

Where $U_1 = U_0 + S$ for $S \in \mathscr{P}(U_{12} - U_0)$ and $U_2 = U_0 + R$ for $R \in \mathscr{P}(U_{12} - U_0)$. $\qquad\square$

Chapter 7

Main Theorem

This section contains the proofs of the main theorems and their consequences.

7.1   Introduction

Let Δ^n be the n-simplex and $\mathrm{sk}_0 \Delta^n$ be its 0-skeleton, that is, $n+1$ points. We use $*$ to denote the topological join. Then, for a space X, $\mathrm{sk}_0 \Delta^0 * X \simeq CX$ and $\mathrm{sk}_0 \Delta^n * X \simeq \bigvee_n \Sigma X$. Thus, we have the cosimplicial space

$$(\mathrm{sk}_0 \Delta^* * X) \simeq CX \overset{\longrightarrow}{\underset{\longleftarrow}{\longleftarrow}} \Sigma X \overset{\longrightarrow}{\underset{\longleftarrow}{\longleftarrow}} \Sigma X \vee \Sigma X \overset{\longrightarrow}{\underset{\cdots}{\vdots}} \cdots$$

Goerss [Goe93] and Hopkins [Hop84a, Hop84b] analyzed the spectral sequence associated to this cosimplicial space and showed that when X is connected, it converges to $\mathbb{Z}_\infty X$, the Bousfield $\mathbb{Z}$-nilpotent completion of a space (for details of the construction and applications, see [BK72]).

One result of this chapter is a new conceptual proof of this theorem. We assume for the moment that X is a connected space. $\mathrm{P}_\infty \mathbb{I}(X)$ is the inverse limit of the Goodwillie Taylor tower of the identity functor, applied to X. We show the following weak equivalence, where $\mathbb{I}$ is the identity functor of spaces:

$$\mathrm{holim}\, \mathrm{sk}_0 \Delta^* * X \sim \mathrm{P}_\infty \mathbb{I}(X).$$

By work of Arone-Kankaanrinta [AK98], $\mathrm{P}_\infty \mathbb{I}(X) \sim \mathbb{Z}_\infty(X)$. This gives the result of Goerss and of Hopkins.

Indeed, we show that $\forall k \geq 0$ and X connected, that $\mathrm{holim}\, \mathrm{sk}_k \Delta^* * X \sim \mathrm{P}_\infty \mathbb{I}(X)$.

More generally, if F is ρ-analytic (see section 2 for a definition) and X is at least m connected for $m \geq -1$, then we have weak equivalences,

$$\mathrm{P}_\infty F(X) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_k \Delta^* * X),$$

for all $k \geq \rho - m$ if $\rho \geq m$, and otherwise it is true for all $k \geq 0$. This arises as a natural corollary of our main

results (specifically Corollary 7.1.5, following from Theorems 7.1.2 and 7.3.1 described below).

We first establish a new model for each iterated approximation, $T_n^k F$. Using this model, we obtain maps $\tau^k : T_{n+1}^k F(X) \to T_n^k F(X)$, and therefore a tower of partial approximations for each k:

$$\cdots T_n^k F(X) \xrightarrow{\tau^k} T_{n-1}^k F(X) \xrightarrow{\tau^k} \cdots T_1^k F(X)$$

We will need a few definitions before we give our model for the $T_n^k F$'s, which will be used in the proof of our main theorem.

Let Δ be the category of finite ordered sets and monotone maps, with objects denoted $[j] = \{0, \dots, j\}$ with the usual order. For a k-cosimplicial space $X^{\vec{\bullet}}$, we denote by $\mathrm{diag}(X^{\vec{\bullet}})$ its diagonal. Let $\Delta_{\leq n}$ be the full subcategory of Δ with objects $[j]$ such that $j \leq n$. The n-coskeleton of X^*, denoted $\mathrm{cosk}^n X^*$, constructed by precomposing with this inclusion and then taking the right Kan extension along the inclusion of the subcategory. We let $\mathrm{cosk}^{\vec{n}}(X^{\vec{\bullet}})$ denote the k-cosimplicial analog, n-coskeleton taken in every dimension.

Theorem 7.1.1. *There is a weak equivalence*

$$T_n^k F(X) \sim \mathrm{holim}_{\Delta \leq nk} \mathrm{diag}(\mathrm{cosk}^{\vec{n}} F((\underbrace{\mathrm{sk}_0 \triangle^\bullet * \cdots * \mathrm{sk}_0 \triangle^\bullet}_{k \text{ copies}}) * X))$$

In particular, as $n \to \infty$, we have as an immediate consequence the following equivalence:

$$\mathrm{holim}_n T_n^k F(X) \sim \mathrm{holim}_\Delta \mathrm{diag} F((\underbrace{\mathrm{sk}_0 \triangle^\bullet * \cdots * \mathrm{sk}_0 \triangle^\bullet}_{k \text{ copies}}) * X)).$$

Theorem 7.1.2 (Main Theorem)**.** *There is a weak equivalence $\forall k \geq 0$:*

$$\mathrm{holim}_n(\cdots T_n^{k+1} F(X) \xrightarrow{\tau^{k+1}} T_{n-1}^{k+1} F(X) \xrightarrow{\tau^{k+1}} \cdots T_1^{k+1} F(X)) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_k \triangle^\bullet * X)$$

The weak equivalences in Theorems 7.3.1 and 7.1.2 are natural in k.

Remark 7.1.3. *As stated, these theorems are not in full generality. The method of proof of Theorem 7.1.2 is to show cofinality of functors that we then apply $F(- * X)$ to, so immediately generalizes to the unbased calculus setting of [BEJM11, BJME10] by replacing $*X$ with $*_f B$, as described in Section 2.6, where we are working with functors from the category of spaces factoring a chosen map $f : A \to B$.*

The equivalence of Theorem 7.1.2 becomes

$$\mathrm{holim}_n(\cdots \mathrm{T}_n^{k+1}F(A) \xrightarrow{\tau^{k+1}} \mathrm{T}_{n-1}^{k+1}F(A) \xrightarrow{\tau^{k+1}} \cdots \mathrm{T}_1^{k+1}F(X)) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_k \Delta^\bullet *_f B)$$

and Theorem 7.3.1 becomes (using Lemma 2.6.1)

$$\mathrm{T}_n^k F(A) \sim \mathrm{holim}_{\Delta \le nk} \mathrm{diag}(\mathrm{cosk}^{\vec{n}} F((\underbrace{\mathrm{sk}_0 \Delta^\bullet * \cdots * \mathrm{sk}_0 \Delta^\bullet}_{k \text{ copies}}) *_f B))$$

We will discuss in depth the maps involved in these towers in section 7.3.1, as we make use of lemmas from that section to make the maps clearer.

Notice that the tower in Theorem 7.1.2 is over the maps $\tau^{k+1} : \mathrm{T}_n^{k+1}F \to \mathrm{T}_{n-1}^{k+1}F$, along the same stage of iteration of *different* T_nF's. This is markedly different than the directed system used to construct the P_nF's, which is over the maps $t_n : \mathrm{T}_n^j F \to \mathrm{T}_n^{j+1}F$, i.e. along iterations of the *same* T_n construction. We can depict our maps in Figure 7.1. P_nF is then the homotopy colimit along the nth column of Figure 7.1, whereas the partial approximation towers in Theorem 7.1.2 are the rows.

Figure 7.1: Partial approximations

Theorem 7.1.2 provides an equivalence between the homotopy limit over the kth row of the diagram in Figure 7.1 and the homotopy limit of F applied to $(\mathrm{sk}_k \Delta^* * X)$.

We now present several consequences of Theorem 7.1.2.

Corollary 7.1.4. [1] *For a given j, there are weak equivalences for all $k \geq 0$*

$$P_j F(X) \sim \text{holim}_\Delta P_j F(\text{sk}_k \Delta^\bullet * X)$$

In particular, this also implies that $P_\infty F(X) \sim \text{holim}_\Delta P_\infty F(\text{sk}_k \Delta^\bullet * X)$.

A functor F is said to be ρ-analytic if its failure to be polynomial of any degree is bounded with a bound that depends on ρ. One important consequence of ρ-analyticity is that for X at least ρ-connected, $F(X) \simeq P_\infty F(X)$.

We will use this fact and the preceding corollary to establish the following:

Corollary 7.1.5. *Let F be a ρ-analytic functor. Let X be a space that nonempty, that is, at least -1-connected. Then we have weak equivalences $\forall r \geq \rho$,*

$$P_\infty F(X) \sim \text{holim}_n(\cdots T_n^{r+1}F(X) \xrightarrow{\tau^{r+1}} T_{n-1}^{r+1}F(X) \xrightarrow{\tau^{r+1}} \cdots T_1^{r+1}F(X)) \sim \text{holim}_\Delta F(\text{sk}_r \Delta^* * X)$$

If we raise the connectivity of X, we may improve this to

$$P_\infty F(X) \sim \text{holim}_n(\cdots T_n^{\rho+1-k}F(X) \xrightarrow{\tau^{\rho+1-k}} T_{n-1}^{\rho+1-k}F(X) \xrightarrow{\tau^{\rho+1-k}} \cdots T_1^{\rho+1-k}F(X)) \sim \text{holim}_\Delta F(\text{sk}_{\rho-k} \Delta^* * X)$$

where $-1 \leq k = conn(X) \leq \rho$.

Proof. The join of two space X and Y has connectivity equal to 2 plus the sum of the connectivities of X and Y. $\text{sk}_k \Delta^m$ is $(k-1)$-connected, so $\text{sk}_k \Delta^m * X$ has connectivity $(k+1)$ higher than that of X, and so will $\text{sk}_k \Delta^\bullet * X$. That is, for X nonempty, $\text{sk}_k \Delta^\bullet * X$ has connectivity at least k.

Therefore, for F a ρ-analytic functor, $F(\text{sk}_k \Delta^\bullet * X) \simeq P_\infty F(\text{sk}_k \Delta^\bullet * X)$. We take homotopy limits on both sides and apply Cor 7.1.4 to get the following weak equivalences

$$P_\infty F(X) \sim \text{holim}_\Delta P_\infty F(\text{sk}_k \Delta^\bullet * X) \sim \text{holim}_\Delta F(\text{sk}_k \Delta^\bullet * X)$$

We follow with Theorem 7.1.2 and obtain all of our stated equivalences. □

This corollary leads us to the following: if F is ρ-analytic, and X is in its radius of convergence (i.e. at least ρ-connected), then

$$F(X) \sim P_\infty F(X) \sim \text{holim}_\Delta F(\text{sk}_0 \Delta^* * X).$$

---

[1] This can be seen as the unstable extension of a stable result which will appear in [BEJM11]

This result may be rephrased as saying that for F an analytic functor, and a space X, $F(X)$ is well approximated by $F(- * X)$ applied to finite nonempty sets.

We note that Cor 7.1.5 is a description of $P_\infty F(X)$ as construction whose only infinite limits are all homotopy inverse limits, whereas before we needed to use both inverse and direct infinite limits.

By a similar proof to that showing that the identity is 1-analytic, it follows that if a functor F commutes with realizations and preserves filtered colimits, then it is 1-analytic. Corollary 7.1.5 then gives us the following:

Corollary 7.1.6. *If F commutes with realizations and preserves filtered colimits, then we have that for all $r \geq 2$, the following equivalence:*

$$P_\infty F(X) \sim \mathrm{holim}_n(\cdots T_n^{r+1} F(X) \xrightarrow{\tau^{r+1}} T_{n-1}^{r+1} F(X) \xrightarrow{\tau^{r+1}} \cdots T_1^{r+1} F(X)).$$

Proposition 7.1.7. *For the identity $\mathbb{I}$ from spaces to spaces, X a connected space, and $\mathbb{Z}_\infty X$ the Bousfield $\mathbb{Z}$-nilpotent completion of X, we have that for all $k \geq 0$, the following weak equivalence*

$$\mathrm{holim}_\Delta(\mathrm{sk}_k \triangle^\bullet * X) \sim P_\infty(\mathbb{I})(X) \sim \mathbb{Z}_\infty X,$$

and when X is already nilpotent, $\mathrm{holim}_\Delta(\mathrm{sk}_k \triangle^\bullet * X) \sim X$.

Proof. Since $\mathbb{I}$ is 1-analytic, Corollary 7.1.5 allows us to conclude that for any nonempty space X, $\mathrm{holim}_\Delta(\mathrm{sk}_k \triangle^\bullet * X) \sim P_\infty(\mathbb{I})(X)$ for all $k \geq 1$. Restricting to X which are 0-connected instead of simply nonempty, i.e. raising the minimum connectivity from -1 to 0, changes this equivalence to hold for all $k \geq 0$. Then, by Arone-Kankaanrinta [AK98, §3], we have that for X connected, $P_\infty(\mathbb{I})(X) \sim \mathbb{Z}_\infty X$, and for X nilpotent, $P_\infty(\mathbb{I})(X) \sim X$. $\qquad\square$

Note on translating between the language of Goerss [Goe93] and our current terminology:
Goerss defines a cosimplicial construction $C(X, X)$ for a nonempty space X which sends $[n]$ to $\bigvee_n \Sigma X$ (taking the empty wedge here as CX). As mentioned earlier, this is equivalent to $(\mathrm{sk}_0 \triangle^* * X)$. Theorem 1.1 in [Goe93] implies that when X is connected, $\mathrm{holim}_\Delta C(X, X) \sim \mathbb{Z}_\infty X$, that is, $\mathrm{holim}_n T_n \mathbb{I}(X) := \mathrm{holim}_\Delta(\mathrm{sk}_0 \triangle^* * X) \sim \mathbb{Z}_\infty X$.

Note on translating between the language of Hopkins and our current terminology:
This is from [Hop84a], section 3, p221-222. He lets C_n be what we call $\mathscr{P}_0([n])$. He defines, for a given space X, a functor F^n, as the homotopy inverse limit of a (punctured) cube. For $A \in \mathscr{P}_0([n]) =: C_n$, the A-indexed

position of this $(n+1)$-cube is the homotopy colimit of X mapping to $|A|$ different copies of a point, which we will explain shortly. He denotes this by $F^n A$. Regarding A as a finite ordered set, we can view $F^n A$ as the homotopy pushout of the following:

$$
\begin{array}{ccccc}
 & & X & & \\
 & \swarrow & \downarrow & & \searrow \\
\{0\} & \{1\} & & \cdots & \{|A|-1\}
\end{array}
$$

We replace these maps by cofibrations (since we are taking a homotopy colimit), giving us that we are pushing out over the following diagram:

$$
\begin{array}{ccccc}
 & & X & & \\
 & \swarrow & \downarrow & & \searrow \\
\{0\} * X & \{1\} * X & & \cdots & \{|A|-1\} * X
\end{array}
$$

That is, the A-indexed position of this $(n+1)$ cube is $A * X$.

Then $F^n := \operatorname{holim}_{A \in \mathscr{P}_0([n])} F^n A \simeq \operatorname{holim}_{U \mathscr{P}_0([n])} U * X$. That is, we have shown that his F^n's exactly the $\mathrm{T}_n \mathbb{I}(X)$'s. He constructs a tower of these F^n's:

$$
(\cdots \to \operatorname{holim} F^n \to \operatorname{holim} F^{n-1} \to \cdots \operatorname{holim} F^1)
$$

which is therefore our $\mathrm{T}_n \mathbb{I}$ tower,

$$
(\cdots \to \mathrm{T}_n \mathbb{I}(X) \to \mathrm{T}_{n-1} \mathbb{I}(X) \to \cdots \mathrm{T}_1 \mathbb{I}(X)).
$$

Theorem 3.2.2 of [Hop84b] is that the homotopy inverse limit of a construction that is equivalent to the tower of F^n's gives $\mathbb{Z}_\infty X$ when X is connected, i.e. that $\operatorname{holim}_n \mathrm{T}_n \mathbb{I}(X) \sim \mathbb{Z}_\infty X$.

If, instead of combining our result with that of Arone-Kankaanrinta, we combine with the result of Goerss and Hopkins, one can view our main result as justification for why the spectral sequences associated to the Taylor Tower of the identity of spaces and that associated to the $\mathbb{Z}$-nilpotent completion of a space abut to the same thing [2].

---

[2] The spectral sequence associated to the Taylor Tower of $\mathbb{I}(X)$ takes as input the collection of $D_n \mathbb{I}(X)$, which were computed by Johnson [Joh95].

7.1.1 Organization

This chapter is organized as follows. Section 7.2 contains an outline of the proof of Theorem 7.1.2. Theorem 7.3.1 , the more geometric and cosimplicial interpretation of T_nF, is proven in section 7.3 and necessary for the proof of Theorem 7.1.2, given in section 7.4.

7.2 Overview of Proof for Main Theorem, 7.1.2

Our first goal is to prove Theorem 7.3.1, which gives a new interpretation of the iterations $T_n^{k+1}F$ in terms of a homotopy limit over $\Delta_{\leq n}$ and joins of $\text{sk}_0 \Delta^*$ with itself. A priori, each $T_n^{k+1}F$ is a homotopy limit over a product of $(k+1)$ copies of $\Delta_{\leq n}$, that is, each partial tower (over the τ^{k+1}'s) is indexed over $(k+1)$ copies of Δ. The other homotopy limit under consideration in the main theorem, $\text{holim}_\Delta F(\text{sk}_k \Delta^* * X)$, is only over one copy of Δ. We use Theorem 7.3.1 to reduce the $(k+1)$st partial Taylor Tower of F to be induced over only one copy of Δ.

We are at this point comparing the two following homotopy limits:

$$\text{holim}_\Delta F((\underbrace{\text{sk}_0 \Delta^* * \cdots * \text{sk}_0 \Delta^*}_{(k+1) \text{ copies}}) * X) \qquad \text{and} \qquad \text{holim}_\Delta F(\text{sk}_k \Delta^* * X)$$

Now that we have two homotopy limits over the same category, we work to show that they are equivalent.

To prove these equivalences, we need to introduce the notion of *homotopy left cofinality*.

Definition 7.2.1 (see [Hir02], Definition 19.6.1 p418)**.** *Let $G : \Delta \to \mathscr{D}$. The functor G is **homotopy left cofinal** if for every object α of $\mathscr{D}$, the simplicial set $n \mapsto Mor_\mathscr{D}(G(n), \alpha)$ is contractible.*

The following consequence of being homotopy left cofinal is what we use to establish our equivalences:

Theorem 7.2.2 (see [Hir02], Theorem 19.6.7 & 16.6.23)**.** *Let M be a simplicial model category, let $\mathscr{C}, \mathscr{D}$ be small categories, and let $G : \mathscr{C} \to \mathscr{D}$ be a functor. If G is **homotopy left cofinal**, then for every object-wise fibrant $\mathscr{D}$-diagram F in M, we have that the following natural map of homotopy limits is a weak equivalence:*

$$\text{holim}_\mathscr{D} F \to \text{holim}_\mathscr{C} F \circ G$$

Note: We will be working only with a simplicial model category M where the objects are all fibrant, the category of spaces with cofibrations the cellular inclusions of CW complexes, so object-wise fibrancy comes for free.

7.3 Proof of Theorem 7.3.1

We will need a few definitions before we give our model for the $T_n^k F$'s, which will be used in the proof of our main theorem.

Let Δ be the category of finite ordered sets and monotone maps, with objects denoted $[j] = \{0, \ldots, j\}$ with the usual order. A cosimplicial space is a covariant functor from Δ to Top. A k-cosimplicial space is a functor from k copies of Δ to Top. Let $\vec{p}$ denote a k-tuple $([p_1], \ldots, [p_k])$ of elements in $\Delta^{\times k}$ and $X^{\vec{p}}$ be the value of a k-cosimplicial space X at that tuple. For a k-cosimplicial space $X^{\vec{\bullet}}$, we denote by $\text{diag}(X^{\vec{\bullet}})$ its diagonal, which is a cosimplicial space obtained by pre-composition with $\Delta \xrightarrow{\Delta} \Delta \times \cdots \times \Delta$.

Let $\Delta_{\leq n}$ be the full subcategory of Δ with objects $[j]$ such that $j \leq n$. Pre-composing a cosimplicial space with the inclusion of this subcategory gives a truncation functor, $X^* \mapsto \text{tr}_n X^*$ which is just X^* for $0 \leq * \leq n$. This truncation has a right adjoint, the right Kan extension along the inclusion of $\Delta_{\leq n}$ into Δ. If we follow the truncation by this right Kan extension, then we have what is called the n-coskeleton of X^*, denoted $\text{cosk}^n X^*$. For $X^{\vec{\bullet}}$ k-cosimplicial, we let $\text{cosk}^{\vec{n}}(X^{\vec{\bullet}})$ denote the k-cosimplicial space that results from taking the n-coskeleton in every dimension.

Theorem 7.3.1. *There is a weak equivalence*

$$T_n^k F(X) \sim \text{holim}_{\Delta_{\leq nk}} \text{diag}(\text{cosk}^{\vec{n}} F((\underbrace{\text{sk}_0 \Delta^{\bullet} * \cdots * \text{sk}_0 \Delta^{\bullet}}_{k \text{ copies}}) * X))$$

In particular, as $n \to \infty$, we have as an immediate consequence the following equivalence:

$$\text{holim}_n T_n^k F(X) \sim \text{holim}_{\Delta} \text{diag} F((\underbrace{\text{sk}_0 \Delta^{\bullet} * \cdots * \text{sk}_0 \Delta^{\bullet}}_{k \text{ copies}}) * X)).$$

We will need several lemmas to give the proof. The first perhaps better deserves to be called a remark, as it is basically true by definition.

Lemma 7.3.2. *The following is a homotopy equivalence of spaces:* $\text{holim}_{\Delta_{\leq n}} X^* \simeq \text{holim}_{\Delta} \text{cosk}^n X^*$, *and likewise for $X^{\vec{*}}$ a k-cosimplicial space,* $\text{holim}_{(\Delta_{\leq n})^{\times k}} X^{\vec{*}} \simeq \text{holim}_{(\Delta)^{\times k}} \text{cosk}^{\vec{n}}(X^{\vec{*}})$.

Proof. Given X^* fibrant, its totalization $\text{Tot}(X) := \text{Hom}_{\text{Top}^{\Delta}}(\Delta^{\bullet}, X)$ is equivalent to $\text{holim}_{\Delta} X$. We can always replace X^* by a fibrant, weakly equivalent cosimplicial space, so we will assume fibrancy, so we can use totalization and homotopy limit interchangeably.

By definition, $\text{Tot}_n(X^*) := \text{Tot}(\text{cosk}_n X^*)$. Therefore, $\text{holim}_{\Delta_{\leq n}} X^* \simeq \text{holim}_{\Delta} \text{cosk}_n X^*$. Homotopy limit of

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multidimensional things can be replaced by iterated homotopy limits in each direction, which gives us the k-cosimplicial result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 7.3.3 ([BEJM11], Lemma 2.8). *Suppose that $X^{\vec{\bullet}}$ is a multi-cosimplicial space with the property that for all $\vec{p}$, the ith cosimplicial space $(X^{\vec{p},i})^*$ is equivalent to its j_i-coskeleton $\mathrm{cosk}^{j_i}(X^{\vec{p},i})^*$ for some $j_i \geq 1$. Then*

$$\mathrm{diag}X^* \simeq \mathrm{cosk}^J \mathrm{diag}X^*$$

where $J = j_1 + \cdots + j_k$.

Lemma 7.3.4 ([Sin09] Theorem 6.7, or [Hop84b], §3.1 Prop 3.1.4.). *Let $\mathcal{G}_n : \mathscr{P}_\emptyset([n]) \to \Delta_{\leq n}$ be the functor which sends a nonempty subset subset S to $[\#S - 1]$ and which sends an inclusion $S \subseteq S'$ to the composite $[\#S - 1] \cong S \subset S' \cong [\#S' - 1]$. $\mathcal{G}_n$ is homotopy left cofinal.*

The immediate consequence of this last lemma (plus Theorem 7.2.2, which outlines the consequence of cofinality) is that we can move between the two models for each $T_n F$ – this can be seen as the base case of our induction that follows. The content of the theorem is showing that it follows that we can use our proposed model for $T_n^k F$.

Given the cofinality of $\mathcal{G}_n$, we get that

$$\mathrm{holim}_{\Delta_{\leq n}} F(\mathscr{Y}_0 * X) := \mathrm{holim}_{\Delta_{\leq n}} F(\mathrm{sk}_0 \Delta^\bullet * X) \simeq \mathrm{holim}_{U \in \mathscr{P}_\emptyset[n]} F(U * X) =: T_n F(X)$$

We define $\mathscr{Y}_k^*$ to be the cosimplicial space sending n to the join of k copies of $\mathrm{sk}_0 \Delta^n$. It is, in particular, a k-cosimplicial space.

The argument proceeds as follows:

$$
\begin{array}{lll}
T_n^k F(X) & \simeq & T_n(T_n(\cdots (T_n F(X) \cdots) \qquad\qquad\qquad\qquad\quad \text{by Definition} \\
& \simeq & \mathrm{holim}_{\Delta_{\leq n}} (\mathrm{holim}_{\Delta_{\leq n}} (\cdots (\mathrm{holim}_{\Delta_{\leq n}} F(\mathscr{Y}_k^* * X) \cdots) \quad \text{By cofinality of } \mathcal{G}_n \text{, Lemma 7.3.4} \\
& \simeq & \mathrm{holim}_{(\Delta_{\leq n})^{\times k}} F(\mathscr{Y}_k^* * X) \qquad\qquad\qquad\qquad\quad \text{Fubini Theorem} \\
& \simeq & \mathrm{holim}_{(\Delta)^{\times k}} \mathrm{cosk}^{n,\dots,n} F(\mathscr{Y}_k^* * X) \qquad\qquad\quad\; \text{Lemma 7.3.2} \\
& \simeq & \mathrm{holim}_\Delta \mathrm{diag}(\mathrm{cosk}^{n,\dots,n} F(\mathscr{Y}_k^* * X)) \qquad\quad\; \text{By } n\text{-cosimplicial Eilenberg-Zilber} \\
& \simeq & \mathrm{holim}_\Delta \mathrm{cosk}^{kn} \mathrm{diag}(\mathrm{cosk}^{n,\dots,n} F(\mathscr{Y}_k^* * X)) \quad \text{By Lemma 7.3.3} \\
& \simeq & \mathrm{holim}_{\Delta_{\leq nk}} \mathrm{diag}(\mathrm{cosk}^{n,\dots,n} F(\mathscr{Y}_k^* * X)) \quad\;\; \text{Lemma 7.3.2}
\end{array}
$$

$\square$

For a statement and proof of the n-cosimplicial Eilenberg-Zilber-Cartier theorem, see [Shi96, Prop 8.1, p201].

7.3.1   The maps $\tau^k : \mathrm{T}_n^k F(X) \to \mathrm{T}_{n-1}^k F(X)$

We now have the lemmas to simplify our discussion of the maps $\mathrm{T}_n^k F \to \mathrm{T}_{n-1}^k F$. There is a natural map $X^* \to \mathrm{cosk}^n X^*$ (and likewise for multi-cosimplicial spaces) and $\mathrm{cosk}^j \mathrm{cosk}^k X^* \simeq \mathrm{cosk}^k X^*$ for all $j \geq k$, so there is a natural map $\mathrm{cosk}^{(n+1)}(X^*) \to \mathrm{cosk}^{(n+1)}(\mathrm{cosk}^n X^*) \simeq \mathrm{cosk}^n(X^*)$. That is, we have a map

$$\mathrm{cosk}^{\vec{(n+1)}} F((\mathrm{sk}_0 \triangle^\bullet \ast \cdots \ast \mathrm{sk}_0 \triangle^\bullet) \ast X) \to \mathrm{cosk}^{\vec{n}} F((\mathrm{sk}_0 \triangle^\bullet \ast \cdots \ast \mathrm{sk}_0 \triangle^\bullet) \ast X).$$

Taking the diagonal also results in a map. We use Lemma 7.3.3 to replace the diagonals by their appropriate coskeleta, getting a map (where k is the number of copies of $\mathrm{sk}_0 \triangle^\bullet$ joined together):

$$\mathrm{cosk}^{k(n+1)} \mathrm{diag}\, \mathrm{cosk}^{\vec{(n+1)}} F((\mathrm{sk}_0 \triangle^\bullet \ast \cdots \ast \mathrm{sk}_0 \triangle^\bullet) \ast X) \to \mathrm{cosk}^{kn} \mathrm{diag}\, \mathrm{cosk}^{\vec{n}} F((\mathrm{sk}_0 \triangle^\bullet \ast \cdots \ast \mathrm{sk}_0 \triangle^\bullet) \ast X).$$

Applying $\mathrm{holim}_\triangle$ and Lemma 7.3.2, we get our maps

$$\mathrm{holim}_{\triangle_{\leq k(n+1)}} \mathrm{diag}(\mathrm{cosk}^{\vec{(n+1)}} F((\mathrm{sk}_0 \triangle^\bullet \ast \cdots \ast \mathrm{sk}_0 \triangle^\bullet) \ast X)) \to \mathrm{holim}_{\triangle_{\leq kn}} \mathrm{diag}(\mathrm{cosk}^{\vec{n}} F((\mathrm{sk}_0 \triangle^\bullet \ast \cdots \ast \mathrm{sk}_0 \triangle^\bullet) \ast X)).$$

These are the maps which we referred to as $\tau^k : \mathrm{T}_{n+1}^k F(X) \to \mathrm{T}_n^k F(X)$ previously.

7.4   Proof of Main Theorem, 7.1.2

We recall the statement of the main theorem, which we will prove in this section:

Theorem 7.1.2: *There is a weak equivalence $\forall k \geq 0$:*

$$\mathrm{holim}_n(\cdots \mathrm{T}_n^{k+1} F(X) \xrightarrow{\tau^{k+1}} \mathrm{T}_{n-1}^{k+1} F(X) \xrightarrow{\tau^{k+1}} \cdots \mathrm{T}_1^{k+1} F(X)) \sim \mathrm{holim}_\triangle F(\mathrm{sk}_k \triangle^\bullet \ast X)$$

We first will reduce the proof of Theorem 7.1.2 to an equivalent involving the homotopy left cofinality of two functors.

We first make a few definitions. For a functor F, we define a modification, $F_X(Y) := F(Y \ast X)$, still viewed

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as an endofunctor of spaces. Let $\mathcal{K}$ be the category of $(k-1)$-connected spaces of CW type. Define functors $\mathscr{X}_k, \mathscr{Y}_k : \Delta \to \mathcal{K}$ such that

$$\mathscr{X}_k(p) = \mathrm{sk}_k \Delta^p$$
$$\mathscr{Y}_k(p) = \underbrace{\mathrm{sk}_0 \Delta^p * \cdots * \mathrm{sk}_0 \Delta^p}_{(k+1)\ \mathrm{copies}}$$

Notice that for $k = 1$, we have $\mathscr{X}_1(p) = K_p$ (the complete graph on $(p+1)$ vertices) and $\mathscr{Y}_1(p) = K_{p+1,p+1}$ (the complete bipartite graph on two sets of $(p+1)$ vertices). See figure 7.2.
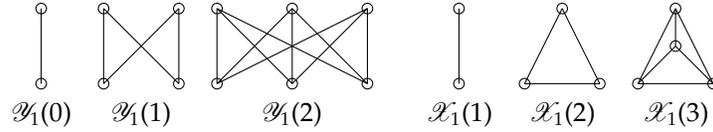

Figure 7.2: $\mathscr{Y}_1$ of 0,1, and 2 and $\mathscr{X}_1$ of 1,2,and 3

Then, $T_n^{k+1}F(X) \simeq \mathrm{holim}_{\Delta_{\leq n}} F(\mathscr{Y}_k * X)$ using Theorem 7.3.1. We can rewrite our homotopy limits more compactly (and suggestively) as:

$$\mathrm{holim}_\Delta F((\underbrace{\mathrm{sk}_0 \Delta^* * \cdots * \mathrm{sk}_0 \Delta^*}_{(k+1)\ \mathrm{copies}}) * X) \simeq \mathrm{holim}_\Delta F_X \circ \mathscr{Y}_k \quad \text{and} \quad \mathrm{holim}_\Delta F(\mathrm{sk}_k \Delta^\bullet * X) \simeq \mathrm{holim}_\Delta F_X \circ \mathscr{X}_k$$

We are now in the situation of comparing two homotopy limits over the same category, Δ, taken of F_X precomposed with a functor, $\mathscr{X}_k$ or $\mathscr{Y}_k$.

Given Definition 7.2.1 and Theorem 7.2.2, we show that $\mathscr{X}_k$ and $\mathscr{Y}_k$ are both homotopy left cofinal for $k \geq 1$, which gives us our two weak equivalences which is equivalent to the statement of Theorem 7.1.2:

$$\mathrm{holim}_\Delta F_X \circ \mathscr{X}_k \xleftarrow{\sim} \mathrm{holim}_{\mathcal{K}} F_X \xrightarrow{\sim} \mathrm{holim}_\Delta F_X \circ \mathscr{Y}_k$$

To show left cofinality of the two functors, we show that for all $Z \in (k-1)$-connected spaces, the simplicial sets $p \mapsto \mathrm{Top}_{cts}(\mathscr{X}_k(p), Z)$ and $p \mapsto \mathrm{Top}_{cts}(\mathscr{Y}_k(p), Z)$ are contractible.

Our proofs will make use of the following Lemma:

Lemma 7.4.1. *Let* $\mathrm{sk}_0 \Delta^\bullet$ *be the cosimplicial space sending n to the 0 skeleton of* Δ^n*, the topological n-simplex. Then for any nonempty space X, the simplicial set* $k \mapsto Top(\mathrm{sk}_0 \Delta^k, X)$ *is contractible by a contracting homotopy.*

Proof. For each k, $\mathrm{sk}_0 \Delta^k$ is the discrete space with $(k+1)$ points. This allows us to write $\mathrm{Top}(\mathrm{sk}_0 \Delta^k, X)$ as $\prod^{k+1} X$. That is, our simplicial set is of the form

$$X \underset{\longleftarrow}{\overset{\longrightarrow}{\rightleftarrows}} X \times X \overset{\longleftarrow}{\underset{\longrightarrow}{\rightrightarrows}} X \times X \times X \cdots$$

The structure maps are

$$
\left\{
\begin{aligned}
d_i : X_{n+1} \to X_n \quad &= \quad \text{projection by deleting the } i\text{th coordinate} \\
& \quad \text{e.g. } d_i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \bar{x}_{\hat{i}} \\
\\
s_i : X_n \to X_{n+1} \quad &= \quad \text{inclusion and diagonal applied to the } i\text{th coordinate} \\
& \quad \text{e.g. } s_i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_{n-1})
\end{aligned}
\right.
$$

Recall (from, e.g.[Dug08]) that for $\mathcal{Y}_\bullet$ a simplicial set augmented by the map $d_0 : \mathcal{Y}_0 \to *$ (i.e., $* =: \mathcal{Y}_{-1}$), a (forward) *contracting homotopy* of $\mathcal{Y}_\bullet$ is given by a collection of maps $\mathcal{S} : \mathcal{Y}_n \to \mathcal{Y}_{n+1}$ for $n \geq -1$ such that for each $y \in \mathcal{Y}_n$, one has

$$
\left\{
\begin{aligned}
d_i(\mathcal{S}y) \quad &= \quad
\begin{cases}
\mathcal{S}(d_i y) & \text{if } 0 \leq i < n \\
y & \text{if } i = n
\end{cases} \\
\\
\mathcal{S}(s_i y) \quad &= \quad s_i(\mathcal{S}y) \qquad\qquad\qquad \text{for } 0 \leq i \leq n
\end{aligned}
\right.
$$

First choose a point $v \in X$. We set $\mathcal{S}(*)$(in our -1st dimension) to be $v \in X$. For n-simplices x for $n > -1$, we define $\mathcal{S}(x) := (x, v)$. That is, if $x = (x_0, \ldots, x_n) \in \prod^{n+1} X$, then $\mathcal{S}(x) = (x_0, \ldots, x_n, v)$. This is our desired contracting homotopy We now check that the above identities hold. First, for $x_0 \in X$, we have that $d_0(\mathcal{S}(x_0)) = d_0(x_0, v) = v$ and $\mathcal{S}(d_0 x_0) = \mathcal{S}(*) = v$. Let us proceed for more general simplices.

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Given $(x_0, \ldots, x_n) \in \prod^{n+1} X$, we have that

for $0 \leq i < n$:

$d_i(\mathcal{S}(x_0, \ldots, x_n)) = d_i(x_0, \ldots, x_n, v) = (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n, v)$

$\mathcal{S}(d_i(x_0, \ldots, x_n)) = \mathcal{S}(x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n, v)$

$$\text{(i.e. } d_i(\mathcal{S}(x_0, \ldots, x_n)) = \mathcal{S}(d_i(x_0, \ldots, x_n)))$$

and $d_n(x_0, \ldots, x_n, v) = (x_0, \ldots, x_n)$

$\mathcal{S}(s_i(x_0, \ldots, x_n)) = \mathcal{S}(x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n, v)$

$s_i(\mathcal{S}(x_0, \ldots, x_n)) = s_i(x_0, \ldots, x_n, v) = (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n, v)$

$$\text{(i.e. } \mathcal{S}(s_i(x_0, \ldots, x_n)) = s_i(\mathcal{S}(x_0, \ldots, x_n)))$$

Therefore, $\mathcal{S}$ is a contracting homotopy of our simplicial set. □

7.4.1 Contractibility of $X_{k,Z}$

This is classical. For each j we have the following isomorphisms of sets:

$$\begin{aligned}
\text{Top}(|\operatorname{sk}_k \Delta^j|, Z) &\cong \text{sSets}(\operatorname{sk}_k \Delta^j, \text{Sing}(Z)) \\
&\cong \text{sSets}(\Delta^j, \operatorname{cosk}_k \text{Sing}(Z)) \\
&:= \operatorname{cosk}_k(\text{Sing}(Z))_j
\end{aligned}$$

Therefore, $X_{k,Z} \cong \operatorname{cosk}_k(\text{Sing}(Z))_\bullet$.

For a simplicial set $Y_\bullet$, the map $Y_\bullet \to \operatorname{cosk}_k Y$ is 1-1 and onto for dimensions $\leq k$, which means that the homotopy groups are the same in dimensions $< k$ (for a reference, see [DK84, §1.2,part (vi)]).

The homotopy groups of $\operatorname{cosk}_k Y$ are trivial in dimensions $\geq k$, for Y fibrant. Our Y is fibrant since the singularization of a space is always a fibrant simplicial set.

We have that $\pi_i \operatorname{cosk}_k(\text{Sing}(Z))_\bullet$ for $i \leq (k-1)$ are trivial, since Z is $(k-1)$ connected. Therefore, for all i, its homotopy groups are trivial, and it is weakly contractible. Its realization is a space of CW type, so by the Whitehead theorem, it is also contractible. □

7.4.2 Contractibility of $Y_{k,Z}$

Recall that we defined $\mathscr{Y}_k^\bullet$ as the join of $(k+1)$ copies of $\operatorname{sk}_0 \Delta^\bullet$ with itself. We prove contractibility of $Y_{k,Z}$ by induction. Our base case, $k = 0$, is Lemma 7.4.1.

Inductive hypothesis: Assume $\text{Top}(\mathscr{Y}_k^\bullet, Z) \simeq *$ for all $k < K$.

We will express $\text{Top}(\mathscr{Y}_K^\bullet, Z)$ as the homotopy pullback contractible simplicial sets, and conclude that it is contractible.

The join of two spaces is the following pushout:

$$
\begin{array}{ccc}
X \times Y & \longrightarrow & CX \times Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & X * Y
\end{array}
$$

We therefore may express the cosimplicial space $\mathscr{Y}_K^* = \text{sk}_0 \Delta^* * \mathscr{Y}_{K-1}^*$ as defined by the following diagram of cosimplicial spaces which is level-wise a pushout of sets:

$$
\begin{array}{ccc}
\mathscr{Y}_{K-1}^\bullet \times \text{sk}_0 \Delta^\bullet & \longrightarrow & C\mathscr{Y}_{K-1}^\bullet \times \text{sk}_0 \Delta^\bullet \\
\downarrow & & \downarrow \\
\mathscr{Y}_{K-1}^\bullet & \longrightarrow & \mathscr{Y}_K^\bullet := \mathscr{Y}_{K-1}^\bullet * \text{sk}_0 \Delta^\bullet
\end{array}
$$

Applying $\text{Top}(, Z)$, we have a square of simplicial sets which is levelwise a pullback of sets.

$$
\begin{array}{ccc}
\text{Top}(\mathscr{Y}_{K-1}^\bullet * \text{sk}_0 \Delta^\bullet, Z) & \longrightarrow & \text{Top}(C(\mathscr{Y}_{K-1}^\bullet) \times \text{sk}_0 \Delta^\bullet, Z) \\
\downarrow & & \downarrow \\
\text{Top}(\mathscr{Y}_{K-1}^\bullet, Z) & \longrightarrow & \text{Top}(\mathscr{Y}_{K-1}^\bullet \times \text{sk}_0 \Delta^\bullet, Z)
\end{array}
$$

Allowing our indices to vary independently, this can be viewed as the diagonal of a diagram of bisimplicial sets. We state a weak version of a theorem of Bousfield and Friedlander which we will apply to conclude that our square is also a *homotopy* pullback square.

Theorem 7.4.2. *[BF78, Theorem B.4] Let*

$$
\begin{array}{ccc}
V & \longrightarrow & X \\
\downarrow & & \downarrow \\
W & \longrightarrow & Y
\end{array}
$$

be a commutative square of bisimplicial sets such that the terms $V_{m,*}$, $W_{m,*}$, $X_{m,*}$ and $Y_{m,*}$ form a homotopy fiber square

for each $m \geq 0$. If $X_{m,*}$ and $Y_{m,*}$ are all connected, then

$$
\begin{array}{ccc}
\mathrm{diag}V & \longrightarrow & \mathrm{diag}X \\
\downarrow & & \downarrow \\
\mathrm{diag}W & \longrightarrow & \mathrm{diag}Y
\end{array}
$$

is a homotopy fiber square.

We will first show that $\mathrm{Top}(C(\mathscr{Y}_{K-1}^m) \times \mathrm{sk}_0 \triangle^\bullet, Z)$ and $\mathrm{Top}(\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \triangle^\bullet, Z)$ are connected for all $m \geq 0$. Note that we have the following isomorphisms:

$$
\begin{aligned}
\mathrm{Top}(C(\mathscr{Y}_{K-1}^m) \times \mathrm{sk}_0 \triangle^\bullet, Z) &\cong \mathrm{Top}(\mathrm{sk}_0 \triangle^\bullet, \hom_{\mathrm{Top}}(C(\mathscr{Y}_{K-1}^m), Z)) \\
\mathrm{Top}(\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \triangle^\bullet, Z) &\cong \mathrm{Top}(\mathrm{sk}_0 \triangle^\bullet, \hom_{\mathrm{Top}}(\mathscr{Y}_{K-1}^m, Z))
\end{aligned}
$$

Then, by Lemma 7.4.1, these are contractible, since we have expressed them as $\mathrm{Top}(\mathrm{sk}_0 \triangle^\bullet, X)$ for X a space, therefore they are connected.

By the following theorem, we may also conclude that $\mathrm{Top}(C(\mathscr{Y}_{K-1}^\bullet) \times \mathrm{sk}_0 \triangle^\bullet, Z)$ and $\mathrm{Top}(\mathscr{Y}_{K-1}^\bullet \times \mathrm{sk}_0 \triangle^\bullet, Z)$ are contractible, as we have shown levelwise contractibility.

Theorem 7.4.3. *[BF78, p119, Theorem B.2] Let $f : X \to Y$ be a map of bisimplicial sets such that $f_{m,*} : X_{m,*} \to Y_{m,*}$ is a weak equivalence for each $m \geq 0$. Then $\mathrm{diag}(f) : \mathrm{diag}X \to \mathrm{diag}Y$ is a weak equivalence.*

Next, we need to show that levelwise we have homotopy pullback squares of simplicial sets. That is, for all m, the following is not just a levelwise pullback of sets but a homotopy pullback of simplicial sets:

$$
\begin{array}{ccc}
\mathrm{Top}(\mathscr{Y}_{K-1}^m * \mathrm{sk}_0 \triangle^\bullet, Z) & \longrightarrow & \mathrm{Top}(C(\mathscr{Y}_{K-1}^m) \times \mathrm{sk}_0 \triangle^\bullet, Z) \\
\downarrow & & \downarrow \\
\mathrm{Top}(\mathscr{Y}_{K-1}^m, Z) & \longrightarrow & \mathrm{Top}(\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \triangle^\bullet, Z)
\end{array}
$$

We will show that the righthand vertical map is a Kan fibration[3], and conclude that our square is a homotopy pullback.

Simplicial sets are a simplicial model category, satisfying Quillen's SM7 axiom, which is as follows[4]:

---

[3]We only need to show one map is a fibration due to right properness of the category of simplicial sets. See the gluing lemma, e.g. in [Sch97], Lemma 1.19.

[4]$\mathrm{Hom}_C(,)$ is the simplicial set from the simplicial model structure, $\mathrm{Hom}_C(X, Y) := C(X \times \triangle^\bullet, Y)$ where $C(X, Y)$ is the hom-set of morphisms in C.

Quillen's SM7 axiom: *Let $\mathscr{C}$ be a simplicial model category, and $\mathrm{Hom}_{\mathscr{C}}(X, Z)$ denote the simplicial set of morphisms between X and Z. If $Y \in \mathscr{C}$ is fibrant and $f : A \to B$ is a cofibration in $\mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(f, Y) : \mathrm{Hom}_{\mathscr{C}}(B, Y) \to \mathrm{Hom}_{\mathscr{C}}(A, Y)$ is a fibration of simplicial sets.*

We may apply several adjunctions to get the following isomorphisms. Note that $\mathrm{sk}_0 \Delta^*$ is equivalent to the levelwise simplicial set skeleton, i.e. $\mathrm{sk}_0 \Delta^\bullet : n \mapsto \mathrm{sk}_0 \Delta^n$, so it is adjoint levelwise to the simplicial coskeleton. We will denote the singularization of a space Z by sZ.

$$
\begin{aligned}
\mathrm{Top}(C\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \Delta^\bullet, Z) \quad &\cong \quad \mathrm{sSet}(C\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \Delta^\bullet, sZ) \\
&\cong \quad \mathrm{sSet}(C\mathscr{Y}_{K-1}^m, \mathrm{Hom}_{sSet}(\mathrm{sk}_0 \Delta^\bullet, Z)) \\
&\cong \quad \mathrm{sSet}(C\mathscr{Y}_{K-1}^m, \mathrm{Hom}_{sSet}(\Delta^\bullet, \mathrm{cosk}_0 sZ)) \\
&\cong \quad \mathrm{sSet}(C\mathscr{Y}_{K-1}^m \times \Delta^\bullet, \mathrm{cosk}_0 sZ) \\
&=: \quad \mathrm{Hom}_{sSet}(C\mathscr{Y}_{K-1}^m, \mathrm{cosk}_0 sZ)
\end{aligned}
$$

and likewise,

$$
\begin{aligned}
\mathrm{Top}(\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \Delta^\bullet, Z) \quad &\cong \quad \mathrm{sSet}(\mathscr{Y}_{K-1}^m \times \mathrm{sk}_0 \Delta^\bullet, sZ) \\
&\cong \quad \mathrm{sSet}(\mathscr{Y}_{K-1}^m \times \Delta^\bullet, \mathrm{cosk}_0 sZ) \\
&=: \quad \mathrm{Hom}_{sSet}(\mathscr{Y}_{K-1}^m, \mathrm{cosk}_0 sZ)
\end{aligned}
$$

We may express our righthand vertical map as

$$
\mathrm{Hom}_{sSet}(C\mathscr{Y}_{K-1}^*, \mathrm{cosk}_0 sZ) \to \mathrm{Hom}_{sSet}(\mathscr{Y}_{K-1}^*, \mathrm{cosk}_0 sZ)
$$

which is $\mathrm{Hom}_{sSet}(\ , \mathrm{cosk}_0 sZ)$ applied to the map $\mathscr{Y}_{K-1}^m \to C\mathscr{Y}_{K-1}^m$. This map is a cofibration of simplicial sets since it is a monomorphism. The singularization of a topological space is a fibrant simplicial set, and as coskeleton is a right Kan extension, it preserves fibrant objects. Therefore, we may apply SM7 and conclude that our map is a fibration of simplicial sets, and our square for each m is a homotopy pullback square. Applying Theorem 7.4.2, we conclude that the following is a homotopy pullback of simplicial sets:

$$
\begin{array}{ccc}
\mathrm{Top}(\mathscr{Y}_{K-1}^\bullet * \mathrm{sk}_0 \Delta^\bullet, Z) & \longrightarrow & \mathrm{Top}(C(\mathscr{Y}_{K-1}^\bullet) \times \mathrm{sk}_0 \Delta^\bullet, Z) \\
\downarrow & & \downarrow \\
\mathrm{Top}(\mathscr{Y}_{K-1}^\bullet, Z) & \longrightarrow & \mathrm{Top}(\mathscr{Y}_{K-1}^\bullet \times \mathrm{sk}_0 \Delta^\bullet, Z)
\end{array}
$$

$\mathrm{Top}(\mathscr{Y}_{K-1}^\bullet, Z)$ is contractible by inductive hypothesis and we have already shown that the right two

simplicial sets are contractible. Now that we have expressed $\mathrm{Top}(\mathscr{Y}^{\bullet}_{K-1} * \mathrm{sk}_0 \triangle^{\bullet}, Z)$ as the homotopy pullback of contractible simplicial sets, we may conclude that it is contractible as well.

$\square$

References

[AK98]   G. Arone and M. Kankaanrinta, <u>A functorial model for iterated snaith splitting with applications to calculus of functors</u>, Stable and unstable homotopy (Toronto, ON, 1996) **19** (1998), 1–30.

[BEJM11] Kristine Bauer, Rosona Eldred, Brenda Johnson, and Randy McCarthy, <u>Geometric models for taylor polynomials of functors</u>, In Preparation, 2011.

[BF78]   A. Bousfield and E. Friedlander, <u>Homotopy theory of γ-spaces, spectra, and bisimplicial sets</u>, Geometric applications of homotopy theory II (1978), 80–130.

[BJME10] Kristine Bauer, Brenda Johnson, Randy McCarthy, and (Rosona Eldred), <u>A cotriple model for calculus in an unbased setting</u>, submitted to Transact., 2010.

[BK72]   A.K. Bousfield and D.M. Kan, <u>Homotopy limits, completions and localizations</u>, Springer, 1972.

[Bou89]   AK Bousfield, <u>Homotopy spectral sequences and obstructions</u>, Israel Journal of Mathematics **66** (1989), no. 1, 54–104.

[DK84]   WG Dwyer and DM Kan, <u>An obstruction theory for diagrams of simplicial sets</u>, Indagationes Mathematicae (Proceedings) **87** (1984), no. 2, 139–146.

[Dug08]   D. Dugger, <u>A primer on homotopy colimits</u>, Preprint, 2008.

[ES52]   S. Eilenberg and N.E. Steenrod, <u>Foundations of algebraic topology</u>, vol. 133, Princeton University Press, 1952.

[Goe93]   P.G. Goerss, <u>Barratt's desuspension spectral sequence and the lie ring analyzer</u>, The Quarterly Journal of Mathematics **44** (1993), no. 1, 43.

[Goo90]   T Goodwillie, <u>Calculus i: The first derivative of pseudoisotopy theory</u>, K-theory (1990), 1–27.

[Goo91]   ———, <u>Calculus ii: analytic functors</u>, K-theory (1991), 295–332.

[Goo03]   ———, <u>Calculus iii: Taylor series</u>, Geometry and Topology (2003), 645–711.

[Goo10]   ———, personal communication, 2010.

[Hir02]   P.S. Hirschhorn, <u>Model categories and their localizations</u>, Mathematical Surveys and Monographs, vol. 99, AMS Bookstore, 2002.

[Hop84a] M J Hopkins, <u>Formulations of cocategory and the iterated suspension</u>, Astérisque **113-114** (1984), 212–226, Algebraic homotopy and local algebra (Luminy, 1982).

[Hop84b] ———, <u>Some problems in topology</u>, Ph.D. thesis, Oxford University, 1984.

[Joh95]   B Johnson, <u>The derivatives of homotopy theory</u>, Transactions of the American Mathematical Society **347** (1995), no. 4, 1295–1321.

[McC]   Randy McCarthy, <u>A filtration for functors which commute with realizations</u>, Unpublished manuscript.

[McC10] R. McCarthy, personal communication, 2010.

[MO02] Andrew Mauer-Oats, Goodwillie calculi, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2002.

[Sch97] S. Schwede, Spectra in model categories and applications to the algebraic cotangent complex, Journal of Pure and Applied Algebra **120** (1997), no. 1, 77–104.

[Seg74] G. Segal, Categories and cohomology theories, Topology **13** (1974), no. 3, 293–312.

[Shi96] B.E. Shipley, Convergence of the homology spectral sequence of a cosimplicial space, American Journal of Mathematics **118** (1996), no. 1, 179–207.

[Sin09] D.P. Sinha, The topology of spaces of knots: cosimplicial models, American Journal of Mathematics **131** (2009), no. 4, 945–980.