STRUCTURAL AND EXTREMAL RESULTS IN GRAPH THEORY

BY

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DISSERTATION

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Abstract

An *H-immersion* is a model of a graph H in a larger graph G. Vertices of H are represented by distinct "branch" vertices in G, while edges of H are represented by edge-disjoint walks in G joining branch vertices. By the recently proved Nash-Williams Immersion Conjecture, any immersion-closed family is characterized by forbidding the presence of H-immersions for a finite number of graphs H. We offer descriptions of some immersion-closed families along with their forbidden immersion characterizations. Our principal results in this area are a characterization of graphs with no $K_{2,3}$-immersion, and a characterization of graphs with neither a $K_{2,3}$-immersion nor a K_4-immersion. We study of the maximum number of edges in an n-vertex graph with no K_t-immersion. For $t \leq 7$, we determine this maximum value. When $5 \leq t \leq 7$, we characterize the graphs with no K_t-immersion having the most edges.

Given an edge-colored graph, a *rainbow subgraph* is a subgraph whose edges have distinct colors. We show that if the edges of a graph G are colored so that at least k colors appear at each vertex, then G contains a rainbow matching of size $\lfloor k/2 \rfloor$. We consider the *rainbow edge-chromatic number* of an edge-colored graph, $\chi'_r(G)$, which we define to be the minimum number of rainbow matchings partitioning the edge set of G. A *d-tolerant* edge-colored graph is one that contains no monochromatic star with $d+1$ edges. We offer examples of d-tolerant n-vertex edge-colored graphs G for which $\chi'_r(G) \geq \frac{d}{2}(n-1)$ and prove that $\chi'_r(G) < d(d+1)n \ln n$ for all such graphs. We study the *rainbow domination number* of an edge-colored graph, $\widehat{\gamma}(G)$, which we define to be the minimum number of rainbow stars covering the vertex set of G. We generalize three bounds on the domination number of

graphs. In particular, we show that $\widehat{\gamma}(G) \leq \left(\frac{d}{d+1}\right) n$ for all d-tolerant n-vertex edge-colored graphs G and characterize the edge-colored graphs achieving this bound.

A *total acquisition move* in a weighted graph G moves all weight from a vertex u to a neighboring vertex v, provided that before this move the weight on v is at least the weight on u. The *total acquisition number*, $a_t(G)$, is the minimum number of vertices with positive weight that remain in G after a sequence of total acquisition moves, starting with a uniform weighting of the vertices of G. We offer an independent proof that $a_t(G) \leq \frac{|V(G)|+1}{3}$ for all graphs with at least two vertices. In addition, we characterize graphs achieving this bound. If $a_t(G) = \frac{|V(G)|+1}{3}$, then $G \in \mathcal{T} \cup \{P_2, C_5\}$, where $\mathcal{T}$ is the family of trees that can be constructed from P_5 by iteratively growing paths with three edges from neighbors of leaves.

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Chapter 1

Introduction

In an increasingly complex world it is vital to understand the networks that surround us. When information is to be disseminated from person to person, it is critical to understand the communities a person is likely to interact with to ensure that the information is dispersed appropriately. For computer viruses or diseases, understanding the paths through which a harmful agent propagates itself is often the first step in stemming its transmission. Graphs can be used to model connections in networks in these and other situations. We refer the reader to Section 1.4 at the end of this chapter for basic graph-theoretic definitions and notation. In this thesis, we focus on results from structural and extremal graph theory through a primarily theoretical perspective.

Graph minors have many algorithmic applications. For a fixed graph H, whether a graph G contains an H-minor can be determined in polynomial-time. If forbidding a finite number of minors characterizes a family of graphs, this can be used to prove the existence of a polynomial-time membership-testing algorithm. The minor order is an enrichment of the subdivision order. In Chapter 2 we study another enrichment of the subdivision order known as the immersion order. Graph immersions have algorithmic applications as well, but the immersion order is not as well understood as the minor order. In Chapter 2, we survey recent results on the immersion order, placing them in the context of classical results on the minor order. We give characterizations of some immersion-closed families and consider a related extremal problem. This is joint work with Jane Butterfield, Stephen Hartke, Kevin Milans, Derrick Stolee, and Paul Wenger.

Results about the size and structure of matchings in graphs are well known. When a graph represents a chemical structure, matchings can represent the possible placement of double bonds. Atoms corresponding to vertices not covered by the matching may be required to bond to other atoms outside of the structure in consideration. This can have drastic effects on the geometric structure and chemical reactivity of a substance. Given an edge-colored graph, a *rainbow subgraph* is a subgraph whose edges have distinct colors. In Chapter 3 we investigate the size of a largest rainbow matching that can be guaranteed to exist. We also bound the minimum number of rainbow matchings required to partition the edge set of a graph. This result is related to the problem of determining the edge-chromatic number of a graph. In Chapter 3 we also consider rainbow stars. We bound the number of rainbow stars necessary to cover the vertices of an edge-colored graph, generalizing classical results on the domination number of a graph. If there are no large monochromatic stars, then stronger bounds can be obtained. In this situation, we characterize the family of edge-colored graphs achieving this stronger bound. This is joint work with Christopher Stocker, Paul Wenger, and Douglas West.

Consider an army dispersed among many cities. The number of soldiers in a city can be viewed as a weight on a vertex in a graph representing a transportation network. The weight on the vertices is to be consolidated through *total acquisition moves* which transfer all of the weight from a vertex u to a neighboring vertex v under the condition that before this move the weight on v is at least as large as the weight on u. The total acquisition number of a weighted graph is the minimum number of vertices which remain with positive weight after a sequence of total acquisition moves. In Chapter 4, we consider the extremal problem of determining the graphs for which the acquisition number is as large as possible. This is joint work with Douglas West.

1.1 Immersions

The first topic we study is related to the classical topic of graph minors. Let G and H be graphs and let F be a subgraph of G. We say that F is an *H-minor* if H can be obtained from a subgraph of F through a sequence of edge contractions. Graph minors were introduced by Wagner [71] when he gave a characterization of planar graphs. As a result of Wagner's description of graphs with no K_5-minor [72], the maximum number of edges in an n-vertex graph with no K_5-minor is known. More recent progress in the study of graph minors includes the celebrated Graph Minor Theorem of Robertson and Seymour [59]. In Chapter 2, we give results in the analagous study of graph immersions, placing them in the context of these classical result in the study of graph minors.

Given a graph H, a subgraph F of a graph G is an *H-immersion* if the vertices of H can be identified with unique "branch" vertices in F in a such a way that the edges of H can be modeled by edge-disjoint walks in F joining corresponding branch vertices. A family of graphs $\mathcal{F}$ is *immersion-closed* if $H \in \mathcal{F}$ whenever H is a subgraph of G and $G \in \mathcal{F}$. Due to the recently proved Nash-Williams Immersion Conjecture [57], immersion-closed families are characterized by forbidding H-immersions for a finite number of graphs H, however, few immersion-closed families are known. In Section 2.3, we describe several immersion-closed families and provide their forbidden immersion characterizations. The *local edge-connectivity* of a pair of vertices $\{u, v\}$ is the minimum number of edges whose deletion separates u and v. The maximum local edge-connectivity of a graph is the maximum local edge-connectivity of its pairs of vertices. We show that the family of graphs for which the maximum local edge-connectivity is bounded by k is immersion-closed and is characterized by forbidding $K_2 \vee \overline{K_k}$-immersions. The principal results in this section are a characterization of graphs with no $K_{2,3}$-immersion, and a characterization of graphs with neither a $K_{2,3}$-immersion nor a K_4-immersion. These families of graphs are characterized in terms of forbidden *subdivisions* whose presence is easier to detect than the forbidden $K_{2,3}$-immersions and K_4-immersions.

In Section 2.4 we consider extremal problems concerning graph immersions. For $k \leq 7$ we show that the maximum number of edges in an n-vertex graph with no K_k-immersions is $(k-2)n - \binom{k-1}{2}$. For $5 \leq k \leq 7$, we characterize the graphs achieving this maximum number of edges. All such graphs consist of independent sets of $n-k+1$ vertices with degree $k-2$ such that the remaining vertices form a $(k-1)$-clique. Using very recent results of DeVos, Dvořák, Fox, McDonald, Mohar, and Sheide [18], we note that almost all graphs G have K_t-immersions, where $t \geq \chi(G)$. This supports the validity of an immersion order analogue of Hadwiger's Conjecture posed by Abu-Khzam and Langston [1].

1.2 Rainbow Subgraphs

Given an edge-colored graph, a *rainbow subgraph* is a subgraph whose edges have distinct colors. Classical results in graph Ramsey theory show the existence of monochromatic subgraphs in edge-colored graphs. In rainbow Ramsey theory, the goal is show the existence of rainbow subgraphs. Conditions must be imposed on the edge-colorings considered since rainbow subgraphs in monochromatically edge-colored graphs can have at most one edge.

The subgraphs considered in Chapter 3 are rainbow matchings and rainbow stars. In an edge-colored graph the *color degree* of a vertex v is the number of colors appearing on the edges incident to v. The *minimum color degree* of a graph is the minimum color degree of its vertices. Wang and Li conjectured that if G is an edge-colored graph with minimum color degree at least k, then G contains a rainbow matching of size $\lceil k/2 \rceil$ [74] . In Section 3.3, we prove that all such graphs have a rainbow matching of size $\lfloor k/2 \rfloor$, proving the conjecture of Wang and Li when k is even. Kostochka and Yancey [40] recently extended the arguments contained in Section 3.3 to complete the proof of the conjecture of Wang and Li.

In Section 3.4 we study a rainbow Ramsey-theoretic version of the edge-chromatic number of a graph. We consider the question of determining the minimum number of rainbow matchings required to partition the edge set of a graph. For an edge-colored graph G, this minimum value is called the *rainbow edge-chromatic number* of G and is denoted $\chi'_r(G)$. We provide examples of properly edge-colored graphs G such that $\chi'_r G > \Delta(G) + 1$. An edge-colored graph is *d-tolerant* if no vertex is incident to more than d edges having the same color. We provide examples of d-tolerant n-vertex edge-colored graphs G such that $\chi'_r(G) \geq \frac{d}{2}(n-1)$. Using the results contained in Section 3.3, we show that all such graphs satisfy $\chi'_r(G) < d(d+1)n \ln n$. The proof involves a generalization of a classical result. We show that if G is a d-tolerant edge-colored graph with *average* color degree c, then G contains a subgraph with minimum color degree more than $c/(d+1)$.

The *domination number* of a graph G is the minimum number of stars needed to cover its vertex set and is denoted $\gamma(G)$. In Section 3.2 we study a rainbow Ramsey-theoretic version of the domination number. We define the *rainbow domination number* of an edge-colored graph to be the minimum number of rainbow stars needed to cover its vertex set. For a graph G, this minimum value is denoted $\widehat{\gamma}(G)$. The study of rainbow domination in edge-colored graphs extends the study of domination in graphs since a graph is 1-tolerant if and only if it properly edge-colored. If G is properly edge-colored, then $\widehat{\gamma}(G) = \gamma(G)$ since all stars in G are rainbow stars. One classical result on domination in graphs states that $\gamma(G) \leq |V(G)| - \Delta(G)$ for all graphs G [7]. We show that if G is an n-vertex graph with maximum color degree k, then $\widehat{\gamma}(G) \leq n - k$, and we provide examples showing that this bound is tight. Another classical result states that $\gamma(G) \leq \frac{|V(G)|}{2}$ for all graphs G [52]. The family of graphs achieving equality in this bound has a simple description. If $\gamma(G) = \frac{|V(G)|}{2}$, then the components of G are C_4 or are such that every non-leaf vertex is adjacent to exactly one leaf [24, 54]. We extend this bound and the characterization of extremal graphs to the context of rainbow domination. For a d-tolerant n-vertex edge-colored graph G, we show that $\widehat{\gamma}(G) \leq \left(\frac{d}{d+1}\right) n$. We offer examples showing that this bound is tight and characterize the d-tolerant n-vertex edge-colored graphs G such that $\widehat{\gamma}(G) = \left(\frac{d}{d+1}\right) n$. If $d \geq 3$, then in each component of such a graph, every non-leaf vertex is adjacent to exactly d leaves via edges of the same color. For $d = 1$ and $d = 2$, the components of such a graph must either have this form or be properly edge-colored copies of C_4 when $d = 1$, or monochromatic copies of C_3 when $d = 2$. Section 3.2 concludes with a generalization to $\widehat{\gamma}(G)$ of bound on $\gamma(G)$ attributed to many authors including Arnautov and Payan [3,6,47,53]. We show that if G is a d-tolerant n-vertex edge-colored graph with minimum degree k, then $\widehat{\gamma}(G) \leq \frac{n(1+\ln(k/d+1))}{k/d+1}$.

1.3 Acquisition

Let G be a weighted graph in which each vertex initially has weight 1. A *total acquisition move* transfers all the weight from a vertex u to a neighboring vertex v under the condition that before this move the weight on v is at least as large as the weight on u. The *total acquisition number* of G, written $a_t(G)$, is the minimum number of vertices with positive weight remaining after a sequence of total acquisition moves. Lampert and Slater [41] proved that the maximum of $a_t(G)$ over connected n-vertex graphs is $\left\lfloor \frac{n+1}{3} \right\rfloor$. In Chapter 4, we offer an alternate proof which allows for the characterization of graphs such that $a_t(G) = \frac{|V(G)|+1}{3}$. Let $\mathcal{T}$ be the family of trees constructed from P_5 by iteratively growing a path with three edges from a neighbor of a leaf. We show that if $a_t(G) = \frac{|V(G)|+1}{3}$, then $G \in \{P_2, C_5\} \cup \mathcal{T}$.

Although the characterization given above is stronger, the most significant result given in Chapter 4 is a characterization of the *trees* achieving the maximum value of the acquisition number. We begin by giving an alternate structural characterization of the trees in $\mathcal{T}$, focusing on vertices having distance at least 2 from all leaves. These vertices play a pivotal role in the proof of the main result. Section 4.3 presents a series of lemmas about the structure of minimal counterexamples. First, we show that graphs in which there are no vertices having distance at least 2 from all leaves cannot be minimal counterexamples. Using the structural characterization of trees in $\mathcal{T}$, we then prove several properties of vertices having distance at least 2 from each leaf in a minimal counterexample. Once the trees achieving the maximum value of the acquisition number are characterized, we extend this characterization to graphs in general. This involves showing that every graph other than P_2, C_5, and the trees in $\mathcal{T}$ has a spanning tree that is not in $\mathcal{T}$.

1.4 Definitions

For integers a and b, we use $[b]$ to denote the set of integers $\{1, \dots, b\}$, and we use $[a, b]$ to denote the set of integers $\{a, a+1, \dots, b-1, b\}$. For a real number r, the *floor* of r, denoted by $\lfloor r \rfloor$, is the largest integer not greater than r. The *ceiling*, denoted by $\lceil r \rceil$, is the smallest integer not less than r. For functions $f\colon \mathbb{N} \to \mathbb{R}$ and $g\colon \mathbb{N} \to \mathbb{R}$, we write $f(n) = o(g(n))$ if for every $\epsilon > 0$ there exists n_0 such that for $n > n_0$ we have $|f(n)| \le \epsilon |g(n)|$. We write $f(n) = \Theta(g(n))$ when there exist positive constants k_1 and k_2 and an integer n_0 such that for $n > n_0$ we have $k_1|g(n)| \le |f(n)| \le k_2|g(n)|$. We write $f(n) \sim g(n)$ when $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. For a subset A' of a set A and a function $f\colon A \to B$, we write $f(A')$ for $\bigcup_{a\in A'} f(a)$ when B is a family of sets, and $f(A')$ for $\sum_{a\in A'} f(a)$ when B is a set of numbers.

A *graph* G consists of a set $V(G)$ of *vertices* and a multiset $E(G)$ of pairs of vertices, called *edges*. The *order* of a graph G is $|V(G)|$. Two vertices u and v are *adjacent* if $\{u, v\}$ is an edge. We write uv for the edge $\{u, v\}$ and say that uv *joins* u and v. A vertex u is a *neighbor* of a vertex v if u and v are adjacent. The *open neighborhood* of a vertex v is the set $N_G(v)$ of neighbors of v in G; the *closed neighborhood*, $N_G[v]$, is the set $N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is the number of edges containing v and is denoted $d_G(v)$. When the graph G is understood it is omitted from subscripts. A vertex v is *isolated* if $d(v) = 0$ and is a *leaf* if $d(v) = 1$. The *minimum* and *maximum degree* of G are the minimum and maximum of the degrees of the vertices of G and are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *average degree* of G is the average degree of the vertices of G. An edge e is *incident* to a vertex v if v is one of the vertices in e. Two edges e and f are *incident* if they share a vertex. A graph is r-*regular* if $d(v) = r$ for each $v \in V(G)$. A graph G is *finite* if $V(G)$ and $E(G)$ are finite sets and is *simple* if no edge is repeated in $E(G)$. Unless otherwise stated, all graphs considered in this thesis are finite and simple. When sets of graphs are considered, we will often use the term *family* instead.

A graph H is *isomorphic* to a graph G if there exists a bijection $f\colon\ V(G) \to V(H)$ such $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$ for all vertices u and v in $V(G)$. We often use the same name for isomorphic graphs, so we say that a graph *is* G if it is isomorphic to G. In this sense, we use K_n to denote any simple graph with n vertices in which each pair of vertices is an edge. A graph is a *complete* graph if it is isomorphic to K_n for some $n \in \mathbb{N}$. Similarly, P_n denotes the isomorphism class of graphs with n vertices that can be indexed so that they are adjacent if and only if their indices differ by 1. A graph is a *path* if it is isomorphic to P_n for some $n \in \mathbb{N}$. A path has two leaves, which are the vertices with the minimum and maximum index. We use C_n to denote any graph with n vertices that can be indexed so that they are adjacent if and only if their indices differ by 1 modulo n. A graph is a *cycle* if it is isomorphic to C_n for some $n \geq 3$.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning* subgraph of G is a subgraph whose vertex set is $V(G)$. The subgraph *induced* by a set U of vertices of G is the subgraph H such that $V(H) = U$ and $E(H)$ contains each edge $e \in E(G)$ such that $e \subseteq U$. When H is the subgraph induced by U we say that U *induces* H. A *clique* is a set of vertices that induces a complete graph. A k-*clique* is a set of k such vertices. An *independent set* is a set of vertices that induces a graph with no edges. A *matching* is a set of non-incident edges. For a subgraph C of G that is a cycle, a *chord* is an edge of G joining vertices non-adjacent in C. A *Hamiltonian cycle* is a spanning cycle. A graph is k-degenerate if each of its subgraphs has minimum degree at most k. A family

$\mathcal{F}$ of subgraphs of G *covers* the vertex set $V(G)$ if for each $v \in V(G)$ there exists an $F \in \mathcal{F}$ such that $v \in V(F)$. Subgraphs are *disjoint* if they share no common vertex, and they are *edge-disjoint* if they share no common edge.

For a graph G and a set of vertices $U \subseteq V(G)$, the graph $G - U$ is the graph induced by $V(G) \setminus U$. For a set of edges $E \subseteq E(G)$, the graph $G - E$ is the graph whose vertex set is $V(G)$ and whose edge set is $E(G) \setminus E$. When U consists of a single vertex u we write $G - u$ for $G - \{u\}$, and when E consists of a single edge e we write $G - e$ for $G - \{e\}$. *Deletion* of a vertex u is the process through which $G - u$ is constructed. *Deletion* of an edge e is the process through which $G - e$ is constructed. When $uv \in E(G)$, *contraction* of the edge uv is the process through which a graph H is constructed by defining $V(H) = (V(G) \cup \{w\}) \setminus \{u, v\}$ and making vertices x and y adjacent in H if they are adjacent in G or if one is w and the other is adjacent to either u or v in G. *Lifting* of incident edges uv and vw in G is the process through which a graph H is constructed by defining $V(H) = V(G)$ and $E(H) = (E(G) \cup \{uw\}) \setminus \{uv, vw\}$.

The *complement* of a graph G is the graph $K_n - E(G)$ and is written $\overline{G}$. The *union* of two subgraphs H_1 and H_2 is the graph whose vertex set is $V(H_1) \cup V(H_2)$ and whose edge set is $E(H_1) \cup E(H_2)$. The *join* of two disjoint graphs G_1 and G_2 is the graph $G_1 \vee G_2$ whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2) \cup (V(G_1) \times V(G_2))$. The *complete bipartite graph* $K_{a,b}$ is the graph $\overline{K_a} \vee \overline{K_b}$. The *complete tripartite graph* $K_{a,b,c}$ is the graph $K_{a,b} \vee \overline{K_c}$.

We say that a path in a graph *joins* u and v if u and v are leaves of the path. Two paths P_1 and P_2 are *internally-disjoint* if every common vertex of P_1 and P_2 is a leaf of both. We say that a path P in G is a *pendant* path if only one vertex of P is incident to an edge not in P, and this vertex is a leaf of P. The *length* of a path is the number of edges it contains. In a graph G, the *distance* between vertices u and v is the smallest length of a path joining them and is denoted $d(u, v)$. A vertex u is in the *center* of a graph G if there is no vertex v

such that $\max\{d(w,v) : w \in V(G)\} < \max\{d(w,u) : w \in V(G)\}$. A *walk* is a set of edges that can be ordered so that if e and f are consecutive then e and f share a vertex. When a walk W is the edge set of a graph this graph has two vertices of odd degree. We say that W *joins* u and v if u and v are these vertices of odd degree.

A graph G is *connected* if for every two vertices u and v there is a path that is a subgraph of G joining u and v. For a graph that is not connected, the *components* are the maximal connected subgraphs. For a connected graph G, a *cut-vertex* is a vertex v such that $G - v$ is not connected. A *cut-edge* is an edge e such that $G - e$ is not connected. A *cut-set* is a set of vertices V such that $G - V$ is not connected. An *edge-cut* is the set of all edges joining vertices in U to vertices in $V(G) \setminus U$ for some subset $U \in V(G)$. For disjoint subsets of vertices U, V and X in a connected graph G, we say that X *separates* U and V if no vertex of U is in the same component of $G - X$ as a vertex of V. For an edge set E we say that E *seperates* U and V if no vertex of U is the same component of $G - E$ as a vertex of V.

A graph is *k-connected* if it has more than k vertices and there exist no cut-sets of size less than k. It is *k-edge-connected* if it has more than k vertices and there exist no edge-cuts of size less than k. For a connected graph G, the *connectivity*, denoted $\kappa(G)$, and *edge-connectivity*, denoted $\kappa'(G)$, are the largest k such that G is k-connected and k-edge-connected, respectively. A *block* in a graph G is a maximal 2-connected subgraph of G. A version of Menger's Theorem [51] states that if G is k-connected, then for any two disjoint subsets of vertices U and V there is a family of k internally-disjoint paths each joining a vertex of U and a vertex of V. Similarly, a graph is k-edge-connected if there is always a family of k edge-disjoint paths, each joining a vertex of U and a vertex of V. A graph is a *forest* if it contains no cycle. A graph is a *tree* if it is a connected forest. A *star* is a tree with n vertices and $n - 1$ leaves. A *cactus* is a graph such that each edge is contained in at most one cycle.

Graphs are often represented by drawings on surfaces. The vertices are represented by points, and an edge uv is represented by a simple continuous curve joining the points representing u and v. Such a drawing of a graph G on a surface S is called an *embedding* of G in S. A graph is *planar* if it can be embedded in the plane is such a way that no curves representing edges intersect each other except at their ends, and for each edge uv the curve representing uv does not contain the points representing any vertices other than u and v. A *planar embedding* is such an embedding of a planar graph. A *toroidal graph* is a graph with such an embedding on the torus. A *face* in an embedding on a surface S is a maximal connected region of S not containing any points representing vertices or any points on curves representing edges. The *unbounded* face in a planar embedding is the face in which the Euclidean distance to the origin is unbounded. A graph is *outerplanar* if it has a planar embedding in which each vertex is contained in the boundary of the unbounded face. An *outerplanar embedding* is such an embedding of an outerplanar graph. A corollary of Euler's Formula [15, 16] states that a planar n-vertex simple graph has at most $3n-6$ edges. An n-vertex *planar triangulation* is a maximal planar subgraph of K_n. All such graphs have $3n-6$ edges.

A *coloring* of a graph G is a function $f\colon V(G) \to \mathbb{N}$. A *$k$-coloring* is a function $f\colon V(G) \to [k]$. For a vertex u, the value of $f(u)$ is called the *color* on u. A *colored graph* is a graph in which the vertices have been assigned colors. A *k-colored graph* is a graph in which each vertex is assigned one of k colors. A coloring f is *proper* if $f(u) \neq f(v)$ for all adjacent vertices u and v. A graph G is *k-colorable* if there exists a proper k-coloring of G. The *chromatic number*, $\chi(G)$, is the least k such that G is k-colorable. The Four Color Theorem [5] states that if G is a planar graph, then G is 4-colorable. An *edge-coloring* of a graph G is a function $g\colon E(G) \to \mathbb{N}$. A *$k$-edge-coloring* is a function $g\colon V(G) \to [k]$. For an edge e, the value of $g(e)$ is called the *color* on e. An *edge-colored graph* is a graph in which the edges have been assigned colors. A *k-edge-colored* graph is a graph in which each edge is assigned one of k colors. The *color classes* of an edge-colored graph are the maximal subsets of edges having the same color. An edge-coloring g is *proper* if $g(e) \neq g(f)$ for all incident edges e and f. A graph G is k*-edge-colorable* if there exists a proper k-edge-coloring of G. The *edge-chromatic number*, $\chi'(G)$, is the least k such that G is k-colorable. Trivially, $\Delta(G) \leq \chi'(G)$. Vizing's Theorem [70] states that $\chi'(G) \leq \Delta(G) + 1$ for every simple graph G.

A *weighting* of a graph G is a function $f\colon V(G) \to \mathbb{R}$. For a vertex v, the value of $f(v)$ is the *weight* on v. Weight x is *transferred* from a vertex u to a vertex v through the process of defining a new weighting f' such that $f'(u) = f(u) - x$, $f'(v) = f(v) + x$, and $f'(w) = f(w)$ for all other vertices. Through such a process, we way that v *acquires* weight x from u.

The probability space $G(n, p)$ is the space of n-vertex graphs such that each edge appears randomly and independently with probability p. A *random graph on* n *vertices generated with edge probality* p is a randomly selected member of $G(n, p)$. An event occurs for *almost all* graphs if the limit as $n \to \infty$ of the probability that it occurs for a random graph on n vertices is 1. It occurs for *almost no* graph if this limit is 0.

A *partially ordered set* is a set S along with a relation $\leq$ on S that is reflexive ($x \leq x$), symmetric ($x \leq y$ implies $y \leq x$), and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$). When $(S, \leq)$ is a partially ordered set, we say that S is *ordered* by the relation $\leq$ and call $\leq$ a *partial order*. We write $a > b$ when $b \leq a$ and $b \neq a$. An *infinite decreasing sequence* in a partially ordered set is a sequence $\{a_i\}_{i=1}^{\infty}$ such that $a_i > a_{i+1}$ for all $i \in \mathbb{N}$. An *ideal* is a set of elements of S such that if $a \in S$ and $b \leq a$, then $b \in S$. The ideal *generated* by a subset T is the smallest ideal containing T as a subset. A *dual ideal* is the complement of an ideal. Two elements a and b of a partially ordered set are *comparable* if $a \leq b$ or $b \leq a$. An *antichain* is a set of incomparable elements.

A *Latin square of order* n is an $n \times n$ array filled with n symbols, each occurring exactly once in each row and exactly once in each column.

Chapter 2

Immersions

2.1 Introduction

An H-*immersion* is a model of a graph H in a larger graph G. Vertices of H are represented by distinct vertices of G, while edges of H are represented by walks in G that share no edges. The following is a more precise definition.

Definition 2.1.1. Let G and H be graphs. A subgraph F of G is an H-*immersion* if there exists an injection $f\colon V(H) \to V(F)$ and a family of edge-disjoint walks $\mathcal{W}$ in F such that there exists a bijection $f'\colon E(H) \to \mathcal{W}$ with the property that for each $uv \in E(H)$, $f'(uv)$ is a walk joining $f(u)$ and $f(v)$.

Equivalently, F is an H-immersion if H can be obtained from a subgraph of F through a sequence of edge lifts and contractions of edges incident to vertices of degree 2. When G contains an H-immersion we write $H \leq_i G$. The class of finite simple graphs will be denoted $\mathcal{G}_i$ when ordered by the *immersion order*, $\leq_i$.

The study of immersions is young and has somewhat paralleled classical results in the study of graph minors. An H-*minor* is another type of embedded substructure in which the vertices of H are represented by connected subgraphs of a larger graph G. In an H-minor, these connected subgraphs are joined by an edge in G when their corresponding vertices are joined by an edge in H. The following is a more formal definition.

Definition 2.1.2. Let G and H be graphs. A subgraph F of G is an H-*minor* if there exists a family $\mathcal{V}$ of disjoint connected subgraphs of F, a bijection $f\colon V(H) \to \mathcal{V}$, and an injection $f'\colon E(H) \to E(F)$ such that $f'(uv)$ is an edge joining vertices of $f(u)$ and $f(v)$.

Equivalently, F is an H-minor if H can be obtained from a subgraph of F through a sequence of arbitrary edge contractions. When G contains an H-minor we write $H \leq_m G$. The class of finite simple graphs will be denoted $\mathcal{G}_m$ when ordered by the *minor order*, $\leq_m$.

Many of the results presented in this chapter will also involve a third type of embedded substructure. An H-subdivision is a model of a graph H in a larger graph G in which vertices of H are represented by distinct vertices of G and edges of H are represented by paths in G that share no internal vertices. The following definition is more precise.

Definition 2.1.3. Let G and H be graphs. A subgraph F of G is an *H-subdivision* if there exists an injection $f\colon V(H) \to V(F)$ and family of internally-disjoint paths $\mathcal{P}$ such that there exists a bijection $f'\colon E(H) \to \mathcal{P}$ with the property that for each $uv \in E(H)$, $f'(uv)$ is a path joining $f(u)$ and $f(v)$.

Equivalently, F is an H-subdivision if H can be obtained from F through a sequence of contractions of edges incident to vertices of degree 2. When G contains an H-subdivision we write $H \leq_s G$. An H-subdivision is simultaneously an H-minor and an H-immersion, so the subdivision relation $\leq_s$ is a more restrictive relation than both the minor relation $\leq_m$ and the immersion relation $\leq_i$.

2.2 Graph Minors and Subdivisions

In this section we survey selected classical results on graph minors. In the next section, we will provide analogous results on graph immersions.

Graph minors were introduced by Wagner when he gave a characterization of planar graphs.

Theorem 2.2.1 ([71]). *A graph is planar if and only if it does not contain a K_5-minor or a $K_{3,3}$-minor.*

The characterization above is a *forbidden minor characterization*. It identifies a family of graphs as those which do not contain one of a set of *forbidden minors*. Many well-studied classes of graphs have forbidden minor characterizations. The following is a relatively complete survey of families with known forbidden minor characterizations.

Proposition 2.2.2. *A graph is a forest if and only if it does not contain a K_3-minor.*

Proposition 2.2.3. *A graph is outerplanar if and only if it does not contain a K_4-minor or a $K_{2,3}$-minor.*

Proposition 2.2.4. *A connected graph is a cactus if and only if it does not contain a $(K_2 \vee \overline{K_2})$-minor.*

One more difficult characterization is that of the "linklessly embeddable" graphs. Two closed curves in $\mathbb{R}^3$ are *linked* if they cannot be continuously distorted while never intersecting each other to produce two non-intersecting circles lying in a plane. A graph G is *linklessly embeddable* if it can be embedded in $\mathbb{R}^3$ in such a way that no two cycles of G are linked. Robertson, Seymour, and Thomas gave a forbidden minor characterization of the linklessly embeddable graphs involving the Petersen family of graphs. Starting with the Petersen graph, depicted in Figure 2.1, the *Petersen family* is the collection of graphs obtained by iteratively performing one of two graph transformations called the $Y - \Delta$ transformation and the $\Delta - Y$ transformation. In the $Y - \Delta$ transformation, a vertex of degree 3 is deleted and its neighbors are made pairwise adjacent. In the $\Delta - Y$ transformation, the edges of a triangle are removed and its vertices are made adjacent to a newly added vertex. The Petersen family is depicted in Figure 2.1 and contains seven graphs including the Petersen graph, K_6, $K_{3,3,1}$, and the graph obtained by deleting an edge from $K_{4,4}$.

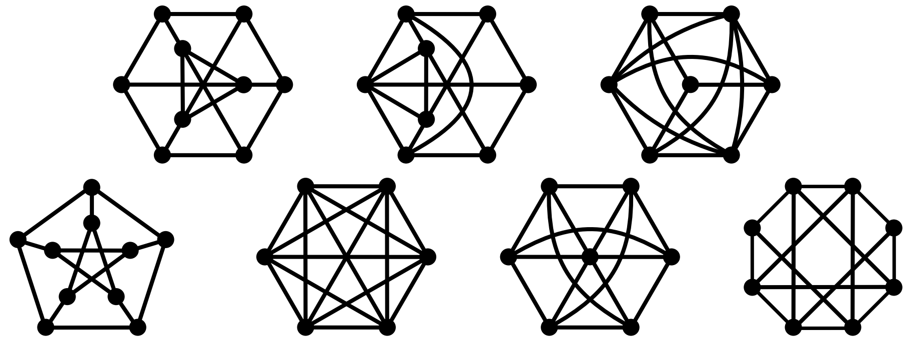


Figure 2.1: The Petersen Family.

Theorem 2.2.5 ([58, 63]). *A graph is linklessly embeddable if and only it does not contain an H-minor for all graphs H in the Petersen family.*

A family of graphs $\mathcal{F}$ is *minor-closed* if it is an ideal in $\mathcal{G}_m$. The five families of graphs above are minor-closed, as each is closed under taking subgraphs and arbitrary edge contractions. As a result, the existence of their forbidden minor characterizations can be inferred from the Graph Minor Theorem of Robertson and Seymour [59]. A *well-partially-ordered class* is a class X equipped with a "well-partial-order" $\leq$. That is, $\leq$ is a partial order under which there is no infinite decreasing sequence, nor an infinite antichain.

Theorem 2.2.6 (The Graph Minor Theorem [59]). *$\mathcal{G}_m$ is a well-partially-ordered class.*

Corollary 2.2.7. *For every minor-closed family $\mathcal{F}$ in $\mathcal{G}_m$, the set of minor-minimal non-members of $\mathcal{F}$ is finite.*

Corollary 2.2.7 is an equivalent phrasing of Theorem 2.2.6. There are no infinite decreasing sequences in $\mathcal{G}_m$, and an antichain A is the set of minor-minimal non-members of the complement of the dual ideal it generates. Theorem 2.2.6 and Corollary 2.2.7 both assert the finiteness of A.

As stated above, the existence of the forbidden minor characterizations listed in Theorem 2.2.1 through Theorem 2.2.5 can be inferred from Theorem 2.2.6, however, its application is non-constructive and does not yield the finite lists of forbidden minors given. In general, the list of forbidden minors characterizing a minor-closed family can be quite extensive and difficult to identify. The family S_γ of graphs embeddable on a surface of fixed genus γ is a minor-closed family. For planar graphs, S_0, there are two forbidden minors, as given by Theorem 2.2.1. For toroidal graphs, S_1, no complete list of forbidden minors is known, but any such list must contain the more than 16,000 minor-minimal non-toroidal graphs found by Myrvold and Chambers [17].

Instead of identifying a forbidden minor characterization of a known family, one can start with a known graph H and determine the family of graphs that contain no H-minor. The graphs that do not have a K_4-minor are the "series-parallel" graphs. A *2-terminal series-parallel graph* is a graph that can be constructed iteratively from copies of K_2 through a sequence of graph composition operations. Each 2-terminal series-parallel graph has two specified vertices called *terminals*; the terminals of K_2 are precisely its individual vertices. Let G_1 and G_2 be 2-terminal series-parallel graphs with pairs of terminals $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively. In the series composition G of G_1 and G_2, the terminals v_1 and u_2 are identified. The terminals of G are then u_1 and v_2. In the parallel composition G of G_1 and G_2, the terminals u_1 and u_2 are identified, while v_1 and v_2 are also identified. The terminals of G are then the vertices resulting from these identifications. A *series-parallel graph* is a graph whose blocks are isolated vertices or 2-terminal series-parallel graphs.

Next, consider the graphs with no K_5-minor. An example of such a graph is the Wagner graph which is the graph obtained by adding an edge joining each pair of vertices at distance 3 in C_8. Wagner desribed the graphs with no K_5-minor using the Wagner graph and "clique-sums". The *j-sum* of graphs G_1 and G_2, each containing j-cliques Q_1 and Q_2, is the graph obtained by identifying the vertices of Q_1 and Q_2 according to an arbitrary bijection and discarding an arbitrary subset of the edges joining these identified vertices.

Theorem 2.2.8 ([72]). *A graph contains no K_5-minor if and only if it can be iteratively* *constructed from planar graphs and copies of the Wagner graph through a series of j-sums, with $j \leq 3$.*

Corollary 2.2.9. *An n-vertex graph with no K_5-minor has at most $3n-6$ edges.*

This corollary follows easily from Euler's Formula [15, 16] and the fact that the bound on the number of edges is preserved during j-sums, with $j \leq 3$. Using the constructive description of series-parallel graphs in a similar manner, it can be shown that a simple n-vertex graph with no K_4-minor has at most $2n-3$ edges. When H has maximum degree at most 3, a graph contains an H-subdivision if and only if it contains an H-minor. Thus, an n-vertex graph with no K_4-subdivision has at most $2n-3$ edges. Extending Corollary 2.2.9 to apply to n-vertex graphs with no K_5-subdivision is not as easy. Nonetheless, Mader showed that the same bound on the number of edges holds in this larger family as well.

Theorem 2.2.10 ([49, 50])**.** *An n-vertex graph with no K_5-subdivision has at most $3n-6$ edges.*

Looser versions of Corollary 2.2.9 and Theorem 2.2.10 state that a graph with average degree at least 6 must contain a K_5-minor and a K_5-subdivision. For any value d less than 6 there is a planar triangulation with average degree at least d but no K_5-minor, and hence no K_5-subdivision, by Theorem 2.2.1. Thus, the smallest average degree that forces the presence of K_5-minors, and K_5-subdivisions, is 6. To extend this result, define $m(t)$ and $s(t)$ to be the smallest average degrees such that a graph with average degree at least $m(t)$ contains a K_t-minor and a graph with average degree at least $s(t)$ contains a K_t-subdivision. The expected average degree necessary to force the random graph to almost surely contain a K_t-minor is $(\alpha+o(1))t\sqrt{\ln(t)}$. Here, $\alpha = (1-\lambda)/2\sqrt{\ln(1/\lambda)}$, where λ is the non-trivial solution to the equation $1-\lambda+2\lambda\ln(\lambda) = 0$. Numerically, $\alpha = 0.319...$. Thomason determined $m(t)$ "more or less exactly" by showing that "random graphs cannot be beaten as extremal examples" [66].

Theorem 2.2.11 ([66])**.** *With $m(t)$ and α defined as above, $m(t) = (\alpha + o(1))t\sqrt{\ln t}$.*

Mader first proved that a graph with average degree at least $(t-1)2^{\binom{t-1}{2}}$ contains a K_t-subdivision [48] (see also Thomassen [67]). Bollobás and Thomason [12] then proved that a graph with minimum degree at least $256t^2$ must contain a K_t-subdivision. While this result does not explicitly give a bound on $s(t)$, one can be obtained by noting that every graph with average degree d contains a subgraph with minimum degree $d/2$. Hence, Bollobás and Thomason showed that $s(t) \leq 512t^2$. Later, they proved that average degree at least $44t^2$ guaranteed the presence of a K_t-subdivision [11]. Thomas and Wollan improved upon a related structural result of Bollobás and Thomason, obtaining the best known bound on $s(t)$ as a corollary [65].

Theorem 2.2.12 ([65])**.** *Every graph with average degree at least $20t^2$ contains a K_t-subdivision. That is, $s(t) \leq 20t^2$.*

Jung observed that when $\ell = t^2/8$, $K_{\ell,\ell}$ contains no K_t-subdivision [33]. This gives that $s(t) > t^2/8$, and, hence, $s(t) = \Theta(t^2)$.

In the previous discussion we have seen that having large average degree guarantees the existence of a K_t-minor. Hadwiger's conjecture is similar in that it asks if another graph property guarantees the existence of a K_t-minor.

Conjecture 2.2.13 (Hadwiger's Conjecture [27])**.** *If a graph G is not $(k-1)$-colorable, then G contains a K_k-minor.*

For $k \leq 3$, the truth of Conjecture 2.2.13 is trivial. Many proofs of Conjecture 2.2.13 when $k = 4$ have been given, including one by Hadwiger in the same article in which the conjecture appears [27]. See also Wagner [73], Woodall [76], the thesis of Dirac [20, 21], and the short proof due to Stiebitz in the survey by Toft [68]. Many of these proofs use structural properties of graphs with no K_4-minors to show that such graphs are 2-degenerate, and, hence, 3-colorable.

When $k = 5$, it can be shown using Theorems 2.2.1 and 2.2.8 that the truth of Conjecture 2.2.13 is equivalent to the Four Color Theorem of Appel and Haken [5]. If a graph G is not 4-colorable, then Conjecture 2.2.13 would imply that G contains a K_5-minor and cannot be planar by Theorem 2.2.1; this is precisely the contrapositive of the Four Color Theorem. Now, let G_1 and G_2 be 4-colored graphs, each containing j-cliques Q_1 and Q_2 that are to be identified in a j-sum. The colors on the vertices of G_2 may be permuted so that the colors on the vertices in Q_2 agree with the colors on the vertices they are to be identified with in Q_1. Now, if G is the graph obtained via this j-sum of G_1 and G_2, then G may be 4-colored by assigning each vertex in G the color it was assigned in the coloring of G_1 or in the permuted coloring of G_2. Now, the Four Color Theorem, Theorem 2.2.8, and the 4-colorability of the Wagner graph imply Conjecture 2.2.13 when $k = 5$. According to Toft [68], it was due to these close connections to the earlier work of Wagner that Hadwiger once expressed in a letter, written in German to Wagner, that "it does not seem completely right to him that the conjecture has been named after him." Hadwiger goes on to imply that his contribution was merely to extend the basic question due to Wagner to larger chromatic numbers.

Conjecture 2.2.13 has been resolved in the case when $k = 6$, while partial results toward the case when $k = 7$ are known. When $k \geq 8$, Conjecture 2.2.13 remains open. Robertson, Seymour, and Thomas [60] proved that, when $k = 6$, Conjecture 2.2.13 is equivalent to Four Color Theorem, as it was in the case when $k = 5$. An *apex graph* is a graph having a vertex whose deletion leaves a planar graph. If G is a planar graph, consider the apex graph $G^* = G \vee K_1$. By Theorem 2.2.1, G^* contains no K_6-minor. Now Conjecture 2.2.13 would imply that G^* is 5-colorable, and that G is 4-colorable. To show that the Four Color Theorem implies Conjecture 2.2.13 in the case when $k = 6$, Robertson, Seymour, and Thomas showed that if G is a minor-minimal graph with respect the property of having no K_6-minor and not being 5-colorable, then G must be an apex graph [60]. The Four Color Theorem implies that all apex graphs are 5-colorable, hence, no counterexample can exist. In the case when

$k = 7$, Jakobsen showed that if a graph is not 6-colorable, then it contains an H_1-minor or an H_2-minor, where H_1 and H_2 are the two non-isomorphic graphs obtained by deleting two edges from K_7 [32]. Kawarabayashi and Toft [37] and Kawarabayashi [36] proved that if a graph G is not 6-colorable, then G must either contain a K_7-minor or both a $K_{3,5}$-minor and a $K_{4,4}$-minor.

Related to Conjecture 2.2.13 is a conjecture usually accredited to Hajòs.

Conjecture 2.2.14. *If a graph G is not $(k-1)$-colorable, then G contains a K_k-subdivision.*

According to Toft [68], Hajós considered this conjecture in the 1940's but never published it. As noted in the paragraph following Corollary 2.2.9, when $\Delta(H) \leq 3$, a graph contains an H-subdivision if and only if it contains an H-minor. Hence, when $k \leq 4$, Conjecture 2.2.14 is equivalent to Conjecture 2.2.13 and consequently has been proved true. For $k \geq 7$, however, Conjecture 2.2.14 has been proved false. As a result, Hajós has been known to deny claim to the now defunct Conjecture 2.2.14. When Conjecture 2.2.14 is attributed to Hajós, his 1961 paper [28] is often cited, although it does not contain the statement of Conjecture 2.2.14. Instead, [28] contains a theorem stating that a graph is not $(k-1)$-colorable if and only if it contains a subgraph that can be constructed from copies of K_k through a sequence of composition operations similar to the j-sums discussed above (see [62]). Possibly the earliest written attribution of Conjecture 2.2.14 to Hajós appears in the review by Tutte [69] of the book "Färbungsprobleme auf flächen und graphen" by Ringel [56]. The book contains the result of [28] which had not yet been published, but it does not contain the statement of Conjecture 2.2.14.

Partial results toward Conjecture 2.2.14 when $k = 5$ are known. Yu and Zickfeld [77] showed that a smallest counterexample to this case of Conjecture 2.2.14 must be 4-connected. Yu and Zickfeld claim this result as a step toward reducing the case of Conjecture 2.2.14 when $k = 5$ to the following conjecture communicated to Yu by Seymour (see [77]) and independently posed by Kelmans [38] in a lecture series: every 5-connected non-planar graph contains a K_5-subdivision. No written record of this conjecture seems to exist outside of work by Yu. When $k = 6$, Conjecture 2.2.14 remains unresolved. When $k \geq 7$, the conjecture is false.

For $k \geq 7$, counterexamples to Conjecture 2.2.14 were constructed by Catlin [14]. Erdős and Fajtlowicz [22] and Bollobás and Catlin [9] then showed that if $0 < p < 1$, then for almost all graphs G generated with constant edge probability p, the largest k such that G contains a K_k-subdivision is $(2 + o(1))\sqrt{n}$, where $n = |V(G)|$. An earlier result of Erdős [23] (see also Grimmett and McDiarmid [26] and Bollobás [8]) stated that for almost all such graphs, the chromatic number of G is at most $(1 + o(1))\frac{n}{\log_b(n)}$, where $b = \frac{1}{1-p}$. This proved that almost every graph is a counterexample to some case of Conjecture 2.2.14. In stark contrast, Bollobás, Catlin, and Erdős [10] showed that for almost all graphs G generated with constant edge-probability p, the largest k such that G contains a K_k-minor is at most $\frac{n}{\sqrt{\log_b(n)-1}}$. This implies that almost no graphs are counterexamples to Conjecture 2.2.13.

2.3 Immersion-Closed Families

In this section we present recent results on immersions and relate these results to the classical results on minors and subdivisions presented in Section 2.2. We begin by recalling the definition of an immersion.

Definition 2.1.1. Let G and H be graphs. A subgraph F of G is an *H-immersion* if there exists an injection $f\colon V(H) \to V(F)$ and a family of edge-disjoint walks $\mathcal{W}$ in F such that there exists a bijection $f'\colon E(H) \to \mathcal{W}$ with the property that for each $uv \in E(H)$, $f'(uv)$ is a walk joining $f(u)$ and $f(v)$.

In an immersion, the vertices in $f(V(H))$ are *branch vertices* of F. The injection f is the *vertex injection* of F, and the bijection f' is the *edge bijection* of F.

The study of immersions has blossomed due to the recent proof of the Nash-Williams Immersion Conjecture by Robertson and Seymour [57]. A family of graphs is *immersion-closed* if it is an ideal in $\mathcal{G}_i$.

Theorem 2.3.1 ([57]). *$\mathcal{G}_i$ is a well-partially-ordered class.*

Corollary 2.3.2. *For every immersion-closed family $\mathcal{F}$ in $\mathcal{G}_i$, the set of immersion-minimal non-members of $\mathcal{F}$ is finite.*

As with minor-closed families, Theorem 2.3.1 guarantees the existence of a finite forbidden immersion characterization for any immersion-closed family $\mathcal{F}$. However, Theorem 2.3.1 does not produce the immersion-minimal non-members of $\mathcal{F}$. In identifying the list of forbidden immersions for $\mathcal{F}$, one would expect to encounter the same difficulties that are encountered in identifying forbidden minors of minor-closed families. This remains to be seen, however, for there are very few immersion-closed families known. Below is a list of some families that are easily seen to be immersion-closed, along with their forbidden immersion characterizations. Algorithmic aspects of related immersion-closed families are studied in [39] and [64].

Proposition 2.3.3. *The family of graphs G such that $\Delta(G) < k$ is immersion-closed and is the family of graphs that do not contain a $K_{1,k}$-immersion. This family is also characterized by forbidding $K_{1,k}$ as a subgraph.*

Proposition 2.3.4. *The family of forests is immersion-closed and is the family of graphs that do not contain K_3-immersions. This family is also characterized by forbidding K_3-minors or K_3-subdivisions.*

Proposition 2.3.5. *The family of cacti is immersion-closed and is the family of graphs that do not contain a $(K_2 \vee \overline{K_2})$-immersion. This family is also characterized by forbidding $(K_2 \vee \overline{K_2})$-minors or $(K_2 \vee \overline{K_2})$-subdivisions.*

To generalize these families, define the *local edge-connectivity*, $\kappa'_G(U,V)$, of a pair of disjoint sets of vertices $\{U,V\}$ in a graph G to be the minimum number of edges whose deletion separates U and V. For a pair of vertices $\{u,v\}$, we write $\kappa'_G(u,v)$ for $\kappa'_G(\{u\},\{v\})$. Define the *maximum local edge-connectivity* of G to be the maximum local edge-connectivity of its pairs of vertices.

Let $\mathcal{K}_k$ be the family of graphs with maximum local-edge-connectivity at most k. A graph is a forest if and only if for each pair of vertices $\{u,v\}$, the deletion of at most one edge separates u and v, so the family of forests is $\mathcal{K}_1$. A graph is a cactus if and only if for any pair of vertices $\{u,v\}$ there is a set of at most two edges whose deletion separates u and v, so the family of cacti is $\mathcal{K}_2$. Next, we will show that $\mathcal{K}_k$ is immersion-closed for any k. The following lemma will be useful.

Lemma 2.3.6. *Let U and V be disjoint sets of vertices in a graph H such that $\kappa'_H(U,V) \geq k$. If a graph G contains an H-immersion F with vertex injection f, then $\kappa'_G(f(U),f(V)) \geq k$.*

Proof. Let f' be the edge bijection of F. By Menger's Theorem [51], there exists k edge-disjoint paths $P_1,\ldots,P_k$ joining U and V in H. For each j, the edges in $f'(E(P_j))$ are the edges of a walk joining $f(U)$ and $f(V)$ in F. For each j, define R_j to be a path joining $f(U)$ and $f(V)$ using only edges in $f'(E(P_j))$. As $P_1,\ldots,P_k$ are edge-disjoint, and the images of distinct edges in H are edge-disjoint walks in F, we conclude that the paths $R_1,\ldots,R_k$ are edge-disjoint paths in G joining $f(U)$ and $f(V)$. Now, any set of edges whose deletion separates $f(U)$ and $f(V)$ must contain an edge of each R_j, so $\kappa'_G(f(U),f(V)) \geq k$. □

Proposition 2.3.7. *The family $\mathcal{K}_k$ is immersion-closed and is also the family of graphs that do not contain a $(K_2 \vee \overline{K_k})$-immersion.*

Proof. Suppose there exist graphs G and H such that $G \in \mathcal{K}_k$ and $H \leq_i G$, but $H \notin \mathcal{K}_k$. Let F be an H-immersion in G with vertex bijection f. By definition, there are vertices u

and v in H such $\kappa'_H(u,v) \geq k+1$. By Lemma 2.3.6, $\kappa'_G(f(u), f(v)) \geq k+1$, however, this contradicts the definition of $\mathcal{K}_k$. We conclude that $\mathcal{K}_k$ is an immersion-closed family.

To see that $\mathcal{K}_k$ is characterized by forbidding $(K_2 \vee \overline{K_k})$-immersions, note that no graph in $\mathcal{K}_k$ may contain a $(K_2 \vee \overline{K_k})$-immersion, since $(K_2 \vee \overline{K_k}) \notin \mathcal{K}_k$. Let G be a graph not in $\mathcal{K}_k$. By definition, there are vertices u and v in G such that $\kappa'_G(u,v) \geq k+1$. By Menger's Theorem [51], there exists $k+1$ edge-disjoint paths $P_1, \ldots, P_{k+1}$ joining u and v in G. Let x_j be the vertex adjacent to u on P_j. The vertices $x_1, \ldots x_k$ are distinct, since the $k+1$ paths share no edges. At most one path has length 1; hence, we may assume without loss of generality that $x_j \neq v$ for $j \in [k]$.

Let $V(K_2 \vee \overline{K_k}) = \{u', v', x'_1, \ldots, x'_k\}$, where $\{x'_1, \ldots, x'_k\}$ is an independent set and u' and v' have degree $k+1$. We may define a vertex injection $f\colon V(K_2 \vee \overline{K_k}) \to V(G)$ by setting $f(u') = u$, $f(v') = v$, and $f(x'_j) = x_j$ for all $j \in [k]$. For each $j \in [k]$, define $f'(u'x'_j)$ to be the edge ux_j and $f(x'_jv')$ to be the x_jv-path $P_j - u$. Lastly, define $f'(u'v')$ to be the path P_{k+1}. By construction, f' is the edge bijection of a $(K_2 \vee \overline{K_k})$-immersion in G.

We conclude that $\mathcal{K}_k$ is precisely the family of graphs which do not contain a $(K_2 \vee \overline{K}_k)$-immersion. □

As noted in Propositions 2.3.4 and 2.3.5, $\mathcal{K}_1$ and $\mathcal{K}_2$ are also minor-closed. This is not the case for $\mathcal{K}_k$ when $k \geq 3$. For a graph G, the maximum local edge-connectivity is bounded by the maximum degree. Hence, for each $k \geq 3$, the family of cubic graphs is contained in the family $\mathcal{K}_k$. For each k, there exists a cubic graph containing a K_{k+2}-minor. However, the maximum local edge-connectivity of K_{k+2} is $k+1$, so $K_{k+2} \notin \mathcal{K}_k$. Therefore, $\mathcal{K}_k$ is not minor-closed when $k \geq 3$.

By Proposition 2.2.3, no outerplanar graph contains a $K_{2,3}$-subdivision and hence no outerplanar graph contains a $(K_2 \vee \overline{K_k})$-subdivision when $k \geq 3$. In Proposition 2.3.9 below, we will show for all k that some outerplanar graph contains a $(K_2 \vee \overline{K_k})$-immersion. As a consequence, $\mathcal{K}_k$ is not characterized by forbidding $(K_2 \vee \overline{K_k})$-subdivisions. Instead, forbidding $(K_2 \vee \overline{K_k})$-subdivisions characterizes the family of graphs with maximum local *vertex*-connectivity bounded by k. The proof is due to argument analogous to the proof of Proposition 2.3.7.

As we have seen, immersion-closed families need not be minor-closed. Conversely, minor-closed families need not be immersion-closed. The next two propositions show that minor-closed families can be ill-behaved with respect to the immersion order.

Proposition 2.3.8. *If H is a finite simple graph, then there exists a planar graph G such that G contains an H-immersion.*

Proof. Let H be embedded in the plane in such a way that no three edges cross at the same point. If two edges u_1v_1 and u_2v_2 cross, then at their intersection point place a vertex w and replace the edges u_1v_1 and u_2v_2 with the edges u_1w, wv_1, u_2w, and wv_2. Doing this for each pair of crossing edges produces a planar embedding of a graph G which is easily seen to contain an H-immersion. □

As a consequence of Proposition 2.3.8, the ideal in $\mathcal{G}_i$ generated by the family of planar graphs is the entire class of finite simple graphs. The same is true for the smaller minor-closed family of outerplanar graphs. The family of bipartite, Hamiltonian, outerplanar graphs is not minor-closed. Still, the ideal generated by this family in $\mathcal{G}_i$ is the entire class of finite simple graphs as well.

Proposition 2.3.9. *If H is a finite simple graph, then there exists a bipartite, Hamiltonian, outerplanar graph G such that G contains an H-immersion.*

Proof. Let $n = 2\lceil |V(H)|/2 \rceil$ so that n is even and at least as large as the number of vertices in H. We begin constructing G by arbitrarily identifying the vertices of H with distinct vertices in C_n via a vertex injection f. The vertices in $f(V(H))$ will become the branch vertices of an H-immersion in G, as indicated with circles in Figure 2.2. Now, we iteratively add paths which model the edges of H. If an edge uv is to be modeled, then the added path will bridge the edges around the exterior cycle to create a path joining $f(u)$ and $f(v)$. As depicted by the gray path in Figure 2.2, if each edge of the exterior cycle is bridged by a path with two added vertices, then the resulting graph remains bipartite, Hamiltonian, outerplanar, and in the end contains an H-immersion. □

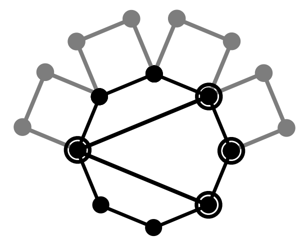


Figure 2.2: A K_4-immersion in a bipartite, Hamiltonian, outerplanar graph G.

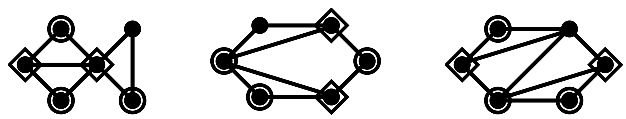
Instead of asking whether a family of graphs is immersion-closed, one can determine the family characterized by forbidding H-immersions for some graph H. The family of graphs that do not contain a K_4-immersion is not well understood. However, there is a linear-time algorithm that determines whether a graph contains a K_4-immersion and, if so, identifies a K_4-immersion [13]. In the next theorem we determine the family of graphs that contain no $K_{2,3}$-immersion. As a corollary, we determine the family of graphs characterized by forbidding both $K_{2,3}$-immersions and K_4-immersions.

The graphs H_1, H_2, and H_3 in Figure 2.3 each contain $K_{2,3}$-immersions. To identify the branch vertices of the $K_{2,3}$-immersion in each, consider the following lemma.

Lemma 2.3.10. *Let F be a smallest H-immersion in a graph G, and let the vertex injection of F be f. For each vertex $a \in V(H)$, the degree of a in H and the degree of $f(a)$ in F have the same parity. Also, a vertex of odd degree in F must be a branch vertex of F.*

Proof. The walks in $\{f'(e)\colon\ e \in E(H)\}$ cover the edges of F, so the edges incident to a vertex x in F start walks from x or occur in pairs in walks passing through x. □

Let u_1 and u_2 be the vertices of degree 3 in $K_{2,3}$, and let v_1, v_2, and v_3 be the vertices of degree 2 in $K_{2,3}$. In each graph in Figure 2.3 there are two vertices of odd degree, indicated with squares. Lemma 2.3.10 suggests that these vertices are the branch vertices $f(u_1)$ and $f(u_2)$. If the branch vertices $f(v_1), f(v_2)$, and $f(v_3)$ are the vertices indicated with circles, then in each immersion, $f'(u_1v_1), f'(u_1v_2), f'(u_2v_1)$, and $f'(u_2v_2)$ cover the edges of a cycle containing both $f(u_1)$ and $f(u_2)$, while $f'(u_1v_3)$ and $f'(u_2v_3)$ cover the remaining edges.



H_1 H_2 H_3

Figure 2.3: $K_{2,3}$-immersions.

Let G be a graph that contains no $K_{2,3}$-immersion. By the transitivity of the immersion order, G cannot contain an H_1-subdivision, an H_2-subdivision, or an H_3-subdivision. If a block B of G is not outerplanar, then B must contain a K_4-subdivision, since it does not contain a $K_{2,3}$-immersion. The graph H obtained by subdividing an edge of K_4 contains $K_{2,3}$ as a subgraph, so B must in fact be K_4 itself. We next show that these necessary conditions characterize the graphs that do not contain $K_{2,3}$-immersions.

Theorem 2.3.11. *A graph G contains no $K_{2,3}$-immersion if and only if G contains no H_j-subdivision for $j \in [3]$, and each block of G is outerplanar or is isomorphic to K_4.*

Proof. Let G be a graph that contains a $K_{2,3}$-immersion but contains no H_j-subdivision for $j \in [3]$, and such that each block of G is outerplanar or is isomorphic to K_4. Deleting an edge from each block of G isomorphic to K_4 produces an outerplanar graph G'. The unbounded face in an outerplanar embedding of G' contains the edges of a Hamiltonian cycle of each 2-connected block of G'. Each of these cycles is a Hamiltonian cycle of the corresponding block in G. By a "chord in a block B", we mean a chord of this Hamiltonian cycle of B. A *leaf-block* of a graph G is a block that contains one cut-vertex of G. Let u_1 and u_2 be the vertices of degree 3 in $K_{2,3}$, and let v_1, v_2, and v_3 be the vertices of degree 2 in $K_{2,3}$. Let F be a smallest $K_{2,3}$-immersion in G, and let the vertex injection and edge bijection of F be f and f', respectively.

There are no cut-edges in $K_{2,3}$, so Lemma 2.3.6 gives that the branch vertices of F are contained in a 2-edge-connected subgraph F' of F. No path joining branch vertices can leave F' via a cut-edge and return; hence, no cut-edge can be traversed by a walk modeling an edge of $K_{2,3}$. Now, there can be no cut-edge in F since F is a smallest $K_{2,3}$-immersion in G. We conclude that each block of F is 2-connected, since every block of a 2-edge-connected graph is 2-connected.

A cut-vertex of F cannot be incident to a chord in any block, since then F would contain an H_1-subdivision. Each block of F is 2-connected, so each cut-vertex x of F therefore has degree 2 in each of the blocks containing it, giving x even degree in F. Let B be the block containing $f(u_1)$. By Lemma 2.3.10, $f(u_1)$ must have odd degree so B has at least one chord. As depicted in Figure 2.4, no vertex has degree more than 4 in B, since then G would contain an H_2-subdivision. Deleting the edges of the Hamiltonian cycle of B leaves a graph B' with maximum degree at most 2 whose edges are precisely the chords in B. There can be no cycle in B', for G contains no H_2-subdivision, as depicted in Figure 2.4.

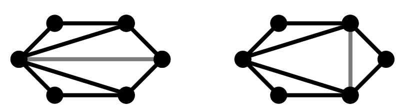
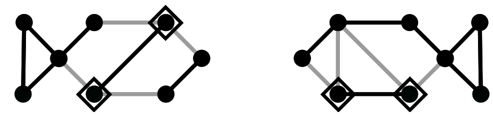


Figure 2.4: Deleting the gray edge produces an H_2-subdivision.

By Lemma 2.3.10, there are exactly two vertices of odd degree in B so B' consists of a single path P and isolated vertices. The path P consists of at most two edges, since F contains no H_3-subdivision. If P has two edges, then the ends of P must be consecutive around the Hamiltonian cycle of B, since G contains no H_2-subdivision. In this case, and in the case where P has one edge, $f(u_1)$ and $f(u_2)$ are adjacent vertices of degree 3 in B which cannot be cut-vertices by Lemma 2.3.10, since all cut-vertices have even degree in F.



P has one edge $\quad$ P has two edges

Figure 2.5: $f(u_1)$ and $f(u_2)$ are indicated by squares

As indicated in Figure 2.5, there are four edges whose deletion separates $\{f(u_1), f(u_2)\}$ from the rest of F. This contradicts Lemma 2.3.6, since if $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, v_3\}$, then $\kappa'_{K_{2,3}}(U, V) = 6$. $\square$

Corollary 2.3.12. *A graph G contains no $K_{2,3}$-immersion and no K_4-immersion if and only if G is outerplanar and contains no H_j-subdivision, for all $j \in [3]$.*

Proof. A graph that does not contain a $K_{2,3}$-immersion or a K_4-immersion cannot contain a $K_{2,3}$-subdivision or a K_4-subdivision and must be outerplanar. Now G cannot contain an H_j-subdivision for any $j \in [3]$ as noted before. For sufficiency, note that if G is an outerplanar graph that contains no H_j-subdivision for $j \in [3]$, then, by Theorem 2.3.11, G contains no $K_{2,3}$-immersion. Hence, we may assume that G contains a K_4-immersion. Let F be the smallest K_4-immersion in G, and let f be the vertex injection of F.

If there is a cut-vertex x separating two branch vertices a and b, then there are two edges incident to x whose deletion separates a and b, since cut-vertices cannot be incident to chords in any block. This contradicts Lemma 2.3.6, since K_4 is 3-connected, so all of the branch vertices of F must be contained in the same block. Let B be the block containing the four branch vertices of F. Since B is an outerplanar block, we may fix an outerplanar embedding $\mathcal{E}$ of B; let a chord in B mean a chord of the unbounded face in $\mathcal{E}$. Because G contains no H_2-subdivision, Lemma 2.3.10 yields that the chords in B form the the edge set of two paths P_1 and P_2 whose leaves are the branch vertices of F. No edge of P_1 may intersect an edge of P_2 in $\mathcal{E}$, so there must exist two edges of the Hamiltonian cycle of B whose deletion separates the branch vertices in $V(P_1)$ from those in $V(P_2)$. This contradicts Lemma 2.3.6, since K_4 is 3-edge-connected. □

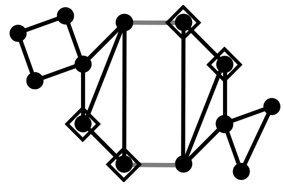


Figure 2.6: Deleting the gray edge separates the branch vertices, indicated with squares.

2.4 Extremal Problems

We now turn our attention to extremal results. A *split graph* is a graph whose vertices can be partitioned into two sets Q and A such that Q is a clique and A is an independent set. Let $\mathcal{T}_k$ be the family of split graphs whose clique has size $k-1$ and whose remaining vertices have degree $k-2$. If F is a K_k-immersion, then the k branch vertices of F must have degree at least $k-1$. If $T \in \mathcal{T}_k$, then T has at most $k-1$ vertices of degree at least $k-1$. We conclude that T does not contain a K_k-immersion. Note that $|E(T)| = (k-2)|V(T)| - \binom{k-1}{2}$ for each $T \in \mathcal{T}_k$.

Recall that if F is an H-subdivision, then F is also an H-immersion. Hence, by Theorem 2.2.10, an n-vertex graph with no K_5-immersion cannot have more than $3n-6$ edges. A graph $T \in \mathcal{T}_5$ has $3|V(T)| - 6$ edges, so this bound is tight. We will show in Theorem 2.4.3 that the graphs in $\mathcal{T}_5$ are the only graphs achieving this bound. DeVos, Kawarabayashi, Mohar, and Okamura recently reproved [19] a result of Lescure and Meyniel [44,45] that will be helpful in proving this characterization.

Theorem 2.4.1 ([19,44,45]). *Fix $k \leq 7$. If G is a graph with minimum degree at least $k-1$, then G contains a K_k-immersion.*

Corollary 2.4.2. *For $k \leq 7$, if G is an n-vertex graph with no K_k-immersion, then G has at most $(k-2)n - \binom{k-1}{2}$ edges.*

Proof. Theorem 2.4.1 implies that a graph with no K_k-immersion must be $(k-2)$-degenerate. The result now follows from the fact that an n-vertex $(k-2)$-degenerate graph has at most $(k-2)(n-(k-1)) + \binom{k-1}{2}$ edges. $\square$

The next theorem implies that if $k \in \{5,6,7\}$, then the graphs in $\mathcal{T}_k$ are the only graphs with no K_k-immersions having this extremal number of edges. Note that an n-vertex graph with $(k-2)n - \binom{k-1}{2}$ edges is a member of $\mathcal{T}_k$ if and only if it has $n-k+1$ independent vertices of degree $k-2$. The only graph in $\mathcal{T}_k$ having $n-k+2$ independent vertices of degree $k-2$ is $K_{k-2} \vee \overline{K_{n-k+2}}$.

Theorem 2.4.3. *Let $k \in \{5,6,7\}$. If G is an n-vertex graph with $(k-2)n - \binom{k-1}{2}$ edges that contains no K_k-immersion, then G has $n-k+1$ independent vertices of degree $k-2$, and $G \in \mathcal{T}_k$.*

Proof. We use induction on n. If $n = k$, then $(k-2)n - \binom{k-1}{2} = \binom{k}{2} - 1$. Now, G must be the graph obtained by deleting an edge from K_k, and this graph is in $\mathcal{T}_k$. For some $n \geq k$, let us assume that there are $n-k+1$ independent vertices of degree $k-2$ in all n-vertex graphs with $(k-2)n - \binom{k-1}{2}$ edges that contain no K_k-immersion. Let G be an $(n+1)$-vertex graph with $(k-2)(n+1) - \binom{k-1}{2}$ edges that contains no K_k-immersion. By Theorem 2.4.1, G has a vertex u with degree at most $k-2$. The degree of u cannot be less than $k-2$, since then $G-u$ is an n-vertex graph with more than $(k-2)n - \binom{k-1}{2}$ edges and by Corollary 2.4.2 must contain a K_k-immersion. We conclude that the degree of u is exactly $k-2$. Now $G-u$ is an n-vertex graph with $(k-2)n - \binom{k-1}{2}$ edges that does not contain a K_k-immersion, so by the inductive hypothesis, $G - u \in \mathcal{T}_k$. We will finish the proof by studying the neighborhood of u in G.

If v and w are non-adjacent vertices in the neighborhood of u, then the graph G^* obtained from $G-u$ by adding the edge vw is an n-vertex graph with $(k-2)n - \binom{k-1}{2} + 1$ edges. By Corollary 2.4.2, this graph contains a K_k-immersion F. The edge vw must be an edge of F, lest $F \subseteq G - u$. As depicted in Figure 2.7, by replacing the edge vw with the edges vu and uw, we obtain a K_k-immersion F' in G, a contradiction. We conclude that the neighborhood of u is complete.

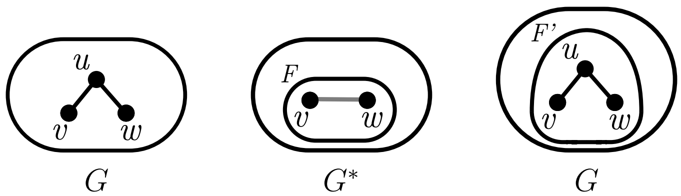


Figure 2.7: F is a K_k-immersion in G^*; F' is a K_k-immersion in G

If $G - u = K_{k-2} \vee \overline{K_{n-k+2}}$, then let V be the set of $n - k + 2$ independent vertices of degree $k - 2$ in $G - u$. The neighborhood of u in G contains at most one vertex of V, since this neighborhood is complete. If the neighborhood of u in G does not contain any vertex of V, then $G = K_{k-2} \vee \overline{K_{n-k+3}}$, and $G \in \mathcal{T}_k$. If the neighborhood of u contains a vertex x in V, then $V - x + u$ is a set of $n - k + 2$ independent vertices of degree $k - 2$ in G. Since G is an $(n+1)$-vertex graph with $(k-2)(n+1) - \binom{k-1}{2}$ edges, the existence of such an independent set requires that $G \in \mathcal{T}_k$.

If $G - u \neq K_{k-2} \vee \overline{K_{n-k+2}}$, then let V be the set of $n - k + 1$ independent vertices of degree $k - 2$ in $G - u$. Let $W = \{w_1, \ldots, w_{k-1}\}$ be the set of vertices of $G - u$ not in V. Note that W is a $(k-1)$-clique in $G - u$ and each vertex in W is adjacent to some vertex in V, lest $G - u = K_{k-2} \vee \overline{K_{n-k+2}}$. All but at most one vertex in the neighborhood of u in G are contained in W, since this neighborhood is complete. Let v be a vertex in V that is adjacent to u in G and let the non-neighbor of v in W be w_{k-1}. Note that $W \cup \{v\}$ induces the graph obtained by deleting one edge from K_k, where the edge missing is vw_{k-1}. If G contains no K_k-immersion, then every path from v and w_{k-1} uses some edge in this induced subgraph. Let v' be a vertex in V adjacent to w_{k-1}. Note that u is adjacent to $k-3$ vertices of W, while v' is adjacent to $k - 2$ vertices of W. Because $|W| = k - 1$ and $k \geq 5$, u and v' must share a neighbor w_j in W. Now $\{vu, uw_j, w_jv', v'w_{k-1}\}$ is the edge set of a path P

joining v and w_{k-1} using no edges in the graph induced by $W \cup \{v\}$, yielding a K_k-immersion in G. We conclude that the neighborhood of u is contained entirely in W, in which case $G \in \mathcal{T}_k$. □

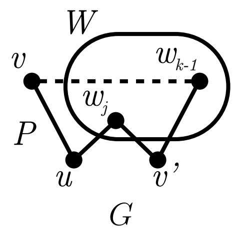


Figure 2.8: The path P models the missing edge in a K_k-immersion in G.

The immersion order analogue of Conjectures 2.2.13 and 2.2.14 is accredited to Abu-Khzam and Langston [1].

Conjecture 2.4.4 ([1])**.** *If G is not $(k-1)$-colorable, then G contains a K_k-immersion.*

Lescure and Meyniel posed a similarly worded conjecture [45]. In their definition of an H-immersion the walk modeling an edge uv may not pass through any branch vertices other than $f(u)$ and $f(v)$. This is a more restrictive definition so their conjecture is stronger than Conjecture 2.4.4. Lescure and Meyniel noted that a graph that is not $(k-1)$-colorable must contain a subgraph with minimum degree $k-1$. When $k \leq 7$, this subgraph must contain a K_k-immersion by Theorem 2.4.1, proving Conjecture 2.4.4. Abu-Khzam and Langston [1] proved that for any fixed k, an immersion-minimal counterexample to Conjecture 2.4.4 must be 4-connected and k-edge-connected.

Recently, DeVos, Dvořák, Fox, McDonald, Mohar, and Sheide [18] proved an immersion order analogue of Theorems 2.2.11 and 2.2.12.

Theorem 2.4.5 ([18])**.** *If G has minimum degree at least $200k$, then G contains a K_k-immersion.*

Although not mentioned explicitly in [18], Theorem 2.4.5 supports the validity of Conjecture 2.4.4 in a very strong way. This theorem easily implies that a graph with average degree at least $400k$ contains a K_k-immersion. For $0 < p < 1$, Chebyshev's inequality gives that almost every graph generated with constant edge probability p has $(1 + o(1))\left(\frac{p}{2}\right)n^2$ edges, where $n = |V(G)|$. By Theorem 2.4.5, almost every such graph contains a K_t-immersion, where $t = (1 + o(1))\left(\frac{p}{800}\right)n$. For almost every such graph G, the chromatic number of G is at most $(1 + o(1))\frac{n}{\log_b(n)}$, where $b = \frac{1}{1-p}$. Hence, almost no graphs are counterexamples to Conjecture 2.4.4, as was the case for Conjecture 2.2.13.

Chapter 3

Rainbow Subgraphs

3.1 Introduction

Given an edge-colored graph, a *rainbow subgraph* is a subgraph whose edges have distinct colors. These subgraphs have also been called *heterochromatic*, *polychromatic*, or *totally multicolored*, but "rainbow" is the most common term. For a vertex v in an edge-colored graph G, the *color degree* is the number of different colors on edges incident to v; we use the notation $\widehat{d}_G(v)$ for this. The *minimum color degree* and *maximum color degree* of G are the minimum and maximum of these values, respectively. The minimum color degree of G will be denoted $\widehat{\delta}(G)$.

A special case of Ramsey's Theorem states that when the edges of a sufficiently large complete graph K_R are each given one of a fixed number of colors, there must be a complete subgraph of fixed size whose edges all receive the same color [55]. These monochromatic subgraphs can be thought of as exhibiting order no matter how disordered the edge-coloring of K_R may be. Motzkin described this phenomenon with the phrase "complete disorder is impossible" [25].

In *rainbow Ramsey theory* one wishes to show that if the distribution of colors in an edge-colored graph satisfies some condition, then there must be a *rainbow* subgraph with some given structure. In contrast with monochromatic subgraphs, these rainbow subgraphs

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exhibit "disorder." Due to this, Jungić, Nešetřil, and Radoičić described the goal of rainbow Ramsey theory with the phrase "complete disorder is unavoidable as well" [34].

In Section 3.3 the rainbow subgraphs we will investigate are rainbow matchings. In Section 3.4 we seek to partition the edge set of a graph into rainbow matchings. In Section 3.2 we seek to cover the vertices of a graph with a family of rainbow stars. A survey of results on rainbow subgraphs, partitions into rainbow subgraphs, and coverings by rainbow subgraphs appears in [35].

3.2 Rainbow Domination

A *dominating set* in a graph G is a set S of vertices such that each vertex in $V(G) - S$ is a neighbor of a vertex in S. The *domination number*, $\gamma(G)$, is the minimum size of a dominating set in G. For a vertex $v \in V(G)$, let R_v be the star whose center is v and whose leaves are the vertices in the neighborhood of v. We say that a vertex u is *dominated* by v if $u \in V(R_v)$. A set S of vertices of G is a dominating set if and only if the family of stars $\{R_v\colon\ v \in S\}$ covers the vertex set of G, so an equivalent definition of the domination number of G is that it is the minimum size of a family of disjoint stars in G that cover $V(G)$.

Many results on domination in graphs give bounds on the domination number in terms of other graph parameters. In this section, we will discuss bounds on $\gamma(G)$ in terms of the number of vertices in a graph and its minimum and maximum degree. Two easy such bounds are given in the next propositions.

Proposition 3.2.1. *If G is a n-vertex graph, then $\gamma(G) \leq n - \Delta(G)$.*

Proposition 3.2.2. *If G is an n-vertex graph with no isolated vertices, then $\gamma(G) \leq n/2$.*

Proposition 3.2.1 is attributed to Berge [7]. To obtain a dominating set of size $n - \Delta(G)$, construct a set containing a vertex v of maximum degree and all vertices that are not adjacent to v. Proposition 3.2.2 is attributed to Ore [52]. The next proposition will provide a proof of Proposition 3.2.2 and will be useful in proving a generalized version of Proposition 3.2.2 as well.

Proposition 3.2.3. *If G is a graph with no isolated vertices, then there is a family $\mathcal{F}$ of disjoint stars covering the vertices of G such that each $F \in \mathcal{F}$ has at least two vertices.*

Proof. Let T be a spanning forest. Let v be a non-leaf vertex of T that is adjacent to a leaf. To construct the desired family of stars in G, add a star F centered at v whose leaves are the leaves of T adjacent to v. Note that F has at least two vertices and that deleting the vertices of F from T cannot leave isolated vertices, since these vertices would have been leaves in T adjacent to v. Now, we may construct the desired family of stars by iteratively adding the star F to $\mathcal{F}$ and applying the same procedure to $T - V(F)$. □

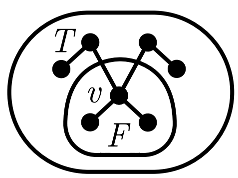

Figure 3.1: Iteratively adding F to $\mathcal{F}$ produces the desired family.

The bounds in Propositions 3.2.1 and 3.2.2 are tight. For graphs H and F, let $H \circ F$ be the graph obtained from the disjoint union of H and $|V(H)|$ copies of F indexed by the vertices of H by making a vertex $v \in V(H)$ adjacent to each vertex in F_v. The resulting graph is called the *corona* of H by F. In $H \circ K_1$ the distance between vertices in distinct copies of K_1 is at least 3, so no two such vertices can be dominated by a single vertex. Any dominating set must have at least $|V(H)|$ vertices and $|V(H)| = |V(H \circ K_1)|/2$. We conclude that if H is connected, then $\gamma(H \circ K_1) = |V(H \circ K_1)|/2$. If H has a vertex of degree $|V(H)| - 1$, then $\Delta(H \circ K_1) = |V(H)| = |V(H \circ K_1)|/2$, so $\gamma(H \circ K_1) = |V(H \circ K_1)| - \Delta(H \circ K_1)$.

The coronas described above offer an almost complete characterization of the graphs G such that $\gamma(G) = |V(G)|/2$. The only connected graph with this property that is not a corona is C_4. This characterization was proved indepently by Payan and Xuong [54] and by Fink, Jacobson, Kinch, and Roberts [24].

An upper bound on the domination number in terms of the minimum degree of a graph was proved independently by Arnautov [6], Payan [53], and Lovász [47]. Later, Alon and Spencer gave a probabilistic proof [3].

Theorem 3.2.4 ([3, 6, 47, 53])**.** *If G is an n-vertex graph, then $\gamma(G) \le \frac{n(1+\ln(\delta(G)+1))}{\delta(G)+1}$.*

This bound has been shown to be tight asymptotically as $\delta(G) \to \infty$. Many authors refer to Alon's "Transversal numbers of uniform hypergraphs" [2] for a proof of the tightness of Theorem 3.2.4. To translate the construction in [2] into the tightness of Theorem 3.2.4, the hyperedges would have to be realizable as the closed neighborhoods of the vertices of a graph. This would require the hypergraphs constructed in [2] to have symmetric incidence matrices and the same number of vertices as hyperedges, but they do not. A more appropriate reference showing the tightness of Theorem 3.2.4 is "High degree graphs contain large-star factors" by Alon and Wormald [4]. In [4] it is shown that if c is fixed and less than 1, and d is sufficiently large, then the expected number of dominating sets of size $(1+o(1))\frac{c\ln d}{d}n$ in a random d-regular n-vertex graph tends to 0 as $n \to \infty$. Since $\frac{1+\ln(d+1)}{d+1} \sim \frac{\ln d}{d}$, with high probability $\gamma(G) = (1+o(1))\frac{n(1+\ln(\delta(G)+1))}{\delta(G)+1}$ for such graphs.

Generalizations of dominating sets and the domination number abound in the literature. In 1990, Hedetniemi and Laskar published the article "Bibliography on domination in graphs and some basic definitions of domination parameters" [31] containing nothing but a brief history, a list of definitions, and 403 references published since 1950. In 1998, the book "Fundamentals of domination in graphs" [30] cites 1,222 references. In 2011, MathSciNet queries find over 1,880 graph theory articles with the words "domination" or "dominating" in the title. Here we discuss a generalization in which we wish to cover the vertices of an edge-colored graph using a family of rainbow stars. We will not allow two rainbow stars in such a family to have the same center, hence we may require these rainbow stars to be disjoint. More formally, we have the following definition.

Definition 3.2.5. Let G be an edge-colored graph. A *rainbow dominating set* in G is a set of vertices containing a central vertex of each star in a family of disjoint rainbow stars covering the vertex set of G. The *rainbow domination number* of G is the minimum size of a rainbow dominating set in G and is denoted $\widehat{\gamma}(G)$. Equivalently, $\widehat{\gamma}(G)$ is the minimum size of a family of disjoint rainbow stars covering the vertices of G.

Observe that if G is an edge-colored graph, then we can cover $\widehat{d_G}(v) + 1$ vertices of G with a rainbow star centered at v. The remaining vertices of G can serve as the centers of degenerate rainbow stars (isomorphic to K_1). This yields the following generalization of Proposition 3.2.1.

Proposition 3.2.6. *If G is an edge-colored n-vertex graph with maximum color degree k, then $\widehat{\gamma}(G) \leq n - k$.*

If G is properly edge-colored, then every star in G is a rainbow star and $\widehat{\gamma}(G) = \gamma(G)$. Also, $\widehat{d_G}(v) = d_G(v)$ for every vertex v, so Proposition 3.2.1 is the special case of Proposition 3.2.6 applied to properly edge-colored graphs. Graphs witnessing the tightness of Proposition 3.2.1 will witness the tightness of Proposition 3.2.6 when their edges are properly colored, but in the next example we construct graphs witnessing the tightness of Proposition 3.2.6 whose edges are far from properly colored.

Example 3.2.7. Fix k, and let G be the split graph $K_k \vee \overline{K_{n-k}}$. Let W be the k-clique and let V be the independent set of size $n-k$. For each $w \in W$ and $v \in V$, give the edge wv color β_w. To complete the edge-coloring of G, give the edges of the complete subgraph induced by W distinct colors that have not already been used. In Figure 3.2, the edges induced by W have been left out for clarity.

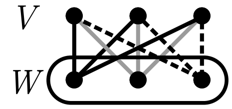

Figure 3.2: The graph G has been edge-colored according to the rules above

By construction, G is an edge-colored graph with minimum and maximum color degree k. No two vertices of V may be covered by the same rainbow star in G since the distance between two such vertices is 2 and any path of length 2 joining them has been colored monochromatically. We conclude that the minimum number of disjoint rainbow stars necessary to cover the vertices of G is at least $|V|$, so $\widehat{\gamma}(G) \geq n-k$. By Proposition 3.2.6, equality holds.

As a consequence of Example 3.2.7, the rainbow domination number of edge-colored graphs cannot be bounded by a constant fraction of the number of vertices. Proposition 3.2.2 and Theorem 3.2.4 do not directly translate to bounds on the rainbow domination number of general edge-colored graphs as Proposition 3.2.1 did. To obtain analogues of Proposition 3.2.2 and Theorem 3.2.4, we must impose restrictions on the edge-colored graphs considered.

The graphs constructed in Example 3.2.7 are far from properly edge-colored; each vertex in the clique is incident to $n-k$ edges having the same color. If the edge-coloring is closer to a proper edge-coloring, then generalizations of Proposition 3.2.2 and Theorem 3.2.4 can be obtained. To measure how close an edge-coloring is to being a proper edge-coloring we count the maximum number of edges of the same color incident to the same vertex. Consider the following definition.

Definition 3.2.8. An edge-colored graph G is *d-tolerant* if no vertex in G is incident to more than d edges having the same color. Equivalently, G does not contain a monochromatic copy of $K_{1,d+1}$.

Heuristically, the vertices of a d-tolerant edge-colored graph can "tolerate" being incident to d edges of the same color, but not $d+1$. Some results in rainbow Ramsey theory impose that in the edge-colored graphs considered no color is used more than a fixed number of times in total [35]. This global condition is stronger than the local condition of being d-tolerant. Hakimi and Kariv [29] studied edge-colorings satisfying a more general constraint. For a function $m\colon V(G) \to \mathbb{N}$, an edge-coloring is m-tolerant if no vertex v is incident to more than $m(v)$ edges having the same color. Hakimi and Kariv [29] studied a generalization of the edge-chromatic number in which they sought to determine the least k such that G has an m-tolerant k-edge-coloring.

Proposition 3.2.9. *Fix $d \geq 1$. If G is an n-vertex d-tolerant edge-colored graph with no isolated vertices, then $\widehat{\gamma}(G) \leq \left(\frac{d}{d+1}\right) n$.*

Proof. By Proposition 3.2.3, there exists a family $\mathcal{F}$ of disjoint stars covering the vertices of G such that each $F \in \mathcal{F}$ has at least 2 vertices. If F is a star in $\mathcal{F}$ with central vertex v_F, then a largest *rainbow* star contained in F has $\widehat{d}_F(v_F)$ leaves. Let $\mathcal{F}'$ be the family of rainbow stars constructed by choosing a largest rainbow substar inside each member of $\mathcal{F}$. Let $s = \sum_{F \in \mathcal{F}} \widehat{d}_F(v_F)$. If $\mathcal{F}$ consists of k stars, then $\mathcal{F}'$ covers $k + s$ vertices with k rainbow stars. To construct a family $\widehat{\mathcal{F}}$ of rainbow stars that covers all vertices of G, extend $\mathcal{F}'$ by adding each vertex not already covered by a rainbow star in $\mathcal{F}'$ as a 1-vertex star. Since, s vertices come "for free" in $\mathcal{F}'$, we have $|\widehat{\mathcal{F}}| = n - s$.

Note that $k \leq s$, since the center of each star $F \in \mathcal{F}$ has color degree at least 1. Each color counted by $\widehat{d}_F(v_F)$ appears on at most d edges of F, so $|V(F)| \leq d \cdot \widehat{d}_F(v_F) + 1$. Summing over the k stars in $\mathcal{F}$ yields $n \leq ds + k \leq (d+1)s$. Now $s \geq n/(d+1)$ and $\widehat{\gamma}(G) \leq |\widehat{\mathcal{F}}| = n - s \leq \left(\frac{d}{d+1}\right) n$. □

An edge-colored graph is properly edge-colored if and only if it is 1-tolerant. We have seen that if G is properly edge-colored, then $\widehat{\gamma}(G)$ is the same as $\gamma(G)$. Thus Proposition 3.2.9 is a generalization of Proposition 3.2.2. When properly edge-colored, the coronas constructed to witness the tightness of Proposition 3.2.2 will witness the tightness of Proposition 3.2.9. For

$d > 1$, a monochromatically edge-colored $K_{1,d}$ will witness the tightness of Proposition 3.2.9. More generally, we define a class of edge-colored graphs which includes all properly edge-colored coronas and each monochromatically edge-colored $K_{1,d}$ and show that this class characterizes the edge-colored graphs witnessing the tightness of Proposition 3.2.9, with one exception for each of $d = 1$ and $d = 2$.

Example 3.2.10. For a graph H and a fixed positive integer d, call $H \circ dK_1$ the d-corona of H. The graph $H \circ dK_1$ is obtained by adding matchings between the vertices of H and each of d independent sets $A_1, \ldots, A_d$ of $|V(H)|$ vertices each. We will say that an edge-colored d-corona $H \circ dK_1$ is a *d-flare* if it is d-tolerant and for each vertex $v \in V(H)$, the edges joining v to its d neighbors in $\bigcup_{i=1}^{d} A_i$ are given the same color. Figure 3.3 depicts a 2-flare and a 3-flare. Note that in a d-flare the color on the edges joining a vertex v of H to its neighbors in $\bigcup_{i=1}^{d} A_i$ may not appear on any edge of H incident to v, but it may appear elsewhere in H.

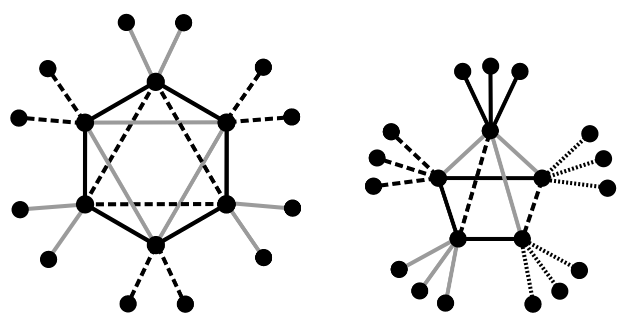

Figure 3.3: A 2-flare and a 3-flare.

In a d-flare $H \circ_d K_1$, no two vertices of $\bigcup_{i=1}^{d} A_i$ may be covered by the same rainbow star since these vertices are either at distance 3 or the only path of length 2 joining them has been colored monochromatically. There are $d|V(H)|$ vertices in $\bigcup_{i=1}^{d} A_i$ and $(d+1)|V(H)|$ vertices in $H \circ_d K_1$, so $\widehat{\gamma}(H \circ dK_1) \geq d|V(H)| = \left(\frac{d}{d+1}\right) |V(H \circ_d K_1)|$. Equality holds by Proposition 3.2.9.

In the next theorem we show that, with the exception of a properly edge-colored C_4

and a monochromatically edge-colored C_3, all connected graphs witnessing the tightness of Proposition 3.2.9 are d-flares. The case $d = 1$ gives the characterization of graphs witnessing the tightness of Proposition 3.2.2 originally proved by Payan and Xuong [54] and by Fink, Jacobson, Kinch, and Roberts [24].

Theorem 3.2.11. *Let G be an n-vertex d-tolerant edge-colored graph with no isolated vertices. If $\widehat{\gamma}(G) = \left(\frac{d}{d+1}\right) n$ and D is a component of G, then D is*

(a) *a d-flare, or*

(b) *a monochromatically edge-colored C_3 if $d = 2$, or*

(c) *a properly edge-colored C_4 if $d = 1$.*

Proof. It will suffice to prove the theorem for connected graphs. Let T be a spanning tree of G, and let v be a vertex that is not a leaf of T. If v is not adjacent to a leaf in T, as in Figure 3.4, then let $C_1, \ldots, C_r$ be the components of $T - v$. We show first that $|V(C_i)| \equiv 0 \pmod{d+1}$ for $i \in [r]$. Otherwise, $\widehat{\gamma}(C_i) \leq \left\lfloor \left(\frac{d}{d+1}\right) |V(C_1)| \right\rfloor < \left(\frac{d}{d+1}\right) |V(C_1)|$ by Proposition 3.2.9, which yields $\widehat{\gamma}(G) \leq \widehat{\gamma}(C_i) + \widehat{\gamma}(G - V(C_i)) < \left(\frac{d}{d+1}\right) |V(C_i)| + \left(\frac{d}{d+1}\right) (|V(G)| - |V(C_i)|) = \left(\frac{d}{d+1}\right) n$. Now $|V(G)| \equiv 1 + \sum_{i=1}^{r} |V(C_i)| \equiv 1 \pmod{d+1}$, which yields $\widehat{\gamma}(G) < \left(\frac{d}{d+1}\right) n$ by Proposition 3.2.9. We conclude that every non-leaf vertex of T must be adjacent to a leaf in T.

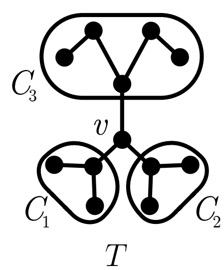

Figure 3.4: The vertex v is not adjacent to a leaf in the spanning tree T.

Let v be adjacent to ℓ leaves in T; thus $T - v$ has ℓ isolated vertices and components $C_1, \ldots, C_r$, each having at least 2 vertices, as in Figure 3.5. Let k be the number of distinct

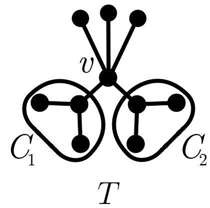

Figure 3.5: The vertex v is adjacent to ℓ leafs in the spanning tree T.

colors on edges joining v and the isolated vertices of $T-v$. Note that $k \geq 1$ and that $k \geq \ell/d$ since each color appears on at most d of these edges. By the definition of k, there is a rainbow star F with $k+1$ vertices centered at v whose leaves are isolated vertices in $T-v$. Taking the $\ell-k$ isolated vertices in $T-v$ not covered by F as degenerate rainbow stars, we find that $\widehat{\gamma}(G) \leq \sum_{i=1}^{r} \widehat{\gamma}(V(C_i)) + \ell - k + 1$. By Proposition 3.2.9, $\widehat{\gamma}(V(C_i)) \leq \left(\frac{d}{d+1}\right) |V(C_i)|$ for each $i \in [r]$. Summing these inequalities gives $\sum_{i=1}^{r} \widehat{\gamma}(V(C_i)) \leq \left(\frac{d}{d+1}\right)(n-\ell-1)$, so $\widehat{\gamma}(G) \leq \left(\frac{d}{d+1}\right)(n-\ell-1)+\ell-k+1$. We conclude that $\left(\frac{d}{d+1}\right)(n-\ell-1)+\ell-k+1 \geq \left(\frac{d}{d+1}\right) n$, which yields $k \leq \frac{\ell+1}{d+1}$. Recalling that $k \geq \ell/d$, we find that $\ell \leq d$. On the other hand, $k \geq 1$ yields $d \leq \ell$. We conclude that $\ell = d$ and that $k = 1$.

Since T and v are arbitrary, we have shown that in every spanning tree T, every non-leaf vertex v is adjacent to exactly d leaves, and the edges joining these leaves to v all have the same color. Thus every spanning tree is a d-flare. If for some fixed spanning tree T there are no edges in G joining leaves of T, then G is a d-flare as well. It remains to show that for some fixed spanning tree T, there can be no edges in G joining leaves of T, unless $G = C_3$ or $G = C_4$.

Let T be a fixed spanning tree of G, and suppose there exists an edge in G joining leaves w_1 and w_2 in T. If w_1 and w_2 share a neighbor in T, then let their common neighbor be u. Note that u is a non-leaf adjacent to at least two leaves. Since T is a d-flare we must have $d \geq 2$, and the edges uw_1 and uw_2 must have the same color. Consider the spanning tree T' obtained from T by replacing the edge uw_1 with w_1w_2, as depicted in Figure 3.6. In T', the vertex w_2 is a non-leaf vertex, which must be adjacent to d leaves. The only neighbors of w_2

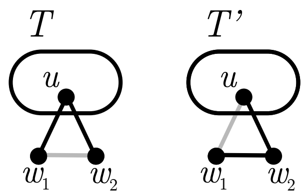

Figure 3.6: The two spanning trees T and T' do not include the gray edges.

in T' are w_1 and u, so $d = 2$ and u must be a leaf in T'. Also, the edges w_1w_2 and w_2u must have the same color and now G is a monochromatically edge-colored C_3.

Suppose now that the leaves w_1 and w_2 of the spanning tree T do not have a common neighbor in T. Let u_1 and u_2 be the neighbors of w_1 and w_2 in T, respectively. Consider the spanning tree T' obtained from T by replacing the edge u_1w_1 with w_1w_2, as depicted in Figure 3.7. Note that w_2 is a non-leaf vertex in T' and the only leaf of T' adjacent to w_2 in T'

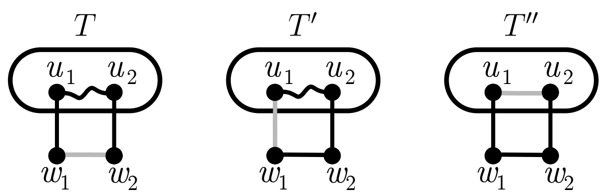

Figure 3.7: The three spanning trees T, T', and T'' do not include the gray edges.

is w_1, so we must have $d = 1$. Adding the edge w_1w_2 to T produces a cycle C. Deleting any edge e of C other than u_1w_1, w_1w_2, and u_2w_2 produces a spanning tree T'' in which w_1 and w_2 are non-leaves. In T'', both w_1 and w_2 must be adjacent to exactly one leaf each. The only possible leaf neighbors of w_1 and w_2 in T'' are u_1 and u_2, respectively. Since u_1 and u_2 are non-leaves in T, we conclude that G has four vertices and that the deleted edge e must be u_1u_2, so $C_4 \subseteq G$. Neither u_1w_2 nor u_2w_1 can be edges of G, since otherwise G contains a spanning tree isomorphic to $K_{1,3}$ which is not a d-corona when $d = 1$. Now $G = C_4$, and since $d = 1$, G is a properly edge-colored. □

We finish this section with a generalization of Theorem 3.2.4. The proof of the next theorem mirrors the probabilistic proof of Theorem 3.2.4 by Alon and Spencer [3]. The asymptotic tightness of Theorem 3.2.4 shows that the next theorem is asymptotically tight in the case of properly edge-colored graphs. We make no attempt to show tightness in other cases.

Theorem 3.2.12. *If G is an n-vertex d-tolerant edge-colored graph with minimum degree k, then* $\widehat{\gamma}(G) \le \frac{n(1+\ln(1+k/d))}{1+k/d}$.

Proof. Let the set of colors on the edges of G be C. For a vertex $v \in V(G)$ and a color $\alpha \in C$, let $N_\alpha(v)$ be the set of vertices adjacent to v via edges of color α. We begin by randomly and uniformly choosing for each vertex v a largest rainbow star $s(v)$ centered at v. The star $s(v)$ can be generated by randomly and uniformly including one neighbor of v from each non-empty set $N_\alpha(v)$. For adjacent vertices v and w, the probability that w is covered by $s(v)$ is at least $1/d$, since at most d edges incident to v can have the color of vw.

Set $p = \frac{\ln(1+k/d)}{1+k/d}$. Form a set $A \subset V(G)$ by including each vertex v with probability p, independently. Let B be the set of vertices not in $\bigcup_{v\in A} s(v)$. The set $A \cup B$ is a rainbow dominating set in G. A vertex w is in B if and only if it is not in A and, for each neighbor v of w, either v is not in A or w is not in $s(v)$. Hence, the probability that w is in B is at most $(1-p)[(1-p)+p(1-1/d)]^k$, since w has at least k neighbors. Noting that $(1-p)[(1-p)+p(1-1/d)]^k = (1-p)(1-p/d)^k \le e^{-p}e^{-kp/d} = e^{-p(1+k/d)} = e^{-\ln(1+k/d)} = \frac{1}{1+k/d}$, we find that the expected size of B is at most $\frac{n}{1+k/d}$. Now the expected size of $A \cup B$ is at most $\frac{n(1+\ln(1+k/d))}{1+k/d}$. We conclude that $\widehat{\gamma}(G) \le \frac{n(1+\ln(1+k/d))}{1+k/d}$. □

3.3 Rainbow Matching

The study of rainbow matchings began with Ryser [61], who conjectured that every Latin square of odd order has a transversal (a set of positions occupied by distinct labels, one in each row and column). When viewed as a colored biadjacency matrix, a Latin square of order n corresponds to a properly n-edge-colored $K_{n,n}$, and a transversal corresponds to a rainbow perfect mathcing. In this view, an equivalent statement of Ryser's conjecture is that when n is odd, every properly n-edge-colored complete bipartite graph $K_{n,n}$ contains a rainbow perfect matching.

Wang and Li [74] studied rainbow matchings in arbitrary edge-colored graphs. They proved that every edge-colored graph G contains a rainbow matching of size at least $\left\lceil \frac{5\widehat{\delta}(G)-3}{12} \right\rceil$. They conjectured that a rainbow matching of size at least $\left\lceil \widehat{\delta}(G)/2 \right\rceil$ can be guaranteed when $\widehat{\delta}(G) \geq 4$ [74]. A properly 3-edge-colored K_4 does not contain a rainbow matching of size 2. However, Li and Xu [46] proved that the conjecture is true for all larger properly edge-colored complete graphs. In this section we show that every edge-colored graph G has a rainbow matching of size at least $\left\lfloor \widehat{\delta}(G)/2 \right\rfloor$, proving the conjecture when $\widehat{\delta}(G)$ is even. This result has been published in The Electronic Journal of Combinatorics [43]. Exploiting this result and the lemmas used in proving it, Kostochka and Yancey proved the conjecture of Wang and Li in its entirety [40]. The main result in this section is Theorem 3.3.1 below.

Theorem 3.3.1. *Every edge-colored graph G has a rainbow matching of size at least $\left\lfloor \widehat{\delta}(G)/2 \right\rfloor$.*

Before proving Theorem 3.3.1, we will develop some notation and a series of lemmas from which the theorem will follow as a corollary.

Let G be an edge-colored graph other than K_4, and let $k = \widehat{\delta}(G)$. If $|V(G)| = k+1$, then G is a properly edge-colored complete graph and has a rainbow matching of size $\lceil k/2 \rceil$, by the result of Li and Xu [46]. Therefore, we may assume that $|V(G)| \geq k+2$.

Let M be a subgraph of G whose edges form a largest rainbow matching. Let $c = k/2 - |E(M)|$, and let the edges of M be $e_1, \ldots, e_{k/2-c}$, with $e_j = u_j v_j$. We may assume throughout that $c \geq 1/2$, since otherwise G has a rainbow matching of size $\lceil k/2 \rceil$. Let H be the subgraph induced by $V(G) - V(M)$, and let $p = |V(H)|$. Note that $|V(G)| = |V(M)| + |V(H)| = k - 2c + p$. Since $|V(G)| \geq k+2$, we conclude that $p \geq 2c+2$.

Let A be the spanning bipartite subgraph of G whose edge set consists of all edges joining $V(M)$ and $V(H)$ (see Figure 3.8). We say that a vertex v is *incident* to a color if some edge

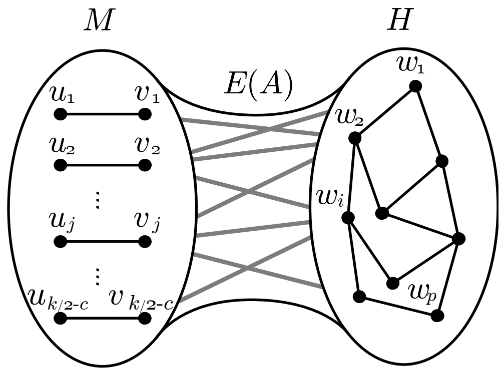

Figure 3.8: $V(M)$ and $V(H)$ partition $V(G)$.

incident to v has that color. A vertex $u \in V(M)$ is incident to at most $|V(M)| - 1$ colors in the subgraph induced by $V(M)$, so u is incident to at least $2c + 1$ colors in A. That is,

$$\widehat{d}_A(u) \geq 2c + 1. \tag{3.1}$$

We say that a color appearing in G is *free* if it does not appear on an edge of M. Let B denote the spanning subgraph of A whose edges have free colors. We prove our results by summing the color degrees in B of the vertices of H. Consider $w \in V(H)$. There are only $k/2 - c$ non-free colors, so w is incident to at least $k/2 + c$ free colors. By the maximality of M, no free color appears in H, so the free colors incident to w occur on edges of B. That is, $\widehat{d}_B(w) \geq k/2 + c$. Summing over $V(H)$ yields

$$\widehat{d}_B(V(H)) \geq p(k/2 + c). \tag{3.2}$$

For $1 \leq j \leq k/2 - c$, let E_j be the subset of edges in $E(B)$ incident to $u_j v_j$. Let B_j be the graph with vertex set $V(H) \cup \{u_j, v_j\}$ and edge set E_j. Note that $\widehat{d}_{B_j}(w) \leq 2$ for $w \in V(H)$.

Lemma 3.3.2. *If at least three vertices in $V(H)$ have positive color degree in B_j, then only one such vertex can have color degree 2 in B_j. Furthermore, $\widehat{d}_{B_j}(V(H)) \leq p + 1$.*

Proof. Let w_1, w_2, and w_3 be vertices of H such that $\widehat{d}_{B_j}(w_1) = \widehat{d}_{B_j}(w_2) = 2$ and $\widehat{d}_{B_j}(w_3) \geq 1$.

By symmetry, we may assume that $w_3v_j \in E(B_j)$. Maximality of M requires the same color on u_jw_1 and v_jw_2. Since $\widehat{d}_{B_j}(w_2) = 2$, the color on u_jw_2 differs from this. Now u_jw_1 or u_jw_2 has a color different from v_jw_3, which yields a larger rainbow matching in G.

Now consider $\widehat{d}_{B_j}(V(H))$. Since $p \geq 2c + 2$, we have $p \geq 3$. If $\widehat{d}_{B_j}(V(H)) \geq p + 2$, then $\widehat{d}_B(w) \leq 2$ for all $w \in V(H)$ requires three vertices as forbidden above. □

If $p = 3$, then color degrees $2, 2, 0$ for the vertices of H in B_j do not contradict Lemma 3.3.2. For $p \geq 4$, the next lemma determines the structure of B_j when $\widehat{d}_{B_j}(V(H)) = p + 1$.

Lemma 3.3.3. *For $p \geq 4$, if $\widehat{d}_{B_j}(V(H)) = p + 1$, then u_j or v_j is adjacent in B_j to $p - 1$ vertices of $V(H)$ via edges of the same color.*

Proof. Since $p + 1 \geq 5$, at least three vertices of H have positive color degree in B_j. Now Lemma 3.3.2 permits only one vertex w such that $\widehat{d}_{B_j}(w)$=2, while $\widehat{d}_{B_j}(w') = 1$ for each other vertex w' in $V(H)$. Let λ_1 and λ_2 be the colors on u_jw and v_jw, respectively. Partition $V(H) - \{w\}$ into two sets by letting $U = N_{B_j}(u_j) - \{w\}$ and $V = N_{B_j}(v_j) - \{w\}$. By the maximality of M, all edges joining u_j to U have color λ_2, and all edges joining v_j to V have color λ_1. If U and V are both nonempty, then replacing u_jv_j with edges to each yields a larger rainbow matching in G. Hence U or V is empty and the other has size $p - 1$. □

Lemma 3.3.4. *If $c \geq 1$, then $\widehat{d}_{B_j}(V(H)) \leq p$ for each j.*

Proof. Since $p \geq 2c + 2$, $c \geq 1$ implies $p \geq 4$. If $\widehat{d}_{B_j}(V(H)) = p + 1$, then Lemma 3.3.3 applies, and u_j or v_j is adjacent via edges of the same color to all but one vertex of H. Now $\widehat{d}_A(u_j) \leq 2$ or $\widehat{d}_A(v_j) \leq 2$, which contradicts Equation 3.1 when $c \geq 1$. □

The proof of Theorem 3.3.1 is now easy.

Theorem 3.3.1 ([43]). *Every edge-colored graph with minimum color degree k has a rainbow matching of size at least $\lfloor k/2 \rfloor$.*

Proof. If the maximum size of a rainbow matching is $k/2-c$, with $c \geq 1$, then Lemma 3.3.4 yields $\widehat{d}_B(V(H)) \leq \sum_{j=1}^{k/2-c} \widehat{d}_{B_j}(V(H)) \leq p(k/2-c)$, which contradicts Equation 3.2. $\square$

The argument of Kostochka and Yancey involves choosing a largest rainbow matching in an ingenious way [40]. With the help of Lemma 3.3.3 and detailed case analysis, they proved the following theorem.

Theorem 3.3.5 ([40])**.** *Every edge-colored graph with minimum color degree k has a rainbow matching of size at least $\lceil k/2 \rceil$.*

3.4 Rainbow Edge-Chromatic Number

Recall that the edge-chromatic number $\chi'(G)$ of a graph G is the minimum number of colors necessary to color the edges of G in such a way that no incident edges have the same color. Each color class in such a coloring must be a matching, so an alternative definition of the edge-chromatic number of G is that it is the minimum number of matchings partitioning the edge set of G. In this section we wish to partition the edge set of an edge-colored graph into *rainbow* matchings. We define the *rainbow edge-chromatic number* $\chi'_r(G)$ of an edge-colored graph G to be the minimum number of rainbow matchings in such a partition of the edge set of G.

Clearly $\chi'_r(G) \geq \chi'(G)$ for any edge-colored graph G, since each partition of $E(G)$ into rainbow matchings is also a partition into uncolored matchings. Now $\chi'_r(G) \geq \Delta(G)$ for any edge-colored graph G, but unlike for the edge-chromatic number, there are edge-colored graphs such that $\chi'_r(G) > \Delta(G) + 1$. Consider the following examples.

Example 3.4.1. Fix d and let $G = K_{dn}$, where n is a positive integer. Starting with a proper (dn)-edge-coloring of G, construct a d-tolerant edge-coloring of G by identifying color classes in d-tuples. The largest rainbow matching in this d-tolerant edge-colored graph has size at most n, since there are now at most n different colors. Partitioning the edge set of G will require at least $|E(G)|/n$ rainbow matchings, so $\chi'_r(G) \geq \frac{d}{2}(dn-1) = \frac{d}{2}(|V(G)|-1) = \frac{d}{2}\Delta(G)$.

Using large values of d in Example 3.4.1 shows that $\chi'_r(G)$ can be much larger than $\Delta(G)+1$. The examples produced are graphs whose edges are far from properly colored, but this is not required. Even properly edge-colored graphs can have $\chi'_r(G) > \Delta(G)+1$. For example, in a properly 3-edge-colored K_4 the largest rainbow matching has one edge, so $\chi'_r(K_4) = 6$, but $\Delta(K_4) = 3$. For more examples we turn to Latin squares and transversals.

As discussed in the previous section, when viewed as a colored biadjacency matrix, a Latin square of order n corresponds to a properly n-edge-colored copy of $K_{n,n}$. Each rainbow matching in this $K_{n,n}$ corresponds to a partial transversal, so the rainbow edge-chromatic number of such a $K_{n,n}$ is the minimum number of partial tranversals partitioning the positions of the Latin square. Ryser conjectured that every Latin square of odd order has a transversal [61], but many Latin squares of even order do not. To see this, let n be odd, and let A and B be latin squares of order n where the labels in A are the numbers in $[n]$ and the labels in B are the numbers in $\{n+1,\ldots,2n\}$. For convenience, let A_1 and A_2 be two copies of A, and let B_1 and B_2 be two copies of B. Let $C = \left(\begin{smallmatrix} A_1 & B_1 \\ B_2 & A_2 \end{smallmatrix}\right)$. It is easy to see that C is a Latin square of order $2n$.

To see that C does not have a transversal, suppose L is such a set of positions and let a_1, a_2, b_1, and b_2 count the number of positions in L corresponding to positions in A_1, A_2, B_1, and B_2, respectively. Each number in $[2n]$ must occur in some position in L, so $a_1 + a_2 = n$ and $b_1 + b_2 = n$. The first n rows of C must contain n positions in L, so $a_1 + b_1 = n$. Similarly $a_2 + b_2 = n$, and by considering columns we find that $a_1 + b_2 = n$ and $a_2 + b_1 = n$. Solving this system of equations gives the unique solution $a_1 = a_2 = b_1 = b_2 = n/2$, but n is odd and the variables a_1, a_2, b_1, and b_2 are integers. We conclude that C does not have a transversal.

The largest partial transversal contained in C has size at most $2n-1$, so at least $\lceil (2n)^2/(2n-1) \rceil$ partial transversals are needed to partition the $(2n)^2$ positions in C into partial transversals. If $K_{2n,2n}$ is the corresponding properly $2n$-edge-colored complete bipartite graph, then $\chi'_r(K_{2n,2n}) \geq \lceil 4n^2/(2n-1) \rceil = 2n+2$. This is again larger than $\Delta(K_{2n,2n})+1$.

In the rest of this section we provide an upper bound on $\chi'_r(G)$. The *average color degree* of G is the average color degree of the vertices of G, *i.e.* $\sum_{v \in V(G)} \widehat{d}(G)/|V(G)|$. The next lemma shows that a graph with large *average* color degree contains a subgraph with large *minimum* color degree.

Lemma 3.4.2. *Let d be a fixed positive integer and let c be a fixed nonzero constant. Let G be a d-tolerant edge-colored graph with average color degree at least c. In G, there is a subgraph H such that $\widehat{\delta}(H) > c/(d+1)$.*

Proof. If $\widehat{\delta}(G) > c/(d+1)$, then we are done, so let v be a vertex in G such that $\widehat{d}_G(v) \leq c/(d+1)$. There can be at most $cd/(d+1)$ edges incident to v, for there are $\widehat{d}_G(v)$ colors incident to v and at most d edges of each color incident to v. If u is a vertex that is adjacent to v, then the color degree of u in $G-v$ is at least $\widehat{d}_G(u)-1$. Now $\sum_{u \in V(G-v)} \widehat{d}_{G-v}(u) \geq \sum_{u \in V(G)} \widehat{d}_G(u) - cd/(d+1) - \widehat{d}_G(v) \geq c|V(G)| - c = c(|V(G)|-1)$. The average color degree of $G-v$ is at least as large as the average color degree of G. To find the desired subgraph H we iteratively delete vertices whose color degrees are at most $c/(d+1)$. Doing this can only increase the average color degree, however, this cannot continue indefinitely. After this process terminates there are no vertices with color degree at most $c/(d+1)$, so the graph constructed has minimum color degree more than $c/(d+1)$. □

The following corollary will be useful in proving the main result in this section.

Corollary 3.4.3. *Let d and n be fixed positive integers, and let G be an n-vertex d-tolerant edge-colored graph. If $|E(G)| = m$, then G contains a rainbow matching of size at least $\lceil m/(nd(d+1)) \rceil$.*

Proof. The color degree sum of the vertices of G is at least $2m/d$, since G is d-tolerant. Now the average color degree of G is at least $2m/(nd)$, and by Lemma 3.4.2 G contains a subgraph

H with minimum color degree at least $2m/(nd(d+1))$. By Theorem 3.3.5, H contains a rainbow matching of size at least $\lceil m/(nd(d+1)) \rceil$. $\square$

Theorem 3.4.4. *Let d and n be fixed positive integers. If G is an n-vertex d-tolerant edge-colored graph, then $\chi'_r(G) < d(d+1)n \ln n$.*

Proof. If G is an n-vertex edge-colored graph, then we may consider G to be a subgraph of an edge-colored copy of K_n by assigning colors to the edges not in G. For the edge-colored complete graph constructed in this way it is clear that $\chi'_r(G) \le \chi'_r(K_n)$. If G is d-tolerant, then assigning distinct colors not already appearing on the edges of G produces a d-tolerant edge-colored K_n as a supergraph. Hence, it suffices to consider a d-tolerant edge-colored copy of K_n.

Let F_0 be d-tolerant edge-colored copy of K_n. We will iteratively construct $F_1, F_2, \dots$ by deleting one rainbow matching from F_{i-1} to obtain F_i. Given F_{i-1} for $i \ge 1$, let $a_{i-1} = |E(F_{i-1})| / \left(\frac{n(n-1)}{2}\right)$; note that $a_0 = 1$. By Corollary 3.4.3, F_{i-1} contains a rainbow matching M_{i-1} with at least $\left\lceil a_{i-1}\left(\frac{n-1}{2d(d+1)}\right)\right\rceil$ edges. Let $F_i = F_{i-1} - M_{i-1}$. Define j to be the first index such that $a_j\left(\frac{n-1}{2d(d+1)}\right) \le 1$. Note that $E(F_j)$ can be partitioned into $|E(F_j)|$ rainbow matchings, each being a single edge. Together with the rainbow matchings M_i for $i \in [0, j-1]$, $E(F_0)$ can be partitioned into $j + |E(F_j)|$ rainbow matchings. It remains only to bound j and $|E(F_j)|$.

Note that for $i \ge 1$, by the definition of a_i we have

$$\begin{aligned}
a_i\left(\frac{n(n-1)}{2}\right) = |E(F_i)| &= |E(F_{i-1})| - |M_{i-1}| \\
&\le a_{i-1}\left(\frac{n(n-1)}{2}\right) - \left\lceil a_{i-1}\left(\frac{n-1}{2d(d+1)}\right)\right\rceil \\
&\le a_{i-1}\left(\frac{n(n-1)}{2}\right) - a_{i-1}\left(\frac{n-1}{2d(d+1)}\right) \\
&= a_{i-1}\left(\frac{n(n-1)}{2}\right)\left(1 - \frac{1}{nd(d+1)}\right).
\end{aligned}$$

Dividing by $\binom{n(n-1)}{2}$, we find $a_i \leq a_{i-1}\left(1 - \frac{1}{nd(d+1)}\right)$. Iterating this inequality and recalling that $a_0 = 1$ yields that

$$a_i \leq \left(1 - \frac{1}{nd(d+1)}\right)^i < e^{\frac{-i}{nd(d+1)}}. \tag{3.3}$$

By the definition of j we have $a_j \leq \frac{2d(d+1)}{n-1}$ and $a_{j-1} > \frac{2d(d+1)}{n-1}$. Using this bound on a_{j-1}, Inequality 3.3 becomes $e^{\frac{-j+1}{nd(d+1)}} > \frac{2d(d+1)}{n-1}$, so $j < nd(d+1)\ln\left(\frac{n-1}{2d(d+1)}\right) + 1$. Now F_0 has a decomposition into $j + a_j\binom{n(n-1)}{2}$ rainbow matchings, and

$$\begin{aligned}
j + a_j\binom{n(n-1)}{2} &< \left(nd(d+1)\ln\left(\frac{n-1}{2d(d+1)}\right) + 1\right) + \frac{2d(d+1)}{n-1}\binom{n(n-1)}{2} \\
&= d(d+1)n\ln(n-1) + nd(d+1) + 1 - nd(d+1)\ln(2d(d+1)) \\
&< d(d+1)n\ln(n-1) \\
&< d(d+1)n\ln(n),
\end{aligned}$$

where the second-to-last inequality follows since we may assume that $n \geq 2$ and $d \geq 1$. We conclude that $\chi'_r(G) < d(d+1)n\ln(n)$. □

Chapter 4

Acquisition

4.1 Introduction

Consider an army dispersed among many cities. We wish to consolidate the troops. Troops move only to neighboring occupied cities, and the number of troops in a move cannot exceed the number already at the destination. Can the troops all move to one city?

We model such situations using graphs with vertex weights. Initially, each vertex has weight 1. An *acquisition move* transfers some weight from a vertex u to a neighboring vertex v, provided that before the move the weight on v is at least the weight on u. The total weight is preserved. Lampert and Slater [41] introduced acquisition in graphs, using *total acquisition moves* that transfer all of the weight from a vertex to a neighbor. Other models involve *partial acquisition moves* in which only a portion of the weight on a vertex is transferred to a neighbor. In this chapter we are concerned only with the total acquisition model and henceforth omit the adjective "total" when referring to acquisition moves.

We refer to a succession of acquisition moves as an *acquisition protocol*. The *residual set* left by an acquisition protocol is the set of vertices that remain with positive weight. The *total acquisition number* (or simply *acquisition number*), written $a_t(G)$, is the minimum possible size of a residual set left by an acquisition protocol (starting from the distribution with weight 1 on each vertex). An acquisition protocol on a graph G is *optimal* if it leaves a residual set of size $a_t(G)$. Lampert and Slater [41] proved that, for $n \geq 2$, the maximum of $a_t(G)$ over connected n-vertex graphs is $\lfloor \frac{n+1}{3} \rfloor$. In this chapter we characterize the graphs

G for which $a_t(G) = \frac{|V(G)|+1}{3}$. In the process we give an independent proof of the result of Lampert and Slater. For many more results concerning total acquisition and related acquisition parameters, see [42] or the thesis of Wenger [75].

4.2 Structure of Three Families of Trees

Trees are of interest for acquisition problems because deleting edges cannot reduce a_t, so among n-vertex graphs a_t is maximized on trees.

Definition 4.2.1. Starting with P_5, let $\mathcal{T}$ be the family of trees constructed by iteratively growing paths with three edges from neighbors of leaves. The process of growing such a path from a neighbor of a leaf will be called *augmentation*.

Note that P_5 can be obtained by augmenting P_2; however, we exclude P_2 from $\mathcal{T}$ for convenience. The following lemma gives elementary properties of trees in $\mathcal{T}$. These properties will be used repeatedly throughout this chapter.

Lemma 4.2.2 (Elementary Properties Lemma)**.** *If $T \in \mathcal{T}$, then*

(**a**) *The distance between any two leaves of T is at least 4.*

(**b**) *The number of leaves in T is $\frac{|V(T)|+1}{3}$.*

Proof. Each property holds for P_5 and is preserved by augmentation. □

In [42] and [75] it is shown that if $T \in \mathcal{T} \cup \{P_2\}$, then $a_t(T) = \frac{|V(T)|+1}{3}$. We provide a proof here to fully justify the main result in this chapter, that C_5 and the trees in $\mathcal{T} \cup \{P_2\}$ are the only graphs G such that $a_t(G) = \frac{|V(G)|+1}{3}$. The following lemma will be useful.

Lemma 4.2.3 (Separation Lemma)**.** *Let x and y be vertices in a tree T. If the unique path joining x and y in T contains a vertex of degree 2 not adjacent to x or y, then the initial weight from x and y cannot reach a common vertex via acquisition moves.*

Proof. Let v be such a vertex on the path joining x and y. For the weight from x and y to reach the same vertex, the weight from x or y must be transferred to v during some acquisition move. The first acquisition move involving v transfers weight 1 to or from it, so this move can only transfer the original weight from a vertex in $N[v]$. After this move, v or one of its neighbors has weight 0. Now there is a vertex of weight 0 separating the weights originally on x and y and it is impossible for these weights to reach the same vertex. □

The vertex v in Lemma 4.2.3 is said to *witness the separation* of the weights from the leaves x and y. We next define a family of trees $\mathcal{T}^*$ with properties that will be useful throughout this chapter. We will refer to a vertex having distance at least 2 from all leaves as an *inner* vertex.

Definition 4.2.4. Let $\mathcal{T}^*$ be the family of trees T containing an inner vertex and such that for every inner vertex v the following properties hold:
(**a**) $d_T(v) = 2$,
(**b**) both components of $T - v$ are in $\mathcal{T} \cup \{P_2\}$, and
(**c**) if w is a neighbor of v, then $T - w$ consists of an isolated vertex and components whose orders are divisible by 3.

We will soon show that $\mathcal{T}^*$ and $\mathcal{T}$ are the same family. Many of the proofs that follow will be inductive. Note that P_5 is the smallest tree satisfying the defining properties of $\mathcal{T}^*$. For larger trees in $\mathcal{T}^*$ the next lemma will provide a mechanism by which we may undo an augmentation so that we may apply an inductive hypothesis. For a tree T, let T' be the tree obtained by deleting all leaves of T, and let $T'' = (T')'$.

Lemma 4.2.5. *If $T \in \mathcal{T}^*$ and $T \neq P_5$, then every leaf of T'' is an inner vertex of T and is contained in a three-edge path attached to the rest of T only at the neighbor of a leaf.*

Proof. Note that $V(T'')$ is not empty since $V(T'')$ contains an inner vertex v of T. If $V(T'')$ contains only v, then note that $d_T(v) = 2$ and that $T - v$ consists of two nontrivial stars, each of which must be P_2 by Properties 4.2.4(b) and 4.2.2(a). In this case $T = P_5$, so we may assume that T'' has more than one vertex.

We next show that the leaves of T'' are inner vertices of T. Let x be a leaf of T'' that is not an inner vertex of T. Let u be a leaf of T adjacent to x. There must be a leaf w' of T' adjacent to x since otherwise x is not a leaf of T''. Now w' must be adjacent to a leaf u' of T since otherwise w' is not a leaf of T' (see Figure 4.1). Now u and u' are leaves of T joined by

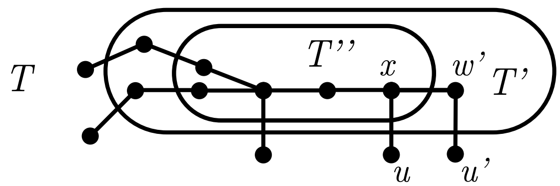

Figure 4.1: No leaf of T'' is adjacent to a leaf of T

the path induced by $\{u, x, w', u'\}$. Now $v \notin \{u, x, w', u'\}$ since $v \in V(T'')$ and $v \neq x$. Now u, x, w', and u' are all contained in the same component of $T - v$. This component must be in $\mathcal{T}$ by Property 4.2.4(b). Now u and u' are leaves at distance 3 in a tree in $\mathcal{T}$ which violates Elementary Property 4.2.2(a). Hence, each leaf of T'' is an inner vertex of T.

Now let v be a leaf of T''. By Property 4.2.4(a), $d_T(v) = 2$, so let w_1 and w_2 be the neighbors of v in T. Note that w_1 and w_2 are each adjacent to exactly one leaf of T by Property 4.2.4(c). Let these leaves be u_1 and u_2, respectively. We may assume that $w_2 \in V(T') \setminus V(T'')$ since if neither neighbor of v is in $V(T') \setminus V(T'')$ then v is not a leaf of T'' (see Figure 4.2). Now the component of $T - v$ containing w_2 must be a non-trivial star,

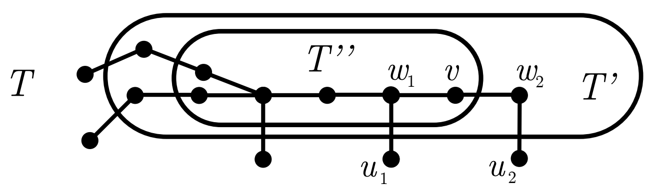

Figure 4.2: The leaves of T'' are inner vertices of T.

and hence must be P_2 by Properties 4.2.4(b) and 4.2.2(a). Now, $\{w_1, v, w_2, u_2\}$ induces a 3-edge path P attached to the rest of T only at w_1, a neighbor of a leaf. $\square$

Lemma 4.2.6. *The families $\mathcal{T}$ and $\mathcal{T}^*$ are the same.*

Proof. Note that P_5 contains an inner vertex satisfying Properties 4.2.4(a)-(c), and when an inner vertex is introduced during an augmentation it satisfies the three properties as well. Each property is preserved for all inner vertices during the augmentation of a tree in $\mathcal{T}$, hence $\mathcal{T} \subseteq \mathcal{T}^*$. To see that $\mathcal{T}^* \subseteq \mathcal{T}$, note that P_5 is the smallest tree satisfying the defining properties of $\mathcal{T}^*$, and $P_5 \in \mathcal{T}$. Hence, let T be a larger tree in $\mathcal{T}^*$. Let v be a leaf of T'' and let P be a three-edge path containing v that is attached to the rest of T at w, a neighbor of a leaf. Let $\widehat{T} = T - V(P)$ and note that $\widehat{T}$ is a component of $T - v$. By Property 4.2.4(b), $\widehat{T} \in \mathcal{T}$. As seen in Figure 4.3, T is the result of augmenting $\widehat{T}$ by growing the path P from w. By the definition of $\mathcal{T}$, we have $T \in \mathcal{T}$ and $\mathcal{T}^* \subseteq \mathcal{T}$. □

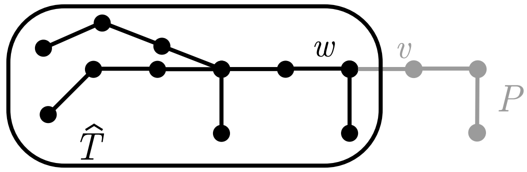

Figure 4.3: T is the result of augmenting $\widehat{T}$ by growing the path P shown in gray.

Theorem 4.2.7. *Let T be a tree. If $T \in \mathcal{T} \cup \{P_2\}$, then $a_t(T) = \frac{|V(T)|+1}{3}$.*

Proof. Clearly P_2 satisfies the claim. To show the upper bound for trees $T \in \mathcal{T}$, we will show by induction on $|V(T)|$ that between any two leaves u and u' there is an inner vertex of degree 2 witnessing the separation of the weights from u and u'. By Lemma 4.2.6 and Properties 4.2.4(a) and (b), T has an inner vertex v such that $d_T(v) = 2$ and the two components of $T - v$ are in $\mathcal{T} \cup \{P_2\}$. If u and u' are contained in the same component of $T - v$, then this component cannot be a copy of P_2 since both u and u' are leaves of T. In this case, u and u' are leaves of a smaller tree in $\mathcal{T}$, so by induction there is an inner vertex of degree 2 witnessing the separation of the weights from u and u'. If u and u' are in opposite componentes of $T - v$, then v is an inner vertex of degree 2 witnessing the separation of the weights from u and u'. By Seperation Lemma 4.2.3, $a_t(T)$ is at least the number of leaves of T, which by Elementary Property 4.2.2(b) is $\frac{|V(T)|+1}{3}$. Equality holds because $a_t(P_5) = 2$ and the weight on a three-vertex path added during an augmentation can be acquired by the central vertex among the three newly added vertices. $\square$

Each tree in $\mathcal{T}$ can be constructed through a sequence of augmentations starting with *some* subgraph isomorphic to P_5. When a vertex is introduced in T it is contained in a four-edge path joining leaves of T. The next lemma will show that *any* four-edge path joining leaves of T may serve as the starting path in a construction of T through a sequence of augmentations. As a result, any vertex of T may be taken as a vertex of the starting P_5 from which T is constructed.

Lemma 4.2.8. *Let T be a tree in $\mathcal{T}$ and let P be a four-edge path joining leaves of T. The tree T can be constructed through a sequence of augmentations starting with P.*

Proof. Note that P_5 is a tree in $\mathcal{T}$ that trivially has the desired property. For a larger tree $T \in \mathcal{T}$ we will again use induction on $|V(T)|$. Let P be a four-edge path between leaves of T. Since $T \neq P_5$, there must be at least two leaves in T''. By Lemma 4.2.5, T contains two pendant three-edge paths attached to T at a neighbor of a leaf. The path P and one of these paths, say P', must be disjoint. Let v' be the inner vertex added during the augmentation that grew P' and let $\widehat{T} = T - V(P')$. Note that $\widehat{T}$ is a component of $T - v'$. Lemma 4.2.6 and Property 4.2.4(b) imply that $\widehat{T} \in \mathcal{T}$. Now P is a four-edge path joining leaves of $\widehat{T}$. By induction $\widehat{T}$ can be constructed through a sequence of augmentations starting with P. The

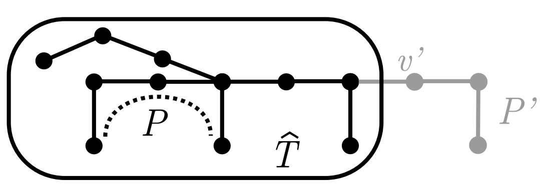

Figure 4.4: T is the result of augmenting $\widehat{T}$ by growing the path P' shown in gray.

original tree T can then be constructed by growing the path P' in one more augmentation, as in Figure 4.4. □

The following three lemmas will be used in the case analysis required to prove the converse of Theorem 4.2.7.

Lemma 4.2.9. *Let T be a tree in $\mathcal{T}$. Let w be a neighbor of a leaf and let v be an inner vertex. There exist acquisition protocols $\mathcal{A}_a$ through $\mathcal{A}_h$ leaving residual sets R_a through R_h with the following properties:*

(**a**) $|R_a| = (|V(T)| + 4)/3$ *and v is left with weight 1,*

(**b**) $|R_b| = (|V(T)| + 4)/3$ *and v is left with weight 2,*

(**c**) $|R_c| = (|V(T)| + 4)/3$ *and v is left with weight 3,*

(**d**) $|R_d| = (|V(T)| + 1)/3$ *and v is left with weight 4,*

(**e**) $|R_e| = (|V(T)| + 4)/3$ *and w is left with weight 1,*

(**f**) $|R_f| = (|V(T)| + 1)/3$ *and w is left with weight 2,*

(**g**) $|R_g| = (|V(T)| + 1)/3$ *and w is left with weight 3, and*

(**h**) $|R_h| = (|V(T)| + 1)/3$ *and w is left with weight 4.*

Proof. By inspection, such acquisition protocols exist for P_5. The existence of such acquisition protocols for a larger tree T in $\mathcal{T}$ will follow from Lemma 4.2.8. Let P be a four-edge path joining leaves of T that contains either v or w. By Lemma 4.2.8, T can be constructed through a sequence of augmentations starting with P. Since $P \simeq P_5$, we can apply the acquisition protocol for P_5 on P that gives v or w the desired weight. Afterwards, the weight on each three-edge path added during an augmentation can be acquired by its central vertex, adding 1 to the size of the residual set and 3 to $|V(T)|$. □

Lemma 4.2.10. *Let T be a tree in $\mathcal{T}$ and let v be an inner vertex of T. If T' is the tree obtained by attaching a leaf to v, then $a_t(T') \leq \frac{|V(T')|-3}{3}$.*

Proof. The proof of this lemma mirrors the previous proof. Let P be a four-edge path joining leaves of T that contains v, and let P' be the tree obtained from P by attaching a leaf to v. It is easy to see by inspection that $a_t(P') = 1 = (|V(P')| - 3)/3$. By Lemma 4.2.8, T' can be constructed through a sequence of augmentations starting with P'. Again, the weight on each three-edge path added during an augmentation can be acquired by the central vertex among the three newly added. □

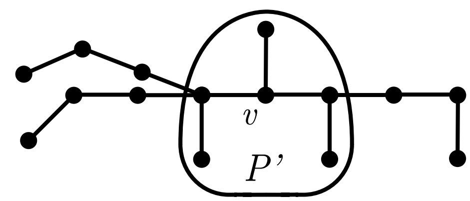

Figure 4.5: The tree T' can be constructed by augmenting P'

Lemma 4.2.11. *Let T be a tree with a vertex x such that $T-x$ consists of p isolated vertices, q isolated edges, and k components in $\mathcal{T}$ such that the edge from x to each component in $\mathcal{T}$ connects x to a vertex that is not a leaf in $T-x$. If $p+q+k \geq 3$, then $a_t(T) \leq \frac{|V(T)|}{3}$.*

Proof. If $p \geq 1$, we begin an acquisition protocol by letting x acquire all weight from the leaves to which it is adjacent. Now, we let x acquire the weight from the q isolated edges in $T-x$. If $p+2q \geq 3$, then x has weight at least 4 and by Lemma 4.2.9(d) and (h), in each of the k components of $T-x$ in $\mathcal{T}$ we may perform an optimal acquisition protocol leaving weight 4 on the neighbor of x. After these acquisition protocols are performed a residual set of size $\frac{|V(T)|-p-2q-1+k}{3}+1$ is left in T. Since x has weight at least 4 and each neighbor of x has weight 4, x can acquire all of the weight left on its neighbors. This reduces the size of the residual set to $\frac{|V(T)|-p-2q-1+k}{3}+1-k = \frac{|V(T)|-p-2q-2k+2}{3}$. Since $p+2q \geq 3$, we find that $a_t(T) \leq \frac{|V(T)|-1}{3}$.

If $p \geq 1$ and $p+2q \leq 2$, then we must have $q=0$ and $k \geq 1$, since $p+q+k \geq 3$. If $k \geq 2$, as is the case when $p=1$, then by Lemma 4.2.9(b) and (f) we may perform an acquisition protocol in one of the components A_1 of $T - x$ in $\mathcal{T}$ leaving a residual set of size at most $\frac{|V(A_1)|+4}{3}$ and weight 2 on the neighbor of x. In the remaining components $A_2, \ldots, A_k$ of $T - x$ in $\mathcal{T}$ we may perform optimal acquisition protocols leaving weight 4 on the neighbor of x by Lemma 4.2.9(d) and (h). Currently, the size of the residual set is at most $\frac{(|V(T)|-2)+(k+3)}{3} + 1$, but x can acquire weight from its neighbor in A_1 and then the weights from its neighbors in $A_2, \ldots, A_k$, reducing the size of the residual set by k. We conclude that $a_t(T) \leq \frac{|V(T)|+4-2k}{3} \leq \frac{|V(T)|}{3}$, since $k \geq 2$. If $p = 2$ and $k = 1$, then after x has acquired weight from the leaves to which it is adjacent we may perform an optimal acquisition protocol in the one component A of $T - x$ in $\mathcal{T}$, leaving weight 4 on the neighbor of x by Lemma 4.2.9(d) and (h). Now the weight from x can be acquired by its neighbor in A leaving a residual set of size $\frac{|V(T)|-2}{3}$.

The case $p = 0$ remains. In this case, begin by having x acquire weight 1 from its neighbor in a smallest component A_1 of $T - x$. By Lemma 4.2.9(a) and (e) and by inspection of acquisition protocols on P_2, an acquisition protocol can be performed concentrating the remaining weight in A_1 on at most $\frac{|V(A_1)|+1}{3}$ vertices. Let A_2 be a second smallest component of $T - x$. We may perform an acquisition protocol on A_2 leaving weight 2 on the neighbor of x. If $A_2 \in \mathcal{T}$ and x is adjacent to an inner vertex of A_2, then by Lemma 4.2.9(b) there is such an acquisition protocol leaving a residual set of size $\frac{|V(A_2)|+4}{3}$ in A_2. Otherwise, a residual set of size $\frac{|V(A_2)|+1}{3}$ may be left in A_2 by Lemma 4.2.9(f) and by inspection of acquisition protocols on P_2. Having x acquire the weight left on its neighbor in A_2 reduces the size the residual set left in A_2 by 1 and increases the weight on x to 4.

By Lemma 4.2.9(e) and (h) and by inspection of acquisition protocols on P_2, in each of the remaining $q + k - 2$ components of $T - x$ we may perform an optimal acquisition protocol leaving weight at most 4 on the neighbor of x. After these protocols are performed a residual set of size at most $1 + \frac{|V(A_1)|+1}{3} + \frac{|V(A_2)|+1}{3} + \frac{(|V(T)|-|V(A_1)|-|V(A_2)|-1)+(q+k-2)}{3}$ remains, with equality only if x is adjacent to an inner vertex of A_2. Simplifying this quantity gives that the residual set has size at most $\frac{|V(T)|+2+q+k}{3}$. Now x can acquire all of the weight left on its neighbors. This reduces the size of the residual set to $\frac{|V(T)|+2+q+k}{3} - (q+k-2)$. Now

$$a_t(T) \leq \frac{|V(T)|+8-2(q+k)}{3}. \tag{4.1}$$

If the conditions for equality are not met, then we save 1 on the size of the residual set left in A_2, and $a_t(T) \leq \frac{|V(T)|+8-2(q+k)}{3} - 1 \leq \frac{|V(T)|-1}{3}$, since $q+k \geq 3$ by hypothesis. If the conditions for equality are met and T is a counterexample to the lemma, then $a_t(T) \geq \frac{|V(T)|+1}{3}$ and $q+k=3$ by Equation 4.1 and by hypothesis. In this case we describe a new acquisition protocol. Let A_2' be the subtree of T induced by $V(A_2) \cup \{x\}$. (See Figure 4.6 below.) Since x is adjacent to an inner vertex of A_2, Lemma 4.2.10 gives that $a_t(A_2') \leq \frac{V(A_2')-3}{3}$. Let A_3 be the remaining component of $T - x$. By Theorem 4.2.7, $a_t(A_1) = \frac{|V(A_1)|+1}{3}$ and $a_t(A_3) = \frac{|V(A_3)|+1}{3}$, since both A_1 and A_3 are in $\mathcal{T} \cup \{P_2\}$ by hypothesis. Now we have $a_t(T) \leq a_t(A_1) + a_t(A_2') + a_t(A_3) \leq \frac{|V(A_1)|+1}{3} + \frac{|V(A_2')|-3}{3} + \frac{|V(A_3)|+1}{3} = \frac{|V(T)|-1}{3}$.

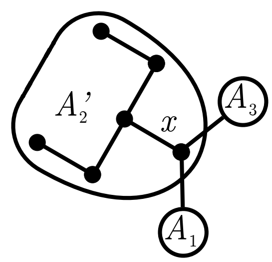

Figure 4.6: The last case of Lemma 4.2.9.

□

To prove the converse of Theorem 4.2.7 we will use the previous lemmas about trees in $\mathcal{T}$ and subsequent lemmas about trees in two related families. Starting with P_4, let $\mathcal{S}$ be the family of trees constructed by iteratively growing a path with *two* edges from a neighbor of a leaf. Starting with a tree $S \in \mathcal{S} \cup \{P_3\}$, let $\mathcal{R}$ be the family of trees constructed by iteratively growing paths with *one* edge from a neighbor of a leaf. Lemma 4.2.12 and Corollary 4.2.13

below provide characterizations of trees in $\mathcal{S}$ and $\mathcal{R}$. Recall from Section 3.2 that $H \circ K_1$ is the graph obtained by adding one pendant edge to each vertex of H.

Lemma 4.2.12. *For a tree S, the following are equivalent:*

(**a**) $S \in \mathcal{S}$

(**b**) *For some subtree S', $S = S' \circ K_1$, that is, S is the corona of S' by K_1.*

(**c**) $S \neq P_2$ *and each vertex that is not a leaf is adjacent to exactly one leaf.*

Proof. Each tree in $\mathcal{S}$ is iteratively constructed from P_4, and $P_4 = P_2 \circ K_1$. The property of being a corona is preserved during the iterative construction of trees in $\mathcal{S}$, so Property (a) implies Property (b). By the definition of corona, Property (b) implies Property (c).

We now use induction on $|V(S)|$. Note that P_4 is the smallest tree satisfying Property (c) and $P_4 \in \mathcal{S}$, so let S be a larger tree satisfying Property (c). Let S' be the tree obtained by deleting all leaves of S, and let w be a leaf in S'. Let u be the unique leaf of S adjacent to w and let $\widehat{S} = S - \{w, u\}$, as in Figure 4.7.

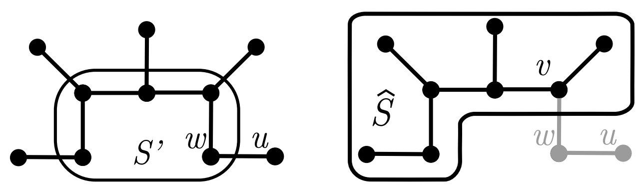

Figure 4.7: S' and $\widehat{S}$ are subtrees of S.

Note that $\widehat{S} = (S' - w) \circ K_1$, so $\widehat{S} \in \mathcal{S}$ by induction. Let v be the neighbor of $w \in S'$. The tree S is constructed from $\widehat{S}$ by growing a two-edge path from v, a non-leaf in $\widehat{S}$. By construction, $S \in \mathcal{S}$. □

Corollary 4.2.13. *A tree R is in $\mathcal{R}$ if and only if $R \neq P_2$ and each vertex that is not a leaf is adjacent to at least one leaf.*

Proof. Each tree in $\mathcal{R}$ is constructed from a tree in $\mathcal{S} \cup \{P_3\}$. Note that P_3 satisfies the desired property and by Lemma 4.2.12, so does each tree in $\mathcal{S}$. This property is preserved when one-edge paths are grown from neighbors of leaves, so the property is satisfied by each tree in $\mathcal{R}$. To prove the converse, note that P_3 is the smallest tree satisfying the desired property and that $P_3 \in \mathcal{R}$. Let R be a larger tree in which each non-leaf is adjacent to at least one leaf. If every non-leaf is adjacent to exactly one leaf, then $R \in \mathcal{S}$ by Lemma 4.2.12. Since $\mathcal{S}$ is a subfamily of $\mathcal{R}$ we have $R \in \mathcal{R}$. Let w be a non-leaf adjacent to leaves u_1 and u_2. In the tree $R-u_1$, each non-leaf vertex is adjacent to at least one leaf, so $R-u_1 \in \mathcal{R}$ by induction. Now R can be constructed from $R-u_1$ by making u_1 adjacent to w, a neighbor of the leaf u_2 in $R-u_1$. By construction, $R \in \mathcal{R}$. □

Inner vertices play an important role in the characterization of trees in $\mathcal{T}$ given in Lemma 4.2.6. Corollary 4.2.13 states that there are no such vertices in a tree in $\mathcal{R}$. In the proof of the converse of Theorem 4.2.7 in Sections 4.3 and 4.4, the possibility that a counterexample is in $\mathcal{R}$ will have to be ruled out. The two lemmas below will be useful.

Lemma 4.2.14. *If S is a tree in $\mathcal{S}$ and S' is the tree obtained by deleting all leaves in S, then $a_t(S) \leq a_t(S')$*

Proof. Begin an acquisition protocol by having each non-leaf acquire the weight from the unique leaf to which it is adjacent. Now, the vertices with positive weight are exactly the vertices in S', and the current assignment of weights is uniform. Any optimal acquisition protocol for S' may be performed leaving a residual set of $a_t(S')$ vertices. □

Lemma 4.2.15 below gives a property of trees in $\mathcal{R}$ analogous to the property of trees in $\mathcal{T}$ given in Definition 4.2.4(b).

Lemma 4.2.15. *If R is a tree in $\mathcal{R}$ and w is a vertex that is not a leaf, then $R-w$ consists of isolated vertices, isolated edges, and components in $\mathcal{R}$.*

Proof. The statement is true if $R = P_3$ or $R = P_4$ and is preserved throughout the iterative construction of a tree in $\mathcal{R}$. □

4.3 Minimal Counterexamples

In this section we prove a series of lemmas that will be used in Section 4.4 in a proof of the converse of Theorem 4.2.7, stated below as Theorem 4.4.1.

Theorem 4.4.1. *Let T be a tree with at least two vertices. If $a_t(T) \geq \frac{|V(T)|+1}{3}$, then $T \in \mathcal{T} \cup \{P_2\}$.*

In each lemma in this section we consider a minimal counterexample, that is, a smallest tree such that $a_t(T) \geq \frac{|V(T)|+1}{3}$, but $T \notin \mathcal{T} \cup \{P_2\}$. If a tree T' is smaller than a minimal counterexample T, then $a_t(T') \leq \frac{|V(T')|+1}{3}$ since this bound holds either by the minimality of T or by Theorem 4.2.7. Also, equality in this bound holds only if $T' \in \mathcal{T} \cup \{P_2\}$. Note that by Lemma 4.2.6 and Corollary 4.2.13 no tree is contained in both $\mathcal{T} \cup \{P_2\}$ and $\mathcal{R}$ since trees in $\mathcal{T}$ contain inner vertices and those in $\mathcal{R}$ do not. We begin by showing that no tree in $\mathcal{R}$ can be a minimal counterexample to Theorem 4.4.1. As a result, there must be an inner vertex in a minimal counterexample to Theorem 4.4.1.

Lemma 4.3.1. *If $R \in \mathcal{R}$, then R is not a minimal counterexample to Theorem 4.4.1.*

Proof. Suppose that R is a minimal counterexample to Theorem 4.4.1. If R is contained in the *subfamily* $\mathcal{S}$, then let R' be the tree obtained by deleting each leaf of R. By the minimality of R and Lemma 4.2.14, $a_t(R) \leq a_t(R') \leq \frac{|V(R')|+1}{3} = \frac{|V(R)|+2}{6} < \frac{|V(R)|}{3}$, since $R \in \mathcal{S}$ implies $|V(R)| \geq 4$. This contradicts the assumption that R is a counterexample.

If $R \in \mathcal{R} \setminus \mathcal{S}$, then let w be a vertex adjacent to at least two leaves. By Lemma 4.2.15, $T - w$ consists of isolated vertices, isolated edges, and components $R_1, \ldots, R_r$ in $\mathcal{R}$. Let $\widehat{V} = V(R) - \bigcup_{i=1}^{r} V(R_i)$ and let $\widehat{R}$ be the graph induced by $\widehat{V}$, as in Figure 4.8. Note that $a_t(\widehat{R}) = 1$, and since w is adjacent to at least two leaves $a_t(\widehat{R}) \leq |V(\widehat{R})|/3$. By the minimality of the counterexample R we have $a_t(R_i) \leq |V(R_i)|/3$ for each $i \in [r]$. Now $\widehat{V}, V(R_1), \ldots, V(R_r)$ partition $V(R)$ and in each part the weight has been acquired by at most $\frac{1}{3}$ of the vertices. We conclude that $a_t(R) \leq \frac{|V(R)|}{3}$ and hence R is not a counterexample. $\square$

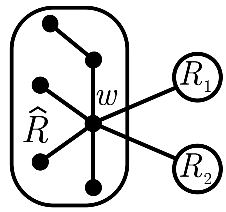

Figure 4.8: $\widehat{R}$ is induced by $V(R) - \bigcup_{i=1}^{r} V(R_i)$

Lemma 4.3.2. *Let T be a minimal counterexample to Theorem 4.4.1. If e is an edge incident to no leaf of T, then $T - e$ contains a component whose order is congruent to 2 modulo 3.*

Proof. Consider a tree T and an edge $e \in E(T)$ such that the conclusion fails. Let A_1 and A_2 be the components of $T - e$. By the minimality of T we have $a_t(A_1) \leq \frac{|V(A_1)|+1}{3}$ and $a_t(A_2) \leq \frac{|V(A_2)|+1}{3}$. Both $a_t(A_1)$ and $a_t(A_2)$ are integers. Since $|V(A_1)| \not\equiv 2 \pmod 3$ and $|V(A_2)| \not\equiv 2 \pmod 3$, we have $a_t(A_1) \leq \frac{|V(A_1)|}{3}$ and $a_t(A_2) \leq \frac{|V(A_2)|}{3}$. Now $a_t(T) \leq a_t(A_1) + a_t(A_2) \leq \frac{|V(A_1)|}{3} + \frac{|V(A_2)|}{3} = \frac{|V(T)|}{3}$, so T is not a counterexample. □

Lemma 4.3.3. *If T is a tree such that $|V(T)| \equiv 0 \pmod 3$, then T is not a minimal counterexample to Theorem 4.4.1.*

Proof. If T is a tree in which every edge is incident to a leaf, then T is a star and $T \in \mathcal{R}$. By Lemma 4.3.1, T is not a counterexample to Theorem 4.4.1. Hence, T has an edge e incident to no leaf. Let A_1 and A_2 be the components of $T - e$. By Lemma 4.3.2, we may assume that $|V(A_1)| \equiv 2 \pmod 3$, in which case $|V(A_2)| \equiv 1 \pmod 3$. By the minimality of T we have $a_t(T) \leq a_t(A_1) + a_t(A_2) \leq \frac{|V(A_1)|+1}{3} + \frac{|V(A_2)|-1}{3} = \frac{|V(T)|}{3}$. □

Lemma 4.3.4. *Let T be a minimal counterexample to Theorem 4.4.1 and let e be an edge incident to no leaf of T. If A is a component of $T - e$ such that $|V(A)| \equiv 2 \pmod 3$, then either $A = P_2$ or $A \in \mathcal{T}$ and e is not incident to a leaf of A.*

Proof. Let $A \neq P_2$. If $A \notin \mathcal{T}$, then by the minimality of T we have $a_t(A) \leq \frac{|V(A)|-2}{3}$ and $a_t(T - V(A)) \leq \frac{|V(T)|-|V(A)|+1}{3}$. Now $a_t(T) \leq a_t(A) + a_t(T - V(A)) \leq \frac{|V(A)|-2}{3} + \frac{|V(T)|-|V(A)|+1}{3} \leq \frac{|V(T)|-1}{3}$. We conclude that $A \in \mathcal{T}$.

Suppose that e is incident to a leaf u of A, and let w be the neighbor of u in A. Now $V(A) - u$ induces a component of $T - uw$, and $|V(A) - u| \equiv 1 \pmod 3$. Let $\widehat{A}$ be the other component of $T - uw$ (see Figure 4.9). Since $|V(T)| \not\equiv 0 \pmod 3$ by Lemma 4.3.3, we must have $|V(\widehat{A})| \not\equiv 2 \pmod 3$, but this contradicts Lemma 4.3.2. □

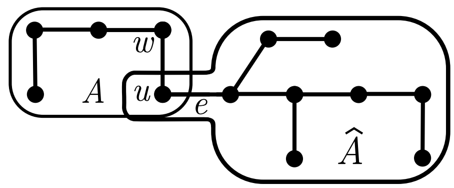

Figure 4.9: The edge e is adjacent to a leaf in A.

Lemma 4.3.5. *If T is a tree such that $|V(T)| \equiv 1 \pmod 3$, then T is not a minimal counterexample to Theorem 4.4.1.*

Proof. By Lemma 4.3.1 and the characterization given in Corollary 4.2.13, we may consider an innver vertex v of T. Let e be an edge incident to v, and let A_1 and A_2 be the components of $T - e$. By Lemma 4.3.2, we may assume that $|V(A_1)| \equiv 2 \pmod 3$, in which case $|V(A_2)| \equiv 2 \pmod 3$. Applying this argument to each edge incident to v gives that each component of $T - v$ has order congruent to 2 modulo 3. By Lemma 4.3.4, $T - v$ consists of isolated edges or components in $\mathcal{T}$ whose leaves are not adjacent to v. The number of such components must be divisible by 3 since $|V(T)| \equiv 1 \pmod 3$. Now Lemma 4.2.11 gives that $a_t(T) \leq \frac{|V(T)|}{3}$, and T is not a counterexample to Theorem 4.4.1. □

We have now shown that the number of vertices in a minimal counterexample to Theorem 4.4.1 must be congruent to 2 modulo 3.

Lemma 4.3.6. *Let T be a minimal counterexample to Theorem 4.4.1 and let v be an inner vertex of T with degree 2. Let A be a component of $T - v$ such that $|V(A)| \equiv 2 \pmod 3$. If w is the neighbor of v in A, then w is a neighbor of a leaf in T.*

Proof. By Lemma 4.3.4, $A \in \mathcal{T} \cup \{P_2\}$. If $A = P_2$ then the neighbor of v in A is a neighbor of a leaf in A and in T, hence, let us assume that $A \neq P_2$. By Lemma 4.3.4, w is not a leaf of A, so all leaves of A are leaves of T. If w is at distance at least 2 from all leaves of T, then w is at distance at least 2 from all leaves of A. Let A' be the tree induced by $V(A) \cup \{v\}$ and let $B = T - V(A')$ as in Figure 4.10. By Lemma 4.2.10, $a_t(A') \leq \frac{|V(A')|-3}{3}$ and by the minimality of T, $a_t(B) \leq \frac{|V(B)|+1}{3}$. Now $a_t(T) \leq a_t(A') + a_t(B) \leq \frac{|V(A')|-3}{3} + \frac{|V(B)|+1}{3} = \frac{|V(T)|-2}{3}$. □

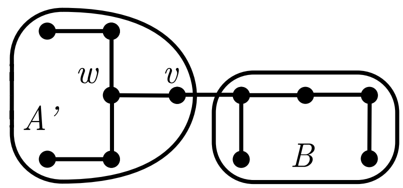

Figure 4.10: The vertex v is adjacent to an inner vertex.

Lemma 4.3.7. *Let T be a minimal counterexample to Theorem 4.4.1 and let v be an inner vertex with degree 2. If the components of $T - v$ are A_1 and A_2, then $|V(A_1)| \equiv 2 \pmod 3$ and $|V(A_2)| \equiv 2 \pmod 3$.*

Proof. If $|V(A_1)|$ and $|V(A_2)|$ are not both congruent to 2 modulo 3 then we may assume that $|V(A_1)| \equiv 1 \pmod 3$ and $|V(A_2)| \equiv 0 \pmod 3$. Now deleting the edge joining v to A_1 as in Figure 4.11 leaves two subtrees whose orders are congruent to 1 modulo 3, contradicting Lemma 4.3.2. □

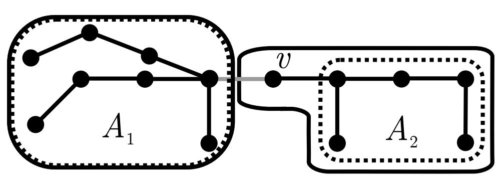

Figure 4.11: Lemma 4.3.2 implies that $|V(A_1)| \equiv 2 \pmod 3$ and $|V(A_2)| \equiv 2 \pmod 3$.

Lemma 4.3.8. *If T is a minimal counterexample to Theorem 4.4.1 and some inner vertex has degree 2, then all inner vertices have degree 2.*

Proof. Let v be an inner vertex of degree 2 and let v' be an inner vertex of degree at least 3. Let the components of $T - v$ be A_1 and A_2. By Lemmas 4.3.7 and 4.3.4, $A_1 \in \mathcal{T} \cup \{P_2\}$ and $A_2 \in \mathcal{T} \cup \{P_2\}$. Without loss of generality we may assume that $v' \in V(A_1)$. Lemma 4.3.4 implies that v is not adjacent to any leaf of A_1, so all leaves of A_1 are leaves of T. Now v' is an inner vertex of A_1 and by Lemma 4.3.6, v is not adjacent to v'. This gives $d_{A_1}(v') \geq 3$, contradicting Property 4.2.4(a). □

4.4 Characterization of acquisition-extremal graphs

Theorem 4.4.1. *Let T be a tree with at least two vertices. If $a_t(T) \geq \frac{|V(T)|+1}{3}$, then $T \in \mathcal{T} \cup \{P_2\}$.*

Proof. By Lemmas 4.3.3 and 4.3.5, it remains only to consider a minimal counterexample T such that $|V(T)| \equiv 2 \pmod 3$. By Lemma 4.3.1 and the characterization given in Corollary 4.2.13, there are inner vertices in T. By Lemma 4.3.8, either all inner vertices have degree 2 or all inner vertices have degree at least 3. Let us first consider the case when each inner vertex has degree 2. Let v be such a vertex, and let the components of $T - v$ be A_1 and A_2. Note that v satisfies the condition given in Definition 4.2.4(a). By Lemmas 4.3.7 and 4.3.4, both A_1 and A_2 are in $\mathcal{T} \cup \{P_2\}$, so v satisfies the condition given in Definition 4.2.4(b). It remains only to show that v satisfies the condition given in Definition 4.2.4(c).

If $A_1 \neq P_2$, then let w be the neighbor of v in A_1. By Lemma 4.3.6, w is adjacent to a leaf in A_1. The non-leaf neighbors of w in A_1 must be an inner vertex of A_1 since otherwise there are leaves in A_1 at distance 3 contradicting Elementary Property 4.2.2(a). Now that w is a neighbor of an inner vertex of A_1, Property 4.2.4(c) gives that $A_1 - w$ consists of an isolated vertex and components whose orders are divisible by 3 (see Figure 4.12 below). Since $|V(A_2) \cup \{v\}|$ is also divisible by 3, $T - w$ satisfies the same condition. A symmetric argument shows if w' is the neighbor of v in A_2, then $T - w'$ satisfies the condition as well. Now v

satisfies the condition given in Definition 4.2.4(c). Since v was arbitrary, Definition 4.2.4 and Theorem 4.2.6 imply that $T \in \mathcal{T}$ and T is not a counterexample to Theorem 4.4.1.

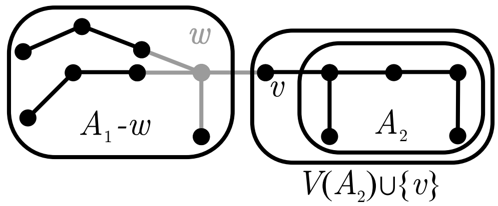

Figure 4.12: The components of $T - w$ are an isolated vertex or have orders divisible by 3.

Now consider the case where each inner vertex has degree at least 3 and let v be such a vertex. Since $|V(T)| \equiv 2 \pmod 3$ and since v is adjacent to no leaves, Lemma 4.3.2 implies that no component of $T - v$ has order congruent to 1 modulo 3. Let the components of $T - v$ be $A_1, \ldots, A_k$ and $B_1, \ldots, B_\ell$, where $|V(A_i)| \equiv 2 \pmod 3$ and $|V(B_j)| \equiv 0 \pmod 3$. By Lemma 4.3.4, $A_i \in \mathcal{T} \cup \{P_2\}$ for each $i \in [k]$. Note that we must have $k \equiv 2 \pmod 3$ since $|V(T)| \equiv 2 \pmod 3$. We claim that if the neighbor u of v in A_i is a leaf of A_i, then $A_i = P_2$. Otherwise the distance between u and each leaf of T is at least 2 but $d_T(u) = 2$. This contradicts the assumption that all inner vertices have degree at least 3.

Let $\widehat{T}$ be the tree obtained by deleting each B_j from T as in Figure 4.13 below. If $k \geq 3$, then Lemma 4.2.11 yields $a_t(\widehat{T}) \leq \frac{|V(\widehat{T})|}{3}$. By the minimality of T, $a_t(B_j) \leq \frac{|V(B_j)|}{3}$ for each j. Now $a_t(T) \leq a_t(\widehat{T}) + \sum_{j=1}^{\ell} a_t(B_j) \leq \frac{|V(\widehat{T})|}{3} + \sum_{j=1}^{\ell} \frac{|V(B_j)|}{3} = \frac{|V(T)|}{3}$ contradicting the assumption that T is a counterexample. We conclude that $k = 2$ and that $\ell \geq 1$ since $d_T(v) \geq 3$.

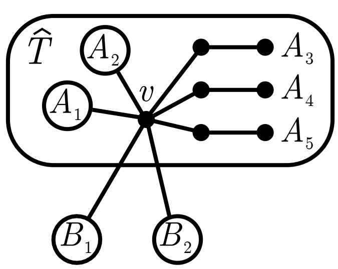

Figure 4.13: If $k \geq 3$, then $a_t(T) \leq \frac{|V(T)|-1}{3}$.

Let w be the vertex of B_1 adjacent to v. There may be leaves of T adjacent to w. Still, Lemma 4.3.2 implies that $T - w$ consists of isolated vertices $u_1, \ldots, u_r$ and components $C_1 \ldots, C_s$ and $D_1, \ldots, D_t$ such that $|V(C_i)| \equiv 2 \pmod 3$ and $|V(D_j)| \equiv 0 \pmod 3$. By Lemma 4.3.4, $C_i \in \mathcal{T} \cup \{P_2\}$ for each $i \in [s]$. If $C_i \neq P_2$ and w is adjacent to a leaf u' of C_i, then in T the distance between u' and each leaf is at least 2 but $d_T(u') = 2$. This contradicts the assumption that all inner vertices have degree at least 3. Since $|V(A_1) \cup V(A_2) \cup \{v\}| \equiv 2 \pmod 3$, we may assume that $s \geq 1$ and that C_1 is the tree induced by $V(A_1) \cup V(A_2) \cup \{v\}$ (see Figure 4.14). If $r + s \geq 3$, then let $\widehat{T}$ be the tree obtained by deleting each D_j from T. By Lemma 4.2.11, $a_t(\widehat{T}) \leq \frac{|V(\widehat{T})|}{3}$. By the minimality of T, $a_t(D_j) \leq \frac{|V(D_j)|}{3}$ for each j, so $a_t(T) \leq a_t(\widehat{T}) + \sum_{j=1}^{t} a_t(D_j) \leq \frac{|V(\widehat{T})|}{3} + \sum_{j=1}^{\ell} \frac{|V(D_j)|}{3} = \frac{|V(T)|}{3}$. Hence, we may assume that $r + s \leq 2$.

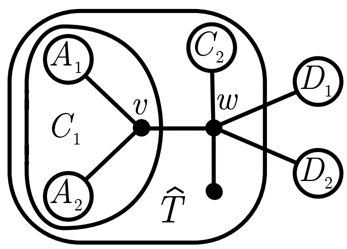

Figure 4.14: If $r + s \geq 3$, then $a_t(T) \leq \frac{|V(T)|-1}{3}$.

Note that $|V(T)| \equiv r+2s+1 \pmod 3$, and $r \leq 1$ since $s \geq 1$. If $r = 1$, then $s \equiv 0 \pmod 3$ since $|V(T)| \equiv 2 \pmod 3$. This contradicts the fact that $r + s \leq 2$. Hence, we may assume $r = 0$ and $s = 2$ (see Figure 4.15). This implies that w is at distance at least 2 from each leaf of T, so by hypothesis, $d_T(w) \geq 3$. Since $d_T(w) = s + t$ and $s = 2$, there must be a component D_1 of $T - w$ whose order is divisible by 3. Let $T' = T - V(D_1)$ as in Figure 4.15. Deleting the edge from D_1 to w leaves components D_1 and T'. Since $|V(T')| \equiv 2 \pmod 3$, Lemma 4.3.4 gives that $T' \in \mathcal{T} \cup \{P_2\}$. Now v is an inner vertex of T' and $d_{T'}(v) = 3$. This contradicts Property 4.2.4(a) and completes the proof. □

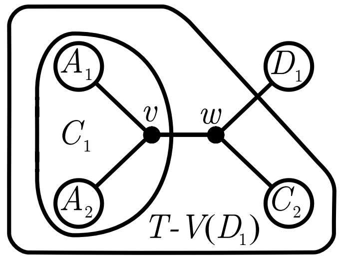

Figure 4.15: Both v and w are at distance at least 2 from all leaves of $T - V(D_1)$.

Corollary 4.4.2 below was originally proved by Lampert and Slater [41]. Theorem 4.4.3 characterizes connected graphs G for which $a_t(G) = \frac{|V(G)|+1}{3}$ and is the main result of this chapter.

Corollary 4.4.2. *If G is a connected graph with at least two vertices, then $a_t(G) \leq \frac{|V(G)|+1}{3}$.*

Proof. Let T be a spanning tree of G. If $T \in \mathcal{T} \cup \{P_2\}$, then by Theorem 4.2.7, $a_t(G) \leq a_t(T) = \frac{|V(G)|+1}{3}$. If $T \notin \mathcal{T} \cup \{P_2\}$, then by Theorem 4.4.1, $a_t(G) \leq a_t(T) < \frac{|V(T)+1}{3}$. □

We have shown that the trees in $\mathcal{T} \cup \{P_2\}$ are extremal when the order is congruent to 2 modulo 3. Our final result shows that C_5 is the only other such graph.

Theorem 4.4.3. *If G is a connected graph with at least two vertices and $a_t(G) = \frac{|V(G)|+1}{3}$, then $G \in \mathcal{T} \cup \{P_2, C_5\}$.*

Proof. If G is a tree, then this follows immediately from Theorem 4.4.1. If G is not a tree, then note that each spanning tree of G must be in $\mathcal{T} \cup \{P_2\}$. It follows easily that if $|V(G)| \leq 5$ and $a_t(G) = \frac{|V(G)|+1}{3}$, then $G \in \{P_2, P_5, C_5\}$, so a counterexample to this theorem must have at least 8 vertices. Let G be a smallest counterexample, and let T be a spanning tree of G. Since G is not a tree, there is an edge e in G that is not an edge of T.

Let T' be the tree obtained by deleting all leaves of T and let T'' be the tree obtained by deleting all leaves of T'. There are $\frac{|V(T)|+1}{3}$ leaves in T and at most $\frac{|V(T)|+1}{3}$ leaves in T', so $|V(T'')| \geq \frac{|V(T)|-2}{3}$. Since $|V(T)| \geq 8$, there are at least two leaves v_1 and v_2 of T''. By

Lemma 4.2.5, both v_1 and v_2 are contained in pendant three-edge paths attached to the rest of T at a neighbor of a leaf. Let $F_1 = \{v_1, w_1, u_1\}$ and $F_2 = \{v_2, w_2, u_2\}$ be the vertices in these paths, respectively, as in Figure 4.16.

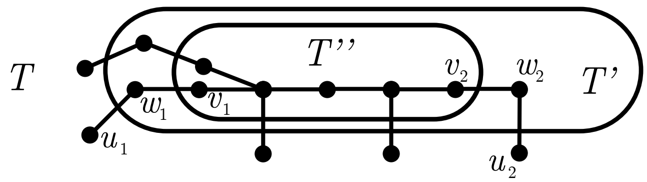

Figure 4.16: There are at least two leaves in the tree T''.

We claim that the edge e must be incident to one vertex in F_1 and one vertex in F_2. If e is not incident to any vertex in F_1, then $G - F_1$ contains a cycle. If $G - F_1$ is not C_5, then by the minimality of G, $a_t(G - F_1) \leq \frac{|V(G-F_1)|-2}{3} = \frac{|V(G)|-5}{3}$. Now F_1 induces P_3 and the weight on F_1 may be acquired by w_1. This gives $a_t(G) \leq \frac{|V(G)|-2}{3}$, so G is not a counterexample. If $G - F_1 = C_5$, then let $\widehat{T}$ be the spanning tree obtained by replacing the edge u_2w_2 with the edge e as depicted in Figure 4.17. The central vertex of $\widehat{T}$ is an inner vertex of degree 3. By Property 4.2.4(a), $\widehat{T} \notin \mathcal{T}$, so G is not a counterexample since it has a spanning tree $\widehat{T}$ not in $\mathcal{T}$. We conclude that e is incident to a vertex in F_1 and a vertex in F_2.

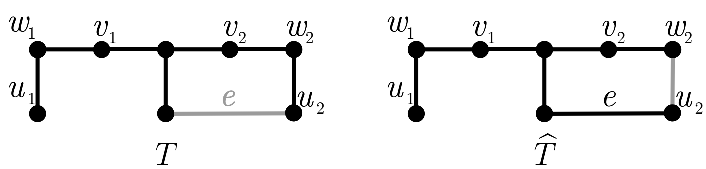

Figure 4.17: The spanning tree $\widehat{T}$ is not in $\mathcal{T} \cup \{P_2\}$

Let f be the edge of T incident to v_1 that is not v_1w_1. If e is incident to w_1, then replacing the edge f with e produces a tree $\widehat{T}$ in which w_1 is adjacent to two leaves u_1 and v_1 as shown in Figure 4.18. These leaves are at distance 2 in $\widehat{T}$, so $\widehat{T}$ is not in $\mathcal{T}$ by Elementary Property 4.2.2(a). Hence, we may assume that e is not incident to w_1, and similarly, e is not incident to w_2.

If e is incident to u_1, then let $\widehat{T}$ be the tree obtained from T by replacing f with e as shown in Figure 4.18. (The resulting tree $\widehat{T}$ is the same if e is incident to v_1.) We must have $\widehat{T} \in \mathcal{T} \cup \{P_2\}$, but the distance between u_1 and each leaf of $\widehat{T}$ is at least 2. By Property 4.2.4(b), both neighbors of u_1 in $\widehat{T}$ must be neighbors of leaves. Now e must be incident to a vertex in $\{u_2, v_2, w_2\}$ that is a neighbor of a leaf in $\widehat{T}$. The only such vertex is w_2, but e cannot be incident to w_2. We conclude that e is not incident to u_1. We have now excluded all cases, so the theorem is proved. □

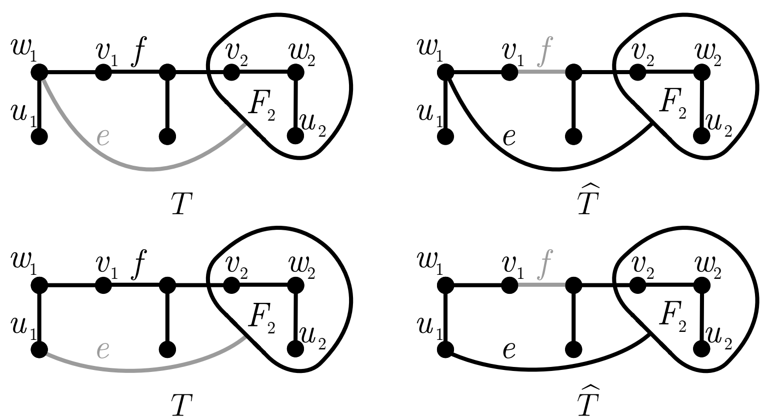

Figure 4.18: The spanning tree $\widehat{T}$ is not in $\mathcal{T} \cup \{P_2\}$

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