PURSUIT-EVASION AND TIME-DEPENDENT GRADIENT FLOW IN SINGULAR SPACES

BY

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DISSERTATION

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Abstract

In this dissertation, we consider an applied problem, namely, pursuit-evasion games. These problems are related to robotics, control theory and computer simulations. We want to find the solution curves of differential equations for pursuit-evasion games, and investigate the properties of solution curves. First, we define $CAT(0)$ and $CAT(K)$ spaces, and explain why they are suitable playing fields, that vastly generalize the usual playing field in the pursuit-evasion literature. Then we prove our existence and uniqueness theorems for continuous pursuit curves in $CAT(K)$ spaces, as well as our convergence estimates and regularity theorem.

Pursuit curves are downward gradient curves for the distance from a moving evader, that is, for a time-dependent gradient flow. We consider not only pursuit curves, but also more general time-dependent gradient flow.

Dedicated to my wife, Eunsil

and my son, Eunjae.

Acknowledgments

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Chapter 1

Introduction

1.1 Motivation

With different domains and different strategies, pursuit-evasion games were considered by many mathematicians, computer scientists and engineers. Those problems are generated from robotics, control theory and computer simulations. With simple strategy assumptions, the main constraints on pursuit-evasion are the geometry and topology of playing domains. Almost always these have been two-dimensional Euclidean domains, or higher-dimensional convex Euclidean domains. Recently came results on surfaces of revolution [16], cones [25], and round spheres [20]. Finally, more general metric spaces including all previous domains were studied as a natural setting, because pursuit-evasion requires neither smoothness nor being locally Euclidean [1] [2]. Those metric spaces are *CAT(K) spaces*. Those playing fields are vastly more general than have been usual in the extensive pursuit-evasion literature.

Roughly, a *CAT(K) space* is a complete metric space such that no triangle is fatter than the triangle with same edge lengths in the model space of constant curvature K (see Definition 2.2.3). Among many references, we mention [12] and [13]. $CAT(K)$ spaces include many important examples. Examples of $CAT(0)$ spaces include convex Euclidean domains, simply connected Riemannian manifolds with non-positive sectional curvature, and trees. Spheres, surfaces of revolution, closed Euclidean domains with smooth boundary supported by spheres [1] and finite-dimensional spherical polyhedra with the link condition of Gromov [15], as well as all $CAT(0)$ spaces, are examples of $CAT(K)$ spaces for $K > 0$.

Recently, continuous-time pursuit games were applied to show the non-existence of shycoupled Brownian motions in many Euclidean domains [11].

On $CAT(K)$ spaces, we will find solution curves of differential equations for pursuit-evasion games. (In particular, this result gives a proof of existence of solutions of differential equations in smooth cases.) Those solution curves are called *continuous pursuit curves*. By the Euler method, discrete solutions are generated geometrically. We show the existence of continuous pursuit curves as the limit of Cauchy sequences obtained by discrete solutions. Here we find Cauchy sequences without an assumption of local compactness for $CAT(K)$ spaces, so we do not need Arzelà-Ascoli Theorem. Uniqueness follows, from estimates for the dependence on initial condition. Moreover, we give estimates on rate of convergence. The regularity properties of continuous pursuit curves that we obtain are new even for previously studied domains, and have important global consequences.

Continuous pursuit curves are negative gradient curves for the *time-dependent* function $dist_{E(t)}$, where E is a given evader curve. Given a point x_0, the negative gradient flow of $dist_{x_0}$ is the geodesic flow toward x_0 as center. Since $E(t)$ is a curve, the continuous pursuit curve of $E(t)$ is a first example of time-dependent gradient curves on $CAT(0)$ spaces.

Time-independent gradient flow has been studied extensively on $CAT(0)$ and related metric spaces [9] [14] [22] [24]. In [24], Mayer found the solution for gradient flows of (time-independent) almost convex functions on $CAT(0)$ spaces using approximation by Cauchy sequences. We will use time-independent gradient curves to generate discrete solutions for the time-dependent case. To prove existence, uniqueness and convergence estimates for gradient curves for time-dependent functions with a convexity property, we formulate appropriate dependence in the time variable. These results in turn feed back to the pursuit-evasion setting to allow multiple evaders or uncertainty in evader position.

1.2 Main results

First we consider pursuit-evasion games. Our pursuer uses the "greedy algorithm", always moving directly toward the evader. We prove the following existence and uniqueness theorem for continuous pursuit curves.

Theorem 1.2.1. *Let X be a $CAT(K)$ space and E be a rectifiable curve with speed ≤ 1 in X. Then there is a unique continuous pursuit curve $P = P(t)$ when $d(P(0), E(0)) < D_K$. So if $K \leq 0$, this result holds for any $P(0)$.*

For the definition of continuous pursuit curves and discrete pursuit curves, look at Definition 3.1.1 and 3.1.2.

Next, we have the following error estimate between the continuous pursuit curve and the discrete pursuit curve.

Theorem 1.2.2. *Let P_n be the discrete pursuit curve with the time gap $1/2^n$ where $P_n(0) = P(0)$ for the continuous pursuit curve P obtained in Theorem 1.2.1. Then for all $t \in [0, T]$, there is $n_0 = n_0(T)$ such that for all $n \geq n_0$,*

$$d(P(t), P_n(t)) \leq \frac{1}{2^n} D' t e^{Dt}$$

with constants D and D' dependent on T and K. Furthermore, if the evader wins, then this is true with constant D and D' dependent on K only.

The continuous pursuit curve we obtain has the following regularity property.

Theorem 1.2.3. *Let X be a $CAT(K)$ space. An initial arc $P(t), 0 \leq t \leq T$ of a continuous pursuit curve P of Theorem 3.1.5 for which $d(P(T), E(T)) > 0$ is a subspace of extrinsic curvature at most*

$$\begin{cases} \frac{32\sqrt{2}}{d(P(T),E(T))} & \text{if } K \leq 0 \\ \frac{8\sqrt{2}\sqrt{K}}{\sin(\sqrt{K} d(P(T),E(T))/4)} & \text{if } K > 0. \end{cases}$$

A curvature bound is regarded as a replacement for $C^{1,1}$ regularity in smooth spaces.

The following two results extend work in [1].

Theorem 1.2.4. *On any $CAT(K)$ space, if the evader wins a continuous-time simple pursuit when $d(E(0), P(0)) \leq D_K$, then $\tau(t)/\sqrt{t}$ is bounded where $\tau(t)$ is the total curvature of the continuous pursuit curve $P|_{[0,t]}$.*

For the definition of the total curvature, look at Definition 2.2.8.

Corollary 1.2.5. *On any $CAT(0)$ space, if the evader wins a continuous-time simple pursuit, then $\sqrt{t}/c(t)$ is asymptotically bounded, where $c(t)$ is the pursuer's circumradius up to time t.*

For multiple evaders, we get a continuous pursuit curve chasing the barycenter point of evaders at each time.

Theorem 1.2.6. *Let X be a $CAT(0)$ space. Given n evaders $E_i = E_i(t)$ with speed ≤ 1 in X, there is a unique continuous pursuit curve $P = P(t)$ chasing the barycenter curve b of evaders.*

Now we turn to general time-dependent gradient flows in $CAT(0)$ spaces. Time-independent gradient flow was considered in $CAT(0)$ spaces by Mayer [24] and in related metric spaces by Ambrosio, Gigli and Savaré [9] (see also [14]). Independently, it was studied in $CAT(K)$ spaces geometrically by Lytchak [22].

We obtain the following result about time-dependent gradient curves.

Theorem 1.2.7. *Let (X, d) be a $CAT(0)$ space. Given $x_0 \in X$ and $t_0 \in \mathbb{R}$, suppose a function $F : \mathbb{R} \times X \to \mathbb{R}$ satisfies*

1) F is locally Lipschitz on X,

2) F is λ-convex on X,

3) F is L-Lipschitz in t,

4) $\exists\ B > 0$ *such that the function* $t \mapsto e(t, x, h)$ *is* Bh*-Lipschitz for any* $x \in X$ *and* $h > 0$.

Then we have a unique time-dependent gradient curve u_{x_0,t_0} *given by* $t \mapsto u_{x_0,t_0}(t_0 + t)$ *of* F *such that* $u_{x_0,t_0}(t_0) = x_0$.

For the definition of the function $t \mapsto e(t, x, h)$, look at Proposition 4.1.14.

Chapter 2

Preliminaries

In this chapter we introduce our main playing domains, spaces of curvature bounded from above, and look over basic definitions and facts about these spaces.

2.1 Curves in metric spaces

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$

1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.

2) $d(x, y) = d(y, x)$.

3) $d(x, y) \leq d(x, z) + d(z, y)$.

We will call (X, d) a metric space. X is *complete* if every Cauchy sequence in X converges. $f : X \to \mathbb{R}$ is said to be C-*Lipschitz* if $|f(x) - f(y)| \leq Cd(x, y)$ for all x, y in X.

A curve $\gamma : I \to X$ is called a *geodesic* if for all $t, t' \in I$, $d(\gamma(t), \gamma(t')) = c|t - t'|$ where c is a constant, the *speed* of the geodesic γ.

$[xy]$ denotes a unit-speed geodesic γ from x to y defined on $[0, t]$ such that $\gamma(0) = x$ and $\gamma(t) = y$ where $t = d(x, y)$. (xy) denotes the geodesic $[xy]$ without the end points x and y.

$\triangle xyz$ denotes the geodesic triangle of geodesics $[xy]$,$[xz]$ and $[yz]$.

A metric space is a *geodesic space* if any two points are joined by a geodesic; and a C-*geodesic space* if any two points with distance $< C$ are joined by a geodesic.

The length $L(\gamma)$ of a curve $\gamma : [a, b] \to X$ is defined by

$$L(\gamma) = \sup \sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i+1}))$$

over all partitions $a = t_1 < t_2 < \cdots < t_{k+1} = b$ of $[a, b]$. γ is *rectifiable* if $L(\gamma) < \infty$.

2.2 Spaces of curvature bounded from above

M_K denotes the 2-dim simply-connected model space of constant curvature K. Then $M_0 = \mathbb{E}^2$, $M_1 = \mathbb{S}^2$ and $M_{-1} = \mathbb{H}^2$. Let d_K be the metric of M_K. For a geodesic $[xy]$ in M_K, the length of $[xy]$, $d_K(x, y)$, is denoted by $|xy|$ when M_K is given clearly. D_K denotes the diameter of M_K. Thus, $D_K = \frac{\pi}{\sqrt{K}}$ if $K > 0$ and $D_K = \infty$ if $K \leq 0$.

Let $\kappa = \sqrt{|K|}$ when $K \neq 0$. If α_1, α_2, α_3 are the angles of a geodesic triangle $\triangle$ in M_K and a_1, a_2, a_3 are the lengths of the opposite sides, then we have the following laws of sines and cosines:

$$\begin{aligned}
\frac{\sinh \kappa a_1}{\sin \alpha_1} = \frac{\sinh \kappa a_2}{\sin \alpha_2} = \frac{\sinh \kappa a_3}{\sin \alpha_3}, & \qquad if\ K < 0 \\
\frac{a_1}{\sin \alpha_1} = \frac{a_2}{\sin \alpha_2} = \frac{a_3}{\sin \alpha_3}, & \qquad if\ K = 0 \\
\frac{\sin \kappa a_1}{\sin \alpha_1} = \frac{\sin \kappa a_2}{\sin \alpha_2} = \frac{\sin \kappa a_3}{\sin \alpha_3}, & \qquad if\ K > 0 \\
\cosh \kappa a_1 = \cosh \kappa a_2 \cosh \kappa a_3 - \sinh \kappa a_2 \sinh \kappa a_3 \cos \alpha_1, & \qquad if\ K < 0 \\
a_1^2 = a_2^2 + a_3^2 - 2a_2 a_3 \cos \alpha_1, & \qquad if\ K = 0 \\
\cos \kappa a_1 = \cos \kappa a_2 \cos \kappa a_3 + \sin \kappa a_2 \sin \kappa a_3 \cos \alpha_1, & \qquad if\ K > 0.
\end{aligned}$$

By the *Gauss-Bonnet formula*,

$$KA = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

where A is the area of $\triangle$.

Definition 2.2.1. *A curve of constant geodesic curvature k in M_K is called a k-*curve. *Thus if $K \geq 0$, a k-curve is a circular arc or geodesic segment.*

A triangle $\triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3$ in M_K is called a *comparison triangle* for $\triangle x_1x_2x_3$ in X if $d_K(\widetilde{x}_i, \widetilde{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. We write

$$\widetilde{\triangle}x_1x_2x_3 = \triangle\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3.$$

Let us define the (Alexandrov) *angle* between two geodesics.

Definition 2.2.2. *Let γ_1, γ_2 be two geodesics in X starting at x. The* (Alexandrov) angle *$\angle_x(\gamma_1, \gamma_2)$ between γ_1 and γ_2 is given by*

$$\angle_x(\gamma_1, \gamma_2) := \limsup_{t_1, t_2 \to 0} \alpha(t_1, t_2)$$

where $\alpha(t_1, t_2)$ is the angle of $\triangle\widetilde{x}\widetilde{y}_1\widetilde{y}_2$ at $\widetilde{x}$ for $y_i = \gamma_i(t_i)$.

Note that we can get $\alpha(t_1, t_2)$ with the law of cosines.

Definition 2.2.3. *Let (X, d) be a metric space and K be a real constant. A D_K-geodesic space X is a* CAT(K) space *if for any geodesic triangle $\triangle xy_1y_2$ of perimeter $< 2D_K$, and its comparison triangle $\triangle\widetilde{x}\widetilde{y}_1\widetilde{y}_2$ in M_K, we have*

$$d(z_1, z_2) \leq d_K(\widetilde{z}_1, \widetilde{z}_2),$$

where z_i is any point on $[xy_i]$ and $\widetilde{z}_i$ is the point on $[\widetilde{x}\widetilde{y}_i]$ such that $d_K(\widetilde{x}, \widetilde{z}_i) = d(x, z_i)$ for $i \in \{1, 2\}$.

If X is a $CAT(K)$ space, then $\alpha(t_1, t_2)$ is non-increasing in both variables. So there exists $\lim_{t \to 0} \alpha(t, t)$ and it is equal to $\angle_x(\gamma_1, \gamma_2)$. For $\triangle xyz$ in X, the *angle* $\angle yxz$ of $\triangle xyz$ at x is the Alexandrov angle between $[xy]$ and $[xz]$.

We can generate a definition of a direction since the triangle inequality holds for angles between two geodesics and $\angle_x(\gamma, \gamma) = 0$ where γ is a geodesic starting at $x \in X$. Two geodesics γ_1 and γ_2 starting at $x \in X$ have the *same direction at* x if $\angle_x(\gamma_1, \gamma_2) = 0$ and we denote this relation by $\gamma_1 \sim \gamma_2$. This is an equivalence relation on the set of geodesics starting at x. Then this set of equivalence classes is a metric space with metric $\angle_x$. Now consider the intrinsic metric d induced from $\angle_x$. Note that if $d([\gamma_1], [\gamma_2]) \leq \pi$, then $d([\gamma_1], [\gamma_2]) = \angle_x([\gamma_1], [\gamma_2])$. The completion of this space with metric d is called the *space of directions at* x and is denoted by Σ_x.

The Euclidean cone over Σ_x is called the *tangent cone* T_x at x; the elements of T_x are pairs $v = (\xi, r)$ where $\xi \in \Sigma_x$, $r \geq 0$ is a real number. We call ξ the direction of v, and r the length of v. All the pairs $(\xi, 0)$ are identified as o_x and o_x is called the *vertex* of T_x. The *norm* on T_x is given by $r = ||(\xi, r)||$, that is, it is the distance from the vertex o_x, and the angle between $(\xi, r), (\eta, s) \in T_x$, when both $r, s \neq 0$, is the same as the angle between ξ, η.

The *inner product* $\langle v, w \rangle$ for $v, w \in T_x$ is defined by $||v|| ||w|| \cos\theta$ where θ is the angle between v and w if $v \neq 0$ and $w \neq 0$. Otherwise, define $\langle v, w \rangle = 0$.

For $f : T_x \to \mathbb{R}$, f is *homogeneous* if for any $r \geq 0$, $f(rv) = rf(v)$ for all $v \in T_x$.

Theorem 2.2.4. *[12] If X is a $CAT(K)$ space, then Σ_x is a $CAT(1)$ space, and T_x is a $CAT(0)$ space.*

Definition 2.2.5. *Let X be a $CAT(K)$ space, and $\gamma : I \to X$ be a rectifiable curve. For $t, t' \in I$ such that $t \leq t'$, let $\xi_{t'}$ be the direction at $\gamma(t)$ of $[\gamma(t)\gamma(t')]$. The curve γ has a* right-side tangent vector $\gamma'(t+) = (\xi, r)$ *at $\gamma(t)$ if there exist*

$$r = \lim_{t' \to t+} \frac{d(\gamma(t'), \gamma(t))}{t' - t},$$

and

$$\xi = \lim_{t' \to t+} \xi_{t'}.$$

Let X be a $CAT(0)$ space, and $d_Y : X \to \mathbb{R}$ be the distance to a complete convex subset Y, defined by

$$x \mapsto d_Y(x) := \min_{y \in Y} d(y, x).$$

Proposition 2.2.6. *[12] The distance function d_Y is convex and for $x \in X$, there is a unique point x_0 of Y such that $d_Y(x) = d(x_0, x)$.*

This point x_0 is called the *footpoint of x in Y.*

Let us introduce Reshetnyak's Majorization Theorem.

Theorem 2.2.7. *[27] Let γ be a closed curve of length $< 2D_K$ in a $CAT(K)$ space X. Then there is a closed curve $\widetilde{\gamma}$ which is the boundary of a convex region $\mathcal{C}$ in M_K and a distance-nonincreasing map $\Phi : \mathcal{C} \to X$ such that the restriction of Φ to $\widetilde{\gamma}$ is an arclength-preserving map onto γ.*

Also we will use the following definitions in our proofs.

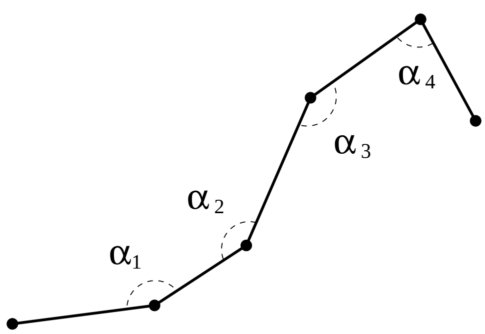

Figure 2.1: A polysegment with six vertices and the total curvature of this polysegment $\sum_{j=1}^{4}(\pi - \alpha_j)$

Definition 2.2.8. *[23] In a $CAT(K)$ space, the* total rotation $\tau^*(\sigma)$ *of a polysegment or polygon (closed polysegment) σ is given by*

$$\sum_j (\pi - \alpha_j)$$

where α_j *are the angles at the interior vertices. The* total curvature $\tau(\gamma)$ *of any curve* γ *is* $\limsup_{\mu_\sigma \to 0} \tau^*(\sigma)$ *where* σ *is a polysegment (or polygon if* γ *is closed) inscribed in* γ *having the maximum edge length* μ_σ*.*

Note $0 \leq \alpha_j \leq \pi$. In [23], it is shown $\tau^*(\sigma) = \tau(\sigma)$ when σ is a polysegment or polygon.

We shall need Helly's Theorem for $CAT(0)$ spaces.

Theorem 2.2.9 (Helly's Theorem)**.** *[21] [6, Th. 29.2] Let* X *be a CAT(0) space and* $\{Y_a\}_{a \in I}$ *be an arbitrary collection of closed bounded convex subsets of* X*.*

If

$$\bigcap_{a \in I} Y_a = \emptyset,$$

then there is an indexed array $a_1, a_2, \ldots, a_n \in I$ *such that*

$$\bigcap_{i=1}^{n} Y_{\alpha_i} = \emptyset.$$

Chapter 3

Continuous pursuit curves in CAT(K) spaces

In this chapter, we first define continuous pursuit curves. Then we prove a theorem of existence and uniqueness of continuous pursuit curves. These pursuit curves have a regularity property as a replacement for $C^{1,1}$ regularity in smooth spaces.

Unless otherwise specified, in this chapter, X always denotes a $CAT(K)$ space.

3.1 Existence of continuous pursuit curves

Given an evader curve and an initial position P_0 of a pursuer, where the pursuer knows the evader's position at every time and chases the evader by moving directly toward the evader, this chasing curve is called the *continuous pursuit curve*.

In order to understand that the pursuer chases the evader by moving directly toward the evader, we consider a simple case. If the evader doesn't move and stays at E_0, the continuous pursuit curve from P_0 is just a geodesic from P_0 to E_0.

Definition 3.1.1. *Let X be a $CAT(K)$ space and $E = E(t)$ be a rectifiable evader curve. A unit-speed continuous curve $P = P(t)$ is a* (simple) continuous pursuit curve *if there is a shortest geodesic γ_t from $P(t)$ to $E(t)$ for every t such that $\gamma_t'(0+)$ is the right-side tangent vector $P'(t+)$.*

Here, we have a time-dependent function $dist_{E(t)}$ defined by $dist_{E(t)}(x) = d(x, E(t))$. If we denote the downward gradient unit vector of the function $dist_{E(t)}$ at $x \in X$ by $v(t, x)$, then $\gamma_t'(0+)$ is equal to $v(t, P(t))$ in Definition 3.1.1. So we may regard the continuous pursuit curve P as a solution of a differential equation $P'(t+) = v(t, P(t))$ with $P(0) = P_0$.

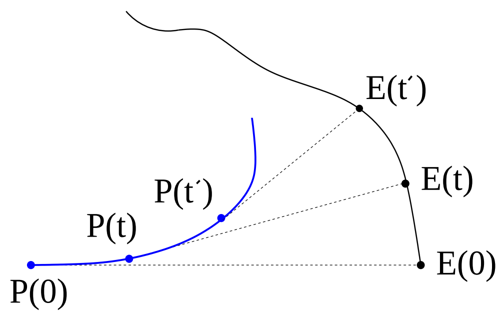

Figure 3.1: A continuous pursuit curve $P = P(t)$ and an evader's curve $E = E(t)$

If the evader's position information is given with a discrete time gap h, we obtain a *discrete pursuit curve*, defined as follows.

To solve an ordinary differential equation $y'(t) = f(t, y(t))$ with $y(0) = y_0$ by the Euler method, we find linear approximation points y_n. A polysegment connecting those points is a discrete approximation curve of the real solution $y = y(t)$. Similarly in our case, in each time interval $[nh, (n+1)h]$ where $n = 0, 1, \cdots$, the pursuer moves directly toward the given evader's position $E(nh)$ with unit-speed along a geodesic from the pursuer to $E(nh)$:

Definition 3.1.2. *A* discrete pursuit curve P_h *with step size* h *is a unit speed curve defined by a chain of geodesic segments having length* h*, such that the geodesic* $[P_h(nh)E(nh)]$ *is an extension of a geodesic* $[P_h(nh)P_h(nh+h)]$ *for all integers* $n \geq 0$*. Thus, if* t *is in* $[nh, nh+h]$*,* $P_h(t)$ *is the point on the geodesic* $[P_h(nh)P_h(nh+h)]$ *such that* $d(P_h(nh), P_h(t)) = t - nh$*.*

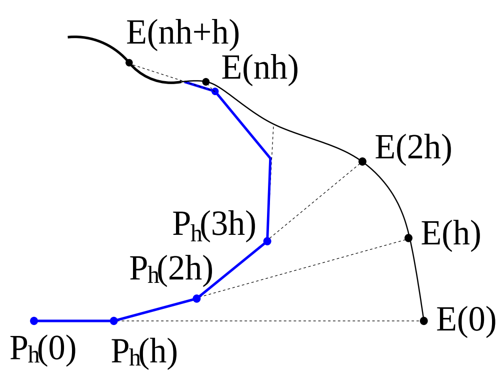

Figure 3.2: A discrete pursuit curve P_h with step size h

For one evader's curve E, now we have two different kinds of pursuit curves. Discrete pursuit curves always exist uniquely when $K \leq 0$. When $K > 0$, we need additional conditions to get uniqueness.

Lemma 3.1.3. *Let X be a $CAT(K)$ space where $K > 0$ and E be a rectifiable curve with speed ≤ 1 in X. If $d(P_h(0), E(0)) < D_K$, then a discrete pursuit curve P_h is unique for a given step size h. In particular, the distance $L_n := d(P_h(nh), E(nh))$ is non-increasing in n.*

Proof. For induction, first we assume that for n, $d(P_h(nh), E(nh)) < D_K$. Then since X is the D_K-geodesic space, there is a unique geodesic $[P_h(nh)E(nh)]$ and we can get a point $P_h(nh+h)$ on this geodesic. Then since $d(P_h(nh+h), E(nh))+h = d(P_h(nh), E(nh)) < D_K$ and $d(E(nh), E(nh+h)) \leq h$, we have

$$d(P_h(nh+h), E(nh+h)) \leq d(P_h(nh), E(nh)) < D_K.$$

Since $d(P_h(0), E(0)) < D_K$, our proof is done. □

Since L_n is non-increasing in n, $\lim_{n\to\infty} L_n$ exists. If this limit is bigger than h, we will say that *the evader wins*. In pursuit-evasion games, it is important to know whether the pursuer captures the evader or not. For continuous pursuit curves, we can define when the evader wins after we prove the following lemma:

Lemma 3.1.4. *Let E be an evader with speed ≤ 1 in a $CAT(0)$ space. Suppose there exists a continuous pursuit curve P of E. If L is the distance function given by $L(t) = d(P(t), E(t))$, then L is non-increasing in t.*

Proof. Let θ be the angle between the tangent vector $E'(t)$ (which exists almost everywhere since E is rectifiable) and the geodesic $[E(t)P(t)]$ at $E(t)$. Since the angle between the right-side tangent $P'(t+)$ and the geodesic $[P(t)E(t)]$ at $P(t)$ is zero, by the First Variation Formula

$$\frac{dL}{dt}(t) = -1 - ||E'(t)|| \cos\theta \leq 0$$

for almost all t. Since

$$|L(t_1) - L(t_2)| \leq d(P(t_1), P(t_2)) + d(E(t_1), E(t_2)),$$

L is 2-Lipschitz (absolutely continuous) on $[a, b]$. So we can use the Fundamental Theorem of Calculus. Thus $L(b) - L(a) = \int_a^b L'(t)dt \leq 0$ if $a \leq b$. □

Since L is non-increasing, there exists a limit $\lim_{t\to\infty} L(t)$. If $\lim_{t\to\infty} L(t) > 0$, then we will say that *the evader wins*.

Next, we need to know when continuous pursuit curves exist. Here is a theorem to answer that question.

Theorem 3.1.5. *Let X be a $CAT(K)$ space and E be a rectifiable curve with speed ≤ 1 in X. Then there is a unique continuous pursuit curve $P = P(t)$ when $d(P(0), E(0)) < D_K$. So if $K \leq 0$, this result holds for any $P(0)$.*

First of all, we explain why we need the condition $d(P(0), E(0)) < \pi/\sqrt{K}$ when K is positive. A Euclidean plane with an open disk of radius $1/\sqrt{K}$ removed is a $CAT(K)$ space. When the initial point $P(0)$ of P is opposite to the initial point $E(0)$ of E with respect to the circle boundary and E moves on the radial line from the center to $E(0)$, then the distance between $P(0)$ and $E(0)$ is $\geq \pi/\sqrt{K}$ and we will have two different continuous pursuit curves starting from $P(0)$.

In order to prove Theorem 3.1.5, we need to get an angle property on the discrete pursuit curve. Taking a time gap $h = 1/2^n$, we will have a set of discrete pursuit curves forming a convergent sequence as n goes to ∞. Also we have the following estimate on rate of convergence:

Theorem 3.1.6. *Let P_n be the discrete pursuit curve with the time gap $1/2^n$ where $P_n(0)$ is equal to $P(0)$ of the continuous pursuit curve P obtained in Theorem 3.1.5. Then for all*

$t \in [0, T]$, there is $n_0 = n_0(T)$ such that for all $n \geq n_0$,

$$d(P(t), P_n(t)) \leq \frac{1}{2^n} D' t e^{Dt}$$

with constants D and D' dependent on T and K. Furthermore, if the evader wins, then this is true with constant D and D' dependent on K only.

3.1.1 Proofs of Theorems 3.1.5 and 3.1.6

Suppose that P_n is the discrete pursuit curve with the time gap $1/2^n$ where $P_n(0)$ is equal to $P(0)$ in Theorem 3.1.5. Then we will discuss how to define P when there exists $B = B(T) > 0$ such that $d(P_n(T), E(T)) > B$ for all but finitely many n.

Lemma 3.1.7. *Assume that there exists $B = B(T) > 0$ such that $d(P_n(T), E(T)) > B$ for all but finitely many n, and $B < D_K - d(P(0), E(0))$ where $d(P(0), E(0)) = d(P_n(0), E(0))$. For $t \in [0, T]$ and n so large that*

$$\begin{cases} \frac{1}{2} > \frac{2h}{B} & \text{if } K \leq 0 \\ \frac{1}{2} > \frac{\sqrt{K} h}{\sin(B\sqrt{K}/2)} & \text{if } K > 0 \ , \end{cases}$$

where $h = 1/2^n$, let j be an integer such that $jh \leq t < (j+1)h$. Let α_t be $\angle E(jh) P_n(t) E(t)$ and $\widetilde{\alpha}_t$ be the corresponding angle of the comparison triangle $\triangle \widetilde{E}(jh) \widetilde{P}_n(t) \widetilde{E}(t)$ of M_K. Then

$$\alpha_t \leq \widetilde{\alpha}_t < Ch$$

where C ($= \frac{2\prime\kappa}{\sqrt{3}\sin(B\kappa/2)}$ if $K > 0$ or $= 4/(\sqrt{3}B)$ if $K \leq 0$) is a constant which is independent of t and n.

Proof. We bound $\widetilde{\alpha}_t$ by using lower and upper bounds on the long sides of the comparison triangle, the upper bound $1/2^n$ on the short side (opposite $\widetilde{\alpha}_t$), and Euclidean geometry or

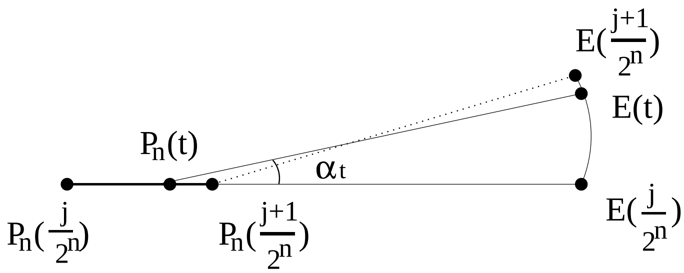

Figure 3.3: A triangle $E(jh)P_n(t)E(t)$ for $t \in [jh, (j+1)h)$

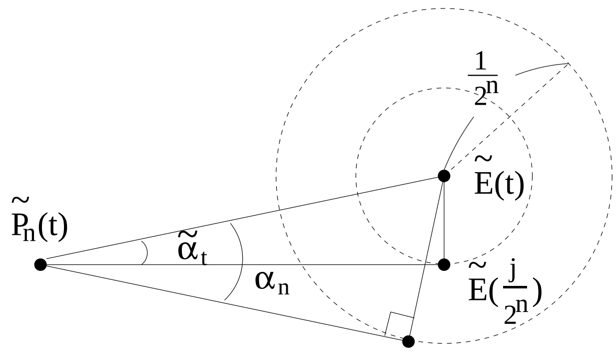

Figure 3.4: A comparison triangle $\triangle \widetilde{E}(jh)\widetilde{P}_n(t)\widetilde{E}(t)$

spherical geometry.

First, we will show it for $K > 0$. Considering a right triangle of M_K which has an edge of length $h = 1/2^n$ and a hypotenuse $\widetilde{P}_n(t)\widetilde{E}(t)$ (see Figure 3.4) gives $\widetilde{\alpha}_t \leq \alpha_n$, where α_n is the angle of the right triangle at $\widetilde{P}_n(t)$.

From Law of Sines in spherical geometry of M_K, we have

$$\frac{\sin(\kappa h)}{\sin \alpha_n} = \frac{\sin[\kappa d\big(P_n(t), E(t)\big)]}{\sin(\pi/2)} \tag{3.1}$$

where $\kappa = \sqrt{K}$. Since

$$d\big(P_n(t), E(t)\big) \geq d\big(P_n(t), E(jh)\big) - h \geq d\big(P_n(jh), E(jh)\big) - 2h > B - 2h > B/2$$

(see Figure 3.3) and $d\big(P_n(t), E(t)\big) \le \pi/\kappa - B$, we have

$$\sin[\kappa d\big(P_n(t), E(t)\big)] \ge \sin(B\kappa/2).$$

Then Equation (3.1) gives

$$\frac{\kappa h}{\sin(B\kappa/2)} \ge \sin\alpha_n.$$

Since the sine function is analytic with nonzero derivative on $[0, \pi/2)$, the inverse function $\sin^{-1}$ is analytic and has a bounded derivative on any closed subinterval of the range $[0, 1)$. For large n, α_n is less than $\pi/2$ since $|\widetilde{P}_n(t)\widetilde{E}(t)|$ is not close to zero and it is not close to π. By the assumption of Lemma 3.1.7, $\sin\alpha_n < 1/2$. The derivative of $\sin^{-1}$ is increasing on [0,1/2], so its maximum at 1/2 is a bound and we can be specific, $C' = 2/\sqrt{3}$. This bound is a Lipschitz constant for $\sin^{-1}$ on $[0, 1/2]$. Therefore

$$\widetilde{\alpha}_t \le \alpha_n \le \sin^{-1}\frac{\kappa h}{\sin(B\kappa/2)} \le \frac{C'\kappa h}{\sin(B\kappa/2)} = C'\Big|\frac{\kappa h}{\sin(B\kappa/2)} - 0\Big|.$$

So Lemma 3.1.7 is true with $C = \frac{C'\kappa}{\sin(B\kappa/2)}$.

Since $\lim_{K\to 0}\frac{C'\kappa}{\sin(B\kappa/2)} = \frac{2C'}{B}$, Lemma 3.1.7 holds for $K \le 0$ with $C = \frac{2C'}{B}$.

□

In order to prove Proposition 3.1.9, we also need the following lemma:

Lemma 3.1.8. *Assume $\triangle zx_1x_2$ is a triangle on M_K with $K > 0$. Let x_i' be the point on $[x_iz]$ at distant h from x_i. If $R_1 \le |zx_i| \le D_K - R_1$ and $h \le R_1/4$, then*

$$|x_1'x_2'| \le (1 + Dh)|x_1x_2|$$

where $0 < R_1 \le D_K/4$ and the constant D is dependent on K and R_1.

Proof. For $0 < R \le R_1$, let Z_R be the subset of M_K such that for any $x \in Z_R$, $R < |zx| < D_K - R$. Let us define a flow map $F_h : Z_{R_1/2} \to M_K$ such that $F_h(x) :=$ the point on $[xz]$

distant $h < R_1/2$ from x. Then we want to show that F_h is $(1+Dh)$-Lipschitz on the closure $\overline{Z}_{R_1}$ where $h \leq R_1/4$.

Let $G : Z_{R_1/2} \times Z_{R_1/2} \times \mathbb{R} \to \mathbb{R}$ be the ratio function given by $G(x_1, x_2, h) := \frac{|(F_h(x_1)F_h(x_2)|}{|x_1x_2|}$, and H_R be the subset of $Z_R \times Z_R$ given by $H_R = \{(x,y) \in Z_R \times Z_R : 0.1R < |xy| < D_K - 0.1R\}$. Because $\partial G/\partial h$ does not exist at pairs where $|xy| = D_K$ and G is not defined on diagonal pairs (x,x), we work in such a region $H_{R_1/2}$ when using G.
Since $d_K(\cdot,\cdot)$ is a smooth function on the open subset $H_{R_1/2}$, G is smooth on $H_{R_1/2} \times [0, R_1/2)$. Thus the derivative $\frac{\partial G}{\partial h}$ is continuous and it has the maximum D_1 on the compact subset $\overline{H}_{R_1} \times [0, R_1/4]$.

By the mean value theorem, we have

$$|G(x,y,h) - G(x,y,0)| \leq D_1 h$$

for all $(x,y,h) \in \overline{H}_{R_1} \times [0, R_1/4]$. Since $G(x,y,0) = 1$, it gives

$$G(x,y,h) \leq 1 + D_1 h.$$

This implies

$$|F_h(x)F_h(y)| \leq (1 + D_1 h)|xy|$$

for all $(x,y) \in \overline{H}_{R_1}$ and for all $h \leq R_1/4$.

We need the following claim:
Claim (proved below). For all $h < R_1/2$, $||dF_h|| \leq 1 + D_2 h$ on the tangent spaces at points of $Z_{0.9R_1}$.

For x and $y \in \overline{Z}_{R_1}$ such that $|xy| \leq 0.1R_1$, $[xy]$ is in $Z_{0.9R_1}$. Since $||dF_h|| \leq 1 + D_2 h$ on the tangent space at $x \in Z_{0.9R_1}$, the length of $F_h([xy])$ is $\leq (1+D_2h)|xy|$. Since $|F_h(x)F_h(y)|$

is at most the length of $F_h([xy])$, then

$$|F_h(x)F_h(y)| \leq (1 + D_2h)|xy|.$$

For x and $y \in \overline{Z}_{R_1}$ such that $D_K \geq |xy| > D_K - 0.1R_1$, we choose a geodesic $[xy]$ with length $|xy|$.

So there is a point y' on $[xy]$ such that $|xy'| = 0.1R_1$ and $y' \in Z_{R_1}$. Then since $[xy'] \subset Z_{0.9R_1}$ and $(y'y) \subset \overline{H}_{R_1}$ because $D_K - 0.2R_1 < |y'y| \leq D_K - 0.1R_1$,

$$\begin{aligned} |F_h(x)F_h(y)| &\leq |F_h(x)F_h(y')| + |F_h(y')F_h(y)| \\ &\leq (1 + D_2h)|xy'| + (1 + D_1h)|y'y|. \end{aligned}$$

Our proof is finished with $D = \max\{D_1, D_2\}$.

□

Proof of claim. We need spherical polar coordinates r, θ with origin z, the attractive pole of F_h. Then the formula for F_h is simply $(r, \theta) \to (r - h, \theta)$. Let $\kappa = \sqrt{K}$. Differentiating with respect to r and θ gives us the identity matrix, which is interpreted as the matrix of dF_h with respect to coordinate bases $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$. But we want the norm of dF_h with respect to distance in the tangent spaces, so we change to orthonormal bases dual to the orthonormal bases of differentials, $(dr, \sin \kappa r \, d\theta)$:

$$(\frac{\partial}{\partial r}, \frac{1}{\sin \kappa r}\frac{\partial}{\partial \theta}) = (E_1, E_2).$$

Then

$$dF_h(E_1(r, \theta)) = E_1(r - h, \theta)$$

and

$$dF_h(E_2(r, \theta)) = \frac{1}{\sin \kappa r}\frac{\partial}{\partial \theta}(r - h, \theta) = \frac{\sin \kappa(r - h)}{\sin \kappa r} E_2(r - 2, \theta).$$

So the new matrix of dF_h is still diagonal, but has eigenvalues $1, A$ where $A = \frac{\sin \kappa(r-h)}{\sin \kappa r}$. Thus the norm = Lipschitz constant of dF_h is $\max\{1, A\}$.

When $A > 1$ then $r > \pi/2\kappa + h$ and the Lipschitz constant is given by the mean value theorem as

$$A = \frac{\sin \kappa r - (\cos x)\kappa h}{\sin \kappa r},$$

where $\kappa(r-h) < x < \kappa r$. Hence $-\cos x < -\cos \kappa r < \cos(0.9\kappa R_1)$, where $D_K - 0.9R_1$ is the upper bound of r, and $\sin \kappa r > \sin(0.9\kappa R_1)$. This gives the estimate for the Lipschitz constant:

$$A < 1 + D_2 h$$

where $D_2 = \kappa \cot(0.9\kappa R_1)$. □

Proposition 3.1.9. *If there exists B as in Lemma 3.1.7, then $d(P_{n+1}(t), P_n(t)) \leq \frac{Ch}{2\sqrt{3}} t e^{Dt}$ for any dyadic rational number $t \in [0, T]$ where $h = 1/2^n$, $B < D_K/4$, C is from Lemma 3.1.7 and $D = D(B, K)$ is from Lemma 3.1.8.*

Proof. First, we will show it for $K > 0$. For any dyadic rational number $t \leq T$, there exists a smallest number n_0 such that $t = \frac{\ell}{2^{n_0}}$. For any integer $n \geq n_0$, let k_n be $2^{n-n_0}\ell$. Thus $t = \frac{\ell}{2^{n_0}} = \frac{k_n}{2^n}$.

For $0 \leq i \leq k_n$, let x_i be the point on $[P_{n+1}(\frac{i-1}{2^n})E(\frac{i-1}{2^n})]$ at distance $1/2^n$ from $P_{n+1}(\frac{i-1}{2^n})$. Then $P_{n+1}(\frac{2i-1}{2^{n+1}})$ is the midpoint between $P_{n+1}(\frac{i-1}{2^n})$ and z_i. (See Figure 3.5)

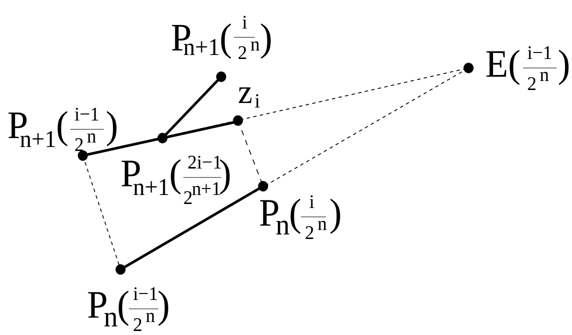

Figure 3.5: Discrete pursuit curves P_n,P_{n+1} and the point z_i

Let d_i be the length of the segment corresponding to $[z_i P_n(\frac{i}{2^n})]$ on the comparison triangle $\triangle \widetilde{P}_{n+1}(\frac{i-1}{2^n})\widetilde{P}_n(\frac{i-1}{2^n})\widetilde{E}(\frac{i-1}{2^n})$. (See Figure 3.5).

By Lemma 3.1.8, letting $R_1 = B$, we have

$$d_i \leq (1 + Dh)d(P_{n+1}(ih - h), P_n(ih - h)).$$

Since X is the $CAT(K)$ space, $d(z_i, P_n(ih)) \leq d_i$ and it gives

$$d(z_i, P_n(ih)) \leq (1 + Dh)d(P_{n+1}(ih - h), P_n(ih - h)).$$

Claim (proved below). $d(P_{n+1}(ih), z_i) \leq \frac{Ch^2}{2\sqrt{3}}$ for all i.

For $i = 1$, since $P_{n+1}(0) = P_n(0)$, z_1 is equal to $P_n(h)$. Thus $d(P_{n+1}(h), P_n(h)) \leq \frac{Ch^2}{2\sqrt{3}}$. Next, for $i = j$, we get $d(P_{n+1}(jh), P_n(jh)) \leq \frac{Ch^2}{2\sqrt{3}} \sum_{i=1}^{j}(1 + Dh)^{i-1}$ because we have a recursive relation, that is,

$$\begin{aligned} d(P_{n+1}(ih), P_n(ih)) &\leq d(P_{n+1}(ih), z_i) + d(z_i, P_n(ih)) \\ &\leq \frac{Ch^2}{2\sqrt{3}} + (1 + Dh)d(P_{n+1}(ih - h), P_n(ih - h)). \end{aligned} \tag{3.2}$$

Since $(1 + Dh) \geq 1$, trivially $\sum_{i=1}^{j}(1 + Dh)^{i-1} \leq j(1 + Dh)^j$. Therefore

$$d(P_{n+1}(jh), P_n(jh)) \leq \frac{Ch^2}{2\sqrt{3}} j(1 + Dh)^j.$$

Letting $j = k_n$, we get

$$d(P_{n+1}(t), P_n(t)) \leq \frac{Ch^2}{2\sqrt{3}} k_n(1 + Dh)^{k_n}.$$

Since $(1 + Dh)^{k_n} = (1 + Dh)^{\frac{1}{Dh}Dt} \leq e^{Dt}$ and $hk_n = t$, we have $d(P_{n+1}(t), P_n(t)) \leq \frac{Ch}{2\sqrt{3}} te^{Dt}$.

□

Proof of claim. Let us show that $d(P_{n+1}(\frac{i}{2^n}), z_i) \leq \frac{Ch^2}{2\sqrt{3}}$.

For $\triangle P_{n+1}(\frac{2i-1}{2^{n+1}})E(\frac{2i-1}{2^{n+1}})E(\frac{2i-2}{2^{n+1}})$, let a be $d(P_{n+1}(\frac{i}{2^n}), z_i)$, φ be $\angle E(\frac{2i-1}{2^{n+1}})P_{n+1}(\frac{2i-1}{2^{n+1}})E(\frac{2i-2}{2^{n+1}})$,

$\tilde{a}$ be the corresponding length in the comparison triangle $\triangle\widetilde{P}_{n+1}(\frac{2i-1}{2^{n+1}})\widetilde{E}(\frac{2i-1}{2^{n+1}})\widetilde{E}(\frac{2i-2}{2^{n+1}})$ and $\widetilde{\varphi}$ be the corresponding angle of the same comparison triangle. Then by spherical geometry of M_K, we get

$$\cos(\kappa\tilde{a}) = \cos^2(\kappa h/2) + \sin^2(\kappa h/2)\cos\widetilde{\varphi}$$

where $\kappa = \sqrt{K}$. By using $\cos\theta = 1 - 2\sin^2\theta/2$, we have

$$\sin^2(\kappa\tilde{a}/2) = \sin^2(\kappa h/2)\sin^2(\widetilde{\varphi}/2).$$

Since the three sines are non-negative with arguments less than $\pi/2$, this gives

$$\sin(\kappa\tilde{a}/2) = \sin(\kappa h/2)\sin(\widetilde{\varphi}/2).$$

This equation implies $\sin(\kappa\tilde{a}/2) \leq \kappa h\widetilde{\varphi}/4$. Since we can apply the inverse sine calculation of Lemma 3.1.7 here,

$$\kappa\tilde{a}/2 \leq (2/\sqrt{3})\kappa h\widetilde{\varphi}/4.$$

Finally, $a \leq \frac{h\widetilde{\varphi}}{\sqrt{3}}$ since $a \leq \tilde{a}$. Since $\widetilde{\varphi} \leq \frac{Ch}{2}$ by Lemma 3.1.7, $a \leq \frac{Ch^2}{2\sqrt{3}}$. □

By Proposition 3.1.9, $P_n(t)$ is uniformly Cauchy for all dyadic rational numbers $t \in [0,T]$ and then $\lim_{n\to\infty} P_n(t)$ exists. Define $P(t) := \lim_{n\to\infty} P_n(t)$. Completing dyadic rational numbers, we get the continuous curve P. Next, we will show that P has unit speed. Also we will get an angle calculation that will be used again in the proof of Theorem 3.1.5.

Proposition 3.1.10.

$$\lim_{t'\to t} \frac{d\big(P(t), P(t')\big)}{|t'-t|} = 1.$$

Proof. Since this proposition deals with a limit value, we can show this for $K = 0$ by taking the limit as $K \to 0$. So let us show this for $K > 0$.

Assume $0 \le t' - t < D_K/2$ and let $h = 1/2^n$. Let ℓ be the integer such that $(\ell - 1)h \le t < \ell h$ and let ℓ' be the integer such that $\ell' h \le t' < (\ell' + 1)h$.

By applying Reshetnyak's Theorem to the polygon $P_n(t)P_n(\ell h) \cdots P_n(\ell' h)P_n(t')$, we get a spherical convex polygon $\bar{P}_n(t)\bar{P}_n(\ell h) \cdots \bar{P}_n(\ell' h)\bar{P}_n(t')$ in M_K with the corresponding edges having the same length. In particular,

$$d\big(P_n(t'), P_n(t)\big) = d_K\big(\bar{P}_n(t'), \bar{P}_n(t)\big). \tag{3.3}$$

Since $0 \le t' - t < D_K/2$, this spherical convex polygon lies in a disk with radius $|t' - t|$ having an area

$$2\pi(1 - \cos(\sqrt{K}|t' - t|)) = 4\pi \sin^2(\sqrt{K}|t' - t|/2).$$

This area is $\le \pi K |t' - t|^2$.

Let $\widetilde{\alpha}_i$ be the angle $\angle \widetilde{x}_i \widetilde{p}_i \widetilde{x}_{i-1}$ of $\triangle \widetilde{x}_i \widetilde{p}_i \widetilde{x}_{i-1}$ where $x_i = E(\ell h + ih)$ and $p_i = P_n(\ell h + ih)$, and $\bar{\alpha}_i$ be the exterior angle at $\bar{p}_i$ of the spherical convex polygon. Define $\bar{\beta} = \sum_{i=0}^{\ell' - \ell} \bar{\alpha}_i$. Then $\bar{\beta}$ is the total curvature of the curve $\bar{P}_n(t)\bar{P}_n(\ell h) \cdots \bar{P}_n(\ell' h)\bar{P}_n(t')$.

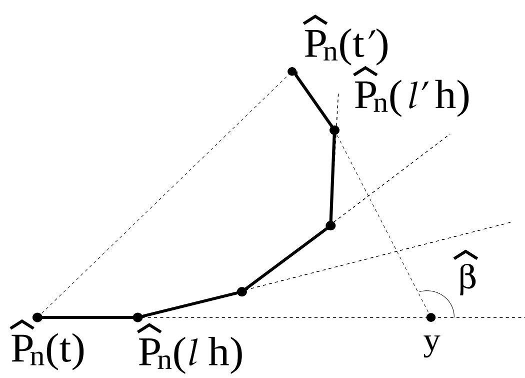

Figure 3.6: Euclidean convex polygon $\hat{P}_n(t)\hat{P}_n(\ell h) \cdots \hat{P}_n(\ell' h)\hat{P}_n(t')$ and the angle $\hat{\beta}$ at y,

For $\triangle \bar{q}\bar{p}_i\bar{p}_{i+1}$ where $q = P_n(t)$ and $p_{\ell' - \ell + 1} = P_n(t')$, let $\triangle \hat{q}\hat{p}_i\hat{p}_{i+1}$ be a Euclidean triangle with same edge lengths. Pasting same labeled edges of Euclidean triangles, we have a fan construction $\hat{P}_n(t)\hat{P}_n(\ell h) \cdots \hat{P}_n(\ell' h)\hat{P}_n(t')$. Define $\hat{\beta}$ as the total curvature of the curve $\hat{P}_n(t)\hat{P}_n(\ell h) \cdots \hat{P}_n(\ell' h)\hat{P}_n(t')$ like $\bar{\beta}$ (See Figure 3.6).

Since each angle of the fan construction is not bigger than the corresponding angle of the spherical convex polygon, then $\angle \hat{p}_0\hat{q}\hat{p}_{\ell'-\ell+1} < \pi$, and this fan construction is convex.

By the Gauss-Bonnet formula, we obtain

$$\begin{aligned} \hat{\beta} - \bar{\beta} &= \hat{\beta} - \sum_{i=0}^{\ell'-\ell} \bar{\alpha}_i \\ &\leq \text{Area of the convex polygon} \\ &\leq \pi K |t' - t|^2. \end{aligned} \tag{3.4}$$

Claim (proved below). $\bar{\alpha}_i \leq \widetilde{\alpha}_i$.

By Lemma 3.1.7 and this claim,

$$\begin{aligned} \bar{\beta} = \sum_{i=0}^{\ell'-\ell} \bar{\alpha}_i &\leq (\ell' - \ell) Ch \\ &\leq C|t' - t|. \end{aligned} \tag{3.5}$$

Then we get $\hat{\beta} \leq C|t' - t| + \pi K |t' - t|^2$ and

$$\cos(\hat{\beta}/2) \geq \cos \tfrac{1}{2}(C|t' - t| + \pi K |t' - t|^2)$$

by assuming that $C|t' - t| + \pi K |t' - t|^2 < \pi$.

Consider the Euclidean triangle which has three vertices y, $\hat{P}_n(t)$, $\hat{P}_n(t')$ and has $\hat{\beta}$ as the exterior angle at y (See Figure 3.6). Comparing this Euclidean triangle with the fan construction,

$$\frac{d_{\mathbb{R}^2}\big(\hat{P}_n(t), \hat{P}_n(t')\big)}{\cos(\hat{\beta}/2)} \geq |y\hat{P}_n(t)| + |y\hat{P}_n(t')| \geq |t' - t|$$

since among all triangles with a given base length $d_{\mathbb{R}^2}\big(\hat{P}_n(t), \hat{P}_n(t')\big)$ and opposite vertex angle $\pi - \hat{\beta}$, the isosceles triangle has the largest sum of lengths of two other sides. Note that we get the second inequality using the nested convex sets property.

Therefore since

$$\frac{d_{\mathbb{R}^2}\big(\hat{P}_n(t), \hat{P}_n(t')\big)}{|t'-t|} \geq \cos(\hat{\beta}/2) \geq \cos\tfrac{1}{2}(C|t'-t| + \pi K|t'-t|^2),$$

we get the following inequality

$$\begin{aligned} 1 - \frac{d_{\mathbb{R}^2}\big(\hat{P}_n(t), \hat{P}_n(t')\big)}{|t'-t|} &\leq 1 - \cos\tfrac{1}{2}(C|t'-t| + \pi K|t'-t|^2) \\ &= 2\sin^2\tfrac{1}{4}(C|t'-t| + \pi K|t'-t|^2) \\ &\leq 2(\tfrac{1}{4}(C|t'-t| + \pi K|t'-t|^2))^2. \end{aligned} \tag{3.6}$$

Since $d_{\mathbb{R}^2}\big(\hat{P}_n(t), \hat{P}_n(t')\big) = d_K\big(\bar{P}_n(t'), \bar{P}_n(t)\big)$, Equation (3.3) with $d\big(P_n(t), P_n(t')\big) \leq |t'-t|$ yields

$$1 - 2(\tfrac{1}{4}(C|t'-t| + \pi K|t'-t|^2))^2 \leq \frac{d\big(P_n(t), P_n(t')\big)}{|t'-t|} \leq 1.$$

Then taking the limit as $n \to \infty$,

$$1 - 2(\tfrac{1}{4}(C|t'-t| + \pi K|t'-t|^2))^2 \leq \frac{d\big(P(t), P(t')\big)}{|t'-t|} \leq 1. \tag{3.7}$$

Hence,

$$\lim_{t' \to t} \frac{d\big(P(t), P(t')\big)}{|t'-t|} = 1.$$

□

Proof of claim. Since our convex polygon has non-decreasing inner distance in M_K by Reshetnyak's Majorization Theorem,

$$\angle p_{i-1}p_ip_{i+1} \leq \angle \bar{p}_{i-1}\bar{p}_i\bar{p}_{i+1}. \tag{3.8}$$

Since X is a $CAT(K)$ space,

$$\angle x_{i-1}p_ix_i \leq \angle \widetilde{x}_{i-1}\widetilde{p}_i\widetilde{x}_i. \tag{3.9}$$

Since $\angle x_{i-1}p_ip_{i+1} = \angle x_{i-1}p_ix_{i+1}$, by (3.8) and (3.9),

$$\begin{aligned}\pi &\leq \angle p_{i-1}p_ip_{i+1} + \angle x_{i-1}p_ip_{i+1}\\ &\leq \angle \bar{p}_{i-1}\bar{p}_i\bar{p}_{i+1} + \angle \widetilde{x}_{i-1}\widetilde{p}_i\widetilde{x}_i.\end{aligned}$$

Hence

$$\begin{aligned}\bar{\alpha}_i &= \pi - \angle \bar{p}_{i-1}\bar{p}_i\bar{p}_{i+1}\\ &\leq \angle \widetilde{x}_{i-1}\widetilde{p}_i\widetilde{x}_i = \widetilde{\alpha}_i.\end{aligned}$$

□

Lemma 3.1.11. *Let $f_{\inf}$ be the function given by $f_{\inf}(t) := \liminf_{n\to\infty} d(P_n(t), E(t))$ and $f_{\sup}$ be the function given by $f_{\sup}(t) := \limsup_{n\to\infty} d(P_n(t), E(t))$. Then $f_{\inf}$ and $f_{\sup}$ are continuous for any time $t \geq 0$. Furthermore, they are non-increasing on $\{t \geq 0 : f_{\inf}(t) > 0\}$.*

Proof. Let $h = 1/2^n$. Since $|d(P_n(t_1), E(t_1)) - d(P_n(t_2), E(t_2))| \leq 2|t_1 - t_2|$, $f_{\inf}$ and $f_{\sup}$ are continuous. For a time t_2, if $f_{\inf}(t_2) > 0$, then $d(P_n(ih), E(ih))$ is non-increasing in $i \leq i_0$ when $i_0h \leq t_2$. For each n, let ℓ' be the integer such that $\ell'h \leq t_2 < (\ell'+1)h$. Then we get

$$d\big(P_n(\ell'h), E(\ell'h)\big) = |t_2 - \ell'h| + d\big(P_n(t_2), E(\ell'h)\big) \geq d\big(P_n(t_2), E(t_2)\big). \tag{3.10}$$

For $t_1 < t_2$, let ℓ be the integer such that $\ell h \leq t_1 < (\ell+1)h$. Note $\ell \leq \ell'$. Then the triangle inequality gives

$$d\big(P_n(t_1), E(t_1)\big) + |t_1 - \ell h| \geq d\big(P_n(t_1), E(\ell h)\big)$$

and

$$d\big(P_n(t_1), E(t_1)\big) + 2|t_1 - \ell h| \geq d\big(P_n(\ell h), E(\ell h)\big).$$

With Equation (3.10),

$$d\big(P_n(t_1), E(t_1)\big) + 2h \geq d\big(P_n(\ell h), E(\ell h)\big) \geq d\big(P_n(\ell' h), E(\ell' h)\big) \geq d\big(P_n(t_2), E(t_2)\big).$$

Taking the liminf or the limsup, our proof is done.

□

Proof of Theorem 3.1.5. If $P(0) = E(0)$, then we define P by $P(t) := E(t)$ for all $t \geq 0$. If $P(0) \neq E(0)$, then either $f_{\inf}(t) > 0$ for all $t \geq 0$ or there is a constant T_0 such that $f_{\inf}(t) > 0$ on $[0, T_0)$ and $f_{\inf}(T_0) = 0$. This follows because $f_{\inf}$ is continuous and is non-increasing on $\{t \geq 0 : f_{\inf}(t) > 0\}$ by Lemma 3.1.11.

For $T < \infty$ in the first case or for $T < T_0$ in the second case, $f_{\inf}(T) > 0$. Let B be $f_{\inf}(T)/2$. Then by Proposition 3.1.9, $P_n(t)$ is uniformly Cauchy for all dyadic rational numbers $t \in [0, T]$ and then $\lim_{n\to\infty} P_n(t)$ exists. Define $P(t) := \lim_{n\to\infty} P_n(t)$. Completing dyadic rational numbers, we get the continuous curve $P(t)$.

Also since

$$\begin{aligned} d(P_m(t), P_n(t)) &\leq d(P_m(t), P_{m-1}(t)) + \cdots + d(P_{n+1}(t), P_n(t)) \\ &\leq \frac{C}{2\sqrt{3}} t e^{Dt} \sum_{i=n}^{m-1} \frac{1}{2^i} \end{aligned}$$

for $m > n$, we have the estimate

$$d(P(t), P_n(t)) \leq \frac{C}{2\sqrt{3}} t e^{Dt} \frac{2}{2^n} \tag{3.11}$$

for all $t \in [0, T]$ by taking the limit as $m \to \infty$. This implies

$$\liminf_{n\to\infty} d(P_n(t), E(t)) = \limsup_{n\to\infty} d(P_n(t), E(t)) = d(P(t), E(t)) \tag{3.12}$$

for all $t \in [0, T]$.

For uniqueness, let $P' = P'(t)$ be a continuous pursuit curve and P'_n be a discrete pursuit curve of P' with $h = 1/2^n$. By Lemma 3.1.8, we have

$$d(P_n(t), P'_n(t)) \leq (1 + Dh)^{k_n} d(P(0), P'(0)) + 2h$$

where $k_n = [2^n t]$. Since $(1 + Dh)^{k_n} \leq e^{Dt}$, we get

$$d(P_n(t), P'_n(t)) \leq e^{Dt} d(P(0), P'(0)) + 2h.$$

Taking the limit as $n \to \infty$,

$$d(P(t), P'(t)) \leq e^{Dt} d(P(0), P'(0)).$$

Thus if $P(0) = P'(0)$, then $P(t) = P'(t)$ for any $t \in [0, T]$.

If $f_{\inf}(t) > 0$ for all $t \geq 0$, then we can define P for all $t \geq 0$ by $P|_{[0,T]}$ as T goes to ∞ since $P|_{[0,T]}$ is unique. Otherwise, we can define P for all $t \in [0, T_0)$ by $P|_{[0,T]}$ as T goes to T_0. By Equation (3.12), we have $f_{\inf}(t) = f_{\sup}(t)$ for all $t \in [0, T_0)$. This gives $f_{\sup}(T_0) = f_{\inf}(T_0) = 0$. This means $\lim_{n\to\infty} d(P_n(T_0), E(T_0)) = 0$. Then we define $P(t) := E(t)$ for all $t \geq T_0$.

Last, we show that the righthand direction of P at $P(t)$ is the same as the direction of $P(t)E(t)$. For this, we have to show that $\angle P(t')P(t)E(t)$ goes to 0 as t' goes to t where $t' \geq t$.

We can assume that t' is an irrational number because of semi-continuity of angles. So there is an integer number ℓ' such that $\ell' h < t' < (\ell' + 1)h$ with $h = 1/2^n$. For this n, let ℓ

be the integer such that $\ell h \leq t < (\ell+1)h$. In order to finish the proof, we will show that

$$\angle P_n(t)P_n(t')E(\ell h) \geq \pi - C_0|t'-t| - C|t'-t| - 2\pi K|t'-t|^2.$$

From the proof of Proposition 3.1.10, we have

$$\angle P_n(t)P_n(t')P_n(\ell' h) \leq \angle \bar{P}_n(t)\bar{P}_n(t')\bar{P}_n(\ell' h)$$

and

$$\angle \bar{P}_n(t)\bar{P}_n(t')\bar{P}_n(\ell' h) - \angle \hat{P}_n(t)\hat{P}_n(t')\hat{P}_n(\ell' h) \leq \pi K|t'-t|^2$$

where $\hat{P}_n$ represents a vertex of the fan construction as in the proof of Proposition 3.1.10. Since $\angle \hat{P}_n(t)\hat{P}_n(t')\hat{P}_n(\ell' h) \leq \hat{\beta}$,

$$\angle P_n(t)P_n(t')P_n(\ell' h) \leq C|t'-t| + 2\pi K|t'-t|^2. \tag{3.13}$$

The proof of Lemma 3.1.7 shows that there is a constant C_0 which is independent of t and n such that $\angle E(\ell' h)P_n(t')E(\ell h) \leq C_0|t'-t|$. Since $P_n(t')$ is on the geodesic $[P_n(\ell' h)E(\ell' h)]$,

$$\begin{aligned} \pi &\leq \angle E(\ell' h)P_n(t')E(\ell h) + \angle P_n(\ell' h)P_n(t')E(\ell h) \\ &\leq C_0|t'-t| + \angle P_n(\ell' h)P_n(t')E(\ell h). \end{aligned} \tag{3.14}$$

Since $\angle P_n(\ell' h)P_n(t')P_n(t) + \angle P_n(t)P_n(t')E(\ell h) \geq \angle P_n(\ell' h)P_n(t')E(\ell h)$,

$$\begin{aligned} \angle P_n(t)P_n(t')E(\ell h) &\geq \angle P_n(\ell' h)P_n(t')E(\ell h) - \angle P_n(\ell' h)P_n(t')P_n(t) \\ &\geq \pi - C_0|t'-t| - \angle P_n(\ell' h)P_n(t')P_n(t) \qquad \text{by (3.14)} \\ &\geq \pi - C_0|t'-t| - C|t'-t| - 2\pi K|t'-t|^2 \quad \text{by (3.13)}. \end{aligned}$$

By semi-continuity of angles, we have

$$\angle P(t)P(t')E(t) \geq \limsup_{n\to\infty} \angle P_n(t)P_n(t')E(\ell h) \geq \pi - C_0|t'-t| - C|t'-t| - 2\pi K|t'-t|^2.$$

Since Gauss-Bonnet formula implies

$$\angle P(t)P(t')E(t) + \angle P(t')P(t)E(t) + \angle P(t')E(t)P(t) \leq \pi + Area[\triangle \tilde{P}(t')\tilde{E}(t)\tilde{P}(t)],$$

then

$$\begin{aligned} \angle P(t')P(t)E(t) &\leq \angle P(t')P(t)E(t) + \angle P(t')E(t)P(t) \\ &\leq \pi - \angle P(t)P(t')E(t) + Area[\triangle \tilde{P}(t')\tilde{E}(t)\tilde{P}(t)] \\ &\leq C_0|t'-t| + C|t'-t| + 2\pi K|t'-t|^2 + Area[\triangle \tilde{P}(t')\tilde{E}(t)\tilde{P}(t)]. \end{aligned}$$

Therefore $\angle P(t')P(t)E(t)$ goes to 0 as t' goes to t where $t' \geq t$. □

Proof of Theorem 3.1.6. The first part is proved by Equation (3.11). For the second part, if the evader wins, then $\lim_{t\to\infty} L(t) > 0$. Then there exists a constant $B > 0$ such that $B < \lim_{t\to\infty} L(t)$. So for any time T, $B < L(T)$ since L is non-increasing in t. Thus $B < L(T)$ implies there is $n_0 = n_0(T)$ such that for all $n \geq n_0$, $d(P_n(T), E(T)) > B$. With this constant B, Equation (3.11) holds with constants C and D dependent on K only. □

3.2 Regularity of continuous pursuit curves

For us to look at our pursuit curve P's regularity, we need a generalized definition of Lipschitz-continuous derivative. In $CAT(K)$ spaces, we have no acceleration vector field of a curve since there is no way of comparing vectors at different points on the curve. Instead, using the inequality of lengths and distances, we study bending intensity. Indeed,

Definition 3.2.1. *[3] Let P be a complete subset of X where X has curvature bounded* *above by K. Given a real number $a \geq 0$, P is* (a,2,ρ)-convex *if there exists $\rho > 0$ such that $d_P(x,y) \leq d_X(x,y) + a d_X(x,y)^3$ for $d_X(x,y) < \rho$.*

Definition 3.2.2. *[3] Let X be a metric space of curvature bounded above by K. P is a* subspace of extrinsic curvature $\leq A$ *in X if there is a length-preserving map $F : P \to X$ between intrinsic metric spaces, where P is complete and*

$$d_P(x,y) \leq d_X(F(x),F(y)) + \frac{A^2}{24} d_X(F(x),F(y))^3 + o(d_X(F(x),F(y))^3)$$

for all x and y having $d_P(x,y)$ sufficiently small.

The coefficient $A^2/24$ is chosen so that if F is a smooth curve in M_K with nonzero velocity, then A is the smallest bound on the geodesic curvature of F.

A bound on extrinsic curvature acts as a replacement for $C^{1,1}$ regularity in smooth spaces [3] [5].

Theorem 3.2.3. *Let X be a $CAT(K)$ space. An initial arc $P(t), 0 \leq t \leq T$ of a continuous pursuit curve P of Theorem 3.1.5 for which $d(P(T),E(T)) > 0$ is a subspace of extrinsic curvature at most*

$$\begin{cases} \frac{32\sqrt{2}}{d(P(T),E(T))} & \text{if } K \leq 0 \\ \frac{8\sqrt{2}\sqrt{K}}{\sin(\sqrt{K}d(P(T),E(T))/4)} & \text{if } K > 0. \end{cases}$$

Example 3.2.4. *We have an example such that P has infinite curvature when P capture E. It is modified slightly from [10, Page 362-363]. In $\mathbb{R}^2$, let $P(0) = (1,0)$ and an evader $E : [0,\sqrt{2}) \to \mathbb{R}^2$ be defined by $E(s) = e^{-t}(-\sin t, \cos t)$ where $t = -\log(1 - s/\sqrt{2})$. Then there is a continuous pursuit curve P given by $P(s) = e^{-t}(\cos t, \sin t)$. We can check that P has unit speed, the curvature of P at s is $1/(\sqrt{2} - s)$ and $\lim_{s \to \sqrt{2}} |P(s)E(s)| = 0$.*

Proof of Theorem 3.2.3. Suppose $K > 0$. From Equation (3.7) in Proposition 3.1.10, we have

$$|t - t'| - \frac{1}{8}|t - t'|^3 (C + \pi K |t - t'|)^2 \leq d(P(t), P(t')) \tag{3.15}$$

where $C = \frac{2\sqrt{K}/\sqrt{3}}{\sin(\sqrt{K}B/2)}$ and B are from Lemma 3.1.7.

Since P is a unit speed curve, then $d_P(x, y)$ represents the arclength of P between x and y. Therefore

$$d_P(x, y) = |t - t'|$$

where $x = P(t)$ and $y = P(t')$. Then Equation (3.15) becomes

$$d_P(x, y) - \frac{1}{8} d_P(x, y)^3 (C + \pi K d_P(x, y))^2 \leq d(x, y). \tag{3.16}$$

Since $d_P(x, y)$ is sufficient small in Definition 3.2.2, we can assume that $d_P(x, y) \leq \min\{\frac{C}{\pi K}, \frac{1}{C}\}$.

If $d_P(x, y) \leq \frac{C}{\pi K}$, then $(C + \pi K d_P(x, y))^2 \leq (2C)^2$. Applying this to Equation (3.16), then we have

$$d_P(x, y) - \frac{C^2}{2} d_P(x, y)^3 \leq d(x, y). \tag{3.17}$$

If $d_P(x, y) \leq \frac{1}{C}$, then $\frac{C^2}{2} d_P(x, y)^3 \leq \frac{1}{2} d_P(x, y)$. This gives

$$\frac{1}{2} d_P(x, y) \leq d_P(x, y) - \frac{C^2}{2} d_P(x, y)^3. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\frac{1}{2} d_P(x, y) \leq d_P(x, y) - \frac{C^2}{2} d_P(x, y)^3 \leq d(x, y).$$

Since $\frac{1}{2} d_P(x, y) \leq d(x, y)$, then from Equation (3.17), we get

$$d_P(x, y) \leq d(x, y) + \frac{C^2}{2} d_P(x, y)^3 \leq d(x, y) + 4C^2 d(x, y)^3.$$

Let B be $d(P(T), E(T))/2$ as in Proof of Theorem 3.1.5. This means that $2\sqrt{24}C = \frac{8\sqrt{2}\sqrt{K}}{\sin(\sqrt{K}d(P(T),E(T))/4)}$ is an upper bound of extrinsic curvature for $K > 0$.

Taking the limit as K goes to 0, $\frac{32\sqrt{2}}{d(P(T),E(T))}$ is an upper bound of extrinsic curvature for $K \leq 0$. So our proof is done. □

From this regularity, also we can find a lower bound of $d(P(0), P(T))$.

Corollary 3.2.5. *Let $K = 0$. If $\frac{16\sqrt{2}}{d(P(T),E(T))}T \leq \pi$, then*

$$d(P(0), P(T)) \geq \frac{2}{k}\sin(kT/2)$$

where k is an upper bound of extrinsic curvature of $P|_{[0,T]}$ obtained in Theorem 3.2.3.

Proof. From Definition 2.2.1, a k-curve in M_K is a curve of constant geodesic curvature k. By [5, Theorem 1.1], for an initial arc $P|_{[0,T]}$, there is k'-curve γ in $\mathbb{R}^2$ having the same arclength T and chordlength $d(P(0), P(T))$ for some k' such that $k' \leq k$.

Suppose that $\gamma : I \to \mathbb{R}^2$ be a curve given by

$$t \mapsto (\frac{1}{k'}\cos t, \frac{1}{k'}\sin t) \tag{3.19}$$

where $I = [-k'T/2, k'T/2]$. Note that we can check the parametrization (3.19) of γ has arclength T.

Let x_1 and x_2 be the end points of γ. Since γ has chordlength $d(P(0), P(T))$, then $|x_1x_2| = d(P(0), P(T))$. This means

$$\frac{2}{k'}\sin(k'T/2) = d(P(0), P(T)).$$

Since $k' \leq k$, $kT \leq 2\pi$ and $\frac{\sin t}{t}$ is decreasing on $[0, \pi]$, we have

$$\frac{2}{k}\sin(kT/2) \leq d(P(0), P(T))$$

where $k = \frac{32\sqrt{2}}{d(P(T),E(T))}$.

□

From it, we obtain a lower bound on the distance between the pursuer positions $P(0)$ and $P(T)$. If the pursuer $P(T)$ leaves the visible range of the base center $P(0)$, the capture is worthless.

3.3 Further results on continuous pursuit curves

3.3.1 Total curvature and circumradius

The *circumradius* of $P|_{[0,t]}$ is $\max_{s\in[0,t]} d(P(s), P(0))$. We need this definition for Lemma 3.3.1.

Lemma 3.3.1. *Let $c(t)$ be the circumradius of $P|_{[0,t]}$ and $c_n(t)$ be the circumradius of $P_n|_{[0,t]}$. Then $c(t) = \lim_{n\to\infty} c_n(t)$.*

Proof. Since P_n converges uniformly to P on $[0, t]$, for every $\epsilon > 0$, $\exists N > 0$ such that $d(P_n(s), P(s)) < \epsilon$ for all $s \in [0, t]$ and $n \geq N$. Note $P(0) = P_n(0)$ for all n. Hence, for any $n \geq N$, the maximum $c_n(t) = d(P_n(s_{n,max}), P(0))$ is within ϵ of $d(P(s_{n,max}), P(0))$. This means $c_n(t) \leq c(t) + \epsilon$. Conversely, for any $n \geq N$, the maximum $c(t) = d(P(s_{max}), P(0))$ is within ϵ of $d(P_n(s_{max}), P(0))$. Therefore, $c(t) \leq c_n(t) + \epsilon$. □

Recently, in [2], Alexander, Bishop and Ghrist considered simple pursuit evasion problems in the discrete-time case on a $CAT(0)$ space.

Theorem 3.3.2. *[1, Th. 8] The pursuer always wins a discrete-time simple pursuit on any compact $CAT(0)$ space.*

In noncompact cases, they show a necessary condition when the evader wins a discrete-time simple pursuit.

Theorem 3.3.3. *[1, Th. 10 and Th. 13] On any $CAT(K)$ space, if the evader wins a discrete-time simple pursuit, then $\tau(t)/\sqrt{t}$ is bounded, where $\tau(t)$ is the total curvature of the discrete pursuit curve $P|_{[0,t]}$.*

Here we can get analogous results for continuous pursuit curves,

Theorem 3.3.4. *On any $CAT(K)$ space, if the evader wins a continuous-time simple pursuit when $d(E(0), P(0)) \leq D_K$, then $\tau(t)/\sqrt{t}$ is bounded where $\tau(t)$ is the total curvature of the continuous pursuit curve $P|_{[0,t]}$.*

Proof. Suppose that $K > 0$. If the evader wins a continuous-time simple pursuit, then $\lim_{t\to\infty} L(t) > 0$ where $L(t) = d(P(t), E(t))$. Let $2B$ be $\lim_{t\to\infty} L(t)$. Since $L(t)$ is non-increasing, $L(t) \geq 2B$ for all t. For each t, there is an integer $n_0 = n_0(t)$ such that $L_n(t) := d(P_n(t), E(t)) \geq B$ for all $n \geq n_0$, where P_n is from Theorem 3.1.6 with step size $h = 1/2^n$. We can assume that $h < B$.

Let α_i be the angle $\angle E(ih)P_n(ih)E(ih - h)$ of P_n and $\widetilde{\alpha}_i$ be the angle of a comparison triangle corresponding to α_i where $1 \leq i \leq [2^n t]$. Note P_n has $[2^n t] + 1$ segments and $[2^n t]$ turning angles up to time t.

Since X is the $CAT(K)$ space, $\alpha_i \leq \widetilde{\alpha}_i$. Then by the definition of $\tau_n(t)$, we have

$$\tau_n(t) = \sum_{i=1}^{[2^n t]} \alpha_i \leq \sum_{i=1}^{[2^n t]} \widetilde{\alpha}_i.$$

By [1, Theorem 13], applying the spherical law of cosines to $\widetilde{\triangle} E(ih)P_n(ih)E(ih - h)$, we have the upper bound of $\widetilde{\alpha}_i^2$. Indeed,

$$\widetilde{\alpha}_i^2 \leq \frac{5Kh(L_n(ih - h) - L_n(ih))}{\sin(\sqrt{K}B)\sin(\sqrt{K}(B - h))}.$$

Taking the sum over i,

$$\sum_{i=1}^{[2^n t]} \widetilde{\alpha}_i^2 \leq \frac{5Kh(L(0)-B)}{\sin(\sqrt{K}B)\sin(\sqrt{K}(B-h))}$$

since $L_n(0) = L(0)$ and $L_n(ih) \geq B$. This inequality gives

$$\frac{\{\tau_n(t)\}^2}{t} = \frac{1}{t}(\sum_{i=1}^{[2^n t]} \widetilde{\alpha}_i)^2 \leq \frac{1}{t}\sum_{i=1}^{[2^n t]} 1^2 \sum_{i=1}^{[2^n t]} \widetilde{\alpha}_i^2 \leq \frac{5K(L(0)-B)}{\sin(\sqrt{K}B)\sin(\sqrt{K}(B-h))}$$

for all t.

Since $P_n|_{[0,t]}$ uniformly converges to $P|_{[0,t]}$ by Theorem 3.1.6, from [18] we can use the inequality $\tau(t) \leq \liminf_{n\to\infty} \tau_n(t)$. Taking the limit as $n \to \infty$, we obtain

$$\frac{\{\tau(t)\}^2}{t} \leq \frac{5K(L(0)-B)}{\sin^2(\sqrt{K}B)}$$

for all t. Taking the limit as K goes to 0, we have $\frac{\{\tau(t)\}^2}{t}$ bounded for any K.

□

Corollary 3.3.5. *On any $CAT(0)$ space, if the evader wins a continuous-time simple pursuit, then $\sqrt{t}/c(t)$ is asymptotically bounded, where $c(t)$ is the pursuer's circumradius up to time t.*

Proof. By [2, Theorem 10], we know the relation between $\tau_n(t)$ and $c_n(t)$. Indeed,

$$\frac{\sqrt{t}}{c_n(t)} \leq \frac{2\sqrt{2}}{\sqrt{t}}\Big(\frac{\tau_n(t)}{\pi/2} + 1\Big).$$

Then Theorem 3.3.4 and Lemma 3.3.1 imply $\sqrt{t}/c(t)$ is bounded for large t. □

3.3.2 Barycenter of multiple evaders

Suppose that X is a $CAT(0)$ space in this section.

When we deal with multiple evaders in pursuit-evasion games, we still have a strategy to get a continuous pursuit curve. For this, we need a definition of the *barycenter* of multiple points.

Since for points $x_1, \cdots, x_n \in X$, the function $x \mapsto \sum_{i=1}^{n} d^2(x, x_i)$ is strictly convex, there exists a unique minimum point of the function by Lemma 3.3.8.

Definition 3.3.6. *Given n points x_i on X, the* barycenter *b of the x_i's is defined to be the minimum point of the function $x \mapsto \sum_{i=1}^{n} d^2(x, x_i)$.*

Let $\mathcal{P}(X)$ denote the set of all probability measures ν on $(X, \mathcal{B}(X))$ with separable support $\text{supp}(\nu) \subset X$ where $\mathcal{B}(X)$ is the set of Borel sets of X.

More generally,

Definition 3.3.7. *[29, Prop. 4.3] For $\nu \in \mathcal{P}(X)$ such that $\int_X d^2(x, y)\nu(dy) < \infty$ for some (hence all) $x \in X$, the minimum point of the function $x \mapsto \int_X d^2(x, y)\nu(dy)$ is called the* barycenter *$b(\nu)$ of ν.*

Since the function $x \mapsto \int_X d^2(x, y)\nu(dy)$ is continuous and strictly convex, the barycenter of ν is well-defined. Indeed, let $f(x) = \int_X d^2(x, y)\nu(dy)$. Choose a base point x_0 and let $B(t)$ be the closed ball of radius $t > 0$ and center x_0; then $B(t)$ is a nonempty bounded convex closed set, so f has a unique minimum point on $B(t)$. We show that for t sufficiently large the minimum point on $B(t)$ is also a minimum point on all of X. Let $x \notin B(t)$. Then

$$d(x, y) \geq |d(x, x_0) - d(x_0, y)| > |t - d(x_0, y)|,$$

$$f(x) > \int_X (t^2 - 2td(x_0, y) + d^2(x_0, y))\nu(dy) = t^2 - 2t \int_X d(x_0, y)\nu(dy) + f(x_0).$$

Hence if $t > 2\int_X d(x_0, y)\nu(dy)$, then

$$f(x) > f(x_0) \geq \min_{B(t)} f$$

and $\min_{B(t)} f$ is a minimum for f on all of X.

Let δ_x be the *Dirac measure* of x given by $\delta_x(A) = 1$ if $x \in A$ or $\delta_x(A) = 0$ otherwise, for any subset A of X.

Lemma 3.3.8. *Let ν be $\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$. Then the barycenter $b(\nu)$ of ν is equal to the barycenter of the x_i's.*

Proof. Since $\nu = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$,

$$\int_X d^2(x,y)\nu(dy) = \frac{1}{n}\sum_{i=1}^{n} d^2(x,x_i).$$

Then the minimum point of the function $x \mapsto \sum_{i=1}^{n} d^2(x,x_i)$ is equal to the barycenter of ν. $\square$

Here we want to give a strategy for chasing multiple evaders. Given n evaders $E_i = E_i(t)$ with speed ≤ 1 in X, we have the barycenter curve $b = b(t)$ defined by $b(t) :=$ the barycenter of $E_i(t)$. Then we want to show that $b = b(t)$ has also speed ≤ 1.

In order to show this, we need a theorem to deal with the distance between $b(t)$ and $b(t')$. From [29], we have the following theorem. This theorem gives an upper bound of the distance between two barycenters by integrating a coupling. Given two probability measures $\nu_1, \nu_2 \in \mathcal{P}(X)$, we call $\mu \in \mathcal{P}(X^2)$ a *coupling* of ν_1 and ν_2 if $\mu(A \times X) = \nu_1(A)$ and $\mu(X \times A) = \nu_2(A)$ for $\forall A \in \mathcal{B}(X)$.

Theorem 3.3.9. *[29, Th. 6.3] Let X be a $CAT(0)$ space. For two probability measures ν_1 and ν_2 satisfying $\int_X d(x,y)\nu_i(dy) < \infty$,*

$$d(b(\nu_1), b(\nu_2)) \leq \int_{X^2} d(x,y)\mu(dxdy)$$

where μ is a coupling of ν_1 and ν_2.

Proposition 3.3.10. *Let X be a $CAT(0)$ space. Given n evaders $E_i = E_i(t)$ with speed ≤ 1 in X, then the barycenter curve $b = b(t)$ is 1-Lipschitz.*

Proof. Let μ be $\frac{1}{n}\sum_{i=1}^{n}\delta_{(E_i(t),E_i(t'))}$. Then μ is the coupling of ν_t and $\nu_{t'}$ where $\nu_t = \frac{1}{n}\sum_{i=1}^{n}\delta_{E_i(t)}$ and $\nu_{t'} = \frac{1}{n}\sum_{i=1}^{n}\delta_{E_i(t')}$.

Since $b(t) = b(\nu_t)$ and $b(t') = b(\nu_{t'})$ by Lemma 3.3.8, by applying ν_t, $\nu_{t'}$ and μ to Theorem 3.3.9, we have

$$d(b(t), b(t')) \leq \frac{1}{n}\sum_{i=1}^{n} d(E_i(t), E_i(t')).$$

Since each evader E_i has speed ≤ 1, $d(E_i(t), E_i(t')) \leq |t - t'|$. Then we have

$$d(b(t), b(t')) \leq |t - t'|.$$

□

This proof shows that the barycenter curve of curves with speed ≤ 1 has also speed ≤ 1. By letting the evader of Theorem 3.1.5 be a barycenter curve b, we obtain

Theorem 3.3.11. *Let X be a $CAT(0)$ space. Given n evaders $E_i = E_i(t)$ with speed ≤ 1 in X, there is a unique continuous pursuit curve $P = P(t)$ chasing the barycenter curve b of evaders.*

Proof. Since b has speed ≤ 1 by Proposition 3.3.10, we take the barycenter curve b as the evader of Theorem 3.1.5. □

Chapter 4

Time-dependent gradient curves

In this chapter, we will generalize our previous results. A continuous pursuit curve is an example of a gradient curve of the time-dependent function $dist_{E(t)}$. Here, we will study the conditions on functions to get time-dependent gradient flows. In this chapter, we always assume that X is a $CAT(0)$ space.

4.1 Time-dependent gradient curves on CAT(0) spaces

4.1.1 Semi-convex functions and their gradient vectors

Definition 4.1.1. *For $\lambda \in \mathbb{R}$, a function $F : X \to \mathbb{R}$ is* λ-convex *if for every geodesic γ with unit speed, the function $s \mapsto F \circ \gamma(s) - \frac{\lambda s^2}{2}$ is convex.*

For $z_0, z_1 \in X$, there is a unique minimal geodesic $[z_0 z_1]$. For $0 \leq s \leq 1$, let z_s be the point on this geodesic such that $sd(z_0, z_1) = d(z_0, z_s)$ and $(1-s)d(z_0, z_1) = d(z_1, z_s)$. Sometimes we will use the notation $(1-s)z_0 + sz_1$ for z_s. If we assume that this geodesic $[z_0 z_1]$ has constant speed $d(z_0, z_1)$, we have to replace s by $sd(z_0, z_1)$ in Definition 4.1.1. Then F is λ-convex if $s \mapsto F \circ \gamma(sd(z_0, z_1)) - \frac{\lambda s^2 d^2(z_0, z_1)}{2}$ is convex. This is equivalent to a following inequality

$$F(z_s) - \frac{\lambda s^2 d^2(z_0, z_1)}{2} \leq (1-s)F(z_0) + s\big(F(z_1) - \frac{\lambda d^2(z_0, z_1)}{2}\big)$$

or

$$F(z_s) \leq (1-s)F(z_0) + sF(z_1) - s(1-s)\frac{\lambda d^2(z_0, z_1)}{2}. \tag{4.1}$$

We call a λ-convex function *very convex* when $\lambda > 0$ and *almost convex* when $\lambda < 0$. A 0-convex function is said to be *convex (on X)*. A convex function F is said to be *strictly convex* if $F(z_s) < (1-s)F(z_0) + sF(z_1)$ for $s \in (0,1)$.

Remark 4.1.2. *Mayer works with* (4.1)*, with the notation $S = -\lambda/2$.*

Proposition 4.1.3. *A geodesic metric space X is $CAT(0)$ if and only if for any x, z_0 and $z_1 \in X$,*

$$d^2(x, z_s) \leq (1-s)d^2(x, z_0) + sd^2(x, z_1) - s(1-s)d^2(z_0, z_1).$$

Briefly, if and only if the function $z \mapsto d^2(x,z)$ is 2-convex.

To define gradient vectors of λ-convex functions, we need *differentials* of λ-convex functions.

In [19], Kleiner showed if a λ-convex function $F : X \to \mathbb{R}$ is L-Lipschitz on a $CAT(0)$ space X, then for every $x \in X$, there is a unique L-Lipschitz function $d_xF : T_x \to \mathbb{R}$ where T_x is the tangent cone of X at x. Furthermore d_xF is convex and homogeneous of degree 1.

Definition 4.1.4 (Differentials of convex functions)**.** *d_xF is called the* differential *of F at x.*

Then we can define *gradient vectors* of λ-convex functions.

Definition 4.1.5. *A tangent vector $v \in T_x$ is called the* downward gradient vector *of F at x if*

1. *$(d_xF)(w) \geq -\langle v, w\rangle$ for all $w \in T_x$, and*

2. *$(d_xF)(v) = -\langle v, v\rangle$.*

We denote v by $\nabla_x(-F)$.

So the geometric meaning of the (downward) gradient vector is that F will be decreased fastest in the direction of this gradient and the length of the gradient vector is the rate at which F decreases in that direction.

Lemma 4.1.6. *Let X be a $CAT(0)$ space. If F is locally Lipschitz and λ-convex on X, then for any point $x \in X$, there is a unique downward gradient vector $\nabla_x(-F) \in T_x$.*

Proof. For uniqueness, if v, v' are two distinct downward gradient vectors of F at x, then

$$||v||^2 = -(d_xF)(v) \leq \langle v, v' \rangle,$$

$$||v'||^2 = -(d_xF)(v') \leq \langle v, v' \rangle.$$

Then these inequalities imply that $||v|| = 0$ if and only if $\langle v, v' \rangle = 0$, hence if and only if $||v'|| = 0$ by the inner product definition. It follows that $v = v' = o_x$. Otherwise if $||v|| > 0$ and $||v'|| > 0$, by the inner product definition, we have

$$||v||^2 \leq ||v|| ||v'|| \cos\theta, \ ||v'||^2 \leq ||v|| ||v'|| \cos\theta,$$

where θ is the angle between v and v'. Therefore

$$1 \leq \cos^2\theta$$

since $||v|| \leq ||v'|| \cos\theta \leq ||v|| \cos^2\theta$. Since $\cos\theta > 0$ because $0 < ||v|| \leq ||v'|| \cos\theta$, we obtain $\cos\theta = 1$ and $\theta = 0$. Thus $v = v'$.

For existence, first if $d_xF \geq 0$ then $\nabla_x(-F)$ is defined to be o_x. Otherwise, let

$$r = \inf_{\eta \in \Sigma_x} (d_xF)(\eta) < 0$$

where Σ_x is the direction space at x. Let S_x be the unit ball $\{w \in T_x | \|w\| \leq 1\}$ of the

$CAT(0)$ space T_x. Since d_xF is Lipschitz and convex on T_x, d_xF attains its infimum on the nonempty bounded convex closed subset S_x. Since d_xF is homogeneous, $\inf_{S_x} d_xF = r$. So we have a minimum direction ξ such that $d_xF(\xi) = r$. Then $v = (\xi, |r|)$ satisfies the definition of the downward gradient vector, as follows:

1. When ξ is the minimum point of d_xF on the closed ball S_x, the convexity of d_xF gives the support inequality

   $$d_xF(\eta) \geq d_xF(\xi)\cos(s) = r\langle \xi, \eta \rangle,$$

   where $\eta \in \Sigma_x$ and $s = d_{\Sigma_x}(\xi, \eta) < \pi$.

   From the support inequality the proof of defining property (1) for the gradient vector v easily follows from the homogeneity of d_xF and $\langle \cdot, \cdot \rangle$:

   For $\eta \in \Sigma_x$ and $s < \pi$,

   $$d_xF(\eta) \geq r\langle \xi, \eta \rangle = -|r|\langle \xi, \eta \rangle = -\langle v, \eta \rangle.$$

   When $d_{\Sigma_x}(\xi, \eta) = \pi$, then the geodesic from ξ to η goes through the origin o_x and the inequality we want is $d_xF(\eta) + d_xF(\xi) \geq 0$, as follows from the convexity of d_xF on that geodesic.

2. $d_xF(v) = |r| d_xF(\xi) = |r| r = -|r|^2 \langle \xi, \xi \rangle = -\langle v, v \rangle$.

□

4.1.2 Time-independent gradient curves

Lytchak defines downward gradient curves [22]. We start by defining an *absolute gradient* of F at x from [24] and [26] for the downward case.

Definition 4.1.7. *For a locally Lipschitz function $F : X \to \mathbb{R}$ and $x \in X$, define the* absolute gradient $|\nabla_- F|(x)$ *of F at x by*

$$|\nabla_- F|(x) := \max\Big\{\limsup_{y \to x} \frac{F(x) - F(y)}{d(x, y)}, 0\Big\}.$$

The following condition is sufficient for the set $\{x \in X : |\nabla_- F|(x) \neq 0\}$ of non-critical points to be open:

Definition 4.1.8. *[22],[26] For a locally Lipschitz function $F : X \to \mathbb{R}$, F has* semi-continuous absolute gradients *if $\liminf_{y \to x} |\nabla_- F|(y) \geq |\nabla_- F|(x)$ for all $x \in X$.*

By Definition 4.1.7, we know:

Lemma 4.1.9. *If F is locally Lipschitz and λ-convex, then $||\nabla_x(-F)|| = |\nabla_- F|(x)$.*

Now, we can give the definition of *gradient curves* on metric spaces.

Definition 4.1.10. *[22] For a function $F : X \to \mathbb{R}$ having semi-continuous absolute gradients, a curve $m : [0, a) \to X$ is called the* (time-independent) gradient curve *of F if for all $t \in [0, a)$,*

$$\lim_{\epsilon \to 0+} \frac{d(m(t + \epsilon), m(t))}{\epsilon} = |\nabla_- F|(m(t)) \tag{4.2}$$

and

$$\lim_{\epsilon \to 0+} \frac{F \circ m(t + \epsilon) - F \circ m(t)}{\epsilon} = -\big(|\nabla_- F|(m(t))\big)^2. \tag{4.3}$$

Next we look at Mayer's Theorem from [24]:

Theorem 4.1.11. *[24, Th. 1.13] Let X be a $CAT(0)$ space. For $x_0 \in X$ and a function $G : X \to \mathbb{R}$, assume that*

1) G is lower semicontinuous,

2) G is λ-convex.

For any $y \in X$*, let*

$$A = -\min\{0, \liminf_{d(x,y)\to\infty} \frac{G(x)}{d^2(x,y)}\},$$

$$I_A = \begin{cases} (0,\infty) & \text{if } A = 0 \ , \\ (0, \frac{1}{16A}] & \text{if } A > 0 \ . \end{cases} \tag{4.4}$$

Then there is a unique curve $m : I_A \to X$ *such that* $\lim_{t\to 0} m(t) = x_0$ *and* $G(m(t)) \leq G(x_0)$ *satisfying Equations* (4.2) *and* (4.3)*.*

Note that A is independent of y because of the triangle inequality.

In Theorem 4.1.11, it may not happen that the curve m has right-side tangent vectors $m'(t+)$ for all $t \in I_A$. But if G is locally Lipschitz instead of being lower semicontinuous, then we obtain the (time-independent) gradient curves of G as in Definition 4.1.10.

Proposition 4.1.12. *Let* $G : X \to \mathbb{R}$ *be locally Lipschitz and* λ*-convex. Then for* $x_0 \in X$*, we have a gradient curve* $m : I_A \to X$ *such that* $m(0) = x_0$*, and* I_A *is as in* (4.4)*. Moreover, there exists a right-side tangent vector* $m'(t+)$ *at* t *and it is equal to* $\nabla_{m(t)}(-G)$ *for all* $t \in I_A$*.*

Proof. In [26, Corollary 131], Plaut has proved essentially the same proposition for the case where X has curvature bounded below. We can modify his proof for the $CAT(0)$ case as follows. In the first part of [26, Corollary 130], which is valid for a $CAT(0)$ space, for any $y, z \in X$, we can take $f = -G$ and let γ be the geodesic from y to z, thus obtaining

$$|\nabla_{-}G|(y) \geq \frac{G(y) - G(z)}{d(y,z)} - |\lambda| d(y,z). \tag{4.5}$$

For any $\epsilon < \frac{1}{2}$, there exist x' such that

$$\frac{G(x) - G(x')}{d(x,x')} > |\nabla_{-}G|(x) - \epsilon \tag{4.6}$$

and $d(x,x') < \epsilon$.

For any y close enough to x satisfying $G(y)-G(x') \geq (1-\epsilon)(G(x)-G(x'))$ and $d(y,x') \leq (1+\epsilon)d(x,x') < 2\epsilon$, then

$$\begin{aligned}
|\nabla_{-}G|(y) &\geq \frac{G(y)-G(x')}{d(y,x')} - |\lambda| d(y,x') \qquad \text{by (4.5)} \\
&\geq \frac{1-\epsilon}{1+\epsilon}\Big(\frac{G(x)-G(x')}{d(x,x')}\Big) - 2|\lambda|\epsilon \\
&\geq \frac{1-\epsilon}{1+\epsilon}|\nabla_{-}G|(x) - (2|\lambda| + \frac{1}{3})\epsilon \quad \text{by (4.6)}.
\end{aligned}$$

Taking $\epsilon \to 0$, we see that G has semi-continuous absolute gradients. Thus, the curve m which we get by Theorem 4.1.11 must be the gradient curve of G with $m(0) = x_0$.

Since we have the unique downward gradient $\nabla_x(-G) \in T_x$ from Lemma 4.1.6, we must show that there exists a right-side tangent vector $m'(t+)$ at t and it is equal to $\nabla_x(-G)$ for all $t \in I_A$ where $x := m(t)$.

Let v be the gradient $\nabla_x(-G)$ and w_i be the tangent $\in T_x$ of a geodesic $[m(t)m(t_i)]$ for any $t_i > t$. Then it follows from (4.2) and (4.3) that as $i \to \infty$,

$$||w_i|| \to |\nabla_{-}G|(x) \ \text{ and } \ d_xG(w_i) \to -(|\nabla_{-}G|(x))^2. \tag{4.7}$$

If $v = o_x$, our proof is done since $||w_i||$ goes to 0. Otherwise, by Definition 4.1.5, we get

$$d_xG(w_i) \geq -\langle w_i, v\rangle = -||w_i|| ||v|| \cos\theta_i \ \text{ and } \ d_xG(v) = -||v||^2$$

where θ_i be the angle between w_i and v. Then we obtain

$$-\frac{d_xG(w_i)}{||w_i|| ||v||} \leq \cos\theta_i.$$

Since $||v|| = |\nabla_{-}G|(x)$ by Definition 4.1.7, the left side becomes 1 by (4.7). Then θ_i goes to zero as $i \to \infty$. Our proof is finished.

□

4.1.3 Results for time-dependent gradient curves

For a function $F = F(t,x)$ on $\mathbb{R}\times X$, let F_t be given by $F_t(x) := F(t,x)$. Then F is λ-*convex* on X if the function F_t is λ-convex.

In this section, let $F = F(t,x)$ be a function on $\mathbb{R}\times X$ which is λ-convex on X. Then we need to define a *step-energy function* of F.

Definition 4.1.13. *Given an initial position $x_0 \in X$, an initial time $t_0 \in \mathbb{R}$ and a time gap $h > 0$, the* step-energy function $E_{t_0,x_0,h} : X \to \mathbb{R}$ *at (t_0, x_0) is defined by*

$$x \mapsto F(t_0, x) + \frac{1}{2h} d^2(x_0, x).$$

If we regard this step-energy function E as a cost function, we have to find a discrete solution minimizing our cost function. The choice of the factor $1/2h$ is what is needed to make the speed of the limit curve of discrete solutions be equal to $|\nabla_- F_t|$, so that limit does not have to be reparametrized to get a gradient curve. In order to get the discrete solution, we have to show that E has a unique minimum point in X.

Proposition 4.1.14. *[24] Suppose that F is λ-convex and locally Lipschitz on X. Then when $-\lambda < \frac{1}{2h}$, $E_{t_0,x_0,h}$ has a unique minimum point on X. We will denote the unique minimum point of the step-energy function $E_{t_0,x_0,h}$ by $e(t_0, x_0, h)$.*

Proof. Since F is λ-convex on X and $d^2(x_0, x)$ is 2-convex, $E_{t_0,x_0,h}$ is $(\lambda + \frac{1}{h})$-convex and $E_{t_0,x_0,2h}$ is $(\lambda + \frac{1}{2h})$-convex. Since $-\lambda < \frac{1}{2h}$, $E_{t_0,x_0,h}$ and $E_{t_0,x_0,2h}$ are strictly convex.

Let $B_r(x)$ be the closed ball $\{y \in X : d(x,y) \le r\}$. Since F is locally Lipschitz on X, for x_0, there is a constant $r_0 > 0$ and $c_3 \ge 0$ such that $|F(t_0, x_0) - F(t_0, y)| \le c_3 d(y, x_0)$ for all $y \in B_{r_0}(x_0)$. This implies that

$$\begin{aligned} E_{t_0,x_0,2h}(y) &\ge F(t_0, y) \\ &\ge F(t_0, x_0) - c_3 d(y, x_0) \end{aligned} \tag{4.8}$$

for all $y \in B_{r_0}(x_0)$, where the first inequality is given by Definition 4.1.13.

Let $f : [0, \infty) \to \mathbb{R}$ be given by

$$f(r) := \inf_{x \in B_r(x_0)} E_{t_0,x_0,2h}(x).$$

Note we can check that f is convex. Therefore for $0 \leq r_0 \leq r$, we get

$$f(r_0) \leq \frac{r_0}{r} f(r) + \frac{r - r_0}{r} f(0).$$

This gives

$$f(0) + \frac{r}{r_0}[f(r_0) - f(0)] \leq f(r).$$

For $y \in B_{r_0}(x_0)^c$ such that $d(y, x_0) = r > r_0$, we have

$$\begin{aligned} E_{t_0,x_0,2h}(y) &\geq f(r) \text{ by the definition of } f \\ &\geq f(0) + d(y, x_0)[f(r_0) - f(0)]/r_0. \end{aligned} \tag{4.9}$$

Here, $f(0) = F(t_0, x_0)$. By (4.8) and (4.9), we have

$$E_{t_0,x_0,2h}(y) \geq c_1 - c_2 d(y, x_0)$$

for all y where $c_1 := F(t_0, x_0) - c_3 r_0$ and $c_2 := [f(r_0) - f(0)]/r_0 \geq 0$. By the definition of $E_{t_0,x_0,2h}$, this inequality becomes

$$E_{t_0,x_0,h}(y) \geq c_1 - c_2 d(y, x_0) + \frac{1}{4h} d^2(y, x_0). \tag{4.10}$$

Let M be $\inf_{y \in X} E_{t_0,x_0,h}(y)$. From (4.10), we have $M \neq -\infty$. For a sequence $\{M_n\}$ strictly decreasing to M, we can see that a sublevel $\{y : E_{t_0,x_0,h}(y) \leq M_n\}$ is bounded by solving $M_n \geq c_1 - c_2 d(y, x_0) + \frac{1}{4h} d^2(y, x_0)$. Since F and $E_{t_0,x_0,h}$ are lower semicontinuous, this sublevel is closed. Since $E_{t_0,x_0,h}$ is strictly convex, this sublevel is convex. Since $\{y : E_{t_0,x_0,h}(y) \leq M_n\}$ is a descending sequence of nonempty convex, closed and bounded sets in the $CAT(0)$ space, by Helly's Theorem, an intersection of this sequence is nonempty. This intersection is a one point set since $E_{t_0,x_0,h}$ is strictly convex.

□

Thus we have three functions $t \mapsto e(t, x_0, h)$, $h \mapsto e(t_0, x_0, h)$ and $x \mapsto e(t_0, x, h)$ for any $x_0 \in X$, $t_0 \in \mathbb{R}$ and $h > 0$. It is important to understand the movement of those three functions when t, h or x is varied. The first two functions from $\mathbb{R}$ to X are illustrated in Figure 4.1.

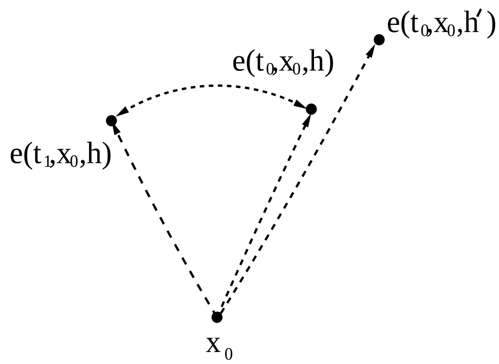

Figure 4.1: Two maps $t \mapsto e(t, x_0, h)$ and $h \mapsto e(t_0, x_0, h)$ from x_0

The last function, which we will call the *discrete flow function*, was studied in detail by Mayer. We have the following result from [24]. Note that [24, Lemma 1.12] can be easily modified to give Lemma 4.1.15, including the case $\lambda > 0$.

Lemma 4.1.15. *For any $t_0 \in \mathbb{R}$ and $-\lambda < \frac{1}{2h}$, the function $x \mapsto e(t_0, x, h)$ is $(1 + \lambda h)^{-1}$-Lipschitz.*

This improvement is obtained by multiplying out the squared difference term and making cancelations.

Suppose that the function $t \mapsto e(t, x, h)$ is Bh-Lipschitz. Then let us see how this condition works on the distance between two gradient curves issuing from the same point.

Proposition 4.1.16. *Given $x_0 \in X$ and $t_0 \in \mathbb{R}$, suppose a function $F : \mathbb{R} \times X \to \mathbb{R}$ satisfies*

1) F is locally Lipschitz on X,

2) F is λ-convex on X,

3) $\exists\ B > 0$ such that the function $t \mapsto e(t, x, h)$ is Bh-Lipschitz for any $x \in X$ and $h > 0$.

Let m_1 be the fixed time t_0 gradient curve of the function $x \mapsto F(t_0, x)$, m_2 be the fixed time $t_0 + a$ gradient curve of the function $x \mapsto F(t_0 + a, x)$ and suppose both curves are defined on $[0, T]$ where $m_1(0) = m_2(0) = x_0$. Then

$$d(m_1(T), m_2(T)) \leq BTae^{-\lambda_0 T}$$

where $\lambda_0 = \min\{0, \lambda\}$.

Proof. For each $n \in \mathbb{N}$ such that $-\lambda < \frac{1}{2h}$ where $h = T/2^n$ and $i = 0, 1, \cdots, 2^n$, we define $x_i^n := e(t_0, x_{i-1}^n, h)$ where $x_0 = x_0^n$, and $y_i^n := e(t_0 + a, y_{i-1}^n, h)$ where $x_0 = y_0^n$.

By induction on i, we will show that $d(x_i^n, y_i^n) \leq Biha(1 + \lambda_0 h)^{-i}$. First, assume that

$$d(x_{i-1}^n, y_{i-1}^n) \leq B(i-1)ha(1 + \lambda_0 h)^{-(i-1)}.$$

Since $(1+\lambda h)^{-1} \leq (1+\lambda_0 h)^{-1}$, if we denote the minimum point $e(t_0, y_{i-1}^n, h)$ by z_i^n, then

$$\begin{aligned}
d(x_i^n, y_i^n) &\leq d(x_i^n, z_i^n) + d(z_i^n, y_i^n) \\
&\leq d(x_i^n, z_i^n) + Bha && \text{by condition 3)} \\
&\leq (1+\lambda h)^{-1} d(x_{i-1}^n, z_{i-1}^n) + Bha && \text{by Lemma 4.1.15} \\
&\leq B(i-1)ha(1+\lambda_0 h)^{-i} + Bha \\
&\leq Biha(1+\lambda_0 h)^{-i} && \text{since } 1 \leq (1+\lambda_0 h)^{-1}.
\end{aligned} \tag{4.11}$$

Setting $i = 2^n$, we have

$$d(x_{2^n}^n, y_{2^n}^n) \leq BTa(1 + \lambda_0 h)^{-2^n}.$$

Taking the limit as $n \to \infty$, we get

$$d(m_1(T), m_2(T)) \le BTae^{-\lambda_0 T}$$

because $x^n_{2^n}$ converges to $m_1(T)$ and $y^n_{2^n}$ converges to $m_2(T)$ by Theorem 4.1.11 and Proposition 4.1.12.

□

Suppose that F is L-Lipschitz in t. Now we show that the gradient curves of $x \mapsto F(t,x)$ are defined on the same interval which is not dependent on t.

Proposition 4.1.17. *Let the function $x \mapsto F(t,x)$ be locally Lipschitz and λ-convex. Suppose the function $t \mapsto F(t,x)$ is L-Lipschitz. Then for each time t, every gradient curve of the function $x \mapsto F(t,x)$ starting at x_0 is defined on I_A where I_A is independent of t.*

Proof. By Proposition 4.1.12, we have the gradient curve of $x \mapsto F(t,x)$ defined on the interval I_A starting at x_0 for any t. The interval I_A is dependent on the constant

$$\liminf_{d(x,y)\to\infty} \frac{F(t,x)}{d^2(x,y)}$$

(see Equation (4.4)). Since $|F(t,x) - F(t',x)| \le L|t-t'|$ and

$$\liminf_{d(x,y)\to\infty} \frac{L|t-t'|}{d^2(x,y)} = 0,$$

I_A is not dependent on t.

□

The assumption that the function $t \mapsto e(t,x,h)$ is Bh-Lipschitz is important. So we need to understand this condition carefully and look for a better form of a condition that might replace it. Here we obtain a necessary condition by seeing how this Lipschitz condition acts on the gradient vectors at x_0.

Theorem 4.1.18. *Let X be a $CAT(0)$ space. Given $x_0 \in X$ and $t_1 \in \mathbb{R}$, suppose a function $F : \mathbb{R} \times X \to \mathbb{R}$ satisfies*

1) F is locally Lipschitz on X,

2) F is λ-convex on X,

3) F is L-Lipschitz in t,

4) $\exists\ B > 0$ such that the function $t \mapsto e(t, x, h)$ is Bh-Lipschitz for any $x \in X$ and $h > 0$.

Let m_1 be the fixed time t_1 gradient curve of the function F_{t_1} given by $F_{t_1}(x) := F(t_1, x)$, m_2 be the fixed time t_2 gradient curve of the function F_{t_2} given by $F_{t_2}(x) := F(t_2, x)$, where both curves are defined on I_A, and $m_1(0) = m_2(0) = x_0$. Then

$$d\big(\nabla_{x_0}(-F_{t_1}), \nabla_{x_0}(-F_{t_2})\big) \le C'|t_1 - t_2|$$

for sufficient small $|t_1 - t_2|$.

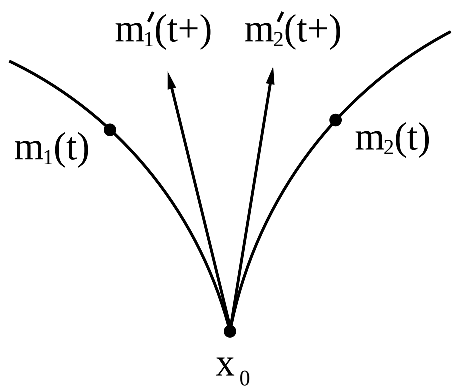

Figure 4.2: Two time-independent gradient curves m_1 and m_2 and tangent vectors

Proof. We get the gradient curves m_1 and m_2 by Proposition 4.1.12. From Proposition 4.1.16, we have

$$\frac{d(m_1(t), m_2(t))}{t} \le B|t_1 - t_2|e^{-\lambda_0 t}$$

for all $t \in I_A$ with $\lambda_0 = \min\{0, \lambda\}$. By the triangle inequality with x_0,

$$\Big|\frac{d(m_1(t), m_1(0))}{t} - \frac{d(m_2(t), m_2(0))}{t}\Big| \leq B|t_1 - t_2|e^{-\lambda_0 t}.$$

Since

$$\lim_{t \to 0+} \frac{d(m_i(t), m_i(0))}{t} = ||\nabla_{x_0}(-F_{t_i})|| \tag{4.12}$$

for $i = 1$ and 2, we have

$$\Big|||\nabla_{x_0}(-F_{t_1})|| - ||\nabla_{x_0}(-F_{t_2})||\Big| \leq B|t_1 - t_2|. \tag{4.13}$$

When one of $||\nabla_{x_0}(-F_{t_i})||$ is zero, we have the zero tangent vector. So Equation (4.13) is enough to prove Theorem 4.1.18. Otherwise, suppose that $||\nabla_{x_0}(-F_{t_1})||$ is not zero. Then there is a constant C such that $||\nabla_{x_0}(-F_{t_1})||$ is bigger than $4C$. If $|t_1 - t_2|$ is less than C/B, then $2C$ is less than $||\nabla_{x_0}(-F_{t_2})||$. By (4.12), for sufficient small t, we get $d(m_1(t), x_0) \geq tC$ and $d(m_2(t), x_0) \geq tC$ where $m_1(0) = m_2(0) = x_0$.

The distance between the direction of m_1 at x_0 and the direction of m_2 at x_0 is the angle

$$\lim_{t \to 0+} \angle_{x_0} m_1(t), m_2(t).$$

Since this angle is less than the angle of a Euclidean triangle with two edge lengths tC, tC and third edge length less than $Bt|t_1 - t_2|e^{-\lambda_0 t}$, we have that the distance of two directions at x_0 is less than $\frac{2B}{C}|t_1 - t_2|$. So the direction at x_0 is $\frac{2B}{C}$-Lipschitz and the righthand side speed at x_0 is B-Lipschitz. Thus the distance of tangent vectors of m_1 and m_2 at x_0 is C'-Lipschitz.

□

Definition 4.1.19. *A locally Lipschitz curve $u : I \to X$ is a* time-dependent gradient curve *of $F = F(t, x)$ if for all $t \in I$, there exists the right-side tangent vector $u'(t+)$ and it is equal* *to the downward gradient vector* $\nabla_{u(t)}(-F_t)$ *at* $u(t)$ *where* $F_t(x) := F(t, x)$.

We are going to obtain time-dependent gradient curves here. For this, we need a set of discrete solutions to converge a continous solution. Look at Figure 4.3.

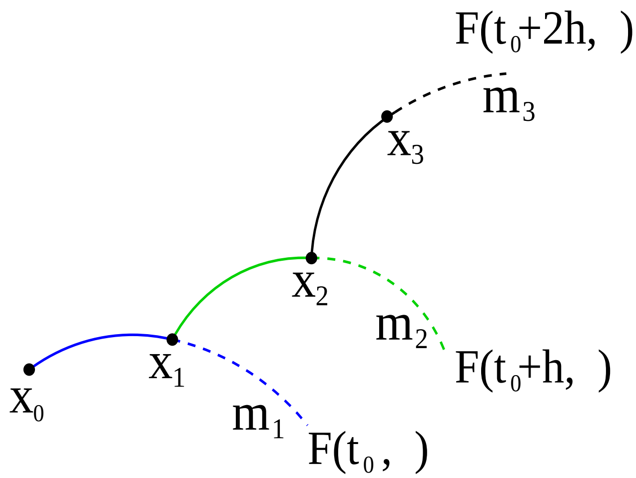

Figure 4.3: A discrete solution curve with step size h

This is our idea. For small $h \in I_A$, we have a gradient curve m_1 of $x \mapsto F(t_0, x)$ up to the time h. So $m_1(0) = x_0$ and $m_1(h) = x_1$ in Figure 4.3. Then we extend this with another gradient curve m_2 of $x \mapsto F(t_0 + h, x)$ up to the time h because we need to capture the effect of time changing. Thus $m_2(0) = x_1$ and $m_2(h) = x_2$ in Figure 4.3. Continuing this way, we will have the discrete solution curve. Then we need to show this curve converges to a limit curve as h goes to zero.

Theorem 4.1.20. *Let* (X, d) *be a* $CAT(0)$ *space. Given* $x_0 \in X$ *and* $t_0 \in \mathbb{R}$*, suppose a function* $F : \mathbb{R} \times X \to \mathbb{R}$ *satisfies*

1) F *is locally Lipschitz on* X*,*

2) F *is* λ*-convex on* X*,*

3) F *is* L*-Lipschitz in* t*,*

4) $\exists\ B > 0$ *such that the function* $t \mapsto e(t, x, h)$ *is* Bh*-Lipschitz for any* $x \in X$ *and* $h > 0$*.*

Then we have a unique time-dependent gradient curve u_{x_0,t_0} *given by* $t \mapsto u_{x_0,t_0}(t_0 + t)$ *of* F *such that* $u_{x_0,t_0}(t_0) = x_0$*.*

Proof. Assume that $h = t/2^n \in I_A$. Let p_i^n be the point on the fixed time $t_0 + (i-1)h$ gradient curve flowing for time h from p_{i-1}^n where $p_0^n = x_0$. Let z_i be the point on the fixed time $t_0 + (i-1)h$ gradient curve flowing for time h from p_{2i-2}^{n+1}. See Figure 4.4 and compare Figure 3.5.

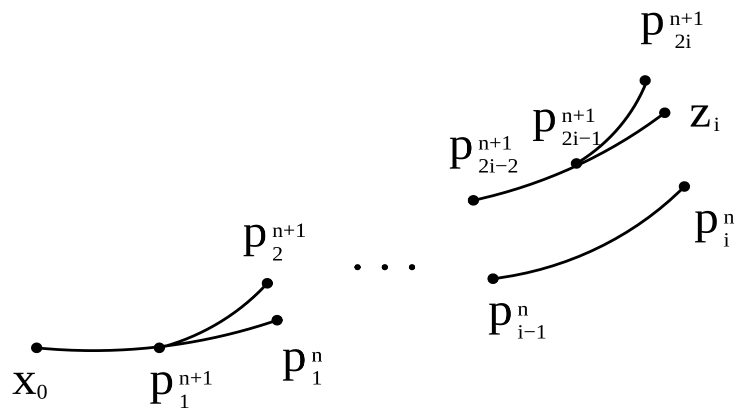

Figure 4.4: Discrete solution curves p^n and p^{n+1} and the point z_i

By induction on i, we will show that $d(p_i^n, p_{2i}^{n+1}) \leq Bih^2 e^{-\lambda_0 h/2} e^{-\lambda_0 (i-1)h}/4$ where $\lambda_0 = \min\{0, \lambda\}$.

First, assume that $d(p_{i-1}^n, p_{2(i-1)}^{n+1}) \leq B(i-1)h^2 e^{-\lambda_0 h/2} e^{-\lambda_0 (i-2)h}/4$. Then

$$
\begin{aligned}
d(p_i^n, p_{2i}^{n+1}) &\leq d(p_i^n, z_i) + d(z_i, p_{2i}^{n+1}) \\
&\leq d(p_i^n, z_i) + Bh^2 e^{-\lambda_0 h/2}/4 && \text{by Proposition 4.1.16} \\
&\leq e^{-\lambda_0 h} d(p_{i-1}^n, p_{2(i-1)}^{n+1}) + Bh^2 e^{-\lambda_0 h/2}/4 && \text{by [24, Theorem2.1]} \\
&\leq e^{-\lambda_0 h} B(i-1)h^2 e^{-\lambda_0 h/2} e^{-\lambda_0 (i-2)h}/4 + Bh^2 e^{-\lambda_0 h/2}/4 \\
&\leq Bih^2 e^{-\lambda_0 h/2} e^{-\lambda_0 (i-1)h}/4
\end{aligned}
\tag{4.14}
$$

Letting i be 2^n, we have

$$d(p_{2^n}^n, p_{2^{n+1}}^{n+1}) \leq B \tfrac{t}{2} \tfrac{t}{2^{n+1}} e^{-\lambda_0 t/2^{n+1}} e^{-\lambda_0 t}. \tag{4.15}$$

This means that we have the Cauchy sequence $\{p_{2^n}^n\}$. So we can define $u_{x_0,t_0}(t_0 + t) :=$

$\lim_{n\to\infty} p^n_{2^n}$.

To finish the proof of Theorem 4.1.20, we need the following two results.

Proposition 4.1.21.

$$d(u_{x_0,t_0}(t_0+t), u_{y_0,t_0}(t_0+t)) \le e^{-\lambda_0 t} d(x_0, y_0)$$

where $\lambda_0 = \min\{0, \lambda\}$.

Proof. Assume that $h = t/2^n \in I_A$. For x_0, let p^n_i be the point on the fixed time $t_0+(i-1)h$ gradient curve flowing for time h from p^n_{i-1} where $p^n_0 = x_0$. For y_0, let q^n_i be the point on the fixed time $t_0+(i-1)h$ gradient curve flowing for time h from q^n_{i-1} where $q^n_0 = y_0$.

Then by [24, Theorem 2.1], $d(p^n_i, q^n_i) \le e^{-\lambda_0 h} d(p^n_{i-1}, q^n_{i-1})$ and $d(p^n_i, q^n_i) \le e^{-\lambda_0 ih} d(x_0, y_0)$. When we let i be 2^n and let n go to ∞, we have

$$d(u_{x_0,t_0}(t_0+t), u_{y_0,t_0}(t_0+t)) \le e^{-\lambda_0 t} d(x_0, y_0).$$

□

Lemma 4.1.22. *Let* $\bar{u}$ *be a reparametization of the limit curve given by* $s \mapsto \bar{u}(s) := u_{x_0,t_0}(t_0+t+s)$. *Then for* $s \in I_A$,

$$d(\bar{u}(s), m(s)) \le B\frac{s^2}{2}e^{-2\lambda_0 s}$$

where $m = m(s)$ *is the gradient curve of the function* $x \mapsto F(t_0+t, x)$ *with* $m(0) = \bar{u}(0) = u_{x_0,t_0}(t_0+t)$ *and* $\lambda_0 = \min\{0, \lambda\}$.

This implies

$$\lim_{s\to 0+} \frac{d(\bar{u}(s), \bar{u}(0))}{s} = ||\nabla_{\bar{u}(0)}(-F_{t_0+t})||$$

where the function F_{t_0+t} *is given by* $F_{t_0+t}(x) := F(t_0+t, x)$.

To finish the proof of Theorem 4.1.20, first look at a distance between two directions. The distance between the direction of $\bar{u}$ at $\bar{u}(0)$ and the direction of m at $\bar{u}(0)$ is the angle

$$\lim_{s \to 0+} \angle_{\bar{u}(0)} \bar{u}(s), m(s).$$

So we have to show that this angle is zero for the proof of Theorem 4.1.20.

Suppose $||\nabla_{\bar{u}(0)}(-F_{t_0+t})||$ is not zero. So there is a constant C such that $||\nabla_{\bar{u}(0)}(-F_{t_0+t})|| > 2C$. For sufficient small t, we get $d(m(t), m(0)) \geq tC$ and $d(\bar{u}(t), \bar{u}(0)) \geq tC$, where $m(0) = \bar{u}(0)$, since

$$\lim_{s \to 0+} \frac{d(m(s), m(0))}{s} = ||\nabla_{\bar{u}(0)}(-F_{t_0+t})|| = \lim_{s \to 0+} \frac{d(\bar{u}(s), \bar{u}(0))}{s}.$$

Since $\angle_{\bar{u}(0)} \bar{u}(t), m(t)$ is less than the angle of a Euclidean triangle with two edge lengths tC, tC and third edge length less than $Bt^2 e^{-2\lambda_0 t}/2$ by Lemma 4.1.22, we have that $\angle_{\bar{u}(0)} \bar{u}(t), m(t)$ is less than $\frac{Bt}{C} e^{-2\lambda_0 t}$. As $t \to 0+$, this becomes zero. Therefore

$$\lim_{t \to 0+} \angle_{\bar{u}(0)} \bar{u}(t), m(t) = 0.$$

□

From the first part in the proof of Theorem 4.1.20, we get the following estimate on rate of convergence:

Corollary 4.1.23.

$$d(u_{x_0,t_0}(t_0 + t), p^n_{2^n}) \leq B \frac{t}{2} \frac{t}{2^n} e^{-\lambda_0 t/2^{n+1}} e^{-\lambda_0 t}$$

where $\lambda_0 = \min\{0, \lambda\}$.

Proof. For m and n such that $m > n$, from Equation (4.15),

$$d(p^n_{2^n}, p^m_{2^m}) \leq d(p^n_{2^n}, p^{n+1}_{2^{n+1}}) + \cdots + d(p^{m-1}_{2^{m-1}}, p^m_{2^m}) \leq B \frac{t}{2} \frac{t}{2^n} e^{-\lambda_0 t/2^{n+1}} e^{-\lambda_0 t}.$$

Then as $m \to \infty$, we have

$$d(u_{x_0,t_0}(t_0+t), p^n_{2^n}) \le B\frac{t}{2}\frac{t}{2^n}e^{-\lambda_0 t/2^{n+1}}e^{-\lambda_0 t}.$$

□

Proof of Lemma 4.1.22. First, we look at x_0 and show the case $t = 0$. Letting $n = 0$ in Corollary 4.1.23, $p^n_{2^n}$ becomes $m(s)$ since $p^n_{2^n}$ is only the time-independent gradient curve of $x \mapsto F(t_0, x)$ when $n = 0$. Thus we have

$$d(\bar{u}(s), m(s)) \le B\frac{s^2}{2}e^{-\lambda_0 s/2}e^{-\lambda_0 s}$$

where $\bar{u}$ is the limit curve given by $s \mapsto \bar{u}(s) := u_{x_0,t_0}(t_0 + s)$.

Second, for any point $u_{x_0,t_0}(t_0 + t)$ on u_{x_0,t_0}, we can do this calculation with $\bar{u}$ given by

$$s \mapsto \bar{u}(s) := u_{x_0,t_0}(t_0 + t + s)$$

since it is the unique limit curve. Then we get the same inequality. Our proof is done. □

4.1.4 An example

We show how Theorem 4.1.20 works. In a Euclidean plane, if we have a moving line segment, then we find a continuous pursuit curve chasing this segment.

Let Y be a line segment with two end points E_1, E_2. Suppose that E_i has speed $\le k$ given by $t \mapsto E_i(t) \in \mathbb{R}^2$. Denote the line segment $[E_1(t)E_2(t)]$ at time t by Y_t. By a continuous pursuit curve, we mean that $P = P(t)$ is the time-dependent gradient curve of the function $F(t, x) = d_{Y_t}(x)$. Note that this is a different algorithm than that used in Theorem 1.2.6, where we chase the midpoint of the segment.

First let us construct a discrete pursuit curve P_h with time size h. Let p_0 be the pursuer's initial position. By Proposition 2.2.6, p_0 has a unique footpoint q_0 in Y_0. Then the first part of P_h will be the line segment $[p_0p_1]$ where p_1 is the point on $[p_0q_0]$ such that $|p_0p_1| = h$ if $|p_0q_0| > h$. Otherwise $p_1 := q_0$. Next we consider the footpoint q_1 of p_1 in Y_h. Then the second part of P_h will be the line segment $[p_1p_2]$ where p_2 is the point on $[p_1q_1]$ such that $|p_1p_2| = h$ if $|p_1q_1| > h$. Otherwise $p_2 := q_1$. Continuing this process, we will get a polysegment $p_0p_1p_2p_3\cdots$.

Proposition 4.1.24. *Let q_t be the footpoint of p in Y_t. Then the step-energy function $E_{t,p,h}(y) = d_{Y_t}(y) + \frac{1}{2h}d^2(p,y)$ has a unique minimum point p' as $e(t,p,h)$ where p' is the point on $[pq_t]$ at distance h from p.*

Proof. Let x_s be the point on $[pq_t]$ at distance s from p. Let C be $|pq_t|$. Thus $d_{Y_t}(x_s) = C - s$. Then we can check that x_s is the footpoint of p in the sublevel given by $d_{Y_t} \le C - s$. This implies that $E_{t,p,h}(x_s) \le E_{t,p,h}(z_s)$ for any point z_s such that $d_{Y_t}(z_s) = C - s$.

Since $E_{t,p,h}(p) < E_{t,p,h}(z)$ for z such that $d_{Y_t}(z) > C$, we only need to find the minimum point on $[pq_t]$. Since $E_{t,p,h}(x_s) = C - s + \frac{1}{2h}s^2$, it has the minimum at p'.

□

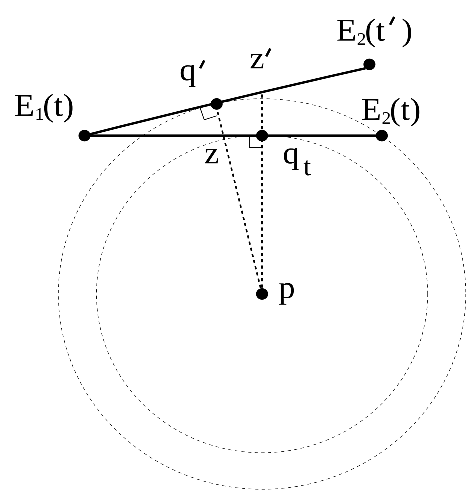

Figure 4.5: The footpoint q_t in $[E_1(t)E_2(t)]$ and the footpoint q' in $[E_1(t)E_2(t')]$

Proposition 4.1.25. *Let q_t be the footpoint of p in Y_t and $q_{t'}$ be the footpoint of p in $Y_{t'}$. If there is a constant B' such that $|E_1(t)E_2(t)| > B'$ for all t and E_i has speed at most k and $|t-t'|$ is less than $B'/(2k)$, then $|q_t q_{t'}|$ is at most*

$$2k|t-t'|\Big(1+\frac{2L}{\sqrt{3}B'}\Big)$$

where $L = \max\{d_{Y_t}(p), d_{Y_{t'}}(p)\}$. This implies that $t \mapsto F(t,p)$ is locally Lipschitz in t.

Proof. First consider two line segments $[E_1(t)E_2(t)]$ and $[E_1(t)E_2(t')]$. Since $|E_2(t)E_2(t')| \leq k|t-t'|$ and $|E_1(t)E_2(t)| > B'$,

$$\sin\angle E_2(t)E_1(t)E_2(t') \leq \frac{k|t-t'|}{B'}.$$

Since $k|t-t'|/B' \leq 1/2$,

$$\angle E_2(t)E_1(t)E_2(t') \leq \frac{2k|t-t'|}{\sqrt{3}B'}$$

because $\sin^{-1}$ has Lipschitz constant $2/\sqrt{3}$ on $[0,1/2]$.

First, we work when $|pq_t| \leq |pq'|$ where q' is the footpoint of p in $[E_1(t)E_2(t')]$. If $q_t = E_2(t)$ and $q' = E_2(t')$,

$$|q_t q'| \leq k|t-t'|(1+\frac{2L}{\sqrt{3}B'})$$

because of $|E_2(t)E_2(t')| \leq k|t-t'|$. This inequality is our goal.

Otherwise, let z be the intersection point of $[pq']$ with the circle having center p and radius L_t where $L_t = |pq_t|$.

If there is a point z' on $[E_1(t)E_2(t')]$ such that z', p and q_t are collinear, then since q' is the footpoint of p in $[E_1(t)E_2(t')]$,

$$|pq'| \leq |pz'|.$$

Then this implies

$$|zq'| = |pq'| - L_t \leq |pz'| - L_t = |q_t z'|.$$

Since $|q_t z'| \leq |E_2(t)E_2(t')| \leq k|t-t'|$, we obtain

$$|zq'| \leq k|t-t'|. \tag{4.16}$$

For $|zq_t|$, we check that the angle at $E_1(t)$ is equal to the angle $\alpha = \angle q_t p z$ because the two right triangles $\triangle E_1(t)q'z$ and $\triangle p q_t z$ have the same angle at z. This angle is less than $2k|t-t'|/(\sqrt{3}B')$. Since the length of the arc between z and q_t on the circle is the radius $L_t\alpha$,

$$|zq_t| \leq \frac{2k|t-t'|L_t}{\sqrt{3}B'}. \tag{4.17}$$

Then by (4.16) and (4.17),

$$|q_t q'| \leq |q_t z| + |zq'| \leq k|t-t'|\Big(1 + \frac{2L}{\sqrt{3}B'}\Big). \tag{4.18}$$

If there is no point z' on $[E_1(t)E_2(t')]$ such that z', p and q_t are collinear, let z'' be the footpoint of q_t in the extension of $[E_1(t)E_2(t')]$. Since $\angle E_2(t')q_tE_2(t) > \pi/2$,

$$|q_t E_2(t')| \leq |E_2(t)E_2(t')| \leq k|t-t'|.$$

This gives

$$|q_t z''| \leq |q_t E_2(t')| \leq k|t-t'|.$$

Since $|q'p| \leq |q_t p| + |q_t z''|$, we have

$$|q'z| \leq |q_t z''| \leq k|t-t'|.$$

Then by (4.17), we get Equation (4.18).

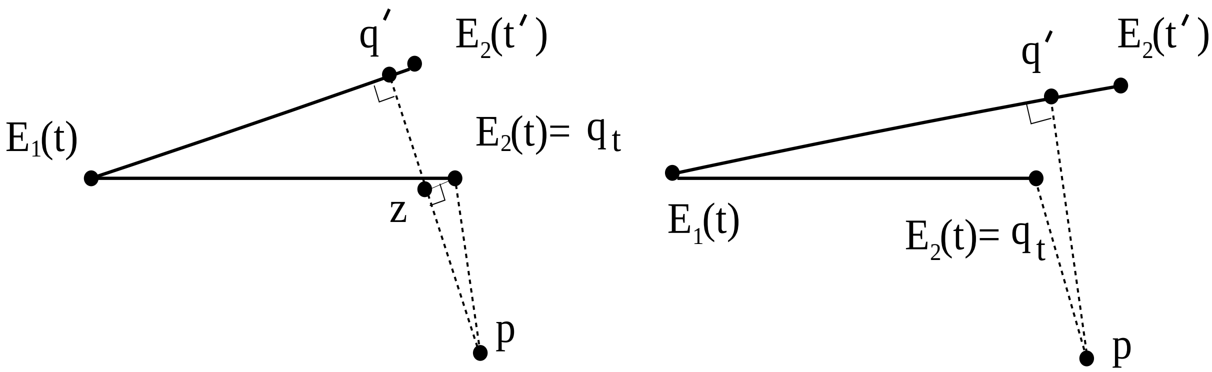

Figure 4.6: The footpoint q_t in $[E_1(t)E_2(t)]$ and the footpoint q' in $[E_1(t)E_2(t')]$ when $\angle E_1(t)q_tp > \pi/2$

If $E_2(t) = q_t$, $\angle E_1(t)q_tp > \pi/2$ and $[pq']$ does not meet $[E_1(t)E_2(t)]$, then

$$|q_tq'| \leq |E_2(t)E_2(t')| \leq k|t-t'|$$

(See Figure 4.6). So (4.18) is true in this case.

If $E_2(t) = q_t$, $\angle E_1(t)q_tp > \pi/2$ and $[pq']$ meets $[E_1(t)E_2(t)]$, let z be the point on $[pq']$ such that $\angle pzq_t = \pi/2$. Then

$$|q'z| \leq |E_2(t)E_2(t')| \leq k|t-t'|$$

and

$$|zq_t| \leq L_t\angle zpq_t.$$

Since $\angle zpq_t$ is less than $\angle E_2(t')E_1(t)E_2(t)$, (4.18) is true in this case.

When $|pq_t| > |pq'|$, in the Figure 4.5, we exchange the positions of $E_2(t)$ and $E_2(t')$. Also exchange the positions of q_t and q'. Then we have the same result.

Similarly, we have

$$|q_{t'}q'| \leq k|t-t'|\Big(1 + \frac{2L}{\sqrt{3}B'}\Big).$$

Thus the distance between q_t and $q_{t'}$ is at most

$$2k|t - t'|\Big(1 + \frac{2L}{\sqrt{3}B'}\Big).$$

□

Proposition 4.1.26. *If there is a constant B' such that $|E_1(t)E_2(t)| > B'$ for all t, and E_i has speed at most k, then there is a continuous pursuit curve P such that $P(0) = p_0$.*

Here, the continuous pursuit curve means that $P = P(t)$ is the time-dependent gradient curve of the function d_{Y_t}.

Proof. For any time T, let L be $\sup_{t\in[0,T]} \max\{|P(0)E_1(t)|, |P(0)E_2(t)|\}$. By Proposition 4.1.25, this means that F is $2k(1+2L/\sqrt{3}B')$-Lipschitz in $t \in [0, T]$ where $F(t, x) := d_{Y_t}(x)$.

By Proposition 2.2.6, F is convex on X.

Let x_t be the footpoint of x in Y_t and z_t be the footpoint of z in Y_t. Suppose $|xx_t| \leq |zz_t|$. Let z' be the point on $[zz_t]$ at distant $|xx_t|$ from z_t. Since z' is the footpoint of z in $d_{Y_t} \leq |xx_t|$,

$$|zz_t| - |xx_t| = |z'z| \leq |xz|.$$

Thus F is 1-Lipschitz on X.

For a constant $C > 0$, let Z_t be the set $\{z \in \mathbb{R}^2 \mid d_{Y_t}(z) > C\}$. If p is in Z_t and $Z_{t'}$, let q_t be the footpoint of p in Y_t and $q_{t'}$ be the footpoint of p in $Y_{t'}$. Suppose $|q_tq_{t'}|$ is smaller than $C/2$. Let p_t be the point on $[pq_t]$ at distant h from p. Then p_t is $e(t, p, h)$ for $F(t, x) = d_{Y_t}(x)$. Since $|pq_t| > C$, $\sin \angle q_tpq_{t'}$ is less than $|q_tq_{t'}|/C$. Since $|q_tq_{t'}|/C \leq 1/2$, this implies

$$\angle q_tpq_{t'} \leq \frac{2|q_tq_{t'}|}{\sqrt{3}C}$$

because $\sin^{-1}$ has Lipschitz constant $2/\sqrt{3}$ on $[0, 1/2]$.

By Proposition 4.1.25, we have a constant C' such that $\angle q_tpq_{t'} \leq C'|t - t'|$. Since

$|pp_t| = h$, there is a constant B such that $|p_t p_{t'}| \leq Bh|t - t'|$. This implies that with this constant B, F satisfies condition 4) in Theorem 4.1.20. Therefore by Theorem 4.1.20, we have a unique continuous pursuit curve P of the moving line segment Y_t up to the distance C from this moving line segment. Since this pursuit curve is unique, letting C go to zero, our proof is finished.

□

4.1.5 Further research

There are many natural problems that arise from my work. These include:

1. Continue the study of time-dependent gradient flow. Find a better condition replacing the condition 4) in Theorem 4.1.20.

2. Extend our work on $CAT(0)$ spaces to $CAT(K)$ spaces for $K > 0$.

3. Develop applications of Theorem 4.1.20 to the problem of multiple evaders.

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