HOLOMORPHIC CHAINS ON THE PROJECTIVE LINE

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2012

Urbana, Illinois

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Abstract

Holomorphic chains on a smooth algebraic curve are tuples of vector bundles on the curve together with the homomorphisms between them. A *type* of a holomorphic chain is a tuple $\mathbf{t}$ of integers consisting of the ranks and degrees of the underlying vector bundles. The definition of holomorphic chains was introduced and their stability were defined by L. Álvarez-Cónsul and O. García-Prada. The moduli spaces of holomorphic chains were constructed using the Geometric Invariant Theory (GIT) by A. H. W. Schmitt. They studied holomorphic chains on a smooth algebraic curve of genus $g \geq 2$. In general it is difficult to describe moduli spaces. The stability of holomorphic chains depends on a real vector parameter in $\mathbb{R}^n$ called $\boldsymbol{\alpha}$-stability ($\boldsymbol{\alpha} \in \mathbb{R}^n$). The region $R^s(\mathbf{t})$ ($R(\mathbf{t})$) is the set of $\boldsymbol{\alpha}$ for which there exists an $\boldsymbol{\alpha}$-(semi)stable holomorphic chain of type $\mathbf{t}$. Thus the moduli spaces depend on the parameter. A rational vector parameter corresponds to a given linearization of the GIT quotient. $\mathbb{R}^n$ is partitioned into locally closed subsets called chambers where the stability does not change. The moduli spaces in each chamber do not change.

The case in which the underlying bundles are all line bundles is simple. The chamber structure for this case is classified. Line bundles on a smooth algebraic curve can be parameterized by a Poincaré line bundle. Considering this, holomorphic chains composed of line bundles with fixed degrees are parameterized by the direct sum of vector bundles when the gaps of degrees between consecutive line bundles are sufficiently large, and the moduli space is identified as the product of the corresponding projective space bundles. Finally, the automorphism group of the holomorphic chains composed of line bundles is $(\mathbb{C}^*)^n$. The variation of $\boldsymbol{\alpha}$ relates to the variation of the GIT quotient for the action of $(\mathbb{C}^*)^n$.

The next step is to look at holomorphic chains on $\mathbb{P}^1$. Vector bundles on $\mathbb{P}^1$ are splitting, by A. Grothendieck. The first case to look at is the holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$. The chamber structure for holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ is identified and that for its dual chain of type $\mathbf{t} = (2, 1; s, 0)$ is the same as for those of type $\mathbf{t}^\vee = (1, 2; 0, -s)$. The chamber is determined by partitioning $\mathbb{R}$. Moreover, the moduli spaces of type $\mathbf{t} = (1, 2; 0, -s)$ can be identified as those of type $\mathbf{t}^\vee = (1, 2; 0, -s)$ by sending a chain to its dual chain. The stability region $R^s(\mathbf{t})$ is a bounded open interval which is partitioned into subinterval chambers. It is relatively easy to describe the moduli spaces corresponding to the leftmost

and rightmost intervals. They are Grassmannian varieties for s even and projective spaces, respectively. The same results are found in different guises. A *coherent system* on a smooth algebraic curve is the pair of a vector bundle and a vector subspace of holomorphic sections. A coherent system can be described as a holomorphic chain. The isomorphism class of a coherent system of type $(2, s, 1)$ can be identified as the isomorphism class of the associated holomorphic chain of type $\mathbf{t} = (2, 1; s, 0)$. A *holomorphic pair* is the pair of a vector bundle and a holomorphic section of it. A holomorphic pair can be described as a holomorphic chain. The isomorphism class of a holomorphic pair of rank two can be identified as the isomorphism class of the associated holomorphic chain of type $\mathbf{t} = (2, 1, ; s, 0)$. Moreover, their stabilities coincide. The stabilities for coherent systems and holomorphic pairs involve real parameters. The parameter α relates to these parameters. P. E. Newstead and H. Lange studied coherent systems on $\mathbb{P}^1$. M. Thaddeus studied holomorphic pairs of rank 2 with a fixed determinant on a smooth algebraic curve of genus $g \geq 2$. His description of the moduli spaces are applicable for any genus. On $\mathbb{P}^1$, if the degree of a vector bundle is fixed, then its determinant is automatically fixed. The moduli spaces of the holomorphic chains of type $\mathbf{t} = (2, 1, ; s, 0)$ on $\mathbb{P}^1$ can be identified as those of the associated coherent systems and holomorphic pairs.

The chamber structure for the holomorphic chains of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ is identified. The stability region $R^s(\mathbf{t})$ is an open subset of $\mathbb{R}^2$ bounded by a parallelogram. The stability region is partitioned into sub-parallelogram chambers. Each edge of the parallelogram has a nonzero slope. Analogous to the leftmost and rightmost interval chambers are bottommost and topmost ones. The corresponding moduli spaces are the product of two Grassimannian varieties for d_0 and d_2 even, and the product of two projective spaces.

A co-Higgs bundle on a smooth algebraic curve is a vector bundle with a Higgs field. A Higgs field is a holomorphic section of the tensor product of the endomorphism bundle of the vector bundle and the dual of the canonical line bundle. A co-Higgs bundle can be described as a holomorphic chain. If two co-Higgs bundles are isomorphic then the associated holomorphic chains are isomorphic. On a smooth algebraic curve, interesting co-Higgs bundle are found only on $\mathbb{P}^1$. S. Rayan classified stable co-Higgs bundles on $\mathbb{P}^1$. He characterized the moduli space of stable co-Higgs bundles of rank 2 and degree odd as a universal elliptic curve with a globally defined equation. The stability of co-Higgs bundles of rank 2 and degree odd is compared with the α-stability of the associated holomorphic chains. The α-stable holomorphic chains associated with co-Higgs bundles of rank 2 and degree -1 are classified. A co-Higgs bundle of rank 2 and degree -1 is stable if and only if the associated holomorphic chain is 3-semistable. A co-Higgs bundle of rank 2 and degree -1 is stable if the associated holomorphic chain is α-stable for $\alpha > 3$.

The moduli spaces of holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ with $s > 2$ on $\mathbb{P}^1$ have natural subspaces with fixed underlying bundles. The underlying bundles are determined by splitting types denoted by $(0, (-d, -e))$ with $s = d + e$. The automorphism group of a chain is non-reductive unless the underlying bundles are semistable. In the non-reductive case, Drézet-Trautmann's non-reductive GIT method applies to the subspaces. Their method involves a tuple of rational parameters $\Lambda = (\lambda_1, \lambda_2, \mu_1)$ called a polarization. Given a splitting type $(0, (-d, -e))$ with $1 \leq e < d \leq \lceil \frac{s}{2} \rceil$ and $d + e = s$, if $d \neq e$, a chain $(\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ is α-stable for some α if and only if the map ϕ is stable with respect to some polarization Λ. The subspace of the fixed splitting type $(0, (-d, -e))$ can be identified as Drézet and Trautmann's non-reductive GIT quotient for some polarization. The parameter α relates to the polarization Λ.

For the dual type $\mathbf{t}^\vee = (2, 1; s, 0)$, Doran and Kirwn's non-reductive GIT method applies to the subspaces. Their non-reductive GIT quotients involve a rational parameter δ. Given a splitting type $((d, e), 0)$ with $1 \leq e < d \leq \lceil \frac{s}{2} \rceil$ and $d + e = s$, if $d - e > 1$, then a chain $(\mathcal{O}(d) \oplus \mathcal{O}(e), \mathcal{O}, ; \phi)$ is α-stable for some α if and only if the map ϕ is δ-stable for some δ. The subspace of the fixed splitting type $((d, e), 0)$ can be identified as Doran and Kirwan's non-reductive GIT quotient for some δ. In the paper it is explained how the parameter α relates to the parameter δ. Moreover, by a symplectic description, Doran and Kirwan's non-reductive GIT quotient is a $\mathbb{P}^e$ bundle over $\mathbb{P}^{e-1}$.

The subspace of fixed splitting type $(0, (-d, -e))$ can be identified as the subspace of fixed splitting type $((d, e), 0)$ by mapping dual chains. If $d - e > 1$ then Drézet-Trautmann's non-reductive GIT quotient is identical to Doran and Kirwan's non-reductive GIT quotient. It is explained how the polarization Λ relates to the parameter δ.

To my family

Acknowledgments

This dissertation would not have been possible without the guidance and help of several individuals who in one way or another contributed to or extended their valuable assistance in the preparation and completion of this study.

I would like to express sincere appreciation to my thesis advisor Steven Bradlow for his guidance. Professor Bradlow introduced me to the topic and gave me insight during weekly meetings.

I would like to thank my committee members Professor William Haboush, Professor Hal Schenck and Professor Thomas Nevins. William Haboush has been a great teacher and friend. Hal Schenck has always encouraged me. Tom Nevins has always been kind and helpful.

I would like to thank Professor Karen Mortensen, whose kind efforts regarding my academic requirements were invaluable.

I would like to thank Professor Sheldon Katz for offering me summer support. During that time I studied an important example.

My thanks goes Kathleen Smith, who came by and gave me a hug when I gave my final examination presentation.

I would like to thank my algebraic geometry colleagues Jinwon Choi and Mee Seong Im who helped me to improve my final presentation.

I would like to thank my former officemates Chih-Chung Liu, Dan Lior, Ryan Rettberg and Anja Bankovic for their friendship and support.

Finally I thank my wife Hyunjung Noh, without whose help I would not have been able to finish my thesis. She supported me in every way, especially in taking care of our children, Jeana, Grace and Joseph. I would also like to thank my kids, with whom I have such happiness.

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Chapter 1

Introduction

In this thesis we describe some moduli spaces of holomorphic chains on $\mathbb{P}^1$. The moduli spaces of vector bundles with and without additional structures have been studied for a long time. In general, it is difficult to describe a moduli space. Holomorphic chains on a smooth algebraic curve were introduced and generally studied for genus $g \geq 2$ in [3]. The next step is to study holomorphic chains on $\mathbb{P}^1$. In this case, all of the vector bundles are direct sums of line bundles, by A. Grothendieck [17], [26]. A line bundle on $\mathbb{P}^1$ is completely determined by its degree, which is an integer. We will use this fact to obtain a description of the moduli spaces of holomorphic chains on $\mathbb{P}^1$.

A holomorphic chain is a certain type of quiver bundle. A quiver is an oriented graph consisting of a set V of vertices and a set A of arrows. If the number $\sharp V$ of vertices is equal to the number $\sharp A$ of arrows plus 1, then the quiver is called a tree. A representation in the category of vector bundles means the assignments of a vector bundle to each vertex and a homomorphism to each arrow. These are called *quiver bundles*. A *type* of a holomorphic chain is a tuple of integers determined by ranks and degrees of the underlying vector bundles. A *type* of a quiver bundle is defined by a tuple of Hilbert polynomials of the underlying vector bundles.

A moduli problem consists of a collection, an equivalence relation, and a concept of family. A moduli space is a variety which can be identified with the set of equivalence classes. In order to be a (coarse) moduli space a family must satisfy a certain property, the so-called local universal property.

A. H. W. Schmitt constructed the moduli spaces of quiver bundles using Geometric Invariant Theory (GIT). More precisely, he constructed the moduli spaces of coherent $\mathcal{O}_X$-modules on a smooth projective variety X [39], [40]. The construction is similar to that of the moduli spaces of vector bundles over a smooth projective algebraic curve. This construction is found in [26]. The moduli spaces of quiver bundles are in general a quasi-projective scheme, but those of holomorphic chains are projective schemes. The stability for holomorphic chains involves a real vector parameter $\boldsymbol{\alpha} \in \mathbb{R}^n$. The rational vector parameter $\boldsymbol{\alpha}$ comes from a given linearization of the GIT quotient. $\mathbb{R}^n$ is partitioned in locally closed regions called *chambers*. The $\boldsymbol{\alpha}$-stability does not change in the chambers, so the moduli spaces are the same for $\boldsymbol{\alpha}$ in the chambers.

Let a reductive group G act linearly on a variety X. "Linear action" means an action given by a linear representation of the group G. Since G acts on X, it also acts on the coordinate ring of X. The ring of invariants is a subring of the coordinate ring. A subring is in general not finitely generated. If G is reductive then the ring of invariants is always finitely generated. We can associate the ring of invariants with a variety. The Geometric Invariant Theory is a method of associating the space of orbits under G with a variety. We exclude some bad orbits which are called unstable points.

In Chapter 2, in addition to holomorphic chains, we summarize the definitions of coherent systems and co-Higgs bundles, which we will then study by associating them with holomorphic chains. The classical GIT and its relation to symplectic quotients is also summarized. These will be used in Chapter 7.

The first case to look at is the holomorphic chains for which the underlying bundles are line bundles. In Chapter 3, we will study this case. This case is simple enough to consider arbitrary genus. The underlying bundles are line bundles, so the universal line bundles can be utilized to parameterize holomorphic chains. The chamber structure for parameters $\boldsymbol{\alpha}$ is identified and the moduli spaces of holomorphic chains on a smooth algebraic curve of arbitrary genus are described In the rest of the chapters we will consider holomorphic chains on $\mathbb{P}^1$.

The second case to look at is the holomorphic 2-chains of type $\mathbf{t} = (1, 2; 0, -s)$ and their dual holomorphic chains of type $\mathbf{t}^\vee = (2, 1; s, 0)$ on $\mathbb{P}^1$. In Chapter 4, we identify the chamber structure for α and describe the moduli spaces for some cases. The variation of the moduli spaces of holomorphic triples (holomorphic 2-chains) of genus $g \geq 2$ with respect to the parameter α is studied in [7]. Some general results in [7] are also true for holomorphic chains on $\mathbb{P}^1$. In particular the following is true for chains on $\mathbb{P}^1$. The moduli space of α-stable holomorphic triples is empty unless α lies in the interval (α_m, α_M). This interval is partitioned into subintervals in which the α-stability does not change, so the moduli spaces are the same for α in the subintervals. It is relatively easy to describe the moduli spaces corresponding to the leftmost and rightmost intervals. We call the rightmost one an *extremal chamber* and the corresponding moduli space an *extremal moduli space*. We describe the extremal moduli spaces for holomorphic 2-chains of type $(2, 1; s, 0)$ on $\mathbb{P}^1$.

A coherent system on a smooth algebraic curve is a pair (E, V) of vector bundles and a vector subspace of its sections. The type of a coherent system (E, V) is the tuple of integers $(\mathrm{rk}(E), \deg(E), \dim(V))$. The stability for coherent systems involves a real parameter $\nu \in \mathbb{R}$.

A coherent system of type $(n, d, 1)$ can be described as a *holomorphic pair*. A holomorphic pair (E, ϕ) is a vector bundle together with a section $\phi \in H^0(E)$ [6, Section 1.1]. A coherent system (E, V) of type $(r, d, 1)$ maps to a holomorphic pair (E, ϕ), where $V = \mathrm{span}\{\phi\}$. Then the isomorphism class of the coherent system can be identified with the isomorphism class of the associated holomorphic pair.

S. Bradlow and G. Daskalopoulos gave a gauge-theoretic construction of the moduli spaces of τ-stable pairs for $\tau \in \mathbb{R}$, [4], [5]. This is summarized in [43, Section 1]. The parameter τ can be reformulated [6, Definition 1.2b] by setting $\sigma = \tau - \mu(E)$, where $\mu(E) = \frac{\deg(E)}{\mathrm{rk(E)}}$. The moduli spaces of σ-stable pairs is empty unless (σ_0, σ_m). This interval is partitioned into subinterval chambers. M. Thaddeus gave a GIT construction of σ-semistable pairs (E, ϕ) of $\mathrm{rk}(E) = 2$ with a fixed determinant $\det(E)$ and $\phi \neq 0$ on a curve of genus $g \geq 2$. He described by blow-ups and blow-downs the variation of the moduli spaces of σ-stable pairs [43]. In this case σ relates to ν by setting $\nu = 2\sigma$. Indeed, a holomorphic pair of rank two with a nonzero section is σ-semistable if and only if its associated coherent system is $\nu = 2\sigma$-semistable.

A. D. King and P. E. Newstead constructed the moduli spaces of coherent systems on an algebraic curve [25]. Then H. Lange and P. E. Newstead studied coherent systems on $\mathbb{P}^1$ [27], [28], [29]. The main results for the coherent systems on $\mathbb{P}^1$ are that the moduli spaces of coherent systems are smooth, irreducible and of the expected dimension. H. Lange and P. E. Newstead found a genus 0 version of Thaddeus's results [29]. Later, Stefano Pasotti and Francesco Prantil studied holomorphic triples on $\mathbb{P}^1$ by comparing them to coherent systems [36]. A holomorphic 2-chain is called a *holomorphic triple*.

In this thesis, the only case we will consider is coherent systems of type $(2, s, 1)$. The holomorphic 2-chains of type $(2, 1; s, 0)$ can be identified with the coherent systems of type $(2, s, 1)$. We describe the variation of the moduli spaces of holomorphic chains of type $(2, 1; s, 0)$ on $\mathbb{P}^1$ using the results for the corresponding coherent systems of type $(2, s, 1)$ on $\mathbb{P}^1$.

In Chapter 5, we study the holomorphic 3-chains of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ on $\mathbb{P}^1$. For 3-chains, the parameter region for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is 2-dimensional. We can define the analogues of the rightmost and leftmost chambers for holomorphic 2-chains. The analogue of the former is also called an extremal chamber, and the corresponding moduli space is called an extremal moduli space. L. Álvarez-Cónsul, O. García-Prada and A. H. W. Schmitt studied this case [3, Sections 5, 6]. They described chamber structure and the extremal moduli spaces. We identify $\boldsymbol{\alpha}$-stable chains, chamber structure and the extremal moduli spaces on $\mathbb{P}^1$.

In Chapter 6, we compare the stability for co-Higgs bundles of rank two with the α-stability for the associated holomorphic chains. A Higgs bundle on an algebraic variety is a vector bundle E with a Higgs field $\phi \in H^0(X, \mathrm{End}(E) \otimes T^\vee)$ such that $\phi \wedge \phi = 0$, where $T^\vee$ is the cotangent bundle of the tangent bundle T on X. A co-Higgs bundle is defined by replacing $T^\vee$ with T. A stable co-Higgs bundle (E, ϕ) is a stable bundle in a restricted sense for which we check bundle stability only for ϕ-invariant subbundles. The moduli space of co-Higgs bundles is constructed in a more general setting [34]. Especially on a curve, interesting co-Higgs bundles are found only on $\mathbb{P}^1$. Indeed for genus $g > 1$, stable co-Higgs bundles are just stable bundles: Higgs fields are 0 in this case. For genus $g = 1$, co-Higgs bundles coincide with Higgs bundles: the canonical line bundle is trivial.

S. Rayan studied co-Higgs bundles on $\mathbb{P}^1$ [38]. He identified an equivalent condition for stable co-Higgs bundles and described the moduli spaces of co-Higgs bundles of rank two. A co-Higgs bundle (E,ϕ) on $\mathbb{P}^1$ can be associated with the holomorphic 2-chain $E\otimes\mathcal{O}(-2)\xrightarrow{\phi}E$. The moduli space of co-Higgs bundles of rank one is simple. Thus we start with rank two. We identify α-stable chains for the associated 2-chains and their relation to stable co-Higgs bundles. We can define natural maps from the moduli spaces of stable co-Higgs bundles to the moduli spaces of associated α-semistable 2-chains for some α. Then we can study the moduli spaces of the co-Higgs bundles by these maps.

Let $C=(\mathcal{O},E;\phi)$ be a holomorphic chain of type $\mathbf{t}=(1,2,;0,-s)$ on $\mathbb{P}^1$. There are countably many underlying vector bundles E of degree $-s$. The vector bundle E is determined by its splitting type. If $E=\mathcal{O}(-d)\oplus\mathcal{O}(-e)$ with $s=d+e$ then the tuple of integers $(0,(-d,-e))$ is called the fixed splitting type of C. If $d\neq e$ then the automorphism group $\mathrm{Aut}(C)=\mathrm{Aut}(\mathcal{O})\times\mathrm{Aut}(\mathcal{O}(-d)\oplus\mathcal{O}(-e))$ is non-reductive. Indeed the moduli space $\mathcal{M}_\alpha(\mathbf{t})$ is the union of natural subspaces:

$$\mathcal{M}_\alpha(\mathbf{t})=\bigcup_{d+e=s}\mathcal{M}_\alpha(-d,-e),$$

where $\mathcal{M}_\alpha(-d,-e)$ is the subspaces of fixed splitting type $(0,(-d,-e))$. This phenomena occur since a vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles.

More generally, the moduli spaces of holomorphic chains on $\mathbb{P}^1$ is a union of natural subspaces of fixed splitting types, i.e. of fixed underlying bundles for holomorphic chains. If we fix a tuple of bundles $(E_i,0\leq i\leq n;\phi_j,1\leq j\leq n)$, then the natural parameter space of holomorphic chains is the vector space

$$\mathrm{Hom}(E_n,E_{n-1})\times\ldots\times\mathrm{Hom}(E_1,E_0).$$

The equivalence of a holomorphic chain $(E_i,0\leq i\leq n;\phi_j,1\leq j\leq n)$ of type $\mathbf{t}=(r_i,0\leq i\leq n;d_j;0\leq j\leq n)$ on $\mathbb{P}^1$ is given by the automorphism group $\mathrm{Aut}(E_0)\times\ldots\times\mathrm{Aut}(E_n)$. We write $E_i=\bigoplus_{1\leq k_i\leq r_i}\mathcal{O}(d_{k_i})$ with $d_1\geq\ldots\geq d_{r_i}$. The automorphism group is non-reductive unless all E_i's are semistable vector bundles. Each $\mathrm{Aut}(E_i)$ is an upper triangular matrix with diagonal $\mathbb{C}^*$, i.e.,

$$\mathrm{Aut}(E_i)=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1r_i}\\0&a_{22}&\cdots&a_{2r_i}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a_{r_ir_i}\end{pmatrix},$$

where $a_{k_i\ell_i} \in H^0(\mathbb{P}^1, \mathcal{O}(d_{k_i} - d_{\ell_i}))$ for all $1 \leq k_i, \ell_i \leq r_i$. If a group acting on a variety is non-reductive, then the ring of invariants is in general not finitely generated. In this case we cannot use classical GIT.

Recently, J. Drézet and G. Trautmann developed a non-reductive method to parameterize maps between two coherent sheaves [13]. B. Doran and F. Kirwan developed the non-reductive Geometric Invariant Theory, which generalizes the Classical Geometric Invariant Theory, [11], [22]. We will apply these methods to holomorphic 2-chains of a fixed splitting type $(-d, -e)$ with $d \neq e$.

Drézet and Trautmann's method: Given the two coherent sheaves E and F, the group $H = \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ acts on $W = \mathrm{Hom}(E, F)$ by conjugation: $(g, h) \cdot w = hwg^{-1}$. The group H is in general non-reductive. The non-reductive group H contains a normal subgroup U called a unipotent radical. Then H is a semidirect product of U and its reductive subgroup $R \cong H/U$. Drézet and Trautmann defined a natural stability notion which generalizes King's stability for the action of R [21]. An element of W is (semi)stable if and only if every element in the U-orbit is (semi)stable for the action of R in the sense of King. The stability involves rational vector parameters called polarizations, which are linearizations on the ample line bundle on the projective space $\mathbb{P}(W)$. They constructed a good (geometric) quotient $W//H$ (W^s/H) using King's quotient [21], and identified conditions for which there exist a good quotient and a geometric quotient. In some cases they classified the polarizations which define good and geometric quotients.

Doran and Kirwan's method: Any affine algebraic group H over $\mathbb{C}$ has a normal subgroup U called a unipotent radical, and H is the semidirect product of the unipotent radical U and the reductive subgroup isomorphic to H/U. Let H act on a projective variety X. Doran and Kirwan defined the quotient $X//U$, which is a gluing of finitely generated pieces, i.e., the variety associated with the finitely generated subrings of the ring of U-invariants. If the ring of U-invariants is finitely generated, then the quotient $X//U$ is projective. When the ring of U-invariants is not finitely generated, we embed as follows:

$$X \hookrightarrow G \times_U X \hookrightarrow \overline{G \times_U X},$$

where G is a reductive group containing U, and $\overline{G \times_U X}$ is a completion. If the GIT quotient $\overline{X//U} = \overline{G \times_U X}//G$ inherits an induced action of H/U, then we can take the quotient $\overline{X//U}//(H/U)$. Kirwan explicitly constructed a non-reductive quotient by U when $U \cong (\mathbb{C}^+)^r$, where $\mathbb{C}^+$ is the additive group $\mathbb{G}_a$ over $\mathbb{C}$ [22].

When a complex reductive group linearly acts on a smooth projective variety, there is a moment map for the action of a maximal compact subgroup. The symplectic quotient of the maximal compact subgroup is homeomorphic to the GIT quotient. Since the non-reductive quotient $\overline{X//U}//(H/U)$ is defined by the two step reductive GIT quotients in order, we can find a symplectic quotient. We can study $\overline{X//U}//(H/U)$ using classical GIT and symplectic geometry.

In chapter 7, we study subpaces of the moduli spaces of holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ in which the splitting type of the bundles is fixed. A moduli space of holomorphic chains on $\mathbb{P}^1$ breaks into a union of these subspaces. For each subspace, we apply non-reductive GIT. Drézet and Trautmann's method is applied for type $\mathbf{t} = (1, 2; 0, -s)$, and Doran and Kirwan's for its dual type $\mathbf{t}^\vee = (2, 1; s, 0)$.

Recall that the parameter region for α is an interval (α_m, α_M), which is partitioned into subinterval chambers. In each subinterval, we prove that there exists a unique splitting type such that its α-stability coincides with Drézet and Trautmann's and Doran and Kirwan's non-reductive stability. Thus, we can identify these unique subspaces with their corresponding non-reductive quotients. We relate Drézet and Trautmann's non-reductive quotients to Doran and Kirwan's by dual chains. Then this defines a bijective morphism by the properties of categorical quotient. Drézet and Trautmann's non-reductive quotients are good quotients, so they are categorical quotients. Drézet and Trautmann's and Doran and Kirwan's non-reductive stability also involves real parameters. We explain how the parameter α relates to these.

In Doran and Kirwan's method, we find symplectic descriptions for the subspaces. They are projective space bundles over projective spaces. The dimensions of the base projective spaces are determined by the splitting types.

Chapter 2

Preliminaries

2.1 Quiver bundles

A *quiver* (V, A, t, h) is an oriented graph consisting of a set of vertices V, a set of arrows A and head and tail maps $h, t : V \to A$. If the number $\sharp V$ of vertices is equal to the number $\sharp A$ of arrows plus 1 then the quiver is called a *tree*. A *representation* in the category of vector bundles over a complex algebraic variety means the assignments of a vector bundle to each vertex and a homomorphism to each arrow. This representation is called a *quiver bundle*.

In this chapter we introduce three objects: holomorphic chains, coherent systems and co-Higgs bundles. The definition of holomorphic chains and general results are found in [3]. A study for σ-stable coherent systems on $\mathbb{P}^1$ is found in [27], [28], [29]. S. Rayan recently studied stable co-Higgs bundle on $\mathbb{P}^1$ [42]. We can associate holomorphic 2-chains (holomorphic triples) with the last two objects. Holomorphic 2-chains on $\mathbb{P}^1$ were studied by associating with coherent systems [36].

Throughout this chapter, let X denote a smooth projective curve over an algebraically closed field unless otherwise stated.

2.1.1 Holomorphic chains

A holomorphic chain is a quiver bundle such that the associated quiver is a certain type of tree (V, A, h, t), where $V = \{0, 1, ..., n\}$, $A = \{a_1, a_2, ..., a_n\}$ and $h, t : V \to A$ with $h(a_i) = i, t(a_i) = i - 1$ for $i = 1, ..., n$. The corresponding diagram is

$$n \to n-1 \to ... \to 1 \to 0.$$

We associate a vector bundle E_i to each vertex $i \in V$ and a homomorphism $\phi_i \in \mathrm{Hom}(E_i, E_{i-1})$ to each arrow $a_i \in A$.

The following definitions are from [3].

Definition 2.1.1. *A holomorphic $(n+1)$-chain on X is a tuple $C = (E_0, ..., E_n; \phi_1, ..., \phi_n)$ of vector* *bundles* $E_i,\ i = 0, ..., n$ *together with homomorphisms* $\phi_j : E_j \to E_{j-1},\ j = 1, ..., n$. *The tuple* $\boldsymbol{t} = (\mathrm{rk}(E_0), ..., \mathrm{rk}(E_n); \deg(E_0), ..., \deg(E_n))$ *is called the type of* C. *We will often write a chain in the form*

$$E_n \xrightarrow{\phi_n} ... \xrightarrow{\phi_1} E_0.$$

A subchain of C *is a holomorphic chain* $C' = (F_0, F_1, ..., F_n; \phi_1|F_1, \phi_2|F_2 ..., \phi_n|F_n)$, *with* $F_i \subseteq E_i$, *a subsheaf such that* $\phi_i(F_i) \subseteq F_{i-1},\ i = 1, 2, ..., n$. *The subchains* $(0, ..., 0)$ *and* $(E_0, ..., E_n)$ *are called trivial subchains. We call* $\mathrm{r}(C) = \sum_{i=0}^n \mathrm{rk}(E_i)$ *and* $\mathrm{d}(C) = \sum_{i=0}^n \deg(E_i)$ *the total rank and total degree of* C *respectively.*

Let $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_n) \in \mathbb{R}^{n+1}$. The $\boldsymbol{\alpha}$-slope of a chain C of type $\boldsymbol{t} = (r_0, ..., r_n; d_0, ..., d_n)$ is defined by the fraction

$$\mu_{\boldsymbol{\alpha}}(C) := \frac{\sum_{i=0}^n (d_i + \alpha_i r_i)}{\sum_{i=0}^n r_i}.$$

Definition 2.1.2. *A holomorphic chain* C *is* $\boldsymbol{\alpha}$-*(semi)stable if*

$$\mu_{\boldsymbol{\alpha}}(C') < (\leq) \mu_{\boldsymbol{\alpha}}(C)$$

for any nontrivial subchain C' *of* C.

Remark 2.1.1. (i) *A subsheaf of a vector bundle over* X *is a vector bundle. The subsheaf is not a subbundle, but it is contained in a subbundle of the same rank who has the maximal degree containing the subsheaf. Thus semistability can be checked by subchains composed of subbundles.*

(ii) *Let* $\boldsymbol{\alpha} = (\alpha_0, ..., \alpha_n)$. *Set* $\boldsymbol{\beta} = (\alpha_0 + \alpha', ..., \alpha_n + \alpha')$. *For any chain of type* $\boldsymbol{t} = (r_0, r_1, ..., r_n; d_0, d_1, ..., d_n)$,

$$\mu_{\boldsymbol{\beta}}(C) = \mu_{\boldsymbol{\alpha}}(C) + \alpha'.$$

This implies that C *is* $\boldsymbol{\alpha}$-*(semi)stable if and only if* C *is* $\boldsymbol{\beta}$-*(semi)stable. So we may assume that* $\alpha_0 = 0$ *and hence* $\boldsymbol{\alpha} \in \mathbb{R}^n$ [3, Remark 2.3, (iii)].

(iii) *A chain* $C = (E_0, ..., E_n; \phi_1, ..., \phi_n)$ *has a corresponding dual chain* $C^\vee = (E_n^\vee, ..., E_0^\vee; \phi_n^\vee, ..., \phi_1^\vee,)$. *Then* C *is* $(\alpha_0, ..., \alpha_n)$-*(semi)stable if and only if* $C^\vee$ *is* $(-\alpha_n, ..., -\alpha_0)$-*(semi)stable.*

Definition 2.1.3. *A region* $R(\boldsymbol{t})$ $(R^s(\boldsymbol{t}))$ *in* $\mathbb{R}^n$ *is the collection of* $\boldsymbol{\alpha}$ *such that there exists an* $\boldsymbol{\alpha}$-*semistable* ($\boldsymbol{\alpha}$-*stable) holomorphic chain of type* $\boldsymbol{t}$. *A chamber is a locally closed subset of* $R(\boldsymbol{t})$ *where the* $\boldsymbol{\alpha}$-*stability condition is independent of* $\boldsymbol{\alpha}$.

A morphism from $(E_0, ..., E_n; \phi_1, ..., \phi_n)$ to $(E'_0, ..., E'_n; \phi'_1, ..., \phi'_n)$ is given by homomorphisms $\psi_i : E_i \to E'_i$ for $i = 0, ..., n$ such that $\phi'_j \circ \psi_j = \psi_{j-1} \circ \phi_j$ for $j = 1, ..., n$, i.e. the following diagram commutes:

$$\begin{array}{ccccccc} E_n & \xrightarrow{\phi_n} & E_{n-1} & \xrightarrow{\phi_{n-1}} & ... & \xrightarrow{\phi_1} & E_0 \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_{n-1}} & & & & \downarrow{\scriptstyle \psi_0} \\ E'_n & \xrightarrow{\phi'_n} & E'_{n-1} & \xrightarrow{\phi'_{n-1}} & ... & \xrightarrow{\phi'_1} & E'_0 \end{array}$$

The holomorphic chains $(E_0, ..., E_n; \phi_1, ..., \phi_n)$ and $(E'_0, ..., E'_n; \phi'_1, ..., \phi'_n)$ are isomorphic if each ψ_i are isomorphisms for $i = 0, ..., n$.

Let $\mathcal{C}(\mu_{\boldsymbol{\alpha}})$ denote the category of $\boldsymbol{\alpha}$-semistable chains of a fixed $\boldsymbol{\alpha}$-slope $\mu_{\boldsymbol{\alpha}}$.

Proposition 2.1.1. *The category $\mathcal{C}(\mu_{\boldsymbol{\alpha}})$ is an artinian and noetherian abelian category. Hence the Jordan-Hölder Theorem holds. The simple objects are precisely the $\boldsymbol{\alpha}$-stable chains.*

(i) (*Jordan-Hölder Theorem*) *If C is an $\boldsymbol{\alpha}$-semistable holomorphic chain of $\boldsymbol{\alpha}$-slope $\mu_{\boldsymbol{\alpha}}$, then C has a Jordan-Hölder filtration*

$$0 = C_0 \subsetneq C_1 \subsetneq ... \subsetneq C_m = C$$

such that $\mu_{\boldsymbol{\alpha}}(C_i/C_{i-1}) = \mu_{\boldsymbol{\alpha}}(C),\ i = 1, 2, ..., m$ and each quotient chain C_i/C_{i-1} is $\boldsymbol{\alpha}$-stable.

(ii) *If C is an $\boldsymbol{\alpha}$-stable holomorphic chain, then* $\mathrm{End}(C) \cong \mathbb{C}$.

Proof. (i) is a chain version of Jordan-Hölder Theorem. The following proof is found in [26]. Let C be chain with $\mu_{\boldsymbol{\alpha}}(C) = \mu_{\boldsymbol{\alpha}}$. If C is $\boldsymbol{\alpha}$-semistable, but not $\boldsymbol{\alpha}$-stable, of $\boldsymbol{\alpha}$-slope $\mu_{\boldsymbol{\alpha}}$, then C has an $\boldsymbol{\alpha}$-stable subchain C_1 of $\mu_{\boldsymbol{\alpha}}(C_1) = \mu_{\boldsymbol{\alpha}}$. If there is no such subchain then we can construct a descending sequence of holomorphic chains in $\mathcal{C}(\mu_{\boldsymbol{\alpha}})$ in which the total ranks strictly decrease. This is impossible. The quotient chain C/C_1 is in $\mathcal{C}(\mu_{\boldsymbol{\alpha}})$ and we can continue the above construction. This terminates in a finite step. We obtain an increasing sequence of holomorphic chains

$$0 = C_0 \subsetneq C_1 \subsetneq ... \subsetneq C_m = C$$

in $\mathcal{C}(\mu_{\boldsymbol{\alpha}})$ such that C_i/C_{i-1} is an $\boldsymbol{\alpha}$-stable chain in $\mathcal{C}(\mu_{\boldsymbol{\alpha}})$.

(ii) is a direct result from [3, Proposition 3.4, (ii)].

□

While the filtration of C is not unique, it has a unique direct sum of chains.

Definition 2.1.4. *The direct sum*

$$\mathrm{gr}(C) = \bigoplus_{i=1}^{m} (C_i/C_{i-1})$$

is called the graduation *of C. Two $\boldsymbol{\alpha}$-semistable chains C and C' are called S-*equivalent *if* $\mathrm{gr}(C) \cong \mathrm{gr}(C')$. *In particular, if C is an $\boldsymbol{\alpha}$-stable holomorphic chain then* $\mathrm{gr}(C) = C$.

Remark 2.1.2. *An object in an abelian category is* simple *if there are precisely two subobjects* 0 *and the object itself.*

Given a type $\boldsymbol{t}$ and a rational parameter $\boldsymbol{\alpha}$, the moduli space of S-equivalence classes of $\boldsymbol{\alpha}$-semistable holomorphic chains of type $\boldsymbol{t}$ is constructed by GIT([39], [40]). It is a projective scheme. Let $\mathcal{M}_{\boldsymbol{\alpha}}(\boldsymbol{t})$ denote the projective moduli scheme. Let $\mathcal{M}^s_{\boldsymbol{\alpha}}(\boldsymbol{t}) \subseteq \mathcal{M}_{\boldsymbol{\alpha}}(\boldsymbol{t})$ denote the open subset which is the moduli space of $\boldsymbol{\alpha}$-stable holomorphic chains of type $\boldsymbol{t}$.

2.1.2 Coherent systems

The following was first defined by J. Le Potier [35]. The definitions are also found in [8], [27].

Definition 2.1.5. *A coherent system of type (r, d, k) on X is a pair (E, V) consisting of a vector bundle E on X of rank r and degree d and a vector subspace $V \subseteq H^0(X, E)$ of dimension k. A coherent subsystem of (E, V) is a coherent system (E', V') such that E' is a subbundle of E and $V' \subseteq V \cap H^0(E')$.*

In particular, a coherent system (E, V) of type $(r, d, 1)$ can be described as a *holomorphic pair* (E, ϕ), where $V = \mathrm{span}\{\phi\}$. The isomorphism class of (E, V) can be identified with the isomorphism class of (E, ϕ).

For any $\nu \in \mathbb{R}$, the ν-slope $\mu_\nu(E, V)$ is defined by

$$\mu_\nu(E, V) := \mu(E) + \nu \frac{k}{r},$$

where $\mu(E)$ is the slope of the vector bundle E.

A morphism from (E, V) to (E', V') is given by a linear map $f : V \to V'$ and a homomorphism $\psi_i : E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes \mathcal{O} & \xrightarrow{\phi} & E \\ \downarrow{\scriptstyle f\otimes 1} & & \downarrow{\scriptstyle \psi} \\ V' \otimes \mathcal{O} & \xrightarrow{\phi'} & E'. \end{array}$$

The coherent systems (E, V) and (E', V') are isomorphic if both f and ψ are isomorphisms.

Definition 2.1.6. *A coherent system (E,V) is ν-(semi)stable if, for every (nonzero) proper coherent subsystem (E',V'),*

$$\mu_\nu(E',V') < (\leq) \mu_\nu(E,V).$$

Proposition 2.1.2. ([8, Proposition 2.2]) *The ν-semistable coherent systems of a fixed ν-slope form an artinian and noetherian abelian category. Hence the Jordan-Hölder Theorem holds. The simple objects are precisely the ν-stable systems.*

(i) (Jordan-Hölder Theorem) *If (E,V) is a σ-semistable coherent systems, then (E,V) has a Jordan-Hölder filtration*

$$0 = (E_0,V_0) \subsetneq (E_1,V_1) \subsetneq \ldots \subsetneq (E_m,V_m) = (E,V)$$

such that $\mu_\nu((E_i,V_i)/(E_{i-1},V_{i-1}) = \mu_\nu(E,V)$ for $i = 1,\ldots,m$, and $(E_i,V_i)/(E_{i-1},V_{i-1})$ is ν-stable.

(ii) *If (E,V) is a ν-stable coherent system, then* $\mathrm{End}(E,V) \cong \mathbb{C}$.

Definition 2.1.7. *The direct sum*

$$\mathrm{gr}(E,V) = \bigoplus_{i=1}^{m}((E_i,V_i)/(E_{i-1},V_{i-1}))$$

is called the graduation *of (E,V). Two ν-semistable coherent systems (E,V) and (E',V') are called S-equivalent if* $\mathrm{gr}(E,V) \cong \mathrm{gr}(E',V')$. *In particular, if (E,V) is a ν-stable coherent system then* $\mathrm{gr}(E,V) = (E,V)$.

A coherent system (E,V) is given by the evaluation map

$$\phi : V \otimes \mathcal{O}_X \to E.$$

Thus a coherent system (E,V) of type (r,d,k) is associated to a 2-chain $(E,\mathcal{O}^{\oplus k};\varphi)$ of type $(r,k;d,0)$ in which the holomorphic structure of the bundle $\mathcal{O}^{\oplus k}$ is fixed. The ν-stability for a coherent system relates to the α-stability for the associated holomorphic 2-chain in the restricted sense [6, Proposition 1.14].

Given type (r,d,k), the moduli space of the S-equivalence classes of ν-semistable coherent systems of type (r,d,k) is constructed by GIT [25]. It is a projective scheme.

2.1.3 Co-Higgs bundles

The following first arose in connection with Hitchin's notions of generalized complex structure. A generalized holomorphic bundle is a co-Higgs bundle [18]. The following definitions are also found in [38].

Definition 2.1.8. *A co-Higgs bundle is a pair (E, ϕ) consisting of a vector bundle E on an algebraic variety X and an endomorphism $\phi \in H^0(EndE \otimes T)$, twisted by the tangent bundle T on X such that $\phi \wedge \phi = 0$ in$H^0(EndE \otimes \wedge^2 T)$. The endomorphism ϕ is called a Higgs field.*

We will restrict co-Higgs bundles to those on curves in Chapter 6. In this case the condition $\phi \wedge \phi = 0$ is trivial and T is the dual $K^\vee$ of the canonical line bundle K on a curve.

A morphism from (E, ϕ) to (E', ϕ') is given by a homomorphism $\psi : E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes K^\vee \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi\otimes 1} \\ E' & \xrightarrow{\phi'} & E' \otimes K^\vee. \end{array}$$

The co-Higgs bundles (E, ϕ) and (E', ϕ') are isomorphic if ψ is an isomorphism.

Definition 2.1.9. *A co-Higgs bundle (E, ϕ) is (semi)stable if*

$$\frac{\deg(F)}{\operatorname{rk}(F)} < (\leq) \frac{\deg(E)}{\operatorname{rk}(E)},$$

for every nonzero proper subbundle $F \subset E$ that is invariant under ϕ (meaning $\phi(F) \subseteq F \otimes K^\vee$).

Proposition 2.1.3. ([38], Section 1, [32], Section 4) *The semistable co-Higgs bundles of a fixed slope form an artinian and noetherian abelian category. Hence the Jordan-Hölder Theorem holds. The simple objects are precisely the stable co-Higgs bundles.*

(i) (Jordan-Hölder Theorem) *If (E, ϕ) is a semistable co-Higgs bundle, then E has a Jordan-Hölder filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq ... \subsetneq E_m = E$$

consisting of ϕ-invariant subbundles such that $\mu(E_i/E_{i-1}) = \mu(E)$ for $i = 1, ..., m$ and each $(E_i/E_{i-1}, \widetilde{\phi|_{E_i}})$ is stable. Here $\widetilde{\phi|_{E_i}} : E_i/E_{i-1} \to (E_i/E_{i-1}) \otimes K^*$ is the induced map from the restriction map $\phi|_{E_i} : E_i \to E_i \otimes K^*$.

(ii) *If (E, ϕ) is a stable co-Higgs bundle, then* $\operatorname{End}(E, \phi) \cong \mathbb{C}$.

Definition 2.1.10. *The direct sum*

$$\operatorname{gr}(E, \phi) = \bigoplus_{i=1}^{m} (E_i/E_{i-1}, \widetilde{\phi|_{E_i}})$$

is called the graduation *of* (E, ϕ). *Two semistable co-Higgs bundles* $(E, \phi), (E', \phi')$ *are called* $S-$equivalent *if* $\mathrm{gr}(E, \phi) \cong \mathrm{gr}(E', \phi')$. *In particular, if* (E, ϕ) *is a stable co-Higgs bundle then* $\mathrm{gr}(E, \phi) = (E, \phi)$.

Let X be a smooth curve of genus g. When $g \geq 2$, there is no stable co-Higgs bundle with a nonzero Higgs field. Thus stable co-Higgs bundles are nothing more than stable vector bundles. When $g = 1$, co-Higgs bundles are Higgs bundles since $K = K^{\vee} = \mathcal{O}$. Higgs bundles are obtained by replacing T with $T^{\vee}$ from co-Higgs bundles. When $g = 0$, there are plenty of stable co-Higgs bundles with nonzero Higgs fields [42, Section 2]. Thus we are interested in co-Higgs bundles on the projective line.

A co-Higgs bundle (E, ϕ) can be associated with a holomorphic 2-chain

$$E \xrightarrow{\phi} E \otimes K^{\vee}.$$

A co-Higgs bundle is a twisted quiver bundle, where the associated quiver is $(V = \{0\}, A = \{a\}, h, t)$, where h and t are the identity map on A. The diagram is as follows:

$$0 \circlearrowleft \otimes K^{\vee} \ .$$

The moduli space of S-equivalence classes of semistable co-Higgs bundle of rank r and degree d on a smooth projective curve can be constructed using results in [34], where $K^{\vee}$ is replaced by an arbitrary line bundle L. for arbitrary line bundle L instead of . It is a quasi-projective scheme. Let $\mathcal{M}(r, d)$ denote the moduli space of co-Higgs bundles of rank r and degree d. Let $\mathcal{M}^s(r, d) \subseteq \mathcal{M}(r, d)$ denote the open subset which is the moduli space of stable co-Higgs bundles of rank r and degree d.

2.2 Classical GIT

We will use the GIT in Chapter 3 and Chapter 7. In Chapter 7, the classical GIT is generalized. The GIT is found in [32], [31], [12], [45] and [22].

Assume that X is a complex projective variety embedded in $\mathbb{P}^n$ and $L = \mathcal{O}_X(1)$ is the pullback of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. If a complex reductive group G acts on X by a representation

$$\rho : G \to \mathrm{GL}(n+1; \mathbb{C})$$

then the action of G induces an action on L. This action induces an action on the homogeneous coordinate ring of X,

$$\mathbb{C}[X] = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}).$$

Since G is reductive, the G-invariant subring $\mathbb{C}[X]^G$ is a finitely generated complex algebra. We define the GIT quotient $X/\!/G$ to be the variety $\mathrm{Proj}(\mathbb{C}[X]^G)$. The inclusion $\mathbb{C}[X]^G \subseteq \mathbb{C}[X]$ defines a rational map $q : X \dashrightarrow \mathrm{Proj}(\mathbb{C}[X]^G)$. The open subset $X^{ss} \subseteq X$ of *semistable* points is the set of points x such that there exists $f \in \mathbb{C}[X]^G$ with $f(x) \neq 0$. Then the rational map q is restricted to a surjective G-invariant map $q : X^{ss} \to X/\!/G$. A point $x \in X^{ss}$ is (properly)*stable* if

(i) there is an open neighborhood such that every G-orbit is closed in it and

(ii) $\dim(G.x) = \dim G$. (The stablizer of x is finite.)

The subset $X^s \subseteq X^{ss}$ of stable points is a G-invariant open subset and the open subset $q(X^s) \subseteq X^{ss}/\!/G$ is an orbit space X^s/G for the action of G on X^s. We summarize this in the following diagram:

$$\begin{array}{ccccc} X^s & \underset{\text{open}}{\subseteq} & X^{ss} & \underset{\text{open}}{\subseteq} & X \\ \downarrow & & \downarrow & & \\ X^s/G & \underset{\text{open}}{\subseteq} & X^{ss}/\!/G & & \end{array}$$

We characterize the subsets X^s and X^{ss} by the following properties.

Proposition 2.2.1. ([12], [11], [22]) (Hilbert-Mumford criteria)

(i) *A point x is semistable(stable) if and only if every element in the G-orbit $G.x$ is semistable(stable) for the action of a fixed maximal torus of G.*

(ii) *A point x with homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ is semistable(stable) for the action of a maximal torus of G acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^{\dim G}$ if and only if the convex hull*

$$\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0(in its interior).

We can extend the definitions of X^s and X^{ss}. Let X be a quasi-projective variety and L a line bundle on X. The linearization of an action of G with respect to L is an action of G on L such that the map $L \to X$ is G-equivariant and the induced map $g : L_x \to L_{g.x},\ x \in X, g \in G$ on each fiber is linear [32], [12].

Definition 2.2.1. *Let a reductive group G act on a quasi-projective variety X with a linearization L on X. Then a point $x \in X$ is semistable with respect to L if there exists a G-invariant section $f \in H^0(X, L^{\otimes m})^G$ for some $m \geq 0$ such that*

(i) *$f(x) \neq 0$ and*

(ii) *the open subset $X_f = \{y \in X : f(y) \neq 0\}$ is affine.*

In addition, if the action of G on X_f is closed and all stabilizers are finite, then x is stable.

Classical GIT and symplectic quotient are related as follows. We will use the symplectic quotient in Chapter 7. The relation between symplectic quotient and GIT is found in [31, Chapter 8], [22] and [45].

A GIT quotient in complex algebraic geometry can be identified with a symplectic quotient. A symplectic manifold is a smooth manifold equipped with a non-degenerate closed 2-form called a symplectic form. Suppose that a compact connected Lie group K acts smoothly on a symplectic manifold X and the action preserves the symplectic form ω. The infinitesimal action of K is a Lie algebra homomorphism from $\mathrm{Lie}(K)$ to the Lie algebra of the smooth vector fields on X: we denote the image of $a \in \mathrm{Lie}(K)$ by $a \mapsto a_x$ for $x \in X$. A moment map for the action of K on X is a smooth map

$$\mu : X \to \mathrm{Lie}(K)^*$$

such that

(i) the map is K-equivariant with respect to the given action of K on X and the coadjoint action of K on $\mathrm{Lie}(K)^*$, and

(ii) the map satisfies $d\mu(x)(\xi).a = \omega_x(\xi, a_x)$ for all $x \in X$, $\xi \in T_x X$ and $a \in \mathrm{Lie}(K)$.

The orbit space $\mu^{-1}(0)/K$ is a symplectic manifold equipped with an induced symplectic form and it is called a *symplectic quotient*.

Now let a complex reductive group G act on a complex smooth projective variety $X \subset \mathbb{P}^n$. The action gives the representation $\rho : G \to GL(n+1; \mathbb{C})$. If $K \subseteq G$ is a maximal compact subgroup, then we can choose appropriate coordinates on $\mathbb{P}^n$ so that the image $\rho(K)$ lies in the unitary subgroup $U(n+1) \subset GL(n+1; \mathbb{C})$. Then the action of K preserves the Fubini-Study form ω on $\mathbb{P}^n$ and we restrict ω to a Kähler form on X. There is a moment map $\mu : X \to \mathrm{Lie}(K)^*$ defined by the formula

$$\mu(x).a = \frac{\overline{\widehat{x}}^t \rho_*(a) \widehat{x}}{2\pi i \|\widehat{x}\|^2}$$

for all $a \in \mathrm{Lie}(K)$, where $\widehat{x} \in \mathbb{C}^{n+1} \setminus \{0\}$ is a representative for $x \in \mathbb{P}^n$, and the representation induces the linear map $\rho_* : \mathrm{Lie}(K) \to \mathrm{Lie}(U(n+1))$.

Remark 2.2.1. *We can extend μ to a map $\mu : X \to \mathrm{Lie}(G)^*$ defined by*

$$\mu(x).a = \mathrm{re}\left(\frac{\overline{\widehat{x}}^t \rho_*(a)\widehat{x}}{2\pi i \|\widehat{x}\|^2}\right)$$

for all $a \in \mathrm{Lie}(G) = \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}$.

In this case we can identify

$$\mu^{-1}(0)/K = X/\!/G \quad \text{and} \quad \mu^{-1}(0)_{\mathrm{reg}}/K = X^s/G,$$

where $\mu^{-1}(0)_{\mathrm{reg}} = \{x \in \mu^{-1}(0) | d\mu(x) : T_xX \to \mathrm{Lie}(K)^* \text{ is surjective}\}$.

Chapter 3

Holomorphic chains composed of line bundles

Holomorphic chains composed of line bundles are of type $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. In this chapter we identify the moduli space $\mathcal{M}^s_\alpha(\mathbf{t})$ of α-stable chains of type $\mathbf{t}$ as direct sum of projective space bundles when the gaps $d_{i-1} - d_i$ are sufficiently large for $i = 1, ..., n$. We also associate the change of $\boldsymbol{\alpha}$ with a variation in the linearization of the action of $(\mathbb{C}^*)^n$. Through this chapter assume that X is a smooth projective curve of genus g. As a corollary, the moduli spaces of holomorphic chains of type $\mathbf{t}$ on $\mathbb{P}^1$ is a product of projective spaces. If the underlying bundles are line bundles then they are stable, so we can find natural parameter spaces for holomorphic chains using a Poincaré line bundle for arbitrary genus g. However, in case that the underlying bundles are vector bundles, we do not have such natural parameter spaces. Indeed there are many $\boldsymbol{\alpha}$-stable holomorphic chains such that underlying bundles are not stable.

3.1 The parameter region for holomorphic chains of type $\mathbf{t} = (1, ..., 1 : d_0, ..., d_n)$

Proposition 3.1.1. *Let $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ be a chain of type $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$ over X. Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$. If C is $\boldsymbol{\alpha}$-(semi)stable then*

$$(n - i + 1)(\alpha_1 + ... + \alpha_{i-1}) - i(\alpha_i + ... + \alpha_n) < (\leq)(d_i + ... + d_n)i - (n - i + 1)(d_0 + ... + d_{i-1}) \qquad (3.1)$$

for $i = 1, ..., n$. In particular, if $i = 1$ then $-(\alpha_1 + ... + \alpha_n) < (\leq)(d_1 + ... + d_n) - nd_0$.

Proof. The subchain

$$C_i = (L_0, ..., L_{i-1}, 0, ..., 0; \phi_1, ..., \phi_{i-1}, 0, ..., 0)$$

is called the i-th standard subchain for $i = 1, ..., n$. In particular $C_1 = (L_0, 0, ..., 0; 0, ..., 0)$. If C is $\boldsymbol{\alpha}$-(semi)stable for some $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, then

$$\mu_{\boldsymbol{\alpha}}(C_i) < (\leq)\mu_{\boldsymbol{\alpha}}(C)$$

for all $i = 1, ..., n$. This implies the inequalities in 3.1. □

Let h_i be the hyperplane determining the closed half space H_i defined by the inequality 3.1, i.e., h_i is defined by the equation

$$(n-i+1)(\alpha_1 + ... + \alpha_{i-1}) - i(\alpha_i + ... + \alpha_n) = (d_i + ... + d_n)i - (n-i+1)(d_0 + ... + d_{i-1}) \tag{3.2}$$

for $i = 1, ..., n$.

Clearly

$$R^s(\mathbf{t}) \subseteq R(\mathbf{t}) \subseteq \bigcap_{i=1}^{n} H_i,$$

where $R^s(\mathbf{t})$ is defined in Definition 2.1.3.

Proposition 3.1.2. *The region $\bigcap_{i=1}^n H_i$ is an n-dimensional convex closed polytope with a vertex $v_0 := (d_0 - d_1, ..., d_0 - d_n)$ in $\mathbb{R}^n$.*

Proof. The equations (3.2) form an $n \times n$ matrix

$$\begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ n-1 & -2 & -2 & \cdots & -2 \\ n-2 & n-2 & -3 & \cdots & -3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-(n-1) & n-(n-1) & n-(n-1) & \cdots & -n \end{pmatrix}$$

with determinant $(-1)^n(n+1)^{n-1}$. The determinant is computed by elementary row operations. Moreover the point v_0 satisfies the n equations. Thus it is the unique solution of the system of the n linear equations in 3.2, i.e., $\bigcap_{i=1}^n h_i = \{v_0\}$.

The region $\bigcap_{i=1}^n H_i$ contains the half line $\{v_0 + t(0, ..., 0, 1) : t \geq 0\}$, so it is nonempty. Let

$$A_i(\boldsymbol{\alpha}) = (n+i-1)(\alpha_1 + ... + \alpha_{i-1}) - i(\alpha_i + ... + \alpha_n)$$
$$b_i = i(d_i + ... + d_n) - (n-i+1)(d_0 + ... + d_{i-1})$$

for $i = 1, ..., n$. Then $A_i(v_0) = b_i$ and $A_i(\boldsymbol{\alpha}) \leq b_i$ for any $\boldsymbol{\alpha} \in \bigcap_{i=1}^n H_i$. If $t \geq 0$ then

$$\begin{aligned} A_i(v_0 + t(0, ..., 0, 1)) &= A_i(v_0) + tA_i((0, ..., 0, 1)) \\ &= b_i + t(-i) \leq b_i. \end{aligned}$$

for $i = 1, ..., n$. Thus the half line $\{v_0 + t(0, ..., 0, 1) | t \geq 0\}$ lies in $\bigcap_{i=1}^{n} H_i$. If v and w lie in $\bigcap_{i=1}^{n} H_i$, then $A_i(v) \leq b_i$ and $A_i(w) \leq b_i$ for all $i = 1, ..., n$. The line segment $\{(1-t)v + tw | 0 \leq t \leq 1\}$ is contained in $\bigcap_{i=1}^{n} H_i$. Indeed,

$$A_i((1-t)v + tw) = (1-t)A_i(v) + tA_i(w) \leq (1-t)b_i + tb_i = b_i.$$

Hence, the region $\bigcap_{i=1}^{n} H_i$ is an n-dimensional convex closed polytope.

□

Proposition 3.1.3. *Let $\boldsymbol{\alpha} \in \mathbb{R}^n$. A chain $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ of type $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$ is $\boldsymbol{\alpha}$-stable if and only if $\phi_i \neq 0$ for all $i = 1, ..., n$ and $\boldsymbol{\alpha}$ lies in the open interior of $\bigcap_{i=1}^{n} H_i$.*

Proof. Suppose that $\phi_i \neq 0$ for all $i = 1, ..., n$ and $\boldsymbol{\alpha}$ lies in the open interior of $\bigcap_{i=1}^{n} H_i$. In this case the n standard subchains are the only subchains composed of subbundles. Say $C_1, ..., C_n$. Thus C is $\boldsymbol{\alpha}$-stable if and only if $\mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_i)$ for $i = 1, ..., n$. Since $\boldsymbol{\alpha}$ lies in the open interior of $\bigcap_{i=1}^{n} H_i$, $\mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_i)$ for $i = 1, ..., n$. Hence C is $\boldsymbol{\alpha}$-stable.

Conversely, if $\phi_j = 0$ for some j, then both C_j and $Q_j = (0, ..., 0, L_j, ..., L_n; 0, ...0, \phi_{j+1}, ..., \phi_n)$ are subchains. Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$. Then the inequalities $\mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_j)$ and $\mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(Q_j)$ imply

$$(n-j+1)(\alpha_1 + ... + \alpha_{j-1}) - j(\alpha_j + ...\alpha_n) < j(d_j + ... + d_n) - (n-j+1)(d_0 + ... + d_{j-1}) \text{ and}$$
$$(n-j+1)(\alpha_1 + ... + \alpha_{j-1}) - j(\alpha_j + ...\alpha_n) > j(d_j + ... + d_n) - (n-j+1)(d_0 + ... + d_{j-1}),$$

respectively. There is no such $\boldsymbol{\alpha}$. Hence, if C is $\boldsymbol{\alpha}$-stable, then $\phi_i \neq 0$ for all $i = 1, ..., n$. Now assume that $\phi_i \neq 0$ for all $i = 1, ..., n$. Then the chain C is $\boldsymbol{\alpha}$-stable if and only if $\mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_i)$ for all $i = 1, ..., n$. This implies that $\boldsymbol{\alpha}$ lies in the interior of $\bigcap_{i=1}^{n} H_i$. □

Remark 3.1.1. *If $R^s(\mathbf{t}) \neq \emptyset$ then it is the open interior of $\bigcap_{i=1}^{n} H_i$, and it is a unique chamber since $\bigcap_{i=1}^{n} H_i$ is convex, so connected. A parameter region $R^s(\mathbf{t})$ is sketched below in Figure 3.1.*

Corollary 3.1.1. *Let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. Then the moduli space $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ is nonempty if and only if $d_{i-1} \geq d_i$ for all $i = 1, ..., n$ and $\boldsymbol{\alpha}$ lies in the interior of $\bigcap_{i=1}^{n} H_i$.*

Proof. If $d_{i-1} \geq d_i$ for all $i = 1, ..., n$, then there is a chain $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ such that each homomorphism $\phi_i : L_i \to L_{i-1}$ is nonzero for $i = 1, ..., n$. The converse is clear. □

3.2 The moduli space of $\boldsymbol{\alpha}$-stable holomorphic chains of type $\mathbf{t} = (1, ..., 1 : d_0, ..., d_n)$

By considering the deformation theory of holomorphic chains, the dimension of $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ is given as follows. They assumed that genus $g \geq 2$ in [3]. The deformation theory of holomorphic chains is from a more general theory, so the following results are true for an arbitrary genus g.

Proposition 3.2.1. [3, Theorem 3.8] *Let $C = (E_0, ..., E_n; \phi_1, ..., \phi_n)$ be a holomorphic chain of type $\mathbf{t} = (r_0, ..., r_n; d_0, ..., d_n)$ over a curve of genus g.*

(i) *If C defines a smooth point in $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$, then*

$$\dim_C \mathcal{M}^s_\alpha(\mathbf{t}) = (g-1)\left(\sum_{i=0}^{n} r_i^2 - \sum_{i=1}^{n} r_i r_{i-1}\right) + \sum_{i=1}^{n}(r_i d_{i-1} - r_{i-1} d_i) + 1;$$

(ii) *If $\phi_i : E_i \to E_{i-1}$ is injective or generically surjective for all $i = 1, ..., n$, then the chain C defines a smooth point of $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$.*

Proposition 3.2.2. *Let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. If $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ is nonempty then it is a smooth projective variety of dimension $g + d_0 - d_n$.*

Proof. By plugging in $r_0 = ... = r_n = 1$, the dimension is $g+d_0-d_n$. If a holomorphic chain $(L_0, ..., L_n; \phi_1, ..., \phi_n)$ is $\boldsymbol{\alpha}$-stable, then $\phi_i \neq 0$ for $i = 1, ..., n$ (Proposition 3.1.3). Thus each ϕ_i is injective (or generically surjective) for $i = 1, ..., n$. □

Lemma 3.2.1. *Let $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ and $C' = (L_0, ..., L_n; \psi_1, ..., \psi_n)$ be holomorphic chains of type $(1, ..., 1; d_0, ..., d_n)$. Then C and C' are isomorphic if and only if $(\phi'_1, ..., \phi'_n) = (\lambda_1\phi_1, ..., \lambda_n\phi_n)$ for some $(\lambda_1, ..., \lambda_n) \in (\mathbb{C}^*)^n$.*

Proof. The automorphism group $\mathrm{Aut}(L_i)$ is isomorphic to $\mathbb{C}^*$ for $i = 0, ..., n$. Two holomorphic chains C and C' are isomorphic if and only if $(\phi'_1, ..., \phi'_n) = (t_0\phi_1 t_1^{-1}, ..., t_{n-1}\phi_n t_n^{-1})$ for some $(t_0, ..., t_n) \in (\mathbb{C}^*)^{n+1}$. Set $\lambda_i = t_{i-1}t_i^{-1}$ for $i = 1, ..., n$. Then $(\phi'_1, ..., \phi'_n) = (\lambda_1\phi_1, ..., \lambda_n\phi_n)$. For the converse, assume that $(\phi'_1, ..., \phi'_n) = (\lambda_1\phi_1, ..., \lambda_n\phi_n)$ for some $(\lambda_1, ..., \lambda_n) \in (\mathbb{C}^*)^n$. Then the tuple $(1, \lambda_1^{-1}, \lambda_1^{-1}\lambda_2^{-1}..., \lambda_1^{-1}...\lambda_n^{-1}) \in (\mathbb{C}^*)^{n+1}$ defines an isomorphism from C to C'. □

Now we construct a parameter space for holomorphic chains of type $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. Let $\mathcal{L}_i$ be a Poincaré line bundle over

$$\mathrm{Pic}^{d_{i-1}-d_i}(X) \times X,$$

where $\mathrm{Pic}^{d_{i-1}-d_i}(X)$ is the moduli space of line bundles of degree $d_{i-1}-d_i$ over X. Let

$$\nu_i : \mathrm{Pic}^{d_{i-1}-d_i}(X) \times X \to \mathrm{Pic}^{d_{i-1}-d_i}(X)$$

be the first projection for $i = 1, ..., n$. We can identify the line bundle $\mathcal{L}_i$ with its sheaf of regular sections. Since $\mathcal{L}_i$ is a coherent sheaf, the direct image $(\nu_i)_*(\mathcal{L}_i)$ is a coherent sheaf over $\mathrm{Pic}^{d_{i-1}-d_i}(X)$.

Remark 3.2.1. *Let $W_d^r = \{L \in \mathrm{Pic}^d(X) : \dim H^0(X, L) \geq r+1\}$ be the Brill-Noether locus. Then the direct images $(\nu_i)_*(\mathcal{L}_i)$ are vector bundles over the locus $W^r_{d_{i-1}-d_i} \setminus W^{r+1}_{d_{i-1}-d_i} \subseteq \mathrm{Pic}^{d_{i-1}-d_i}(X)$ which have exactly $r+1$ linearly independent sections if the locus is nonempty. The fiber of the vector bundle at L is $H^0(X, L)$.*

Consider the pullback sheaf

$$\mathcal{E}_i = f_i^*((\nu_i)_*(\mathcal{L}_i))$$

for $i = 1, ..., n$, where f_i is the composition map

$$f_i : \mathrm{Pic}^{d_0}(X) \times ... \times \mathrm{Pic}^{d_n}(X) \to \mathrm{Pic}^{d_{i-1}}(X) \times \mathrm{Pic}^{d_i}(X) \to \mathrm{Pic}^{d_{i-1}-d_i}(X)$$

defined by $(L_0, ..., L_n) \mapsto (L_{i-1}, L_i) \mapsto L_{i-1}L_i^{-1}$.

Remark 3.2.2. *Each $\mathcal{E}_i$ is a vector bundle over the locus*

$$f_i^{-1}(W^r_{d_{i-1}-d_i} \setminus W^{r+1}_{d_{i-1}-d_i}) \subseteq \mathrm{Pic}^{d_0}(X) \times ... \times \mathrm{Pic}^{d_n}(X)$$

with fiber $H^0(X, L_{i-1}L_i^{-1})$ at $(L_0, ..., L_n)$.

From now on let $Z = \mathrm{Pic}^{d_0}(X) \times ... \times \mathrm{Pic}^{d_n}(X)$.

Lemma 3.2.2. *Let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. If $d_i - d_{i-1} > \max\{-1, 2g-2\}$ for $i = 1, ..., n$, then the total space of the direct sum $\bigoplus_{i=1}^n \mathcal{E}_i$ of vector bundles naturally parameterizes the holomorphic chains of type* $\mathbf{t}$

Proof. By the definition of Poincaré line bundle, $\mathcal{L}_i$ is a line bundle of degree $d_{i-1} - d_i$ on each fiber of ν_i for $i = 1, ..., n$. Since $d_i - d_{i-1} > \max\{-1, 2g-2\}$, $\mathcal{E}_i$ is a vector bundle over Z with fiber

$$H^0(X, L_0L_1^{-1}) \times ... \times H^0(X, L_{n-1}L_n^{-1})$$

at $(L_0, ..., L_n)$ for $i = 1, ..., n$. This is a vector space of all chains $L_n \overset{\phi_n}{\to} L_{n-1} \overset{\phi_{n-1}}{\to} ... \overset{\phi_1}{\to} L_0$ with a fixed $(L_0, ..., L_n)$. □

Theorem 3.2.1. *Let X be a curve of genus g and let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. If $d_{i-1} - d_i > \max\{-1, 2g-2\}$ for $i = 1, ..., n$, then $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) \neq \emptyset$, and we can identify*

$$\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbf{P}(\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{E}_n),$$

where $\mathbf{P}(\mathcal{E}_i) = \mathbf{Proj}(\bigoplus_{d\geq 0} \mathrm{Sym}^d((\mathcal{E}_i)^\vee)$ is a projective space bundle over Z with fiber $\mathbb{P}^{d_{i-1}-d_i-g}$ for $i = 1, ..., n$.

Proof. By Lemma 3.2.1., the automorphism group $(\mathbb{C}^*)^n$ naturally acts on $\bigoplus_{i=1}^n \mathcal{E}_i$ by the formula

$$(\lambda_1, ..., \lambda_n).(L_0, ..., L_n; \phi_1, ..., \phi_n) = (L_0, ..., L_n; \lambda_1\phi_1, ..., \lambda_n\phi_n)$$

for $(\lambda_1, ..., \lambda_n) \in (\mathbb{C}^*)^n$ and $(L_0, ..., L_n; \phi_1, ..., \phi_n) \in \bigoplus_{i=1}^n \mathcal{E}_i$. Then two chains C and C' are isomorphic if and only if they both lie in the same $(\mathbb{C}^*)^n$-orbit. Since a chain $(L_0, ..., L_n; \phi_1, ..., \phi_n)$ is $\boldsymbol{\alpha}$-stable for some $\boldsymbol{\alpha}$ if and only if $\phi_i \neq 0$ for $i = 1, ..., n$, we can identify

$$\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbf{P}(\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{E}_n).$$

Each direct image $(\nu_i)_*\mathcal{L}_i$ is a vector bundle of rank $d_{i-1} - d_i + 1 - g$ for $i = 1, ..., n$. The pullback $\mathcal{E}_i = f_i^*((\nu_i)_*\mathcal{L}_i)$ is also a vector bundle of the same rank. Thus the fiber of $\mathbf{P}(\mathcal{E}_i)$ is $\mathbb{P}^{d_{i-1}-d_i-g}$ for $i = 1, ..., n$. □

Corollary 3.2.1. *Let $X = \mathbb{P}^1$ and let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. If $d_{i-1} - d_i \geq 0$ for $i = 1, ..., n$, then $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) \neq \emptyset$ and*

$$\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbb{P}^{d_0-d_1} \times ... \times \mathbb{P}^{d_{n-1}-d_n}.$$

Proof. Each direct image $(\nu_i)_*\mathcal{L}_i$ is the vector space $H^0(X, \mathcal{O}(d_{i-1} - d_i))$, and the pullback $\mathcal{E}_i = f_i^*((\nu_i)_*\mathcal{L}_i)$ is the same vector space. Hence the projective space bundle $\mathbf{P}(\mathcal{E}_i)$ is the projective space $\mathbb{P}^{d_{i-1}-d_i}$, and we can identify

$$\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbf{P}(\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{E}_n) = \mathbb{P}^{d_0-d_1} \times ... \times \mathbb{P}^{d_{n-1}-d_n}.$$

□

Proposition 3.2.3. *Let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. If $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ is nonempty then it generically fibers over the intersection*

$$\bigcap_{i=1}^n f_i^{-1}(W_{d_{i-1}-d_i}) \subseteq Z$$

with the generic fiber $\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_n}$, *where* $m_i = \min_{r \geq 0}\{r : W^r_{d_{i-1}-d_i} \neq W^{r+1}_{d_{i-1}-d_i}\}$ *for* $i = 1, ..., n$.

Proof. Let $m_i = \min_{r \geq 0}\{r : W^r_{d_{i-1}-d_i} \neq W^{r+1}_{d_{i-1}-d_i}\}$ for $i = 1, ..., n$. Then $W_{d_{i-1}-d_i} = W^{m_i}_{d_{i-1}-d_i}$ and the direct image $(\nu_i)_*\mathcal{L}_i$ is a vector bundle over the open subset $W^{m_i}_{d_{i-1}-d_i} \setminus W^{m_i+1}_{d_{i-1}-d_i}$ of $W_{d_{i-1}-d_i}$ for $i = 1, ..., n$. Each pullback $\mathcal{E}_i = f_i^*((\nu_i)_*\mathcal{L}_i)$ is a vector bundle over $f_i^{-1}(W^{m_i}_{d_{i-1}-d_i} \setminus W^{m_i+1}_{d_{i-1}-d_i})$, so the direct sum $\bigoplus_{i=1}^n \mathcal{E}_i$ is a vector bundle over the intersection $Z' = \bigcap_{i=1}^n f_i^{-1}(W^{m_i}_{d_{i-1}-d_i} \setminus W^{m_i+1}_{d_{i-1}-d_i})$.

By an argument similar to the proof of Theorem 3.2.1, the following fiber product parameterizes the isomorphism classes of holomorphic chains:

$$\mathbf{P}(\mathcal{E}'_1) \times_{Z'} ... \times_{Z'} \mathbf{P}(\mathcal{E}'_n),$$

where $\mathcal{E}'_i = \mathcal{E}_i|_{Z'}$ for $i = 1, ..., n$. Z' is an open subset of $\bigcap_{i=1}^n f_i^{-1}(W_{d_{i-1}-d_i})$; thus $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ generically fibers over $\bigcap_{i=1}^n f_i^{-1}(W_{d_{i-1}-d_i})$. Clearly the fiber is $\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_n}$. □

3.3 The moduli space of $\boldsymbol{\alpha}$-semistable holomorphic chains of type $\mathbf{t} = (1, ..., 1 : d_0, ..., d_n)$ with $d_0 \geq ... \geq d_n$

Let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. Recall that if $d_{i-1} - d_i \geq 0$ for $i = 1, ..., n$, then $R^s(\mathbf{t})$ is nonempty and the open interior of $\bigcap_{i=1}^n H_i$. In this case we can easily see that $R(\mathbf{t}) = \bigcap_{i=1}^n H_i$. Recall also that the open interior of $\bigcap_{i=1}^n H_i$ is the unique chamber for $\boldsymbol{\alpha}$-stable chains. There are lower dimensional chambers for $\boldsymbol{\alpha}$-semistable chains.

Proposition 3.3.1. *Let* $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. *Assume that* $d_{i-1} - d_i \geq 0$ *for* $i = 1, ..., n$. *There are an* $\binom{n}{j}$ *number of* j*-dimensional chambers, which is determined by* $n - j$ *standard hyperplanes for* $0 \leq j < n$. *Moreover, the* j*-dimensional chambers are*

$$\left(\bigcap_{i=1}^n H_i\right) \bigcap \left((h_{i_1} \cap ... \cap h_{i_{n-j}}) \setminus \bigcup_{k \notin \{i_1, ..., i_{n-j}\}} (h_{i_1} \cap ... \cap h_{i_{n-j}} \cap h_k)\right)$$

for $1 \leq i_1 < ... < i_{n-j} \leq n$. *We denote the* j*-dimensional chamber by* $\mathcal{C}_{i_1 ... i_{n-j}}$.

Proof. There are particular subchains which determine each chamber $\mathcal{C}_{i_1 ... i_{n-j}}$. Let $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ be a chain of type $\mathbf{t}$. Then C is $\boldsymbol{\alpha}$-semistable for $\boldsymbol{\alpha} \in \mathcal{C}_{i_1 ... i_{n-j}}$ if and only if $\phi_i \neq 0$ for $i \notin \{i_1, ..., i_{n-j}\}$. The subchains of the chain C are the standard subchains $C_{i_1}, ..., C_{i_{n-j}}$ together with the quotient chains $Q_{i_1} = C/C_{i_1}, ..., Q_{i_{n-j}} = C/C_{i_{n-j}}$. They precisely determine the chamber $\mathcal{C}_{i_1 ... i_{n-j}}$. □

Theorem 3.3.1. *Let X be a curve of genus g and let $\mathbf{t}=(1,...,1;d_0,...,d_n)$. If $d_{i-1}-d_i>\max\{-1,2g-2\}$ for $i=1,...,n$, then for $\boldsymbol{\alpha}\in\mathcal{C}_{i_1\dots i_{n-j}}$, we can identify*

$$\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t})=\mathbf{P}(\mathcal{E}_1)\times_Z ...\times_Z \mathbf{P}(\mathcal{E}_{i_1-1})\times_Z \mathbf{P}(\mathcal{E}_{i_1+1})\times_Z ...\times_Z \mathbf{P}(\mathcal{E}_{i_{n-j}-1})\times_Z \mathbf{P}(\mathcal{E}_{i_{n-j}+1})\times_Z ...\times_Z \mathbf{P}(\mathcal{E}_n),$$

where $\mathbf{P}(\mathcal{E}_i)$ is a projective space bundle over $\mathrm{Pic}^{d_0}(X)\times...\times\mathrm{Pic}^{d_n}(X)$ with fiber $\mathbb{P}^{d_{i-1}-d_i-g}$ for $i=1,...,n$. Moreover it has $\dim\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t})=d_0-d_n-\sum_{k=1}^{n-j}\left(d_{i_k}-d_{i_{k+1}}\right)+g$.

Proof. Assume that $\boldsymbol{\alpha}\in\mathcal{C}_{i_1\dots i_{n-j}}$. Then an $\boldsymbol{\alpha}$-semistable chain $(L_0,...,L_n;\phi_1,...,\phi_n)$ is S-equivalent to the chain $(L'_0,...,L'_n;\phi'_1,...,\phi'_n)$ such that

$$\begin{cases}\phi'_i=0 & \text{if } i\in\{i_1,...,i_{n-j}\};\\ \phi'_i=\phi_i & \text{if } i\notin\{i_1,...,i_{n-j}\}.\end{cases}$$

The space of chains $(L'_0,...,L'_n;\phi'_1,...,\phi'_n)$ such that

$$\begin{cases}\phi'_i=0 & \text{if } i\in\{i_1,...,i_{n-j}\};\\ \phi'_i=\phi_i & \text{if } i\notin\{i_1,...,i_{n-j}\}\end{cases}$$

can be parameterized by the total space of the vector bundle $\bigoplus_{i=1}^n\mathcal{E}'_i$ such that

$$\begin{cases}\mathcal{E}'_i=\mathrm{Pic}^{d_0}(X)\times...\times\mathrm{Pic}^{d_n}(X) & \text{if } i\in\{i_1,...,i_{n-j}\};\\ \mathcal{E}'_i=\mathcal{E}_i & \text{if } i\notin\{i_1,...,i_{n-j}\}.\end{cases}$$

Thus, $\mathbf{P}(\mathcal{E}'_{i_1})=...=\mathbf{P}(\mathcal{E}'_{i_{n-j}})=\emptyset$. Hence, we have the above identification. □

Corollary 3.3.1. *Let $X=\mathbb{P}^1$ and let $\mathbf{t}=(1,...,1;d_0,...,d_n)$. If $d_{i-1}-d_i\geq 0$ for $i=1,...,n$, then for $\boldsymbol{\alpha}\in\mathcal{C}_{i_1\dots i_{n-j}}$, we can identify*

$$\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t})=\mathbb{P}^{d_0-d_1}\times...\times\mathbb{P}^{d_{i_1-1}-d_{i_1}}\times\mathbb{P}^{d_{i_1+1}-d_{i_1+2}}\times...\times\mathbb{P}^{d_{i_{n-j}-1}-d_{i_{n-j}}}\times\mathbb{P}^{d_{i_{n-j}+1}-d_{i_{n-j}+2}}\times...\times\mathbb{P}^{d_{n-1}-d_n}.$$

Moreover it has $\dim\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t})=d_0-d_n-\sum_{k=1}^{n-j}\left(d_{i_k}-d_{i_{k+1}}\right)$.

Proof. The proof is similar to that of Theorem 3.1.1. We replace $\mathrm{Pic}^{d_0}(X)\times...\times\mathrm{Pic}^{d_n}(X)$ with a point pt. □

Proposition 3.3.2. *Let* $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$*. Assume that* $d_{i-1} - d_i \geq 0$ *for* $i = 1, ..., n$*. If* $\boldsymbol{\alpha} \in \mathcal{C}_{i_1 \dots i_{n-j}}$*, then* $\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t})$ *is nonempty and generically fibers over*

$$\bigcap_{i=1}^{n} f_i^{-1}(W_{d_{i-1}-d_i}) \subseteq Z$$

with a generic fiber $\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_{i_1-1}} \times \mathbb{P}^{m_{i_1+1}} \times .... \times \mathbb{P}^{m_{i_{n-j}-1}} \times \mathbb{P}^{m_{i_{n-j}+1}} \times ... \times \mathbb{P}^{m_n}$*, where* $m_{i_k} = \min_{r \geq 0}\{r : W^r_{d_{i_{k-1}}-d_{i_k}} \neq W^{r+1}_{d_{i_{k-1}}-d_{i_k}}\}$ *for* $k = 1, ..., n-j$*.*

Proof. The proof is similar to that of Theorem 3.3.1 by replacing $Z = \mathrm{Pic}^{d_0}(X) \times ... \times \mathrm{Pic}^{d_n}(X)$ with $\bigcap_{i=1}^{n} f_i^{-1}(W_{d_{i-1}-d_i})$. □

Example 3.3.1. (i)(A chamber structure) *Let* $\mathbf{t} = (1, 1, 1; d_0, d_1, d_2)$*. Assume that* $d_0 - d_1 \geq \max\{-1, 2g-2\}$ *and* $d_1 - d_2 \geq \max\{-1, 2g-2\}$*. The chambers are determined by the type* $\mathbf{t}$*. Then*

$$R(\mathbf{t}) = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 | \alpha_1 + \alpha_2 \geq 2d_0 - d_1 - d_2, \alpha_1 - 2\alpha_2 \leq -d_0 - d_1 + 2d_2\}.$$

$R^s(\mathbf{t})$ *is the open interior of* $R(\mathbf{t})$*,* $\mathcal{C}_1 = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 | \alpha_1 + \alpha_2 \geq 2d_0 - d_1 - d_2\} \setminus \{(d_0 - d_1, d_0 - d_2)\}$*,* $\mathcal{C}_2 = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 | \alpha_1 - 2\alpha_2 \leq -d_0 - d_1 + 2d_2\} \setminus \{(d_0 - d_1, d_0 - d_2)\}$ *and* $\mathcal{C}_{12} = \{(d_0 - d_1, d_0 - d_2)\}$*.*

(ii)(The moduli space for each chamber)

$$\begin{aligned}
\boldsymbol{\alpha} \in R^s(\mathbf{t}) &\longleftrightarrow \mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbf{P}(\mathcal{E}_1) \times_Z \mathbf{P}(\mathcal{E}_2), \\
\boldsymbol{\alpha} \in \mathcal{C}_1 &\longleftrightarrow \mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbf{P}(\mathcal{E}_1), \\
\boldsymbol{\alpha} \in \mathcal{C}_2 &\longleftrightarrow \mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t}) = \mathbf{P}(\mathcal{E}_2), \\
\boldsymbol{\alpha} \in \mathcal{C}_{12} &\longleftrightarrow \mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t}) = Z = \mathrm{Pic}^{d_0}(X) \times \mathrm{Pic}^{d_1}(X) \times \mathrm{Pic}^{d_2}(X).
\end{aligned}$$

The chamber structure is sketched in

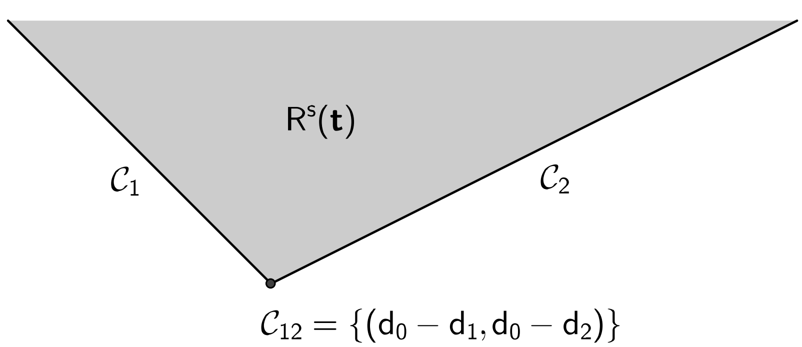


Figure 3.1: The chamber structure for chains of type $\mathbf{t} = (1, 1, 1; d_0, d_1, d_2)$.

3.4 A variation of GIT and the parameter $\boldsymbol{\alpha}$

Let $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. Assume that $d_{i-1} - d_i > \max_{1\le i\le n}\{-1, 2g-2\}$ for $i = 1, ..., n$. In the previous section we identified the moduli space $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ with a fiber product of projective space bundles. The projective space bundles in this case also can be constructed by a GIT of the action of $(\mathbb{C}^*)^n$. By Lemma 3.2.1 the group $(\mathbb{C}^*)^n$ acts on the total space of the vector bundle $\bigoplus_{i=1}^n \mathcal{E}_i$.

Lemma 3.4.1. *We can naturally embed the total space of* $\bigoplus_{i=1}^n \mathcal{E}_i$ *into the fiber product* $\mathbf{P}(\mathcal{O}\oplus\mathcal{E}_1)\times_Z ... \times_Z \mathbf{P}(\mathcal{O}\oplus\mathcal{E}_n)$. *Then the* $(\mathbb{C}^*)^n$*-action on* $\bigoplus_{i=1}^n \mathcal{E}_i$ *naturally extends to an action on* $\mathbf{P}(\mathcal{O}\oplus\mathcal{E}_1)\times_Z ... \times_Z \mathbf{P}(\mathcal{O}\oplus\mathcal{E}_n)$. *Here* $\mathbf{P}(\mathcal{O}\oplus\mathcal{E}_i) = \mathrm{Proj}(\bigoplus_{d\ge 0} Sym^d((\mathcal{O}\oplus\mathcal{E}_i)^\vee))$ *for* $i = 1, ..., n$.

Proof. The embedding $\bigoplus_{i=1}^n \mathcal{E}_i \hookrightarrow \mathbf{P}(\mathcal{E}_1\oplus\mathcal{O}) \times_Z ... \times_Z \mathbf{P}(\mathcal{E}_n\oplus\mathcal{O})$ is given by the formula

$$(L_0, ..., L_n; \phi_1, ..., \phi_n) \mapsto (L_0, ..., L_n; [1 : \phi_1], ..., [1 : \phi_n])$$

for $(L_0, ..., L_n; \phi_1, ..., \phi_n) \in \bigoplus_{i=1}^n \mathcal{E}_i$. Then the $(\mathbb{C}^*)^n$-action on $\bigoplus_{i=1}^n \mathcal{E}_i$ naturally extends to an action on $\mathbf{P}(\mathcal{O}\oplus\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O}\oplus\mathcal{E}_n)$. This action is given by the formula

$$(\lambda_1, ..., \lambda_n).(L_0, ..., L_n; [1 : \phi_1], ..., [1 : \phi_n]) = (L_0, ..., L_n; [1 : \lambda_1\phi_1], ..., [1 : \lambda_n\phi_n])$$

for $(\lambda_1, ..., \lambda_n) \in (\mathbb{C}^*)^n$. $\square$

The variety $\mathbf{P}(\mathcal{O}\oplus\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O}\oplus\mathcal{E}_n)$ fibers over $Z = \mathrm{Pic}^{d_0}(X) \times ... \times \mathrm{Pic}^{d_n}(X)$. Each fiber F of this scheme is a product of projective spaces. We can vary the $(\mathbb{C}^*)^n$-action on F by multiplying by the characters of $(\mathbb{C}^*)^n$.

Since $\mathrm{Pic}(F) \cong \mathbb{Z}^m$ for some $m \le n$ and $\mathcal{X}((\mathbb{C}^*)^n) = \mathbb{Z}^n$, the abelian group of $(\mathbb{C}^*)^n$-linearized line bundles is isomorphic to $\mathrm{Pic}^G(F) = \mathbb{Z}^m \times \mathbb{Z}^n$. Let $G = (\mathbb{C}^*)^n$. Indeed there is an exact sequence of abelian groups

$$0 \to Ker(f) \cong \mathcal{X}(G) \to \mathrm{Pic}^G(F) \xrightarrow{f} \mathrm{Pic}(F) \to \mathrm{Pic}(G) = 0,$$

where f is the forgetful map (See [12, Theorem 7.2]). Here $\mathcal{X}(G)$ is the group of characters $G \to \mathbb{C}^*$ and $\mathrm{Pic}^G(F)$ is the group of G-linearized line bundles. A G-linearized line bundle $\mathcal{O}_{\mathbb{P}^{m_1}}(i_1) \otimes ... \otimes \mathcal{O}_{\mathbb{P}^{m_n}}(i_n)$ on F defines a G-equivariant embedding, i.e., Veronese embedding followed by a Segre embedding, into a projective space, where $F = \mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_n}$. A linearization is defined by the formula

$$(\lambda_1, ..., \lambda_n).((a_1, \phi_1), .., (a_n, \phi_n)) = \lambda_1^{-q_1} ... \lambda_n^{-q_n}((a_1, \lambda_1\phi_1), ..., (a_n, \lambda_n\phi_n))$$

for some $(q_1, ..., q_n) \in \mathcal{X}((\mathbb{C}^*)^n) = \mathbb{Z}^n$. Denote the linearized line bundle by $L_{(q_1,...,q_n),(i_1,...,i_n)}$.

The stability doesn't change when we raise $L_{(q_1,...,q_n),(i_1,...,i_n)}$ to the r-th power: we replace $L_{(q_1,...,q_n),(i_1,...,i_n)}$ with $L_{(rq_1,...,rq_n),(ri_1,...,ri_n)}$. We may assume that $L = L_{((q_1,...,q_n),(1,...,1))}$ with $(q_1, ..., q_n) \in \mathbb{Q}^n$ when we choose $(i_1, ..., i_n) = (1, ..., 1)$. This is so called a fractional linearization[22, 12]. The linearization varies as $(q_1, ..., q_n)$ varies over $\mathbb{Q}^n$, and it relates with the parameter $\boldsymbol{\alpha} \in \mathbb{R}^n$ of chains.

Using the Hilbert-Mumford Criteria in Proposition 2.2.1, we find a natural isomorphism

$$\mathbf{P}(\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{E}_n) \cong (\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)) /\!/_\chi (\mathbb{C}^*)^n$$

for some $\chi \in \mathcal{X}((\mathbb{C}^*)^n)$. Indeed, let $\chi = (q_1, ..., q_n) \in \mathcal{X}((\mathbb{C}^*)^n) = \mathbb{Z}^n$. If $\chi = (q_1, ..., q_n)$ lies in the n-dimensional open cube $(0, 1)^n \subset \mathbb{R}^n$, then

$$\mathbf{P}(\mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{E}_n) \cong (\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)) /\!/_\chi (\mathbb{C}^*)^n.$$

Proposition 3.4.1. *Let* $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. *Assume that* $d_{i-1} - d_i > \max\{-1, 2g - 2\}$ *for* $i = 1, ..., n$. *A chain* $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ *is* $\boldsymbol{\alpha}$*-semistable for* $\boldsymbol{\alpha} \in \mathcal{C}_{i_1...i_{n-j}}$ *if and only if* $(L_0, ..., L_n; [1 : \phi_1], ..., [1 : \phi_n])$ *is* χ*-semistable in* $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)$ *for* $\chi \in I_1 \times ... \times I_n$, *where*

$$\begin{cases} I_i = \{0\} & \text{if } i \in \{i_1, ..., i_{n-j}\}; \\ I_i = (0, 1) & \text{if } i \notin \{i_1, ..., i_{n-j}\}. \end{cases}$$

We denote $I_1 \times ... \times I_n$ *by* $I_{i_1...i_{n-j}}$.

Proof. Since the group $(\mathbb{C}^*)^n$ acts trivially on the base of $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)$, then it is enough to consider the action on a fiber. The fiber is the product of projective spaces. If $\boldsymbol{\alpha} \in \mathcal{C}_{i_1...i_{n-j}}$, then a chain $C = (L_0, ..., L_n; \phi_1, ..., \phi_n)$ is $\boldsymbol{\alpha}$-semistable if and only if $\phi_i \neq 0$ for $i \notin \{i_1, ..., i_{n-j}\}$. This is equivalent to the condition on that its image $(L_0, ..., L_n; [1 : \phi_1], ..., [1 : \phi_n])$ in $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)$ is χ-semistable for $\chi \in I_{i_1...i_{n-j}}$. Indeed, the element $([1 : \phi_1], ..., [1 : \phi_n])$ in the fiber is χ-semistable for $\chi \in I_{i_1...i_{n-j}}$.

□

Proposition 3.4.2. *Let* $\mathbf{t} = (1, ..., 1; d_0, ..., d_n)$. *Assume that* $d_{i-1} - d_i > \max\{-1, 2g - 2\}$ *for* $i = 1, ..., n$. *If* $\boldsymbol{\alpha} \in \mathcal{C}_{i_1...i_{n-j}}$, *then we can identify*

$$\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t}) = (\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z ... \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)) /\!/_\chi (\mathbb{C}^*)^n$$

for $\chi \in I_{i_1 \dots i_{n-j}}$.

Proof. Let $\chi \in I_{i_1 \dots i_{n-j}}$. We first identify the χ-semistable points in $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z \dots \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)$. If a point $x = (L_0, \dots, L_n; [a_1 : \phi_1], \dots, [a_n : \phi_n])$ is χ-semistable then by the Hilbert-Mumford Criteria, $a_1 \neq 0$,..., $a_n \neq 0$, so we may assume that $x = (L_0, \dots, L_n; [1 : \phi_1], \dots, [1 : \phi_n])$. Thus χ-semistable points are from the image of the embedding $\bigoplus_{i=1}^n \mathcal{E}_i \hookrightarrow \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z \dots \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)$. By Lemma 3.4.1, a chain $C = (L_0, \dots, L_n; \phi_1, \dots, \phi_n)$ is $\boldsymbol{\alpha}$-semistable for $\boldsymbol{\alpha} \in \mathcal{C}_{i_1 \dots i_{n-j}}$ if and only if its image is χ-semistable for $\chi \in I_{i_1 \dots i_{n-j}}$.

For the quotient, it is enough to look at the fiber of $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z \dots \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)$. A fiber at $(L_0, \dots, L_n)$ is

$$F := \mathbb{P}\big(\mathbb{C} \oplus H^0(L_0 L_1^{-1})\big) \times \dots \times \mathbb{P}\big(\mathbb{C} \oplus H^0(L_{n-1} L_n^{-1})\big).$$

Then the group $(\mathbb{C}^*)^n$ acts on F by the formula

$$(t_1, \dots, t_n).\big([1 : \phi_1], \dots, [1 : \phi_n]\big) = \chi(t_1, \dots, t_n)\big([1 : t_1\phi_1], \dots, [1 : t_n\phi_n]\big)$$

for $(t_1, \dots, t_n) \in (\mathbb{C}^*)^n$ and $([1 : \phi_1], \dots, [1 : \phi_n]) \in F$. In the proof of Proposition 3.4.1 we find

$$F^{ss,\chi} = \{\big([1 : \phi_1], \dots, [1 : \phi_n]\big) | \phi_i \neq 0 \text{ for } i \notin \{i_1, \dots, i_{n-j}\}\}$$

with

$$\begin{aligned} F/\!/_\chi(\mathbb{C}^*)^n = {} & \mathbb{P}H^0(L_0 L_1^{-1}) \times \dots \times \mathbb{P}H^0(L_{i_1-2} L_{i_1-1}^{-1}) \times \mathbb{P}H^0(L_{i_1} L_{i_1+1}^{-1}) \\ & \times \dots \times \mathbb{P}H^0(L_{i_{n-j}-2} L_{i_{n-j}-1}^{-1}) \times \mathbb{P}H^0(L_{i_{n-j}} L_{i_{n-j}+1}^{-1}) \times \dots \times \mathbb{P}H^0(L_{n-1} L_n^{-1}). \end{aligned}$$

By Theorem 3.3.1, this is the fiber of $\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t})$ at $(L_0, \dots, L_n)$. Hence, we can identify

$$\mathcal{M}_{\boldsymbol{\alpha}}(\mathbf{t}) = \big(\mathbf{P}(\mathcal{O} \oplus \mathcal{E}_1) \times_Z \dots \times_Z \mathbf{P}(\mathcal{O} \oplus \mathcal{E}_n)\big) /\!/_\chi (\mathbb{C}^*)^n.$$

□

Chapter 4

Holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$

In this chapter we study holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ over $\mathbb{P}^1$. We describe the moduli spaces of holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ by relating them to the corresponding ones of coherent systems and holomorphic pairs.

A holomorphic chain $(\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ of type $\mathbf{t} = (1, 2; 0, -s)$ can be regarded as a holomorphic pair $(\mathcal{O}(d) \oplus \mathcal{O}(e), \phi^\vee)$, where $\phi^\vee$ is the dual map of ϕ, i.e. $\phi^\vee \in H^0(\mathcal{O}(d) \oplus \mathcal{O}(e))$. The isomorphism class of the holomorphic chain of type $\mathbf{t}$ can be identified with the isomorphism class of the associated holomorphic pair. If $\phi \neq 0$ then the holomorphic pair can be described as a coherent system $(\mathcal{O}(d) \oplus \mathcal{O}(e), V)$ of type $(2, s, 1)$, where $V = \text{span}\{\phi^\vee\}$. The isomorphism class of the holomorphic pair can be identified with the isomorphism class of the associated coherent system.

M. Thaddeus studied holomorphic pairs (E, ϕ) of $\text{rk}(E) = 2$ with a fixed determinant $\det(E)$ and $\phi \neq 0$ on a smooth projective complex curve of genus $g \geq 2$ [43]. The stability for holomorphic pairs involves a parameter $\sigma \in \mathbb{R}$. The allowed range of σ is an interval (σ_0, σ_m). The moduli spaces of σ-stable pairs are empty unless $\sigma \in (\sigma_0, \sigma_m)$. This interval is partitioned into subintervals of length 2 in which the stability does not change. One of his main results is a description of the variation of the moduli spaces with respect to the parameters $\sigma \in \mathbb{R}$. Thaddeus described the variation by blow-ups and blow-downs. The σ-stability relates to ν-stability for the associated coherent system by setting $\nu = 2\sigma$.

The determinant of a vector bundle E on $\mathbb{P}^1$ is the line bundle of degree $\deg(E)$. If we fixed a degree of E then the determinant of E is automatically fixed. Thus we can apply Thaddeus' results to holomorphic chains of type $\mathbf{t} = (2, 1; -s, 0)$.

Later, P. E. Newstead and H. Lange studied coherent systems on $\mathbb{P}^1$ [27], [28], [29]. A genus-0 version of Thaddeus's result is found in [29]. Thaddeus also found that the moduli space corresponding to the rightmost interval is a projective space. Newstead and Lange found in their study for coherent systems on $\mathbb{P}^1$ that the same result is true for the rightmost interval and the moduli space corresponding to the leftmost interval is a Grassmannian [29].

A coherent system of type $(2, s, 1)$ can be associated with a holomorphic 2-chain of type $\mathbf{t} = (2, 1; s, 0)$.

A parameter ν for coherent systems is related to a parameter α for associated 2-chains by the formula in [36, Proposition 3.1] or in [6, Proposition 1.14].

In Section 4.1, we identify α-stable holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ and the chamber structure in the space of allowed α-values.

In Section 4.2, we describe the moduli spaces of holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ for the leftmost and rightmost intervals.

In Section 4.3, we relate holomorphic chains of type $\mathbf{t} = (2, 1; s, 0)$ to the associated holomorphic pairs and the associated coherent systems. Then we describe the moduli spaces of holomorphic chains of type $\mathbf{t} = (2, 1; s, 0)$ by this relation.

4.1 Holomorphic chains of type $\mathbf{t} = (1, 2, ; 0, -s)$ with $s > 1$

Let $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ be a holomorphic chain of type $(1, 2; 0, -s)$. We write degrees in decreasing order, i.e., $-d \geq -e$. If $s \leq 1$ then C cannot be α-stable, so we assume that $s > 1$. Indeed, we have the following result.

Proposition 4.1.1. *Let $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$. If C is α-stable for some α then $d > 0$.*

Proof. If $d < 0$ then the subchain $(0, \mathcal{O}(-d); 0)$ is, at the same time, a quotient chain of C by a subchain $(\mathcal{O}, \mathcal{O}(-e); \phi|_{\mathcal{O}(-e)})$, so the α-stability implies that both $\mu_\alpha(C) > \mu_\alpha(0, \mathcal{O}(-d); 0)$ and $\mu_\alpha(C) < \mu_\alpha(0, \mathcal{O}(-d); 0)$. This is a contradiction. Hence C cannot be α-stable when $d < 0$.

Suppose that $d = 0$, i.e., $C = (\mathcal{O}, \mathcal{O} \oplus \mathcal{O}(-s); \phi)$. In this case ϕ is onto or $\phi(\mathcal{O}) = 0$. If ϕ is onto then the kernel of the map $Ker(\phi) = \mathcal{O}(-s)$, so the subchain $(\mathcal{O}, \mathcal{O}; \phi|_{\mathcal{O}})$ is, at the same time, a quotient chain of C by a subchain $(0, \mathcal{O}(-s); 0)$. For the same reason, C cannot be α-stable. If $\phi(\mathcal{O}) = 0$ then $(0, \mathcal{O})$ is a subchain, so the subchain $(\mathcal{O}, \mathcal{O}(-s); \phi|_{\mathcal{O}(-s)})$ is at the same time a quotient chain of C by $(0, \mathcal{O})$. For the same reason, C cannot be α-stable. Hence C cannot be α-stable when $d = 0$. □

Remark 4.1.1. *By Proposition 4.1.1, if a holomorphic chain of type $\mathbf{t} = (1, 2, ; 0, -s)$ is α-stable then $d > 0$, so $s = d + e \geq 2d > 1$. This is a result in [27, Remark 5.2] in a different guise.*

Theorem 4.1.1. *Let $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ be a chain of type $(1, 2; 0, -s)$ with $0 < d \leq e$. Then C is α-stable if and only if*

(i) $\mu_\alpha(C) > \mu_\alpha(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)})$ *and*

(ii) $\mu_\alpha(C) > \mu_\alpha((0, Ker(\phi); 0))$.

Proof. The proper subchains composed of line bundles are

$$(\mathcal{O}, 0; 0), (\mathcal{O}, L; \phi|_L), (0, Ker(\phi); 0),$$

where L is a line subbundle of $\mathcal{O}(-d) \oplus \mathcal{O}(-e)$. Since $\mathcal{O}(-d)$ is the unique maximal line subbundle of $\mathcal{O}(-d) \oplus \mathcal{O}(-e)$, then $\mu_\alpha(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)}) \geq \mu_\alpha(\mathcal{O}, L; \phi|_L)$, so $\mu_\alpha(C) > \mu_\alpha(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)}) \geq \mu_\alpha(\mathcal{O}, L; \phi|_L)$. Thus the α-stability can be checked for the subchain $(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)})$ together with the other two subchains $(\mathcal{O}, 0; 0)$ and $(0, Ker(\phi); 0)$.

Now we compare the α for $(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)})$ with that for $(\mathcal{O}, 0; 0)$. We compute as follows:

$$\mu_\alpha(C) > \mu_\alpha(\mathcal{O}, 0; 0) \Leftrightarrow \alpha > \frac{s}{2} \text{ and } \mu_\alpha(C) > \mu_\alpha(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)}) \Leftrightarrow \alpha > 2s - 3d.$$

Since $2s - 3d - \frac{s}{2} = \frac{3(e-d)}{2} \geq 0$, the α for $(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)})$ is bigger than that for $(\mathcal{O}, 0; 0)$. Therefore, the α-stability can be checked for the subchain $(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)})$ together with the subchain $(0, Ker(\phi); 0)$. □

Let $\phi : \mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O}$ be a homomorphism with $0 < d \leq e$. Write

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix},$$

where $\phi_1 \in H^0(\mathcal{O}(d))$ and $\phi_2 \in H^0(\mathcal{O}(e))$. Then ϕ_1 and ϕ_2 can be identified with homogeneous polynomials of two variables of degree d and e, respectively.

Proposition 4.1.2. *Let $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ be a chain of type $\mathbf{t} = (1, 2; 0, -s)$ with $0 < d \leq e$. Let $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$. If $\alpha \in (2s - 3d, -s - 3\deg(Ker(\phi)))$, then C is α-stable if and only if $\phi_1 \neq 0$ and $\phi_1 \nmid \phi_2$, which means that ϕ_1 does not divide ϕ_2.*

Proof. Let $\alpha \in (2s - 3d, -s - 3\deg(Ker(\phi)))$. Consider the exact sequence

$$0 \to Ker(\phi) \to \mathcal{O}(-d) \oplus \mathcal{O}(-e) \to Im(\phi) \to 0.$$

Suppose that $\phi_1 \neq 0$ and $\phi_1 \nmid \phi_2$. Then $-d < \deg(Im(\phi)) \leq 0$, so $-s \leq \deg(Ker(\phi)) < d - s$. By Theorem 4.1.1., C is α-stable if and only if $\mu_\alpha(C) > \mu_\alpha(\mathcal{O}, \mathcal{O}(-d); \phi|_{\mathcal{O}(-d)})$ and $\mu_\alpha(C) > \mu_\alpha((Ker(\phi), 0; 0))$ if and only if $\alpha \in (2s - 3d, -s - 3\deg(Ker(\phi)))$. Furthermore, the difference between the end points is found as follows:

$$-s - 3\deg(Ker(\phi)) - (2s - 3d) = -3s + 3d - 3\deg(Ker(\phi)) > -3s + 3d - 3(d - s) = 0.$$

Thus the interval $(2s-3d, -s-3\deg(Ker(\phi)))$ is nonempty. Hence C is α-stable.

Conversely, if $\phi_1 = 0$ then $Ker(\phi)$ contains $\mathcal{O}(-d)$, so $\deg(Ker(\phi)) \leq -d$. The right end of the interval $(2s-3d, -s-3\deg(Ker(\phi)))$ is less than its left end:

$$-s - 3\deg(Ker(\phi)) < -s + 3d = -s + 3(s-e) = 2s - 3e < 2s - 3d.$$

Thus the interval $(2s-3d, -s-3\deg(Ker(\phi)))$ is empty. This is a contradiction.

If $\phi_1 \mid \phi_2$ then $\deg(Im(\phi)) = -d$, so $\deg(Ker(\phi)) = -e$. Since $d + e = s$, the right end is the same as the left end:

$$-s - 3\deg(Ker(\phi)) = -s + 3e = -s + 3(s-d) = 2s - 3d.$$

Thus the interval $(2s-3d, -s-3\deg(Ker(\phi)))$ is empty too. This is a contradiction. Consequently, $\phi_1 \neq 0$ and $\phi_1 \nmid \phi_2$. □

Lemma 4.1.1. *Let* $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ *be a chain of type* $\mathbf{t} = (1,2;0,-s)$ *with* $0 < d \leq e$. *If* C *is* α*-stable for some* α*, then*

$$\alpha_0 < \alpha < 2s,$$

where

$$\alpha_0 = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s+3}{2} & \text{if } s \text{ is odd.} \end{cases}$$

Proof. **Case 1,** s is odd:

Since $0 < d \leq e$, the maximum of d is $d_{\max} = \frac{s-1}{2}$. Since $\deg(Im(\phi)) \leq 0$, then $\deg(Ker(\phi)) \geq -s$. If $d = d_{\max}$ and $\deg(Ker(\phi)) = -s$, then the interval $(2s-3d, -s-3\deg(Ker(\phi))) = (\frac{s+3}{2}, 2s)$.

Case 2, s is even:

Since $0 < d \leq e$, the maximum of d is $d_{\max} = \frac{s}{2}$. Since $\deg(Im(\phi)) \leq 0$, then $\deg(Ker(\phi)) \geq -s$. If $d = d_{\max}$ and $\deg(Ker(\phi)) = -s$, then the interval $(2s-3d, -s-3\deg(Ker(\phi))) = (\frac{s}{2}, 2s)$.

If $\alpha \notin (\alpha_0, 2s)$ then there is no α-stable chain of type $\mathbf{t}$.

□

Proposition 4.1.3. *Let* $\mathbf{t} = (1,2;0,-s)$. *The* 1*-dimensional chambers are the disjoint union of open intervals:*

$$(\alpha_0, \alpha_0+3) \bigsqcup (\alpha_0+3, \alpha_0+6) \bigsqcup \ldots \bigsqcup (2s-6, 2s-3) \bigsqcup (2s-3, 2s),$$

where

$$\alpha_0 = \begin{cases} \frac{s}{2} & \text{if } s \text{ is even;} \\ \frac{s+3}{2} & \text{if } s \text{ is odd.} \end{cases}$$

Proof. Let $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ be a chain of type $\mathbf{t}$ with $0 < d \leq e$. The length of the interval $(2s-3d, -s-3\deg(Ker(\phi)))$ is $-s-3\deg(Ker(\phi))-2s+3d = 3(-s-\deg(Ker(\phi))+d)$, which is a multiple of 3. By Lemma 4.1.1, the 1-dimensional chambers are the open subintervals $(\alpha_0+3k, \alpha_0+3k+3) \subseteq (\alpha_0, 2s)$.

Case 1, s is odd:

Since $0 < d \leq e$, the splitting types for the α-stable chains are

$$(-d,-e) = \left(-\frac{s-1}{2}, -\frac{s+1}{2}\right), \left(-\frac{s-3}{2}, -\frac{s+3}{2}\right), ..., \left(-\frac{s-(s-2)}{2}, -\frac{s+(s-2)}{2}\right) = (-1, -(s-1)).$$

If the chain $C = (\mathcal{O}, \mathcal{O}(-\frac{s-(2k-1)}{2}) \oplus \mathcal{O}(-\frac{s+2k-1}{2}); \phi)$ is α-stable for $1 \leq k \leq \frac{s-1}{2}$, then

$$\alpha \in (2s-3d, -s-3\deg(Ker(\phi))) = \left(\frac{s+6k-3}{2}, -s-3\deg(Ker(\phi))\right) = \left(\frac{s+3}{2}+3k-3, -s-3\deg(Ker(\phi))\right).$$

In this case $-\frac{s-(2k-1)}{2} < \deg(Im(\phi)) \leq 0$, so $-s \leq \deg(Ker(\phi)) < \frac{-s-(2k-1)}{2}$. Let $\sharp Z(\phi)$ be the number of common zeroes of ϕ_1 and ϕ_2, where $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$. Then $\sharp Z(\phi) = s + \deg(Ker(\phi))$, so $0 \leq \sharp Z(\phi) < \frac{s-(2k-1)}{2}$. The chain C is α-stable

$$\text{if and only if} \quad \alpha \in \begin{cases} (\frac{s+3}{2}+3k-3, 2s) & \text{if } \sharp Z(\phi) = 0; \\ (\frac{s+3}{2}+3k-3, 2s-3) & \text{if } \sharp Z(\phi) = 1; \\ & \vdots \\ (\frac{s+3}{2}+3k-3, \frac{s+3}{2}+3k) & \text{if } \sharp Z(\phi) = \frac{s-(2k-1)}{2} - 1. \end{cases}$$

Case 2, s is even:

Since $0 < d \leq e$, the splitting types for the α-stable chains are

$$(-d,-e) = \left(-\frac{s}{2}, -\frac{s}{2}\right), \left(-\frac{s-2}{2}, -\frac{s+2}{2}\right), ..., \left(-\frac{s-(s-2)}{2}, -\frac{s+(s-2)}{2}\right) = (-1, -(s-1)).$$

If the chain $C = (\mathcal{O}, \mathcal{O}(-\frac{s-(2k-2)}{2}) \oplus \mathcal{O}(-\frac{s+2k-2}{2}); \phi)$ is α-stable for $1 \leq k \leq \frac{s-1}{2}$, then

$$\alpha \in (2s-3d, -s-3\deg(Ker(\phi))) = \left(\frac{s}{2}+3k-3, -s-3\deg(Ker(\phi))\right).$$

In this case $-\frac{s-(2k-2)}{2} < \deg(Im(\phi)) \leq 0$, so $-s \leq \deg(Ker(\phi)) < \frac{-s-(2k-2)}{2}$. Then $\sharp Z(\phi) = s + \deg(Ker(\phi))$, so $0 \leq \sharp Z(\phi) < \frac{s-(2k-2)}{2}$. The chain C is α-stable

$$\text{if and only if} \quad \alpha \in \begin{cases} (\frac{s}{2}+3k-3, 2s) & \text{if } \sharp Z(\phi) = 0; \\ (\frac{s}{2}+3k-3, 2s-3) & \text{if } \sharp Z(\phi) = 1; \\ & \vdots \\ (\frac{s}{2}+3k-3, \frac{s}{2}+3k) & \text{if } \sharp Z(\phi) = \frac{s-(2k-2)}{2} - 1. \end{cases}$$

□

4.2 The moduli spaces of holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$ with $s > 1$

It is relatively easy to identify the moduli spaces of α-stable chains corresponding to the left and the right extreme chambers. If $s \leq 1$ then the moduli space $\mathcal{M}^s_\alpha(\mathbf{t})$ is empty. We assume that $s > 1$.

We obtain the following result from [3, Theorem 3.8].

Proposition 4.2.1. *Let $\mathbf{t} = (1, 2; 0, -s)$ with $s > 1$. If α lies in the allowed range, then the moduli space $\mathcal{M}^s_\alpha(\mathbf{t})$ is nonempty smooth of dimension $s - 2$.*

Proof. Let $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \phi)$ represent a point in $\mathcal{M}^s_\alpha(\mathbf{t})$. Since C is α-stable, ϕ is generically surjective, so by [3, Theorem 3.8, v)], C defines a smooth point. Thus $\mathcal{M}^s_\alpha(\mathbf{t})$ is smooth. The dimension is obtained from the general formula [3, Theorem 3.8, iv)]. □

The following proof can be generalized to the case $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ in Chapter 5.

Proposition 4.2.2. *Let $C = (\mathcal{O}, E; \phi)$ be a chain of type $\mathbf{t} = (1, 2; 0, -s)$. Then the right extreme chamber in the interval $R^s(\mathbf{t})$ is the subinterval $(2s-3, 2s)$. If $\alpha \in (2s-3, 2s)$ then we can identify*

$$\mathcal{M}^s_\alpha(\mathbf{t}) = \Big(\mathrm{Ext}^1((0, \mathcal{O}; 0), (\mathcal{O}, \mathcal{O}(-s); 0)) \setminus 0\Big)/(\mathbb{C}^*)^2 \cong \mathbb{P}^{s-2}.$$

Proof. If $\alpha \in (2s-3, 2s)$, then the proof of Proposition 4.1.3 shows that C is α-stable if and only if $\sharp Z(\phi) = 0$, which means that ϕ is onto. The holomorphic chain C be expressed as nontrivial extensions:

$$
\begin{array}{cccccccc}
 & & & 0 & & 0 & & . \\
 & & & \downarrow & & \downarrow & & \\
C' : & & & Ker(\phi) & \xrightarrow{0} & \mathcal{O} & & \\
 & & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle 1} & & \\
C : & Ker(\phi) & \xrightarrow{\iota} & E & \xrightarrow{\phi} & \mathcal{O} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\
C'' : & & & \mathcal{O} & \longrightarrow & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & &
\end{array}
$$

Since ϕ is onto, $Ker(\phi) \cong \mathcal{O}(-s)$. Let $C'' = (0, \mathcal{O}; 0)$ and $C' = (\mathcal{O}, \mathcal{O}(-s); 0)$. Then by [3, Proposition 3.1], the first extension of the holomorphic chains is isomorphic to the first hypercohomology of the 2-step complex $F^\bullet(C'', C')$:

$$\mathrm{Ext}^1(C'', C') \cong \mathbb{H}^1(F^\bullet(C'', C').$$

There is an exact sequence

$$0 \to \mathbb{H}^0(F^\bullet(C'', C')) \to H^0(F^0) \to H^0(F^1) \to \mathbb{H}^1(F^\bullet(C'', C')) \to H^1(F^0) \to H^1(F^1) \to \mathbb{H}^2(F^\bullet(C'', C')) \to 0.$$

The vector bundles F^0 and F^1 are the following:

$$
\begin{aligned}
F^0 &= \mathcal{H}om(0, \mathcal{O}) \oplus \mathcal{H}om(\mathcal{O}, \mathcal{O}(-s)) \cong \mathcal{O}(-s), \\
F^1 &= \mathcal{H}om(\mathcal{O}, \mathcal{O}) \cong \mathcal{O}.
\end{aligned}
$$

Thus $H^0(F^0) = H^1(F^1) = 0$ and we have

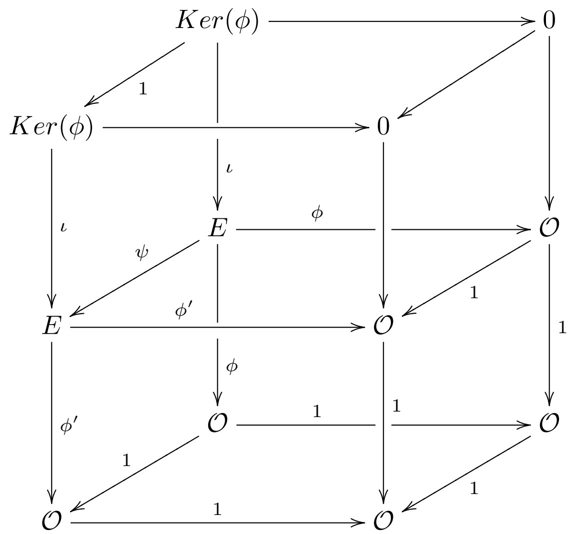
$$\mathbb{H}^1(F^\bullet(C'', C')) \cong H^0(F^1) \oplus H^1(F^0) = H^0(\mathcal{O}) \oplus H^1(\mathcal{O}(-s)) \cong H^0(\mathcal{O}) \oplus H^0(\mathcal{O}(s-2)).$$

Two such holomorphic chains are isomorphic if and only if extension classes differ by an element of $(\mathbb{C}^*)^2$.

This implies the following:

$$\mathcal{M}_\alpha^s(\mathbf{t}) = \left(\mathrm{Ext}^1((0,\mathcal{O};0),(\mathcal{O},\mathcal{O}(-s);0)) \setminus 0\right)/(\mathbb{C}^*)^2$$
$$= \mathbb{P}H^0(\mathcal{O}) \times \mathbb{P}H^0(\mathcal{O}(s-2)) \cong \{pt\} \times \mathbb{P}^{s-2} \cong \mathbb{P}^{s-2}.$$

More precisely, the isomorphisms for holomorphic chains are given by the group $\mathrm{Aut}(\mathcal{O}) \times \mathrm{Aut}(E)$. The isomorphism for extensions are given by $\{1\} \times \{\psi \in \mathrm{Aut}(E) | \psi \text{ is } Ker(\phi)\text{-invariant}\}$. Indeed, the following diagram commutes.



Thus two such holomorphic chains are isomorphic if and only if extension classes differ by an element of $\mathrm{Aut}(\mathcal{O}) \times \mathrm{Aut}(Ker(\phi)) \cong \mathbb{C}^* \times \mathbb{C}^*$.

□

The following result will be restated in following section after we associate with coherent systems.

Proposition 4.2.3. *Let C be a chain of type $\mathbf{t} = (1,2;0,-s)$ with s even. The left extreme chamber is $(\frac{s}{2}, \frac{s}{2}+3)$. If $\alpha \in (\frac{s}{2}, \frac{s}{2}+3)$ then we can identify*

$$\mathcal{M}_\alpha^s(\mathbf{t}) = Gr\left(2, \frac{s}{2}+1\right).$$

Proof. By Proposition 4.1.3, C is α-stable if and only if $C = (\mathcal{O}, \mathcal{O}(\frac{-s}{2}) \oplus \mathcal{O}(\frac{-s}{2}); \phi)$, and $0 \le \sharp Z(\phi) \le \frac{s}{2} - 1$. This is equivalent to the condition that $\phi_1 \neq 0$ and $\phi_1 \nmid \phi_2$, where $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$. Furthermore, this is equivalent to the condition that ϕ_1 and ϕ_2 are linearly independent in the vector space $H^0(\mathcal{O}(\frac{s}{2}))$. The automorphism group $GL(2;\mathbb{C})$ acts on the vector space of chains $H^0(\mathcal{O}(\frac{s}{2}))\oplus H^0(\mathcal{O}(\frac{s}{2}))$. Thus we can identify $\mathcal{M}^s_\alpha(\mathbf{t}) = Gr\left(2, \frac{s}{2}+1\right)$. □

We described the moduli space for the rightmost chamber by extension of chains in Proposition 4.2.1. Now we describe a particular subspace of this extremal moduli space. We assumed $s>1$ in the beginning of the section. If $s=2$ then the moduli space is $\mathcal{M}^s_\alpha(\mathbf{t}) = pt$ with $\mathbf{t}=(1,2;0,-s)$.

Proposition 4.2.4. *Let $C = (\mathcal{O}, \mathcal{O}(-1)\oplus\mathcal{O}(-s+1);\phi)$ a holomorphic chain of type $\mathbf{t}=(1,2;0,-s)$ with $s>2$. If α lies in the rightmost chamber $(2s-3,2s)$, then $\sharp Z(\phi)=0$, and we can identify*

$$\mathcal{M}^s_\alpha(-1,-s+1) = \mathbb{P}^1,$$

where $\mathcal{M}^s_\alpha(-1,-s+1)$ is the subspace of $\mathcal{M}^s_\alpha(\mathbf{t})$ consisting of the isomorphism classes of holomorphic chains of fixed splitting type $(\mathcal{O}, \mathcal{O}(-1)\oplus\mathcal{O}(-s+1))$.

Proof. If $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$ is α-stable for $\alpha\in(2s-3,2s)$ then by Proposition 4.1.3 that $\sharp Z(\phi)=0$. This is equivalent to saying that $\phi_1\neq 0$ and $\phi_1 \nmid \phi_2$. Define a map $\mathcal{M}^s_\alpha(-1,-s+1)\to\mathbb{P}^1$ by $[\phi]\mapsto[\phi_1]$. Since the two chains ϕ and ϕ' are isomorphic if and only if $\phi'_1=\lambda\phi_1$ and $\phi'_2=\phi_1\psi+\mu\phi_2$ for some $\lambda,\mu\in\mathbb{C}^*$ and ψ a homogeneous polynomial, the map is well defined. Indeed if we let $\psi=\frac{\phi'_2-\mu\phi_2}{\phi_1}$, where $\mu=\frac{\phi'_2(z)}{\phi_2(z)}$ with $\phi_1(z)=0$, then $\phi'_2=\phi_1\psi+\mu\phi_2$. The map is bijective. It has an inverse map $\mathbb{P}^1\to\mathcal{M}^s_\alpha(-1,-s+1)$, which is given by $[a:b]\mapsto[\psi_1,\psi_2]$ for any ψ_2, where $\psi_1=ax+by$.

The inverse morphism is well-defined. Indeed, $[\psi_1,\psi_2]\sim[\lambda\psi_1,\psi'_2]$ for any $\lambda\in\mathbb{C}^*$ and ψ'_2 with $\psi_1\nmid\psi'_2$, since $\begin{pmatrix} \lambda\psi_1 & \psi'_2 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}\begin{pmatrix} \lambda & \psi' \\ 0 & \mu \end{pmatrix}$, where $\psi'=\frac{\psi'_2-\mu\psi_2}{\psi_1}$ with $\mu=\frac{\psi'_2(z')}{\psi_2(z')}$ and z' is the zero of ψ_1. □

Corollary 4.2.1. *Let $\mathbf{t}=(1,2;0,-3)$. If $\alpha\in(3,6)$ then we can identify $\mathcal{M}^s_\alpha(\mathbf{t})=\mathbb{P}^1$.*

Proof. It follows from the fact that $\mathcal{M}^s_\alpha(\mathbf{t}) = \mathcal{M}^s_\alpha(-1,-2)$. □

4.3 Holomorphic chains of type $\mathbf{t}=(2,1;s,0)$ and coherent systems of type $(2,s,1)$ with $s>1$

If $s\leq 1$ then there are no α-stable chains and ν-stable coherent systems. A holomorphic chains of type $(2,1;s,0)$ is the dual chain of the holomorphic chain of type $(1,2;0,-s)$. Let C be a chain of type $(2,1;s,0)$ and let $C^\vee$ be the dual chain of C.

Lemma 4.3.1. *Let C be a holomorphic 2-chain. Then C is α-stable if and only if $C^\vee$ is α-stable. Moreover their chamber structure is the same.*

Proof. By Remark 2.1.1 we have the following. C is $(0,\alpha)$-stable if and only if $C^\vee$ is $(-\alpha,0)$-stable. This equivalent to the condition that $C^\vee$ is $(0,\alpha)$-stable. We identify $(0,\alpha)$ with α. The parameter α remains the same. □

Proposition 4.3.1. *Let $(E,\mathcal{O};\phi)$ be a holomorphic chain of type $\mathbf{t}=(2,1;s,0)$. If $\phi\neq 0$ then the holomorphic chain can be identified with a coherent system of type $(2,s,1)$.*

Proof. The corresponding coherent system is (E,V), where $V=\mathrm{span}\{\phi\}$. The isomorphism is given by $\mathrm{Aut}(E)$. □

The following proposition is from the more general results of [6, Proposition 1.14](or [36, Proposition 3.1]). But in the above case, α-stable holomorphic chains coincide with the associated σ-stable coherent systems after modifying the parameters.

Proposition 4.3.2. *Let $(\mathcal{O}(d)\oplus\mathcal{O}(e),\mathcal{O};\phi)$ be a chain of type $(2,1;s,0)$ with $\phi\neq 0$. Then $(\mathcal{O}(d)\oplus\mathcal{O}(e),V)$ is the associated coherent system of type $(2,s,1)$, where $V=\mathrm{span}\ \{\phi\}$. Let α and ν be such that*

$$\alpha=\frac{s}{2}+\frac{3\nu}{2},\quad \nu=-\frac{s}{3}+\frac{2\alpha}{3}.$$

Then the following are equivalent:

(i) *the holomorphic chain $(\mathcal{O}(d)\oplus\mathcal{O}(e),\mathcal{O};\phi)$ is α-stable.*

(ii) *the coherent system $(\mathcal{O}(d)\oplus\mathcal{O}(e),\phi)$ is ν-stable.*

We write for brevity

$$\mathcal{M}_k:=\mathcal{M}_\alpha(2,1;s,0)$$

for any α in the range $2s-3k-3<\alpha<2s-3k$, where $0\leq k<\lfloor\frac{s}{2}\rfloor$.

For the last moduli space , the results are from [29, Proposition 6.1].

Proposition 4.3.3. (a): *If s is even then we can identify $\mathcal{M}_{\frac{s}{2}-1}=Gr\left(2,\frac{s}{2}+1\right)$.* (b): *If s is odd then we can identify $\mathcal{M}_{\frac{s-3}{2}}$ with a $\mathbb{P}^{\frac{s-3}{2}}$-bundle over $\mathbb{P}^{\frac{s-1}{2}}$.*

Proof. For (a) by Proposition 4.3.1, let $\frac{s}{2}<\alpha<\frac{s}{2}+3$ and $0<\nu<2$. Then we can identify $\mathcal{M}_{\frac{s}{2}-1}=\mathcal{M}_\alpha(2,1;s,0)$ with the moduli space of ν-stable coherent systems of type $(2,s,1)$. By [29, Proposition

6.1], the moduli space of coherent systems of type $(2, s, 1)$ is isomorphic to $Gr\left(2, \frac{s}{2}+1\right)$. Hence $\mathcal{M}_{\frac{s}{2}-1} = Gr\left(2, \frac{s}{2}+1\right)$.

For (b) by Proposition 4.3.1, let $\frac{s+3}{2} < \alpha < \frac{s+3}{2}+3$ and $1 < \nu < 3$. Then we can identify $\mathcal{M}_{\frac{s-3}{2}} = \mathcal{M}_\alpha(2,1;s,0)$ with the moduli space of coherent systems of type $(2,s,1)$. By [29, Proposition 6.1], the moduli space of coherent systems of type $(2,s,1)$ is a $\mathbb{P}^{\frac{s-3}{2}}$-bundle over $\mathbb{P}^{\frac{s-1}{2}}$. Hence $\mathcal{M}_{\frac{s-3}{2}}$ is a $\mathbb{P}^{\frac{s-3}{2}}$-bundle over $\mathbb{P}^{\frac{s-1}{2}}$. □

For the first moduli space, the results are from [29, Proposition 6.2] or in [43, Section 3].

Proposition 4.3.4. *We can identify* $\mathcal{M}_0 = \mathbb{P}^{s-2}$.

Proof. By Proposition 4.3.1, let $2s-3 < \alpha < 2s$ and $s-2 < \nu < s$. Then we can identify $\mathcal{M}_0 = \mathcal{M}_\alpha(2,1;s,0)$ with the moduli space of ν-stable coherent systems of type $(2,s,1)$. By [29, Proposition 6.1], the moduli space of ν-stable coherent systems of type $(2,s,1)$ is isomorphic to $\mathbb{P}^{s-2}$. Hence $\mathcal{M}_0 = \mathbb{P}^{s-2}$. □

The following result is found in [43].

Theorem 4.3.1. *There exist t nonempty moduli spaces* $\mathcal{M}_0, ..., \mathcal{M}_{t-1}$, *where* $t = \lfloor \frac{s}{2} \rfloor$. *Let* $s \geq 4$. *Then the family of moduli spaces are birationally equivalent and it can be described by blowups and blowdowns, i.e.,* $\mathcal{M}_i$ *is the blow-down of the blow-up* $\widetilde{\mathcal{M}_i}$ *of* $\mathcal{M}_{i-1}$ *for* $i = 1, ..., t-1$. *In sum, we have the following diagram:*

$$\begin{array}{ccccccccccc} & \widetilde{\mathcal{M}_1} & & \widetilde{\mathcal{M}_2} & & \widetilde{\mathcal{M}_3} & & \cdots & & \widetilde{\mathcal{M}_{t-1}} & \\ \swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & & & \cdots & & & \mathcal{M}_{t-1} \end{array}.$$

Proof. If $s = 3$ then $t = 0$, i.e., there is only one moduli space $\mathcal{M}_0$. Assume that $s \geq 4$. By Proposition 4.3.1 we can identify the family $\{\mathcal{M}_i | i = 0, ..., t-1\}$ with the associated family of the moduli spaces of ν-stable coherent systems. The family of the moduli spaces of ν-stable coherent systems can be described by blow-ups and blow-downs [43]. Hence, the family $\{\mathcal{M}_i | i = 0, ..., t-1\}$ has the same description. □

Chapter 5

Holomorphic chains of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$

5.1 Holomorphic chains of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$

In this section we classify $\boldsymbol{\alpha}$-stable chains of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ and the chamber structure for the stability parameter $\boldsymbol{\alpha}$.

The parameter region for a holomorphic chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ is estimated in [3, Proposition 5.3]. All subchains can be computed for a given holomorphic chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ on $\mathbb{P}^1$. We use the fact that vector bundles on $\mathbb{P}^1$ split into line bundles and line bundles are determined by their degrees. Thus the parameter region can be precisely determined in terms of degrees.

Lemma 5.1.1. *Let $C = (E_0, E_1, E_2; \phi_1, \phi_2) = (\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(\ell) \oplus \mathcal{O}(m); \phi_1, \phi_2)$ be a holomorphic chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ with $i \geq j$ and $\ell \geq m$. If $j \leq d_1$ or $d_1 \leq \ell$, then C cannot be $\boldsymbol{\alpha}$-stable for any $\boldsymbol{\alpha} \in \mathbb{R}^2$.*

Proof. If $j < d_1$ or $d_1 < \ell$ then the subchain $(\mathcal{O}(j), 0, 0; 0, 0)$ is a quotient chain of C by a subchain $(\mathcal{O}(i), E_1, E_2; \phi_1, \phi_2)$ or the subchain $(0, 0, \mathcal{O}(\ell); 0, 0)$ is a quotient chain of C by a subchain $(E_0, E_1, \mathcal{O}(m); \phi_1, \phi_2)$, respectively. Thus C cannot be $\boldsymbol{\alpha}$-stable.

Suppose that $j = d_1$ or $d_1 = \ell$. If $j = d_1$ then $\phi_1(\mathcal{O}(d_1)) \subseteq \mathcal{O}(i)$ or $\phi_1(\mathcal{O}(d_1)) \cong \mathcal{O}(j)$. So the subchain $(\mathcal{O}(j), 0, 0; 0, 0)$ is a quotient chain of C by a subchain $(\mathcal{O}(i), E_1, E_2; \phi_1, \phi_2)$ or the subchain $(\mathcal{O}(i), 0, 0; 0, 0)$ is a quotient chain of C by a subchain $(\mathcal{O}(j), E_1, E_2; \phi_1, \phi_2)$, respectively. If $d_1 = \ell$ then $\phi_2(\mathcal{O}(\ell)) = 0$ or $\phi_2(\mathcal{O}(\ell)) = \mathcal{O}(d_1)$. So the subchain $(0, 0, \mathcal{O}(\ell); 0, 0)$ is a quotient chain of C by a subchain $(E_0, E_1, \mathcal{O}(m); \phi_1, \phi_2)$ or the subchain $(0, 0, \mathcal{O}(m); 0, 0)$ is a quotient chain of C by a subchain $(E_0, E_1, \mathcal{O}(\ell); \phi_1, \phi_2)$, respectively. Thus C cannot be $\boldsymbol{\alpha}$-stable.

Consequently, C cannot be $\boldsymbol{\alpha}$-stable.

□

Proposition 5.1.1. *Let $C = (E_0, E_1, E_2; \phi_1, \phi_2) = (\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(\ell) \oplus \mathcal{O}(m); \phi_1, \phi_2)$ be a holomorphic chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ with $i \geq j$ and $\ell \geq m$. Suppose that $0 \leq \sharp Z(\phi_1) < j - d_1$ and*

$0 \leq \sharp Z(\phi_2) < d_1 - \ell$. *Then* C *is* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$*-stable if and only if*

$$\begin{aligned}
&\text{(I)} \quad \alpha_1 + 2\alpha_2 < 4d_0 - d_1 - d_2 - 5i_1,\\
&\text{(II)} \quad \alpha_1 + 2\alpha_2 > -d_0 - d_1 - d_2 + 5i,\\
&\text{(III)} \quad \alpha_1 - 3\alpha_2 < -d_0 - d_1 + 4d_2 - 5\ell,\\
&\text{(IV)} \quad \alpha_1 - 3\alpha_2 > -d_0 - d_1 - d_2 + 5k_2,
\end{aligned}$$

where $k_2 = \deg(Ker(\phi_2))$ *and* $i_1 = \deg(\overline{Im(\phi_1)})$. *Here* $\overline{Im(\phi_1)}$ *is the subbundle of* E_0 *containing the subsheaf* $Im(\phi_1)$. *Moreover* $d_1 \leq i_1 < j$ *and* $d_2 - d_1 \leq k_2 < m$.

Proof. Suppose that $0 \leq \sharp Z(\phi_1) < j - d_1$ and $0 \leq \sharp Z(\phi_2) < d_1 - \ell$. Then $\phi_1 \neq 0$ and $\phi_2 \neq 0$. All rank types of nontrivial subchains are

$$\begin{aligned}
(r_0, r_1, r_2) = {} & (0,0,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1),\\
& (1,1,2), (2,0,0), (2,0,1), (2,1,0), (2,1,1).
\end{aligned}$$

We label the following subchains as the labeling of the lines corresponding these in [3, FIGURE 6]. The lines determined by $(II)'$ and $(III)'$ are parallel to those determined by (II) and (III), respectively. The inequalities determined by (II) and (III) imply those determined by $(II)'$ and $(III)'$, respectively. The $\boldsymbol{\alpha}$-stability is in fact determined by the four subchains (I), (II), (III) and (IV). Those subchains without labels does not affect the $\boldsymbol{\alpha}$-stability of the chain. The subchains composed of subbundles for each rank type are

(IV):$(0, 0, Ker(\phi_2); 0, 0)$, (II):$(\mathcal{O}(i), 0, 0; 0, 0)$, $(\mathcal{O}(i), 0, Ker(\phi_2); 0, 0)$, $(\overline{Im(\phi_1)}, E_1, 0; \phi_1, 0)$, $(\overline{Im(\phi_1)}, E_1, \mathcal{O}(\ell); \phi_1, \phi_2|_{\mathcal{O}(\ell)})$, (I):$(\overline{Im(\phi_1)}, E_1, E_2; \phi_1, \phi_2)$, $\text{(II)}'$: $(E_0, 0, 0; 0, 0)$, $(E_0, 0, Ker(\phi_2); 0, 0)$, $\text{(III)}'$: $(E_0, E_1, 0; \phi_1, 0)$, (III):$(E_0, E_1, \mathcal{O}(\ell); \phi_1, \phi_2|_{\mathcal{O}(\ell)})$,

respectively.

Let $L_{(II)}$ and $L_{(II)'}$ be the line which determines the half-spaces defined by (II) and $(II)'$, respectively. Then $L_{(II)}$ and $L_{(II)'}$ have the same slope, and the y-intercept of $L_{(II)}$ is bigger than or equal to that of $L_{(II)'}$. Both half-spaces open upward. Thus the half-space for $L_{(II)}$ contains that for $L'_{(II)}$.

Let $L_{(III)}$ and $L_{(III)'}$ be the line which determines the half-spaces defined by (III) and $(III)'$, respectively. Then $L_{(III)}$ and $L_{(III)'}$ have the same slope, and the y-intercept of $L_{(III)}$ is bigger than or equal to that of $L_{(III)'}$. Both half-spaces open upward. Thus the half-space for $L_{(III)}$ contains that for $L'_{(III)}$.

Let $L_{(I)}$ and $L_{(IV)}$ be the line which determines the half-spaces defined by (I) and (IV), respectively. Then the half-spaces for $L_{(I)}$ and $L_{(IV)}$ open downward. The four lines $L_{(I)}$, $L_{(II)}$, $L_{(III)}$ and $L_{(IV)}$ determine a parallelogram which is the intersection of those four half-spaces.

Finally, the half-spaces defined by the rest of the four subchains $(\mathcal{O}(i), 0, Ker(\phi_2); 0, 0)$, $(\overline{Im(\phi_1)}, E_1, 0; \phi_1, 0)$, $(\overline{Im(\phi_1)}, E_1, \mathcal{O}(\ell); \phi_1, (\phi_2)|_{\mathcal{O}(\ell)})$ and $(E_0, 0, Ker(\phi_2); 0, 0)$ contain the parallelogram.

Thus, C is $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$-stable if and only if

$$\text{(I)} \quad \mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_1 = (\overline{Im(\phi_1)}, E_1, E_2; \phi_1, \phi_2)),$$

$$\text{(II)} \quad \mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_2 = (\mathcal{O}(i), 0, 0; 0, 0)),$$

$$\text{(III)} \quad \mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_3 = (E_0, E_1, \mathcal{O}(\ell); \phi_1, \phi_2|_{\mathcal{O}(\ell)})),$$

$$\text{(IV)} \quad \mu_{\boldsymbol{\alpha}}(C) > \mu_{\boldsymbol{\alpha}}(C_4 = (0, 0, Ker(\phi_2); 0, 0)),$$

where $\overline{Im(\phi_1)}$ is the subbundle of E_0 containing the subsheaf $Im(\phi_1)$.

Now we show that if $0 \leq \sharp Z(\phi_1) < j - d_1$ and $0 \leq \sharp Z(\phi_2) < d_1 - \ell$, then the region bounded by the parallelogram is nonempty.

The parallelogram is between $L_{(I)}$ and $L_{(II)}$, i.e., the region determined by

$$-d_0 - d_1 - d_2 + 5i < \alpha_1 + 2\alpha_2 < 4d_0 - d_1 - d_2 - 5i_1.$$

The degree of the line subbundle containing $Im(\phi_1)$ is $i_1 = \deg(\overline{Im(\phi_1)}) = d_1 + \sharp Z(\phi_1)$, so

$$4d_0 - d_1 - d_2 - 5i_1 - (-d_0 - d_1 - d_2 + 5i) = 5(j - i_1) = 5(j - d_1 - \sharp Z(\phi_1)).$$

Since $0 \leq \sharp Z(\phi_1) < j - d_1$ by assumption, the difference is positive. Thus, $L_{(I)}$ and $L_{(II)}$ define a nonempty region between them.

The parallelogram is between $L_{(III)}$ and $L_{(IV)}$, i.e., the region determined by

$$-d_0 - d_1 - d_2 + 5k_2 < \alpha_1 - 3\alpha_2 < -d_0 - d_1 + 4d_2 - 5\ell.$$

We have $\deg(Im(\phi_2)) = d_1 - \sharp Z(\phi_2)$. Then $k_2 = \deg(Ker(\phi_2)) = d_2 - \deg(Im(\phi_2)) = d_2 - d_1 + \sharp Z(\phi_2)$. Thus

$$-d_0 - d_1 + 4d_2 - 5\ell - (-d_0 - d_1 - d_2 + 5k_2) = 5(m - k_2) = 5(d_1 - \ell - \sharp Z(\phi_2)).$$

Since $0 \leq \sharp Z(\phi_2) < d_1 - \ell$ by assumption, the difference is positive. Thus, $L_{(III)}$ and $L_{(IV)}$ define a nonempty region between them.

Consequently, the chain C is $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$-stable if and only if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ lies in the parallelogram determined by the four lines $L_{(I)}$, $L_{(II)}$, $L_{(III)}$ and $L_{(IV)}$.

□

Remark 5.1.1. *Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ be a holomorphic chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$. Then the parallelogram depends on the splitting type of the vector bundles E_0 and E_2 and the number of common zeroes $\sharp Z(\phi_1)$ and $\sharp Z(\phi_2)$.*

Proposition 5.1.2. *Let $C = (E_0, E_1, E_2; \phi_1, \phi_2) = (\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(\ell) \oplus \mathcal{O}(m); \phi_1, \phi_2)$ be a holomorphic chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ with $i \geq j$ and $\ell \geq m$. Assume that $j > d_1 > \ell$. Then the chain C is $\boldsymbol{\alpha}$-stable for some $\boldsymbol{\alpha}$ if and only if $0 \leq \sharp Z(\phi_1) < j - d_1$ and $0 \leq \sharp Z(\phi_2) < d_1 - \ell$.*

Proof. By Lemma 5.1.1, if $j \leq d_1$ or $d_1 \leq \ell$ then C cannot be $\boldsymbol{\alpha}$-stabe, so we assume that $j > d_1 > \ell$.

If $0 \leq \sharp Z(\phi_1) < j - d_1$ and $0 \leq \sharp Z(\phi_2) < d_1 - \ell$, then the parallelogram defined in Proposition 5.1.1 is nonempty. Thus there is an $\boldsymbol{\alpha}$ such that C is $\boldsymbol{\alpha}$-stable. For the converse, suppose that C is $\boldsymbol{\alpha}$-stable for some $\boldsymbol{\alpha}$. The numbers of common zeroes $\sharp Z(\phi_1)$ and $\sharp Z(\phi_2)$ lie in the following range:

$$0 \leq \sharp Z(\phi_1) \leq j - d_1, \quad 0 \leq \sharp Z(\phi_2) \leq d_1 - \ell$$

or $\sharp Z(\phi_1) = i - d_1$ or $\sharp Z(\phi_2) = d_1 - m$.

If $\sharp Z(\phi_1) = i - d_1$, then $\deg(\overline{Im(\phi_1)}) = d_1 + \sharp Z(\phi_1) = d_1 + (i - d_1) = i$, so $\overline{Im(\phi_1)} \cong \mathcal{O}(i)$. Thus the subchain $(\overline{Im(\phi_1)}, E_1, E_2; \phi_1, \phi_2)$ is the quotient chain of C by the subchain $(L, 0, 0; 0, 0)$ with $L \cong \mathcal{O}(j)$. The chain C cannot be $\boldsymbol{\alpha}$-stable.

If $\sharp Z(\phi_2) = d_1 - m$, then $\deg(Ker(\phi_2)) = d_2 - \deg(Im(\phi_2)) = d_2 - (d_1 - \sharp Z(\phi_1)) = d_2 - d_1 + (d_1 - m) = d_2 - m = (\ell + m) - m = \ell$, so $Ker(\phi_2) \cong \mathcal{O}(m)$. Thus the subchain $(0, 0, Ker(\phi_2); 0, 0)$ is $(E_0, E_1, M; \phi_1, \phi_2|_M)$ with $M \cong \mathcal{O}(m)$. The chain C cannot be $\boldsymbol{\alpha}$-stable.

The cases $\sharp Z(\phi_1) = j - d_1$, $\sharp Z(\phi_2) = d_1 - \ell$ can be excluded by Lemma 5.1.1.

Thus if C is $\boldsymbol{\alpha}$-stable for some $\boldsymbol{\alpha}$, then $0 \leq \sharp Z(\phi_1) < j - d_1$ and $0 \leq \sharp Z(\phi_2) < d_1 - \ell$.

□

Definition 5.1.1. *We define a region $R^s((i, j), d_1, (\ell, m)) \subseteq \mathbb{R}^2$ to be the region in which there is an $\boldsymbol{\alpha}$-stable chain of the fixed splitting type $(\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(\ell) \oplus \mathcal{O}(m))$.*

Proposition 5.1.3. *Let $((i,j), d_1, (\ell, m))$ be a fixed splitting type for the holomorphic chains of type* $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$.

(i) $R^s((i,j), d_1, (l, m)) \neq \emptyset$ *if and only if* $j > d_1 > \ell$.

(ii) *We have the following inclusions:*

$$R^s((i,j), d_1, (\ell, m)) \supseteq R^s((i{+}1, j{-}1), d_1, (\ell, m)) \quad \textit{and} \quad R^s((i,j), d_1, (\ell, m)) \supseteq R^s((i,j), d_1, (\ell{+}1, m{-}1)).$$

(iii) *The region* $R^s((i,j), d_1, (l, m))$ *defined by* (I-IV) *is partitioned into chambers by the inequalities* (II) *and* (III), *and*

$$\text{(I)}_p \quad \alpha_1 + 2\alpha_2 < 4d_0 - 6d_1 - d_2 - 5p,$$

$$\text{(IV)}_q \quad \alpha_1 - 3\alpha_2 > -d_0 - 6d_1 + 4d_2 + 5q,$$

where $0 \leq p < j - d_1$ *and* $0 \leq q < d_1 - \ell$.

Proof. (i): it is clear by Proposition 5.1.2.

(ii): The splitting types involve the inequalities (II) and (III). The numbers i_1 and k_2 involve inequalities (I) and (IV). The region determined by (II) and (III) for (i,j) contains the region determined by (II) and (III) for $(i+1, j-1)$. The lower bounds for i_1 and k_2 are fixed by the type $\mathbf{t}$ but the upper bound for i_1 decreases as (i,j) moves to $(i+1, j-1)$. Similar for k_2. Thus, the region determined by (I) and (IV) for (ℓ, m) contains the region determined by (I) and (IV) for $(\ell+1, m-1)$.

(iii): (II) and (III) are determined by the fixed splitting type $((i,j), d_1, (\ell, m))$. But (I) and (IV) change as $i_1 = \deg(\overline{Im(\phi_1)})$ and $k_2 = \deg(Ker(\phi_2))$ change, respectively. Let $p = \sharp Z(\phi_1)$ and $q = \sharp Z(\phi_2)$. Then $i_1 = d_1 + p$ and $k_2 = d_2 - d_1 + q$. Thus by plugging in $i_1 = d_1 + p$ and $k_2 = d_2 - d_1 + q$, we get (I)_p and (IV)_q.

□

Finally, we identify the parameter region $R^s(\mathbf{t})$ which is defined in Section 2.1.1.

Theorem 5.1.1. *Let* $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$. *Then*

(i) $R^s(\mathbf{t}) \neq \emptyset$ *if and only if* $\lfloor \frac{d_0}{2} \rfloor > d_1 > \lceil \frac{d_2}{2} \rceil$, *where* $\lfloor x \rfloor$ *is the largest integer* $\leq x$ *and* $\lceil x \rceil$ *is the smallest integer* $\geq x$.

(ii) $R^s(\mathbf{t}) = R^s((\lceil \frac{d_0}{2} \rceil, \lfloor \frac{d_0}{2} \rfloor), d_1, (\lceil \frac{d_2}{2} \rceil, \lfloor \frac{d_2}{2} \rfloor))$.

Proof. (i): (Only if) let $C = (\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(\ell) \oplus \mathcal{O}(m); \phi_1, \phi_2)$ be an $\boldsymbol{\alpha}$-stable chain for some $\boldsymbol{\alpha} \in R^s(\mathbf{t})$ with $i \geq j$ and $\ell \geq m$. By Proposition 5.3.1 (i), $j > d_1 > \ell$. Since $\frac{d_0}{2}$ is the average of i and j, $\frac{d_0}{2} \geq j > d_1$, so $\frac{d_0}{2} > d_1$. Similarly, $d_1 > \ell = d_2 - m \geq d_2 - \frac{d_2}{2} = \frac{d_2}{2}$, so $d_1 > \frac{d_2}{2}$.

(If) assume that $\lfloor \frac{d_0}{2} \rfloor > d_1 > \lceil \frac{d_2}{2} \rceil$. We find a splitting type $((i,j), d_1, (\ell, m))$ such that $R^s((i,j), d_1, (\ell, m)) \neq \emptyset$. Set $(i,j) = (\lceil \frac{d_0}{2} \rceil, \lfloor \frac{d_0}{2} \rfloor)$ and $(\ell, m) = (\lceil \frac{d_2}{2} \rceil, \lfloor \frac{d_2}{2} \rfloor)$. Then $j = \lfloor \frac{d_0}{2} \rfloor > d_1 > \lceil \frac{d_2}{2} \rceil = \ell$, so by Proposition 5.1.3 (i), $R^s((i,j), d_1, (\ell, m)) \neq \emptyset$.

(ii): By the inclusion in Proposition 5.1.3 (ii), we have

$$R^s(\mathbf{t}) = \bigcup_{\substack{i+j=d_0 \\ \ell+m=d_2}} R^s\big((i,j), d_1, (\ell, m)\big) = R^s\Bigg(\bigg(\left\lceil \frac{d_0}{2} \right\rceil, \left\lfloor \frac{d_0}{2} \right\rfloor\bigg), d_1, (\left\lceil \frac{d_2}{2} \right\rceil, \left\lfloor \frac{d_2}{2} \right\rfloor\bigg)\Bigg).$$

It is clear that the region for $\boldsymbol{\alpha}$-stable holomorphic chains of type $\mathbf{t}$ is the union of all of the regions for $\boldsymbol{\alpha}$-stable holomorphic chains of fixed splitting types. The region $R^s((i,j), d_1, (\ell, m))$ is an empty set except for finitely many fixed splitting types $((i,j), d_1, (\ell, m))$ with $i + j = d_0$ and $\ell + m = d_2$. Thus, it is a finite union.

□

Remark 5.1.2. *The region $R^s(\mathbf{t})$ is partitioned into chambers, which are the regions bounded by smaller parallelograms. The partitioning is determined by Proposition 5.1.3* (ii).

5.2 The moduli spaces of $\boldsymbol{\alpha}$-stable holomorphic chains of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$

In this section we describe the extremal moduli spaces for the extremal chamber $\mathcal{C}_\infty$ which is an analogue of the rightmost interval for 2-chains. If $C = (\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(k) \oplus \mathcal{O}(\ell); \phi_1, \phi_2)$ is $\boldsymbol{\alpha}$-stable for some $\boldsymbol{\alpha} \in \mathcal{C}_\infty$ then the degrees i, j, k and ℓ are as far apart as possible. We also describe another extremal moduli spaces for the extremal chamber $\mathcal{C}_0$ which is an analogue of the leftmost interval for 2-chains. If the chain C is $\boldsymbol{\alpha}$-stable for some $\boldsymbol{\alpha} \in \mathcal{C}_\infty$ then the degrees i, j, k and ℓ are as close as possible.

Proposition 5.2.1. *Let $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$. If $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ is nonempty, then it is smooth of dimension $d_0 - d_2 - 4$.*

Proof. Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ represent a point in $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$. Since C is $\boldsymbol{\alpha}$-stable, ϕ_1 is injective and ϕ_2 is generically surjective, so by [3, Theorem 3.8, v)], C defines a smooth point. The dimension is obtained from the general formula [3, Theorem 3.8, iv)]. □

Definition 5.2.1. *Let* $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$*. We define an extremal chamber* $\mathcal{C}_\infty$ *to be the region*

$$R^s((d_0 - d_1 - 1, d_1 + 1), d_1, (d_1 - 1, d_2 - d_1 + 1)).$$

Proposition 5.2.2. *The extremal chamber* $\mathcal{C}_\infty$ *is the intersection*

$$\bigcap_{R^s((i,j),d_1,(k,\ell))\neq\emptyset} R^s((i,j), d_1, (k,\ell))$$

The stability region $R^s(\mathbf{t})$ *is bounded by a parallelogram. The chamber* $\mathcal{C}_\infty$ *is at the top vertex of the parallelogram. The region* $R^s(\mathbf{t})$ *is sketched in the Figure 5.1 at the end of Section 5.2.*

Proof. By 5.1.3 (ii), $\mathcal{C}_\infty = \bigcap_{R^s((i,j),d_1,(k,\ell))\neq\emptyset} R^s((i,j), d_1, (k,\ell))$ □

The extremal moduli spaces in $\mathcal{C}_\infty$ are described [3, Section 6.3]. Underlying vector bundles are described by extensions and the extremal moduli spaces are smooth connected projective variety of dimension $d_0 - d_2 - 4$. On $\mathbb{P}^1$, the moduli spaces can be described by extensions of holomorphic chains.

Theorem 5.2.1. *Let* $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ *be an* $\boldsymbol{\alpha}$ *chain of type* $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ *for* $\boldsymbol{\alpha} \in \mathcal{C}_\infty$*. Then we can identify*

$$\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) = \Big(\mathrm{Ext}^1((Coker(\phi_1), E_1, E_1; 0, 1), (E_1, 0, Ker(\phi_2); 0, 0)) \setminus 0\Big)/(\mathbb{C}^*)^3 \cong \mathbb{P}^{d_0-2d_1-2} \times \mathbb{P}^{2d_1-d_2-2}.$$

Proof. If $\alpha \in \mathcal{C}_\infty$, then by Proposition 5.1.3, C is $\boldsymbol{\alpha}$-stable if and only if $p = \sharp Z(\phi_1) = 0$ and $q = \sharp Z(\phi_2) = 0$. This means that $\overline{Im(\phi_1)} = Im(\phi_1)$ and ϕ_2 is onto. The holomorphic chain C can be expressed as nontrivial extensions:

$$\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
C': & & & Ker(\phi_2) & \xrightarrow{0} & 0 & \xrightarrow{0} & E_1 & & \\
 & & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi_1} & & \\
C: & Ker(\phi_2) & \xrightarrow{\iota} & E_2 & \xrightarrow{\phi_2} & E_1 & \xrightarrow{\phi_1} & E_0 & \xrightarrow{\pi} & Coker(\phi_1) \\
 & & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi} & & \\
C'': & & & E_1 & \xrightarrow{1} & E_1 & \xrightarrow{0} & Coker(\phi_1) & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}.$$

Since $\overline{Im(\phi_1)} = Im(\phi_1) \cong E_1$, $Coker(\phi_1) = E_0/Im(\phi_1) \cong \mathcal{O}(d_0 - d_1)$. Since ϕ_2 is onto, $Ker(\phi_2) \cong \mathcal{O}(d_2 - d_1)$.

Let $C'' = (Coker(\phi_1), E_1, E_1; 0, 1)$ and $C' = (E_1, 0, Ker(\phi_2); 0, 0)$. Then by [3, Proposition 3.1], the first extension of the holomorphic chains is isomorphic to the first hypercohomology of the 2-step complex $F^\bullet(C'', C')$:

$$\mathrm{Ext}^1(C'', C') \cong \mathbb{H}^1(F^\bullet(C'', C').$$

There is an exact sequence

$$0 \to \mathbb{H}^0(F^\bullet(C'', C')) \to H^0(F^0) \to H^0(F^1) \to \mathbb{H}^1(F^\bullet(C'', C')) \to H^1(F^0) \to H^1(F^1) \to \mathbb{H}^2(F^\bullet(C'', C')) \to 0.$$

The vector bundles F^0 and F^1 are the following:

$$F^0 = \mathcal{H}om(Coker(\phi_1), E_1) \oplus \mathcal{H}om(E_1, 0) \oplus \mathcal{H}om(E_1, Ker(\phi_2)) = \mathcal{O}(2d_1 - d_0) \oplus \mathcal{O}(d_2 - 2d_1),$$
$$F^1 = \mathcal{H}om(E_1, E_1) \oplus \mathcal{H}om(E_1, 0) = \mathcal{O}.$$

Clearly $H^1(F^1) = 0$. Since C is $\boldsymbol{\alpha}$-stable, $2d_1 - d_0 < 0$ and $d_2 - 2d_1 < 0$, so $H^0(F^0) = 0$. Thus we have

$$\mathbb{H}^1(F^\bullet(C'', C')) \cong H^0(F^1) \oplus H^1(F^0) = H^0(\mathcal{O}) \oplus H^0(\mathcal{O}(d_0 - 2d_1 - 2) \oplus \mathcal{O}(2d_1 - d_2 - 2)).$$

Two such holomorphic chains are isomorphic if and only if extension classes differ by an element of $(\mathbb{C}^*)^3$.

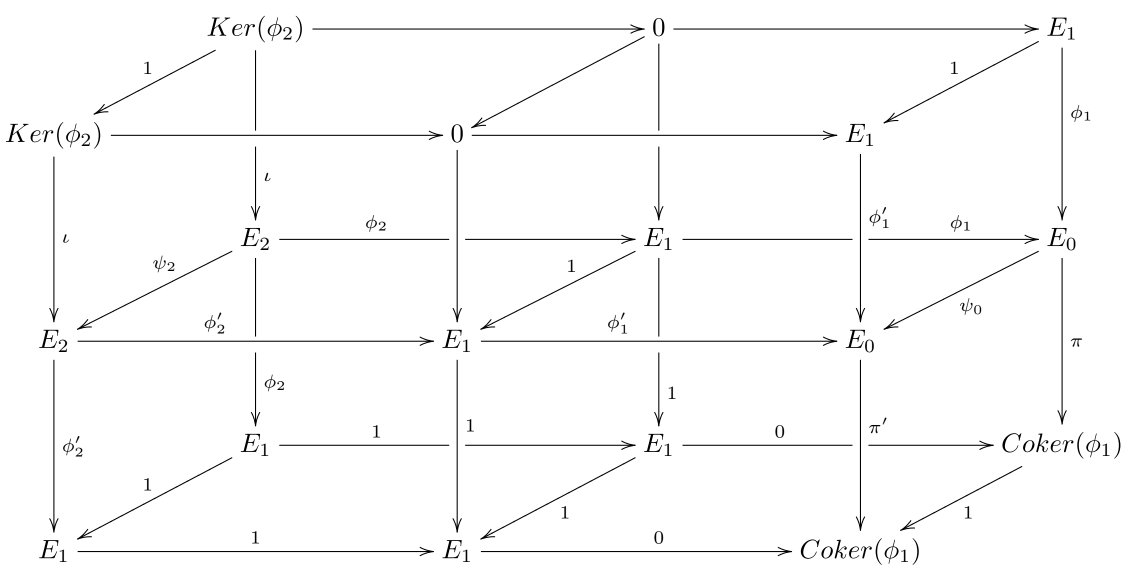
This implies that

$$\begin{aligned}\mathcal{M}^s_\alpha(\mathbf{t}) &= \left(\mathrm{Ext}^1((Coker(\phi_1), E_1, E_1; 0, 1), (E_1, 0, Ker(\phi_2); 0, 0)) \setminus 0\right)/(\mathbb{C}^*)^3 \\ &= \mathbb{P}H^0(\mathcal{O}) \oplus \mathbb{P}H^0(\mathcal{O}(d_0 - 2d_1 - 2) \oplus \mathbb{P}\mathcal{O}(2d_1 - d_2 - 2)) \\ &\cong \{pt\} \times \mathbb{P}^{d_0 - 2d_1 - 2} \times \mathbb{P}^{2d_1 - d_2 - 2} \cong \mathbb{P}^{d_0 - 2d_1 - 2} \times \mathbb{P}^{2d_1 - d_2 - 2}.\end{aligned}$$

More precisely, the isomorphisms for holomorphic chains are given by the group $\mathrm{Aut}(E_0) \times \mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)$. The isomorphism for extensions are given by the subgroup

$$\{\psi_0 \in \mathrm{Aut}(E_0) | \psi_0 \text{ is } Im(\phi_1)\text{-invariant}\} \times \{1\} \times \{\psi_2 \in \mathrm{Aut}(E_2) | \psi_2 \text{ is } Ker(\phi_2)\text{-invariant}\}.$$

Indeed, the following diagram commutes.



Thus two such holomorphic chains are isomorphic if and only if extension classes differ by an element of $\mathrm{Aut}(Im(\phi_1)) \times \mathrm{Aut}(E_1) \times \mathrm{Aut}(Ker(\phi_2)) \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$.

□

Remark 5.2.1. (*Conjecture*) *If we can find an analogous of* $\mathcal{C}_\infty$ *for* $(2n+1)$*-chains of type* $\mathbf{t} = (2, 1, 2, 1, ..., 2, 1, 2; d_0, d_1, ..., d_{2n})$, *then Theorem 5.2.1 can be generalized. The statement is as follows:*

If $\boldsymbol{\alpha} \in \mathcal{C}_\infty$ then

$$\mathcal{M}^s_\alpha(\mathbf{t}) = (Ext^1(C'', C') \setminus 0)/(\mathbb{C}^*)^{2n+1} \cong \mathbb{P}^{d_0-2d_1-2} \times \mathbb{P}^{2d_1-d_2-2} \times ... \times \mathbb{P}^{d_{2n-2}-2d_{2n-1}-2} \times \mathbb{P}^{2d_{2n-1}-d_{2n}-2},$$

where

$$C' = (E_1, 0, Ker(\phi_2), 0, Ker(\phi_4), ..., Ker(\phi_{2n-2}), 0, Ker(\phi_{2n}); 0, 0, 0, 0, ...0, 0) \quad \textit{and}$$
$$C'' = (Coker(\phi_1), E_1, E_1, E_3, E_3, ...E_{2n-3}, E_{2n-3}, E_{2n-1}, E_{2n-1}; 0, 1, \phi_2\phi_3, 1, ..., 1, \phi_{2n-2}\phi_{2n-1}, 1).$$

The proof is similar to that of Theorem 5.2.1.

Definition 5.2.2. *Let $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$. We define another extremal chamber $\mathcal{C}_0$ which is a partitioned region lying in $R^s\left(\left(\frac{d_0}{2}, \frac{d_0}{2}\right), d_1, \left(\frac{d_2}{2}, \frac{d_2}{2}\right)\right)$, which is determined by $(\mathrm{I})_p$ and $(\mathrm{IV})_q$ with $p = \frac{d_0}{2} - d_1 - 1$ and $q = d_1 - \frac{d_2}{2} - 1$, i.e. $\mathcal{C}_0$ is determined by*

$$\frac{3}{2}d_0 - d_1 - d_2 < \alpha_1 + 2\alpha_2 < \frac{3}{2}d_0 - d_1 - d_2 + 5$$
$$-d_0 - d_1 + \frac{3}{2}d_2 - 5 < \alpha_1 - 3\alpha_2 < -d_0 - d_1 + \frac{3}{2}d_2.$$

The stability region $R^s(\mathbf{t})$ is bounded by a parallelogram. The chamber $\mathcal{C}_0$ is at the bottom vertex of the parallelogram. The region $R^s(\mathbf{t})$ is sketched in the Figure 5.1 at the end of Section 5.2.

The moduli spaces in the extremal chamber $\mathcal{C}_0$ were not studied in [3]. The moduli spaces are products of two Grassmannian.

Proposition 5.2.3. *Let $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ with d_0 and d_2 even. If $\boldsymbol{\alpha} \in \mathcal{C}_0$ then we can identify*

$$\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) = Gr\left(2, H^0\left(\mathcal{O}\left(\frac{d_0}{2} - d_1\right)\right)\right) \times Gr\left(2, H^0\left(\mathcal{O}\left(d_1 - \frac{d_2}{2}\right)\right)\right).$$

Proof. If $C = (\mathcal{O}(i) \oplus \mathcal{O}(j), \mathcal{O}(d_1), \mathcal{O}(\ell) \oplus \mathcal{O}(m); \phi_1, \phi_2)$ is an $\boldsymbol{\alpha}$-stable chain of type $\mathbf{t} = (2, 1, 2; d_0, d_1, d_2)$ for $\boldsymbol{\alpha} \in \mathcal{C}_0$, then $((i, j), d_1, (\ell, m)) = ((\frac{d_0}{2}, \frac{d_0}{2}), d_1, (\frac{d_2}{2}, \frac{d_2}{2}))$. Indeed by Proposition 5.1.2, $0 \le p = \sharp Z(\phi_1) < j - d_1$ and $0 \le q = \sharp Z(\phi_2) < d_1 - \ell$. Since C is $\boldsymbol{\alpha}$-stable for $\boldsymbol{\alpha} \in \mathcal{C}_0$, $p = \frac{d_0}{2} - d_1 - 1$ and $q = d_1 - \frac{d_2}{2} - 1$. By plugging in p and q, we get $j = \frac{d_0}{2}$ and $\ell = \frac{d_2}{2}$. Thus, the splitting type is fixed by the $\boldsymbol{\alpha}$-stability. Moreover the range of the numbers of common zeroes $\sharp Z(\phi_1)$ and $\sharp Z(\phi_2)$ are as follows:

$$0 \le \sharp Z(\phi_1) < \frac{d_0}{2} - d_1 \quad \text{and} \quad 0 \le \sharp Z(\phi_2) < d_1 - \frac{d_2}{2}.$$

Two such chains are isomorphic if and only if they differ by $GL(2;\mathbb{C}) \times GL(2;\mathbb{C})$.

□

Remark 5.2.2. (*Conjecture*) *If we can find an analogous of* $\mathcal{C}_0$ *for* $(2n+1)$*-chains of type* $\mathbf{t} = (2,1,2,1,...,2,1,2;d_0,d_1,...,d_{2n})$ *, then Proposition 5.2.3 can be generalized. The statement is as follows: Let* $\mathbf{t} = (2,1,2,1,...,2,1,2;d_0,d_1,...,d_{2n})$ *be a type of* $(2n+1)$*-chains with* d_{2k} *even for* $k = 0,1,...,n$ *and let* $\mathcal{C}_0$ *be an analogue of the extremal chamber for 3-chains. If a chain* C *of type* $\mathbf{t}$ *is* $\boldsymbol{\alpha}$*-stable for* $\boldsymbol{\alpha} \in \mathcal{C}_0$*, then*

$$C = \Big(\mathcal{O}(\frac{d_0}{2}) \oplus \mathcal{O}(\frac{d_0}{2}), \mathcal{O}(d_1), \mathcal{O}(\frac{d_2}{2}) \oplus \mathcal{O}(\frac{d_2}{2}), ..., \mathcal{O}(d_{2n-1}), \mathcal{O}(\frac{d_{2n}}{2}) \oplus \mathcal{O}(\frac{d_{2n}}{2})\Big).$$

Moreover we can identify

$$\mathcal{M}^s_\alpha(\mathbf{t}) = Gr\Big(2, H^0\big(\mathcal{O}(\frac{d_0}{2} - d_1)\big)\Big) \times \frac{V_{12} \times V_{23}}{GL(2;\mathbb{C})} \times \frac{V_{34} \times V_{45}}{GL(2;\mathbb{C})} \times ... \times Gr\Big(2, H^0\big(\mathcal{O}\big(d_{2n-1} - \frac{d_{2n}}{2}\big)\big)\Big),$$

where $V_{12} = \{(\phi_1,\phi_2) \in H^0\big(\mathcal{O}\big(d_1 - \frac{d_2}{2}\big)\big)^{\oplus 2} | \phi_1$ *and* ϕ_2 *are linearly independent* $\}$ *and* $V_{23} = \{(\psi_1,\psi_2) \in H^0\big(\mathcal{O}\big(\frac{d_2}{2} - d_3\big)\big)^{\oplus 2} | \psi_1$ *and* ψ_2 *are linearly independent* $\}$*.* $V_{2t-1,2t}$ *and* $V_{2t,2t+1}$ *are similarly defined for* $t = 2,...,n-1$*.*

Remark 5.2.3. *The chamber structure is sketched in*

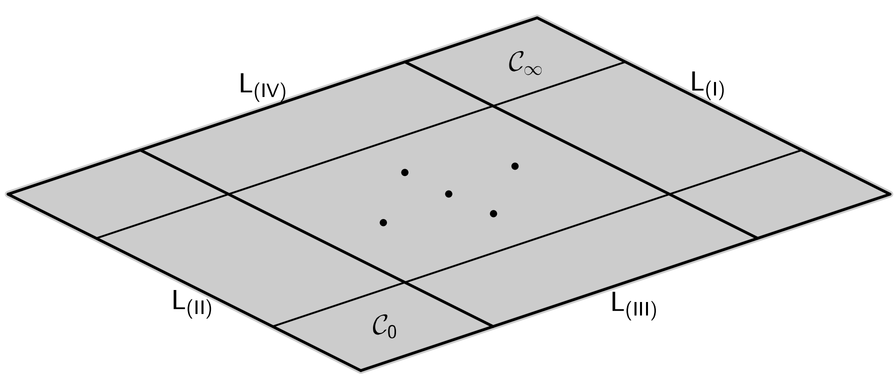


Figure 5.1: The chamber structure for chains of type $\mathbf{t} = (2,1,2;d_0,d_1,d_2)$.

Chapter 6

Holomorphic 2-chains and co-Higgs bundles of rank 2

In this chapter we study the relation between co-Higgs bundles of rank 2 and their associated 2-chains. We relate α-stability to stability for co-Higgs bundles. A co-Higgs bundle of rank 2 and degree d can be described as a holomorphic chain of type $(2, 2; d + 4, d)$. If two co-Higgs bundles are isomorphic then the associated holomorphic chains are isomorphic. The isomorphism class of a co-Higgs bundle is a subset of the isomorphism class of the associated holomorphic chain. Two such subsets are in the same isomorphism class of the associated holomorphic chain if and only if they differ by the automorphism group of the underlying vector bundle.

6.1 Co-Higgs bundles of rank 2

In this section we summarize the results from [34] and [38].

N. Nitsure constructed a coarse moduli scheme for S-equivalence classes of semistable pairs (E, ϕ) of rank r and degree d on a smooth projective curve X, where $\phi \in \mathrm{Hom}(X, EndE \otimes L)$ for a fixed line bundle L. Let $\mathcal{M}(r, d, L)$ denote the quasi-projective coarse moduli scheme. It contains an open subscheme $\mathcal{M}^s(r, d, L)$ of stable pairs.

Proposition 6.1.1. [34] *If $\deg(L) > \deg(K)$ then the dimension of the moduli scheme at a stable pair (E, ϕ) is*

$$\dim_{(E,\phi)}\mathcal{M}(r, d, L) = r^2\deg(L) + 1 + \dim H^1(X, L).$$

Theorem 6.1.1. [34] *For rank 2, the moduli scheme $\mathcal{M}(2, d, L)$ is connected for any d and L.*

If $X = \mathbb{P}^1$ then $K = \mathcal{O}(-2)$. Thus if $L = \mathcal{O}(2)$ then Nitsure's construction applies and the resulting moduli scheme is a moduli scheme of co-Higgs bundles on $\mathbb{P}^1$. Recall that $\mathcal{M}(r, d)$ denotes the moduli space of semistable co-Higgs bundles on $\mathbb{P}^1$.

Theorem 6.1.2. [38] *A rank r vector bundle $E = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus ... \oplus \mathcal{O}(m_r)$ on $\mathbb{P}^1$, where $m_1 \geq m_2 \geq ... \geq m_r$, admits a semistable $\phi \in H^0(\mathbb{P}^1, EndE \otimes \mathcal{O}(2))$ if and only if $m_i \leq m_i + 2$ for all $i = 1, ..., r-1$.*

For $r = 2$, Ryan only considered the moduli spaces $\mathcal{M}(r, 0)$ and $\mathcal{M}(r, -1)$. We can recover the co-Higgs bundles of other degrees by tensoring the elements of these two spaces by $\mathcal{O}(\pm 1)^{\otimes n}$.

For $r = 2$, by Theorem 6.1.2, if $\deg(E) = 0$ then E admits a semistable Higgs field if and only if $E \cong \mathcal{O} \oplus \mathcal{O}$ or $E \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. If $\deg(E) = -1$ then E admits a semistable Higgs field if and only if $E \cong \mathcal{O} \oplus \mathcal{O}(-1)$ [38, Section 5].

The moduli spaces $\mathcal{M}(2, 0)$ and $\mathcal{M}(2, -1)$ are 9-dimensional connected quasi-projective schemes (Proposition 6.1.1 and Theorem 6.1.1). We consider the moduli space of odd degree $\mathcal{M}(2, -1)$ in the following section.

6.2 Co-Higgs bundles of rank 2 and odd degree and holomorphic 2-chains

For rank 1 co-Higgs bundles, we have a simple result.

Proposition 6.2.1. *For $\boldsymbol{\alpha} > 2$, the moduli space $\mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t})$ of holomorphic chains of type $\mathbf{t} = (1, 1; d + 2, d)$ is nonempty and isomorphic to $\mathbb{P}^2$. The moduli space $\mathcal{M}(1, d)$ of associated stable co-Higgs bundles of rank 1 and degree d is $H^0(\mathbb{P}^1, \mathcal{O}(2)) \cong \mathbb{C}^3$. The natural map $(\mathcal{O}(d), \phi) \mapsto (\mathcal{O}(d + 2), \mathcal{O}(d); \phi)$ defines the rational map*

$$\mathcal{M}(1, d) \cong H^0(\mathcal{O}(2)) \dashrightarrow \mathcal{M}^s_{\boldsymbol{\alpha}}(\mathbf{t}) \cong \mathbb{P}^2.$$

Proof. By Corollary 3.2.2, $\mathcal{M}^s_{\boldsymbol{\alpha}}(1, 1; d + 2, d) \cong \mathbb{P}^2$. The underlying bundle for each co-Higgs bundle of rank 1 and degree d is $\mathcal{O}(d)$, and Higgs fields are elements of $H^0(\mathcal{O}(2))$. Clearly all of the co-Higgs bundles of rank 1 are stable. The equivalence of two co-Higgs bundles is given by the conjugation action of the automorphism group $\mathrm{Aut}(\mathcal{O}(d)) = \mathbb{C}^*$, so this action is trivial. Thus, $\mathcal{M}(1, d) = \{pt\} \times H^0(\mathcal{O}(2))$. Let $(\mathcal{O}(d), \phi)$ be an element of $\mathcal{M}(1, d)$. Since $\phi \in H^0(End(\mathcal{O}(d)) \otimes \mathcal{O}(2))$, ϕ can be identified as a homomorphism $\phi : \mathcal{O}(d) \to \mathcal{O}(d + 2)$. This defined the natural map. If $\phi = 0$ then the associated holomorphic chain $(\mathcal{O}(d + 2), \mathcal{O}(d); 0)$ is α-unstable for any α. The map is undefined at $(\mathcal{O}(d), 0)$. Two Higgs fields are associated with the isomorphic holomorphic chains if and only if they differ by $\mathbb{C}^*$. Hence, the map is the rational map $H^0(\mathcal{O}(2)) \dashrightarrow \mathbb{P}^2$ defined by $ax^2 + bxy + cy^2 \mapsto [ax^2 + bxy + cy^2]$, where $[ax^2 + bxy + cy^2]$ means the hypersurface of degree 2 in $\mathbb{P}^1$ and x and y are projective coordinates for $\mathbb{P}^1$. ☐

We will only consider co-Higgs bundles of rank 2 and degree -1. The underlying bundle is fixed, which is $\mathcal{O} \oplus \mathcal{O}(-1)$ (Section 6.1), its associated holomorphic chain is $(\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$.

Definition 6.2.1. *[3, Section 5.2] A holomorphic 2-chain $(E_0, E_1; \phi)$ is called rank maximal if ϕ has a generically maximal rank, i.e., is generically surjective or injective.*

We find a necessary condition that the associated chains are α-semistable.

Theorem 6.2.1. *If a holomorphic chain $C = (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$ is α-semistable (α-stable) for some α, then*

$$\begin{cases} \alpha \geq (>)3 \quad \textit{and} \quad i_2 \leq (<)1 & \textit{if } C \textit{ is rank maximal}, \\ 3 \leq (<)\alpha \leq (<)\min\{1-2k, 5-2i_1\} \quad \textit{and} \quad i_2 \leq (<)1 & \textit{if } C \textit{ is not rank maximal}, \end{cases} \tag{6.1}$$

where $k = \deg(Ker(\phi))$, $i_1 = \deg(\overline{Im(\phi)})$ and $i_2 = \deg(\overline{Im(\phi|_{\mathcal{O}})})$.

Proof. Let $C = (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$ be a holomorphic chain of type $\mathbf{t} = (2, 2; 3, -1)$. If $\phi = 0$ then C is α-unstable. We assume that $\phi \neq 0$. Then there are six rank types of proper subchains:

$$(0,1), (1,0), (1,1), (1,2), (2,0), (2,1).$$

The subchains composed of subbundles are

$$(0, Ker(\phi); 0), \quad (\mathcal{O}(2), 0; 0), \quad (\overline{Im(\phi|_L)}, L; \phi|_L),$$
$$(\overline{Im(\phi)}, \mathcal{O} \oplus \mathcal{O}(-1); \phi), \quad (\mathcal{O}(2) \oplus \mathcal{O}(1), 0; 0), \quad \text{and} \quad (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O}; \phi|_{\mathcal{O}}),$$

where L is a line subbundle of $\mathcal{O} \oplus \mathcal{O}(-1)$. Then $L \cong \mathcal{O}(a)$ for $a \leq 0$ [41].

Both $\mu_\alpha(C) \geq \mu_\alpha(\mathcal{O}(2), 0; 0)$ and $\mu_\alpha(C) \geq \mu_\alpha(\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O}; \phi|_{\mathcal{O}})$ imply that $\alpha \geq 3$, and the inequality $\mu_\alpha(C) \geq \mu_\alpha(\mathcal{O}(2) \oplus \mathcal{O}(1), 0; 0)$ implies that $\alpha \geq 2$. Thus we choose $(\mathcal{O}(2), 0; 0)$ to check α-stability.

The subchain $(\overline{Im(\phi|_L)}, \mathcal{O}; \phi|_L)$ varies with respect to line subbundle L. The inequality $\mu_\alpha(C) \geq \mu_\alpha(\overline{Im(\phi|_L)}, L; \phi|_L)$ implies that

$$\frac{1+\alpha}{2} \geq \frac{\deg(L) + \deg(\overline{Im(\phi|_L)}) + \alpha}{2},$$

so $\deg(\overline{Im(\phi|_L)}) \leq 1 - \deg(L)$. If $\deg(L) \leq -1$ then $\deg(\overline{Im(\phi|_L)}) \leq 2$. Since $\overline{Im(\phi|_L)}$ is a line subbundle of $\mathcal{O}(2) \oplus \mathcal{O}(1)$, we must have $\deg(\overline{Im(\phi|_L)}) \leq 2$. If $\deg(L) = 0$ then $L = \mathcal{O}$. Thus, we get the subchain $(\overline{Im(\phi|_{\mathcal{O}})}, \mathcal{O}; \phi|_{\mathcal{O}})$ to check the α-stability. The other two subchains are $(0, Ker(\phi); 0)$ and $(\overline{Im(\phi)}, \mathcal{O} \oplus \mathcal{O}(-1); \phi)$.

Thus, if C is α-semistable, then

$$\begin{aligned} &\text{(i)} \quad \mu_\alpha(C) \geq \mu_\alpha(\mathcal{O}(2), 0; 0), \\ &\text{(ii)} \quad \mu_\alpha(C) \geq \mu_\alpha(0, Ker(\phi); 0), \\ &\text{(iii)} \quad \mu_\alpha(C) \geq \mu_\alpha(\overline{Im(\phi|_{\mathcal{O}})}, \mathcal{O}; \phi|_{\mathcal{O}}), \\ &\text{(iv)} \quad \mu_\alpha(C) \geq \mu_\alpha(\overline{Im(\phi)}, \mathcal{O} \oplus \mathcal{O}(-1); \phi). \end{aligned}$$

If C is not rank maximal then (i)-(iv) imply that $3 \leq \alpha \leq \min\{1-2k, 5-2i_1\}$ and $i_2 \leq 1$.

If C is rank maximal then $Ker(\phi) = 0$ and $\overline{Im(\phi)} = \mathcal{O}(2) \oplus \mathcal{O}(1)$, so $(0, Ker(\phi); 0) = (0, 0; 0)$ and $(\overline{Im(\phi)}, \mathcal{O} \oplus \mathcal{O}(-1); \phi) = (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$, which are trivial subchains. Thus (ii) and (iv) disappear, and (i) and (iii) imply that $\alpha \geq 3$ and $i_2 \leq 1$.

For the α-stability we replace $\leq$ with $<$. □

Remark 6.2.1. *The converse of Theorem 6.2.1 is true if the map $\phi \neq 0$ and there exists an α satisfying the inequalities in* (6.1)*, then the holomorphic chain $C = (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$ is α-semistable* (α*-stable*)*.*

Theorem 6.2.2. *Let $C = (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$ be a holomorphic chain. Write $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in H^0(\mathcal{O}(2))$, $b \in H^0(\mathcal{O}(3))$ and $c \in H^0(\mathcal{O}(1))$. Then C is α-semistable for some α if and only if $c \neq 0$.*

Proof. (Only if): Suppose that C is α-semistable for some α. If $c = 0$ then $(\mathcal{O}(2), \mathcal{O}; \phi|_{\mathcal{O}})$ is an α-destablizing subchain for any α, so C is α-unstable for any α. Thus $c \neq 0$.

(If): Suppose that $c \neq 0$. Then the proof of Theorem 6.2.1 shows that the four inequalities (i)-(iv) determine α for which C is α-semistable. Since $c \neq 0$, $\phi(\mathcal{O}) \nsubseteq \mathcal{O}(2)$, so $i_2 = \deg(\overline{Im(\phi|_{\mathcal{O}})}) \leq 1$. If C is not rank maximal then (ii) and (iv) need to be checked for α-semistability for C. Since $c \neq 0$, $\phi(\mathcal{O}) \neq 0$ and $Im(\phi) \nsubseteq \mathcal{O}(2)$, then $k = \deg(Ker(\phi)) \leq -1$ and $i_1 = \deg(\overline{Im(\phi)}) \leq 1$, respectively. Thus $\min\{1-2k, 5-2i_1\} \geq 3$. Finally, (i) implies $\alpha \geq 3$. Consequently, the interval determined by (i)-(iv) is nonempty.

□

Proposition 6.2.2. *Let $(\mathcal{O} \oplus \mathcal{O}(-1), \phi)$ be a co-Higgs bundle of rank 2 and degree -1. Every semistable Higgs field ϕ is stable. Let $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in H^0(\mathcal{O}(2))$, $b \in H^0(\mathcal{O}(3))$ and $c \in H^0(\mathcal{O}(1))$. Then ϕ is a stable Higgs field if and only if $c \neq 0$.*

Proof. This is from [38, Section 6]. $\square$

Proposition 6.2.3. *A co-Higgs bundle $(\mathcal{O} \oplus \mathcal{O}(-1), \phi)$ is stable if and only if the associated holomorphic chain $C = (\mathcal{O}(2) \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(-1); \phi)$ is 3-semistable.*

Proof. Let $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in H^0(\mathcal{O}(2))$, $b \in H^0(\mathcal{O}(3))$ and $c \in H^0(\mathcal{O}(1))$. If the associated holomorphic chain C is 3-semistable then $c \neq 0$, and by Theorem 6.2.2 ϕ is a stable Higgs field. For the converse suppose that ϕ is a stable Higgs field. By Proposition 6.2.2, $c \neq 0$. The proof of Theorem 6.2.2 shows that the associated holomorphic chain C is at least 3-semistable.

$\square$

Let $(\mathcal{O} \oplus \mathcal{O}(-1), \phi)$ be a co-Higgs bundle. Write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in H^0(\mathcal{O}(2))$, $b \in H^0(\mathcal{O}(3))$ and $c \in H^0(\mathcal{O}(1))$. We can identify a, b, c and d with homogeneous polynomials of two variables. Let $\sharp Z(a, c)$ be the number of common zeroes of the polynomials a and c.

Theorem 6.2.3. *Let $(\mathcal{O} \oplus \mathcal{O}(-1), \phi)$ be a co-Higgs bundle with $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in H^0(\mathcal{O}(2))$, $b \in H^0(\mathcal{O}(3))$ and $c \in H^0(\mathcal{O}(1))$. Let $C = (\mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O}(2) \oplus \mathcal{O}(1); \phi)$ be the associated chain. Then ϕ is a stable Higgs field and C is α-unstable for $\alpha > 3$ if and only if ϕ is not rank maximal and $c \neq 0$.*

Proof. (Only if): Since ϕ is a stable Higgs field, by Proposition 6.2.2 $c \neq 0$. If ϕ is rank maximal then C is α-semistable for $\alpha \geq 3$ by the proof of Theorem 6.2.2. Thus, ϕ is not rank maximal.

(If): Suppose that ϕ is not rank maximal and $c \neq 0$. Since $c \neq 0$, ϕ is a stable co-Higgs field by Proposition 6.2.2, and $\sharp Z(a, c) = 0$ or 1.

Case 1: $\sharp Z(a, c) = 0$.

In this case $i_2 = \deg(\overline{Im(\phi|_{\mathcal{O}})}) = 0$. Thus it satisfies $i_2 \leq 1$. In this case ϕ is of the form

$$\begin{pmatrix} a & ad' \\ c & cd' \end{pmatrix},$$

where $a \in H^0(\mathcal{O}(2)) \setminus 0$ and $c, d' \in H^0(\mathcal{O}(1))$. The vector bundle $\mathcal{O} \oplus \mathcal{O}(-1)$ is the trivial extension of $Im(\phi)$ by $Ker(\phi)$, i.e.

$$0 \to Ker(\phi) \cong \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O}(-1) \xrightarrow{\phi} Im(\phi) \cong \mathcal{O} \to 0.$$

Since $Im(\phi)$ is generated by the section $\begin{pmatrix} a \\ c \end{pmatrix}$, $Im(\phi)$ is a line subbundle, i.e. $\overline{Im(\phi)} = Im(\phi) \cong \mathcal{O}$. We have $k = \deg(Ker(\phi)) = -1$ and $i_1 = \deg(\overline{Im(\phi)}) = 0$, so $\min\{1-2k, 5-2i_1\} = 3$. Therefore, by Theorem 6.2.1, C is α-unstable for $\alpha > 3$.

Case 2: $\sharp Z(a,c) = 1$.

In this case $i_2 = \deg(\overline{Im(\phi|_{\mathcal{O}})}) = 1$. Thus it satisfies $i_2 \leq 1$. In this case ϕ is of the form

$$\begin{pmatrix} a'c & a'd \\ c & d \end{pmatrix},$$

where $a', c \in H^0(\mathcal{O}(1)) \setminus 0$ and $d \in H^0(\mathcal{O}(2))$. We have

$$Im(\phi) \cong \begin{cases} \mathcal{O}(1) & \text{if } \sharp Z(c,d) = 0, \\ \mathcal{O} & \text{if } \sharp Z(c,d) = 1. \end{cases}$$

If $\sharp Z(c,d) = 0$ then $Im(\phi)$ is generated by the section $\begin{pmatrix} a' \\ 1 \end{pmatrix}$, so $Im(\phi)$ is a line subbundle, i.e. $\overline{Im(\phi)} = Im(\phi) \cong \mathcal{O}(1)$. Thus $k = \deg(Ker(\phi)) = -2$, and $i_1 = \deg(\overline{Im(\phi)}) = 1$, so $\min\{1-2k, 5-2i_1\} = 3$. Therefore, by Theorem 6.2.1, C is α-unstable for $\alpha > 3$.

If $\sharp Z(c,d) = 1$ then $Im(\phi)$ is not generated by the section $\begin{pmatrix} a' \\ 1 \end{pmatrix}$, so $Im(\phi)$ is not a subbundle, and $\overline{Im(\phi)} \cong \mathcal{O}(1)$. Thus $k = \deg(Ker(\phi)) = -1$, and $i_1 = \deg(\overline{Im(\phi)}) = 1$, so $\min\{1-2k, 5-2i_1\} = 3$. Therefore, by Theorem 6.2.1, C is α-unstable for $\alpha > 3$.

□

Theorem 6.2.4. *Let $(\mathcal{O} \oplus \mathcal{O}(-1), \phi)$ be a co-Higgs bundle with $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in H^0(\mathcal{O}(2))$, $b \in H^0(\mathcal{O}(3))$ and $c \in H^0(\mathcal{O}(1))$. Let $C = (\mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O}(2) \oplus \mathcal{O}(1); \phi)$ be the associated chain. The following statements are equivalent:*

(i) *A Higgs field ϕ is stable and C is α-stable for $\alpha > 3$;*

(ii) *C is α-stable for $\alpha > 3$;*

(iii) *ϕ is rank maximal and $\sharp Z(a,c) = 0$.*

Proof. (i) ⇒ (ii): It is clear.

(ii) $\Rightarrow$ (i): If C is α-stable for $\alpha > 3$, then by Theorem 6.2.2 $c \neq 0$, so by Proposition 6.2.2 the Higgs field ϕ is stable.

(ii) $\Rightarrow$ (iii): Suppose that C is α-stable for $\alpha > 3$. By Theorem 6.2.1, $i_2 = \deg(\overline{Im(\phi|_{\mathcal{O}})}) < 1$. Thus $\sharp Z(a, c) = i_2 = 0$. If ϕ is not rank maximal then by Theorem 6.2.1, $\min\{1 - 2k, 5 - 2i_1\} > 3$. Then $k < -1$ and $i_1 < 1$

Consider the exact sequence

$$0 \to Ker(\phi) \to \mathcal{O} \oplus \mathcal{O}(-1) \ \to Im(\phi) \to 0.$$

Then $i_1 = \deg(\overline{Im(\phi)}) \geq \deg(Im(\phi)) = -\deg(Ker(\phi)) - 1 = -k - 1 > 0$. Thus $0 < i_1 < 1$. This is a contradiction. Hence ϕ is rank maximal.

(iii) $\Rightarrow$ (ii): Suppose that ϕ is rank maximal and $\sharp Z(a, c) = 0$. Clearly $\phi \neq 0$. We only need to check the inequality $i_2 = \deg(\overline{Im(\phi|_{\mathcal{O}})}) < 1$. Since $\sharp Z(a, c) = 0$, $i_2 = 0 < 1$. Thus, by Remark 6.2.1, C is α-stable for $\alpha > 3$.

□

Chapter 7

Non-reductive Geometric Invariant Theory

7.1 Subspaces of the moduli spaces of holomorphic chains on $\mathbb{P}^1$

Let $\mathbb{P}^1 := \mathbb{P}(V)$, where V is a two-dimensional vector space over the complex numbers $\mathbb{C}$. Let (E_0, E_1) be a pair of vector bundles of type $\mathbf{t} = (r_0, r_1; d_0, d_1)$ on $\mathbb{P}^1$. Then the moduli space $\mathcal{M}_{\boldsymbol{\alpha}}(\boldsymbol{t})$ contains a natural subspace $\mathcal{M}_{\boldsymbol{\alpha}}(E_0, E_1)$. The subspace $\mathcal{M}_{\boldsymbol{\alpha}}(E_0, E_1)$ is the set of S-equivalence classes of $\boldsymbol{\alpha}$-semistable holomorphic chains with a fixed (E_0, E_1). This space contains an open subset $\mathcal{M}^s_{\boldsymbol{\alpha}}(E_0, E_1)$ of $\boldsymbol{\alpha}$-stable holomorphic chains with the fixed (E_0, E_1). The point in $\mathcal{M}^s_{\boldsymbol{\alpha}}(E_0, E_1)$ is a $(\mathrm{Aut}(E_0) \times \mathrm{Aut}(E_1))$-orbits. Indeed the automorphism group acts on the vector space $\mathrm{Hom}(E_1, E_0)$ by $g_0 \varphi g_1^{-1}$ for $(g_0, g_1) \in \mathrm{Aut}(E_0) \times \mathrm{Aut}(E_1)$ and $\varphi \in \mathrm{Hom}(E_1, E_0)$. The group $(\mathrm{Aut}(E_0) \times \mathrm{Aut}(E_1))$ is non-reductive unless E_0 and E_1 are semistable vector bundles. Thus we need to use non-reductive GIT methods.

Drézet and Trautmann studied the space of homomorphism $\mathrm{Hom}(E_1, E_0)$ by the action of $(\mathrm{Aut}(E_0) \times \mathrm{Aut}(E_1))$ [13]. We compare their non-reductive quotients with the subspaces $\mathcal{M}_{\boldsymbol{\alpha}}(E_0, E_1)$. We only consider holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$. We summarize their method in the following section.

For this non-reductive action, we have similar results as [22, Section 3]. The vector space $\mathrm{Hom}(E_1, E_0)$ is a natural parameter space for $\mathcal{M}_{\boldsymbol{\alpha}}(E_0, E_1)$. The automorphism group $\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_0)$ acts linearly on $\mathrm{Hom}(E_1, E_0)$. Since we have the isomorphisms

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \phi\ } & E_0 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \lambda} \\ E_1 & \xrightarrow{\ \lambda\phi\ } & E_0 \end{array}$$

for any $\lambda \in \mathbb{C}^*$, we can replace $\mathrm{Hom}(E_1, E_0)$ with the projective space $W = \mathbb{P}\mathrm{Hom}(E_1, E_0)$. Since the group $\mathbb{C}^*$ of homotheties acts trivially, we can replace $\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_0)$ with the quotient group

$$H = (\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_0))/\mathbb{C}^*.$$

The affine algebraic group H acts linearly on the projective space W.

Now set

$$E_1 = \mathcal{O}(d_{11}) \oplus ... \oplus \mathcal{O}(d_{1r_1}) \quad \text{and} \quad E_0 = \mathcal{O}(d_{01}) \oplus ... \oplus \mathcal{O}(d_{0r_0}),$$

where $d_{11} \geq ... \geq d_{1r_1}, d_{01} \geq ... \geq d_{0r_0}$. Then the $r_0 \times r_1$ matrix (a_{ij}) represents the homomorphism $\phi : E_1 \to E_0$, where $a_{ij} \in H^0(\mathbb{P}^1, \mathcal{O}(d_{0i} - d_{1j})) = S^{d_{0i}-d_{1j}}(V^*)$ for $1 \leq i \leq r_0, 1 \leq j \leq r_1$. The tautological family $\mathcal{C}$ parameterized by W consists of the line bundle $\mathcal{O}_W(1)$ and the indexed set

$$A = (a_{ij}^{kl} : k, l \geq 0, k + l = d_{0i} - d_{1j})$$

of sections of $\mathcal{O}_W(1)$. The family $\mathcal{C}$ parameterized by W has the following two properties:

(i) the chains $\mathcal{C}_s, \mathcal{C}_t$ parameterized by the $r_0 \times r_1$ matrices s and t of homogeneous polynomials are isomorphic if and only if s and t are in the same orbit of the action of automorphism group H.

(ii) The family $\mathcal{C}$ parameterized by W has the local universal property. In other words, any family of chains is locally equivalent to the pullback of $\mathcal{C}$ along a morphism to W.

Hence constructing a coarse moduli space of homomorphisms $E_1 \to E_0$ is equivalent to constructing an orbit space of the action of H on W [32, Proposition 2.13].

Remark 7.1.1. *Similarly we can define a family of holomorphic $(n+1)$-chains. For a fixed tuple $(E_0, ..., E_n)$ of vector bundles, let*

$$W_1 = \mathbb{P}\mathrm{Hom}(E_1, E_0), ..., W_n = \mathbb{P}\mathrm{Hom}(E_n, E_{n-1}).$$

Then there is a tautological family parameterized by the product of the projective spaces $W_1 \times ... \times W_n$.

Definition 7.1.1. *The tuple of numbers $((d_{01}, ..., d_{0r_0}), (d_{11}, ..., d_{1r_1}))$ is called the splitting of type for the chain $(E_0, E_1; \phi) = (\mathcal{O}(d_{01}) \oplus ... \oplus \mathcal{O}(d_{0r_0}), \mathcal{O}(d_{11}) \oplus ... \oplus \mathcal{O}(d_{1r_1}); \phi)$.*

Example 7.1.1. *Given α, the moduli space of α-stable chains of a type $\mathbf{t} = (1, 2; 0, -s)$ is a union of subspaces:*

$$\mathcal{M}_\alpha^s(1, 2; 0, -s) = \bigcup_{d+e=s} \mathcal{M}_\alpha^s(-d, -e),$$

where $\mathcal{M}_\alpha^s(-d, -e)$ is the subspace of fixed splitting type $(0, (-d, -e))$, i.e. $\mathcal{M}_\alpha^s(-d, -e) = \mathcal{M}_\alpha^s(\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e))$.

7.2 Drézet and Trautmann's method

In this section, we review the results in [13]. Drézet and Trautmann used results of King in [21, Proposition 3.1] (or [13, Section 3.1]) to show that there were a good quotient and a geometric quotient of the natural conjugation action on the affine space of homomorphisms between two coherent sheaves over X. They proved sufficient conditions for the existence of such quotients.

We briefly summarize results of King [13, Section 3.1]. Let Q be a finite set, $\Gamma \subset Q \times Q$ a subset such that the union of the two projections of Γ is Q. Let

$$W_0 = \bigoplus_{(i,j)\in\Gamma} \mathrm{Hom}(M_i \otimes V_{ij}, M_j),$$

where M_i and V_{ij} are nonzero finite dimensional vector spaces for i and $j \in \Gamma$. The reductive group $G_0 = \prod_{\alpha} GL(M_\alpha)$ acts on W_0 by conjugation: $(g_i) \cdot (w_{ji}) = (g_j w_{ji} (g_i \otimes id)^{-1})$, where id is the $\dim(V_{ij}) \times \dim(V_{ij})$ identity matrix. King defined stability for the action of G_0 on W_0 using the character χ of G_0 defined by $\prod_{i\in\Gamma} \det(g_i)^{-e_i}$.

Definition 7.2.1. (i) *An element $x \in W_0$ is χ-semistable if there exists an integer $n \geq 1$ and a polynomial $f \in \mathbb{C}[W_0]$ such that $f(w) \neq 0$ and $f(gw) = \chi^n(g) f(w)$ for every $w \in W_0$ and $g \in G_0$ (f is called a χ^n-invariant polynomial).*

(ii) *The point is χ-stable if moreover $\dim(G_0 x) = \dim(G_0/\mathbb{C}^*)$ and the action of G_0 on $\{w \in W_0 : f(w) \neq 0\}$ is closed.*

King showed the following results in [21]:

Proposition 7.2.1. (i) *A point $w = (w_{ij}) \in W_0$ is χ-(semi)stable if and only if for each family (M_i') of subspaces $M_i' \subset M_i$ which is neither (0) nor (M_i) itself such that*

$$w_{ji}(M_i' \otimes V_{ij}) \subset M_j'$$

for each $(i, j) \in \Gamma$, we have

$$\sum_{i\in Q} e_i \dim(M_i')(\leq) < 0.$$

(ii) *There exist a good quotient $W_0//G_0$ and a geometric quotient W_0^s/G_0.*

Let $\mathcal{E}, \mathcal{F}$ be two coherent sheaves. Then the algebraic group $G := G_L \times G_R = \mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ acts on the affine space $W := \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ by $(g, h).w = h \circ w \circ g^{-1}$. In general, $\mathcal{E}$ and $\mathcal{F}$ are decomposed into simple sheaves such that G is not reductive. We thus cannot use King's result directly.

Let $\mathcal{E}, \mathcal{F}$ be direct sums

$$\mathcal{E} = \bigoplus_{1\le i\le r} M_i \otimes \mathcal{E}_i, \quad \mathcal{F} = \bigoplus_{1\le \ell\le s} N_\ell \otimes \mathcal{F}_\ell,$$

where M_i, N_ℓ are finite dimensional vector spaces and $\mathcal{E}_i, \mathcal{F}_\ell$ are simple sheaves, i.e.,$\mathrm{End}(\mathcal{E}_i) = \mathrm{End}(\mathcal{F}_\ell) = \mathbb{C}$ such that $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_m) = 0$ for $i > j$ and $\ell > m$. Then the group $\mathrm{Aut}(\mathcal{E})$ is the group of matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ r_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix},$$

where $g_i \in \mathrm{GL}(M_i)$ and $u_{j,i} \in \mathrm{Hom}(M_i, M_j \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j))$. The group $\mathrm{Aut}(\mathcal{F})$ is similar.

The group G contains a reductive subgroup

$$G_{\mathrm{red}} = G_{L,\mathrm{red}} \times G_{R,\mathrm{red}} = \prod GL(M_i) \times \prod GL(N_\ell)$$

such that $G = G_{\mathrm{red}} \ltimes H$, where $H = H_L \times H_R$ is the unipotent radical of G. The subgroup $H_L \subseteq G_L$ is defined by the condition $g_i = id_{M_i}$. The subgroup $H_R \subseteq G_R$ is similarly defined.

Drézet and Trautmann defined a polarization which is a refinement of the character χ of G_{red} defined by $\chi(g,h) = \prod \det(g_i)^{a_i} \prod \det(h_\ell)^{-b_i}$ and used it to restate King's stability for the action of G_{red}; then they defined a notion of stability for the action of the full group G.

Definition 7.2.2. *A polarization Λ is a tuple of rational numbers $(\lambda_1, \lambda_2, ..., \lambda_r, -\mu_1, -\mu_2, ..., -\mu_s) \in \mathbb{Q}^{r+s}$ such that $\sum \lambda_i m_i = \sum \mu_\ell n_\ell = 1$, where $m_i = \mathrm{dim} M_i, n_\ell = \mathrm{dim} N_\ell$.*

A family of subspaces $M_i' \subseteq M_i$, $N_\ell' \subseteq N_\ell$ is called *admissible* if not all subspaces are zero and we do not have $M_i' = M_i$, $N_\ell' = N_\ell$ for all i, ℓ.

Definition 7.2.3. (*A numerical criterion*)

(i) *An element $\varphi \in W$ is $(G_{\mathrm{red}}, \Lambda)$-(semi)stable if*

$$\sum_{i=1}^{r} \lambda_i m_i' < (\le) \sum_{\ell=1}^{s} \mu_\ell n_\ell'$$

for each admissible family of subspaces $M_i' \subseteq M_i, N_\ell' \subseteq N_\ell$ such that $\varphi(\bigoplus_{i=1}^r M_i' \otimes \mathcal{E}_i) \subset \bigoplus_{\ell=1}^s M_\ell' \otimes \mathcal{F}_\ell$.

(ii) *An element* $\varphi \in W$ *is* (G,Λ)*-(semi)stable if every element in the* H*-orbit* Hw *is* $(G_{\mathrm{red}},\Lambda)$*-(semi)stable. We denote by* $W^{ss}(G,\Lambda)$ *and* $W^{s}(G,\Lambda)$ *the sets of* (G,Λ)*-semistable, respectively* (G,Λ)*-stable points in* W*.*

Remark 7.2.1. *There is a priori restriction on polarizations. By a general result, if* $W^{ss}(G,\Lambda) \neq \emptyset$ $(W^{s}(G,\Lambda) \neq \emptyset)$ *then*

$$\lambda_i \geq (>)0 \quad \textit{and} \quad \mu_\ell \geq (>)0, \quad \textit{for all} \quad i, \ell.$$

Since the group G is not reductive, they embed the action of G on W into an action of $\boldsymbol{G}$ on $\boldsymbol{W}$ of a reductive group $\boldsymbol{G}$ such that the diagram commutes:

$$\begin{array}{ccc} G \times W & \longrightarrow & W \\ {\scriptstyle (\theta,\zeta)}\downarrow & & {\scriptstyle \zeta}\downarrow \\ \boldsymbol{G} \times \boldsymbol{W} & \longrightarrow & \boldsymbol{W}, \end{array}$$

where $\theta : G \hookrightarrow \boldsymbol{G}$, $\zeta : W \hookrightarrow \boldsymbol{W}$. Now we briefly explain the embedding. Set the vector spaces:

$$H_{\ell i} = \mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_\ell), \quad A_{ji} = \mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i), \quad B_{k\ell} = \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_k)$$

and

$$P_i = \mathrm{Hom}(\mathcal{E}_\ell, \mathcal{E}) = \bigoplus_{j=1}^{r} M_j \otimes A_{ji}, \quad Q_\ell = \mathrm{Hom}(\mathcal{F}, \mathcal{F}_\ell)^* = \bigoplus_{m=1}^{\ell} N_m \otimes B^*_{\ell m}.$$

Set

$$\boldsymbol{W} = \boldsymbol{W}_L \oplus \mathrm{Hom}(P_1, Q_s \otimes H_{s1}) \oplus \boldsymbol{W}_R,$$

where

$$\boldsymbol{W}_L = \bigoplus_{i=2}^{r} \mathrm{Hom}(P_i \otimes A_{i,i-1}, P_{i-1}),$$
$$\boldsymbol{W}_R = \bigoplus_{\ell=1}^{s-1} \mathrm{Hom}(Q_{\ell+1} \otimes B_{\ell+1,\ell}, Q_\ell).$$

They also introduce an associated polarization on $\boldsymbol{W}$

$$\widetilde{\Lambda} = (\alpha_1, \alpha_2, ..., \alpha_r, -\beta_1, -\beta_2, ..., -\beta_s).$$

The map ζ is defined by

$$W \xrightarrow{\zeta} \boldsymbol{W}, \quad w \mapsto ((\xi_2, ..., \xi_r), \gamma(w), (\eta_1, ..., \eta_{\ell-1})),$$

where

$$P_i \otimes A_{i,i-1} \xrightarrow{\xi_i} P_{i-1} \quad \text{and} \quad Q_{\ell+1} \otimes B_{\ell+1,\ell} \xrightarrow{\eta_\ell} Q_\ell$$

are the canonical morphisms on each component $M_j \otimes A_{ji}$ of P_i. The map ξ_i is the map

$$(M_j \otimes A_{ji}) \otimes A_{i,i-1} \to M_j \otimes A_{j,i-1}$$

induced by the composition map of the space A. The map η_ℓ is defined similarly, and $\gamma(w)$ is the matrix $(\gamma_{\ell i}(w))$, for which each $\gamma_{\ell i}(w)$ is the composed linear map

$$M_i \otimes A_{i1} \to N_\ell \otimes H_{\ell i} \otimes A_{i1} \to N_\ell \otimes H_{\ell 1} \to N_\ell \otimes B^*_{s\ell} \otimes H_{s1}.$$

Set

$$\boldsymbol{G} = \boldsymbol{G}_L \times \boldsymbol{G}_R, \quad \text{with} \quad \boldsymbol{G}_L = \prod_{i=1}^{r} \mathrm{GL}(P_i), \boldsymbol{G}_R = \prod_{\ell=1}^{s} \mathrm{GL}(Q_\ell).$$

Then the natural action of $\boldsymbol{G}$ on $\boldsymbol{W}$ is given by

$$g_{i-1} \circ x_{i-1,i} \circ (g_i \otimes id)^{-1}, h_s \circ \psi \circ (g_1 \otimes id)^{-1}, h_\ell \circ y_{\ell,\ell+1} \circ (h_{\ell+1} \otimes id)^{-1}$$

for

$$x_{i-1,i} \in \mathrm{Hom}(P_i \otimes A_{i,i-1}, P_{i-1}), \quad \psi \in \mathrm{Hom}(P_1 \otimes H^*_{s,1}, Q_s),$$
$$y_{\ell,\ell+1} \in \mathrm{Hom}(Q_{\ell+1} \otimes B_{\ell+1,\ell}, Q_\ell).$$

The map θ consists of two maps $\theta_L : G_L \to \boldsymbol{G}_L$ and $\theta_R : G_R \to \boldsymbol{G}_R$. The map θ_L is the map $(\theta_{L,1}, \theta_{L,2}, ..., \theta_{L,r})$. Let $g \in G_L$,

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ r_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix},$$

with $g_i \in \mathrm{GL}(M_i)$ and $u_{j,i} \in \mathrm{Hom}(M_i, M_j \otimes A_{ji})$. Then $\theta_{L,i}(g) \in GL(P_i)$ is defined by the matrix

$$\theta_{L,i}(g) = \begin{pmatrix} \widetilde{g}_i & 0 & \cdots & 0 \\ \widetilde{u}_{i+1,i} & \widetilde{g}_{i+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widetilde{r}_{r,i} & \cdots & \widetilde{u}_{r,r-1} & \widetilde{g}_r \end{pmatrix}$$

with respect to the decomposition of P_i with the following components: $\widetilde{g}_j = g_j \otimes id$ on $M_j \otimes A_{ji}$. For $i \leq j \leq k$ the map $\widetilde{u}_{kj}$ is the composition

$$M_j \otimes A_{ji} \to M_k \otimes A_{kj} \otimes A_{ji} \to M_k \otimes A_{ki}.$$

The second component θ_R is defined similarly.

Set

$$p_i = \dim(P_i), \quad q_m = \dim(Q_m), \quad a_{ji} = \dim(A_{ji}), \quad b_{ml} = \dim(B_{ml}).$$

The associated polarization is defined by the conditions

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{r,1} & \cdots & a_{r,r-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix},$$

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_s \end{pmatrix} = \begin{pmatrix} 1 & b_{2,1} & \cdots & b_{s,1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{s,s-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix}.$$

Then we have

$$1 = \sum_{i=1}^{r} \lambda_i m_i = \sum_{i=1}^{r} \alpha_i p_i, \quad 1 = \sum_{\ell=1}^{s} \mu_\ell n_\ell = \sum_{\ell=1}^{s} \beta_\ell q_\ell.$$

Lemma 7.2.1. *We have the inclusion*

$$\zeta^{-1}(\boldsymbol{W}^s(\boldsymbol{G}, \widetilde{\Lambda})) \subset W^s(G, \Lambda) \quad \textit{and} \quad \zeta^{-1}(\boldsymbol{W}^{ss}(\boldsymbol{G}, \widetilde{\Lambda})) \subset W^{ss}(G, \Lambda).$$

They showed the following statements:

Proposition 7.2.2. *([13, Proposition 6.1.1])*

i) *If* $\zeta^{-1}(\boldsymbol{W}^s(\boldsymbol{G},\widetilde{\Lambda})) = W^s(G,\Lambda)$*, then there exists a geometric quotient* $W^s(G,\Lambda)/G$*, which is a non-singular quasi-projective variety.*

ii) *If in addition* $\zeta^{-1}(\boldsymbol{W}^{ss}(\boldsymbol{G},\widetilde{\Lambda})) = W^{ss}(G,\Lambda)$ *and* $(\bar{Z}\setminus Z)\cap \boldsymbol{W}(\boldsymbol{G},\widetilde{\Lambda}) = \emptyset$*, then there exists a good quotient* $W^{ss}(G,\Lambda)//G$*, which is a normal projective variety such that* $W^s(G,\Lambda)/G \subseteq W^{ss}(G,\Lambda)//G$ *is an open subset, where* Z *is the saturation* $Z = \boldsymbol{G}\zeta(W)\subset \boldsymbol{W}$ *of the image of* W*.*

7.2.1 An application for holomorphic chains of type $(1,2)$ on $\mathbb{P}^1$

We use Drézet and Trautmann's method to study holomorphic chains on $\mathbb{P}^1$. Their method is applied to 2-chains. We assume that $\mathbb{P}^1 = \mathbb{P}(V)$, where V is a two-dimensional vector space over the field $\mathbb{C}$ of complex numbers.

In this case a holomorphic chain is

$$\mathcal{O}(-d)\oplus\mathcal{O}(-e)\to\mathcal{O}, \quad d>e\geq 0.$$

Then $G = \mathrm{Aut}(\mathcal{O}(-d)\oplus\mathcal{O}(-e))\times\mathrm{Aut}(\mathcal{O}) = \begin{pmatrix} \mathbb{C}^* & 0 \\ S^{d-e}(V^\vee) & \mathbb{C}^* \end{pmatrix}\times\mathbb{C}^*$ and $W = \mathrm{Hom}(\mathcal{O}(-d)\oplus\mathcal{O}(-e),\mathcal{O})$. Here the reductive part of G is $G_{\mathrm{red}} = (\mathbb{C}^*\times\mathbb{C}^*)\times\mathbb{C}^*$ and the unipotent radical of G is $H = (\mathbb{C}^+)^{d-e+1}$. Since $d\neq e$, $M_1 = M_2 = \mathbb{C}$ and clearly $N_1 = \mathbb{C}$. A polarization Λ is a triple:

$$\Lambda = (\lambda_1,\lambda_2,-\mu_1), \quad \text{with} \quad \mu_1 = 1 \quad \text{and} \quad \lambda_1+\lambda_2 = 1.$$

Let $\lambda = \lambda_1$. Then $\lambda_2 = 1-\lambda_1 = 1-\lambda$ and $\Lambda = (\lambda, 1-\lambda, -1)$. Since $\lambda_1\geq 0$ and $\lambda_2\geq 0$, $0\leq\lambda\leq 1$.

Recall that

$$H_{11} = \mathrm{Hom}(\mathcal{O}(-d),\mathcal{O}) = S^d(V^\vee), \quad H_{12} = \mathrm{Hom}(\mathcal{O}(-e),\mathcal{O}) = S^e(V^\vee)$$

and

$$\begin{aligned}
A_{21} &= \mathrm{Hom}(\mathcal{O}(-d), \mathcal{O}(-e)) = S^{d-e}(V^{\vee}),\\
P_1 &= \mathrm{Hom}(\mathcal{O}(-d), \mathcal{O}(-d) \oplus \mathcal{O}(-e)) = M_1 \oplus (M_2 \otimes A_{21}),\\
P_2 &= \mathrm{Hom}(\mathcal{O}(-e), \mathcal{O}(-d) \oplus \mathcal{O}(-e)) = M_2,\\
\mathbf{W} &= \mathrm{Hom}(P_2 \otimes A_{21}, P_1) \oplus \mathrm{Hom}(P_1, H_{11} \otimes N_1).
\end{aligned}$$

Write $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$, where $\varphi' \in \mathrm{Hom}(\mathcal{O}(-d), \mathcal{O})$ and $\varphi'' \in \mathrm{Hom}(\mathcal{O}(-e), \mathcal{O})$, and we consider a matrix

$$X = \begin{pmatrix} x^{d-e} & x^{d-e-1}y & \cdots & y^{d-e} \end{pmatrix}$$

whose entries form a basis of A_{21}. The embedding $W \hookrightarrow \boldsymbol{W}$ is given by the affine map

$$\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \mapsto \left(\begin{pmatrix} 0 \\ X^T \end{pmatrix}, \begin{pmatrix} \varphi' & \varphi'' X \end{pmatrix} \right).$$

The group $\boldsymbol{G}$ is given by

$$\begin{aligned}
\boldsymbol{G} &= GL(P_1) \times GL(P_2) \times GL(N_1) = GL(\mathbb{C} \oplus S^{d-e}(V^{\vee})) \times GL(\mathbb{C}) \times GL(\mathbb{C})\\
&\cong GL(d-e+2) \times \mathbb{C}^* \times \mathbb{C}^*,
\end{aligned}$$

and the embedding $G \hookrightarrow \boldsymbol{G}$ is given by

$$\left(\begin{pmatrix} g_1 & 0 \\ u_{2,1} & g_2 \end{pmatrix}, \begin{pmatrix} h_1 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} g_1 & 0 \\ \widetilde{u}_{2,1} & g_2 I_{d-e+1} \end{pmatrix} \times \begin{pmatrix} g_2 \end{pmatrix}, \begin{pmatrix} h_1 \end{pmatrix} \right),$$

where $\widetilde{u}_{2,1}$ is given by the composition

$$M_1 \otimes \mathbb{C} \to M_2 \otimes S^{d-e}(V^{\vee}) \otimes \mathbb{C} \to M_2 \otimes S^{d-e}(V^{\vee}).$$

If $u_{2,1} = c_0 x^{d-e} + c_1 x^{d-e-1} y + \cdots + c_{d-e} y^{d-e}$ then $\widetilde{u}_{2,1} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{d-e} \end{pmatrix}^T$.

The induced polarization $\widetilde{\Lambda} = (\lambda, 1 - \lambda - (d-e+1)\lambda, -1) = (\lambda, 1 - (d-e+2)\lambda, -1)$.

For example, if $d - e = 1$, then

$$\mathbf{W} = \mathrm{Hom}(\mathbb{C} \otimes S^1(V^{\vee}), \mathbb{C} \oplus S^1(V^{\vee})) \oplus \mathrm{Hom}(\mathbb{C} \oplus (\mathbb{C} \otimes S^1(V^{\vee})), S^d(V^{\vee}) \otimes \mathbb{C}).$$

The embedding $W \hookrightarrow \mathbf{W}$ is given by

$$\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \mapsto \left(\begin{pmatrix} 0 \\ x \\ y \end{pmatrix}, \begin{pmatrix} \varphi' & \varphi'' x & \varphi'' y \end{pmatrix} \right)$$

and

$$\boldsymbol{G} = GL(\mathbb{C} \oplus S^1(V^\vee)) \times GL(\mathbb{C}) \times GL(\mathbb{C}) \cong GL(3) \times GL(1) \times GL(1).$$

In this case, the induced polarization $\widetilde{\Lambda} = (\lambda, 1 - 3\lambda, -1)$.

While we cannot completely classify the polarizations which allow the good quotient $W^{ss}//G$, we can identify all possible polarizations Λ such that there exists a (G, Λ)-semistable point in W.

Proposition 7.2.3. *Let* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \in \mathrm{Hom}(\mathcal{O}(-d) \oplus \mathcal{O}(-e), \mathcal{O})$ *with* $d > e \geq 0$*. If* $0 < \lambda < 1$ *then the morphism* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ *is* (G, Λ)*-stable with* $\Lambda = (\lambda, 1 - \lambda, -1)$ *if and only if* $\varphi'' \neq 0$ *and* φ'' *does not divide* φ'*.*

Proof. Recall that φ is $(G_{\mathrm{red}}, \Lambda)$-stable if and only if for each admissible family of subspaces $M_1' \subseteq M_1$, $M_2' \subseteq M_2, N_1' \subseteq N_1$ such that $\varphi((M_1' \otimes \mathcal{O}(-d)) \oplus M_2' \otimes \mathcal{O}(-e)) \subseteq N_1' \otimes \mathcal{O}$,

$$\lambda m_1' + (1 - \lambda) m_2' < n_1'.$$

Let H be the unipotent radical of G. Then a morphism φ is (G, Λ)-stable if and only if every element in $H\varphi$ is $(G_{\mathrm{red}}, \Lambda)$-stable.

Since $\dim(M_1) = \dim(M_2) = \dim(N_1) = 1$, there are six admissible families of subspaces:

$$(M_1', M_2', N_1') = (0, 0, N_1), (0, M_2, N_1), (M_1, 0, N_1), (M_1, 0, 0), (0, M_2, 0), (M_1, M_2, 0).$$

Case 1: $N_1' = N_1$.

Any morphism φ in this case satisfies the inclusion

$$\varphi((M_1' \otimes \mathcal{O}(-d)) \oplus (M_2' \otimes \mathcal{O}(-e))) \subset N_1' \otimes \mathcal{O} = \mathcal{O}.$$

Since $0 < \lambda < 1$, then $\lambda m_1' + (1 - \lambda) m_2' < 1$ for $(m_1', m_2') = (0, 0), (0, 1), (1, 0)$. Hence every morphism φ satisfies

$$\lambda m_1' + (1 - \lambda) m_2' < 1$$

for the first three admissible families of subspaces $(M_1', M_2', N_1') = (0, 0, N_1), (0, M_2, N_1), (M_1, 0, N_1)$.

Case 2: $N_1' = 0$.

In this case we have $\varphi((M_1' \otimes \mathcal{O}(-d)) \oplus (M_2' \otimes \mathcal{O}(-e))) \nsubseteq N_1' \otimes \mathcal{O} = 0$ unless $\varphi' = 0$ or $\varphi'' = 0$. The zero morphism $\varphi = 0$ is $(G_{\text{red}}, \Lambda)$-unstable. If $\varphi = 0$ then the admissible family of subspaces $(M_1, M_2, 0)$ satisfies the inclusion

$$\varphi((M_1 \otimes \mathcal{O}(-d)) \oplus (M_2 \otimes \mathcal{O}(-e))) \subset N_1' \otimes 0 = 0,$$

but it destabilizes. Indeed $\lambda m_1' + (1-\lambda) m_2' = \lambda + 1 - \lambda = 1 < 0 = n_1'$. This is a contradiction. Hence $\varphi = 0$ must be (G, Λ)-unstable.

We have two other cases: i) $\varphi' \neq 0$ and $\varphi'' = 0$ and ii) $\varphi' = 0$ and $\varphi'' \neq 0$.

In case i), $(0, M_2, 0)$ is the only admissible family of subspaces such that

$$\varphi(M_2 \otimes \mathcal{O}(-e)) \subset 0.$$

Since $0 < \lambda < 1$, the morphism $(\varphi', 0)$ is $(G_{\text{red}}, \Lambda)$-unstable. Indeed $\lambda m_1' + (1-\lambda) m_2' = 1 - \lambda < n_1' = 0$. This contradicts the inequality $0 < \lambda < 1$. Thus $(\varphi', 0)$ must be (G, Λ)-unstable.

In case ii), $(M_1, 0, 0)$ is the only admissible family of subspaces such that

$$\varphi(M_1 \otimes \mathcal{O}(-d)) \subset 0.$$

Since $0 < \lambda < 1$, the morphism $(0, \varphi'')$ is $(G_{\text{red}}, \Lambda)$-unstable. Indeed $\lambda m_1'' = \lambda < n_1' = 0$. This contradicts the inequality $0 < \lambda < 1$. Thus $(0, \varphi'')$ must be (G, Λ)-unstable.

Thus, the results in **Case 1** together with those in **Case 2** imply the following: If $\varphi'' = 0$ then by the results of **Case 2**, φ is (G, Λ)-unstable. If φ'' divides φ' then $\begin{pmatrix} 0 & \varphi'' \end{pmatrix}$ is an element of $H\varphi$. Consequently, by the results of **Case 2**, φ is (G, Λ)-unstable.

For the converse, assume $\varphi'' \neq 0$ and φ'' does not divide φ'. Every element of $H\varphi$ is given by the matrix multiplication

$$\begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix} = \begin{pmatrix} \varphi' + \varphi''\psi & \varphi'' \end{pmatrix}$$

for $\psi \in S^{d-e}(V^\vee)$. Since φ'' does not divide φ', then $\varphi' + \varphi''\psi \neq 0$ for any ψ. The only admissible family of subspaces is $(M_1', M_2', N_1') = (0, 0, N_1), (0, M_2, N_1), (M_1, 0, N_1)$. By the results of **Case 1**, φ is (G, Λ)-stable.

$\square$

Proposition 7.2.4. *Let* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \in \mathrm{Hom}(\mathcal{O}(-d) \oplus \mathcal{O}(-e), \mathcal{O})$ *with* $d > e \geq 0$*. If* $\lambda = 0$ *then the morphism* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ *is* (G, Λ)*-semistable with* $\Lambda = (\lambda, 1-\lambda, -1)$ *if and only if* $\varphi'' \neq 0$*.*

Proof. If $\varphi'' = 0$ then the admissible family of subspaces $(0, M_2, 0)$ destabilizes. Indeed $\lambda m_1' + (1-\lambda) m_2' = 1 \leq n_1' = 0$. This is a contradiction. For the converse, assume that $\varphi'' \neq 0$. Then there are four admissible families of subspaces:

$$(M_1', N_2', N_1') = (0, 0, N_1), (0, M_2, N_1), (M_1, 0, N_1), (M_1, 0, 0).$$

Since $(m_1', m_2', n_1') = (0,0,1), (0,1,1), (1,0,1), (1,0,0)$, then $\lambda m_1' + (1-\lambda) m_2' = m_2' \leq n_1'$. Thus $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ is $(G_{\mathrm{red}}, \Lambda)$-semistable. Since $H\varphi = \left\{ \begin{pmatrix} \varphi' + \varphi''\psi & \varphi'' \end{pmatrix} | \psi \in H \right\}$, every element of $H\varphi$ has nonzero φ''. Consequently, $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ is (G, Λ)-semistable.

$\square$

Proposition 7.2.5. *Let* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \in \mathrm{Hom}(\mathcal{O}(-d) \oplus \mathcal{O}(-e), \mathcal{O})$ *with* $d > e \geq 0$*. If* $\lambda = 1$ *then the morphism* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ *is* (G, Λ)*-semistable with* $\Lambda = (\lambda, 1-\lambda, -1)$ *if and only if* φ'' *does not divide* φ'*.*

Proof. If φ'' divides φ' then $\begin{pmatrix} 0 & \varphi'' \end{pmatrix}$ is an element in $H\varphi = \left\{ \begin{pmatrix} \varphi' + \varphi''\psi & \varphi'' \end{pmatrix} | \psi \in H \right\}$. The admissible family of subspaces $(M_1, 0, 0)$ destabilizes. Indeed $\lambda m_1' + (1-\lambda) m_2' = 1 \leq n_1' = 0$. This is a contradiction. For the converse, assume that φ'' does not divide φ'. Then every element $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \in H\varphi$ has nonzero φ'. Thus there are four admissible families of subspaces:

$$(M_1', N_2', N_1') = (0, 0, N_1), (0, M_2, N_1), (M_1, 0, N_1), (0, M_2, 0).$$

Since $(m_1', m_2', n_1') = (0,0,1), (0,1,1), (1,0,1), (0,1,0)$, then $\lambda m_1' + (1-\lambda) m_2' = m_1' \leq n_1'$. Thus every element $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix} \in H\varphi$ is $(G_{\mathrm{red}}, \Lambda)$-semistable, so $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ is (G, Λ)-semistable.

$\square$

Remark 7.2.2. *A morphism* $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$ *is* (G, Λ)*-semistable*

$$\text{if and only if} \begin{cases} \varphi'' \neq 0 \quad \text{and} \quad \varphi'' \nmid \varphi' & \text{if} \quad 0 < \lambda < 1; \\ \varphi'' \neq 0 & \text{if} \quad \lambda = 0; \\ \varphi'' \nmid \varphi' & \text{if} \quad \lambda = 1; \end{cases}$$

and if and only if $\varphi \neq 0$.

Proposition 7.2.6. *If* $\Lambda = (\lambda, 1-\lambda, -1)$ *is a polarization satisfying the condition* $0 < \lambda < \frac{1}{d-e+2}$ *then* $W^{ss}(G,\Lambda)$ *admits a geometric quotient* $W^{ss}(G,\Lambda)//G$*, which is a quasi-projective variety.*

Proof. By [30, 3.3], the conditions on the polarization $\Lambda = (\lambda, 1-\lambda, -1)$ imply the inequality $0 < \lambda < \frac{1}{d-e+2}$. □

Recall that the chamber structure of holomorphic chains of type $\mathbf{t} = (1,2;0,s)$ with $s > 1$ is determined by the following inequalities:

$$\alpha_0 < \alpha_1 < \cdots < \alpha_m = 2s,$$

where $|\alpha_i - \alpha_{i-1}| = 3$ for all $i = 1, ..., m$. For a fixed total degree $s = d+e$ and $\alpha \in [\alpha_0, 2s]$, we have

$$\mathcal{M}_\alpha(\mathbf{t}) = \bigcup_{d+e=s, d\geq e} \mathcal{M}_\alpha(-d,-e),$$

where $(-d,-e)$ denotes the splitting type.

We can compare (G,Λ)-stability with α-stability.

Theorem 7.2.1. *A holomorphic chain* $(\mathcal{O}, \mathcal{O}(-d)\oplus\mathcal{O}(-e);\varphi)$ *with* $0 < e \leq d$ *is* α*-stable for some* α *if and only if* φ *is* (G,Λ)*-stable for* $\Lambda = (1-\lambda, \lambda, -1)$ *with* $0 < \lambda < 1$.

Proof. Let $\varphi = (\varphi' \ \varphi'')$. Note that the chain $\mathcal{O}(-d)\oplus\mathcal{O}(-e) \stackrel{(\varphi' \ \varphi'')}{\longrightarrow} \mathcal{O}$ can be written as

$$\mathcal{O}(-e)\oplus\mathcal{O}(-d)) \stackrel{(\varphi'' \ \varphi')}{\longrightarrow} \mathcal{O}.$$

A holomorphic chain $(\mathcal{O}, \mathcal{O}(-d)\oplus\mathcal{O}(-e);\varphi)$ is α-stable for some α if and only if $\varphi' \neq 0$ and φ' does not divide φ'' and, by Proposition 7.2.3, if and only if φ is (G,Λ)-stable for $\Lambda = (\lambda, 1-\lambda, -1)$ with $0 < \lambda < 1$. □

Theorem 7.2.2. *If* $\alpha_{i-1} < \alpha < \alpha_i$ *then there exists a unique splitting type* $(-d_i, -e_i)$ *such that we can identify*

$$\mathcal{M}_\alpha(-d_i,-e_i) = W^{ss}(G,\Lambda)//G,$$

where $W = \mathrm{Hom}(\mathcal{O}(-d_i)\oplus\mathcal{O}(-e_i), \mathcal{O})$ *and* $W^s(G,\Lambda)//G$ *is a geometric quotient for* $\Lambda = (\lambda, 1-\lambda, -1)$ *with* $0 < \lambda < \frac{1}{d_i - e_i + 2}$*. Moreover,*

$$(-d_i,-e_i) = \begin{cases} \left(\frac{-s+1}{2} - i, \frac{-s-1}{2} + i\right), & \text{if } s = d_i + e_i \text{ is odd;} \\ \left(\frac{-s}{2} - i + 1, \frac{-s}{2} + i - 1\right), & \text{if } s = d_i + e_i \text{ is even and } d_i \neq e_i. \end{cases}$$

Proof. Let $(-d,-e)$ be a splitting type with $0 < e \geq d$.

Case 1: s is odd:

If a holomorphic chain $C = (\mathcal{O}, \mathcal{O}(-d) \oplus \mathcal{O}(-e); \varphi)$ is α-semistable for $\alpha_{i-1} < \alpha < \alpha_i$ then $d - e \leq 2i - 1$. Let $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$. If $d - e = 2i - 1$ then C is α-semistable for $\alpha_{i-1} < \alpha < \alpha_i$ if and only if $\varphi' \neq 0$ and φ' does not divide φ''. This is equivalent to the condition that φ is (G, Λ)-semistable for $\Lambda = (\lambda, 1 - \lambda, -1)$ with $0 < \lambda < 1$. Since $d + e = s$, we have $(-d, -e) = \left(\frac{-s+1}{2} - i, \frac{-s-1}{2} + i\right)$, which we denote by $(-d_i, -e_i)$.

Case 2: s is even:

If a holomorphic chain $C = (\mathcal{O}, \mathcal{O}(d) \oplus \mathcal{O}(e); \varphi)$ is α-semistable for $\alpha_{i-1} < \alpha < \alpha_i$ then $d - e \leq 2i - 2$. Let $\varphi = \begin{pmatrix} \varphi' & \varphi'' \end{pmatrix}$. If $d - e = 2i - 2$ and $i \neq 1$, then C is α-semistable for $\alpha_{i-1} < \alpha < \alpha_i$ if and only if $\varphi' \neq 0$ and φ' does not divide φ''. This is equivalent to the condition that φ is (G, Λ)-semistable for $\Lambda = (\lambda, 1 - \lambda, -1)$ with $0 < \lambda < 1$. Since $d + e = s$, we have $(-d, -e) = \left(\frac{-s}{2} - i + 1, \frac{-s}{2} + i - 1\right)$, which we denote by $(-d_i, -e_i)$.

By Proposition 7.2.6, if $0 < \lambda < \frac{1}{d_i - e_i + 2}$ then $W^{ss}(G, \Lambda)$ admits the geometric quotient $W^{ss}(G, \Lambda)//G$. If $\alpha_{i-1} < \alpha < \alpha_i$ then $W^{ss}(G, \Lambda) \subseteq W$ is the open subset of α-semistable holomorphic chains of fixed splitting type $(-d_i, -e_i)$. Then we can identify $\mathcal{M}_\alpha(-d_i, -e_i) = W^{ss}(G, \Lambda)//G$. □

Corollary 7.2.1. *If the total degree s is odd and $\alpha_0 < \alpha < \alpha_1$, then $W^{ss}(G, \Lambda) \subseteq W$ is the open subset of α-semistable holomorphic chains of type $\mathbf{t} = (1, 2; 0, -s)$, and we can identify*

$$\mathcal{M}_\alpha(\mathbf{t}) = W^{ss}(G, \Lambda)//G,$$

where $\Lambda = (\lambda, 1 - \lambda, -1)$ with $0 < \lambda < \frac{1}{3}$.

Proof. Since $i = 1$, $d - e = 1$. Since $d + e = s$, we must have $(-d, -e) = (\frac{-s-1}{2}, \frac{-s+1}{2}) = (-d_1, -e_1)$. In other words, if $\alpha_0 < \alpha < \alpha_1$ then $(-d_1, -e_1)$ is the only splitting type of α-semistable chains. Hence $W^{ss}(G, \Lambda) \subseteq W$ is the open subset of α-semistable holomorphic chains of type $(1, 2; 0, -s)$. Since $d_1 - e_1 = \frac{-s+1}{2} - \frac{-s-1}{2} = 1$, there is a geometric quotient $W^{ss}(G, \Lambda)//G$ for $\Lambda = (\lambda, 1 - \lambda, -1)$ with $0 < \lambda < \frac{1}{d_1 - e_1 + 2} = \frac{1}{3}$, and we can identify $\mathcal{M}_\alpha(1, 2; 0, -s) = \mathcal{M}_\alpha(-d_1, -e_1) = W^{ss}(G, \Lambda)//G$.

□

7.3 Doran and Kirwan's method

In this section, we review the results from [11] and [22]. Doran and Kirwan developed the Geometric Invariant Theory for the linear actions of any affine algebraic groups. Let an affine algebraic group H act linearly on a complex projective variety X. If H is not reductive, then the ring of invariants is not, in general, a finitely generated algebra. So we cannot use the classical GIT.

Any affine algebraic group H contains a unipotent radical $U \trianglelefteq H$ such that H/U is a reductive group. The unipotent radical is a maximal, closed, connected unipotent subgroup.

The main step of Doran and Kirwan's method is to embed U into a reductive group G as a closed subgroup and to transfer the U-action to the induced G-action on the quasi-projective variety

$$G \times_U X,$$

which is defined by the action $u.(g,x) = (gu^{-1}, u.x)$ for $u \in U, (g,x) \in G \times X$([37, Theorem 4.19]). The action of G on $G \times_U X$ is given by the formula $g.[h,x] = [gh,x]$ for $g \in G$ and $[h,x] \in G \times_U X$.

Doran and Kirwan's method works in two steps. We define the quotients $X^{s,U}/U$, $X//U$ by transferring the U-action to a G-action of a reductive group containing U. Next, if a completion $\overline{X//U} = \overline{G \times_U X}/G$ is sufficiently canonically chosen to induce the action of H/U, then

$$\overline{X//U}//(H/U)$$

is a projective completion of the geometric quotient

$$X^{s,H}/H = (X^{s,H}/U)/(H/U),$$

where $X^{s,H} \subseteq X^{s,U}$ is the inverse image of the open subset of (H/U)-stable points by the map

$$X^{s,U} \to X^{s,U}/U \subseteq X//U \subseteq \overline{X//U}.$$

Let an affine algebraic group H act linearly on a complex projective variety X with respect to an ample line bundle L. The action of the unipotent radical $U \trianglelefteq H$ is studied in [11]. We review definitions and results.

The inclusion $\mathbb{C}[X]^U \subseteq \mathbb{C}[X]$ implies that a rational map of schemes

$$q : X \dashrightarrow \operatorname{Proj}(\mathbb{C}[X]^U),$$

where $\mathbb{C}[X]$ is the homogeneous coordinate ring $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$.

Definition 7.3.1. (*Naively semistable*) *A point x is naively semistable if there is an invariant section*

$f \in H^0(X, L^{\otimes m})$ for some $m > 0$ such that $f(x) \neq 0$. The set of naively semistable points X^{nss} is the domain of the definition of q.

Let $I = \bigcup_{m>0} H^0(X, L^{\otimes m})^U$. For $f \in I$ let $X_f \subseteq X$ be the U-invariant affine open subset where f does not vanish.

Definition 7.3.2. (*Locally trivial stable*)

$$X^{lts} = \bigcup_{f \in I^{lts}} X_f,$$

where $I^{lts} = \{f \in I | \mathbb{C}[X_f]^U$ is finitely generated and $X_f \to \mathrm{Spec}(\mathbb{C}[X_f]^U)$ is a locally trivial geometric quotient$\}$.

Definition 7.3.3. (*Finitely generated semistable*)

$$X^{ss,fg} = \bigcup_{f \in I^{fg}} X_f,$$

where $I^{fg} = \{f \in I | \mathbb{C}[X_f]^U$ is finitely generated$\}$.

Definition 7.3.4. (*Enveloping quotient*) *The enveloping quotient of X is the union*

$$X//U = \bigcup_{f \in I^{fg}} \mathrm{Spec}(\mathbb{C}[X_f]^U),$$

where $\mathbb{C}[X_f]$ is the affine coordinate ring of X_f.

Remark 7.3.1. *The image $q(X^{ss,fg})$ of the map $q : X^{ss,fg} \subseteq X^{nss} \to \mathrm{Proj}(\mathbb{C}[X]^U)$ is a dense constructible subset of the enveloping quotient $X//U$.*

Proposition 7.3.1. *[22, Proposition 3.4] The enveloping quotient $X//U$ is a quasi-projective variety with an ample line bundle which pulls back to a positive tensor power of L under the map $X^{ss,fg} \to X//U$. If $\mathbb{C}[X]^U$ is finitely generated then $X//U = \mathrm{Proj}(\mathbb{C}[X]^U)$, where $\mathbb{C}[X] = \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$.*

Now suppose that G is a complex reductive group containing U as a closed subgroup.

Definition 7.3.5. (*Mumford stable and semistable*) *Let H act on a quasi-projective variety X. Let $i : X \hookrightarrow G \times_U X$ be the closed immersion given by $x \mapsto [e, x]$. Define $X^{ms} := i^{-1}((G \times_U X)^s), X^{mss} := i^{-1}((G \times_U X)^{ss})$, where $(G \times_U X)^s((G \times_U X)^{ss})$ is the open subset of stable (semistable) points for the induced G-action (Definition 7.1.1).*

Remark 7.3.2. *[11, Lemma 5.1.7, Proposition 5.1.10] We have $X^{ms} = X^{mss}$ and $X^{ms} = X^{lts}$.*

Definition 7.3.6. (*Finite fully separating set*) *A finite separating set of invariants for the linear action of U in X is a collection of invariant sections $\{f_1, f_2, ..., f_n\}$ of positive tensor power of L such that for any two points $x, y \in X$, $f(x) = f(y)$ for all invariant sections f of $L^{\otimes k}$ and all $k > 0$ if and only if*

$$f_i(x) = f_i(y) \quad \text{for all } i = 1, 2..., n.$$

If G is any reductive group containing U, a finite separating set S of invariant sections of positive tensor power of L is a finite fully separating set of invariants for the linear U-action on X if

(i) *for every $x \in X^{ms})$ there exists $f \in S$ with associated G-invariant F over $G \times_U X$ such that $x \in (G \times_U X)_F$ and $(G \times_U X)_F$ is affine; and*

(ii) *for every $x \in X^{fg}$ there exists $f \in S$ such that $x \in X_f$ and S generates $\mathbb{C}[X_f]^U$.*

Remark 7.3.3. *This definition is independent of the choice of G(see, [11, Proposition 5.1.9]).*

Definition 7.3.7. (*Reductive envelope*) *Let X be a quasi-projective variety with a linear U-action with respect to an ample line bundle L on X. Let G be a complex reductive group containing U. Then a G-equivariant projective completion $\overline{G \times_U X}$ together with a G-linearization L' which restricts to the U-linearization L on X is a reductive envelope of the U-action on X if every U-invariant f in some finite fully separating set of invariants S for the U-action on X extends to a G-invariant section of a tensor power of L' over $\overline{G \times_U X}$.*

Definition 7.3.8. (*Fine reductive envelope and ample reductive envelope*) *If there exists such an S for which every $f \in S$ extends to a G-invariant section F over $\overline{G \times_U X}$ such that $\overline{G \times_U X}_F$ is affine then $(\overline{G \times_U X}, L')$ is called a fine reductive envelope. If L' is ample then it is called an ample reductive envelope.*

Definition 7.3.9. (*Strong reductive envelope*) *Let $D_1, D_2, ..., D_r$ denote the codimension 1 components of the boundary of $G \times_U X$ in $\overline{G \times_U X}$. If every $f \in S$ extends to a G-invariant F over $\overline{G \times_U X}$ which vanishes on each D_j, then $(\overline{G \times_U X}, L')$ is called a strong reductive envelope.*

Definition 7.3.10. (*Completely semistable and stable*) *Let X be a projective variety with a linear U-action and a reductive envelope $(\overline{G \times_U X}, L')$. Let*

$$X \overset{i}{\hookrightarrow} G \times_U X \overset{j}{\hookrightarrow} \overline{G \times_U X}.$$

$$\text{Then} \quad X^{\bar{s}} = (j \circ i)^{-1}(\overline{G \times_U X}^s), \quad X^{\bar{s}s} = (j \circ i)^{-1}(\overline{G \times_U X}^{ss}).$$

Remark 7.3.4. *[11, Remark 5.2.12] If $\overline{G \times_U X}$ is normal and $(\overline{G \times_U X}, L')$ is a fine strong reductive envelope, then $X^{\bar{s}}$ and $X^{\bar{s}s}$ are independent of the choice of $(\overline{G \times_U X}, L')$.*

Theorem 7.3.1. (*Main Theorem 1 in [11]*) *Let X be a normal projective variety with a linear U-action with respect to an ample line bundle L, for U a connected unipotent group, and let $(\overline{G \times_U X}, L')$ be any fine reductive envelope. Then*

$$X^{\overline{s}} \subseteq X^{lts} = X^{ms} = X^{mss} \subseteq X^{ss,fg} \subseteq X^{\overline{ss}} = X^{nss}.$$

The stable sets $X^{\overline{s}}$ and $X^{lts} = X^{ms} = X^{mss}$ admit quasi-projective geometric quotients given by restrictions of the composition

$$X \overset{i}{\hookrightarrow} G \times_U X \overset{j}{\hookrightarrow} \overline{G \times_U X}^{ss} \to \overline{G \times_U X}//G.$$

The enveloping quotient $X//U$ is an open subvariety of $\overline{G \times_U X}//G$ with an ample line bundle which pulls back to a positive tensor power of L under the map $X^{ss,fg} \to X//U$.

Theorem 7.3.2. (*Main Theorem 2 in [11]*) *If $\overline{G \times_U X}$ is normal and $(\overline{G \times_U X}, L')$ is a fine strong reductive envelope for the linear U-action on X, then*

$$X^{\overline{s}} = X^{lts} = X^{ms} = X^{mss} \subseteq X^{ss,fg} = X^{\overline{ss}} = X^{nss}.$$

Remark 7.3.5. *If $\overline{G \times_U X}$ is normal and $(\overline{G \times_U X}, L')$ is a fine strong reductive envelope for the linear U-action on X, then $X//U = \overline{G \times_U X}//G$.*

Definition 7.3.11. *[11, Definition 5.3.7] Let X be a complex projective variety equipped with a linear U-action. We say a point $x \in X$ is stable if $x \in X^{ms} (= X^{lts})$ and that it is semistable if $x \in X^{ss,fg}$.*

Now we review Kirwan's results about choosing reductive envelopes in [22].

A complex connected affine algebraic group H contains a unipotent radical U(a connected normal subgroup) such that the quotient group H/U is reductive. Since U has a canonical series of normal subgroups

$$\{e\} = U_0 \trianglelefteq U_1\ unlhd ... \trianglelefteq U_k = U$$

with each successive quotient isomorphic to $(\mathbb{C}^+)^r$ for some r, we hope to construct quotient successively by unipotent groups of the form $(\mathbb{C}^+)^r$ and then finally by the reductive group H/U. So Kirwan concentrated on the case $U \cong (\mathbb{C}^+)^r$.

There exits a natural embedding $U = (\mathbb{C}^+)^r \subseteq G = SL(r+1;\ mathbbC)$ as a closed subgroup. The embedding is given by the map

$$(u_1, u_2, ..., u_r) \mapsto \begin{pmatrix} 1 & \cdots & 0 & u_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & u_r \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We have an isomorphism:

$$G/U \cong \{A \in \mathbb{C}^{(r+1)\times r} | \mathrm{rk}(A) = r\},$$

where $\mathbb{C}^{(r+1)\times r}$ is the affine space of $(r+1)\times r$ matrices. The group G naturally acts on G/U by the left multiplication:

$$g.A = gA$$

for $g \in G$ and $A \in G/U$. The isomorphism is induced by the map $\pi : G \to \mathbb{C}^{(r+1)\times r}$, which is defined by deleting the last column of g for $g \in G$. The quotient variety G/U is isomorphic to a quasi-affine variety in $\mathbb{C}^{(r+1)\times r}$. The image of π is the set of $(r+1)\times r$ matrices of rank r whose complement is the determinantal variety of rank is at most $r-1$. The complement has codimension 2. It follows that $U = (\mathbb{C}^+)^r$ is a Grosshans subgroup of $G = \mathrm{SL}(r+1;\mathbb{C})$, and

$$\mathbb{C}[G]^U \cong \mathbb{C}[G/U] \cong \mathbb{C}[\mathbb{C}^{(r+1)\times r}]$$

is finitely generated [15].

Proposition 7.3.2. *[22, Section 4.1] Let X be a normal projective variety with a linear action of $U = (\mathbb{C}^+)^r$ with respect to an ample line bundle L. If the linear U-action on X extends to a linear action of $G = SL(r+1;\mathbb{C})$, then $\mathbb{C}[X]^U$ is finitely generated, and the natural completion $\overline{G \times_U X} = \mathbb{P}^{r(r+1)} \times X$ equipped with line bundle $\mathcal{O}_{\mathbb{P}^{r(r+1)}}(N) \otimes L$ is an ample strong (so fine) reductive envelope for sufficiently large $N > 0$ with $X//U = (\mathbb{P}^{r(r+1)} \times X)//G$.*

Proof. Since the U-action on X extends to an action of $G = SL(r+1;\mathbb{C})$, $G \times_U X \cong (G/U) \times X$ by the map $[g, x] \mapsto [gU, gx]$. Then by the Borel transfer theorem [12, Lemma 4.1]

$$\mathbb{C}[X]^U \cong \mathbb{C}[G \times_U X]^G \cong (\mathbb{C}[G/U] \otimes \mathbb{C}[X])^G$$

is finitely generated (see also, [16]). By [11, Lemma 5.3.14], $(\mathbb{P}^{r(r+1)} \times X, \mathcal{O}_{\mathbb{P}^{r(r+1)}}(N) \otimes L)$ is an ample strong reductive envelope for sufficiently large $N > 0$. Finally by Theorem 7.3.2, $X//U = (\mathbb{P}^{r(r+1)} \times X)//G$. $\square$

Remark 7.3.6. *In this case,* $X^s = X^{\overline{s}}$ *and* $X^{ss} = X^{\overline{ss}}$.

Kirwan constructed two more projective completions of $G \times_U X$. Let P be a parabolic subgroup

$$P = U \rtimes GL(r;\mathbb{C}) \subseteq G = SL(r+1;\mathbb{C})$$

with Levi subgroup $GL(r;\mathbb{C})$ embedded in $SL(r+1;\mathbb{C})$ as

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & \det g^{-1} \end{pmatrix}.$$

Remark 7.3.7. (i) *We have*

$$G \times_U X \cong G \times_P (P \times_U X), \quad [g,x] \mapsto [g,[e,x]]$$

where $P/U \cong GL(r;\mathbb{C})$ *and* $G/P \cong \mathbb{P}^r$.

(ii) *If the action of* U *extends to an action of* G, *then* $\overline{P \times_U X} = \overline{P/U} \times X = \mathbb{P}^{r^2} \times X$ *is the closure of* $P \times_U X$ *in* $\overline{G \times_U X} = \overline{G/U} \times X = \mathbb{P}^{r(r+1)} \times X$.

(iii) *So there is a birational morphism*

$$G \times_P \overline{P \times_U X} \to \overline{G \times_U X}, \quad [g,y] \mapsto g.y.$$

Let $\widehat{L}^{(N)}$ *be the pullback, via this birational morphism, of the line bundle* $\mathcal{O}_{\mathbb{P}^{r(r+1)}}(N) \otimes L$. *This line bundle is not ample but the tensor product* $\widehat{L}_\epsilon := \widehat{L}^{(N)} \otimes \mathcal{O}_{\mathbb{P}^r}(\epsilon)$ *is ample for sufficiently large* N, *where* $\mathcal{O}_{\mathbb{P}^r}(\epsilon)$ *is the pullback via the morphism*

$$G \times_P \overline{(P \times_U X)} \to G/P \cong \mathbb{P}^r$$

of the fractional line bundle $\mathcal{O}_{\mathbb{P}^r}(\epsilon)$.

(iv) *Let* $\widetilde{P \times_U X} = \widetilde{\mathbb{P}^{r^2}} \times X$, *where* $\widetilde{\mathbb{P}^{r^2}}$ *is the wonderful compactification of* $P/U = GL(r;\mathbb{C})$ *given by* *a blowing up of* $\mathbb{P}^{r^2}$. *There is another birational morphism*

$$G \times_P \widetilde{P \times_U X} \to G \times_P \overline{P \times_U X} \to \overline{G \times_U X}.$$

Let $\widetilde{L}^{(N)}$ *be the pullback, via this birational morphism, of the line bundle* $\mathcal{O}_{\mathbb{P}^{r(r+1)}}(N) \otimes L$ *via this birational morphism. Similarly we can define* $\widetilde{L}_\epsilon$.

Definition 7.3.12.

$$\widehat{X/\!/U} := G \times_P (\overline{P \times_U X})/\!/_{\widehat{L}_\epsilon} G, \quad \widetilde{X/\!/U} := G \times_P (\widetilde{P \times_U X})/\!/_{\widetilde{L}_\epsilon} G$$

for sufficiently small $\epsilon > 0$. *Both are projective completions of* X^s/U.

In general, the linear action of $U = (\mathbb{C}^+)^r$ does not extend to an action of $G = SL(r+1;\mathbb{C})$. In this case, Kirwan associated a projective variety Y_m containing X as an embedding. Here the index m is a positive integer. Kirwan showed that if m is sufficiently divisible and N is sufficiently large(depending on m), then the linear U-action on X has a reductive envelope $\overline{G \times_U X}^m$ which is the closure of $G \times_U X$ embedded in $\mathbb{P}^{r(r+1)} \times Y_m$ as $G(\{\iota\} \times X)$, where $\iota = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{(r+1)\times r} \subset \mathbb{P}^{r(r+1)}$.

Similarly we can define the following quotients.

Definition 7.3.13.

$$\widehat{X/\!/U}^m := G \times_P (\overline{P \times_U X}^m)/\!/_{\widehat{L}_\epsilon} G, \quad \widetilde{X/\!/U}^m := G \times_P (\widetilde{P \times_U X}^m)/\!/_{\widetilde{L}_\epsilon} G$$

for sufficiently small $\epsilon > 0$. *Both are projective completions of* X^s/U.

7.3.1 An application for holomorphic chains of type $(2,1)$ on $\mathbb{P}^1$

We apply Doran and Kirwan's method for holomorphic 2-chains of type $(2,1;s,0)$, which is the dual chain of type $(1,2;0,-s)$. The chamber structure of type $(2,1;s,0)$ coincides with that of dual type $(1,2;0,-s)$. Indeed, a chain C is $(0,\alpha)$-semistable if and only if the dual chain $C^\vee$ is $(-\alpha,0)$-semistable if and only if $C^\vee$ is $(0,\alpha)$-semistable. Hence, C is α-semistable if and only if $C^\vee$ is α-semistable.

The moduli problem of the space of homomorphisms

$$\mathcal{O} \to \mathcal{O}(d) \oplus \mathcal{O}(e), \quad d > e \geq 0$$

is equivalent to constructing a quotient by the action of the automorphism group

$$\mathrm{Aut}(\mathcal{O}) \times \mathrm{Aut}(\mathcal{O}(d) \oplus \mathcal{O}(e)) = \mathbb{C}^* \times \begin{pmatrix} \mathbb{C}^* & S^{d-e}(V^*) \\ 0 & \mathbb{C}^* \end{pmatrix} \cong \mathbb{C}^* \times [(\mathbb{C}^*)^2 \ltimes (\mathbb{C}^+)^{d-e+1}].$$

Since the center $Z = \mathbb{C}^* \times \mathbb{C}^*$ of the automorphism group acts trivially on $W = \mathbb{P}\mathrm{Hom}(\mathcal{O}, \mathcal{O}(d) \oplus \mathcal{O}(e))$, we can replace the automorphism group with $H = [\mathrm{Aut}(\mathcal{O}) \times \mathrm{Aut}(\mathcal{O}(d) \oplus \mathcal{O}(e))]/Z$. Then

$$H \cong \begin{pmatrix} \mathbb{C}^* & S^{d-e}(V^*) \\ 0 & 1 \end{pmatrix} \cong \mathbb{C}^* \ltimes (\mathbb{C}^+)^{d-e+1}.$$

Let $r = d - e + 1$. Then $H \cong \mathbb{C}^* \ltimes (\mathbb{C}^+)^r$, whose unipotent radical is $U = (\mathbb{C}^+)^r$. We modify the natural embedding $U \subset G = SL(r+1; \mathbb{C})$ from the previous section. We use the following embedding:

$$(u_1, u_2, ..., u_r) \mapsto \begin{pmatrix} 1 & u_1 & \cdots & u_r \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then we have an isomorphism as in the previous section:

$$G/U \cong \{A \in \mathbb{C}^{r \times (r+1)} | \mathrm{rk}(A) = r\},$$

where $\mathbb{C}^{r \times (r+1)}$ is the affine space of $r \times (r+1)$ matrices. The isomorphism is given by the map

$$\pi : G = SL(r+1; \mathbb{C}) \to \mathbb{C}^{r \times (r+1)},$$

which is defined by deleting the first row of g for $g \in G$. The image $\pi(U)$ is the $r \times (r+1)$ matrix ι, where

$$\iota = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The group G naturally acts on G/U by right multiplication of the transpose of g:

$$g.A = Ag^t$$

for $g \in G$ and $A \in G/U$. The quotient space G/U is isomorphic to a quasi-affine variety in $\mathbb{C}^{r\times(r+1)}$. The quasi-affine variety is the complement of the determinantal variety of rank at most $r-1$. The codimension of the determinantal variety is 2. It follows that $U = (\mathbb{C}^+)^r$ is a Grosshans subgroup of $G = SL(r+1;\mathbb{C})$, and

$$\mathbb{C}[G]^U \cong \mathbb{C}[G/U] \cong \mathbb{C}[\mathbb{C}^{r\times(r+1)}]$$

is finitely generated [15].

There is a natural embedding

$$U = (\mathbb{C}^+)^r \subset SL(r+1;\mathbb{C}) \subset GL(r+1;\mathbb{C})$$

as a closed subvariety. The action of U on W does not extend to the action of $SL(r+1;\mathbb{C})$. We can naturally embed W into a projective variety $\mathbf{W}$ on which the action of U naturally extends to an action of $SL(r+1;\mathbb{C})$.

The natural embedding is the following:

$$W = \mathbb{P}\begin{pmatrix} S^d(V^*) \\ S^e(V^*) \end{pmatrix} \hookrightarrow \mathbf{W} := \mathbb{P}\begin{pmatrix} S^d(V^*) \\ \vdots \\ S^d(V^*) \end{pmatrix}, \quad \begin{pmatrix} s \\ s' \end{pmatrix} \mapsto \begin{pmatrix} s \\ x^{r-1}s' \\ x^{r-2}ys' \\ \vdots \\ y^{r-1}s' \end{pmatrix},$$

where the projective coordinates x and y form a basis for V^*.

Then the $U = (\mathbb{C}^+)^r$ action on W extends to a natural $G = SL(r+1;\mathbb{C})$ action on $\mathbf{W}$, which is given by

$$g.w = \begin{pmatrix} g_{11} & \cdots & g_{1,r+1} \\ \vdots & \ddots & \vdots \\ g_{1,r+1} & \cdots & g_{r+1,r+1} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{r+1} \end{pmatrix}$$

for $g = (g_{ij}) \in G$ and $w = (w_i)^t \in \mathbf{W}$. Thus we have the sequence of embedding

$$W \hookrightarrow \mathbf{W} \hookrightarrow G \times_U \mathbf{W} \cong G/U \times \mathbf{W} \hookrightarrow \overline{G/U} \times \mathbf{W} \cong \mathbb{P}^{r(r+1)} \times \mathbf{W}$$

with

$$\mathbf{W}/\!/U = (\mathbb{P}^{r(r+1)} \times \mathbf{W})/\!/G \text{ and } \widetilde{\mathbf{W}/\!/U} = (G \times_P (\widetilde{\mathbb{P}^{r^2}} \times \mathbf{W}))/\!/G,$$

where $\mathbb{P}^{r(r+1)} = \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^{r\times(r+1)})$ and $\mathbb{P}^{r^2} = \mathbb{P}(\mathbb{C} \oplus \mathbb{C}^{r\times r})$. Here the linearization on $\mathbb{P}^{r(r+1)} \times \mathbf{W}$ is $\mathcal{O}_{\mathbb{P}^{r(r+1)}}(N) \otimes \mathcal{O}_{\mathbf{W}}(1)$ for $N \gg 0$, and the linearization on $\mathbb{P}^{r^2} \times \mathbf{W}$ is $\mathcal{O}_{\mathbb{P}^{r^2}}(N) \otimes \mathcal{O}_{\mathbf{W}}(1)$.

Remark 7.3.8.

$$\mathbb{P}^{r(r+1)} = \{[a_0 : A] | a_0 \in \mathbb{C}, A \textit{ is a } r \times (r+1) \textit{ matrix}\}$$

is a completion of the affine space $\mathbb{C}^{r\times(r+1)}$ *of* $r \times (r+1)$ *matrices.*

We identify the semistable points for the action of G on $\mathbb{P}^{r(r+1)} \times \mathbf{W}$. We write

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss,G} = \bigsqcup_{0\le q\le r} (\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss,G}_q,$$

where $(\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss,G}_q = \{[a_0 : A] \in (\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss,G} : \mathrm{rk}(A) = q\}$.

The following proposition is similar to the results in [22, Section 5.1].

Proposition 7.3.3.

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss,G}_q = \bigsqcup_{\iota_q} G \times_{U_q} (\{[1 : \iota_q]\} \times \mathbf{W}^{ss,q}),$$

where $\iota_q \in \mathbb{C}^{r\times(r+1)}$*, is the unique representative in each* G *orbit of* A *with* $\mathrm{rk}(A) = q$ *such that its first* q

columns from the second column are linearly independent and all other columns are zero, i.e.,

$$\iota_q = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1,q+1} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2,q+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{r2} & \cdots & a_{r,q+1} & 0 & \cdots & 0 \end{pmatrix};$$

and the subgroup $U_q \subset SL(r+1)$ is the stabilizer of ι_q,i.e.,

$$U_q = \left\{ g \in G \middle| g = \begin{pmatrix} g_{11} & 0 & \cdots & 0 & g_{1,q+1} & \cdots & g_{1,r+1} \\ g_{21} & 1 & \cdots & 0 & g_{2,q+1} & \cdots & g_{2,r+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{q+1,1} & 0 & \cdots & 1 & g_{q+1,q+1} & \cdots & g_{r+1,q+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{r+1,1} & 0 & \cdots & 0 & g_{r+1,q+1} & \cdots & g_{r+1,r+1} \end{pmatrix} \right\};$$

and

$$\mathbf{W}^{ss,q} = \{w \in \mathbf{W} : uw \in \mathbf{W}^{ss,T_q} \textit{ for all } u \in U_q\}.$$

Here $T_q = \{\operatorname{diag}(t_i) \in T | t_2 = ... = t_{q+1} = 1\}$ is a subtorus of the standard maximal torus T of $G = SL(r+1;\mathbb{C})$.

In particular,

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_0^{ss,G} = \{[1 : \mathbf{0}_{r\times(r+1)}]\} \times \mathbf{W}^{ss,0} = \{[1 : \mathbf{0}_{r\times(r+1)}]\} \times \mathbf{W}^{ss,G}$$

and

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_r^{ss,G} = G \times_{U^t} (\{[1 : \iota]\} \times \mathbf{W}^{ss,r}),$$

where $\iota := \iota_r = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ *and $U_r = U^t = \{u^t | u \in U\}$.*

Proof. Let $T \subset G = SL(r+1;\mathbb{C})$ be the standard maximal torus. The weights of the action of T on $\mathbb{P}^{r(r+1)}$ with respect to $\mathcal{O}_{\mathbb{P}^{r(r+1)}}(1)$ are

$$0,\ \chi_1,\ ...,\ \chi_r,\ \chi_{r+1} = -\chi_1 - \chi_2 - ... - \chi_r,$$

where $\chi_1,\ \ldots,\ \chi_{r+1}$ are the standard representations of $G = SL(r+1;\mathbb{C})$ on $\mathbb{C}^{r+1}$. The weights of the action of T on $\mathbf{W}$ with respect to $\mathcal{O}_{\mathbf{W}}(1)$ are

$$\chi_1,\ \ldots,\ \chi_r,\ \chi_{r+1} = -\chi_1 - \chi_2 - \ldots - \chi_r.$$

If $N \gg 0$ then we have $a_0 \neq 0$ for any semistable element $([a_0 : A], w)$. We may assume that $a_0 = 1$.

By the Hilbert-Mumford criteria, $([1 : A], w)$ is semistable if and only if $(g.[1 : A], g.w)$ is T-semistable for all $g \in G$, and $([1 : A], w)$ is T-semistable if and only if 0 lies in the convex hull of the set of weights

$$\{\chi_i : w_i \neq 0\} \cup \{N\chi_1 + \chi_i : w_i \neq 0 \text{ and } A_1 \neq 0\} \cup \ldots \cup \{N\chi_{r+1} + \chi_i : w_i \neq 0 \text{ and } A_{r+1} \neq 0\},$$

where A_i is the i-th column of A. The result follows. □

Remark 7.3.9. *The abelian group U is isomorphic to U^t by the map $u \mapsto u$ for $u \in U$.*

Now we consider the action of $H = U \rtimes \mathbb{C}^*$ on W, where $\mathbb{C}^*$ acts by conjugation on $\mathrm{Lie}(U) = \mathbb{C}^r$ with weights all equal to 1. The action of $\mathbb{C}^*$ on W is given by

$$t.\begin{pmatrix} s \\ s' \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s \\ s' \end{pmatrix} = \begin{pmatrix} ts \\ s' \end{pmatrix}$$

for $t \in \mathbb{C}^*$ and $\begin{pmatrix} s \\ s' \end{pmatrix} \in W$.

The action of H on W extends to a linear action on $\mathbf{W}$, which is the restriction of the $GL(r+1;\mathbb{C})$ action on $\mathbf{W}$ via the embedding of H in $GL(r+1;\mathbb{C})$. This is given by

$$(u_1, \ldots, u_r, t) \mapsto \begin{pmatrix} t & u_1 & \cdots & u_r \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

If we twist this action by δ times the standard character of $\mathbb{C}^*$, then we get a fractional linearization of the action of H on $\mathbf{W}$ which extends the fractional linearization $\mathcal{L}_\delta$ on W. We also get an action of $\mathbb{C}^* = H/U$ on $\mathbb{P}^{r(r+1)} = \overline{G/U}$ via

$$t.[a_0 : A] \mapsto [a_0 : tA].$$

Let $\mathbb{C}^*$ be a one-parameter subgroup in $GL(r+1;\mathbb{C})$ and let T be the standard maximal torus in $G = SL(r+1;\mathbb{C})$. Then $\mathbb{C}^*T \cong \mathbb{C}^* \times T$ is the maximal torus of $GL(r+1;\mathbb{C})$. Indeed, the isomorphism $\mathbb{C}^* \times T \to \mathbb{C}^*T$ is given by

$$(t, \text{diag}(\tau_1, \tau_2, ..., \tau_{r+1})) \mapsto \text{diag}(t\tau_1, \tau_2, ..., \tau_{r+1})$$

for $t \in \mathbb{C}^*$ and $\text{diag}(\tau_1, \tau_2, ..., \tau_{r+1}) \in T$. Thus the action of $GL(r+1, \mathbb{C})$ on $\mathbb{P}^{r(r+1)} \times \mathbf{W}$ is given by

$$g.([a_0 : A], w) = (g.[a_0 : A], g.w) = ([a_0 : \det(g)Ag^t], \det(g)^\delta gw)$$

for $g \in GL(r+1;\mathbb{C})$ and $([a_0 : A], w) = \mathbb{P}^{r(r+1)} \times \mathbf{W}$.

Remark 7.3.10. *[12, Theorem 7.2] Let $GL(r;\mathbb{C})$ act on a normal projective variety. Since the Picard group* $\text{Pic}(GL(r;\mathbb{C}))$ *is trivial, the linearization of a fixed line bundle is precisely determined by the group of characters. The only character of the group $GL(r;\mathbb{C})$ is the power of the determinant character. Thus, for a fixed line bundle, the linearization on the line bundle is determined by the power of the determinant character.*

Lemma 7.3.1.

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss, GL(r+1;\mathbb{C}), \delta} = \emptyset$$

unless $-\frac{r+3}{r+1} \leq \delta \leq -\frac{1}{r+1}$.

Proof. Let $\text{diag}(t_1, ..., t_{r+1}) \in \mathbb{C}^*T \subset GL(r+1;\mathbb{C})$. Then $t = t_1 ... t_{r+1}$. The weights of the action of $\mathbb{C}^*T$ on $\mathbb{P}^{r(r+1)}$ are

$$0, \quad 2\chi_1 + \chi_2 + ... + \chi_{r+1}, \quad \chi_1 + 2\chi_2 + ... + \chi_{r+1}, \quad ... \quad , \chi_1 + \chi_2 + ... + 2\chi_{r+1},$$

and the fractional weights of the action of $\mathbb{C}^*T$ on $\mathbf{W}$ are

$$\chi_1 + \delta(\chi_1 + ... + \chi_{r+1}), \quad \chi_2 + \delta(\chi_1 + ... + \chi_{r+1}), \quad ... \quad , \chi_{r+1} + \delta(\chi_1 + ... + \chi_{r+1}),$$

where $\chi_1, ..., \chi_{r+1}$ are the standard representations of $GL(r+1;\mathbb{C})$ on $\mathbb{C}^{r+1}$.

The convex hull lies between the following two hyperplanes in $\mathbb{R}^{r+1}$:

$$x_1 + ... + x_{r+1} = (r+1)\delta + 1, \quad x_1 + ... + x_{r+1} = (r+1)\delta + r + 3.$$

They intersect with the line $\{\lambda(1, ..., 1) \in \mathbb{R}^{r+1} : \lambda \in \mathbb{R}\}$ at $a(\delta) = \delta + \frac{1}{r+1}$ and $b(\delta) = \delta + \frac{r+3}{r+1}$, respectively.

Consider the line segment

$$(1-\lambda)(a(\delta), ..., a(\delta)) + \lambda(b(\delta), ..., b(\delta))$$

for $0 \le \lambda \le 1$, which lies between the two hyperplanes. Hence, the convex hull cannot contain the origin unless

$$a(\delta) \le 0 \le b(\delta).$$

This implies that $-\frac{r+3}{r+1} \le \delta \le -\frac{1}{r+1}$.

□

We also write

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_q^{ss,GL(r+1;\mathbb{C}),\delta} = \{([1:A], w) \in (\mathbb{P}^{r(r+1)} \times \mathbf{W})^{ss,GL(r+1;\mathbb{C}),\delta} | \mathrm{rk}(A) = q\}, \quad 0 \le q \le r.$$

Lemma 7.3.2. *If* $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$, *then*

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_q^{ss,GL(r+1;\mathbb{C}),\delta} = \emptyset$$

for all $q < r$ *and if* $q = r$, *then*

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_r^{ss,GL(r+1;\mathbb{C}),\delta} \cong GL(r+1;\mathbb{C}) \times_H \mathbf{W}',$$

where

$$\mathbf{W}' = \{w \in \mathbf{W} | \text{ the entries of } w \text{ are linearly independent } \}.$$

Proof. The $2r$ weights

$$\chi_1 + \delta \sum_{i=1}^{r+1} \chi_i, \; \dots, \chi_r + \delta \sum_{i=1}^{r+1} \chi_i, \; \chi_1 + (\delta+1) \sum_{i=1}^{r+1} \chi_i + \chi_{r+1}, \; \dots, \chi_r + (\delta+1) \sum_{i=1}^{r+1} \chi_i + \chi_{r+1}$$

determine a unique plane

$$x_1 + ... + x_r - \frac{r}{2} x_{r+1} = \frac{r\delta}{2} + 1.$$

which intersects with the line $\{\lambda(1, ..., 1) \in \mathbb{R}^{r+1} : \lambda \in \mathbb{R}\}$ at $c(\delta) = \delta + \frac{2}{r}$. Then $a(\delta) \le c(\delta) \le b(\delta)$. If $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$ then $c(\delta) < 0 < b(\delta)$. Let $x = ([1:A], w) \in \mathbb{P}^{r(r+1)} \times \mathbf{W}$. If $\mathrm{rk}(A) < r$, then an element $g_1.A = Ag_1^t$ has at most $r-1$ nonzero columns for some $g_1 \in GL(r+1;\mathbb{C})$. Then x is unstable for $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$. If the entries of $w \in \mathbf{W}$ are linearly dependent then an element $g_2.w = g_2 w$ has at most

$r-1$ nonzero entries for some $g_2 \in GL(r+1;\mathbb{C})$. Then x is unstable for $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$. Thus

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_q^{ss,GL(r+1;\mathbb{C}),\delta} = \begin{cases} \emptyset & \text{if } q < r, \\ (\mathbb{P}^{r(r+1)} \times \mathbf{W})_r^{ss,GL(r+1;\mathbb{C}),\delta} \cong GL(r+1;\mathbb{C}) \times_H \mathbf{W}' & \text{if } q = r, \end{cases}$$

where

$$\mathbf{W}' = \{w \in \mathbf{W} | \text{ the entries of } w \text{ are linearly independent } \}.$$

Indeed if $q = r$, then

$$(\mathbb{P}^{r(r+1)} \times \mathbf{W})_r^{ss,GL(r+1;\mathbb{C}),\delta} \cong (GL(r+1;\mathbb{C})/H) \times \mathbf{W}' \cong GL(r+1;\mathbb{C}) \times_H \mathbf{W}'.$$

□

Proposition 7.3.4. *If $-1 < \delta < -\frac{2}{r}$ with $r > 2$, then the ring of invariants $\mathbb{C}[W]^{H,\delta}$ is finitely generated, and $W//_\delta H = \mathrm{Proj}(\mathbb{C}[W]^{H,\delta}) = W^{ss,H,\delta}/\sim_H$, where $W^{ss,H,\delta} = \{w \in W | uw \in W^{ss,\delta,\mathbf{C}^*}$ for all $u \in U\}$.*

Proof. If $-1 < \delta < -\frac{2}{r}$ with $r > 2$, then $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$. Since

$$\begin{aligned} \mathbb{C}[\mathbf{W}]^{H,\delta} &= \mathbb{C}[GL(r+1;\mathbb{C}) \times_H \mathbf{W}]^{GL(r+1;\mathbb{C}),\delta} \\ &= \mathbb{C}[\overline{GL(r+1;\mathbb{C}) \times_H \mathbf{W}}]^{GL(r+1;\mathbb{C}),\delta} \\ &= \mathbb{C}[\mathbb{P}^{r(r+1)} \times \mathbf{W}]^{GL(r+1;\mathbb{C}),\delta}, \end{aligned}$$

then $\mathbb{C}[\mathbf{W}]^{H,\delta}$ is finitely generated, and hence, by the naturality property[22, Proposition 4.17], the invariant ring $\mathbb{C}[W]^{H,\delta}$ is the quotient ring of $\mathbb{C}[\mathbf{W}]^{H,\delta}$, and it is also finitely generated. Recall that the embedding $W \hookrightarrow \mathbf{W}$ is given by

$$\begin{pmatrix} s & s' \end{pmatrix}^t \mapsto \begin{pmatrix} s & x^{r-1}s' & x^{r-2}ys' & \cdots & y^{r-1}s' \end{pmatrix}^t$$

for $\begin{pmatrix} s & s' \end{pmatrix}^t \in W$, where x and y form projective coordinates for $\mathbb{P}^1$. Then

$$\begin{aligned} &\begin{pmatrix} s & s' \end{pmatrix}^t \in W^{ss,\delta,H} \quad \text{for} \quad -1 < \delta < -\frac{2}{r} \\ \Leftrightarrow\, & s' \neq 0 \quad \text{and} \quad s' \text{ does not divide } s \\ \Leftrightarrow\, & \begin{pmatrix} s & x^{r-1}s' & x^{r-2}ys' & \cdots & y^{r-1}s' \end{pmatrix}^t \in \mathbf{W}'. \end{aligned}$$

Hence $W//_\delta H = \operatorname{Proj}(\mathbb{C}[W]^{H,\delta}) = W^{ss,H,\delta}/\sim_H$. □

Recall that the chamber structure of holomorphic chains of type $\mathbf{t} = (2, 1; s, 0)$ with $s > 0$ is determined by the following inequalities:

$$\alpha_0 < \alpha_1 < \cdots < \alpha_m = 2s,$$

where $\alpha_i - \alpha_{i-1} = 3$ for all $i = 1, ..., m$. For a fixed total degree $s = d + e$ and $\alpha \in [\alpha_0, 2s]$, we have

$$\mathcal{M}_\alpha(\mathbf{t}) = \bigcup_{d+e=s} \mathcal{M}_\alpha(d, e),$$

where $\mathcal{M}_\alpha(d, e)$ is the subspace of fixed splitting type $((d, e), 0)$.

We can compare Kirwan's δ-stability with α-stability.

Theorem 7.3.3. *If $\alpha_{i-1} < \alpha < \alpha_i$ with $i > 1$, then there exists a unique splitting type $((d_i, e_i), 0)$ such that $r = d_i - e_i + 1 > 2$ and we can identify*

$$\mathcal{M}_\alpha(d_i, e_i) = W//_\delta H \tag{7.1}$$

for $-1 < \delta < -\frac{2}{r}$, where $W = \operatorname{Hom}(\mathcal{O}, \mathcal{O}(d_i) \oplus \mathcal{O}(e_i))$. Moreover,

$$(d_i, e_i) = \begin{cases} \left(\frac{s-1}{2} + i, \frac{s+1}{2} - i\right), & \text{if } s = d_i + e_i \text{ is odd;} \\ \left(\frac{s}{2} + i - 1, \frac{s}{2} - i + 1\right), & \text{if } s = d_i + e_i \text{ is even and } d_i \neq e_i. \end{cases}$$

Proof. Let $((d, e), 0)$ be a splitting type with $d \geq e$.

Case 1, s is odd:

If a holomorphic chain $C = (\mathcal{O}(d) \oplus \mathcal{O}(e), \mathcal{O}; \phi)$ is α-semistable for $\alpha_{i-1} < \alpha < \alpha_i$, then $d - e \leq 2i - 1$. If $d - e = 2i - 1$ then C is α-semistable if and only if $[\phi] \in W^{H,\delta}$ for $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$. Since $d + e = s$, we have $(d, e) = \left(\frac{s-1}{2} + i, \frac{s+1}{2} - i\right)$ which we denote by (d_i, e_i).

Case 2, s is even:

If $C = (\mathcal{O}(d) \oplus \mathcal{O}(e), \mathcal{O}; \phi)$ is α-semistable for $\alpha_{i-1} < \alpha < \alpha_i$ then $d - e \leq 2i - 2$. If $d - e = 2i - 2$ and $i \neq 1$, then the chain C is α-semistable if and only if $[\phi] \in W^{H,\delta}$ for $-\frac{r+3}{r+1} < \delta < -\frac{2}{r}$. Since $d + e = s$, we have $(d, e) = \left(\frac{s}{2} + i - 1, \frac{s}{2} - i + 1\right)$ which we denote by (d_i, e_i).

If $i > 1$ then $r = d_i - e_i + 1 = 2i > 2$ (s odd) or $2i - 1 > 2$ (s even), and thus $W^{ss,H,\delta}$ is nonempty. By Proposition 7.3.4, if $-1 < \delta < -\frac{2}{r}$, then $W^{ss,H,\delta}$ admits a GIT quotient $W//_\delta H$. If $\alpha_{i-1} < \alpha < \alpha_i$, then $W^{ss,H,\delta} \subseteq W$ is the open subset of α-semistable holomorphic chains of fixed splitting type (d_i, e_i).

Consequently, we can identify $\mathcal{M}_\alpha(d_i, e_i) = W/\!/_\delta H$.

□

7.3.2 A symplectic description

Kirwn's non-reductive quotient is a successive two reductive quotients. The following is in [22, Section 5.4]. Let a non-reductive complex affine algebraic group H act linearly on a complex projective variety X. H has a nontrivial unipotent radical U and $H = U \rtimes (H/U)$. H/U is isomorphic to a reductive subgroup of H. In Kirwan's successive quotient, the first quotient is $\widetilde{X/\!/U} = \widetilde{G \times_U X}/\!/G$, where G is a reductive group containing U and $\widetilde{G \times_U X}$ is a blowup of a completion of a quasi-affine variety $G \times_U X$. There are 'moment-map-like' descriptions of the quotients $\widetilde{X/\!/U} = \widetilde{G \times_U X}/\!/G$ and $\widetilde{X/\!/U}/\!/(H/U)$. In our case $H = U \rtimes \mathbb{C}^*$ with $U = (\mathbb{C}^+)^r$.

We apply the symplectic quotient to holomorphic chains of type $(2,1)$. Since

$$W/\!/_\delta H \subseteq \mathbf{W}/\!/_\delta H = (\mathbb{P}^{r(r+1)} \times \mathbf{W})/\!/_\delta GL(r+1;\mathbb{C}),$$

there is a moment map

$$\mu_{U(r+1)} : \mathbb{P}^{r(r+1)} \times \mathbf{W} \to \mathrm{Lie}(U(r+1))^*,$$

which induces another 'moment map' $\mu_H : W \to \mathrm{Lie}(H)^*$. The map μ_H is defined by

$$\mu_H(w).a = \mathrm{re}\left(\frac{\overline{\widehat{w}}^t \rho_*(a)\widehat{w}}{2\pi i \|\widehat{w}\|^2}\right)$$

for all $a \in \mathrm{Lie}(H)$, where $\widehat{w} \in \mathbb{C}^{d+e+2} \setminus \{0\}$ is a representative for $w \in W = \mathbb{P}^{d+e+1}$.

Proposition 7.3.5. *We can identify*

$$W/\!/_\delta H = \frac{\mu_H^{-1}(-\delta)}{S^1},$$

where S^1 is a maximal compact subgroup of H.

Proof. The proof is the same as the one in [22, Section 5.4]. □

Let $[\phi] \in W = \mathbb{P}\mathrm{Hom}(\mathcal{O}, \mathcal{O}(d) \oplus \mathcal{O}(e)) \cong \mathbb{P}^{d+e+1}$ with $d > e$. We write $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$, where

$\phi_1 \in S^d(V^*)$ and $\phi_2 \in S^e(V^*)$. The ϕ_1 and ϕ_2 can be represented by homogeneous polynomials:

$$\phi_1 = a_0x^d + a_1x^{d-1}y + ... + a_dy^d,$$
$$\phi_2 = b_0x^e + b_1x^{e-1}y + ... + b_ey^e,$$

where x and y form a basis of V^*. Here the coefficients of the homogeneous polynomials form the projective coordinates of the projective space W, i.e., $[a_0 : ... : a_d : b_0 : ... : b_e] \in W$.

The action of $H = \mathrm{Aut}(\mathcal{O}(d) \oplus \mathcal{O}(e)) \cong (\mathbb{C}^+)^{d-e+1} \rtimes \mathbb{C}^*$ on W is given by

$$(u_0, ..., u_{d-e}, t).(a_0, ..., a_d, b_0, ..., b_e) = (ta_0+u_0b_0, ta_1+u_0b_1+u_1b_0, ..., ta_j+\sum_{i+k=j} u_ib_k, ..., ta_d+u_{d-e}b_e, b_0, ..., b_e)$$

for $(u_0, ..., u_{d-e}, t) \in H$ and $(a_0, ..., a_d, b_0, ..., b_e) \in W$. Hence the H-action on W is given by the representation:

$$\rho : H \to GL(d+e+2; \mathbb{C}), \quad (u_0, ..., u_{d-e}, t) \mapsto \begin{pmatrix} tI_{d+1} & A \\ 0_{(e+1)\times(d+1)} & I_{e+1} \end{pmatrix},$$

where I_{d+1} and I_{e+1} are identity matrices of size $d+1$ and $e+1$ respectively, $0_{(e+1)\times(d+1)}$ is the $(e+1)\times(d+1)$ zero matrix, and A is the $(d+1) \times (e+1)$ matrix with entries $A_{ii} = u_0, A_{i,i+1} = u_1, ..., A_{i,i+d-e} = u_{d-e}$ for $1 \leq i \leq e+1$.

The representation ρ induces the linear map:

$$\rho_* : \mathrm{Lie}(H) \to \mathrm{Lie}(GL(d+e+2; \mathbb{C})), \quad (p_0, ..., p_{d-e}, q) \mapsto \begin{pmatrix} qI_{d+1} & B \\ 0_{(e+1)\times(d+1)} & 0_{(e+1)\times(e+1)} \end{pmatrix},$$

where $B_{ii} = p_0, B_{i,i+1} = p_1, ..., B_{i,i+d-e} = p_{d-e}$ for $1 \leq i \leq e+1$.

We replace ρ with $\widetilde{\rho} = t^\delta \rho$. Then $\widetilde{\rho}_* = \delta I_{d+e+2} + \rho_*$ such that

$$\widetilde{\rho}_*(p_0, ..., p_{d-e}, q) = \delta q I_{d+e+2} + \rho_*(p_0, ..., p_{d-e}, q).$$

The twisted representation $\widetilde{\rho}$ induces a moment map:

$$\begin{aligned}
\widetilde{\mu}(w).h &= \mathrm{re}\left(\frac{\overline{\hat{w}}^t\widetilde{\rho}_*(h)\hat{w}}{2\pi i\|\hat{w}\|^2}\right)\\
&= \mathrm{re}\left(\frac{\overline{\hat{w}}^t(\delta q I_{d+e+2}+\rho_*(h))\hat{w}}{2\pi i\|\hat{w}\|^2}\right)\\
&= \mathrm{re}\left(\frac{\overline{\hat{w}}^t(\delta q I_{d+e+2})\hat{w}+\overline{\hat{w}}^t(\rho_*(h))\hat{w}}{2\pi i\|\hat{w}\|^2}\right)\\
&= \mathrm{re}\left(\frac{\delta q\|\hat{w}\|^2+\overline{\hat{w}}^t\rho_*(h)\hat{w}}{2\pi i\|\hat{w}\|^2}\right)\\
&= \mathrm{re}\left(\frac{\delta q}{2\pi i}\right)+\mathrm{re}\left(\frac{\overline{\hat{w}}^t\rho_*(h)\hat{w}}{2\pi i\|\hat{w}\|^2}\right)\\
&= \delta\mathrm{re}\left(\frac{q}{2\pi i}\right)+\mu(w).h
\end{aligned}$$

for $h=(p_0,...,p_{d-e},q)\in\mathrm{Lie}(H)$ and $w\in W$.

Hence $w\in\widetilde{\mu}^{-1}(0)$ if and only if $w\in\mu^{-1}(-\delta)$.

The inverse image $\mu^{-1}(-\delta)\subset W=\mathbb{P}^{d+e+1}$ is given by the equation

$$\delta\mathrm{re}\left(\frac{q}{2\pi i}\right)+\mu(w).h=0$$

for all $h=(p_0,...,p_{d-e},q)\in\mathrm{Lie}(H)$.

Lemma 7.3.3. *The inverse image $\mu^{-1}(-\delta)=\{w\in W|\mu(w)=\delta\}\subset W=\mathbb{P}^{d+e+1}$ is given by the $d-e+2$ equations*

$$\begin{aligned}
(\delta+1)\|\phi_1\|^2+\delta\|\phi_2\|^2&=0,\\
\overline{a_0}b_0+...+\overline{a_e}b_e&=0,\\
\overline{a_1}b_0+...+\overline{a_{e+1}}b_e&=0,\\
&\vdots\\
\overline{a_{d-e}}b_0+...+\overline{a_d}b_e&=0,
\end{aligned}$$

where $\|\phi_1\|^2=\sum_{i=0}^d|a_i|^2$, $\|\phi_2\|^2=\sum_{j=0}^e|b_j|^2$ and $\hat{w}=(\phi_1,\phi_2)=(a_0,a_1,...,a_d,b_0,b_1,...,b_e)$.

Proof.

$$
\begin{aligned}
&\delta \text{re}\left(\frac{q}{2\pi i}\right) + \mu(w).h = 0 \\
\Leftrightarrow\ &\delta \text{re}\left(\frac{q}{2\pi i}\right) + \text{re}\left(\frac{\overline{\hat{w}}^t \rho_*(h)\hat{w}}{2\pi i\|\hat{w}\|^2}\right) = 0 \\
\Leftrightarrow\ &\delta \frac{\text{Im}(q)}{2\pi} + \frac{\text{Im}\left(\overline{\hat{w}}^t \rho_*(h)\hat{w}\right)}{2\pi\|\hat{w}\|^2} = 0 \\
\Leftrightarrow\ &\delta\|\hat{w}\|^2 \text{Im}(q) + \text{Im}\left(\overline{\hat{w}}^t \rho_*(h)\hat{w}\right) = 0 \\
\Leftrightarrow\ &\delta\|\hat{w}\|^2 \text{Im}(q) + \text{Im}\left(q\sum_{i=0}^{d}|a_i|^2 + p_0\sum_{i=0}^{e}\overline{a_i}b_i + ... + p_{d-e}\sum_{i=d-e}^{d}\overline{a_i}b_i\right) = 0 \\
\Leftrightarrow\ &\delta\|\hat{w}\|^2 \text{Im}(q) + \text{Im}(q)\sum_{i=0}^{d}|a_i|^2 + \text{Im}\left(p_0\sum_{i=0}^{e}\overline{a_i}b_i + ... + p_{d-e}\sum_{i=d-e}^{d}\overline{a_i}b_i\right) = 0 \\
\Leftrightarrow\ &\delta(\|\phi_1\|^2 + \|\phi_2\|^2)\text{Im}(q) + \text{Im}(q)\|\phi_1\|^2 + +\text{Im}\left(p_0\sum_{i=0}^{e}\overline{a_i}b_i + ... + p_{d-e}\sum_{i=d-e}^{d}\overline{a_i}b_i\right) = 0 \\
\Leftrightarrow\ &((\delta+1)\|\phi_1\|^2 + \delta\|\phi_2\|^2)\text{Im}(q) + \text{Im}\left(p_0\sum_{i=0}^{e}\overline{a_i}b_i + ... + p_{d-e}\sum_{i=d-e}^{d}\overline{a_i}b_i\right) = 0 \\
\Leftrightarrow\ &((\delta+1)\|\phi_1\|^2 + \delta\|\phi_2\|^2)\text{Im}(q) + \text{Im}\left(p_0\sum_{i=0}^{e}\overline{a_i}b_i\right) + ... + \text{Im}\left(p_{d-e}\sum_{i=d-e}^{d}\overline{a_i}b_i\right) = 0 \\
\Leftrightarrow\ &((\delta+1)\|\phi_1\|^2 + \delta\|\phi_2\|^2)\text{Im}(q) + \text{Re}(p_0)\text{Im}\left(\sum_{i=0}^{e}\overline{a_i}b_i\right) + \text{Im}(p_0)\text{Re}\left(\sum_{i=0}^{e}\overline{a_i}b_i\right) + ...+ \\
&+ \text{Re}(p_{d-e})\text{Im}\left(\sum_{i=d-e}^{d}\overline{a_i}b_i\right) + \text{Im}(p_{d-e})\text{Re}\left(\sum_{i=d-e}^{d}\overline{a_i}b_i\right) = 0.
\end{aligned}
$$

Since the last equation is satisfied for all $h = (p_0, ..., p_{d-e}, q)$, then the inverse image $\mu^{-1}(-\delta)$ is the cut locus of the following $d - e + 2$ equations

$$
(\delta+1)\|\phi_1\|^2 + \delta\|\phi_2\|^2 = 0, \quad \sum_{i=0}^{e}\overline{a_i}b_i = 0, \quad ... \quad , \sum_{i=d-e}^{d}\overline{a_i}b_i = 0,
$$

where $w = (\phi_1, \phi_2) = (a_0, ..., a_d, b_0, ..., b_e)$. ∎

Lemma 7.3.4. *If a vector* $(x_1, ..., x_n) \in \mathbb{C}^n$ *is nonzero, then the following* $m \times (n+m-1)$ *matrix*

$$\begin{pmatrix} x_1 & \cdots & x_n & 0 & \cdots & 0 \\ 0 & x_1 & \cdots & x_n & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_1 & \cdots & x_n \end{pmatrix}$$

has the full rank m*, i.e., the row vectors are linearly independent.*

Proof. Since $(x_1, ..., x_n) \neq 0$, there is a minimum i such that $x_i \neq 0$. The $m \times m$ submatrix

$$\begin{pmatrix} x_i & x_{i+1} & \cdots & * \\ 0 & x_i & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_i \end{pmatrix}$$

has determinant $x_i^m \neq 0$. Thus the rank of the $m \times (n+m-1)$ matrix is m.

□

Proposition 7.3.6. *If* $-1 < \delta < -\frac{2}{r}$ *with* $r = d - e > 2$*, then the quotient* $\mu_H^{-1}(-\delta)/S^1$ *is a* $\mathbb{P}^{e-1}$*-bundle over the projective space* $\mathbb{P}^e$*.*

Proof. Let $f : \mathbb{P}^e \to \mathrm{Gr}(e, \mathbb{C}^{d+1})$ be the map defined by

$$b = [b_0 : b_1 : ... : b_e] \mapsto NS(M_b)$$

for $b = [b_0 : b_1 : ... : b_e] \in \mathbb{P}^e$, where M_b is the $(d-e+1) \times (d+1)$ matrix

$$M_b = \begin{pmatrix} \overline{b_0} & \cdots & \overline{b_e} & 0 & \cdots & 0 \\ 0 & \overline{b_0} & \cdots & \overline{b_e} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \overline{b_0} & \cdots & \overline{b_e} \end{pmatrix},$$

and $NS(M_b)$ is the null space of M_b. The rank-nullity theorem says that $\mathrm{rank}(M_b) + \mathrm{nullity}(M_b) = d+1$. By the Lemma 7.3.4, $\mathrm{rank}(M_b) = d-e+1$, and so $\dim NS(M_b) = (d+1)-(d-e+1) = e$. Thus the map f is well defined. The null space is independent of the choice of the representative of $b = [b_0 : b_1 : ... : b_e]$. Consider the universal bundle $\mathcal{U}_e$ over $\mathrm{Gr}(e, \mathbb{C}^{d+1})$. The fiber of the pullback $f^*(\mathcal{U}_e)$ at the point $b = [b_0 : b_1 : ... : b_e]$ is the vector space $NS(M_b)$.

Let $b = [b_0 : b_1 : ... : b_e]$ be fixed. Then the null space $NS(M_b)$ is a linear subspace of $\mathbb{C}^{d+1}$ of real codimension $2(d-e+1)$. Since $-1 < \delta < 0$, the locus in $\mathbb{C}^{d+1}$, defined by the equation

$$(\delta + 1)\sum_{i=0}^{d} |a_i|^2 + \delta \sum_{i=0}^{e} |b_i|^2 = 0,$$

is the $(2d+1)$-sphere S^{2d+1}. Indeed we can write

$$\sum_{i=0}^{d} |a_i|^2 = -\frac{\delta}{\delta+1}\sum_{i=0}^{e} |b_i|^2.$$

Thus

$$NS(M_b) \bigcap \left\{ [a_0 : a_1 : ... : a_d] | \sum_{i=0}^{d} |a_i|^2 = -\frac{\delta}{\delta+1}\sum_{i=0}^{e} |b_i|^2 \right\} \cong S^{2e-1}.$$

Hence we can identify $\mu_H(-\delta)$ with an S^{2e-1}-bundle over $\mathbb{P}^e$.

Recall that the action of S^1 on W is given by

$$e^{i\theta}.[a_0 : a_1 : ... : a_d : b_0 : b_1 : ... : b_e] = [e^{i\theta}a_0 : e^{i\theta}a_1 : ... : e^{i\theta}a_d : b_0 : b_1 : ... : b_e].$$

The action of S^1 on $\mu_H(-\delta) \subset W$ is the restriction of this action. Thus the fiber of $\mu_H(-\delta)/S^1$ is $S^{2e-1}/S^1 \cong \mathbb{P}^{e-1}$. Consequently, we can identify

$$\mu_H(-\delta)/S^1 = \mathbb{P}(f^*(\mathcal{U}_e)).$$

□

The quotient in (7.1) has the following symplectic description.

Corollary 7.3.1. *If $\alpha_{i-1} < \alpha < \alpha_i$ with $i > 1$, then there exists a unique splitting type $((d_i, e_i), 0)$ such that we can identify*

$$\mathcal{M}_\alpha(d_i, e_i) = W//_\delta H = \frac{\mu^{-1}(-\delta)}{S^1} = \mathbb{P}(f^*(\mathcal{U}_{e_i}))$$

for $-1 < \delta < -\frac{2}{d_i - e_i + 1}$, where the quotient $W//_\delta H$ is the one for holomorphic chains of fixed splitting type $((d_i, e_i), 0)$, i.e. $W = \mathbb{P}Hom(\mathcal{O}, \mathcal{O}(d_i) \oplus \mathcal{O}(e_i))$.

7.4 Drézet-Trautmann's method and Doran-Kirwan's method for holomorphic chains of type $(1,2)$

In this section we compare Drézet-Trautmann's method with that of the Doran-Kirwan for holomorphic chains of type $(1,2)$.

Assume that $d > e \geq 0$. The automorphism group $G = \mathrm{Aut}(\mathcal{O}(-d) \oplus \mathcal{O}(-e)) \times \mathrm{Aut}(\mathcal{O})$ acts on the vector space $W = \mathrm{Hom}(\mathcal{O}(-d) \oplus \mathcal{O}(-e), \mathcal{O})$. Let a tuple $\Lambda = (\lambda, 1-\lambda, -1)$ be a polarization for $\lambda \in \mathbb{Q}$. In section 7.2.1, we identified (G,Λ)-semistable points of W. By the results in [30], there is a geometric quotient $W^{ss}(G,\Lambda)//G$ for $0 < \lambda < \frac{1}{d-e+2}$ which is a quasi-projective variety.

In Doran-Kirwan's method we consider the projective space $\mathbb{P}(W^\vee)$, where $W^\vee$ is the vector space $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(d) \oplus \mathcal{O}(e))$ of the dual morphisms. The group $H = \begin{pmatrix} \mathbb{C}^* & S^{d-e}(V^\vee) \\ 0 & 1 \end{pmatrix}$ acts on $\mathbb{P}(W^\vee)$. In section 7.2.3, we showed that there is a non-reductive GIT quotient $\mathbb{P}(W^\vee)//_\delta H$ for $-1 < \delta < -\frac{2}{d-e+1}$ with $d - e > 1$. The rational number δ is determined by the fractional character of the subgroup $\mathbb{C}^* \subset H$.

Proposition 7.4.1. *Let $\varphi^\vee \in W^\vee$ be the dual of the morphism $\varphi \in W$. Then $\varphi \in W^{ss}(G,\Lambda)$ for $\Lambda = (\lambda, 1-\lambda, -1)$ with $0 < \lambda < 1$ if and only if $[\varphi^\vee] \in \mathbb{P}(W^\vee)^{H,\delta}$ for $-1 < \delta < 0$.*

Proof. Let $\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \end{pmatrix}$, where φ_1 and φ_2 are sections of $\mathcal{O}(d)$ and $\mathcal{O}(e)$ respectively. Then φ is (G,Λ)-semistable for $\Lambda = (\lambda, 1-\lambda, -1)$ with $0 < \lambda < 1$ if and only if φ_1 does not divide φ_2. This is equivalent to saying that $[\varphi^\vee]$ is (H,δ)-semistable for $-1 < \delta < 0$.

□

Remark 7.4.1. *The existence of those non-reductive stable chains does not imply the existence of the non-reductive quotients. The non-reductive quotients are defined as subvarieties of the reductive GIT quotients of bigger projective varieties. If the parameters λ and δ satisfies certain conditions then the reductive GIT quotients are nonempty and it allows us to define the non-reductive quotients.*

Proposition 7.4.2. *If $d - e > 1$, then there is a natural bijective morphism*

$$W^{ss}(G,\Lambda)//G \to \mathbb{P}(W^\vee)//_\delta H$$

for $\Lambda = (\lambda, 1-\lambda, -1)$ with $0 < \lambda < \frac{1}{d-e+2}$ and $-1 < \delta < -\frac{2}{d-e+1}$.

Proof. If $d - e = 1$ then the parameter δ does not exist. The natural composition map

$$W^{ss}(G, \Lambda) \to \mathbb{P}(W^{\vee}) \to \mathbb{P}(W^{\vee})/\!/_{\delta} H$$

defined by $\varphi \mapsto [\varphi^{\vee}] \mapsto H[\varphi^{\vee}]$, is an onto morphism. The quotient $W^{ss}(G, \Lambda)/\!/G$ is a geometric quotient, so it is a categorical quotient. By the universal property of categorical quotients, there is a unique morphism $W^{ss}(G, \Lambda)/\!/G \to \mathbb{P}(W^{\vee})/\!/_{\delta} H$ such that the following diagram commutes:

$$\begin{array}{ccc} W^{ss}(G,\Lambda) & \longrightarrow & W^{ss}(G,\Lambda)/\!/G \\ & \searrow \quad \swarrow & \\ & \mathbb{P}(W^{\vee})/\!/_{\delta} H & \end{array}$$

□

Remark 7.4.2. *$W^{ss}(G, \Lambda) = W^{s}(G, \Lambda)$ for $\Lambda = (\lambda, 1 - \lambda, -1)$ with $0 < \lambda < 1$, and $\mathbb{P}(W^{\vee})^{ss,H,\delta} = \mathbb{P}(W^{\vee})^{s,H,\delta}$ for $-1 < \delta < 0$.*

Let C be a holomorphic chain of type $\mathbf{t} = (1, 2; 0, -s)$ with $s > 1$ on $\mathbb{P}(V) = \mathbb{P}^1$, where V is a 2-dimensional vector space over $\mathbb{C}$. Then its dual chain $C^{\vee}$ is of type $\mathbf{t}^{\vee} = (2, 1; s, 0)$. Both types have the same chamber structure for α. Thus, we can identify $\mathcal{M}^{s}_{\alpha}(\mathbf{t}) = \mathcal{M}^{s}_{\alpha}(\mathbf{t}^{\vee})$ by the dual map $(\mathcal{O}, E; \phi) \mapsto (E^{\vee}, \mathcal{O}; \phi^{\vee})$.

Recall the chamber structure for α:

Proposition 7.4.3.

$$R^{s}(\mathbf{t}) = \left(\bigcup_{1 \le k \le m} (\alpha_{k-1}, \alpha_k) \right) \bigcup \{\alpha_1\} \bigcup, ..., \bigcup \{\alpha_{m-1}\},$$

where $m = \lfloor \frac{s}{2} \rfloor$, $\alpha_m = 2s$ and $\alpha_k - \alpha_{k-1} = 3$ for all $k = 1, ..., m$.

Proof. This is from Proposition 4.1.3. The number of 1-dimensional chambers is $m = \frac{1}{3}(2s - \alpha_0)$. Since each subinterval has equal length 3, the number m is obtained by dividing the total length $2s - \alpha_0$ by 3. If s is even then $\alpha_0 = \frac{s}{2}$, so $m = \frac{s}{2}$. If s is odd then $\alpha_0 = \frac{s+3}{2}$, so $m = \frac{s-1}{2}$. Thus $m = \lfloor \frac{s}{2} \rfloor$. □

Remark 7.4.3. *If α lies in a 1-dimensional chamber (α_{p-1}, α_p) then there is no strictly α-semistable holomorphic chain.*

Proposition 7.4.4. *For $\alpha \in (\alpha_{p-1}, \alpha_p)$, the vector bundle E of degree $-s$ allows p splitting types such that* *the holomorphic chain* $(\mathcal{O}, E; \phi)$ *is* α*-stable. Indeed*

$$E = \mathcal{O}(-s+1+(m-k)) \oplus \mathcal{O}(-1-(m-k))$$

for $k = 1, ..., p$.

Proof. If a holomorphic chain $(\mathcal{O}, E; \phi)$ is α-stable for $\alpha \in (\alpha_{p-1}, \alpha_p)$, then by Proposition 4.1.2, $\phi_1 \neq 0$ and $\phi_1 \nmid \phi_2$, where $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$. We find that ϕ has at most $m-p$ common zeroes, i.e. $\sharp Z(\phi) \leq m-p$. Let $(-d, -e)$ be a splitting type for E with $0 < d \leq e$. Then we find that

$$m - (m-p) + 1 = p + 1 \leq d \leq m,$$

and $e = s - d$. Thus we have the result.

□

Proposition 7.4.5. *For* $\alpha \in (\alpha_{p-1}, \alpha_p)$*, the moduli space* $\mathcal{M}_\alpha^s(\mathbf{t})$ *is the union*

$$\bigcup_{1 \leq k \leq p} \mathcal{M}_\alpha^s(-s+1+(m-k), -1-(m-k)),$$

where $\mathcal{M}_\alpha^s(-s+1+(m-k), -1-(m-k))$ *is the subspace lying in* $\mathcal{M}_\alpha^s(\mathbf{t})$ *of the fixed splitting type* $(-s+1+(m-k), -1-(m-k))$.

Proof. This is clear from Proposition 7.4.4. □

Theorem 7.4.1. *For* $\alpha \in (\alpha_{p-1}, \alpha_p)$*, there is a unique splitting type* $(d_p, e_p) = (-s+1+(m-p), -1-(m-p))$*, and if* $0 < \lambda < \frac{1}{s-2m+2p}$ *and* $-1 < \delta < -\frac{2}{s-2m+2p-1}$*, then we can identify*

$$\begin{aligned} \mathcal{M}_\alpha^s(-s+1+(m-p), -1-(m-p)) &= W^{ss}(G, \Lambda)//G \\ &= \mathbb{P}(W^\vee)//_\delta H \quad \textit{(if } p > 1\textit{)}, \end{aligned}$$

where

$$W = \mathrm{Hom}(E, \mathcal{O}), \quad W^\vee = \mathrm{Hom}(\mathcal{O}, E^\vee),$$

$$G = \mathrm{Aut}(E) \times \mathrm{Aut}(\mathcal{O}), \quad \Lambda = (\lambda, 1-\lambda, -1) \quad \textit{and}$$

$$H = \mathrm{Aut}(E)/\mathbb{C}^* = \begin{pmatrix} \mathbb{C}^* & S^{s-2m+2p-2}(V^\vee) \\ 0 & 1 \end{pmatrix} \cong \mathbb{C}^* \ltimes (\mathbb{C}^+)^{s-2m+2p-1}.$$

Proof. This result is from the combination of Theorem 7.2.2 and Theorem 7.3.3. □

Corollary 7.4.1. *Let* $\mathbf{t} = (1, 2; 0, -s)$*. If* $\alpha \in (\alpha_0, \alpha_1)$ *then we can identify*

$$\mathcal{M}^s_\alpha(\mathbf{t}) = W^{ss}(G, \Lambda)//G$$

Proof. If $\alpha \in (\alpha_0, \alpha_1)$ then $\mathcal{M}^s_\alpha(\mathbf{t}) = \mathcal{M}^s_\alpha(-s + m, -m)$. Since $p = 1$, by Theorem 7.4.1, $\mathcal{M}^s_\alpha(\mathbf{t}) = \mathcal{M}^s_\alpha(-s + m, -m) = W^{ss}(G, \Lambda)//G$. □

Proposition 7.4.6. *If* $p > 1$ *then by the symplectic description we can identify* $\mathcal{M}^s_\alpha(s-1-(m-p), 1+(m-p))$ *as* $\mathbb{P}^{m-p}$*-bundle over* $\mathbb{P}^{m-p+1}$*.*

Proof. This is from Proposition 7.3.6. □

Let $\mathbf{t} = (1, 2; 0, -s)$. Recall that if $s < 2$ then $\mathcal{M}_\alpha(\mathbf{t}) = \emptyset$ and if $s = 2$ then $\mathcal{M}_\alpha(\mathbf{t}) = pt$.

Proposition 7.4.7. *Let* $\mathbf{t} = (2, 1; -s, 0)$ *with* $s > 2$*. The extremal moduli space* $\mathcal{M}^s_\alpha(\mathbf{t})$ *for the extremal chamber* $(2s-3, 2s)$ *is* $\mathbb{P}^{s-2}$*. The unique subspace* $\mathcal{M}^s_\alpha(-s+1, -1)$ *of* $\mathbb{P}^{s-2}$ *is* $\mathbb{P}^1$*. If* $s > 3$ *then corresponding Doran and Kirwan's non-reductive quotient is also* $\mathbb{P}^1$ *by the symplectic description.*

Proof. By Proposition 4.2.2, the extremal moduli space $\mathcal{M}^s_\alpha(\mathbf{t})$ for the extremal chamber $(2s-3, 2s)$ is $\mathbb{P}^{s-2}$. By Proposition 4.2.4, we can identify $\mathcal{M}^s_\alpha(-s + 1, -1) = \mathbb{P}^1$.

If $s = 3$ then there is a unique 1-dimensional chamber $(\alpha_0, \alpha_1) = (3, 6)$, so there is no parameter δ in the application of Doran and Kirwan's quotients. Assume that $s > 3$. In Doran and Kirwan's quotients, we consider the dual holomorphic chain of type $\mathbf{t}^\vee = (1, 2; 0, s)$. We can identify $\mathcal{M}^s_\alpha(-s+1, -1)$ as $\mathcal{M}^s_\alpha(s-1, 1)$ by mapping dual chains. The unique splitting type is determined by Theorem 7.3.3, so if $\alpha \in (2s - 3, 2s)$ then it is $(s - 1, 1)$. Hence by Proposition 7.4.6, $\mathcal{M}^s_\alpha(-s + 1, -1) = \mathcal{M}^s_\alpha(s - 1, 1) = \mathbb{P}^1$. □

Let $\alpha_p^- = \alpha_p - \epsilon$ for sufficiently small $\epsilon > 0$. Thus $\alpha_p^- \in (\alpha_{p-1}, \alpha_p)$. The subspaces are described as follows:

$$\begin{array}{ccccccc}
\mathcal{M}^s_{\alpha_1^-}(-s+m,-m) \supset & \mathcal{M}^s_{\alpha_2^-}(-s+m,-m) & \supset & \cdots & \supset & \mathcal{M}^s_{\alpha_m^-}(-s+m,-m) \\
 & \mathcal{M}^s_{\alpha_2^-}(-s+m-1,-m+1) & \supset & \cdots & \supset & \mathcal{M}^s_{\alpha_m^-}(-s+m-1,-m+1) \\
 & & \ddots & & & \vdots \\
 & & & \mathcal{M}^s_{\alpha_{m-1}^-}(-s+2,-2) & \supset & \mathcal{M}^s_{\alpha_m^-}(-s+2,-2) \\
 & & & & & \mathcal{M}^s_{\alpha_m^-}(-s+1,-1).
\end{array}$$

Remark 7.4.4. *The following subspaces can be identified as Drézet and Trautmann's non-reductive quotients:*

$$\mathcal{M}^s_{\alpha_1^-}(-s+m,-m),\quad \mathcal{M}^s_{\alpha_2^-}(-s+m-1,-m+1),\quad \ldots\quad ,\mathcal{M}^s_{\alpha_m^-}(-s+1,-1).$$

Note that if $s=2m$ then $\mathcal{M}^s_{\alpha_1^-}(-s+m,-m)$ is a reductive quotient, which is a Grassmannian variety. The subspaces can be identified as flip loci in Theorem 4.3.1. for $\alpha_2,...,\alpha_m$ since they can be identified as the subspaces for dual chains. This is described in the following.

The subspaces for dual chains are as follows:

$$\begin{array}{ccccc}
\mathcal{M}^s_{\alpha_1^-}(s-m,m)\supset & \mathcal{M}^s_{\alpha_2^-}(s-m,m) & \supset & \cdots & \supset\ \mathcal{M}^s_{\alpha_m^-}(s-m,m) \\
 & \mathcal{M}^s_{\alpha_2^-}(s-m+1,m-1) & \supset & \cdots & \supset \mathcal{M}^s_{\alpha_m^-}(s-m+1,m-1) \\
 & & \ddots & & \vdots \\
 & & & \mathcal{M}^s_{\alpha_{m-1}^-}(s-2,2)\supset & \mathcal{M}^s_{\alpha_m^-}(s-2,2) \\
 & & & & \mathcal{M}^s_{\alpha_m^-}(s-1,1).
\end{array}$$

Remark 7.4.5. *The following subspaces are flip loci in Theorem 4.3.1.*

$$\mathcal{M}^s_{\alpha_m^-}(s-1,1),\quad \mathcal{M}^s_{\alpha_{m-1}^-}(s-2,2),\quad \ldots\quad ,\quad \mathcal{M}^s_{\alpha_2^-}(s-m+1,m-1).$$

Note that in Theorem 4.3.1, the moduli spaces $\mathcal{M}^s_\alpha(\mathbf{t})$ is labeled from the rightmost chamber. The flip loci can be identified as Doran and Kirwn's non-reductive quotients

$$\mathcal{S}_k := \mathbb{P}\mathrm{Hom}(\mathcal{O},\mathcal{O}(s-k,k))//_\delta H,$$

where $H\cong(\mathbb{C}^+)^{s-2k+1}\rtimes\mathbb{C}^*$ for $k=1,...,m-1$, respectively. By a symplectic description they are $\mathbb{P}^{k-1}$-bundles over $\mathbb{P}^k$ for $k=1,...,m-1$, respectively. Finally each non-reductive quotient $\mathcal{S}_k$ contains the subspaces

$$\mathcal{M}^s_{\alpha_t^-}(s+t-m-1,-t+m+1)$$

for $t=k+1,...,m$, respectively.

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