NEW RESULTS IN STOCHASTIC MOVING BOUNDARY PROBLEMS

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Abstract

Moving boundary problems arise in many areas of science and engineering and they are of great importance in the areas of partial differential equations (PDEs) since they characterize phase change phenomena where a system has two phases such as solid and liquid. However, unlike other PDEs in a prescribed region such as heat equation on a bounded domain, moving boundary problems are difficult to solve theoretically or numerically since we consider partial differential equations in one or two phases and at the same time need to trace the positions of the interface. Thus, they provide deep mathematical challenges.

There is a vast literature on deterministic moving boundary problems. In addition, random perturbations of partial differential equations (e.g. stochastic heat equations) have been studied extensively. However, there has not been much attention paid to random perturbations of moving boundary problems. In this thesis, we consider random perturbations of two kinds of one-dimensional moving boundary problems: the Stefan problem, which describes the melting of the ice, and a free boundary problem proposed by Ludford and Stewart and studied by Caffarelli and Vazquez.

In the first part, we consider a one-dimensional Stefan problem perturbed by a multiplicative noise. The noise is Brownian in time but smoothly correlated in space. We first define a weak solution then transform this problem into a nonlinear stochastic partial differential equation (SPDE) with a fixed boundary condition. We characterize the domain of existence and prove existence and uniqueness of a solution.

The second part deals with a random perturbation of a moving boundary problem proposed by Ludford and Stewart and studied by Cafferelli and Vazquez. The random perturbation is a single Brownian motion and the moving boundary condition is different from the Stefan boundary condition. We consider existence and uniqueness of a solution and focus on numerical analysis of the problem. As for the stochastic Stefan problem, we use the transformation which transforms the stochastic moving boundary problem to a nonlinear SPDE which has a fixed spatial domain. Our numerical approximations are based on the nonlinear transformed SPDE. We use the explicit finite difference method and the Euler-Maruyama scheme to discretize time and space respectively. We also investigate the convergence theory.
To my family.
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Chapter 1

Introduction

Suppose there is a semi-infinite sheet of ice occupying the interval \((-\infty, 0)\) and water occupying the interval \((0, \infty)\). The temperature of the ice is identically 0 whereas the temperature of the water is strictly positive. As time goes on, we can see a phase-change from the ice to the water (or from the water to the ice) due to a discontinuity of heat flux. In other words, the position of the interface between the ice and the water changes in time. This is one of the canonical moving boundary problems, so called the Stefan problem. A mathematical formulation of this problem is as follows:

\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) \quad x > \beta(t)
\]

\[
\lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t, x) = -\rho \dot{\beta}(t)
\]

\[
u(0, x) = u_0(x) \quad x \in \mathbb{R}
\]

\[
\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid u(t, x) > 0\} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > \beta(t)\}
\]

Here \(u_0(x) > 0\) for \(x > 0\) and \(u_0 \equiv 0\) for \(x \leq 0\), and \(\rho \neq 0\) is a fixed constant. In addition, \(u\) and \(\beta\) represent temperature and interface respectively. The first equation in (1.0.1) says that \(u\) follows a heat equation as long as \(x > \beta(t)\) (the water region). The last condition in (1.0.1) is the positivity condition, which implies that \(u(t, x)\) is strictly positive as long as \(x > \beta(t)\) and \(u = 0\) at the interface. This means that there is only one interface which separates two phases. The second equation in (1.0.1) is called the Stefan boundary condition, the boundary condition at the interface which comes from the energy balance. In fact, the positivity condition and the Stefan boundary condition tell us how the boundary is moving; by the positivity condition, the first partial derivative of \(u\) with respect to \(x\) at the interface is strictly positive (or negative) for all \(t > 0\) corresponding to the sign of \(\rho\), which implies that \(\beta(t)\) is strictly decreasing (or increasing) as \(t\) increases. The important thing in this problem is that the interface between the ice and the water is a priori unknown, thus it provides deep mathematical challenges not only in the areas of existence, uniqueness and regularity but also numerical analysis of moving boundary problems. In fact, it
is important to develop numerical analysis of moving boundary problems since exact solutions of moving boundary problems are very limited in applications.

As the Stefan problem characterizes phase change phenomena in a solid-liquid system, moving boundary problems arise in many areas such as material science, chemical reactions, molecular biology, and also finance. In addition, moving boundary problems have been formulated by deterministic partial differential equations (e.g. (1.0.1)) and there is a vast literature on deterministic moving boundary problems. In reality, however, there exists always noise, which makes it difficult to describe these phenomena precisely by deterministic models. Even though heat equations perturbed by various noises have been investigated extensively, there has been fairly little written on the effect of noise on moving boundary problems (see [BDP02], [CLM06], and [KM10]; see also the work on the stochastic porous medium equation in [BDPR09, DPR04a, DPR04b, DPRRW06, Kim06]). Furthermore, to the best of my knowledge, nothing has been done on numerical analysis of random perturbations of moving boundary problems (see [Cra84, EOS82] for deterministic moving boundary problems and references therein).

The focus of this work is to investigate the effect of noise on moving boundary problems theoretically and numerically. Consider the following equation:

\[ \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \alpha u(t,x) + u(t,x)d\zeta(t) \quad x > \beta(t) \]

\[ \lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t,x) = -\rho \beta(t) \]

\[ u(0,x) = u_\circ(x) \quad x \in \mathbb{R} \]

\[ \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid u(t,x) > 0\} = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > \beta(t)\}, \]

where \(\zeta(t)\) is a random field which is Brownian in time but smoothly correlated in space and \(\beta(t)\) represents a moving boundary.

This is the Stefan problem perturbed by multiplicative noise which is Brownian in time but which is correlated in space. In fact, the multiplicative term \(u\) in front of \(d\zeta(t)\) is a natural nonlinearity since this implies that the last requirement in (1.0.2) (the positivity condition) holds. In other words, due to the multiplicative noise, there is only one interface which separates two phases ice and water. Furthermore, the term \(\alpha u\) implies that (1.0.2) is invariant under Ito and Stratonovich formulations (see Remark 2.3.1).

In Chapter 2, we consider existence and uniqueness of a solution of (1.0.2). We first use a weak formulation to define a weak solution and obtain the Stefan boundary condition from the weak solution. The main ingredient to show existence and uniqueness is a transformation which transforms (1.0.2) into a nonlinear
 stochastic partial differential equation (SPDE) on the half space \([0, \infty)\) with a fixed boundary condition at \(x = 0\) (the Dirichlet boundary condition). This transformation and inverse transformation (which transforms a nonlinear SPDE back to (1.0.2)) can be carried out by using the Stefan boundary condition and assuming some regularity on the moving boundary. Finally, using a Picard-type iteration, we show existence and uniqueness of the nonlinear SPDE on the half space, which results in existence and uniqueness of a weak solution of (1.0.2).

In Chapter 3, we investigate the following equation:

\[
du(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x)dt + \alpha u(t, x)dt + u(t, x) \circ dB_t \quad x > \beta(t)
\]

\[
\lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t, x) = 1
\]

\[
u(0, x) = \nu_0(x) \quad x \in \mathbb{R}
\]

\[
\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid u(t, x) > 0\} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > \beta(t)\},
\]

where \(B_t\) is the standard Brownian motion.

This is a random perturbation of a moving boundary problem proposed by Ludford and Steward (Ste85) and studied by Caffarelli and Vazquez (CV95). Without noise, this problem appears in combustion theory in the analysis of the propagation of equidiffusional premixed flames with high activation energy (see Vaz96 for more detailed description). The main difference between this problem and the Stefan problem is the condition at the interface: the Stefan boundary condition itself tells us how the interface is moving whereas the condition here (the second equation in (1.0.3)) implicitly defines an evolution equation for a moving boundary.

The first part of Chapter 3 overviews the results on existence, uniqueness and regularity of a solution of (1.0.3) in KMS. Here we also use a weak formulation and obtain a representation of a weak solution. Using this representation, we can get an evolution equation for the moving boundary \(\beta(t)\). Note that since our noise is a single Brownian motion so that it has the same effect on each state space for a fixed time \(t\), we have a smooth solution \(u(t, x)\) in space. In order to show existence and uniqueness, we also use a transformation,
\( \tilde{u}(t, x) = u(t, x + \beta(t)) + e^{-x} \) and \( \tilde{u} \) satisfies the nonlinear SPDE:

\[
\begin{align*}
   d\tilde{u}(t,x) &= \left\{ \frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) + \tilde{\beta}(\tilde{u}(t,x) - e^{-x}) - e^{-x} \right. \\
   &\quad\quad\left. + \left( \tilde{u}(t,x) - e^{-x} \right) \frac{\partial \tilde{u}}{\partial x}(t,x) + e^{-x} \right\} dt \\
   &\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\Quad...
Chapter 2
A Stochastic Stefan Problem

2.1 Introduction

In this chapter we consider a random perturbation of the one-dimensional Stefan problem. First fix a probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) and assume that \(\zeta\) is a random field which is Brownian in time but which is correlated in space (we will rigorously define \(\zeta\) in Section 2.2). We consider the stochastic partial differential equation (SPDE)

\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \alpha u(t, x) + u(t, x) \, d\zeta_t(x) \quad x > \beta(t)
\]

\[
\lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t, x) = -\varrho \dot{\beta}(t)
\]

\[
u(0, x) = u_0(x) \quad x \in \mathbb{R}
\]

\(\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid u(t, x) > 0\} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > \beta(t)\} \).

Here the constant \(\varrho\) is not 0 and the constant \(\alpha \in \mathbb{R}\) is fixed (we shall later see why it is natural to include this term). We also assume that the initial condition \(u_0 \in C(\mathbb{R})\) satisfies some specific properties:

- \(u_0 \equiv 0\) on \(\mathbb{R}_-\), \(u_0 > 0\) on \((0, \infty)\), and \(\lim_{x \searrow 0} \frac{du_0}{dx}(x)\) exists.

- \(u_0\) and its first three derivatives exist on \((0, \infty)\) and are square-integrable (on \((0, \infty)\)).

The last requirement in (2.1.1) means that the boundary between \(u \equiv 0\) and \(u > 0\) is exactly the graph of \(\beta\). In other words, there is only one moving boundary \((x = \beta(t))\) which separates two phases \((u \equiv 0\) and \(u > 0)\) and \(u(t, x)\) is strictly positive as long as \(x > \beta(t)\) (the positivity condition). Here, both this positivity condition and the Stefan boundary condition tell us how the boundary is moving; if \(\varrho > 0\), \(\beta(t)\) is strictly decreasing as \(t\) increases because \(\frac{du}{dx}\) at the interface from the right is strictly positive.

Since (2.1.1) is a differential form, it is not yet clear whether (2.1.1) makes sense. Differential equations are pointwise statements. Stochastic differential equations are in fact shorthand representations of corresponding
integral equations; pointwise statements typically do not make sense. In addition, we do not know whether the partial derivatives of $u$ with respect to $x$ is well-defined because of a random perturbation. Therefore we consider a weak formulation to define a solution of (2.1.1) (see Section 2.3).

Our goal here is to study the effect of noise on the Stefan problem, more precisely, existence and uniqueness. First we obtain the Stefan boundary condition, which is a pointwise statement, from a weak solution, which is a statement about stochastic integrals (see Section 2.4). Using the Stefan boundary condition, we transform (2.1.1) into a nonlinear SPDE on a half space with the Dirichlet boundary condition and vice versa (see Section 2.5). This transformed SPDE is nonlocal and nonlinear. We use a Picard-type iteration to show existence and uniqueness of the transformed SPDE, which results in existence and uniqueness of a solution of the stochastic Stefan problem (see Section 2.6). Here other transformations can be carried out in order to show the last requirement of (2.1.1). This result is joint work with Richard B. Sowers and Zhi Zheng ([KZS]).

2.2 Noise

We define the noise which is Brownian in time but smoothly correlated in space. The smoothness in space is important since it results in the smoothness of the solution of the stochastic Stefan problem in space. Fix $\eta \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\eta^{(n)} \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $n \in \{1, 2, 3\}$ and such that

$$\int_{y \in \mathbb{R}} \eta^2(y) dy = 1. \quad (2.2.1)$$

Let $W$ be a Brownian sheet. For $t \geq 0$ and $x \in \mathbb{R}$, define

$$\zeta_t(x) \overset{\text{def}}{=} \int_0^t \int_{y \in \mathbb{R}} \eta(x - y)W(ds, dy)$$

Then $\zeta$ is a zero-mean Gaussian field with covariance structure given by

$$\mathbb{E}[\zeta_t(x)\zeta_s(y)] = (t \wedge s) \int_{z \in \mathbb{R}} \eta(x - z)\eta(y - z) dz$$
for all \( s \) and \( t \) in \( \mathbb{R}_+ \) and \( x \) and \( y \) in \( \mathbb{R} \). Thus for each \( x \in \mathbb{R} \), \( t \mapsto \zeta_t(x) \) is a Brownian motion, and for each \( t > 0 \), \( x \mapsto \zeta_t(x) \) is in \( C^2 \) with derivative given by

\[
\zeta_t^{(n)}(x) = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \eta^{(n)}(x - y)W(ds, dy)
\]

for all \( t \geq 0 \) and \( x \in \mathbb{R} \) for \( n \in \{0, 1, 2\} \).

Let’s next understand integration against \( \zeta \). Let \( \{\beta(t) \mid t \geq 0\} \) be an \( \mathbb{R} \)-valued predictable and continuous function. Let \( \{f(t) \mid t \geq 0\} \) be an \( \mathbb{R} \)-valued predictable and continuous function, which is also bounded. We define

\[
\int_{s=0}^{t} f(s) d\zeta_s(\beta(s)) \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{0 \leq j \leq [tN]} f\left(\frac{j}{N}\right) \{\zeta_{(j+1)/N}(\beta(j/N)) - \zeta_{j/N}(\beta(j/N))\}
\]

\[
= \int_{s=0}^{t} \int_{y \in \mathbb{R}} f(s, y)W(ds, dy).
\]

this being a limit in \( L^2 \). Thanks to (2.2.1),

\[
\int_{s=0}^{t} d\zeta_s(\beta(s)) = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \eta(\beta(s) - y)W(ds, dy)
\]

is a Brownian motion. Therefore we can define a Brownian motion \( B^x_t \) for each fixed \( x \) as

\[
B^x_t \overset{\text{def}}{=} \int_{s=0}^{t} d\zeta_s(x + \beta(s)). \tag{2.2.2}
\]

Note also that \( d\zeta_t(\beta(t)) \) is not the total derivative of \( \zeta_t(\beta(t)) \); i.e.,

\[
\zeta_t(\beta(t)) - \zeta_0(\beta(0)) \neq \int_{s=0}^{t} d\zeta_s(\beta(s)).
\]

To understand the total derivative, we must also include the spatial variation of \( \zeta_t \):

\[
d[\zeta_t(\beta(t))] = \int_{y \in \mathbb{R}} \eta(\beta(t) - y)W(dy, dt) + \int_{s=0}^{t} \int_{y \in \mathbb{R}} \dot{\eta}(\beta(t) - y)\dot{\beta}(t)W(dy, ds)dt.
\]

This implies that

\[
\zeta_t(\beta(t)) - \zeta_0(\beta(0)) = \int_{s=0}^{t} d\zeta_s(\beta(s)) + \int_{s=0}^{t} \dot{\zeta}_s(\beta(s))\dot{\beta}(s)ds. \tag{2.2.3}
\]
2.3 Weak formulation and Main Theorem

In order to restate (2.1.1) as a statement of stochastic integrals, we first consider when \( \dot{\zeta}_t(x) \) is replaced by a smooth function \( b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \); the Wong-Zakai result (cf. [KS91, Section 5.2D]) implies that this is a reasonable approximation of an SPDE with Stratonovich integration against the noise; we can then convert this into the desired SPDE with Ito integration. Namely, consider the PDE

\[
\frac{\partial v}{\partial t}(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x) + \alpha_s v(t, x) + v(t, x)b(t, x) \quad x > \beta_\circ(t)
\]

\[
\lim_{x \searrow \beta_\circ(t)} \frac{\partial v}{\partial x}(t, x) = -\varrho \beta_\circ(t)
\]

\[
\quad v(0, x) = u_\circ(x). \quad x \in \mathbb{R}
\]

\[
\{ (t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid v(t, x) > 0 \} = \{ (t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > \beta_\circ(t) \}.
\]

where \( \alpha_s \defeq \alpha - \frac{1}{2} \) (we will see that this corresponds to the Stratonovich analogue of (2.1.1)). This will be our starting point.

For a deterministic Stefan problem, the enthalpy formulation can be used to define a weak solution (see [EO82]). That is, define the enthalpy function \( E(v) \) as

\[
E(v) \defeq \begin{cases}
  v + \rho & \text{if } v > 0 \\
  [0, \rho] & \text{if } v = 0 \\
  -v & \text{if } v < 0,
\end{cases}
\]

and then the enthalpy formulation follows as: for any \( \varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}) \)

\[
\int_{x \in \mathbb{R}} E(v(t, x)) \varphi(t, x)dx = \int_{x \in \mathbb{R}} E(v_0(x)) \varphi(0, x)dx
\]

\[
+ \int_s^t \int_{x \in \mathbb{R}} E(v(s, x)) \frac{\partial \varphi}{\partial s}(s, x)dxds + \int_{s=0}^t \int_{x \in \mathbb{R}} v(s, x) \left\{ \frac{\partial^2 \varphi}{\partial x^2}(s, x) + \alpha_s \varphi(s, x) \right\} dxds
\]

\[
+ \int_{s=0}^t \int_{x \in \mathbb{R}} v(s, x) \varphi(s, x)b(s, x)dxds. \quad (2.3.2)
\]

Here the enthalpy function is a multivalued operator. Thus, instead of using the enthalpy formulation, we consider a weak formulation (see [Fri64, Ch. 8]). They are in fact equivalent (see Remark 2.3.2). Fix
\( \varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}) \). Assume that \( \beta_0 \) is continuously differentiable. Define

\[
V_\varphi(t) \overset{\text{def}}{=} \int_{x \in \mathbb{R}} v(t, x) \varphi(t, x) \, dx = \int_{x = \beta_0(t)}^\infty v(t, x) \varphi(t, x) \, dx.
\]

Now we consider \( \dot{V}_\varphi(t) \). Using (2.3.1) and the fact that \( v(t, \beta_0(t)) = 0 \), we have that

\[
\dot{V}_\varphi(t) = \int_{x = \beta_0(t)}^\infty \left\{ \frac{\partial v}{\partial t}(t, x) \varphi(t, x) + v(t, x) \frac{\partial \varphi}{\partial t}(t, x) \right\} \, dx - v(t, \beta_0(t)) \varphi(t, \beta_0(t)) \dot{\beta}_0(t)
\]

Using integration by parts, the Stefan boundary condition and the fact that \( v(t, \beta_0(t)) = 0 \), we obtain that

\[
\int_{x = \beta_0(t)}^\infty \frac{\partial^2 v}{\partial x^2}(t, x) \varphi(t, x) \, dx = \lim_{x \searrow \beta_0(t)} \left\{ -\frac{\partial v}{\partial x}(t, x) \varphi(t, x) + v(t, x) \frac{\partial \varphi}{\partial x}(t, x) \right\} + \int_{x = \beta_0(t)}^\infty v(t, x) \frac{\partial^2 \varphi}{\partial x^2}(t, x) \, dx
\]

Recombining things we get the standard formula that

\[
\dot{V}_\varphi(t) = \int_{x \in \mathbb{R}} v(t, x) \left\{ \frac{\partial \varphi}{\partial t}(t, x) + \frac{\partial^2 \varphi}{\partial x^2}(t, x) + \alpha_s \varphi(t, x) \right\} \, dx
\]

Replacing \( b \) by our noise, we should have the following formulation: that for any \( \varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}) \) and any \( t > 0 \),

\[
\int_{x \in \mathbb{R}} u(t, x) \varphi(t, x) \, dx = \int_{x \in \mathbb{R}} u_\circ(x) \varphi(0, x) \, dx
\]

+ \[ \int_{s=0}^t \int_{x \in \mathbb{R}} u(s, x) \left\{ \frac{\partial \varphi}{\partial s}(s, x) + \frac{\partial^2 \varphi}{\partial x^2}(s, x) + \alpha_s \varphi(s, x) \right\} \, dx \, ds
\]

+ \[ \int_{x \in \mathbb{R}} \int_{s=0}^t u(s, x) \varphi(s, x) \circ d\zeta(s) \, dx + \theta \int_{s=0}^t \varphi(s, \beta_0(s)) \dot{\beta}_0(s) \, ds \]
The Ito formulation of this would be that

\[
\int_{x \in \mathbb{R}} u(t, x)\varphi(t, x)\,dx = \int_{x \in \mathbb{R}} u_0(x)\varphi(0, x)\,dx \\
+ \int_0^t \int_{x \in \mathbb{R}} u(s, x) \left\{ \frac{\partial \varphi}{\partial s}(s, x) + \frac{\partial^2 \varphi}{\partial x^2}(s, x) + \alpha \varphi(s, x) \right\} \,dx\,ds \\
+ \int_{x \in \mathbb{R}} \int_{s=0}^t u(s, x)\varphi(s, x)\,d\zeta(s)\,dx + \varrho \int_{s=0}^t \varphi(s, \beta_0(s))\dot{\beta}_0(s)\,ds.
\]

**Remark 2.3.1.** The structure of the SPDE [2.1.1] is invariant under Ito and Stratonovich formulations; this is the motivation for including \(\alpha\) in [2.1.1].

**Remark 2.3.2.** This weak formulation is equivalent to the enthalpy formulation [2.3.2] for the deterministic Stefan problem. Indeed, a simple calculation shows that

\[
\varrho \int_{s=0}^t \int_{x \in \mathbb{R}} \frac{\partial \varphi}{\partial s}(s, x)\,dx\,ds = \varrho \left\{ \int_{x=\beta_0(t)}^\infty \varphi(t, x)\,dx - \int_{x=0}^\infty \varphi(0, x)\,dx \right\} + \varrho \int_{s=0}^t \varphi(s, \beta_0(s))\dot{\beta}_0(s)\,ds.
\]

Thus,

\[
\int_{x \in \mathbb{R}} E(v(t, x))\varphi(t, x)\,dx - \int_{x \in \mathbb{R}} E(v_0(x))\varphi(0, x)\,dx - \int_{s=0}^t \int_{x \in \mathbb{R}} E(v(s, x)) \frac{\partial \varphi}{\partial s}(s, x)\,dx\,ds \\
= \int_{x \in \mathbb{R}} v(t, x)\varphi(t, x)\,dx - \int_{x \in \mathbb{R}} v_0(x)\varphi(0, x)\,dx + \varrho \left\{ \int_{x=\beta_0(t)}^\infty \varphi(t, x)\,dx - \int_{x=0}^\infty \varphi(0, x)\,dx \right\} \\
- \int_{s=0}^t \int_{x \in \mathbb{R}} v(s, x) \frac{\partial \varphi}{\partial s}(s, x)\,dx\,ds + \varrho \int_{s=0}^t \int_{x \in \mathbb{R}} \frac{\partial \varphi}{\partial s}(s, x)\,dx\,ds - \varrho \int_{s=0}^t \varphi(s, \beta_0(s))\dot{\beta}_0(s)\,ds,
\]

which implies the equivalence.

We can now formally define a weak solution of [2.1.1]. In this definition, we allow for blowup. Define \(\mathcal{F}_t \overset{\text{def}}{=} \sigma\{W(s, y) : s \leq t, y \in \mathbb{R}\}\) for all \(t \geq 0\).

**Definition 2.3.3.** A weak solution of [2.1.1] is a nonnegative predictable path \(\{u(t, \cdot) : 0 \leq t < \tau\} \subset C(\mathbb{R}) \cap L^1(\mathbb{R})\), where \(\tau\) is a predictable stopping time with respect to \(\{\mathcal{F}_t\}_{t>0}\), such that for any \(\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})\)
and any finite stopping time $\tau' < \tau$,

$$\int_{x \in \mathbb{R}} u(\tau', x) \varphi(\tau', x) dx = \int_{x \in \mathbb{R}} u_0(x) \varphi(0, x) dx$$

$$+ \int_{s=0}^{\tau'} \int_{x \in \mathbb{R}} u(s, x) \left\{ \frac{\partial \varphi}{\partial s}(s, x) + \frac{\partial^2 \varphi}{\partial x^2}(s, x) + \alpha \varphi(s, x) \right\} dx ds$$

$$+ \int_{x \in \mathbb{R}} \int_{s=0}^{\tau'} u(s, x) \varphi(s, x) d\xi_s(x) dx + \rho \int_{s=0}^{\tau'} \varphi(s, \beta(s)) \dot{\beta}(s) ds$$

and where

$$\{(t, x) \in [0, \tau) \times \mathbb{R} \mid u(t, x) > 0\} = \{(t, x) \in [0, \tau) \times \mathbb{R} \mid x > \beta(t)\}$$

where $\beta(t)$ is a semimartingale.

Our main existence and uniqueness theorems are the following. The arguments leading up to these results will come together in Section 2.6.

**Theorem 2.3.4** (Existence). There exists a predictable path $\{u(t, \cdot) \mid 0 \leq t < \tau\} \subset C(\mathbb{R}) \cap L^1(\mathbb{R})$ which satisfies (2.3.3), and $u(t, \cdot) \in C^1[\beta(t), \infty)$ for all $t \in [0, \tau)$ and

$$\tau \leq \inf \left\{ t \geq 0 : \left| \frac{\partial u}{\partial x}(t, \beta(t)) \right| = \infty \right\}.$$ 

Furthermore, if $u(t, \cdot) \in C^2[\beta(t), \infty)$ for all $t \in [0, \tau)$, then it satisfies (2.3.4).

**Proof.** Combine Lemmas 2.6.8 and 2.6.9

We also have uniqueness.

**Theorem 2.3.5** (Uniqueness). Suppose that $\{u_1(t, \cdot) ; 0 \leq t < \tau_1\}$ and $\{u_2(t, \cdot) ; 0 \leq t < \tau_2\}$ are two solutions of (2.1.1). Assume that for $i \in \{1, 2\}$, the map $x \mapsto u_i(t, x + \beta(t))$ has three generalized square-integrable derivatives on $(0, \infty)$. Then $u_1(t, \cdot) = u_2(t, \cdot)$ for $0 \leq t < \min\{\tau_1, \tau_2\}$.

**Proof.** The proof follows from Lemma 2.6.10

### 2.4 The Stefan Boundary Condition

In this section, we obtain the Stefan boundary condition which is a pointwise statement from a weak solution of a stochastic Stefan problem. This requires some regularity on the boundary $\beta(t)$. In fact, regularity of moving boundary problems is an incredibly challenging area (see [CS05]). Therefore we can
make some headway. Namely, if we assume enough regularity for the boundary, we can get better control of
the sense in which the boundary behavior holds. First we rewrite (2.3.3) by using heat kernels. Define
\[ p_\circ(t,x) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{x^2}{4t} \right] \quad t > 0, x \in \mathbb{R} \]
\[ p_\pm(t,x,y) \overset{\text{def}}{=} \{ p_\circ(t,x-y) \pm p_\circ(t,x+y) \} e^{\alpha t} = \{ p_\circ(t,x-y) \pm p_\circ(t,-x-y) \} e^{\alpha t} \quad t > 0, x,y \in \mathbb{R} \]
the second representation of \( p_\pm \) stems from the fact that \( p_\circ \) is even in its second argument. We then have
that
\[ \partial p_\pm / \partial t (t,x,y) = \partial^2 p_\pm / \partial y^2 (t,x,y) + \alpha p_\pm (t,x,y) \quad t > 0, x,y \in \mathbb{R} \]
\[ \lim_{t \downarrow 0} p_\pm(t,\cdot,x) = \delta_x \pm \delta_{-x}; \quad x \in \mathbb{R} \setminus \{0\} \]  
(2.4.1)
the relevant distinction between \( p_+ \) and \( p_- \) is their behavior at \( x = 0 \), namely,
\[ \partial^{2n-1} p_+ / \partial x^{2n-1}(t,0,y) = \partial^{2n} p_- / \partial x^{2n}(t,0,y) = 0 \]
for all \( n \in \mathbb{N} \overset{\text{def}}{=} \{1,2,\ldots\} \), all \( t > 0 \) and all \( y \in \mathbb{R} \). This will come up in the arguments of Lemma 2.4.2 and Lemma 2.5.1. The following lemma says that we can have a pointwise statement of a weak solution. This is similar to obtaining mild solutions from weak solutions in partial differential equations. Define \( C_0(\mathbb{R}) \) is a set of continuous functions which asymptotically vanishes at infinity.

**Lemma 2.4.1.** Suppose that \( \{u(t,\cdot) \mid 0 \leq t < \tau\} \subset C_0(\mathbb{R}) \cap C^1([\beta(t),\infty)) \) is a weak solution of (2.1.1).
Suppose also that \( \beta \) is continuously differentiable and \( \{\mathcal{F}_t \}_{t \geq 0} \)-adapted. Then \( u(t,x+\beta(t)) \) satisfies the integral equation
\[ u(t,x+\beta(t)) = \int_{y=0}^{\infty} p_+(t,x,y)u_\circ(y)dy \]
\[ + \int_{s=0}^{t} \int_{y=0}^{\infty} p_+(t-s,x,y) \partial u / \partial x(s,y+\beta(s))\dot{\beta}(s)dyds \]
\[ + \int_{y=0}^{\infty} \int_{s=0}^{t} p_+(t-s,x,y)u(s,y+\beta(s))d\kappa_s(y+\beta(s))dy \]
\[ + g(t) \]
\[ \int_{s=0}^{t} p_+(t-s,x,0)\dot{\beta}(s)ds \]
(2.4.2)
for all \( t < \tau \) and \( x > 0 \).
Proof. Fix $x > 0$ and $T > 0$. For $t \in [0, \tau \wedge T)$, define

$$U^T(t) \overset{\text{def}}{=} \int_{y=0}^{\infty} u(t, y + \beta(t))p_+(T - t, x, y)dy.$$ 

Using Definition 2.3.3, we have

$$U^T(t) = \int_{y=0}^{\infty} u(t, y + \beta(t))p_+(T - t, x, y - \beta(t))dy$$

$$= \int_{y \in \mathbb{R}} u(t, y)p_+(T - t, x, y - \beta(t))dy$$

$$= U^T(0) + U^T_1(t) + U^T_2(t)$$

where

$$U^T_1(t) = \int_{r=0}^{t} \int_{y \in \mathbb{R}} u(r, y) \left\{ \frac{\partial p_+}{\partial t}(T - r, x, y - \beta(r)) + \frac{\partial^2 p_+}{\partial y^2}(T - r, x, y - \beta(r)) + \alpha p_+(T - r, x, y - \beta(r)) - \frac{\partial p_+}{\partial y} (T - r, x, y - \beta(r))\dot{\beta}(r)\right\} dydr$$

$$+ \varrho \int_{r=0}^{t} p_+(T - r, x, \beta(r) - \beta(r))\dot{\beta}(r)dr,$$

$$U^T_2(t) = \int_{y \in \mathbb{R}} \int_{r=0}^{t} u(r, y)p_+(T - r, x, y - \beta(r))d\zeta_r(y)dy$$

Integration by parts and (2.4.1) imply that

$$dU^T_1(t) = \int_{y \in \mathbb{R}} u(t, y) \left\{ -\frac{\partial p_+}{\partial t}(T - t, x, y - \beta(t)) + \frac{\partial^2 p_+}{\partial y^2}(T - t, x, y - \beta(t)) + \alpha p_+(T - t, x, y - \beta(t)) - \frac{\partial p_+}{\partial y} (T - t, x, y - \beta(t))\dot{\beta}(t)\right\} dydt$$

$$+ gp_+(T - t, x, 0)\dot{\beta}(t)dt$$

$$= - \left\{ \int_{y=0}^{\infty} u(t, y) \frac{\partial p_+}{\partial y}(T - t, x, y - \beta(t))dy \right\} \dot{\beta}(t)dt + gp_+(T - t, x, 0)\dot{\beta}(t)dt$$

$$= \left\{ \int_{y=\beta(t)}^{\infty} \frac{\partial u}{\partial y}(t, y)p_+(T - t, x, y - \beta(t))dy \right\} \dot{\beta}(t)dt + gp_+(T - t, x, 0)\dot{\beta}(t)dt$$

$$= \left\{ \int_{y=0}^{\infty} \frac{\partial u}{\partial y}(t, y + \beta(t))p_+(T - t, x, y)dy \right\} \dot{\beta}(t)dt + gp_+(T - t, x, 0)\dot{\beta}(t)dt$$
We have here used the fact that \( u(t, \beta(t)) = 0 \). Thus

\[
U^T_1(t) = \int_{s=0}^{t} \left\{ \int_{y=0}^{\infty} p_+(T - s, x, y) \frac{\partial u}{\partial y}(s, y + \beta(s)) \hat{\beta}(s) dy \right\} ds + \varrho \int_{s=0}^{t} p_+(T - s, x, 0) \hat{\beta}(s) ds.
\]

Now we consider \( U^T_2(t) \). Using the stochastic Fubini theorem [Wal86 Theorem 2.6], we obtain

\[
U^T_2(t) = \int_{y \in \mathbb{R}} \int_{r=0}^{t} u(r, y) p_+(T - r, x, y - \beta(r)) d\zeta_r(y) dy
\]

\[
= \int_{y \in \mathbb{R}} \int_{r=0}^{t} u(r, y) p_+(T - r, x, y - \beta(r)) \int_{z \in \mathbb{R}} \eta(y - z) W(dr, dz) dy
\]

\[
= \int_{r=0}^{t} \int_{z \in \mathbb{R}} \left\{ \int_{y \in \mathbb{R}} u(r, y) p_+(T - r, x, y - \beta(r)) \eta(y - z) dy \right\} W(dr, dz)
\]

\[
= \int_{r=0}^{t} \int_{z \in \mathbb{R}} \left\{ \int_{y=\beta(r)}^{\infty} u(r, y) p_+(T - r, x, y - \beta(r)) \eta(y - z) dy \right\} W(dr, dz)
\]

\[
= \int_{y=0}^{\infty} \int_{r=0}^{t} u(r, y + \beta(r)) p_+(T - r, x, y) \eta(y + \beta(r) - z) dy \int_{r, x, y} W(dr, dz)
\]

Combine things to get that

\[
U^T(t) = U^T(0) + \int_{s=0}^{t} \left\{ \int_{y=0}^{\infty} p_+(T - s, x, y) \frac{\partial u}{\partial y}(s, y + \beta(s)) \hat{\beta}(s) dy \right\} ds
\]

\[
+ \int_{y=0}^{\infty} \int_{s=0}^{t} p_+(T - s, x, y) u(s, y + \beta(s)) d\zeta_s(y + \beta(s)) dy
\]

\[
+ \varrho \int_{s=0}^{t} p_+(T - s, x, 0) \hat{\beta}(s) ds.
\]

Now let \( T - t \) to get the claimed result.

Note that (2.4.2) is not an explicit formula for \( u \) since the right-hand side of (2.4.2) depends on \( \beta \). However, using (2.4.2), we can show the following lemma. The value of Lemma 2.4.2 is that if \( \beta \) is continuously differentiable, then the Stefan boundary condition of (2.1.1) holds pointwise.

**Lemma 2.4.2.** Suppose that \( \{u(t, \cdot) \mid 0 \leq t < \tau\} \subset C_0(\mathbb{R}) \cap C^1([\beta(t), \infty)) \) is a weak solution of (2.1.1).

Assume also that

\[
E \left[ \sup_{0 \leq t < \tau} \int_{x=\beta(t)}^{\infty} \left( \frac{\partial^{n} u}{\partial x^{n}}(t, x) \right)^2 dx \right] < \infty.
\]  

(2.4.3)

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If \( \beta \) is continuously differentiable, then

\[
\lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t, x) = -\varrho \dot{\beta}(t)
\]

for all \( t \in [0, \tau) \).

**Proof.** We first rewrite (2.4.2) to see the boundary behavior at the interface more clearly. If \( \{u(t, \cdot) \mid 0 \leq t < \tau \} \) is a weak solution of (2.1.1) and \( 0 < t < \tau \), set

\[
U_1(t, x) \overset{\text{def}}{=} \int_{y=0}^{\infty} p_+(t, x, y)u_0(y)dy
\]

\[
U_2(t, x) \overset{\text{def}}{=} \int_{s=0}^{t} e^{\alpha(t-s)} \int_{y=0}^{\infty} p_+(t-s, y + \beta(s))\dot{\beta}(s)dy ds
\]

\[
U_3(t, x) \overset{\text{def}}{=} \int_{s=0}^{t} \int_{z \in \mathbb{R}} e^{\alpha(t-s)} \int_{y=0}^{\infty} p_+(t-s, y + \beta(s))u(s, y + \beta(s))\eta(y + \beta(s) - z)dy W(ds dz)
\]

\[
U_4(t, x) \overset{\text{def}}{=} \varrho \int_{s=0}^{t} p_+(t-s, x, 0)\dot{\beta}(s)ds.
\]

Let \( \varepsilon > 0 \). Note that

\[
\int_{s=0}^{t} \int_{y=0}^{\infty} p_+(t-s, \varepsilon, y)\frac{\partial u}{\partial x}(s, y + \beta(s))\dot{\beta}(s)dy ds = U_2(t, \varepsilon) + U_2(t, -\varepsilon)
\]

\[
\int_{y=0}^{\infty} \int_{s=0}^{t} p_+(t-s, \varepsilon, y)u(s, y + \beta(s))d\zeta(y, \beta(s) - z)dy = U_3(t, \varepsilon) + U_3(t, -\varepsilon)
\]

Thus

\[
u(t, \beta(t) + \varepsilon) = U_1(t, \varepsilon) + U_2(t, \varepsilon) + U_2(t, -\varepsilon) + U_3(t, \varepsilon) + U_3(t, -\varepsilon) + U_4(t, \varepsilon)
\]

and hence

\[
\frac{\partial u}{\partial x}(t, \beta(t) + \varepsilon) = \frac{\partial U_1}{\partial x}(t, \varepsilon) + \frac{\partial U_2}{\partial x}(t, \varepsilon) - \frac{\partial U_2}{\partial x}(t, -\varepsilon) + \frac{\partial U_3}{\partial x}(t, \varepsilon) - \frac{\partial U_3}{\partial x}(t, -\varepsilon) + \frac{\partial U_4}{\partial x}(t, \varepsilon).
\]

Since \( \frac{\partial p_+}{\partial x}(t, 0, y) = 0 \), we have

\[
\lim_{\varepsilon \searrow 0} \frac{\partial U_1}{\partial x}(t, \varepsilon) = 0.
\]

Next note that

\[
\frac{\partial p_+}{\partial x}(t, x) = -\frac{1}{2\sqrt{\pi}} \frac{x}{t^{3/2}} \exp \left[ -\frac{x^2}{4t} \right].
\]
Thus

\[
\frac{\partial U_2}{\partial \varepsilon}(t, \varepsilon) = \int_{s=0}^{t} e^{\alpha(t-s)} \int_{y=0}^{\infty} \frac{\partial p_{\alpha}}{\partial x}(t-s, y+\varepsilon) \frac{\partial u}{\partial x}(s, y + \beta(s)) \beta(s) dy ds
\]

\[
= -\frac{1}{2\sqrt{4\pi}} \int_{s=0}^{t} e^{\alpha(t-s)} \int_{y=0}^{\infty} \frac{y + \varepsilon}{(t-s)^{3/2}} \exp\left[\frac{(y + \varepsilon)^2}{4(t-s)}\right] \frac{\partial u}{\partial x}(s, y + \beta(s)) \beta(s) dy ds
\]

\[
= -\frac{1}{2\sqrt{4\pi}} \int_{s=0}^{t} e^{\alpha(t-s)} \int_{y=0}^{\infty} y \exp\left[-\frac{y^2}{4}\right] \frac{\partial u}{\partial x}(s, y\sqrt{t-s} - \varepsilon + \beta(s)) \beta(s) dy ds
\]

Dominated convergence then implies that

\[
\lim_{\varepsilon \to 0} \frac{\partial U_2}{\partial \varepsilon}(t, \varepsilon) = -\frac{1}{2\sqrt{4\pi}} \int_{s=0}^{t} e^{\alpha(t-s)} \int_{y=0}^{\infty} y \exp\left[-\frac{y^2}{4}\right] \frac{\partial u}{\partial x}(s, y\sqrt{t-s} + \beta(s)) \beta(s) dy ds
\]

Turning to \( U_3 \), we can integrate by parts and using the fact that \( u(t, \beta(t)) = 0 \), we get that

\[
\frac{\partial U_3}{\partial \varepsilon}(t, \varepsilon) = \int_{s=0}^{t} \int_{z \in \mathbb{R}} e^{\alpha(t-s)} \int_{y=0}^{\infty} \frac{\partial p_{\alpha}}{\partial x}(t-s, y+\varepsilon) u(s, y + \beta(s)) \eta(y + \beta(s) - z) dy W(ds dz)
\]

\[
= -\int_{s=0}^{t} \int_{z \in \mathbb{R}} e^{\alpha(t-s)} \int_{y=0}^{\infty} p_{\alpha}(t-s, y+\varepsilon) \left\{ \frac{\partial u}{\partial x}(s, y + \beta(s)) \eta(y + \beta(s) - z) + u(s, y + \beta(s)) \eta(y + \beta(s) - z) \right\} dy W(ds dz).
\]

Since

\[
\sup_{\varepsilon \in (0, 1)} \int_{t=0}^{T} \int_{y=0}^{\infty} p_{\alpha}^2(t, y+\varepsilon) dy dt < \infty
\]

for all \( T > 0 \), we can use dominated convergence and (2.4.3) to get that

\[
\lim_{\varepsilon \to 0} \frac{\partial U_3}{\partial \varepsilon}(t, \varepsilon) = -\int_{s=0}^{t} \int_{z \in \mathbb{R}} e^{\alpha(t-s)} \int_{y=0}^{\infty} p_{\alpha}(t-s, y) \left\{ \frac{\partial u}{\partial x}(s, y + \beta(s)) \eta(y + \beta(s) - z) + u(s, y + \beta(s)) \eta(y + \beta(s) - z) \right\} dy W(ds dz).
\]

Finally we consider \( U_4 \). Since \( p_+(t, x, 0) = 2e^{\alpha t} p_{\alpha}(t, x) \), we get that

\[
\frac{\partial p_+}{\partial x}(t, x) = -\frac{e^{\alpha(t-s)}}{\sqrt{4\pi}} \frac{x}{t^{3/2}} \exp\left[-\frac{x^2}{4t}\right].
\]
Therefore

\[ \frac{\partial U_4}{\partial \varepsilon}(t, \varepsilon) = -\varrho \int_{s=0}^{t} e^{\alpha(t-s)} \frac{\varepsilon}{\sqrt{4\pi(t-s)^3/2}} \exp \left[ -\frac{\varepsilon^2}{4(t-s)} \right] \dot{\beta}(s) ds \]

\[ = -\frac{2\varrho}{\sqrt{4\pi}} \int_{u=\varepsilon/\sqrt{t}}^{\infty} e^{\alpha^2/u^2} \exp \left[ -\frac{u^2}{4} \right] \dot{\beta}(t - \varepsilon^2/u^2) du \]

Since \( \dot{\beta} \) is continuously differentiable, dominated convergence ensures that

\[ \lim_{\varepsilon \downarrow 0} \frac{\partial U_4}{\partial x}(t, \varepsilon) = -\frac{2\varrho}{\sqrt{4\pi}} \int_{u=0}^{\infty} \exp \left[ -\frac{u^2}{4} \right] du = -\varrho \dot{\beta}(t). \]

Collecting things together, we get the claim. \(\square\)

### 2.5 A Transformation

Moving boundary problems are difficult since the boundary is \textit{a priori} unknown. For one-dimensional problems, however, we can shift the boundary in order to obtain a fixed boundary condition. Let’s again return to our deterministic PDE (2.3.1). For all \( t \geq 0 \) and \( x \in \mathbb{R} \), define \( \tilde{v}(t, x) = v(t, x + \beta_\circ(t)) \); then \( v(t, x) = \tilde{v}(t, x - \beta_\circ(t)) \). Assuming that \( \beta_\circ \) is differentiable, we have that for \( x > 0 \) and \( t > 0 \),

\[ \frac{\partial \tilde{v}}{\partial t}(t, x) = \frac{\partial v}{\partial t}(t, x + \beta_\circ(t)) + \frac{\partial v}{\partial x}(t, x + \beta_\circ(t)) \dot{\beta}_\circ(t) \]

\[ \frac{\partial \tilde{v}}{\partial x}(t, x) = \frac{\partial v}{\partial x}(t, x + \beta_\circ(t)) \]

\[ \frac{\partial^2 \tilde{v}}{\partial x^2}(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x + \beta_\circ(t)). \]

We can combine these equations and use the PDE for \( v \) to rewrite the evolution of \( \tilde{v} \) as

\[ \frac{\partial \tilde{v}}{\partial t}(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x + \beta_\circ(t)) + \alpha_x v(t, x + \beta_\circ(t)) + v(t, x + \beta_\circ(t)) b(t, x + \beta_\circ(t)) \]

\[ + \frac{\partial v}{\partial x}(t, x + \beta_\circ(t)) \dot{\beta}_\circ(t) \]

\[ = \frac{\partial^2 \tilde{v}}{\partial x^2}(t, x) + \alpha_x \tilde{v}(t, x) + \frac{\partial \tilde{v}}{\partial x}(t, x) \dot{\beta}_\circ(t) + \tilde{v}(t, x) b(t, x + \beta_\circ(t)). \]

(2.5.1)
Inserting the boundary condition that \(-\varrho \dot{\beta}_s(t) = \lim_{x \to \beta(t)} \frac{\partial v}{\partial x}(t, x)\) back into (2.5.1), we have that
\[
\frac{\partial \tilde{v}}{\partial t}(t, x) = \frac{\partial^2 \tilde{v}}{\partial x^2}(t, x) + \alpha_s \tilde{v}(t, x) - \frac{1}{\varrho} \frac{\partial \tilde{v}}{\partial x}(t, 0) \frac{\partial \tilde{v}}{\partial x}(t, x) + \tilde{v}(t, x) b(t, x + \beta_o(t)) \quad t > 0, x > 0
\]
\[
\tilde{v}(t, 0) = 0 \quad t > 0
\]
\[
\tilde{v}(0, x) = u_o(x) \quad x > 0
\]
\[
\dot{\beta}_s(t) = -\frac{1}{\varrho} \frac{\partial \tilde{v}}{\partial x}(t, 0) \quad t > 0.
\]

Replacing \(b\) by \(\zeta\) and \(\alpha_s\) by \(\alpha\), we should be able to write down a nonlinear SPDE for \(\tilde{u}(t, x) = u(t, x + \beta(t))\).

We now get the following.

**Lemma 2.5.1.** Suppose that \(\{u(t, \cdot) \mid 0 \leq t < \tau\} \subset C_0(\mathbb{R}) \cap L^1(\mathbb{R})\) is a solution of (2.1.1) such that \(u(t, \cdot) \in C^1([\beta(t), \infty))\) and \(\frac{\partial u}{\partial x}(t, \cdot) \in L^1([\beta(t), \infty))\) for each \(0 \leq t < \tau\) and (2.4.3) holds. Suppose also that \(\beta\) is continuously differentiable and \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted. Then \(\tilde{u}(t, x) = u(t, x + \beta(t))\) satisfies the integral equation
\[
\tilde{u}(t, x) = \int_{y=0}^{\infty} p_-(t, x, y) u_o(y)dy
\]
\[
- \frac{1}{\varrho} \int_{s=0}^{t} \int_{y=0}^{\infty} p_-(t-s, x, y) \frac{\partial \tilde{u}}{\partial x}(s, 0) \frac{\partial \tilde{u}}{\partial x}(s, y)dyds
\]
\[
+ \int_{y=0}^{\infty} \int_{s=0}^{t} p_-(t-s, x, y) \tilde{u}(s, y) d\zeta(s + \beta(s))dy
\]
for all \(t \in [0, \tau)\) and \(x > 0\) where
\[
\beta(t) = -\frac{1}{\varrho} \int_{s=0}^{t} \frac{\partial \tilde{u}}{\partial x}(s, 0)ds
\]
for all \(t \in [0, \tau)\).

**Proof.** The proof is very similar to that of Lemma 2.4.2. Fix \(x > 0\) and \(T > 0\). For \(t \in [0, \tau \wedge T)\), define
\[
U^T(t) \overset{\text{def}}{=} \int_{y=0}^{\infty} \tilde{u}(t, y) p_-(T-t, x, y)dy.
\]

Using Definition 2.3.3 we have
\[
U^T(t) = \int_{y=0}^{\infty} u(t, y + \beta(t)) p_-(T-t, x, y)dy = \int_{y=\beta(t)}^{\infty} u(t, y) p_-(T-t, x, y - \beta(t))dy
\]
\[
= \int_{y \in \mathbb{R}} u(t, y) p_-(T-t, x, y - \beta(t))dy
\]
\[
= U^T(0) + U^T_1(t) + U^T_2(t)
\]
where

\[
U_T^1(t) = \int_{r=0}^{t} \int_{y \in \mathbb{R}} u(r,y) \left\{ \frac{\partial p}{\partial t}(T - r, x, y - \beta(r)) + \frac{\partial^2 p}{\partial y^2}(T - r, x, y - \beta(r)) + \alpha p(T - r, x, y - \beta(r)) \beta'(r) \right\} dydr
+ \varrho \int_{r=0}^{t} p(T - r, x, y - \beta(r)) \beta'(r) dr,
\]

\[
U_T^2(t) = \int_{y \in \mathbb{R}} \int_{r=0}^{t} u(r,y) p(T - r, x, y - \beta(r)) d\zeta_r(y) dy.
\]

Let’s consider the differential of $U_T^1$. Using integration by parts, (2.4.1) and the fact that $p(T - t, x, 0) = 0$, we get that

\[
dU_T^1(t) = - \left\{ \int_{y=\beta(t)}^{\infty} u(t,y) \frac{\partial p}{\partial y}(T - t, x, y - \beta(t)) dx \right\} \dot{\beta}(t) dt
= \left\{ \int_{y=\beta(t)}^{\infty} \frac{\partial}{\partial y}(t,y)p(T - t, x, y - \beta(t)) dx \right\} \dot{\beta}(t) dt
= \left\{ \int_{y=0}^{\infty} \frac{\partial}{\partial y}(t,y + \beta(t))p(T - t, x, y) dx \right\} \dot{\beta}(t) dt
= \left\{ \int_{y=0}^{\infty} \frac{\partial \tilde{u}}{\partial y}(t,y)p(T - t, x, y) dx \right\} \dot{\beta}(t) dt.
\]

We have here used the fact that $u(t, \beta(t)) = 0$. Combining the characterization of $\dot{\beta}$ as in Lemma 2.4.2, we get that

\[
\dot{\beta}(t) = -\frac{1}{\varrho} \frac{\partial \tilde{u}}{\partial x}(t,0).
\]

Thus

\[
U_T^1(t) = -\frac{1}{\varrho} \int_{s=0}^{t} \left\{ \int_{y=0}^{\infty} p(T - s, x, y) \frac{\partial \tilde{u}}{\partial x}(s,0) \frac{\partial \tilde{u}}{\partial y}(s,y) dy \right\} ds.
\]
Now we consider $U^T_2(t)$. Again using the stochastic Fubini theorem, we obtain

$$U^T_2(t) = \int_{y \in \mathbb{R}} \int_{r=0}^{t} u(r, y)p_-(T - r, x, y - \beta(r))d\zeta_r(y)dy$$

$$= \int_{y \in \mathbb{R}} \int_{r=0}^{t} u(r, y)p_-(T - r, x, y - \beta(r))\int_{z \in \mathbb{R}} \eta(y - z)W(dr, dz)dy$$

$$= \int_{r=0}^{t} \int_{z \in \mathbb{R}} \left\{ \int_{y \in \mathbb{R}} u(r, y)p_-(T - r, x, y - \beta(r))\eta(y - z)dy \right\} W(dr, dz)$$

$$= \int_{r=0}^{t} \int_{z \in \mathbb{R}} \left\{ \int_{y=\beta(r)}^{\infty} u(r, y)p_-(T - r, x, y - \beta(r))\eta(y - z)dy \right\} W(dr, dz)$$

$$= \int_{r=0}^{t} \int_{z \in \mathbb{R}} \left\{ \int_{y=0}^{\infty} u(r, y + \beta(r))p_-(T - r, x, y)\eta(y + \beta(r) - z)dy \right\} W(dr, dz)$$

$$= \int_{y=0}^{\infty} \int_{r=0}^{t} u(r, y + \beta(r))p_-(T - r, x, y)d\zeta_r(y + \beta(r))dy$$

$$= \int_{y=0}^{\infty} \int_{r=0}^{t} \tilde{u}(r, y)p_-(T - r, x, y)d\zeta_r(y + \beta(r))dy.$$

Combine things to get that

$$U^T(t) = U^T(0) - \frac{1}{\varrho} \int_{s=0}^{t} \left\{ \int_{y=0}^{\infty} \frac{\partial \tilde{u}(s, 0)}{\partial x} \frac{\partial \tilde{u}(s, y)}{\partial y} p_+(T - s, x, y)dy \right\} ds$$

$$+ \int_{y=0}^{\infty} \int_{s=0}^{t} \tilde{u}(s, y)p_+(T - s, x, y)d\zeta_r(y + \beta(s))dy.$$  

Now let $T \searrow t$ to get the claimed result. \qed

We can also obtain a SPDE for $\tilde{u}$. By (2.4.1) and (2.5.2), we have that for $0 < t < \tau$ and $x > 0$

$$d\tilde{u}(t, x) = \int_{y=0}^{\infty} \left\{ \frac{\partial^2 p_-(t, x, y) + \alpha p_-(t, x, y)}{\partial x^2} u_0(y)dy dt ight\}$$

$$- \frac{1}{\varrho} \int_{s=0}^{t} \int_{y=0}^{\infty} \left\{ \frac{\partial^2 p_-}{\partial x^2}(t - s, x, y) + \alpha p_-(t - s, x, y) \right\} \frac{\partial \tilde{u}(s, 0)}{\partial x}(s, y)dy ds dt$$

$$+ \int_{y=0}^{\infty} \int_{s=0}^{t} \left\{ \frac{\partial^2 p_-}{\partial x^2}(t - s, x, y) + \alpha p_-(t - s, x, y) \right\} \tilde{u}(s, y)d\zeta_r(y + \beta(s))dy dt$$

$$- \frac{1}{\varrho} \frac{\partial \tilde{u}(t, 0)}{\partial x}(t, x)dt + \tilde{u}(t, x)d\zeta(x + \beta(t)).$$
Since \( p_-(t, 0, y) = 0 \), we have the following SPDE

\[
\begin{align*}
\tilde{d}\tilde{u}(t, x) &= \left\{ \frac{\partial^2 \tilde{u}}{\partial x^2}(t, x) + \alpha \tilde{u}(t, x) - \frac{1}{\varrho} \frac{\partial \tilde{u}}{\partial x}(t, 0) \frac{\partial \tilde{u}}{\partial x}(t, x) \right\} dt \\
&\quad + \tilde{u}(t, x) d\zeta_t(x + \beta(t)) \quad 0 < t < \tau, x > 0 \\
\tilde{u}(t, 0) &= 0 \quad 0 < t < \tau \\
\tilde{u}(0, x) &= u_c(x) \quad x > 0 \\
\dot{\beta}(t) &= -\frac{1}{\varrho} \frac{\partial \tilde{u}}{\partial x}(t, 0) \quad 0 < t < \tau
\end{align*}
\] (2.5.3)

We can also find a converse to Lemma 2.5.1.

**Lemma 2.5.2.** Suppose that \( \{ \tilde{u}(t, \cdot) | 0 \leq t < \tau \} \subset C^0(\mathbb{R}_+) \cap C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) for \( 0 \leq t < \tau \) satisfies (2.5.2) and is strictly positive for \( x > 0 \). Set

\[
\begin{align*}
\beta(t) &= -\frac{1}{\varrho} \int_{s=0}^{t} \frac{\partial \tilde{u}}{\partial x}(s, 0) \, ds \quad 0 \leq t < \tau \\
u(t, x) &\overset{\text{def}}{=} \begin{cases}
\tilde{u}(t, x - \beta(t)) & x \geq \beta(t), 0 \leq t < \tau \\
0 & x < \beta(t), 0 \leq t < \tau
\end{cases}
\end{align*}
\] (2.5.4)

and define

Then \( \{ u(t, \cdot) | 0 \leq t < \tau \} \) is a weak solution of (2.1.1).

**Proof.** Fix \( \varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}) \) and define for \( 0 \leq t < \tau \),

\[
U(t) \overset{\text{def}}{=} \int_{x \in \mathbb{R}} \varphi(t, x) u(t, x) dx = \int_{x=\varphi(t)}^{\infty} \varphi(t, x) \tilde{u}(t, x - \beta(t)) dx = \int_{x=0}^{\infty} \varphi(t, x + \beta(t)) \tilde{u}(t, x) dx
\]

To see the evolution of \( U \), we fix \( \delta > 0 \) and define

\[
\begin{align*}
\tilde{u}_\delta(t, x) &\overset{\text{def}}{=} \int_{y=0}^{\infty} p_-(\delta, x, y) \tilde{u}(t, y) dy \quad x \geq 0 \\
u_\delta(t, x) &\overset{\text{def}}{=} \begin{cases}
\tilde{u}_\delta(t, x - \beta(t)) & x \geq \beta(t), 0 \leq t < \tau \\
0 & x < \beta(t), 0 \leq t < \tau
\end{cases}
\end{align*}
\]
We rewrite $\tilde{u}_\delta$. Using (2.5.2) and the semigroup property of the heat kernel, we get

\[ \tilde{u}_\delta(t, x) = \int_{y=0}^{\infty} p_-(t + \delta, x, y) u_s(y) dy \]

\[ - \frac{1}{\varrho} \int_0^t \int_{y=0}^{\infty} p_-(t + \delta - s, x, y) \frac{\partial u}{\partial x}(s, 0) \frac{\partial u}{\partial x}(s, y) dy ds \]

\[ + \int_{y=0}^{\infty} \int_{s=0}^{t} p_-(t + \delta - s, x, y) u_s(y) d\zeta_s(y + \beta(s)) dy. \]

(2.5.6)

Now we define

\[ U_1^\delta(t) \overset{\text{def}}{=} \int_{y=0}^{\infty} \varphi(t, x + \beta(t)) \tilde{u}_\delta(t, x) dx = U_1^\delta(t) + U_2^\delta(t) + U_3^\delta(t) \]

where

\[ U_1^\delta(t) = \int_{s=0}^{t} \int_{y=0}^{\infty} \int_{x=0}^{\infty} \varphi(t, x + \beta(t)) p_-(t + \delta - s, x, y) \xi(s, y) dy dx ds \]

\[ U_2^\delta(t) = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{s=0}^{t} \varphi(t, x + \beta(t)) p_-(t + \delta - s, x, y) \tilde{u}_s(y) d\zeta_s(y + \beta(s)) dy dx \]

\[ U_3^\delta(t) = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \varphi(t, x + \beta(t)) p_-(t + \delta, x, y) \tilde{u}_0(y) dy dx \]

where

\[ \xi(t, x) = -\frac{1}{\varrho} \frac{\partial u}{\partial x}(t, 0) \frac{\partial u}{\partial x}(t, x). \]

We also note that we can rewrite the evolution of $\beta$ as

\[ \dot{\beta}(t) = -\frac{1}{\varrho} \frac{\partial u}{\partial x}(t, \beta(t)), \quad t \in [0, \tau) \]
Thus

\[
dU_1^s(t) = \left( \int_{x=0}^{t} \int_{y=0}^{\infty} \int_{x=0}^{\infty} \left\{ \frac{\partial \varphi}{\partial t}(t, x + \beta(t))p_-(t + \delta - s, x, y) + \frac{\partial \varphi}{\partial x}(t, x + \beta(t))p_-(t + \delta - s, x, y) \gamma(t) \right\} \xi(s, y) dy \, dx \, ds \right) dt \\
+ \left( \int_{x=0}^{t} \int_{y=0}^{\infty} \varphi(t, x + \beta(t))p_-(\delta, x, y) \xi(t, y) dy \, dx \right) dt
\]

\[
= \left( \int_{x=0}^{t} \int_{y=0}^{\infty} \int_{x=0}^{\infty} \left\{ \frac{\partial \varphi}{\partial t}(t, x + \beta(t))p_-(t + \delta - s, x, y) + \frac{\partial \varphi}{\partial x}(t, x + \beta(t))p_-(t + \delta - s, x, y) \right\} \xi(s, y) dy \, dx \, ds \right) dt \\
+ \left( \int_{x=0}^{t} \int_{y=0}^{\infty} \varphi(t, x + \beta(t))p_-(\delta, x, y) \xi(t, y) dy \, dx \right) dt.
\]

Similar calculations show that

\[
dU_2^s(t) = \left( \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{x=0}^{t} \left\{ \frac{\partial \varphi}{\partial t}(t, x + \beta(t))p_-(t + \delta - s, x, y) \right\} \ddot{u}(s, y) d\zeta_s(y + \beta(s)) dy \, dx \right) dt \\
+ \left( \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{x=0}^{t} \frac{\partial \varphi}{\partial x}(t, x + \beta(t))p_-(t + \delta - s, x, y) \ddot{u}(s, y) d\zeta_s(y + \beta(s)) dy \, dx \right) dt \\
+ \left( \int_{x=0}^{\infty} \int_{y=0}^{\infty} \varphi(t, x + \beta(t))p_-(\delta, x, y) \ddot{u}(t, y) d\zeta_t(y + \beta(t)) dy \right) dt
\]

and finally

\[
dU_3^s(t) = \left( \int_{x=0}^{\infty} \int_{y=0}^{\infty} \left\{ \frac{\partial \varphi}{\partial t}(t, x + \beta(t))p_-(t + \delta, x, y) \right\} \ddot{u}_o(y) dy \, dx \right) dt \\
+ \left( \int_{x=0}^{\infty} \int_{y=0}^{\infty} \frac{\partial \varphi}{\partial x}(t, x + \beta(t))p_-(t + \delta, x, y) \ddot{u}_o(y) dy \, dx \right) dt.
\]
Adding these expressions together and using (2.5.6), we get that

\[
U^\delta(t) - U^\delta(0) = \int_{s=0}^{t} \left( \int_{x=0}^{\infty} \left( \frac{\partial \varphi}{\partial t} + \alpha \varphi \right) (s, x + \beta(s)) \tilde{u}_\delta(s, x) dx \right.
\]

\[\]
\[+ \int_{x=0}^{\infty} \varphi(s, x + \beta(s)) \frac{\partial^2 u_\delta}{\partial x^2}(s, x + \beta(s)) dx + \int_{x=0}^{\infty} \frac{\partial \varphi}{\partial x}(s, x + \beta(s)) u_\delta(s, x + \beta(s)) dx \beta(s) \right) ds
\]

\[\]
\[+ \int_{x=0}^{t} \left( \int_{y=0}^{\infty} \varphi(s, x + \beta(s)) p_-(\delta, x, y) \xi(s, y) dy dx \right) ds
\]

\[\]
\[+ \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{s=0}^{t} \varphi(s, x + \beta(s)) p_-(\delta, x, y) \tilde{u}(s, y) d\zeta(y + \beta(s)) dy dx
\]

By definition of \( p_- \), we conclude that \( u_\delta(s, \beta(s)) = 0 \). In addition, we also have that

\[
\lim_{\delta \to 0} \frac{\partial \tilde{u}_\delta}{\partial x}(t, x) = \lim_{\delta \to 0} \int_{y=0}^{\infty} \frac{\partial p_-}{\partial x}(\delta, x, y) \tilde{u}(t, y) dy
\]

\[\]
\[= \lim_{\delta \to 0} \int_{y=0}^{\infty} \frac{\partial p_+}{\partial y}(\delta, x, y) \tilde{u}(t, y) dy
\]

\[\]
\[= \lim_{\delta \to 0} \int_{y=0}^{\infty} p_+(\delta, x, y) \frac{\partial \tilde{u}}{\partial y}(t, y) dy
\]

\[\]
\[= \frac{\partial \tilde{u}}{\partial x}(t, x).
\]

Upon letting \( \delta \to 0 \) and rearranging things, we indeed get a weak solution of (2.1.1).
2.6 The Transformed Nonlinear SPDE

By section 2.5 it is enough to prove existence and uniqueness of a solution of the transformed nonlinear
SPDE (2.5.2). However, it is not easy to handle (2.5.2) since it has the nonlocal nonlinear term \( \frac{\partial}{\partial x} u(t,0) \).
In addition, we have a different noise at each position and the shift by \( \beta \) in the evaluation of the integral
against \( \zeta \) complicates things. In order to control these difficulties, we will use truncation functions.

Let’s first set up a functional framework in which we can use Picard-type iterations. As usual, \( C_0^\infty(R_+) \) is
the collection of infinitely smooth functions on \( [0, \infty) \) which asymptotically vanishes at infinity. Define next

\[
C_0^{\infty, \text{odd}}(R_+) \equiv \left\{ \varphi \in C_0^\infty(R_+) \mid \varphi^{(n)}(0) = 0 \text{ for all even } n \in \mathbb{N} \right\};
\]

in other words, \( C_0^{\infty, \text{odd}}(R_+) \) are those elements of \( C_0^\infty(R_+) \) which can be extended to an odd element of
\( C^\infty(R) \) (namely, consider the map \( y \mapsto \text{sgn}(y) \varphi(|y|) \)). For all \( \varphi \in C_0^\infty(R_+) \), define

\[
\| \varphi \|_H \overset{\text{def}}{=} \sqrt{\sum_{i=0}^{2} \int_{x \in (0, \infty)} |\varphi^{(i)}(x)|^2 \, dx}.
\]

Let \( H \) be the closure of \( C_0^{\infty}(R_+) \) with respect to \( \| \cdot \|_H \) and let \( H_{\text{odd}} \) be the closure of \( C_0^{\infty, \text{odd}}(R_+) \) with
respect to \( \| \cdot \|_H \). We also define

\[
\| \varphi \|_L \overset{\text{def}}{=} \sqrt{\int_{x \in (0, \infty)} |\varphi(x)|^2 \, dx}
\]

for all square-integrable functions on \( R_+ \). Of course \( H \) and \( H_{\text{odd}} \) are Hilbert spaces (\( H \) is more commonly
written as \( H^2 \); i.e., it is the collection of functions on \( R_+ \) which possess two weak square-integrable derivatives).
The important aspect of \( H \) is the following fairly standard result.

Lemma 2.6.1. We have that \( H \subset C_0(R_+) \cap C^1(R_+) \). More precisely, for any \( \varphi \in H \), we have that

\[
\sup_{x \in R_+} \left| \varphi^{(i)}(x) \right| \leq 2\| \varphi \|_H.
\]

Finally, for \( i \in \{0, 1\} \), \( \varphi^{(i)}(0) \overset{\text{def}}{=} \lim_{x \to 0} \varphi^{(i)}(x) \) is well-defined.

Proof. The fact that \( H \subset C^1 \) is well-known: [Eva98]. Fix \( \varphi \in C_0^\infty(R_+) \), \( x \in (0, \infty) \), and \( i \in \{0, 1\} \). We then
have that
\[
\frac{\partial^i \varphi}{\partial x^i}(x) = \int_{s=x}^{x+1} \frac{\partial^i \varphi}{\partial x^i}(s) ds - \int_{s=x}^{x+1} \left\{ \frac{\partial^i \varphi}{\partial x^i}(s) - \frac{\partial^i \varphi}{\partial x^i}(x) \right\} ds
\]
\[
= \int_{s=x}^{x+1} \frac{\partial^i \varphi}{\partial x^i}(s) ds - \int_{s=x}^{x+1} s \frac{\partial^i+1 \varphi}{\partial x^{i+1}}(s) ds = \int_{s=x}^{x+1} \frac{\partial^i \varphi}{\partial x^i}(s) ds - \int_{r=x}^{x+1} (x+1-r) \frac{\partial^i+1 \varphi}{\partial x^{i+1}}(r) dr
\]

Thus
\[
\left| \frac{\partial^i \varphi}{\partial x^i}(x) \right| \leq \sqrt{\int_{s=x}^{x+1} \left| \frac{\partial^i \varphi}{\partial x^i}(s) \right|^2 ds} + \sqrt{\int_{s=x}^{x+1} \left| \frac{\partial^i+1 \varphi}{\partial x^{i+1}}(s) \right|^2 ds} \leq \| \varphi \|_H.
\]

Of course we also have that
\[
\left| \varphi^{(i)}(x) - \varphi^{(i)}(y) \right| \leq \left| \int_{s=x}^{y} \varphi^{(i+1)}(s) ds \right| \leq \| \varphi \|_H \sqrt{|x-y|}
\]
so the stated limits at \( x = 0 \) exist. Furthermore,
\[
\| \varphi(M) \| \leq \sqrt{\int_{s=M}^{\infty} \left| \frac{\partial \varphi}{\partial x}(s) \right|^2 ds} + \sqrt{\int_{s=M}^{\infty} \left| \frac{\partial \varphi}{\partial x}(s) \right|^2 ds},
\]
which implies \( H \in C_0(\mathbb{R}_+) \).

Now we define the Dirichlet heat semigroup, which will be used in Picard iteration analysis. For \( \varphi \in C_0^\infty(\mathbb{R}_+), t > 0, \) and \( x > 0, \) define
\[
(T_t \varphi)(x) \overset{\text{def}}{=} \int_{y=0}^{\infty} p_-(t, x, y) \varphi(y) dy.
\]

**Lemma 2.6.2.** For each \( t > 0, \) \( T_t \) has a unique extension from \( C_0^\infty(\mathbb{R}_+) \) to \( H \) such that \( T_t H \subset H_{\text{odd}} \) and such that \( \| T_t f \|_H \leq e^{\alpha t} \| f \|_H \) for all \( f \in H. \) Secondly, there is a \( K_A > 0 \) such that
\[
\| T_t \hat{f} \|_H \leq \frac{K_A}{e^{3/4}} \| f \|_H
\]
for all \( f \in H_{\text{odd}} \cap C^3(\mathbb{R}_+). \)

**Proof.** The proof relies upon a combination of fairly standard calculations.
To begin, fix $\varphi \in C_0^\infty(\mathbb{R}_+)$ and define

$$u(t, x) \overset{\text{def}}{=} \int_{y \in \mathbb{R}} p_\sigma(t, x - y) \text{sgn}(y) \varphi(|y|) dy = \int_{y = 0}^{\infty} p_\sigma(t, x - y) \varphi(y) dy - \int_{y = -\infty}^{0} p_\sigma(t, x - y) \varphi(-y) dy = \int_{y = 0}^{\infty} \{p_\sigma(t, x - y) - p_\sigma(t, x + y)\} \varphi(y) dy.$$

Thus $u(t, x) = (T_t \varphi)(x)$ for $x > 0$, and since $p_\sigma$ is even in its second argument,

$$u(t, -x) = \int_{y \in \mathbb{R}} p_\sigma(t, -x - y) \text{sgn}(y) \varphi(|y|) dy = \int_{y \in \mathbb{R}} p_\sigma(t, x + y) \text{sgn}(y) \varphi(|y|) dy$$

$$= -\int_{y \in \mathbb{R}} p_\sigma(t, x - y) \text{sgn}(y) \varphi(|y|) dy = -u(t, x)$$

so in fact $u(t, \cdot)$ is odd. Thus we indeed have that $\frac{\partial^n u}{\partial x^n}(t, 0) = 0$ for all even $n \in \mathbb{N}$; thus $T_t \varphi \in H_{\text{odd}}$.

A standard calculation shows that $T_t$ is a contraction on $H$. Indeed, for each nonnegative integer $n$,

$$\frac{d}{dt} \int_{x \in \mathbb{R}} \left| \frac{\partial^n u}{\partial x^n}(t, x) \right|^2 dx = 2 \int_{x \in \mathbb{R}} \frac{\partial^{n+2} u}{\partial x^{n+2}}(t, x) \frac{\partial^n u}{\partial x^n}(t, x) dx = -2 \int_{x \in \mathbb{R}} \left| \frac{\partial^{n+1} u}{\partial x^{n+1}}(t, x) \right|^2 dx \leq 0$$

and thus

$$\int_{x = 0}^{\infty} \left| \frac{\partial^n u}{\partial x^n}(t, x) \right|^2 dx = \frac{1}{2} \int_{x \in \mathbb{R}} \left| \frac{\partial^n u}{\partial x^n}(t, x) \right|^2 dx \leq \frac{1}{2} \int_{x \in \mathbb{R}} \left| \frac{\partial^n u}{\partial x^n}(0, x) \right|^2 dx$$

(2.6.1)

Summing these inequalities up for $n \in \{0, 1, 2\}$, we see that $\|T_t \varphi\|_H^2 \leq \|\varphi\|_H^2$ for all $\varphi \in C_0^\infty(\mathbb{R}_+)$. This implies that $T_t$ is a contraction on $C_0^\infty(\mathbb{R}_+)$ and has the claimed extension.

To proceed, fix $\varphi \in C_{0, \text{odd}}(\mathbb{R}_+)$. Note that thus $y \mapsto \varphi(|y|)$ is continuous. Define

$$v(t, x) = \int_{y = 0}^{\infty} \{p(t, x - y) - p_\sigma(t, x + y)\} \varphi^{(1)}(y) dy = \int_{y \in \mathbb{R}} p_\sigma(t, x - y) \text{sgn}(y) \varphi^{(1)}(|y|) dy$$

We can now fairly easily conclude from (2.6.1) with $n = 0$ that

$$\int_{x = 0}^{\infty} v^2(t, x) dx \leq \int_{x = 0}^{\infty} \left| \varphi^{(1)}(x) \right|^2 dx.$$
Differentiating and integrating by parts as needed, we get that

\[
\frac{\partial v}{\partial x}(t, x) = \int_{y \in \mathbb{R}} \frac{\partial p_\circ}{\partial x}(t, x - y) \text{sgn}(y) \varphi^{(1)}(|y|) dy
\]

\[
= -2 p_\circ(t, x) \varphi^{(1)}(0) - \int_{y \in \mathbb{R}} p_\circ(t, x - y) \varphi^{(2)}(|y|) dy
\]

\[
\frac{\partial^2 v}{\partial x^2}(t, x) = -2 \frac{\partial p_\circ}{\partial x}(t, x) \varphi^{(1)}(0) - \int_{y \in \mathbb{R}} \frac{\partial p_\circ}{\partial x}(t, x - y) \varphi^{(2)}(|y|) dy
\]

\[
= -2 \frac{\partial p_\circ}{\partial x}(t, x) \varphi^{(1)}(0) - \int_{y \in \mathbb{R}} p_\circ(t, x - y) \text{sgn}(y) \varphi^{(3)}(|y|) dy.
\]

We now note that there is a \( K > 0 \) such that

\[
\left| \frac{\partial p_\circ}{\partial x}(t, x) \right| \leq \frac{K}{\sqrt{t}} p_\circ(2t, x)
\]

for all \( t > 0 \) and \( x \in \mathbb{R} \). Thus

\[
\left| \frac{\partial v}{\partial x}(t, x) \right| \leq 2 p_\circ(t, x) \left| \varphi^{(1)}(0) \right| + \int_{y \in \mathbb{R}} p_\circ(t, x - y) \left| \varphi^{(2)}(|y|) \right| dy
\]

\[
\left| \frac{\partial^2 v}{\partial x^2}(t, x) \right| \leq \frac{2K}{\sqrt{t}} p_\circ(2t, x) \left| \varphi^{(1)}(0) \right| + \int_{y \in \mathbb{R}} p_\circ(t, x - y) \left| \varphi^{(3)}(|y|) \right| dy.
\]

Note now that

\[
\sqrt{\int_{x=0}^{\infty} p_\circ^2(t, x) dx} = \sqrt{\frac{1}{4 \pi t} \int_{x=0}^{\infty} \frac{1}{\sqrt{\pi t}} \exp \left[ -\frac{x^2}{t} \right] dx} \leq \frac{1}{(4\pi t)^{1/4}}.
\]

Combining this and (2.6.1) with \( n = 0 \), we get that

\[
\sqrt{\int_{x \in \mathbb{R}} \left| \frac{\partial v}{\partial x}(t, x) \right|^2 dx} \leq \frac{2}{(4\pi t)^{1/4}} \left| \varphi^{(1)}(0) \right| + \sqrt{2 \int_{y=0}^{\infty} |\varphi^{(2)}|^2 dy}
\]

\[
\sqrt{\int_{x \in \mathbb{R}} \left| \frac{\partial^2 v}{\partial x^2}(t, x) \right|^2 dx} \leq \frac{2K}{\sqrt{t}(8\pi t)^{1/4}} \left| \varphi^{(1)}(0) \right| + \sqrt{2 \int_{y \in \mathbb{R}} |\varphi^{(3)}(y)|^2 dy}.
\]

Combine things together to get the last claim.

Let’s now define truncation functions. Fix \( L > 0 \), which we will use a truncation parameter. Let \( \Psi_L \in C^\infty(\mathbb{R}; [0, 1]) \) be monotone decreasing such that \( \Psi_L(x) = 1 \) if \( |x| \leq L \) and \( \Psi_L(x) = 0 \) if \( |x| \geq L + 1 \) (and thus \( |\Psi_L| \leq 1 \)). In other words, \( \Psi_L \) is a cutoff function with support of width \( L + 1 \). This truncation function allows us to control the difficulties coming from the nonlocal nonlinear term and the noise term; Picard iterations in general allow only linear growth of various coefficients.
Define
\[ \tilde{u}_n^L(t, x) = \int_{y=0}^{\infty} p_-(t, x, y)u_\omega(y)dy \]
for all \( t > 0 \) and \( x \in \mathbb{R} \) and recursively define
\[ \beta_n^L(t) \overset{\text{def}}{=} -\frac{1}{\varrho} \int_{s=0}^{t} \frac{\partial \tilde{u}_n^L(s, 0)}{\partial x} \, ds \quad t > 0 \]
\[ \tilde{u}_{n+1}^L(t, x) = \int_{y=0}^{\infty} p_-(t, x, y)u_\omega(y)dy \]
\[ - \frac{1}{\varrho} \int_{s=0}^{t} \int_{y=0}^{\infty} p_-(t-s, y) \frac{\partial \tilde{u}_n^L(s, 0)}{\partial x} \frac{\partial \tilde{u}_n^L(s, y)}{\partial x} \Psi_L \left( \|\tilde{u}_n^L(s, \cdot)\|_H \right) \, dy \, ds \]
\[ + \int_{s=0}^{t} \int_{y=0}^{\infty} p_-(t-s, y)\tilde{u}_n^L(s, y) \Psi_L \left( \|\tilde{u}_n^L(s, \cdot)\|_H \right) \, d\zeta_s(y + \beta_n^L(s))dy \quad t > 0, x > 0 \]
(2.6.2)

For each \( n \in \mathbb{N} \), \( \{\tilde{u}_n^L(t, \cdot); t \geq 0\} \) is a well-defined, adapted, and continuous path in \( H_{\text{odd}} \).

Another convenience will be to rewrite the \( ds \) part of (2.6.2). Set
\[ \tilde{\Psi}_n^\varrho(\psi) \overset{\text{def}}{=} -\frac{1}{\varrho} \psi(0)\Psi_L(\|\psi\|_H) \quad \text{and} \quad \tilde{\Psi}_n^b(\psi) \overset{\text{def}}{=} \Psi_L(\|\psi\|_H) \]
for all \( \psi \in H \). Then
\[ -\frac{\partial \tilde{u}_n^L(t, x)}{\partial x} \frac{1}{\varrho} \frac{\partial \tilde{u}_n^L(t, 0)}{\partial x} \Psi_L \left( \|\tilde{u}_n^L(t, \cdot)\|_H \right) = \frac{\partial \tilde{u}_n^L(t, x)}{\partial x} \tilde{\Psi}_n^\varrho(\tilde{u}_n^L(t, \cdot)) \]
\[ \Psi_L \left( \|\tilde{u}_n^L(t, \cdot)\|_H \right) = \tilde{\Psi}_n^b(\tilde{u}_n^L(t, \cdot)) \]
for all \( n \in \mathbb{N} \). For \( \psi \) and \( \eta \) in \( H \), let’s also define
\[ (D\tilde{\Psi}_n^\varrho)(\psi, \eta) \overset{\text{def}}{=} -\frac{1}{\varrho} \psi(0)\Psi_L(\|\psi\|_H) - \frac{1}{\varrho} \psi(0)\tilde{\Psi}_n^\varrho(\|\psi\|_H) \frac{\psi, \eta}_H \]
\[ \Psi_L(\|\psi\|_H) \]
\[ (D\tilde{\Psi}_n^b)(\psi, \eta) \overset{\text{def}}{=} \tilde{\Psi}_n^b(\|\psi\|_H) \frac{\psi, \eta}_H. \]

**Lemma 2.6.3.** For each \( \psi \) and \( \eta \) in \( H \), \((D\tilde{\Psi}_n^\varrho)(\psi, \eta)\) is the Gâteaux derivative of \( \tilde{\Psi}_n^\varrho \) at \( \psi \) in the direction of \( \eta \) and similarly \((D\tilde{\Psi}_n^b)(\psi, \eta)\) is the Gâteaux derivative of \( \tilde{\Psi}_n^b \) at \( \psi \) in the direction of \( \eta \). Furthermore, there is a \( K_B > 0 \) such that
\[ |(D\tilde{\Psi}_n^\varrho)(\psi, \eta)| \leq K_B \chi_{[0, L+1]}(\|\psi\|_H)\|\eta\|_H \]
\[ |(D\tilde{\Psi}_n^b)(\psi, \eta)| \leq K_B \chi_{[0, L+1]}(\|\psi\|_H)\|\eta\|_H \]
for all $\psi$ and $\eta$ in $H$ and $L > 0$.

Proof. By the definition of the Gâteaux derivative, the first claim is true. The second claim is quite straightforward. 

For each $n \in \mathbb{N}$, we now define $\tilde{w}_n^L(t, x) \stackrel{\text{def}}{=} \tilde{u}_{n+1}^L(t, x) - \tilde{u}_n^L(t, x)$ for all $x \geq 0$ and $t \geq 0$. Clearly sup$_{0 \leq t \leq T} \mathbb{E} \left[ \|\tilde{u}_1^L\|^2_{H^n} \right] < \infty$ for all $T > 0$. We then write that

$$
\tilde{w}_{n+1}^L(t, x) = \sum_{j=1}^{5} A_j^{(n)}(t, x)
$$

where

- $A_1^{(n)}(t, x) = \int_{\lambda=0}^{1} \int_{s=0}^{t} \left( \int_{y=0}^{\infty} p_-(t-s, x, y) \frac{\partial \tilde{w}_n^L}{\partial x}(s, y) dy \right) \tilde{\Psi}_n^a \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot) \right) ds d\lambda$
- $A_2^{(n)}(t, x) = \int_{\lambda=0}^{1} \int_{s=0}^{t} \left( \int_{y=0}^{\infty} p_-(t-s, x, y) \left\{ \frac{\partial \tilde{u}_n^L}{\partial x}(s, y) + \lambda \frac{\partial \tilde{w}_n^L}{\partial x}(s, y) \right\} dy \right) \times  D \tilde{\Psi}_n^a \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot), \tilde{w}_n^L(s, \cdot) \right) ds d\lambda$
- $A_3^{(n)}(t, x) = \int_{\lambda=0}^{1} \int_{s=0}^{t} \int_{y=0}^{\infty} p_-(t-s, x, y) \tilde{w}_n^L(s, y) \times \tilde{\Psi}_n^b \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot) \right) d\zeta_s \left( y + \beta_n^L(s) + \lambda (\beta_{n+1}^L(s) - \beta_n^L(s)) \right) dy d\lambda$
- $A_4^{(n)}(t, x) = \int_{\lambda=0}^{1} \int_{s=0}^{t} \int_{y=0}^{\infty} p_-(t-s, x, y) \left( \tilde{u}_n^L(s, y) + \lambda \tilde{w}_n^L(s, y) \right) \times D \tilde{\Psi}_n^b \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot), \tilde{w}_n^L(s, \cdot) \right) d\zeta_s \left( y + \beta_n^L(s) + \lambda (\beta_{n+1}^L(s) - \beta_n^L(s)) \right) dy d\lambda$
- $A_5^{(n)}(t, x) = \int_{\lambda=0}^{1} \int_{s=0}^{t} \int_{y=0}^{\infty} p_-(t-s, x, y) \tilde{w}_n^L(s, y) \times \left( \beta_{n+1}^L(s) - \beta_n^L(s) \right) d\eta \left( y - z + \beta_n^L(s) + \lambda (\beta_{n+1}^L(s) - \beta_n^L(s)) \right) W(ds, dz) d\lambda$

Note that the $\tilde{w}_n^L$'s and $\tilde{w}_n^L$'s are all in $H_{\text{odd}}$.

To bound $A_1^{(n)}$ and $A_2^{(n)}$, we use the fact that $t^{-3/4}$ is locally integrable. More precisely,

$$
\int_{s=0}^{t} \frac{1}{(t-s)^{3/4}} ds = 4t^{1/4}
$$

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for all $t > 0$. Thus
\[
E\left[\|A_1^{(n)}(t, \cdot)\|_H^2\right] \leq E\left[\int_{s=0}^t \left\| T_{t-s} \frac{\partial \tilde{u}_n^L}{\partial x}(s, \cdot)\right\|_H^2 ds\right]
\leq K_A^2 E\left[\int_{s=0}^t \left\| \tilde{u}_n^L(s, \cdot)\right\|_H^2 ds\right] \leq 4K_A^2 t^{1/4} E\left[\int_{s=0}^t \left\| \tilde{u}_n^L(s, \cdot)\right\|_H^2 ds\right].
\]

Similarly, we have that
\[
E\left[\|A_2^{(n)}(t, \cdot)\|_H^2\right] \leq K_B^2 E\left[\int_{s=0}^t \left\| T_{t-s} \left( \frac{\partial \tilde{u}_n^L}{\partial x}(s, \cdot) + \lambda \frac{\partial \tilde{w}_n^L}{\partial x}(s, \cdot)\right)\right\|_H^2 \chi_{[0, L+1]} \left(\left\| \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot)\right\|_H^2 \right) ds\right]
\leq K_A^2 K_B^2 E\left[\int_{s=0}^t \left(\frac{L+1}{(t-s)^{3/4}}\right)^2 \left\| \tilde{u}_n^L(s, \cdot)\right\|_H^2 ds\right] \leq 4K_A^2 K_B^2 (L+1)^{2} t^{1/4} E\left[\|\tilde{u}_n^L(s, \cdot)\|_H^2\right] ds.
\]

To bound $A_3^{(n)}$, $A_4^{(n)}$, and $A_5^{(n)}$, we first rewrite them. For $z \in \mathbb{R}$, define $\eta_z(y) \overset{\text{def}}{=} \eta(y - z)$ for all $y \in \mathbb{R}$.
Then

\[ A_3^{(n)}(t, x) = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \left\{ \int_{\lambda=0}^{1} \int_{y=0}^{\infty} p_{t-s, x, y} \bar{u}^{L}_{n}(s, y) \tilde{\Psi}^{b}_{t}(\bar{u}^{L}_{n}(s, \cdot) + \lambda \bar{w}^{L}_{n}(s, \cdot)) \times \eta (y - z + \beta^{L}_{n}(s) + \lambda (\beta^{L}_{n+1}(s) - \beta^{L}_{n}(s))) \, dy \, d\lambda \right\} W(ds, dz) \]

\[ = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \left\{ \int_{\lambda=0}^{1} T_{1-s} \left( \bar{u}^{L}_{n}(s, \cdot) \eta_{z} - \beta^{L}_{n}(s) - \lambda (\beta^{L}_{n+1}(s) - \beta^{L}_{n}(s)) \right)(x) \times \tilde{\Psi}^{b}_{t}(\bar{u}^{L}_{n}(s, \cdot) + \lambda \bar{w}^{L}_{n}(s, \cdot)) \right\} W(ds, dz) \]

\[ A_4^{(n)}(t, x) = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \left\{ \int_{\lambda=0}^{1} T_{1-s} \left( \bar{u}^{L}_{n}(s, y) + \lambda \bar{w}^{L}_{n}(s, y) \right) D\tilde{\Psi}^{b}_{t}(\bar{u}^{L}_{n}(s, \cdot) + \lambda \bar{w}^{L}_{n}(s, \cdot), \bar{w}^{L}_{n}(s, \cdot)) \times \eta (y - z + \beta^{L}_{n}(s) + \lambda (\beta^{L}_{n+1}(s) - \beta^{L}_{n}(s))) \, dy \, d\lambda \right\} W(ds, dz) \]

\[ = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \left\{ \int_{\lambda=0}^{1} T_{1-s} \left( \bar{u}^{L}_{n}(s, y) + \lambda \bar{w}^{L}_{n}(s, y) \right) D\tilde{\Psi}^{b}_{t}(\bar{u}^{L}_{n}(s, \cdot) + \lambda \bar{w}^{L}_{n}(s, \cdot), \bar{w}^{L}_{n}(s, \cdot)) \times \eta (y - z + \beta^{L}_{n}(s) + \lambda (\beta^{L}_{n+1}(s) - \beta^{L}_{n}(s))) \, dy \, d\lambda \right\} W(ds, dz) \]

\[ A_5^{(n)}(t, x) = \int_{s=0}^{t} \int_{y \in \mathbb{R}} \left\{ \int_{\lambda=0}^{1} T_{1-s} \left( \bar{u}^{L}_{n}(s, y) + \lambda \bar{w}^{L}_{n}(s, y) \right) \tilde{\Psi}^{b}_{t}(\bar{u}^{L}_{n}(s, \cdot) + \lambda \bar{w}^{L}_{n}(s, \cdot)) \times (\beta^{L}_{n+1}(s) - \beta^{L}_{n}(s)) \eta (y - z + \beta^{L}_{n}(s) + \lambda (\beta^{L}_{n+1}(s) - \beta^{L}_{n}(s))) \, dy \, d\lambda \right\} W(ds, dz) \]

We will use the following bound on the interaction between \( \eta \) and the \( H \)-norm.

**Lemma 2.6.4.** There is a \( K > 0 \) such that

\[ \int_{y \in \mathbb{R}} \| f \eta^{(k)}_{y} \|_{H}^{2} \, dy \leq K \| f \|_{H}^{2} \]

for all \( f \in H \) and \( k \in \{0, 1\} \).

**Proof.** The structure of \( \eta \) ensures that there is an \( \hat{\eta} \in L^{2}(\mathbb{R}) \) such that \( |\eta^{(n)}(x)| \leq \hat{\eta}(x) \) for all \( n \in \{0, 1, 2, 3\} \) and \( x \in \mathbb{R} \). Thus for all \( x \in \mathbb{R} \), \( k \in \{0, 1\} \), and \( n \in \{0, 1, 2\} \),

\[ \left| \left( f \eta^{(k)}_{y} \right)^{(n)}(x) \right| = \left| \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x) \eta^{(k+n-j)}(x-y) \right| \leq \sum_{j=0}^{n} \binom{n}{j} |f^{(j)}(x)| \hat{\eta}(x-y) \leq 2 \sum_{j=0}^{2} |f^{(j)}(x)| \hat{\eta}(x-y) \]

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Thus

\[ \sqrt{\int_{y \in \mathbb{R}} \| f^{(k)}(y) \|^2_H dy} \leq \sqrt{\sum_{j=0}^{2} \int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} |f^{(j)}(x)|^2 \hat{\eta}^2(x - y) dx dy} \]

\[ \leq 6 \| \hat{\eta} \|_{L^2(\mathbb{R})} \sqrt{\sum_{j=0}^{2} \int_{x \in \mathbb{R}} |f^{(j)}(x)|^2 dx}. \]

The claim follows.

Fix \( T > 0 \) and set \( K_1 \overset{\text{def}}{=} \exp(2|\alpha|T) \). Let’s first bound \( A_3^{(n)} \). For \( k \in \{0, 1, 2\} \),

\[ \frac{\partial^k A_3^{(n)}}{\partial x^k}(t, x) = \int_{s=0}^{t} \int_{x \in \mathbb{R}} \left\{ \int_{\lambda=0}^{1} \frac{\partial^k T_{t-s} \left( \tilde{\omega}^L_n(s, \cdot) \eta \xi - \beta^L_n(s) - \lambda(\beta_{n+1}^L(s) - \beta^L_n(s)) \right)}{\partial x^k} \right\}
\[ \times \tilde{\Psi}_L^b \left( \tilde{\omega}_n^L(s, \cdot) \right) + \lambda \tilde{\omega}_n^L(s, \cdot)) d\lambda \right\} W(ds, dz); \]

thus for \( 0 \leq t \leq T \)

\[ \mathbb{E} \left[ \int_{x=0}^{\infty} \left| \frac{\partial^k A_3^{(n)}}{\partial x^k}(t, x) \right|^2 dx \right]
\[ = \int_{s=0}^{t} \int_{x \in \mathbb{R}} \int_{x=0}^{\infty} \mathbb{E} \left\{ \int_{\lambda=0}^{1} \frac{\partial^k T_{t-s} \left( \tilde{\omega}^L_n(s, \cdot) \eta \xi - \beta^L_n(s) - \lambda(\beta_{n+1}^L(s) - \beta^L_n(s)) \right)}{\partial x^k} \right\}
\[ \times \tilde{\Psi}_L^b \left( \tilde{\omega}_n^L(s, \cdot) \right) + \lambda \tilde{\omega}_n^L(s, \cdot)) d\lambda \right\}^2 dx dz ds
\[ \leq \int_{\lambda=0}^{1} \int_{s=0}^{t} \int_{x \in \mathbb{R}} \mathbb{E} \left[ \int_{x=0}^{\infty} \left| \frac{\partial^k T_{t-s} \left( \tilde{\omega}^L_n(s, \cdot) \eta \xi - \beta^L_n(s) - \lambda(\beta_{n+1}^L(s) - \beta^L_n(s)) \right)}{\partial x^k} \right|^2 dx \right] dz ds d\lambda
\[ \leq \int_{\lambda=0}^{1} \int_{s=0}^{t} \int_{x \in \mathbb{R}} \mathbb{E} \left[ \left\| T_{t-s} \left( \tilde{\omega}^L_n(s, \cdot) \eta \xi - \beta^L_n(s) - \lambda(\beta_{n+1}^L(s) - \beta^L_n(s)) \right) \right\|_H^2 \right] dz ds d\lambda
\[ \leq K_1 \int_{\lambda=0}^{1} \int_{s=0}^{t} \int_{x \in \mathbb{R}} \mathbb{E} \left[ \left\| \tilde{\omega}^L_n(s, \cdot) \eta \xi - \beta^L_n(s) - \lambda(\beta_{n+1}^L(s) - \beta^L_n(s)) \right\|_H^2 \right] dz ds d\lambda
\[ \leq K_1 K \int_{s=0}^{t} \mathbb{E} \left[ \left\| \tilde{\omega}_n^L(s, \cdot) \right\|_H^2 \right] ds d\lambda
\[ \leq K_1 K \int_{s=0}^{t} \mathbb{E} \left[ \left\| \tilde{\omega}_n^L(s, \cdot) \right\|_H^2 \right] ds \]
The bound on $A_4^{(n)}$ is similar.

\[
\frac{\partial^k A_4^{(n)}}{\partial x^k}(t, x) = \int_{s=0}^{t} \int_{z \in \mathbb{R}} \int_{x = 0}^{1} \left\{ \int_{\lambda = 0}^{1} \frac{\partial^k T_{t-s} \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot) \right) \eta_{z-\beta_0^L(s)-\lambda(\beta_{n+1}^L(s)-\beta_0^L(s))}}{\partial x^k} \right\} \times D\tilde{\Psi}_n^L \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot), \tilde{w}_n^L(s, \cdot) \right) d\lambda \right\} W(ds, dz).
\]

Consequently for $0 \leq t \leq T$

\[
\mathbb{E} \left[ \int_{x=0}^{\infty} \left| \frac{\partial^k A_4^{(n)}}{\partial x^k}(t, x) \right|^2 dx \right] = \int_{s=0}^{t} \int_{z \in \mathbb{R}} \int_{x = 0}^{1} \int_{\lambda = 0}^{1} \mathbb{E} \left[ \left( \int_{x = 0}^{1} \frac{\partial^k T_{t-s} \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot) \right) \eta_{z-\beta_0^L(s)-\lambda(\beta_{n+1}^L(s)-\beta_0^L(s))}}{\partial x^k} \right) \right] \times D\tilde{\Psi}_n^L \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot), \tilde{w}_n^L(s, \cdot) \right) d\lambda dx dz ds
\]

\[
\leq \int_{\lambda = 0}^{1} \int_{s=0}^{t} \int_{z \in \mathbb{R}} \left\| T_{t-s} \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot) \right) \eta_{z-\beta_0^L(s)-\lambda(\beta_{n+1}^L(s)-\beta_0^L(s))} \right\|^2_H \times D\tilde{\Psi}_n^L \left( \tilde{u}_n^L(s, \cdot) + \lambda \tilde{w}_n^L(s, \cdot), \tilde{w}_n^L(s, \cdot) \right) d\lambda ds dz
\]

\[
\leq \int_{\lambda = 0}^{1} \int_{s=0}^{t} \int_{z \in \mathbb{R}} \mathbb{E} \left[ \left\| \tilde{u}_n^L(s, y) + \lambda \tilde{w}_n^L(s, y) \right\|^2_H \times \tilde{\Psi}_n^L(s, \cdot) \right] ds dz d\lambda
\]
To bound $A_5^{(n)}$, we first bound $\beta_{n+1}^L - \beta_n^L$. We have that

$$|\beta_{n+1}^L(t) - \beta_n^L(t)| \leq \frac{1}{\varrho} \int_{s=0}^{t} \left| \frac{\partial \hat{u}_{n+1}^L}{\partial x}(s,0) - \frac{\partial \hat{u}_n^L}{\partial x}(s,0) \right| ds$$

$$\leq \frac{2}{\varrho} \int_{s=0}^{t} \|\hat{u}_n^L(s,\cdot)\|_H ds \leq \frac{2}{\varrho} \sqrt{t \int_{s=0}^{t} \|\hat{w}_n^L(s,\cdot)\|_H^2 ds}$$

For $k \in \{0, 1, 2\}$, we then have that

$$\frac{\partial^k A_5^{(n)}}{\partial x^k}(t,x) = \int_{s=0}^{t} \int_{z \in \mathbb{R}} \int_{\lambda=0}^{1} \frac{\partial^k T_{t-s}}{\partial x^k} \left( (\hat{u}_n^L(s,\cdot) + \lambda \hat{w}_n^L(s,\cdot)) \frac{\hat{u}_n^L(s,\cdot) - \beta_n^L(s) - \lambda(\beta_{n+1}^L(s) - \beta_n^L(s))}{\lambda} \right) (x)$$

$$\times \hat{\Psi}_L^b (\hat{u}_n^L(s,\cdot) + \lambda \hat{w}_n^L(s,\cdot)) d\lambda \left( \beta_{n+1}^L(s) - \beta_n^L(s) \right) W(ds,dz).$$
Hence for $0 \leq t \leq T$

\[
\mathbb{E} \left[ \int_0^\infty \left| \frac{\partial^k A_5^{(n)}}{\partial x^k} (t, x) \right|^2 \right]
\]

\[
= \int_{s=0}^t \int_{z\in \mathbb{R}} \mathbb{E} \left[ \int_0^\infty \left\{ \int_{\lambda=0}^1 \frac{\partial^k T_{t-s}}{\partial x^k} \left( \tilde{\nu}_n^L (s, \cdot) + \lambda \tilde{\nu}_n^L (s, \cdot) \right) \frac{\partial^\lambda \beta_n^L (s)}{\partial \beta_n^L (s)} \right\} (x) \right]^2 dx \, ds \, dz
\]

\[
\leq \int_{\lambda=0}^1 \int_{t-s}^t \int_{z\in \mathbb{R}} \mathbb{E} \left[ \left\| T_{t-s} \left( \tilde{\nu}_n^L (s, \cdot) + \lambda \tilde{\nu}_n^L (s, \cdot) \right) \frac{\partial^\lambda \beta_n^L (s)}{\partial \beta_n^L (s)} \right\|^2_H \right] ds \, dz \, d\lambda
\]

\[
\leq K_1 \int_{\lambda=0}^1 \int_{t-s}^t \int_{z\in \mathbb{R}} \mathbb{E} \left[ \left\| \tilde{\nu}_n^L (s, \cdot) + \lambda \tilde{\nu}_n^L (s, \cdot) \right\|^2_H \right] ds \, dz \, d\lambda
\]

\[
\leq K_1 K(L+1)^2 \int_{t-s}^t \int_{z\in \mathbb{R}} \mathbb{E} \left[ \left\| \tilde{\nu}_n^L (s, \cdot) \right\|^2_H \right] ds \, dz
\]

\[
\leq \frac{4K_1 K(L+1)^2 t^2}{\rho^2} \int_{s=0}^t \mathbb{E} \left[ \left\| \tilde{\nu}_n^L (s, \cdot) \right\|^2_H \right] ds.
\]

**Lemma 2.6.5.** For each $T > 0$, we have that $\sum_{n=1}^\infty \sup_{0 \leq t \leq T} \mathbb{E} \left[ \left\| \tilde{\nu}_{n+1}^L - \tilde{\nu}_n^L \right\|_H \right] < \infty$. Thus $\mathbb{P}$-a.s.,

\[
\tilde{u}_L(t, \cdot) \overset{d}{=} \lim_{n \to \infty} \tilde{u}_n^L (t, \cdot)
\]

exists as a limit in $C([0, T]; H)$ and $u_L$ satisfies the integral equation

\[
\tilde{u}_L(t, x) = \int_{y=0}^\infty p_-(t, x, y) \tilde{u}_n(y) dy
\]

\[
- \frac{1}{\rho} \int_{s=0}^t \int_{y=0}^\infty p_-(t-s, x, y) \frac{\partial \tilde{u}_L}{\partial x} (s, 0) \frac{\partial \tilde{u}_L}{\partial x} (s, y) \Psi_L \left( \| \tilde{u}_L (s, \cdot) \|_H \right) dy \, ds
\]

\[
+ \int_{y=0}^\infty \int_{s=0}^t p_-(t-s, x, y) \tilde{u}_L (s, y) \Psi_L \left( \| \tilde{u}_L (s, \cdot) \|_H \right) \, d\zeta_s (y + \beta_L (s)) dy.
\]

where

\[
\beta_L (t) \overset{d}{=} - \frac{1}{\rho} \int_{s=0}^t \frac{\partial \tilde{u}_L}{\partial x} (s, 0) ds.
\]
Proof. See also [Wal86, Lemma 3.3]. Fixing $T > 0$ we collect the above calculations to see that there is a $K_{T,L} > 0$ such that
\[
\mathbb{E}[\|\tilde{w}_{n+1}^L(t,\cdot)\|_H^2] \leq K_{T,L} \int_0^t \frac{\mathbb{E}[\|\tilde{w}_n^L(s,\cdot)\|_H^2]}{(t-s)^{3/4}} ds
\]
for all $t \in [0,T]$. Iterating this, we get that
\[
\mathbb{E}[\|\tilde{w}_n^L(t,\cdot)\|_H^2] \leq K_{T,L}^{\frac{n-1}{4}} \prod_{j=0}^{n-2} B(1 + j/4, 1/4) \sup_{0 \leq t \leq T} \sqrt{\mathbb{E}[\|\tilde{w}_1^L\|_H^2]}
\]
where $B$ is the standard Beta function and thus that
\[
\sqrt{\mathbb{E}[\|\tilde{w}_n^L(t,\cdot)\|_H^2]} \leq K_{T,L}^{\frac{n-1}{8}} \prod_{j=0}^{n-2} B(1 + j/4, 1/4)^{1/2} \sup_{0 \leq t \leq T} \sqrt{\mathbb{E}[\|\tilde{w}_1^L\|_H^2]}
\]
To show that the terms on the right are summable, we use the ratio test. It suffices to show that
\[
\lim_{n \to \infty} K_{T,L}^{1/2} \frac{n-2}{4} \left( B \left( 1 + \frac{n-2}{4}, 1/4 \right) \right)^{1/2} = 0.
\] (2.6.4)
We calculate that
\[
B(1 + n/4, 1/4) = \int_0^1 s^{n/4}(1-s)^{-3/4} ds = \int_{s=0}^{1/2} s^{n/4}(1-s)^{-3/4} ds + \int_{s=1/2}^1 s^{n/4}(1-s)^{-3/4} ds
\]
\[
\leq n^{3/4} \int_{s=0}^{1/2} s^{n/4} ds + \int_{s=1-1/n}^1 (1-s)^{-3/4} ds
\]
\[
\leq \frac{4n^{3/4}}{n+4} + 4 \left( \frac{1}{n} \right)^{1/4}.
\]
This implies (2.6.4). The rest of the proof follows by Jensen’s inequality and standard calculations. \hfill \Box

We can finally show uniqueness.

Lemma 2.6.6. The solution of (2.6.3) is unique.

Proof. Let $u_1$ and $u_2$ be two solutions. Define $\tilde{w} \overset{\text{def}}{=} u_1 - u_2$. By calculations as above we get that
\[
\mathbb{E}[\|\tilde{w}(t,\cdot)\|_H^2] \leq K_{T,L} \int_0^t (t-s)^{-3/4} \mathbb{E}[\|\tilde{w}(s,\cdot)\|_H^2] ds.
\]
We can now use Gronwall’s inequality for $L > 0$.

Lemma 2.6.7. The solution $\tilde{u}(t, \cdot)$ of (2.6.3) is strictly positive for $0 \leq t \leq \tau_{L}$ and $x \geq 0$.

Proof. We first show non-negativity. Let $\tilde{u}_{\alpha}^{L}(t, x) \overset{\text{def}}{=} \tilde{u}^{L}(t, x) e^{-\alpha t}$. Since we have that $\Psi_{L}(\|\tilde{u}^{L}(t, \cdot)\|_{H}) = 1$ for $0 \leq t \leq \tau_{L}$, from (2.6.3) we obtain that for $0 \leq t \leq \tau_{L}$

$$d\tilde{u}_{\alpha}^{L}(t, x) = \left( \frac{\partial^{2} \tilde{u}_{\alpha}^{L}(t, x)}{\partial x^{2}} - \frac{1}{\alpha} e^{\alpha t} \frac{\partial \tilde{u}_{\alpha}^{L}(t, 0)}{\partial x}(t, 0) \frac{\partial \tilde{u}_{\alpha}^{L}(t, x)}{\partial x}(t, x) \right) dt + \tilde{u}_{\alpha}^{L}(t, x) dB_{t}^{x}$$

$$\tilde{u}_{\alpha}^{L}(0, x) = 0$$

$$\tilde{u}_{\alpha}^{L}(t, 0, x) = u_{0}(x) > 0,$$

where $B_{t}^{x}$ is defined as in (2.2.3). We will then follow the approach used in [Cho08] to show non-negativity of $\tilde{u}_{\alpha}^{L}$ which implies non-negativity of $\tilde{u}^{L}$. To start, fix a nonnegative and nonincreasing $\eta \in C^{\infty}(\mathbb{R})$ such that $\eta(u) = 2$ if $u \leq -1$ and $\eta(u) = 0$ if $u \geq 0$. Define

$$\varphi(u) \overset{\text{def}}{=} \int_{r=0}^{u} \int_{s=0}^{r} \eta(s) ds dr \quad u \in \mathbb{R}$$

$$38$$
Finally define \( \varphi_\varepsilon(u) \equiv \varepsilon^2 \varphi \left( \frac{u}{\varepsilon} \right) \) for all \( u \in \mathbb{R} \). Fixing \( x \in \mathbb{R}_+ \) and applying Itô’s formula to \( \{ \varphi_\varepsilon(\tilde{u}_0^L(t,x)) : 0 \leq t < \tau_L \} \), we have that

\[
\varphi_\varepsilon(\tilde{u}_0^L(t \land \tau_L, x)) - \varphi_\varepsilon(u_0(x)) = \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \frac{\partial^2 \tilde{u}_0^L(s,x)}{\partial x^2} ds \\
- \frac{1}{\varrho} \int_{s=0}^{t \land \tau_L} \frac{\partial \tilde{u}_0^L(s,x)}{\partial x} e^{\alpha s} \left\{ \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \frac{\partial \tilde{u}_0^L(s,x)}{\partial x} \right\} ds \\
+ \frac{1}{2} \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) (\tilde{u}_0^L(s,x))^2 ds + \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \tilde{u}_0^L(s,x) dB_s^x.
\]

Here \( \varphi_\varepsilon(u_0(x)) = 0 \) since \( u_0 \geq 0 \). Then (2.6.5) implies that

\[
\varphi_\varepsilon(\tilde{u}_0^L(t \land \tau_L, x)) e^{-t \land \tau_L} - \varphi_\varepsilon(u_0(x)) = \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \frac{\partial^2 \tilde{u}_0^L(s,x)}{\partial x^2} e^{-s} ds \\
- \frac{1}{\varrho} \int_{s=0}^{t \land \tau_L} \frac{\partial \tilde{u}_0^L(s,x)}{\partial x} e^{(\alpha-1)s} \left\{ \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \frac{\partial \tilde{u}_0^L(s,x)}{\partial x} \right\} ds \\
+ \frac{1}{2} \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) (\tilde{u}_0^L(s,x))^2 e^{-s} ds + \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \tilde{u}_0^L(s,x) e^{-s} dB_s^x \\
- \int_{s=0}^{t \land \tau_L} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) e^{-s} ds.
\]

Next fix a nonincreasing \( \varpi \in C^\infty(\mathbb{R}_+) \) such that \( \varpi(x) = 1 \) for \( x \leq 1 \) and \( \varpi(x) = 0 \) for \( x \geq 2 \). For each \( N \in \mathbb{N} \), define \( \varpi_N(x) \equiv \varpi(x/N) \). Let’s now do several things. Let’s multiply (2.6.6) with \( \varpi_N \). Let’s then integrate in space, and finally take expectations. We get that

\[
\mathbb{E} \left[ \int_{x=0}^{\infty} \varphi_\varepsilon(\tilde{u}_0^L(t \land \tau_L, x)) \varpi_N(x) e^{-t \land \tau_L} dx \right] = \mathbb{E} \left[ \int_{s=0}^{t \land \tau_L} A_{1N}^\varepsilon(s) e^{-s} ds \right] \\
- \frac{1}{\varrho} \mathbb{E} \left[ \int_{s=0}^{t \land \tau_L} \frac{\partial \tilde{u}_0^L(s,x)}{\partial x} e^{(\alpha-1)s} A_{2N}^\varepsilon(s) e^{-s} ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_{s=0}^{t \land \tau_L} A_{3N}^\varepsilon(s) e^{-s} ds \right]
\]

where

\[
A_{1N}^\varepsilon(s) \equiv \int_{x=0}^{\infty} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \frac{\partial^2 \tilde{u}_0^L(s,x)}{\partial x^2} \varpi_N(x) dx \\
A_{2N}^\varepsilon(s) \equiv \int_{x=0}^{\infty} \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \frac{\partial \tilde{u}_0^L(s,x)}{\partial x} \varpi_N(x) dx \\
A_{3N}^\varepsilon(s) \equiv \int_{x=0}^{\infty} \left\{ \varphi_\varepsilon(\tilde{u}_0^L(s,x)) (\tilde{u}_0^L(s,x))^2 - 2 \varphi_\varepsilon(\tilde{u}_0^L(s,x)) \right\} \varpi_N(x) dx.
\]

We need some bounds on \( \varphi_\varepsilon \). Define \( \| \cdot \|_C \) as the sup norm over \( \mathbb{R} \). First note that \( \varphi(u) - 2\chi_{\mathbb{R}_-} = \)
\[ \eta(u) - 2\chi_{\mathbb{R}_-}. \] This implies that for all \( u \in \mathbb{R} \),

\[ |\phi(u) - 2\chi_{\mathbb{R}_-}| \leq \|\eta - 2\|c \chi_{[-1,0]}(u), \quad |\phi(u) - 2u\chi_{\mathbb{R}_-}| \leq \|\eta - 2\|c \]

\[ |\phi(u)| - u^2\chi_{\mathbb{R}_-} \leq \|\eta - 2\|c|u|; \quad (2.6.8) \]

the first bound is direct, and the second two follow by integration. Note also that since \( \eta \) is bounded,

\[ |\phi(u)| \leq \|\eta\|c, \quad |\phi(u)| \leq \|\eta\|c|u|, \quad \text{and} \quad |\phi(u)| \leq \frac{1}{2}\|\eta\|c|u|^2. \quad (2.6.9) \]

Let’s understand the behavior of the various terms of (2.6.7) as \( N \to \infty \) and then \( \varepsilon \to 0 \). From the last bound of (2.6.8), we have that \( \lim_{\varepsilon \to 0} \phi(\varepsilon)(x) = x^2\chi_{\mathbb{R}_-}(x) \). Thanks to the last bound of (2.6.9), we can use dominated convergence and thus conclude that

\[ \lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{E} \left[ \int_{x=0}^{\infty} \phi(\varepsilon)(\frac{\tilde{u}_n^L(t \wedge \tau_L, x)}{\varepsilon}) \omega_N(x)e^{-t \wedge \tau_L}dx \right] = \mathbb{E} \left[ \int_{x=0}^{\infty} \left(\frac{\tilde{u}_n^L(t \wedge \tau_L, x)}{\varepsilon}\right)^2 \chi_{\mathbb{R}_-}(\frac{\tilde{u}_n^L(t \wedge \tau_L, x)}{\varepsilon})e^{-t \wedge \tau_L}dx \right]. \]

We next consider \( A_{1}^{N,\varepsilon}(s) \). Integrating by parts and using the boundary conditions at \( x = 0 \), we have that

\[ A_{1}^{N,\varepsilon}(s) = -\int_{x=0}^{\infty} \eta \left(\frac{\tilde{u}_n^L(s, x)}{\varepsilon}\right)^2 \omega_N(x)dx - \frac{1}{N} \int_{x=0}^{\infty} \varepsilon \phi(\varepsilon)(\frac{\tilde{u}_n^L(s, x)}{\varepsilon}) \frac{\partial \tilde{u}_n^L}{\partial x}(s, x) \omega \left(\frac{x}{N}\right)dx. \]

The first term is nonpositive since \( \eta \) and \( \omega \) are nonnegative. We can also see that

\[ \left| \frac{1}{N} \int_{x=0}^{\infty} \varepsilon \phi(\varepsilon)(\frac{\tilde{u}_n^L(s, x)}{\varepsilon}) \frac{\partial \tilde{u}_n^L}{\partial x}(s, x) \omega \left(\frac{x}{N}\right)dx \right| \leq \frac{2}{N} \|\omega\|c\|\eta\|c \int_{x=0}^{\infty} \left| \frac{\tilde{u}_n^L(s, x)}{\varepsilon} \frac{\partial \tilde{u}_n^L}{\partial x}(s, x) \right|dx \]

\[ \leq \frac{1}{N} \|\omega\|c\|\eta\|c \int_{x=0}^{\infty} \left| \frac{\tilde{u}_n^L(s, x)}{\varepsilon} \right|^2 dx + \left| \frac{\partial \tilde{u}_n^L}{\partial x}(s, x) \right|^2 dx. \]

Thus

\[ \lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{E} \left[ \int_{s=0}^{t \wedge \tau_L} A_{1}^{N,\varepsilon}(s)e^{-s}ds \right] \leq 0. \]

Thirdly, another integration by parts gives us that

\[ A_{2}^{N,\varepsilon}(s) = -\frac{1}{N} \int_{x=0}^{\infty} \varepsilon \phi(\varepsilon)(\tilde{u}_n^L(s, x)) \omega \left(\frac{x}{N}\right)dx. \]
and thus

$$|A_2^{N,\varepsilon}(s)| \leq \frac{1}{2N} \|\phi\|_C \|\eta\|_C \int_{x=0}^{\infty} |\tilde{u}_\alpha^L(s,x)|^2 dx.$$  

Thus

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{E} \left[ \int_{s=0}^{t \wedge \tau_L} \frac{\partial \tilde{u}_\alpha^L}{\partial x}(s,0)e^{\alpha s} A_2^{N,\varepsilon}(s)e^{-s} ds \right] = 0.$$  

Let’s finally bound $A_3^{N,\varepsilon}$. Note that for every $u \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \tilde{\varphi}_\varepsilon(u) = 2u^2 - 2\varphi_\varepsilon(u) = 2u^2 - 2u^2 \chi_{\mathbb{R}_-}(u) = 0.$$  

In light of the first and last bounds of (2.6.9), we can use dominated convergence to see that

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{E} \left[ \int_{s=0}^{t \wedge \tau_L} A_3^{N,\varepsilon}(s) e^{-s} ds \right] = 0.$$  

Combining things together, we finally get that

$$\mathbb{E} \left[ \int_{x=0}^{\infty} \left( \tilde{u}_\alpha^L(t \wedge \tau_L,x) \right)^2 \chi_{\mathbb{R}_-}(\tilde{u}_\alpha^L(t \wedge \tau_L,x)) e^{-t \wedge \tau_L} dx \right] \leq 0.$$  

This implies non-negativity of $\tilde{u}_L$. Positivity can be shown by some transformation using (2.2.3) and a strong maximum principle for parabolic equations (c.f. Theorem 2, page 309, [McOwen96]) (see Lemma 5.8 in [KZS] for details).

Let’s now see what happens as $L \not\to \infty$. Define the random time

$$\tau \overset{\text{def}}{=} \lim_{L \to \infty} (\tau_L \wedge L).$$  

Let’s also define

$$\tilde{u}(t,x) \overset{\text{def}}{=} \lim_{L \to \infty} \tilde{u}_L(t \wedge \tau_L,x), \quad t \geq 0, x \geq 0$$

$$\beta(t) \overset{\text{def}}{=} -\frac{1}{\rho} \int_{s=0}^{t} \frac{\partial \tilde{u}_\alpha^L}{\partial x}(s,0) ds, \quad 0 \leq t < \tau.$$  

**Lemma 2.6.8.** We have that

$$\lim_{t \to \tau} \|\tilde{u}(t,\cdot)\|_H = \infty.$$  

Define $u$ as in (2.5.4)–(2.5.5). Then $\{u(t,\cdot) \mid 0 \leq t < \tau\}$ is a weak solution of (2.1.1).
Proof. Fixing $L' > L$ we have from the uniqueness claim of Lemma 2.6.6 that $\tilde{u}^{L'}(t, \cdot) = \tilde{u}^{L}(t, \cdot)$ for $0 \leq t \leq \tau_L$. Thus $\tau_{L'} \geq \tau_L$ for all $L' > L$, and so $\tau = \lim_{L \to \infty} \tau_L = \lim_{L \to \infty} (\tau_L \wedge L)$ and $\tau$ is predictable. We also have that $\tilde{u}(t, \cdot) = \lim_{L \to \infty} \tilde{u}^{L}(t, \cdot)$ for $0 \leq t < \tau$. From this, Lemma 2.6.7 and Lemma 2.5.2 we conclude that $\{u(t, \cdot) \mid 0 \leq t < \tau\}$ as defined by (2.5.4)–(2.5.5) indeed is a weak solution of (2.1.1). The characterization of $\|\tilde{u}(t, \cdot)\|_H$ at $\tau−$ is obvious.

In fact, we have a more explicit characterization of $\tau$.

Lemma 2.6.9. We have that

$$\lim_{t \uparrow \tau} \left| \frac{\partial \tilde{u}}{\partial x}(t, 0) \right| = \infty.$$  

Proof. For each $L > 0$, define

$$\tau'_L \overset{\text{def}}{=} \inf \left\{ t \in [0, \tau) \mid \left| \frac{\partial \tilde{u}}{\partial x}(t, 0) \right| \geq L \right\}. \quad (\inf \emptyset = \tau)$$

Thus in fact $\tau > \tau'_L$ and hence

$$\left| \frac{\partial \tilde{u}}{\partial x}(\tau'_L, 0) \right| = L.$$ 

Consequently

$$\lim_{L \to \infty} \left| \frac{\partial \tilde{u}}{\partial x}(\tau'_L, 0) \right| = \infty.$$ 

Since $\tau'_L < \tau$, we of course also have that $\lim_{L \to \infty} \tau'_L \leq \tau$. On the other hand, $\|\tilde{u}(t, \cdot)\|_H$ may become large for many reasons other than $\left| \frac{\partial \tilde{u}}{\partial x}(\tau'_L, 0) \right|$ becoming large, so necessarily $\tau \leq \lim_{L \to \infty} \tau'_L$. Putting things together, we get that $\lim_{L \to \infty} \tau'_L = \tau$. The claimed result now follows. □

To finish things off, we prove uniqueness.

Lemma 2.6.10 (Uniqueness). If $\{\tilde{u}(t, \cdot) \mid 0 \leq t < \tau\} \subset H$ and $\{\tilde{u}'(t, \cdot) \mid 0 \leq t < \tau'\} \subset H$ are two solutions of (2.5.2), then $u(t, \cdot) = u'(t, \cdot)$ for $0 \leq t < \min\{\tau, \tau'\}$.

Proof. For each $L > 0$, define

$$\sigma_L \overset{\text{def}}{=} \inf \left\{ t \in [0, \tau \wedge \tau') : \left| \frac{\partial \tilde{u}}{\partial x}(t, 0) \right| \geq L \text{ or } \left| \frac{\partial \tilde{u}'}{\partial x}(t, 0) \right| \geq L \right\}. \quad (\inf \emptyset = \tau \wedge \tau')$$

Then $\tau \wedge \tau' = \lim_{L \to \infty} \sigma_L$. We can use standard uniqueness theory to conclude that $\tilde{u}$ and $\tilde{u}'$ coincide on $[0, \sigma_L]$, and we then let $L \not\to \infty$. □
2.7 Simulations

Simulating moving boundary problems directly (e.g. simulating the SPDE (2.1.1)) is not an easy task since we need to find solutions of (stochastic) partial differential equations and at the same time we need to trace the positions of the interface. In other words, unequal space intervals may be needed in order to account for the moving boundary. Here we can avoid this difficulty since we have the explicit formula for the solution \( u \) in Lemma 2.5.2. That is,\[ u(t, x) \triangleq \begin{cases} \tilde{u}(t, x - \beta(t)) & x \geq \beta(t), 0 \leq t < \tau \\ 0 & x < \beta(t), 0 \leq t < \tau, \end{cases} \]
where \((\tilde{u}, \beta)\) is a solution of the SPDE
\[
d\tilde{u}(t, x) = \left\{ \frac{\partial^2 \tilde{u}}{\partial x^2}(t, x) + \alpha \tilde{u}(t, x) - \frac{1}{\varrho} \frac{\partial \tilde{u}}{\partial x}(t, 0) \frac{\partial \tilde{u}}{\partial x}(t, x) \right\} dt \\
+ \tilde{u}(t, x) d\zeta_t(x + \beta(t)) & t > 0, x > 0 \tag{2.7.1}
\]
\[
\tilde{u}(t, 0) = 0 & t > 0 \\
\tilde{u}(0, x) = u_\circ(x) & x > 0 \\
\dot{\beta}(t) = -\frac{1}{\varrho} \frac{\partial \tilde{u}}{\partial x}(t, 0) & t > 0 \\
\beta(0) = 0.
\]

Therefore we first need to solve the SPDE (2.7.1) numerically in order to obtain the moving boundary \( \beta(t) \) and then the solution \( u(t, x) \). We first discretize space by using the explicit finite difference scheme. Here we can also approximate \( \eta \) by simple functions which converge to \( \eta \) in \( L^2(\mathbb{R}) \) (see [Wal86]). As a result, we can have an approximation of \( \zeta_t(x) \). Note that \( \zeta_t(x) \) is a Brownian motion for each fixed \( x \), however, it is spatially correlated. Now we use the Euler-Maruyama type method to discretize time (see [Gai96, Hig02]). Then we can get a numerical solution of (2.7.1). Since there is a stability issue for parabolic PDE, we note that \( \Delta t/(\Delta x)^2 < 1/2 \), where \( \Delta t \) is a time step and \( \Delta x \) is a space step. Figure ?? comparers a solution of the deterministic Stefan problem, which contains no noise term \( u\tilde{d}\zeta_t(x) \) in (2.1.1), and one realization of a solution of the stochastic Stefan problem (2.1.1) (here we understand (2.1.1) in the Ito sense). Here we use
\alpha = 0.5, \rho = 0.5\) and the initial condition

\[ u_0(x) = \begin{cases} 
\frac{x + x^2}{1 + (\frac{x}{4})^4} & \text{if } x \geq 0 \\
0 & \text{else}
\end{cases} \]

Figure 2.1: (a) A solution of the deterministic Stefan problem (b) A solution of the stochastic Stefan problem

We can clearly see from both (a) and (b) that there are two phases separated by the black line, which is the moving boundary, and how \( u \) is changing on the colored region where \( u > 0 \). Since our noise is \( ud\zeta(t) \) and \( u \) is getting close to 0 when \( x \) approaches the moving boundary, we can expect that the effect of noise is not significant at the moving boundary. This may be justified by Figure 2.1. Furthermore, we can see that the boundary is moving left in both (a) and (b). This just follows from the positivity of the solution \( u(t, x) \) for \( x \geq \beta(t) \) and the Stefan boundary condition of (2.1.1).
Chapter 3

Numerical Analysis of a Stochastic Moving Boundary Value Problem

3.1 Introduction

We consider a stochastic perturbation of a free boundary problem proposed by Ludford and Stewart and studied by Caffarelli and Vazquez. Our goal in this chapter is to investigate how noise can effect a motion of the moving boundary theoretically and numerically. First fix a probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) and assume that \(B\) is a Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\). Now we consider

\[
du(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x)dt + \alpha u(t, x)dt + u(t, x) \circ dB_t \quad x > \beta(t)
\]

\[
\lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t, x) = 1
\]

\[
u(0, x) = \nu_0(x) \quad x \in \mathbb{R}
\]

\[
\{(t, x) \in \mathbb{R}^+ \times \mathbb{R} \mid u(t, x) > 0\} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R} \mid x > \beta(t)\}.
\]

In (3.1.1), \(\circ dB_t\) represents Stratonovich integration. As in Chapter 2, the structure of the SPDE (3.1.1) is invariant under Ito and Stratonovich formulation due to the \(\alpha u(t, x)\) term. In addition, the last requirement in (3.1.1) means that there is only one interface which separates two phases \(u \equiv 0\) and \(u > 0\). Here, the second equation in (3.1.1) is a boundary condition at the interface; compared to the Stefan boundary condition, the evolution equation for the moving boundary \(x = \beta(t)\) in (3.1.1) is implicitly defined (see Section 3.2.1).

We first consider existence and uniqueness of a solution of (3.1.1) in Section 3.2. This is similar to Chapter 2, i.e., we use a weak formulation to define a solution then using a transformation, we show existence and uniqueness of a solution of (3.1.1) (we first show existence and uniqueness of a solution of the transformed
SPDE (3.2.6) and then of (3.1.1)). We also consider regularity in space and obtain an evolution equation for the moving boundary \( x = \beta(t) \) (see Lemma 3.2.3).

In the remaining part of this chapter we consider numerical approximations of (3.1.1). Rather than approximating (3.1.1) directly, we use the transformed SPDE (3.2.6) in order to use a fixed spatial grid (note that (3.2.6) has a fixed boundary condition). By the inverse transformation which transforms (3.2.6) to (3.1.1) (Section 3.2.8), we can have numerical approximations of a solution of (3.1.1). However, there are several obstacles here; first, a solution of the transformed SPDE (3.2.6) may blow up in a finite time; second, our problem is defined on the semi-infinite interval \([0, \infty)\), which is unbounded; third, there is a nonlocal nonlinear term in (3.2.6). To avoid the first problem, we will use a stopping time which stops the process once it reaches a fixed large number. For the second problem, we will use truncation and impose the Dirichlet boundary condition at the right end point. We will consider this truncation problem in Section 3.3 for the heat equation first and then the stochastic moving boundary value problem (3.1.1) (more precisely the transformed SPDE (3.2.6)). The last problem can be handled by first approximating the velocity of the moving boundary, that is, the second partial derivative of a solution of the transformed SPDE (3.2.6) with respect to space variable \( x \) (see Section 3.4). The results of this section are joint work with Carl Mueller and Richard B. Sowers (KMS and KS).

3.2 A Stochastic Moving Boundary Value Problem

In this section, we overview results on a stochastic moving boundary value problem in KMS.

3.2.1 Weak Formulation and Regularity

Let \( \mathcal{F}_t \equiv \sigma\{B_s; 0 \leq s \leq t\} \) for all \( t \geq 0 \); then \( B \) is a Brownian motion with respect to \( \{\mathcal{F}_t\}_{t>0} \) and stochastic integration against \( B \) will be with respect to this filtration. In addition, let \( \alpha \) be a fixed constant and \( \hat{\alpha} \equiv \alpha + 1/2 \).

**Definition 3.2.1.** A weak solution of (3.1.1) is a predictable path \{\( u(t, \cdot) \mid 0 \leq t < \tau \) \} in \( C(\mathbb{R}) \cap L^1(\mathbb{R}) \), where \( \tau \) is a predictable stopping time with respect to \( \{\mathcal{F}_t\}_{t>0} \), such that for any \( \varphi \in C_\infty^\infty(\mathbb{R}_+ \times \mathbb{R}) \) and any
finite stopping time $\tau' < \tau$,

$$\int_{x \in \mathbb{R}} u(\tau', x) \varphi(\tau', x) dx = \int_{x \in \mathbb{R}} u_0(x) \varphi(0, x) dx + \int_{t=0}^{\tau'} \int_{x \in \mathbb{R}} u(r, x) \left\{ \frac{\partial \varphi}{\partial t}(r, x) + \frac{\partial^2 \varphi}{\partial x^2}(r, x) + \hat{\alpha}(r, x) \right\} dx dr$$

$$+ \int_{r=0}^{\tau'} \int_{x \in \mathbb{R}} u(r, x) \varphi(r, x) dx dB_r - \int_{r=0}^{\tau'} \varphi(r, \beta(r)) dr$$

and where

$$\{(t, x) \in [0, \tau) \times \mathbb{R} \mid u(t, x) > 0\} = \{(t, x) \in [0, \tau) \times \mathbb{R} \mid x > \beta(t)\}.$$  

This definition makes sense when Brownian motion is replaced by a smooth function $b(t)$. Indeed, suppose that $v(t, x)$ is a solution of the following PDE:

$$\frac{\partial v}{\partial t}(t, x) = \frac{\partial^2 v}{\partial x^2} + \alpha v(t, x) + v(t, x) b(t) \quad x > \beta_0(t)$$

$$\lim_{x \searrow \beta_0(t)} \frac{\partial v}{\partial x}(t, x) = 1$$

(3.2.1)

$$\{ (t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid v(t, x) > 0\} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > \beta_0(t)\}.$$  

Then we have for a fixed $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

$$\frac{\partial (v \varphi)}{\partial t}(t, x) = \frac{\partial v}{\partial t}(t, x) \varphi(t, x) + v(t, x) \frac{\partial \varphi}{\partial t}(t, x).$$

Using (3.2.1), integration by parts and the facts that $v(t, \beta_0(t)) = 0$ and $\lim_{x \searrow \beta_0(t)} \frac{\partial v}{\partial x}(t, x) = 1$, we have

$$\int_{x \in \mathbb{R}} v(t, x) \varphi(t, x) dx - \int_{x \in \mathbb{R}} u_0(x) \varphi(t, x) dx = \int_{s=0}^{t} \int_{x \in \mathbb{R}} \frac{\partial v}{\partial s}(s, x) \varphi(s, x) dx ds + \int_{s=0}^{t} \int_{x \in \mathbb{R}} v(s, x) \frac{\partial \varphi}{\partial s}(s, x) dx ds$$

$$= - \int_{s=0}^{t} \lim_{x \searrow \beta_0(s)} \left\{ \frac{\partial v}{\partial x}(s, x) \varphi(s, x) \right\} + v(s, \beta_0(s)) \frac{\partial \varphi}{\partial s}(s, \beta_0(s)) ds + \int_{s=0}^{t} \int_{x \in \mathbb{R}} v(s, x) \frac{\partial^2 \varphi}{\partial x^2}(s, x) dx ds$$

$$+ \int_{s=0}^{t} \int_{x \in \mathbb{R}} \alpha v(s, x) \varphi(s, x) dx ds + \int_{s=0}^{t} \int_{x \in \mathbb{R}} v(s, x) dB(s) ds + \int_{s=0}^{t} \int_{x \in \mathbb{R}} v(s, x) \frac{\partial \varphi}{\partial s}(s, x) dx ds.$$  

Our claim follows from the Wong-Zakai result when $b(t)$ approximates Brownian motion.
Before we consider regularity, we first define some useful functions. Define
\[ p_\circ(t, x) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{x^2}{4t} \right] \quad t > 0, x \in \mathbb{R} \]
\[ p_\pm(t, x, y) \overset{\text{def}}{=} \{p_\circ(t, x - y) \pm p_\circ(t, x + y)\} e^{\alpha t} = \{p_\circ(t, x - y) \pm p_\circ(t, -x - y)\} e^{\alpha t} \quad t > 0, x, y \in \mathbb{R} \]
\[ \hat{p}_\pm(t, x, y) \overset{\text{def}}{=} \{p_\circ(t, x - y) \pm p_\circ(t, x + y)\} e^{\hat{\alpha} t} = \{p_\circ(t, x - y) \pm p_\circ(t, -x - y)\} e^{\hat{\alpha} t}; \quad t > 0, x, y \in \mathbb{R} \]

The second representations of \( p_\pm \) and \( \hat{p}_\pm \) stem from the fact that \( p_\circ \) is even in its second argument. The distinction between \( p_\pm \) and \( \hat{p}_\pm \) naturally lies in the distinction between Ito and Stratonovich calculations.

Let us now consider regularity. Since our noise is a single Brownian motion, there is the same effect of noise on each state once we fix time \( t \). Therefore we can expect that the solution can be smooth in space. In order to see this, we need some regularity for the boundary.

**Lemma 3.2.2.** Let \( u \) be a weak solution of (3.1.1) and assume that \( \beta \) is continuous. If \( 0 < t < \tau \) and \( x > \beta(t) \), then
\[ u(t, x) = -\int_{s=0}^{t} e^{B_t - B_s} p_\pm(t - s, x - \beta(t), \beta(s) - \beta(t)) ds + e^{B_t} \int_{y \in \mathbb{R}} p_\pm(t, x - \beta(t), y - \beta(t)) u_\circ(y) dy. \quad (3.2.3) \]

Furthermore, \( u(t, \cdot) \) is \( C^\infty \) on \((\beta(t), \infty)\).

**Proof.** See Lemma 3.1 in [KMS].

Here (3.2.3) is not explicit since the right hand side depends on \( u \) and \( \beta \). However, we can obtain the boundary condition at the interface and an evolution equation for the moving boundary from (3.2.3) if we assume more regularity for \( \beta(t) \). First, we consider deterministic PDE (3.2.1) again in order to see which equation we can expect for an evolution equation for the moving boundary. By the definition \( v(t, \beta_\circ(t)) = 0 \), differentiating \( v(t, \beta_\circ(t)) \) (and using an approximation just to the right of \( \beta_\circ \)) we get that
\[ \frac{\partial v}{\partial t}(t, \beta_\circ(t)) + \frac{\partial v}{\partial x}(t, \beta_\circ(t)) \dot{\beta}_\circ(t) = 0. \]

Using (3.2.1) and the boundary conditions, we can obtain the following:
\[ \dot{\beta}_\circ(t) = -\left\{ \frac{\partial^2 v}{\partial x^2}(t, \beta_\circ(t)) + \alpha v(t, \beta_\circ(t)) \right\} = -\frac{\partial^2 v}{\partial x^2}(t, \beta_\circ(t)). \]

For the SPDE (3.1.1) we should have the same result (since the noise term vanishes at the boundary).
Lemma 3.2.3. Let \( \{u(t, \cdot) \mid 0 \leq t < \tau\} \) be a solution of \((3.1.1)\). If \( \beta \) is continuously differentiable, then
\[
\lim_{x \searrow \beta(t)} \frac{\partial u}{\partial x}(t, x) = 1 \quad \text{and} \quad \lim_{x \searrow \beta(t)} \frac{\partial^2 u}{\partial x^2}(t, x) = -\dot{\beta}(t)
\] (3.2.4)
for all \( t \in [0, \tau) \).

Proof. See Lemma 3.2 in [KMS]

3.2.2 Existence and Uniqueness

We first state our main theorems:

Theorem 3.2.4 (Existence). A solution of \((3.1.1)\) exists. Furthermore, \( u(t, \cdot) \in C^2[\beta(t), \infty) \) for all \( t \in [0, \tau) \) and
\[
\tau \leq \inf \left\{ t \geq 0 : \left| \frac{\partial^2 u}{\partial x^2}(t-, \beta(t)) \right| = \infty \right\}.
\]

Proof. The proof follows from Lemmas \(3.2.10\)

We also have uniqueness.

Theorem 3.2.5 (Uniqueness). Suppose that \( \{u_1(t, \cdot) ; 0 \leq t < \tau_1\} \) and \( \{u_2(t, \cdot) ; 0 \leq t < \tau_2\} \) are two solutions of \((3.1.1)\). Assume that for \( i \in \{1, 2\} \), the map \( x \mapsto u_i(t, x - \beta_i(t)) \) has three generalized square-integrable derivatives on \( (0, \infty) \). Then \( u_1(t, \cdot) = u_2(t, \cdot) \) for \( 0 \leq t < \min\{\tau_1, \tau_2\} \).

Proof. The proof follows from Lemma \(3.2.11\)

The proof of the existence and uniqueness heavily depends on the transformation which transforms \((3.1.1)\) into a nonlinear SPDE with the Neumann boundary condition at \( x = 0 \) and vice versa.

Lemma 3.2.6. Suppose that \( \{u(t, \cdot) \mid 0 \leq t < \tau\} \subset C(\mathbb{R}) \cap L(\mathbb{R}) \) is a weak solution of \((3.1.1)\). Suppose also that \( \beta \) is continuously differentiable and \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted. Then \( \hat{u}(t, x) = u(t, x + \beta(t)) + e^{-x} \) satisfies the integral equation
\[
\hat{u}(t, x) = \int_{y=0}^{\infty} \hat{p}_+(t, x, y) \left( u_\circ(y) + e^{-y} \right) dy + \int_{s=0}^{t} \int_{y=0}^{\infty} \hat{p}_+(t-s, x, y) \left\{ \left( \frac{\partial \hat{u}}{\partial x}(s, y) + e^{-y} \right) \left( 1 - \frac{\partial^2 \hat{u}}{\partial x^2}(s, 0) \right) - (\dot{\beta} + 1)e^{-y} \right\} dy \, ds \quad (3.2.5)
\]
\[
+ \int_{s=0}^{t} \int_{y=0}^{\infty} \hat{p}_+(t-s, x, y) \{ \hat{u}(s, y) - e^{-y} \} dy \, dB_s
\]
for all \( t > 0 \) and \( x > 0 \).
Proof. See Lemma 3.3 in [KMS].

Of course (3.2.5) is equivalent to the SPDE
\[
\begin{align*}
\dot{\tilde{u}}(t,x) &= \left\{ \frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) + \hat{\alpha} \left( \tilde{u}(t,x) - e^{-x} \right) - e^{-x} + \left( \frac{\partial \tilde{u}}{\partial x}(t,x) + e^{-x} \right) \left( 1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(t,0) \right) \right\} dt \\
&\quad + \left( \tilde{u}(t,x) - e^{-x} \right) dB_t & t > 0, x > 0\\
\frac{\partial \tilde{u}}{\partial x}(t,0) &= 0 & t > 0\\
\tilde{u}(0, x) &= \tilde{u}_0(x) = u_0(x) + e^{-x} & x > 0
\end{align*}
\] (3.2.6)

Remark 3.2.7. This transformed SPDE (3.2.6) can be also seen easily from our deterministic PDE (3.2.1) if we replace $B_t$ by a smooth function $b(t)$.

Now we state the converse of Lemma 3.2.6.

Lemma 3.2.8. Suppose that \{\tilde{u}(t, \cdot) \mid 0 \leq t < \tau\} \subset C^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) satisfies (3.2.5). Set
\[
\beta(t) = \int_{s=0}^{t} \left\{ 1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(s,0) \right\} ds & 0 \leq t < \tau
\] (3.2.7)
and define
\[
\begin{cases}
\tilde{u}(t,x - \beta(t)) - \exp[-(x - \beta(t))] & x \geq \beta(t), 0 \leq t < \tau \\
0 & x < \beta(t), 0 \leq t < \tau
\end{cases}
\] (3.2.8)
Then \{u(t, \cdot) \mid 0 \leq t < \tau\} is a weak solution of (3.1.1).

Proof. See Lemma 3.5 in [KMS].

By Lemma 3.2.6 and Lemma 3.2.8 it is enough to show the existence and uniqueness of a solution of (3.2.5). Let’s set up a functional framework in which we can carry out a Picard-type iteration. As usual, $C^\infty_0(\mathbb{R}_+)$ is the collection of infinitely smooth functions on $[0, \infty)$ whose support is bounded. Define next
\[
C^\infty_{0, \text{even}}(\mathbb{R}_+) \overset{\text{def}}{=} \left\{ \varphi \in C^\infty_0(\mathbb{R}_+) \mid \varphi^{(n)}(0) = 0 \text{ for all odd } n \in \mathbb{N} \right\};
\]
in other words, $C^\infty_{0, \text{even}}(\mathbb{R}_+)$ are those elements of $C^\infty_0(\mathbb{R}_+)$ which can be extended to an even element of
\( C^\infty(\mathbb{R}) \) (namely, consider the map \( y \mapsto \varphi(|y|) \)). For all \( \varphi \in C^\infty_0(\mathbb{R}_+) \), define

\[
\|\varphi\|_H \overset{\text{def}}{=} \sqrt{\sum_{i=0}^{\infty} \int_{x \in (0, \infty)} |\varphi^{(i)}(x)|^2 \, dx}.
\]  

(3.2.9)

Let \( H \) be the closure of \( C^\infty_0(\mathbb{R}_+) \) with respect to \( \| \cdot \|_H \) and let \( H_{\text{even}} \) be the closure of \( C^\infty_{0,\text{even}}(\mathbb{R}_+) \) with respect to \( \| \cdot \|_H \) (\( H \) is more commonly written as \( H^3 \); i.e., it is the collection of functions on \( \mathbb{R}_+ \) which possess three weak square-integrable derivatives).

Since there is the nonlinear term \( \frac{\partial^2 u}{\partial x^2}(t, 0) \) in the drift which makes it difficult to do a Picard iteration, we use truncation. Define a truncation function \( \Psi_L \) for each fixed \( L > 0 \) as \( \Psi_L \in C^\infty(\mathbb{R}; [0, 1]) \) such that \( \Psi_L(x) = 1 \) if \( |x| \leq L \) and \( \Psi_L(x) = 0 \) if \( |x| > L + 1 \). Set

\[
\tilde{u}^L_t(t, x) = \int_{y=0}^{\infty} \tilde{p}_+(t, x, y) \tilde{u}_0(y) \, dy
\]

for all \( t > 0 \) and \( x \in \mathbb{R} \) and recursively define

\[
\tilde{u}_{n+1}^L(t, x) = \int_{y=0}^{\infty} \tilde{p}_+(t, x, y) \tilde{u}_0(y) \, dy
\]

\[
+ \int_{s=0}^{t} \int_{y=0}^{\infty} \tilde{p}_+(t - s, x, y) \left\{ \left( \frac{\partial \tilde{u}_n^L}{\partial x}(s, y) + e^{-y} \right) \left( 1 - \frac{\partial^2 \tilde{u}_n^L}{\partial x^2}(s, 0) \right) \Psi_L \left( \| \tilde{u}_n^L(s, \cdot) \|_H \right) \right\}
\]

\[
- (\hat{\alpha} + 1)e^{-y} \right\} dy \, ds
\]

\[
+ \int_{s=0}^{t} \int_{y=0}^{\infty} \tilde{p}_+(t - s, x, y) \left\{ \tilde{u}_n^L(s, y) - e^{-y} \right\} dy dB_s.
\]

For each \( n \in \mathbb{N} \), \( \{ \tilde{u}_n^L(t, \cdot); t \geq 0 \} \) is a well-defined, adapted, and continuous path in \( H_{\text{even}} \). In addition, we can show that there is a limit \( \tilde{u}^L \in H_{\text{even}} \).

**Lemma 3.2.9.** For each \( T > 0 \), we have that \( \sum_{n=1}^{\infty} \sup_{0 \leq t \leq T} \mathbb{E} \left[ \| \tilde{u}_{n+1}^L - \tilde{u}_n^L \|_H \right] < \infty \). Thus \( \mathbb{P} \text{-a.s.,} \)

\( \tilde{u}^L(t, \cdot) \overset{\text{def}}{=} \lim_{n \to \infty} \tilde{u}_n^L(t, \cdot) \) exists as a limit in \( C([0, T]; H) \) and \( \tilde{u}^L \) satisfies the integral equation

\[
\tilde{u}^L(t, x) = \int_{y=0}^{\infty} \tilde{p}_+(t, x, y) \tilde{u}_0(y) \, dy
\]

\[
+ \int_{s=0}^{t} \int_{y=0}^{\infty} \tilde{p}_+(t - s, x, y) \left\{ \left( \frac{\partial \tilde{u}^L}{\partial x}(s, y) + e^{-y} \right) \left( 1 - \frac{\partial^2 \tilde{u}^L}{\partial x^2}(s, 0) \right) \Psi_L \left( \| \tilde{u}^L(s, \cdot) \|_H \right) \right\}
\]

\[
- (\hat{\alpha} + 1)e^{-y} \right\} dy \, ds
\]

\[
+ \int_{s=0}^{t} \int_{y=0}^{\infty} \tilde{p}_+(t - s, x, y) \left\{ \tilde{u}^L(s, y) - e^{-y} \right\} dy dB_s.
\]

For \( t > 0 \) and \( x > 0 \)

Furthermore, the solution of (3.2.10) is unique.

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Proof. See Lemma 4.4 and Lemma 4.5 in \[KMS\].

Let’s now see what happens as $L \nearrow \infty$. Define the random times

\[
\tau_L \overset{\text{def}}{=} \inf\{t \geq 0 : \|\hat{u}^L(t, \cdot)\|_H \geq L\} \quad L > 0
\]

\[
\tau \overset{\text{def}}{=} \lim_{L \to \infty} (\tau_L \land L).
\]

Let’s also define

\[
\hat{u}(t, x) \overset{\text{def}}{=} \lim_{L \to \infty} \hat{u}^L(t \wedge \tau_L, x), \quad t \geq 0, x \geq 0
\]

**Lemma 3.2.10.** We have that

\[
\lim_{t \nearrow \tau} \|\hat{u}(t, \cdot)\|_H = \infty \quad \text{and} \quad \lim_{t \nearrow \tau} \left|\frac{\partial^2 \hat{u}}{\partial x^2}(t, 0)\right| = \infty
\]

Define $u$ as in (3.2.7)–(3.2.8). Then $\{u(t, \cdot) \mid 0 \leq t < \tau\}$ is a weak solution of (3.1.1).

Proof. See Lemma 4.6 and Lemma 4.6 in \[KMS\].

To finish things off, we prove uniqueness.

**Lemma 3.2.11** (Uniqueness). If $\{\hat{u}(t, \cdot) \mid 0 \leq t < \tau\} \subset H$ and $\{\hat{u}'(t, \cdot) \mid 0 \leq t < \tau'\} \subset H$ are two solutions of (3.2.5), then $\hat{u}(t, \cdot) = \hat{u}'(t, \cdot)$ for $0 \leq t < \min\{\tau, \tau'\}$.

Proof. See Lemma 4.7 in \[KMS\].

### 3.3 Truncation

When finding numerical approximations of partial differential equations or stochastic partial differential equations in infinite domains, specific techniques are required due to the unboundedness of the domains (see \[Giv92\] and references therein). One of the most typical methods is to truncate domains. In this section we first consider a truncation method for the heat equation on the semi-infinite interval $[0, \infty)$. In other words, we truncate the domain and impose the Dirichlet boundary condition at the right end point. We first truncate the initial condition so that we can impose the Dirichlet boundary condition at the right end. Then we will show the truncated heat equation approximates the heat equation on $[0, \infty)$ as the right end is getting larger and larger. The second part of this section concerns the transformed SPDE (3.2.6), which is also defined on the semi-infinite interval $[0, \infty)$. We also truncate the initial condition and the domain so
that we can impose the Dirichlet boundary condition at the right end. The approaches to the error analysis for the heat equation and (3.1.1) are, however, somewhat different; for the heat equation we construct some heat kernel and use it to show that the error is small; for (3.1.1) we find the error in a more direct way which uses the nonlinear SPDE (3.2.6) and the truncated SPDE (3.3.15). Norms used in the error analysis for the heat equation and (3.1.1) are also different.

### 3.3.1 Heat Equation

Consider the following heat equation:

\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) \quad t > 0, x > 0 \\
\frac{\partial u}{\partial x}(t, 0) = 0 \quad t > 0 \\
u(t, x) = u_\circ(x) \quad x \geq 0
\]

(3.3.1)

for some initial condition $u_\circ$ and we assume that $u_\circ \in L^2(\mathbb{R}_+)$.

Our main task is to approximate $u$ by a PDE on a finite spatial domain; that will allow us to use a standard spatial grid. Let us first truncate the initial condition. For each $M > 0$ define

\[ u^M_\circ(x) \overset{\text{def}}{=} u_\circ(x) \chi_{[0,M]}(x) \text{ for } x \geq 0. \]

Since $u_\circ \in L^2(\mathbb{R}_+)$ by assumption,

\[ \lim_{M \to \infty} \|u_\circ - u^M_\circ\|_{L^2} = 0. \]

We let $\tilde{u}^{1,M}$ as the solution of

\[
\frac{\partial \tilde{u}^{1,M}}{\partial t}(t, x) = \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, x) \quad t > 0, x > 0 \\
\frac{\partial \tilde{u}^{1,M}}{\partial x}(t, 0) = 0 \quad t > 0 \\
\tilde{u}^{1,M}(t, x) = u^M_\circ(x) \quad x \geq 0
\]

Standard energy estimates thus give us that

\[ \lim_{M \to \infty} \sup_{t \geq 0} \|u(t, \cdot) - \tilde{u}^{1,M}(t, \cdot)\|_{L^2} \leq \lim_{M \to \infty} \|u_\circ - u^M_\circ\|_{L^2} = 0. \]

Now we truncate the semi-infinite interval $[0, \infty)$ by enforcing Dirichlet boundary conditions. Let $\tilde{u}^{2,M}$
satisfy

\[
\frac{\partial \tilde{u}^{2,M}}{\partial t}(t, x) = \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t, x) \quad t > 0, \ 0 < x < 2M
\]

\[
\frac{\partial \tilde{u}^{2,M}}{\partial x}(t, 0) = 0 \quad t > 0
\]

\[
\tilde{u}^{2,M}(t, 2M) = 0 \quad t > 0
\]

\[
\tilde{u}^{2,M}(t, x) = u_\circ^M(x) \quad x \geq 0
\]

Our main goal is to show that \(\tilde{u}^{2,M}\) is a good approximation of \(\tilde{u}^{1,M}\). Set \(\tilde{v}^M = \tilde{u}^{2,M} - \tilde{u}^{1,M}\). Then \(\tilde{v}^M\) satisfies

\[
\frac{\partial \tilde{v}^M}{\partial t}(t, x) = \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, x) \quad t > 0, \ 0 < x < 2M
\]

\[
\frac{\partial \tilde{v}^M}{\partial x}(t, 0) = 0 \quad t > 0
\]

\[
\tilde{v}^M(t, M) = -\tilde{u}^{1,M}(t, 2M) \quad t > 0
\]

\[
\tilde{v}^M(t, x) = 0 \quad 0 \leq x \leq 2M.
\]

We will show that \(\tilde{v}^M\) is small for large \(M\) by solving (3.3.2) explicitly. Let’s first find an explicit expression for \(\tilde{u}^{1,M}\). Define

\[
p(t, x) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{x^2}{4t} \right] \quad t > 0, \ x \in \mathbb{R} \quad (3.3.3)
\]

We then define

\[
\tilde{p}(t, x, y) \overset{\text{def}}{=} p(t, x - y) + p(t, x + y) \quad t > 0, \ x, y \in \mathbb{R}_+.
\]

Here \(p\) and \(\tilde{p}\) are simply heat kernel and Neumann heat kernel. Then we can have an explicit expression for \(\tilde{u}^1\) as

\[
\tilde{u}^{1,M}(t, x) = \int_{y=0}^{\infty} \tilde{p}(t, x, y) u_\circ^M(y) dy = \int_{y=0}^{M} \tilde{p}(t, x, y) u_\circ(y) dy \quad t > 0, \ x \in \mathbb{R}_+.
\]

Thus

\[
\tilde{u}^{1,M}(t, 2M) = \int_{y=0}^{M} \tilde{p}(t, 2M, y) u_\circ(y) dy \quad t > 0.
\]

Note that for \(t \in (0, T]\)

\[
|\tilde{p}(t, 2M, y)| \leq \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{M^2}{t} \right] \leq \frac{K}{M} \quad \text{where}
\]

\[
K \overset{\text{def}}{=} \sup_{z > 0} \frac{1}{\sqrt{4\pi}} \exp \left[ -z^2 \right] < \infty.
\]

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Therefore, for \( t \geq 0 \), we have

\[
|\tilde{u}^{1,M}(t, 2M)| \leq \left| \int_{y=0}^{M} \tilde{p}(t, 2M, y) u_o(y) dy \right| \leq \frac{K}{M} \int_{y=0}^{M} |u_o(y)| dy \leq K \sqrt{\frac{1}{M} \int_{y=0}^{M} |u_o(y)|^2 dy} \leq \frac{K}{\sqrt{M}} \|u_o\|_{L^2}.
\]

(3.3.4)

This says that the inhomogeneity of \( \tilde{v}^M \) is small, so \( \tilde{v}^M \) itself should be small.

Let’s try to solve \( \tilde{v} \) as explicitly as possible. To do so, let’s construct a heat kernel. Consider the PDE

\[
\begin{align*}
\frac{\partial w}{\partial t}(t, x) &= \Delta w(t, x) \quad t > 0, \ 0 < x < 2M \\
\frac{\partial w}{\partial x}(t, 0) &= 0 \quad t > 0 \\
w(t, M) &= f(t) \quad t > 0 \\
w(0, x) &= 0 \quad 0 \leq x \leq 2M,
\end{align*}
\]

where \( f \) is a continuous function with \( f(0) = 0 \). Suppose we can write

\[
w(t, x) = \int_{s=0}^{t} K(t - s, x) f(s) ds
\]

(3.3.5)

for some kernel \( K \). Then \( \tilde{v}^M \) of (3.3.2) will be given by

\[
\tilde{v}^M(t, x) = -\int_{s=0}^{t} K(t - s, x) \tilde{u}^{1,M}(s, 2M) ds.
\]

If we can get a useful bound on \( K \), then we should be able to see that \( \tilde{v}^M \) is small by (3.3.4). Let us construct \( K \). We use the heat kernel \( p(t, x) \) defined in (3.3.3) in order to construct \( K \). Since \( p \) satisfies the heat equation, so does \( \frac{\partial p}{\partial x} \). Note that for any continuous \( \varphi \in C(\mathbb{R}_+) \),

\[
\int_{s=0}^{t} \left\{ -\frac{\partial p}{\partial x}(t - s, \varepsilon) \right\} \varphi(s) ds = \frac{1}{\sqrt{4\pi}} \int_{s=0}^{t} \frac{\varepsilon}{(t-s)^{3/2}} \exp \left[ -\frac{\varepsilon^2}{4(t-s)} \right] \varphi(t-s) ds
\]

\[
= \frac{2}{\sqrt{4\pi}} \int_{u=\varepsilon/\sqrt{t}}^{\infty} e^{-u^2/4} \varphi \left( t - \frac{\varepsilon^2}{u^2} \right) du
\]

(3.3.6)

\[
= \frac{2}{\sqrt{4\pi}} \int_{u=\varepsilon/\sqrt{t}}^{\infty} e^{-u^2/4} \varphi \left( t - \frac{\varepsilon^2}{u^2} \right) du.
\]

Thus we get

\[
\lim_{\varepsilon \to 0} \int_{s=0}^{t} \left\{ -\frac{\partial p}{\partial x}(t - s, \varepsilon) \right\} \varphi(s) ds = \varphi(t)
\]

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Define now
\[ \mathfrak{t}(t, x) \overset{\text{def}}{=} -\frac{\partial p}{\partial x}(t, x - 2M) + \frac{\partial p}{\partial x}(t, x + 2M) \quad t > 0, \ 0 \leq x \leq 2M. \]

Clearly \( \mathfrak{t} \) satisfies the heat equation. In addition, since \( p(t, \cdot) \) is even, \( \frac{\partial^2 p}{\partial x^2}(t, \cdot) \) is also even. Hence
\[ \frac{\partial \mathfrak{t}}{\partial x}(t, 0) = -\frac{\partial^2 p}{\partial x^2}(t, -2M) + \frac{\partial^2 p}{\partial x^2}(t, 2M) = 0 \quad t > 0. \tag{3.3.7} \]

We also have from a simple calculation that \( \lim_{t \searrow 0} \mathfrak{t}(t, x) = 0 \) for all \( x \in [0, 2M] \). Finally, thanks to (3.3.6) we know that
\[ \lim_{x \nearrow 2M} \int_{s=0}^{t} \mathfrak{t}(t-s, x) \varphi(s) ds = \varphi(t) + \int_{s=0}^{t} \frac{\partial p}{\partial x}(t, 4M) \varphi(s) ds \tag{3.3.8} \]
for all \( \varphi \in C(\mathbb{R}^+) \).

Suppose that \( K \) of (3.3.5) is of the form
\[ K(t, x) = \mathfrak{t}(t, x) + \int_{s=0}^{t} \mathfrak{t}(t-s, x) k(s) ds. \]

Let us now make this work. Define
\[ \tilde{w}(t, x) \overset{\text{def}}{=} \int_{s=0}^{t} \mathfrak{t}(t-s, x) f(s) ds + \int_{s=0}^{t} \left( \int_{r=0}^{t-s} \mathfrak{t}(t-s-r, x) \tilde{k}(r) dr \right) f(s) ds \quad \text{for } t > 0, \ 0 < x < 2M \tag{3.3.9} \]
and
\[ \tilde{w}(0, x) \overset{\text{def}}{=} 0 \quad \text{for } 0 \leq x \leq 2M \quad \text{and} \quad \tilde{w}(t, 2M) \overset{\text{def}}{=} \lim_{x \nearrow 2M} \tilde{w}(t, x) \quad \text{for } t > 0. \]

By linearity, \( \tilde{w} \) clearly satisfies the heat equation. Thanks to (3.3.7), we have that \( \frac{\partial \tilde{w}}{\partial x}(t, 0) = 0 \) for \( t > 0 \). We also have that \( \lim_{t \searrow 0} \tilde{w}(t, x) = 0 \) for \( 0 \leq x \leq 2M \). It only remains to arrange things so that \( \lim_{x \nearrow 2M} \tilde{w}(t, x) = f(t) \). From (3.3.8) we have that
\[ \lim_{x \nearrow 2M} \tilde{w}(t, x) = f(t) + \int_{s=0}^{t} \frac{\partial p}{\partial x}(t-s, 4M) f(s) ds + \int_{s=0}^{t} k(t-s) f(s) ds + \int_{s=0}^{t} \left\{ \int_{r=0}^{t-s} \frac{\partial p}{\partial x}(t-s-r, 4M) \tilde{k}(r) dr \right\} f(s) ds. \]

We want this to hold for all \( f \) and all \( s \), and we want that \( \lim_{x \nearrow M} \tilde{w}(t, x) = f(t) \). In other words, we want that
\[ \tilde{k}(t-s) = -\frac{\partial p}{\partial x}(t-s, 4M) - \int_{r=0}^{t-s} \frac{\partial p}{\partial x}(t-s-r, 4M) \tilde{k}(r) dr. \]
for all $t > s > 0$ or equivalently

\[ \tilde{k}(t) = -\frac{\partial p}{\partial x}(t, 4M) - \int_{r=0}^{t} \frac{\partial p}{\partial x}(t-r, 4M) \tilde{k}(r) dr \]

for all $t > 0$. This can be solved iteratively. Define

\[ \tilde{k}_0(t) \overset{\text{def}}{=} 0 \]

\[ \tilde{k}_{n+1}(t) \overset{\text{def}}{=} -\frac{\partial p}{\partial x}(t, 4M) - \int_{r=0}^{t} \frac{\partial p}{\partial x}(t-r, 4M) \tilde{k}_n(r) dr. \]

Then of course

\[ \tilde{k}_{n+1}(t) - \tilde{k}_n(t) = -\int_{r=0}^{t} \frac{\partial p}{\partial x}(t-r, 4M) \left\{ \tilde{k}_n(r) - \tilde{k}_{n-1}(r) \right\} dr. \]

Fix a finite time horizon $T > 0$. It is fairly easy to see that there is a $K_T > 0$ such that

\[ \left| \frac{\partial p}{\partial x}(t, 4M) \right| \leq K_T \]

for all $t \in (0, T]$. Therefore, It follows easily from induction that

\[ \left| \tilde{k}_n(t) - \tilde{k}_{n-1}(t) \right| \leq \frac{K_T (K_T t)^{n-1}}{(n-1)!} \quad (3.3.10) \]

for all $n \in \{1, 2, \ldots\}$ and $t \in (0, T]$. From (3.3.10), we also can see that $\tilde{k}_n(0) = 0$ for each $n \in \{1, 2, \ldots\}$

Thus $\tilde{k}$ does exist and furthermore

\[ |\tilde{k}(t)| \leq K_T e^{K_T t} \]

for all $t \in [0, T]$. Taking absolute values in (3.3.6) and using (3.3.9) and the fact that $\tilde{w} = w$, we have that

\[ |w(t, x)| \leq \left\{ 1 + TK_T e^{K_T T} \right\} \|f\|_{C[0,T]} \]

for all $0 \leq t \leq T$ and $x \in [0, 2M]$. Here $\| \cdot \|_{C[0,T]}$ is the sup norm over the interval $[0, T]$.

Let’s now return to $\tilde{v}^M$ of (3.3.2). Combining things together, we have that

\[ \sup_{t \in [0,T]} \| \tilde{u}^{1,M}(t,x) - \tilde{u}^{2,M}(t,x) \| \leq \frac{K}{\sqrt{M}} \left\{ 1 + TK_T e^{K_T T} \right\} \|u_0\|_{L^2}. \]
3.3.2 The Stochastic Moving Boundary Problem

In Section 3.2 a solution of (3.1.1) can be obtained from a solution of the transformed SPDE (3.2.6) (see Lemma 3.2.8). This implies that numerical approximations of (3.2.6) lead to numerical approximations of (3.1.1). Therefore we focus on the transformed SPDE (3.2.6). For convenience, define $E(x) \equiv e^{-x}$ and revisit (3.2.6):

$$d\tilde{u}(t,x) = \left\{ \frac{\partial^2 \tilde{u}(t,x)}{\partial x^2} + \tilde{\alpha}(\tilde{u}(t,x) - E(x)) - E(x) + \left( \frac{\partial \tilde{u}(t,x)}{\partial x} + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(t,0) \right) \right\} dt$$

$$+ (\tilde{u}(t,x) - E(x)) dB_t \quad 0 < t < \tau, \; x > 0$$

$$\frac{\partial \tilde{u}}{\partial x}(t,0) = 0 \quad 0 < t < \tau$$

$$\tilde{u}(0,x) = \tilde{u}_0(x) = u(x) + E(x), \quad x > 0$$

where $\tau = \lim_{L \to \infty} \tau_L$ where in turn

$$\tau_L \equiv \inf \{ t \in [0,T] : \|u(t,\cdot)\|_H \geq L \} = (\inf \emptyset \equiv T)$$

for each $L > 0$. Here we assume that $\tilde{u}_0$ is in the Sobolev class $H^2([0,\infty))$. This assumption is stronger than before since we consider an SPDE for the second order partial derivative of $\tilde{u}$ with respect to $x$.

Our goal is to replace (3.3.11) with an SPDE on a finite spatial interval. As for the heat equation, we first truncate the initial condition. Let $\xi \in C^\infty(\mathbb{R}_+;\mathbb{R}_+)$ be a monotone decreasing function such that $\xi \equiv 1$ on $[0,1)$, $\xi \geq 0$ on $(0,2)$, and $\xi \equiv 0$ on $[2,\infty)$. For each $M > 0$, define

$$\tilde{u}^M_0(x) \equiv \xi \left( \frac{x}{M} \right) \tilde{u}_0(x), \quad x \geq 0$$

Then it is clear that $\lim_{M \to \infty} \|\tilde{u}_0 - \tilde{u}^M_0\|_H = 0$. For each $M > 0$ and $L > 0$, define now

$$d\tilde{u}^{1,M}(t,x) = \left\{ \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t,x) + \tilde{\alpha}(\tilde{u}^{1,M}(t,x) - E(x)) - E(x) + \left( \frac{\partial \tilde{u}^{1,M}}{\partial x}(t,x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t,0) \right) \right\} dt$$

$$+ (\tilde{u}^{1,M}(t,x) - E(x)) dB_t \quad 0 < t < \tau^0_M, \; x > 0$$

$$\frac{\partial \tilde{u}^{1,M}}{\partial x}(t,0) = 0 \quad 0 < t < \tau^0_M$$

$$\tilde{u}^{1,M}(0,x) = \tilde{u}^M_0(x) \quad x > 0$$

(3.3.12)
where $\tau^a_M \overset{\text{def}}{=} \lim_{L \to \infty} \tau^a_{M,L}$, where in turn

$$\tau^a_{M,L} = \inf \{ t \in [0, T] : \| \tilde{u}^M(t, \cdot) \|_H \geq L \} \quad (\inf \emptyset \overset{\text{def}}{=} T)$$

Our first claim is that $\tilde{u}^{1,M}$ is a good approximation of $\tilde{u}$. Here we will use the standard space $H$ which is the closure of $C_0^\infty(\mathbb{R}_+)$ with respect to $\| \cdot \|_H$ defined in (3.2.9).

**Lemma 3.3.1.** For any $\varphi \in H$, we have that

$$\sup_{x \in \mathbb{R}_+} \left| \varphi^{(i)}(x) \right| \leq 2\| \varphi \|_H.$$  

**Proof.** See Lemma 4.1 in [KMS].

**Proposition 3.3.2.** Fix $L > 0$. We have that

$$\lim_{M \to \infty} \mathbb{P} \{ \tau_L < T, \tau^a_{M,L-1} = T \} = 0$$

and for each $\delta > 0$ we have that

$$\lim_{M \to \infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq T \wedge \tau_L \wedge \tau^a_{M,L-1}} \| \tilde{u}(t, \cdot) - \tilde{u}^{1,M}(t, \cdot) \|_H \geq \delta \right\} = 0$$

**Proof.** Define

$$v^M \overset{\text{def}}{=} \tilde{u} - \tilde{u}^{1,M},$$

we have that

$$dv^M(t,x) = \left( \frac{\partial^2 v^M}{\partial x^2}(t,x) + \hat{a}v^M(t,x) + A(t,x) \right) dt + v^M(t,x) dB_t \quad 0 < t < \tau_L \wedge \tau^a_{M,L-1} \ x > 0$$

$$\frac{\partial v^M}{\partial x}(t,0) = 0 \quad 0 < t < \tau_L \wedge \tau^a_{M,L-1}$$

$$v^M(0,x) = u_0(x) \left\{ 1 - \xi \left( \frac{x}{M} \right) \right\} \quad x > 0,$$

where

$$A(t,x) \overset{\text{def}}{=} \left( \frac{\partial \tilde{u}}{\partial x}(t,x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}}{\partial x^2}(t,0) \right) - \left( \frac{\partial \tilde{u}^{1,M}}{\partial x}(t,x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t,0) \right).$$

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Thus

\[ d\|v^M(t, \cdot)\|^2_H = \left( \frac{\partial^2 v^M}{\partial x^2}(t, \cdot), v^M(t, \cdot) \right)_H + \hat{\alpha}\|v^M(t, \cdot)\|^2_H dt + 2 \langle v^M(t, \cdot), A(t, \cdot) \rangle_H dt \\
+ 2\|v^M(t, \cdot)\|^2_B dt + \|v^M(t, \cdot)\|^2_H dt. \]

For any \(\varphi \in C_0^{\infty}\) and any \(i \in \{0, 1, 2, 3\}\), we have that either \(\varphi^i(0) = 0\) or \(\varphi^{i+1}(0) = 0\). Thus

\[ 2 \left( \frac{\partial^2 v^M}{\partial x^2}(t, \cdot), v^M(t, \cdot) \right)_H = -2 \left\| \frac{\partial v^M}{\partial x}(t, \cdot) \right\|^2_H. \]

Secondly,

\[ 2 \langle v^M(t, \cdot), A(t, \cdot) \rangle_H = 2 \left( v^M(t, \cdot), \frac{\partial v^M}{\partial x}(t, \cdot) \right)_H \int_{\lambda=0}^1 \left( 1 - \frac{\partial^2 \tilde{u}_{1,M}^1}{\partial x^2}(t, 0) - \lambda \frac{\partial^2 v^M}{\partial x^2}(t, 0) \right) d\lambda \]
\[ - 2 \frac{\partial^2 v^M}{\partial x^2}(t, 0) \int_{\lambda=0}^1 \left( v^M(t, \cdot), \frac{\partial \tilde{u}_{1,M}^1}{\partial x}(t, \cdot) + \lambda \frac{\partial v^M}{\partial x}(t, \cdot) + E(\cdot) \right) _H d\lambda. \]

We note that if \(0 \leq t \leq \tau_L \wedge \tau_{M,L-1}^3\), then

\[ \|\tilde{u}(t, \cdot)\|_H \leq L, \quad \|\tilde{u}_{1,M}^1(t, \cdot)\|_H \leq L - 1 \leq L, \quad \text{and} \quad \sup_{\lambda \in [0, 1]} \|\tilde{u}(t, \cdot) + \lambda v^M(t, \cdot)\|_H \leq L, \]

the last inequality following by convexity from the first two. Thus, by Lemma 3.3.1 we know that

\[ \left| \int_{\lambda=0}^1 \left( 1 - \frac{\partial^2 \tilde{u}_{1,M}^1}{\partial x^2}(t, 0) - \lambda \frac{\partial^2 v^M}{\partial x^2}(t, 0) \right) d\lambda \right| \leq 1 + 2L. \]

Next decomposing \(\langle \cdot, \cdot \rangle_H\) into its different terms, we also have that

\[ 2 \left| \int_{\lambda=0}^1 \left\{ \sum_{i=0}^2 \int_{x \in \mathbb{R}^+} \frac{\partial^i v^M}{\partial x^i}(t, x) \left( \frac{\partial^{i+1} \tilde{u}_{1,M}^1}{\partial x^{i+1}}(t, x) + \lambda \frac{\partial^{i+1} v^M}{\partial x^{i+1}}(x) + \frac{d^{i+1} E}{dx^{i+1}}(x) \right) dx \right\} d\lambda \right| \]
\[ \leq 2\|v^M(t, \cdot)\|_H \int_{\lambda=0}^1 \|\tilde{u}_{1,M}^1(t, \cdot) + \lambda v^M(t, \cdot) + E(\cdot)\|_H d\lambda \leq 2\|v^M(t, \cdot)\|_H (L + \|E\|_H) \]
Finally, since \( v^M(t, \cdot) \) is in the \( H \)-closure of \( C^\infty_{\text{even}}(\mathbb{R}_+) \), we have that

\[
2 \left| \int_{\lambda=0}^{1} \left\{ \int_{x \in \mathbb{R}_+} \frac{\partial^4 v^M}{\partial x^4}(t, x) \left( \frac{\partial^4 \tilde{u}^1_{1, M}}{\partial x^4}(t, x) + \lambda \frac{\partial^4 v^M}{\partial x^4}(t, x) + \frac{d^4 E}{dx^4}(x) \right) dx \right\} d\lambda \right| \\
= 2 \left\{ \int_{x \in \mathbb{R}_+} \frac{\partial^4 v^M}{\partial x^4}(t, x) \left( \frac{\partial^4 \tilde{u}^1_{1, M}}{\partial x^4}(t, x) + \lambda \frac{\partial^4 v^M}{\partial x^4}(x) + \frac{d^4 E}{dx^4}(x) \right) dx \right\} d\lambda \\
\leq 2 \left\| \frac{\partial v^M}{\partial x}(t, \cdot) \right\|_H (L + \|E\|_H).
\]

Combining things together, we get that

\[
2 \langle v^M(t, \cdot), A(t, \cdot) \rangle_H \leq 2 \left\| \frac{\partial v^M}{\partial x}(t, \cdot) \right\|_H \|v^M(t, \cdot)\|_H (1 + 2L) \\
+ 8\|v^M(t, \cdot)\|^2_H (L + \|E\|_H) + 8\|v^M(t, \cdot)\|_H \left\| \frac{\partial v^M}{\partial x}(t, \cdot) \right\|_H (L + \|E\|_H) \\
\leq 2 \left\| \frac{\partial v^M}{\partial x}(t, \cdot) \right\|^2_H + ((1 + 2L)^2 + 8(L + \|E\|_H) + 16(L + \|E\|_H)^2) \|v^M(t, \cdot)\|^2_H.
\]

Thus, we have

\[
\frac{d}{dt} \left\| v^M(t, \cdot) \right\|^2_H \leq K_{B,L} \|v^M(t, \cdot)\|^2_H dt + 2\|v^M(t, \cdot)\|^2_H dB_t
\]

where

\[
K_{B,L} \overset{\text{def}}{=} \hat{\alpha} + ((1 + 2L)^2 + 8(L + \|E\|_H) + 16(L + \|E\|_H)^2) + 1.
\]

Now define

\[
Z^M_t \overset{\text{def}}{=} \|v^M(t, \cdot)\|^2_H e^{-K_{B,L} t},
\]

then we have

\[
dZ^M_t \leq 2Z^M_t dB_t.
\]

For any \( \delta \in (0, 1) \), stopping \( Z_{t \wedge T \wedge \tau_{L} \wedge \tau_{\tilde{M}, L}^\infty} \) when it exceeds \( \delta \) and using Markov’s inequality, we get that

\[
P \left\{ \sup_{0 \leq t \leq T \wedge \tau_{L} \wedge \tau_{\tilde{M}, L}^\infty} \|v^{M,L}(t, \cdot)\|^2_H \geq \delta \right\} \leq P \left\{ \sup_{0 \leq t \leq T \wedge \tau_{L} \wedge \tau_{\tilde{M}, L}^\infty} Z^M_t \geq \delta e^{-K_{B,L} T} \right\} \\
\leq \frac{e^{K_{B,L} T}}{\delta} Z^M_0 = \frac{e^{K_{B,L} T}}{\delta} \|u_0 - v^M_0\|^2_H.
\]
Secondly, if \( \tau_L < T \) and \( \tau_{M,L-1}^a = T \), we must have that \( \|v^{M,L}(\tau_L, \cdot)\| \geq 1 \). Thus

\[
\lim_{M \to \infty} P\{\tau_L < T, \tau_{M,L-1}^a \geq T\} = 0.
\]

The conclusions follow.

Next we show that \( \tilde{u}^{1,M}(t, x) \) and its partial derivatives with respect to \( x \) asymptotically vanish as \( x \) goes to infinity.

**Lemma 3.3.3.** Fix \( T > 0 \). We have that

\[
\lim_{M \to \infty} \sum_{k=0}^4 \mathbb{E} \int_{t=0}^{T \wedge \tau_{M,L}} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, 2M) \right|^2 dt = 0.
\]

**Proof.** We first bound the partial derivatives of \( \tilde{u}^{1,M}(t, x) \); for \( t \leq \tau_{M,L} \)

\[
\left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, 2M) \right|^2 = \int_{x=2M}^{2M+1} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, x) \right|^2 dx + \int_{x=2M}^{2M+1} \left( \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, 2M) \right|^2 - \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, x) \right|^2 \right) dx
\]

\[
\leq \int_{x=2M}^{2M+1} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, x) \right|^2 dx + \int_{x=2M}^{2M+1} \left( \right) dx
\]

\[
\leq 2 \int_{x=2M}^{\infty} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, x) \right|^2 dx + 2 \int_{x=2M}^{\infty} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, x) \right|^2 dx.
\]

Since we have that \( \|\tilde{u}^{1,M}(t, \cdot)\|_H \leq L \) for \( t \leq \tau_{M,L}^a \), the dominated convergence implies that

\[
\lim_{M \to \infty} \sum_{k=0}^2 \mathbb{E} \int_{t=0}^{T \wedge \tau_{M,L}} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k} (t, 2M) \right|^2 dt = 0.
\]

For \( k \in \{3, 4\} \), we consider an evolution equation for \( \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, x) \). Let \( \tilde{v}^M(t, x) \) def \( = \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, x) \). By (3.3.12), we can write the evolution of \( \tilde{v}^M \) as

\[
d\tilde{v}^M(t, x) = \left\{ \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, x) + \frac{\partial \tilde{v}^M}{\partial x}(t, x) (1 - \tilde{v}^M(t, 0)) \right\} dt
\]

\[
- E(x) \{ \dot{\alpha} + \tilde{v}^M(t, 0) \} dt + (\tilde{v}^M(t, x) - E(x)) dB_t, \quad 0 < t < \tau_{M,L}^a.
\]

Thus, we have that \( 0 < t < \tau_{M,L}^a \).
\[ d\|v^M(t,\cdot)\|_H^2 = 2\left\langle \dddot{v}^M(t,\cdot), \frac{\partial^2 v^M(t,\cdot)}{\partial x^2}(t,\cdot) \right\rangle_H dt + 2\left\langle \dddot{v}^M(t,\cdot), \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\rangle_H (1 - \dddot{v}^M(t,0)) dt \\
- 2\left( \dddot{a} + \dddot{v}^M(t,0) \right) \langle \dddot{v}^M(t,\cdot), E(\cdot) \rangle_H + \langle \dddot{v}^M(t,\cdot), \dddot{v}^M(t,\cdot) + E(\cdot) \rangle_H dB_t \]

Since \( \dddot{v}^M \) is also in \( H \)-closure of \( C^\infty_{\text{even}}(\mathbb{R}_+) \), we have that
\[
\left\langle \frac{\partial^2 \dddot{v}^M}{\partial x^2}(t,\cdot), \dddot{v}^M(t,\cdot) \right\rangle_H = -\left\| \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\|_H^2.
\]

Using the fact that \( |\dddot{v}^M(t,0)| \leq 2L \) for \( t < \tau_{M,L}^a \), we obtain that
\[
2\left(1 - \dddot{v}^M(t,0)\right) \langle \dddot{v}^M(t,\cdot), E(\cdot) \rangle_H \leq (1 + 2L)^2 \|\dddot{v}^M(t,\cdot)\|_H^2 + \left\| \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\|_H^2 \\
2\left( \dddot{a} + \dddot{v}^M(t,0) \right) \langle \dddot{v}^M(t,\cdot), E(\cdot) \rangle_H \leq (|\dddot{a}| + 2L)^2 \|\dddot{v}^M(t,\cdot)\|_H^2 + \|E(\cdot)\|_H^2 \\
\|\dddot{v}^M(t,\cdot) - E(\cdot)\|_H^2 \leq 2\|\dddot{v}^M(t,\cdot)\|_H^2 + 2\|E(\cdot)\|_H^2.
\]

Combining things together, we get that
\[
d\|\dddot{v}^M(t,\cdot)\|_H^2 \leq -\left\| \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\|_H^2 dt + K_L \|\dddot{v}^M(t,\cdot)\|_H^2 dt + 2\|E(\cdot)\|_H^2 dt + 2\left\langle \dddot{v}^M(t,\cdot), \dddot{v}^M(t,\cdot) - E(\cdot) \right\rangle_H dB_t,
\]

where \( K_L \equiv (1 + 2L)^2 + (|\dddot{a}| + 2L)^2 + 2. \)

Setting \( Z_t \equiv \|\dddot{v}^M(t,\cdot)\|_H^2 e^{-K_L t} \), we have that
\[
dZ_t \leq -e^{-K_L t} \left\| \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\|_H^2 dt + 2e^{-K_L t}\|E(\cdot)\|_H^2 dt + 2e^{-K_L t} \left\langle \dddot{v}^M(t,\cdot), \dddot{v}^M(t,\cdot) - E(\cdot) \right\rangle_H dB_t.
\]

From this, we obtain that
\[
\mathbb{E} \int_{t=0}^{T \land \tau_{M,L}^a} e^{-K_L t} \left\| \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\|_H^2 dt \leq \|\dddot{v}_0^M(\cdot)\|_H^2 + 2 \int_{t=0}^{T} e^{-K_L t} \|E(\cdot)\|_H^2 dt,
\]

where \( \dddot{v}_0^M \equiv \dddot{v}_0^M \). In order words, there exists constant \( K_{L,T} > 0 \) such that
\[
\mathbb{E} \int_{t=0}^{T \land \tau_{M,L}^a} \left\| \frac{\partial \dddot{v}^M}{\partial x}(t,\cdot) \right\|_H^2 dt < K_{L,T} < \infty. \quad (3.3.14)
\]

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Finally using (3.3.13) and the dominated convergence theorem which is guaranteed by (3.3.14), we get that

$$\lim_{M \to \infty} \sum_{k=3}^{4} \mathbb{E} \int_{t=0}^{T \wedge \tau_{M,L}^{b,M}} \left| \frac{\partial^k \tilde{u}^{1,M}}{\partial x^k}(t,2M) \right|^2 dt = 0.$$  

Let’s next enforce Dirichlet boundary conditions. Namely, consider the SPDE

$$d\tilde{u}^{2,M}(t,x) = \left\{ \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t,x) + \hat{a}(\tilde{u}^{2,M}(t,x) - E(x)) - E(x) + \left( \frac{\partial \tilde{u}^{2,M}}{\partial x}(t,x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t,0) \right) \right\} dt$$

$$+ \left( \tilde{u}^{2,M}(t,x) - E(x) \right) dB_t \quad 0 < t < \tau_{M}^{b}, \ 0 < x < 2M$$

$$\frac{\partial \tilde{u}^{2,M}}{\partial x}(t,0) = 0 \quad 0 < t < \tau_{M}^{b}$$

$$\tilde{u}^{2,M}(t,2M) = 0 \quad 0 < t < \tau_{M}^{b}$$

$$\tilde{u}^{2,M}(0,x) = \tilde{u}_a^M(x). \quad 0 \leq x \leq 2M$$

(3.3.15)

where \( \tau_{M}^{b} \defeq \lim_{L \to \infty} \tau_{M,L}^{b} \) where in turn

$$\tau_{M,L}^{b} \defeq \inf \left\{ t \in [0,T] : \left| \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t,0) \right| \geq L \right\} \quad (\inf \emptyset \defeq T).$$

Let’s again write down an appropriate Hilbert space. For each \( \varphi \in C^\infty[0,2M] \), define

$$\| \varphi \|_{H_M} \defeq \sqrt{\sum_{i=0}^{3} \int_{x \in (0,2M)} |\varphi^{(i)}(x)|^2 \, dx}.$$ 

Let \( H_M \) be the closure of \( C^\infty[0,2M] \) with respect to \( \| \cdot \|_{H_M} \) and let \( H_{even,M} \) be the closure of

$$C_{even}^\infty[0,2M] \defeq \left\{ \varphi \in C^\infty[0,2M] : \varphi^{(i)}(0) = 0 \text{ for all odd } i \in \mathbb{N} \text{ and } \varphi^{(i)}(2M) = 0 \text{ for all even } i \in \mathbb{N} \right\}$$

with respect to \( H_M \). We let \( \langle \cdot, \cdot \rangle_{H_M} \) be the inner product associated with \( \| \cdot \|_{H_M} \).

**Lemma 3.3.4.** For any \( \varphi \in H_M \), we have that

$$\sup_{0 \leq x \leq 2M} \left| \varphi^{(i)}(x) \right| \leq 2\| \varphi \|_{H_M}.$$ 

**Proof.** The proof is the same as that of Lemma 3.3.1.  

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We claim that $\tilde{u}^{2,M}$ is sufficiently regular up to time $\tau_{M,L}^b$.

**Lemma 3.3.5.** Define

$$K_a \overset{\text{def}}{=} 3|\tilde{\alpha}| + 3 + 2(1 + 2L)^2$$

$$K_b \overset{\text{def}}{=} |\tilde{\alpha}| + 3.$$

Then

$$E \left[ \int_{s=0}^{\tau_{M,L}^b} \left\| \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(s, \cdot) \right\|^2_{H_M} ds \right] \leq (\|\tilde{u}_0\|^2_{H_M} + K_b\|E\|^2_{H_M} T) e^{K_a T}.$$

**Proof.** This is very similar to the proof of Lemma 3.3.3. We start by writing that

$$d\|\tilde{u}^{2,M}(t, \cdot)\|^2_{H_M} = 2 \left\langle \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t, \cdot), \tilde{u}^{2,M}(t, \cdot) \right\rangle_{H_M} dt + 2\tilde{\alpha} \left\langle \tilde{u}^{2,M}(t, \cdot), \tilde{u}^{2,M}(t, \cdot) - E \right\rangle_{H_M} dt$$

$$- 2 \left\langle \tilde{u}^{2,M}(t, \cdot), E \right\rangle_{H_M} dt + A_t dt + 2\|\tilde{u}^{2,M}(t, \cdot) + E\|^2_{H_M} dB_t + ||\tilde{u}^{2,M}(t, \cdot) + E\|^2_{H_M} dt.$$

where

$$A_t = 2 \left\langle \tilde{u}^{2,M}(t, \cdot), \frac{\partial \tilde{u}^{2,M}}{\partial x}(t, \cdot) + E \right\rangle_{H_M} \left(1 - \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t, 0)\right)$$

Since $\tilde{u}^{2,M}$ is in $H_{\text{even},M}$, we have that

$$\left\langle \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t, \cdot), \tilde{u}^{2,M}(t, \cdot) \right\rangle_{H_M} = -\left\| \frac{\partial \tilde{u}^{2,M}}{\partial x}(t, \cdot) \right\|_{H_M}^2.$$

We also have that

$$\left| 2\tilde{\alpha} \left\langle \tilde{u}^{2,M}(t, \cdot), \tilde{u}^{2,M}(t, \cdot) - E \right\rangle_{H_M} \right| \leq |\tilde{\alpha}| \left\{ 3\|\tilde{u}^{2,M}(t, \cdot)\|^2_{H_M} + \|E\|^2_{H_M} \right\}$$

$$\left| 2 \left\langle \tilde{u}^{2,M}(t, \cdot), E \right\rangle_{H_M} \right| \leq \|\tilde{u}^{2,M}(t, \cdot)\|^2_{H_M} + \|E\|^2_{H_M}$$

$$\|\tilde{u}^{2,M}(t, \cdot) + E\|^2_{H_M} \leq 2\|\tilde{u}^{2,M}(t, \cdot)\|^2_{H_M} + 2\|E\|^2_{H_M}.$$ 

Furthermore, we can obtain that for $0 \leq t \leq \tau_{M,L}^b$,

$$|A_t| \leq 2\|\tilde{u}^{2,M}(t, \cdot)\|_{H_M} \left\| \frac{\partial \tilde{u}^{2,M}}{\partial x}(t, \cdot) + E \right\|_{H_M} (1 + 2L)$$

$$\leq \left\| \frac{\partial \tilde{u}^{2,M}}{\partial x}(t, \cdot) \right\|_{H_M}^2 + \|E\|^2_{H_M} + 2\|\tilde{u}^{2,M}(t, \cdot)\|^2_{H_M} (1 + 2L)^2.$$
Combining things together, we get that
\[
d\|\tilde{u}^{2,M}(t, \cdot)\|^2_{H^M} \leq -\left\| \frac{\partial \tilde{u}^{2,M}}{\partial x}(t, \cdot) \right\|^2_{H^M} dt + K_a \|\tilde{u}^{2,M}(t, \cdot)\|^2_{H^M} dt + K_b \|E\|^2_{H^M} dt + 2\|\tilde{u}^{2,M}(t, \cdot) + E\|_{H^M}^2 dB_t.
\]

Setting \(Z_t \overset{\text{def}}{=} \|\tilde{u}^{2,M}(t, \cdot)\|^2_{H^M} e^{-K_a t}\), we have that
\[
dZ_t \leq -\left\| \frac{\partial \tilde{u}^{2,M}}{\partial x}(t, \cdot) \right\|^2_{H^M} e^{-K_a t} dt + K_b \|E\|^2_{H^M} e^{-K_a t} dt + 2e^{-K_a t} \|L^{M}(t, \cdot) + E\|_{H^M} dB_t.
\]
Thus
\[
\mathbb{E} \left[Z_{\tau_{M,L}} - \|\tilde{u}^{2,M}\|^2_{H^M} + \int_{s=0}^{\tau_{M,L}} \left\| \frac{\partial \tilde{u}^{2,M}}{\partial x}(s, \cdot) \right\|^2_{H^M} e^{-K_a s} ds \right] \leq K_b \|E\|^2_{H^M} T
\]
which implies the stated result.

Even though Lemma 3.3.5 says that \(\tilde{u}^{2,M}\) is sufficiently regular up to time \(\tau_{M,L}^b\), it is not easy to show the continuity of derivatives of \(\tilde{u}^{2,M}\) in time; one can show the continuity of \(\frac{\partial \tilde{u}^{2,M}}{\partial x}(t, x), 0 \leq i \leq 2\) by using (3.3.15) and the Kolmogorov continuity theorem, but we may need more regularity to show the continuity in time for more derivatives of \(\tilde{u}^{2,M}\). Thus we make the following assumption. Assumption 3.3.6 will be used in Section 3.4 in order to show the convergence. This seems a reasonable assumption since random fluctuations only come from the single Brownian motion \(B_t\) which only depends on time. In fact, a solution \(u(t, \cdot)\) of the stochastic moving boundary problem (3.1.1) is \(C^\infty\) on \((\beta(t), \infty)\) (see Lemma 3.1 in [KMS]).

**Assumption 3.3.6.** We assume that \(\frac{\partial \tilde{u}^{2,M}}{\partial x}(t, x)\) are continuous with respect to \(t\) almost surely for each fixed \(x\) and \(0 \leq i \leq 4\).

Now we can show that when \(M\) is large, \(\tilde{u}^{1,M}\) and \(\tilde{u}^{2,M}\) are close to each other up to the stopping time \(\tau_{M,L} \overset{\text{def}}{=} \tau_{M,L}^a \wedge \tau_{M,L}^b\).

**Proposition 3.3.7.** Fix \(L > 0\) and \(T > 0\). Then for each \(\delta > 0\) we have that
\[
\lim_{M \to \infty} P \left\{ \sup_{0 \leq t \leq T \wedge \tau_{M,L}} \|\tilde{u}^{1,M}(t, \cdot) - \tilde{u}^{2,M}(t, \cdot)\|_{H^M} \geq \delta \right\} = 0
\]
Proof. Let \( \tilde{v}^M(t,x) \defeq \tilde{u}^1_M(t,x) - \tilde{u}^2_M(t,x) \). By (3.3.12) and (3.3.15), we have that

\[
d\tilde{v}^M(t,x) = \left\{ \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t,x) + \hat{\alpha}v^M(t,x) + A(t,x) \right\} dt + \tilde{v}^M(t,x)dB_t \quad 0 < t \leq \tau_{M,L}, 0 < x < 2M
\]

where

\[
A(t,x) \defeq \left( \frac{\partial \tilde{u}^1_M}{\partial x}(t,x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}^1_M}{\partial x^2}(t,0) \right) - \left( \frac{\partial \tilde{u}^2_M}{\partial x}(t,x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}^2_M}{\partial x^2}(t,0) \right)
\]

\[
= - \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t,0) \left( \frac{\partial \tilde{u}^1_M}{\partial x}(t,x) + E(x) \right) + \left( 1 - \frac{\partial^2 \tilde{u}^2_M}{\partial x^2}(t,0) \right) \frac{\partial \tilde{v}^M}{\partial x}(t,x).
\]

Therefore, we get that for \( 0 < t \leq \tau_{M,L} \)

\[
d\|\tilde{v}^M(t,\cdot)\|_{H^2_M}^2 = 2 \left\langle \tilde{v}^M(t,\cdot), \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t,\cdot) \right\rangle_{H^2_M} dt - 2 \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t,0) \left\langle \tilde{v}^M(t,\cdot), \frac{\partial \tilde{u}^1_M}{\partial x}(t,\cdot) + E(\cdot) \right\rangle_{H^2_M} dt
\]

\[
+ (2\hat{\alpha} + 1)\|\tilde{v}^M(t,\cdot)\|_{H^2_M}^2 dt + 2 \left( 1 - \frac{\partial^2 \tilde{u}^2_M}{\partial x^2}(t,0) \right) \left\langle \tilde{v}^M(t,\cdot), \frac{\partial \tilde{v}^M}{\partial x}(t,\cdot) \right\rangle_{H^2_M} dt + 2\|\tilde{v}^M(t,\cdot)\|_{H^2_M}^2 dB_t.
\]

For any \( \varphi \in C^\infty_{0,\text{even}} \) and any \( i \in \{0, 1, 2, 3\} \), we have that either \( \varphi^i(0) = 0 \) or \( \varphi^{i+1}(0) = 0 \). In addition, for any \( \psi \in C^\infty_{\text{even}}([0,2M]) \) and any \( i \in \{0, 1, 2, 3\} \), we have that either \( \psi^i(0) = 0 \) or \( \psi^{i+1}(0) = 0 \) and \( \psi^i(2M) = 0 \).
or \( \dot{\psi}^{i+1}(2M) = 0 \). Thus,

\[
2 \left\langle \tilde{v}^M(t, \cdot), \frac{\partial^2 \tilde{u}^M}{\partial x^2}(t, \cdot) \right\rangle_{H_M} = -2 \left\| \frac{\partial \tilde{u}^M}{\partial x}(t, \cdot) \right\|_{H_M}^2 + \sum_{k=0}^3 \frac{\partial^k \tilde{u}^M}{\partial x^k}(t, 2M) \frac{\partial^{k+1} \tilde{u}^M}{\partial x^{k+1}}(t, 2M)
\]

\[
= -2 \left\| \frac{\partial \tilde{u}^M}{\partial x}(t, \cdot) \right\|_{H_M}^2 + \frac{\partial \tilde{v}^M}{\partial x}(t, 2M) \left( \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, 2M) + \frac{\partial^4 \tilde{u}^{1,M}}{\partial x^4}(t, 2M) \right)
\]

\[
\leq -2 \left\| \frac{\partial \tilde{u}^M}{\partial x}(t, \cdot) \right\|_{H_M}^2 + \| \tilde{v}(t, \cdot) \|_{H_M} \left( | \tilde{u}^{1,M}(t, 2M) | + | \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, 2M) | \right)
\]

\[
\leq - \left\| \frac{\partial \tilde{u}^M}{\partial x}(t, \cdot) \right\|_{H_M}^2 + \| \tilde{v}(t, \cdot) \|_{H_M}^2
\]

\[
+ \left( | \tilde{u}^{1,M}(t, 2M) |^2 + | \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, 2M) |^2 + | \frac{\partial^4 \tilde{u}^{1,M}}{\partial x^4}(t, 2M) |^2 \right). 
\]

Note that for \( 0 \leq t \leq \tau_{M,L} \)

\[
\left| \frac{\partial^2 \tilde{u}^M}{\partial x}(t, 0) \right| \leq 2L \wedge \left\| \tilde{v}^M(t, \cdot) \right\|_{H_M}, \quad \| \tilde{u}^{1,M}(t, \cdot) \|_{H_M} \leq L, \quad \left| \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t, 0) \right| \leq L.
\]

The first and third inequalities come from a simple modification of Lemma 3.3.4 and the second inequality is by definition of \( \tau_{M,L} \). Using these inequalities, we have

\[
2 \left( 1 - \frac{\partial^2 \tilde{u}^{2,M}}{\partial x^2}(t, 0) \right) \left\langle \tilde{v}^M(t, \cdot), \frac{\partial \tilde{u}^M}{\partial x}(t, \cdot) \right\rangle_{H_M} \leq 2(1 + L)\| \tilde{v}^M(t, \cdot) \|_{H_M} \left\| \frac{\partial \tilde{u}^M}{\partial x}(t, \cdot) \right\|_{H_M}
\]

\[
\leq 2(1 + L)^2 \| \tilde{v}^M(t, \cdot) \|_{H_M}^2 + \frac{1}{2} \left\| \frac{\partial \tilde{v}^M}{\partial x}(t, \cdot) \right\|_{H_M}^2.
\]

Now we consider \( \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, 0) \left\langle \tilde{v}^M(t, \cdot), \frac{\partial \tilde{u}^{1,M}}{\partial x}(t, \cdot) + E(\cdot) \right\rangle_{H_M} \); for \( 0 \leq k \leq 2 \) and \( t \leq \tau_{M,L} \)

\[
\left| \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, 0) \right| \left\langle \frac{\partial^k \tilde{v}^M}{\partial x^k}(t, \cdot), \frac{\partial^{k+1} \tilde{u}^{1,M}}{\partial x^{k+1}}(t, \cdot) + \frac{\partial^k E}{\partial x^k}(\cdot) \right\rangle_{L^2(\mathbb{R}_+)} \leq \| \tilde{v}^M(t, \cdot) \|_{H_M} \left( \| \tilde{u}^{1,M}(t, \cdot) \|_{H_M} + \| E(\cdot) \|_{H_M} \right)
\]

\[
\leq (L + \| E(\cdot) \|_{H_M}) \| \tilde{v}^M(t, \cdot) \|_{H_M}^2.
\]
for $k = 3$ and $0 \leq t \leq \tau_{M,L}$

$$
\left| \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, 0) \right| \left( \left\| \frac{\partial^3 \tilde{v}^M}{\partial x^3}(t, \cdot) \right\|_{L^2(\mathbb{R}^+)} + \left\| \frac{\partial^4 \tilde{v}^M}{\partial x^4}(t, \cdot) \right\|_{L^2(\mathbb{R}^+)} \right) \leq \left| \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, 0) \right| \left( \left\| \frac{\partial^3 \tilde{v}^M}{\partial x^3}(t, \cdot) \right\|_{L^2(\mathbb{R}^+)} + \left\| \frac{\partial^4 \tilde{v}^M}{\partial x^4}(t, \cdot) \right\|_{L^2(\mathbb{R}^+)} \right) + \left\| \frac{\partial^2 \tilde{v}^M}{\partial x^2}(t, 0) \right\|_{L^2(\mathbb{R}^+)} \left\| E(\cdot) \right\|_{L^2(\mathbb{R}^+)} \left\| \frac{\partial^3 \tilde{v}^M}{\partial x^3}(t, \cdot) \right\|_{L^2(\mathbb{R}^+)}.
$$

Since $\left\| E(\cdot) \right\|_{H_M} \leq \left\| E(\cdot) \right\|_H \leq 2$, we put things together to get that

$$
d\left\| \tilde{v}^M(t, \cdot) \right\|_{H_M}^2 \leq K_L \left\| \tilde{v}^M(t, \cdot) \right\|_{H_M}^2 dt + 2\left\| \tilde{v}^M(t, \cdot) \right\|_{H_M}^2 dB_t + 3 \left( \left\| \tilde{u}^{1,M}(t, 2M) \right\|^2 + \left\| \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, 2M) \right\|^2 + \left\| \frac{\partial^3 \tilde{u}^{1,M}}{\partial x^3}(t, 2M) \right\|^2 + \left\| \frac{\partial^4 \tilde{u}^{1,M}}{\partial x^4}(t, 2M) \right\|^2 \right),
$$

where $K_L \overset{\text{def}}{=} 2(1 + L)^2 + 2(\tilde{\alpha} + 1) + 5L + 2$.

Now we let $\tilde{Z}_t \overset{\text{def}}{=} \left\| \tilde{v}^M(t, \cdot) \right\|_{H_M}^2 e^{-K_L t}$. Then we have

$$
d\tilde{Z}_t \leq 2\tilde{Z}_t dB_t + 3e^{-K_L t} \left( \left\| \tilde{u}^{1,M}(t, 2M) \right\|^2 + \left\| \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, 2M) \right\|^2 + \left\| \frac{\partial^3 \tilde{u}^{1,M}}{\partial x^3}(t, 2M) \right\|^2 + \left\| \frac{\partial^4 \tilde{u}^{1,M}}{\partial x^4}(t, 2M) \right\|^2 \right).
$$

Fix each $\delta > 0$. Since $\tilde{v}^M(0, \cdot) = 0$ we get that

$$
P \left\{ \sup_{0 \leq t \leq \tau_{M,L}} \left\| \tilde{v}^M(t, \cdot) \right\|_{H_M} \geq \delta \right\} \leq P \left\{ \sup_{0 \leq t \leq \tau_{M,L}} \tilde{Z}_t \geq \delta e^{-K_L t} \right\} \leq \frac{3e^{K_L T}}{\delta} \mathbb{E} \int_{t=0}^{T \wedge \tau_{M,L}} \left( \left\| \tilde{u}^{1,M}(t, 2M) \right\|^2 + \left\| \frac{\partial^2 \tilde{u}^{1,M}}{\partial x^2}(t, 2M) \right\|^2 + \left\| \frac{\partial^3 \tilde{u}^{1,M}}{\partial x^3}(t, 2M) \right\|^2 + \left\| \frac{\partial^4 \tilde{u}^{1,M}}{\partial x^4}(t, 2M) \right\|^2 \right) dt
$$

We can now use Lemma 3.3.3 to complete the proof. \qed
3.4 Convergence

In this section, we provide numerical approximations of $\tilde{u}^{2,M}$ which is a solution of the nonlinear SPDE (3.3.15) on the finite interval $[0, 2M]$ for a fixed $M > 0$. More specifically we obtain numerical approximations by using the explicit finite difference method and Euler-Maruyama method. We also see that the approximations converge to $\tilde{u}^{2,M}$ as the sizes of a time step and a spatial grid go to 0 in a certain way (the stability condition). Just for convenience, let $\tilde{u}^M(t, x) \equiv \tilde{u}^{2,M}(t, x)$.

3.4.1 Numerical Approximations

We revisit (3.3.15):

$$
\begin{align*}
&d\tilde{u}^M(t, x) = \left\{ \frac{\partial^2 \tilde{u}^M}{\partial x^2}(t, x) + \alpha (\tilde{u}^M(t, x) - E(x)) - E(x) + \left( \frac{\partial \tilde{u}^M}{\partial x}(t, x) + E(x) \right) \left( 1 - \frac{\partial^2 \tilde{u}^M}{\partial x^2}(t, 0) \right) \right\} dt \\
&\quad + (\tilde{u}^M(t, x) - E(x)) dB_t, \quad 0 < t < \tau^u_M, \ 0 < x < 2M \\
&\frac{\partial \tilde{u}^M}{\partial x}(t, 0) = 0, \quad 0 < t < \tau^u_M \\
&\tilde{u}^M(t, 2M) = 0, \quad 0 < t < \tau^u_M \\
&\tilde{u}^M(0, x) = \tilde{u}^M_0(x), \quad 0 \leq x \leq 2M \\
&\tag{3.4.1}
\end{align*}
$$

where $E(x) \equiv e^{-x}$ and $\tau^u_M \equiv \lim_{L \to \infty} \tau^u_{M,L}$ where in turn

$$
\tau^u_{M,L} \equiv \inf \left\{ t \in [0, T] : \|\tilde{u}^M(t, \cdot)\|_{H^M} \geq L \right\} \quad \text{inf} \ 0 \ \equiv \ T.
$$

Remark 3.4.1. We note that the stopping time $\tau^u_M$ is more strict than $\tau^b_M$ which only stops the process once the speed of the boundary exceeds a fixed number $L > 0$. This is mainly because we need to control the first partial derivative of $\tilde{u}^M$.

The following lemma allows us to use the Dirichlet boundary condition at $x = 0$ for numerical approximations. In fact, this is easily expected since the solution $u(t, x)$ of (3.1.1) at the interface is 0 and $\tilde{u}(t, x) \equiv u(t, x + \beta(t)) + e^{-x}$.

Lemma 3.4.2. For $0 < t < \tau^u_M$, we have

$$
\tilde{u}^M(t, 0) = 1.
$$
Proof. Using the facts that \( \tilde{u}_M(t, \cdot) \in H_M \) and \( \frac{\partial \tilde{u}_M}{\partial x}(t, 0) = 0 \), we can have that, as \( x \) goes to 0,

\[
\tilde{u}_M(t, 0) - \tilde{u}_o(0) = \int_{s=0}^{t} \left( \frac{\partial \tilde{u}_M}{\partial x}(s, 0) - 1 \right) ds + \int_{s=0}^{t} \left( \frac{\partial \tilde{u}_M}{\partial x}(s, 0) - 1 \right) dB_s.
\]

Thus the claim follows from the fact that \( \tilde{u}_o(0) = 1 \).

Going back to (3.4.1), there is the nonlocal nonlinear term \( \tilde{u}_{xx}^M(t, 0) \) in the drift, which makes us difficult to deal with \( \tilde{u}_M \) directly. Thus we first consider \( \tilde{u}_{xx}^M \), i.e., we first analyze numerical approximations of \( \tilde{u}_{xx}^M \) then by using the approximation of \( u_{xx}^M \), we can have numerical approximations of \( u^M \) and show the convergence.

Define \( \tilde{v}_M \) as the even extension of \( \tilde{v}_M \) to handle numerical convergence (neumann boundary conditions can allow us to do that). Define

\[
\tilde{v}_M(t, x) = \tilde{v}_M(t, |x|) \quad \text{for} \quad -2M < x < 2M.
\]

Then \( \tilde{v}_M \) follows the SPDE

\[
d\tilde{v}_M(t, x) = \left\{ \frac{\partial^2 \tilde{v}_M}{\partial x^2}(t, x) + \hat{\alpha} \left( \tilde{v}_M(t, x) - E(x) \right) - E(x) + \frac{\partial \tilde{v}_M}{\partial x}(t, x) + E(x) \right\} (1 - \tilde{v}_M(t, 0)) dt + (\tilde{v}_M(t, x) - E(x)) dB_t, 0 < t < \tau_v, 0 < x < 2M
\]

\[
\frac{\partial \tilde{v}_M}{\partial x}(t, 0) = 0 \quad 0 < t < \tau_v
\]

\[
\tilde{v}_M(t, 2M) = 0 \quad 0 < t < \tau_v
\]

\[
\tilde{v}_M(0, x) = \tilde{v}_o(x), \quad 0 \leq x \leq 2M
\]

where \( \tau_v \) is defined as

\[
\tau_v = \lim_{L \to \infty} \tau_{v, L}
\]

where in turn

\[
\tau_{v, L} = \inf \left\{ t \in [0, T] : \| \tilde{v}_M(t) \|_H \geq L \right\}
\]

We use the even extension of \( \tilde{v}_M \) to handle numerical convergence (neumann boundary conditions can allow us to do that). Define

\[
\tilde{v}_M(t, x) = \tilde{v}_M(t, |x|) \quad \text{for} \quad -2M < x < 2M.
\]

Then \( \tilde{v}_M \) follows the SPDE
\[ \dot{v}_{2M}^M (t, x) = \left\{ \frac{\partial^2 \tilde{v}_{2M}^M (t, x)}{\partial x^2} + \tilde{\alpha} \left( \tilde{v}_{2M}^M (t, x) - E(|x|) \right) - E(|x|) + \left( \sgn(x) \frac{\partial \tilde{v}_{2M}^M (t, x)}{\partial x} \right) \right\} dt + \left( \tilde{v}_{2M}^M (t, x) - E(|x|) \right) dB_t \quad 0 < t < \tau_M^x, \ -2M < x < 2M \] (3.4.3)

\[ \hat{v}_{2M}^M (t, \pm 2M) = 0 \quad 0 < t < \tau_M^x \]

\[ \hat{v}_{2M}^M (0, x) = \tilde{u}_o^M (|x|) \quad -2M \leq x \leq 2M. \]

Now we first construct numerical approximations of \( \hat{v}_{2M}^M \). Using those approximations, we construct numerical approximations of \( \bar{v}_{2M}^M \). Let \( \varepsilon > 0 \) and \( \delta_x > 0 \). Also let \( M \overset{\text{def}}{=} [2M/\varepsilon] \) and \( T \overset{\text{def}}{=} [T/\delta_x] \). Now define \( x_j \overset{\text{def}}{=} j \varepsilon \) and \( t_n \overset{\text{def}}{=} n \delta_x \) for \( -M \leq j \leq M \) and \( 0 \leq n \leq T \). Therefore the sizes of spatial grid and time step are \( \varepsilon \) and \( \varepsilon \) respectively and \( \varepsilon \) depends on \( \varepsilon \). Let’s discretize \( \frac{\partial}{\partial x} \) and \( \frac{\partial^2}{\partial x^2} \) on mesh \( \varepsilon Z_M \), where \( Z_M \overset{\text{def}}{=} \{-M, -M + 1, \cdots, 0, \cdots, M - 1, M\} \) by using the finite difference method: for any vector \( v = (v_{-M}, v_{-M+1}, \cdots, v_0, \cdots, v_M) \)

\[
(\nabla_x v)(j) \overset{\text{def}}{=} \begin{cases} 
0, & \text{if } j = \pm M \text{ or } j = 0 \\
(v(j) - v(j-1))/\varepsilon & \text{if } 0 < j < M \\
(v(j) - v(j+1))/\varepsilon & \text{if } -M < j < 0
\end{cases}
\]

and

\[
(\Delta_x v)(j) \overset{\text{def}}{=} \begin{cases} 
0, & \text{if } j = \pm M \\
v(j+1) - 2v(j) + v(j-1)/\varepsilon^2 & \text{if } -M < j < M.
\end{cases}
\]

Let’s also define a discrete time \( \tau_{\delta_x} (t) \overset{\text{def}}{=} \delta_x [t/\delta_x] \). Using the Euler-Maruyama scheme to discretize time, we can obtain discrete approximations \( \tilde{v}_e^M \)'s of \( \hat{v}_{2M}^M \) as

\[
\tilde{v}_e^M (t_{n+1}, x_j) = \tilde{v}_e^M (t_n, x_j) + \Delta_x \tilde{v}_e^M (t_n, x_j) \delta_x + \alpha (\tilde{v}_e^M (t_n, x_j) - E(|x_j|) \delta_x - E(|x_j|) \delta_x \\
+ (1 - \tilde{v}_e^M (t_n, 0)) (\nabla_x \tilde{v}_e^M (t_n, x_j) + E(|x_j|)) \delta_x \\
+ (\tilde{v}_e^M (t_n, x_j) - E(|x_j|)) (B_{t_{n+1}} - B_{t_n}) , \quad -M < j < M
\]

\[ \tilde{v}_e^M (t_n, x_j) = 0, \quad j = \pm M \]

\[ \tilde{v}_e^M (0, x_j) = \tilde{u}_o^M (x_j), \quad j \in Z_M. \]
It is clear that $\tilde{v}_\varepsilon^M$ is an even function. To compare this to (3.4.3), we consider the following SDE:

\[
d\hat{u}_\varepsilon^M(t, x_j) = (\Delta \hat{v}_\varepsilon^M)(\tau_{\delta_\varepsilon}(t), x_j)dt + \alpha \left( \hat{v}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), x_j) - E(x_j) \right) dt - E(x_j)dt \\
+ (1 - \hat{v}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), 0)) \left( \nabla \hat{v}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), x_j) + E(x_j) \right) dt \\
+ (\hat{v}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), x_j) - E(x_j)) dB_t, \quad -\bar{M} < j < \bar{M}
\]  

(3.4.4)

Then we have $\hat{v}_\varepsilon^M(t_n, \cdot) = \tilde{v}_\varepsilon^M(t_n, \cdot)$. Now let us construct numerical approximations $\hat{u}_\varepsilon^M$ of $\tilde{u}_\varepsilon^M$ by using Lemma 3.4.2 and $\hat{v}_\varepsilon^M$:

\[
d\hat{u}_\varepsilon^M(t, x_j) = (\Delta \hat{v}_\varepsilon^M)(\tau_{\delta_\varepsilon}(t), x_j)dt + \alpha \left( \hat{u}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), x_j) - E(x_j) \right) dt - E(x_j)dt \\
+ (1 - \hat{u}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), 0)) \left( \nabla \hat{v}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), x_j) + E(x_j) \right) dt \\
+ (\hat{u}_\varepsilon^M(\tau_{\delta_\varepsilon}(t), x_j) - E(x_j)) dB_t, \quad 0 < j < \bar{M}
\]  

(3.4.5)

We note that the speed of the moving boundary, i.e., $|\hat{u}_\varepsilon^M(t, 0)|$, can blow up in a finite time. Thus $|\hat{v}_\varepsilon^M(t, 0)|$ may also blow up if $\hat{v}_\varepsilon^M$ is a good approximation of $\hat{u}_\varepsilon^M$. Hence we consider (3.4.3) and (3.4.4) up to the stopping time $\tau^\hat{v}_\varepsilon^M_L$ where

\[
\tau^\hat{v}_\varepsilon^M_L \overset{\text{def}}{=} \inf \{ t \in [0, T] : |\hat{v}_\varepsilon(t, 0)| \geq L \} \quad (\inf \emptyset \overset{\text{def}}{=} T).
\]

### 3.4.2 Convergence

Our main task in this section is to show that $\hat{u}_\varepsilon^M$ is a good approximation of $\tilde{u}_\varepsilon^M$ when $\varepsilon$ and $\delta_\varepsilon$ are small for fixed $M > 0$. We first let $\varphi_x \overset{\text{def}}{=} \partial_x\varphi$ and $\varphi_{xx} \overset{\text{def}}{=} \partial^2_x\varphi$ for any $C^2$ function $\varphi$ in $x$. Now we define the appropriate norm for numerical convergence.
Definition 3.4.3. For a vector \( u = (u_0, u_1, \ldots, u_M) \), define a 2-norm \( \| u \|_2 \) as

\[
\| u \|_2 \overset{\text{def}}{=} \sqrt{\sum_{j=0}^{M} u_j^2}.
\]

For a vector \( v = (v_{-M}, v_{-M+1}, \ldots, v_{M-1}, v_M) \), we also use the same notation \( \| \cdot \|_2 \) (without loss of generality), i.e.

\[
\| v \|_2 \overset{\text{def}}{=} \sqrt{\sum_{j=-M}^{M} v_j^2}.
\]

We can now state the convergence theorem.

Theorem 3.4.4 (Convergence Theorem). Fix \( L > 0 \) and \( M > 0 \) and suppose

\[
\sup_{0 < \delta \varepsilon < \varepsilon < 1} \frac{\delta \varepsilon}{\varepsilon^2} < \frac{1}{2}, \tag{3.4.6}
\]

where \( \varepsilon \) and \( \delta \varepsilon \) are the sizes of spatial grid and time step respectively. Suppose also that Assumption 3.3.6 holds. Then we have that for any \( N \leq \bar{T} \),

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \| \varepsilon \tilde{M}^u(t_N \wedge \tau_L, \cdot) - \varepsilon \tilde{M}^u(t_N \wedge \tau_L, \cdot) \|_2^2 \right] = 0,
\]

where \( \tau_L \overset{\text{def}}{=} \tau_{M,L}^u \wedge \tau_{M,L}^v \wedge \tau_{L_{\varepsilon}}^v \).

Note that the norm \( \varepsilon \| \cdot \|_2^2 \) measures average value over the interval \([0, M]\) or \([-M, M]\), i.e., simply a discretization of the standard \( L^2 \) norm.

Proof. The proof will be given at the end of this section.

We first consider the difference \( \mathcal{E}_v^\varepsilon(t, x) \overset{\text{def}}{=} \tilde{v}^M(t, x) - \tilde{v}_\varepsilon^M(t, x) \) for \( t \leq \tau_L \) and \( x \in \mathbb{Z}_M \). By (3.4.3) and (3.4.4), we obtain

\[
\mathcal{E}_v^\varepsilon(t, x) = \left\{ \Delta_x \mathcal{E}_v^\varepsilon(\tau_{\delta \varepsilon}(t), x) + \left(1 - \tilde{v}_\varepsilon^M(\tau_{\delta \varepsilon}(t), 0)\right) (\nabla_x \mathcal{E}_v^\varepsilon(\tau_{\delta \varepsilon}(t), x)) \right\} dt
\]

\[
+ \hat{\alpha} \mathcal{E}_v^\varepsilon(\tau_{\delta \varepsilon}(t), x) dt - (\text{sgn}(x) \tilde{v}_\varepsilon^M(t, x) + E(|x|) \mathcal{E}_v^\varepsilon(\tau_{\delta \varepsilon}(t), 0)) dt
\]

\[
+ \left\{ \tilde{v}^M(t, x) - \tilde{v}_\varepsilon^M(\tau_{\delta \varepsilon}(t), x) \right\} dB_t + R_v^\varepsilon(t, x) dt, \quad x \neq \pm M
\]

\[
\mathcal{E}_v^\varepsilon(t, \pm M) = 0,
\]

\[
\mathcal{E}_v^\varepsilon(0, x) = 0.
\]
where

\[
R_n^\varepsilon(t, x) \overset{\text{def}}{=} \left\{ \tilde{v}^M_{xx}(t, x) - \Delta_x \tilde{v}^M(t, x), \tilde{v}^M(t, x), \tilde{v}^M(\tau_{\delta_0}(t), x) \right\} + \tilde{\alpha} \left\{ \tilde{v}^M(t, x) - \tilde{v}^M(\tau_{\delta_0}(t), x) \right\} \\
+ \left\{ 1 - \tilde{v}^M(\tau_{\delta_0}(t), 0) \right\} \left\{ \text{sgn}(x) \tilde{v}^M(t, x) - \nabla_x \tilde{v}^M(\tau_{\delta_0}(t), x) \right\} \\
- \left\{ \text{sgn}(x) \tilde{v}^M(t, x) + \mathcal{E}(|x|) \right\} \left\{ \tilde{v}^M(t, 0) - \tilde{v}^M(\tau_{\delta_0}(t), 0) \right\}.
\]

Just for convenience, we let \( \tilde{E}^n_j \overset{\text{def}}{=} \mathcal{E}^n_j(t_n, x_j) \) and let \( \mathcal{E}(x) \overset{\text{def}}{=} \mathcal{E}(|x|) \). Then we can obtain

\[
\tilde{E}^{n+1}_j = \tilde{E}^n_j + \Delta_x \tilde{E}^n_j \delta_x + \tilde{\alpha} \tilde{E}^n_j \delta_x + (1 - \tilde{v}^M(t_n, 0)) \nabla_x \tilde{E}^n_j \delta_x - \tilde{E}^n_0 \int_{t_n}^{t_{n+1}} (\text{sgn}(x_j) \tilde{v}^M(s, x_j) + \mathcal{E}(x)) ds \\
+ \int_{t_n}^{t_{n+1}} (\tilde{v}^M(s, x_j) - \tilde{v}^M(\tau_{\delta_0}(s), x_j)) dB_s + \int_{t_n}^{t_{n+1}} R_n^\varepsilon(s, x_j) ds. \tag{3.4.7}
\]

Now we show that the term \( R_n^\varepsilon \) is small.

**Lemma 3.4.5.** We have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \varepsilon \| R_n^\varepsilon(s, \cdot) \|_2^2 \, ds = 0.
\]

**Proof.** We first consider space discretization, i.e., \( \tilde{v}^M_{xx} - \Delta_x \tilde{v}^M \) and \( \tilde{v}^M - \nabla_x \tilde{v}^M \). By using Taylor’s theorem, we have that for \( t \leq \tau_L \)

\[
| \tilde{v}^M_{xx}(t, x) - \Delta_x \tilde{v}^M(t, x) | \leq \sqrt{\varepsilon} \| \tilde{v}^M(t, \cdot) \|_H \\
| \text{sgn}(x) \tilde{v}^M_{xx}(t, x) - \nabla_x \tilde{v}^M(t, x) | \leq \sqrt{\varepsilon} \| \tilde{v}^M(t, \cdot) \|_H,
\]

which implies that

\[
\mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \| \tilde{v}^M_x(\tau_{\delta_0}(s), \cdot) - \Delta_x \tilde{v}^M(\tau_{\delta_0}(s), \cdot) \|_2^2 \right) ds \leq 2MTL^2 \varepsilon \\
\mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \| \tilde{v}^M_x(\tau_{\delta_0}(s), \cdot) - \nabla_x \tilde{v}^M(\tau_{\delta_0}(s), \cdot) \|_2^2 \right) ds \leq 2MTL^2 \varepsilon.
\]

Here we used the facts that \( \| \tilde{v}^M(t, \cdot) \|_H \leq L \) for \( t \leq \tau_L \) and \( t_N \leq T \). Let’s now consider time discretization.

By Assumption 3.3.6 we can have that for each fixed \( x \in \mathbb{R} \)

\[
\lim_{\varepsilon \to 0} \int_{s=0}^{t} \left| \frac{\partial^2 \tilde{v}^M}{\partial x^2}(s, x) - \frac{\partial^2 \tilde{v}^M}{\partial x^2}(\tau_{\delta_0}(s), x) \right|^2 ds = 0, \tag{3.4.8}
\]

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whenever \( t \leq \tau_L \) and \( 0 \leq i \leq 2 \). Since \( \left| \frac{\partial^i \tilde{v}_M}{\partial x^i}(t,x) \right| \leq 2L \) for \( t \leq \tau_L \), the dominated convergence theorem implies that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \left\| \frac{\partial^i \tilde{v}_M}{\partial x^i}(s,\cdot) - \frac{\partial^i \tilde{v}_M}{\partial x^i}(\tau_s,\cdot) \right\|_2^2 \right) = 0.
\]
Furthermore, (3.4.8) and the dominated convergence theorem again imply that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left| \tilde{v}_M(s,0) - \tilde{v}_M(\tau_\varepsilon(s),0) \right|^2 \left( \varepsilon \left\| \tilde{v}_M^\varepsilon(s,\cdot) + \bar{E}_\varepsilon \right\|_2^2 \right) ds = 0.
\]
Combine all the things to finish the proof.

Next we want to control the nonlocal nonlinear term, i.e., \( \tilde{E}_0^n = \tilde{u}_M(t,0) - \hat{v}_M^\varepsilon(t,0) \). This can be done by the following lemma, since for any vector \( v = (v_{-M}, \ldots, v_0, \ldots, v_M) \) with \( v_{-M} = 0 \) we have
\[
|v_0|^2 \leq M \| \nabla v \|_2^2 \varepsilon.
\]  

**Lemma 3.4.6.** Suppose (3.4.6) holds. Then we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \varepsilon \left\| \nabla (\tilde{u}_M^\varepsilon(t,0) - \hat{v}_M^\varepsilon(t,0)) \right\|_2^2 \right) \delta_\varepsilon = \lim_{\varepsilon \to 0} \mathbb{E} \left( \varepsilon \left\| \nabla \tilde{E}_\varepsilon^n \right\|_2^2 \right) \delta_\varepsilon = 0,
\]
where \( \tilde{E}_\varepsilon^n \overset{\text{def}}{=} (\tilde{E}_n^{-M}, \tilde{E}_n^{-M+1}, \ldots, \tilde{E}_n^M) \).

**Proof.** We first let \( \nu \overset{\text{def}}{=} \delta_\varepsilon / \varepsilon^2 \). Now let us consider the difference of 2-norms of \( \tilde{E}_\varepsilon^{n+1} \) and \( \tilde{E}_\varepsilon^n \), i.e.,
\[
\| \tilde{E}_\varepsilon^{n+1} \|_2^2 - \| \tilde{E}_\varepsilon^n \|_2^2 = \sum_{j=-M}^M \left[ (\tilde{E}_j^{n+1})^2 - (\tilde{E}_j^n)^2 \right] = \sum_{i=1}^6 A_i,
\]
where

\[
A_1 \overset{\text{def}}{=} \sum_{j=-M}^{M} \left( \tilde{\nu}_j^{n+1} - \tilde{\nu}_j^n \right) \tilde{\nu}_j^n + \sum_{j=-M}^{M} \tilde{\nu}_j^{n+1} \left( \Delta_t \tilde{\nu}_j^n \delta \epsilon \right)
\]

\[
A_2 \overset{\text{def}}{=} \hat{\alpha} \sum_{j=-M}^{M} \left( \tilde{\nu}_j^n \tilde{\nu}_j^{n+1} \right) \delta \epsilon
\]

\[
A_3 \overset{\text{def}}{=} (1 - \tilde{\nu}_e^M (t_n, 0)) \sum_{j=-M}^{M} \left( \tilde{\nu}_j^{n+1} \nabla_c \tilde{\nu}_j^n \right) \delta \epsilon
\]

\[
A_4 \overset{\text{def}}{=} -\tilde{\epsilon}_0^n \sum_{j=-M}^{M} \tilde{\nu}_j^{n+1} \int_{t_n}^{t_{n+1}} (\text{sgn}(x_j) \tilde{\nu}_e^M(s, x_j) + E(x_j)) \, ds
\]

\[
A_5 \overset{\text{def}}{=} \sum_{j=-M}^{M} \tilde{\epsilon}_j^{n+1} \int_{t_n}^{t_{n+1}} R_c^e(s, x_j) \, ds
\]

\[
A_6 \overset{\text{def}}{=} \sum_{j=-M}^{M} \tilde{\epsilon}_j^{n+1} \int_{t_n}^{t_{n+1}} (\text{sgn}(x_j) \tilde{\nu}_e^M(s, x_j) - \hat{\nu}_e^M(\tau_{\epsilon}(s), x_j)) \, dB_s.
\]

\(A_1\) can be rewritten as

\[
A_1 = \sum_{j=-M}^{M} \left( \tilde{\nu}_j^{n+1} - \tilde{\nu}_j^n \right) \tilde{\nu}_j^n + \sum_{j=-M}^{M} \tilde{\nu}_j^{n+1} \left( \nu \tilde{\nu}_j^n_{j-1} - 2 \nu \tilde{\nu}_j^n + \nu \tilde{\nu}_j^{n+1} \right)
\]

\[= (1 - 2 \nu) \sum_{j=-M}^{M} \tilde{\nu}_j^n \tilde{\nu}_j^{n+1} - \sum_{j=-M}^{M} \left( \tilde{\nu}_j^n \right)^2 + \nu \sum_{j=-M}^{M} \left( \tilde{\nu}_j^n_{j-1} + \tilde{\nu}_j^{n+1} \right) \tilde{\nu}_j^{n+1}.\]

Since we can have \(\sum_{j=-M}^{M} \tilde{\nu}_j^n \Delta_t \tilde{\nu}_j^n = -\left\| \nabla_c \tilde{\nu}_j^n \right\|_2^2\) from the direct computation, we get from (3.4.7) that

\[
(1 - 2 \nu) \sum_{j=-M}^{M} \tilde{\nu}_j^n \tilde{\nu}_j^{n+1} - \sum_{j=-M}^{M} \left( \tilde{\nu}_j^n \right)^2
\]

\[= -2 \nu \left\| \tilde{\nu}_j^n \right\|_2^2 - (1 - 2 \nu) \left\| \nabla_c \tilde{\nu}_j^n \right\|_2^2 \delta \epsilon + (1 - 2 \nu) \hat{\alpha} \left\| \tilde{\nu}_j^n \right\|_2^2 \delta \epsilon + (1 - 2 \nu)(1 - \tilde{\nu}_e^M(t_n, 0)) \sum_{j=-M}^{M} \left( \tilde{\nu}_j^n \nabla_c \tilde{\nu}_j^n \right) \delta \epsilon
\]

\[+ (1 - 2 \nu) \tilde{\nu}_0^n \sum_{j=-M}^{M} \left\{ \tilde{\nu}_j^n \int_{s=t_n}^{t_{n+1}} (\text{sgn}(x_j) \tilde{\nu}_e^M(s, x_j) + E(x_j)) \, ds \right\} + (1 - 2 \nu) \sum_{j=-M}^{M} \left\{ \tilde{\nu}_j^n \int_{s=t_n}^{t_{n+1}} R_c^e(s, x_j) \, ds \right\}
\]

\[+ (1 - 2 \nu) \sum_{j=-M}^{M} \left\{ \tilde{\nu}_j^n \int_{s=t_n}^{t_{n+1}} (\hat{\nu}_e^M(s, x_j) - \tilde{\nu}_e^M(\tau_{\epsilon}(s), x_j)) \, dB_s \right\}.
\]

Thus, using the Cauchy-Schwartz inequality, (3.4.9), the facts that \(|\tilde{\nu}_e^M(t, 0)| \leq 2L\), \(\|\hat{\nu}_e^M(t,\cdot)\|_H \leq L\) for
t ≤ τ_L, and 2ab ≤ a^2 + b^2, we get that

\[
(1 - 2\nu)(1 - \hat{v}_c^M(t_n, 0)) \sum_{j = -M}^M \left( \hat{E}_j \nabla_x \hat{E}_j \right) \delta_\varepsilon \leq \frac{(1 - 2\nu)^2}{5} \left\| \nabla_x \hat{E}^n \right\|_2^2 \delta_\varepsilon + 5(1 + 2L)^2 \left\| \hat{E}^n \right\|_2^2 \delta_\varepsilon,
\]

\[
(1 - 2\nu) \hat{E}_0^n \sum_{j = -M}^M \left\{ \hat{E}_j^n \int_{s = t_n}^{t_{n+1}} |\hat{v}_x^M(s, x_j) + \hat{E}(x_j)| \, ds \right\} \leq (1 - 2\nu) \left\| \hat{E}_0^n \right\|_2 \int_{s = t_n}^{t_{n+1}} \left\| \hat{E}^n \right\|_2 \left\| \hat{v}_x^M(s, \cdot) + \hat{E}(\cdot) \right\|_2 \, ds
\]

\[
\leq \frac{(1 - 2\nu)^2}{5M\varepsilon} \left\| \hat{E}_0^n \right\|_2^2 + \frac{5M^2 L^2}{2} \hat{E}_0^n \int_{s = t_n}^{t_{n+1}} \left( \varepsilon \left\| \hat{v}_x^M(s, \cdot) + \hat{E}(\cdot) \right\|_2^2 \right) \, ds
\]

\[
(1 - 2\nu) \sum_{j = -M}^M \left\{ \hat{E}_j^n \int_{s = t_n}^{t_{n+1}} R_c^n(s, x_j) \, ds \right\} \leq (1 - 2\nu) \left\| \hat{E}_0^n \right\|_2^2 \delta_\varepsilon + \frac{1}{\delta_\varepsilon} \sum_{j = -M}^M \left( \int_{s = t_n}^{t_{n+1}} \left\| R_c^n(s, \cdot) \right\|_2 \, ds \right)^2
\]

\[
\leq (1 - 2\nu)^2 \left\| \hat{E}_0^n \right\|_2^2 \delta_\varepsilon + \int_{s = t_n}^{t_{n+1}} \left\| R_c^n(s, \cdot) \right\|_2 \, ds.
\]

Furthermore, using the fact that 2ab ≤ a^2 + b^2 again, we get that

\[
\nu \sum_{j = -M}^M \left( \hat{E}_{j+1} - \hat{E}_{j-1} \right) \hat{E}_j \leq \nu \left\| \hat{E}^{n+1} \right\|_2^2 + \nu \left\| \hat{E}^n \right\|_2^2.
\]

Now if we take expectation of A1 and combine all the things, then there exists a constant K_1(M, L) > 0 such that

\[
\mathbb{E}[A_1] \leq \mathbb{E} \left\{ \nu \left( \left\| \hat{E}^{n+1} \right\|_2^2 - \left\| \hat{E}^n \right\|_2^2 \right) + \left(1 - 2\nu\right)^2 \left(1 - \frac{2\nu}{5}\right) \left\| \nabla_x \hat{E}^n \right\|_2^2 \delta_\varepsilon + K_1(M, L) \left\| \hat{E}^n \right\|_2^2 \delta_\varepsilon
\]

\[
\quad + \int_{s = t_n}^{t_{n+1}} \left\| R_c^n(s, \cdot) \right\|_2 \, ds \right\}.
\]

Let us consider A_i's where 2 ≤ i ≤ 5. Similar to A1, there exists a constant K_2(L, \nu) > 0 and K_3(M, L, \nu) > 0 such that

\[
A_2 \leq |\hat{a}| \left\| \hat{E}^{n+1} \right\|_2 \delta_\varepsilon + |\hat{a}| \left\| \hat{E}^n \right\|_2 \delta_\varepsilon
\]

\[
A_3 \leq \frac{(1 - \hat{v}_c^M(t_n, 0))^2}{(1 - 2\nu)^2} \left\| \hat{E}^{n+1} \right\|_2 \delta_\varepsilon + \frac{(1 - 2\nu)^2}{5} \left\| \nabla_x \hat{E}^n \right\|_2^2 \delta_\varepsilon \leq K_2(M, L, \nu) \left\| \hat{E}^{n+1} \right\|_2 \delta_\varepsilon + \frac{(1 - 2\nu)^2}{5} \left\| \nabla_x \hat{E}^n \right\|_2 \delta_\varepsilon
\]

\[
A_4 \leq \frac{(1 - 2\nu)^2}{5} \left\| \nabla_x \hat{E}^n \right\|_2 \delta_\varepsilon + \frac{5M(L + 1)^2}{2} \left\| \hat{E}^n \right\|_2 \delta_\varepsilon \leq \frac{(1 - 2\nu)^2}{5} \left\| \nabla_x \hat{E}^n \right\|_2 \delta_\varepsilon + K_3(M, L, \nu) \left\| \hat{E}^{n+1} \right\|_2 \delta_\varepsilon
\]

\[
A_3 \leq \left\| \hat{E}^{n+1} \right\|_2 \delta_\varepsilon + \int_{s = t_n}^{t_{n+1}} \left\| R_c^n(s, \cdot) \right\|_2 \, ds.
\]

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Here (3.4.6) ensures that $1/(1-2\nu)$ is bounded by some constant. Now let’s consider $A_6$. Using (3.4.7), we can obtain

$$A_6 = \sum_{j=-M}^{M} \left\{ \left( \tilde{\xi}_j^n + \Delta_t \tilde{\xi}_j^n \delta_t + \lambda \tilde{\xi}_j^n \delta_t + (1 - \tilde{\nu}_\varepsilon^M(t_n, 0)) \nabla_{x} \tilde{\xi}_j^n \delta_t \right) \int_{s=t_n}^{t_{n+1}} \left( \tilde{\nu}_\varepsilon^M(s, x_j) - \tilde{\nu}_\varepsilon^M(\tau_{\delta_t}(s), x_j) \right) dB_s \right\}

- \tilde{\xi}_0^n \sum_{j=-M}^{M} \left\{ \int_{s=t_n}^{t_{n+1}} (\text{sgn}(x_j)) \tilde{\nu}_\varepsilon^M(s, x_j) + \tilde{E}(x) \right) ds \int_{s=t_n}^{t_{n+1}} \left( \tilde{\nu}_\varepsilon^M(s, x_j) - \tilde{\nu}_\varepsilon^M(\tau_{\delta_t}(s), x_j) \right) dB_s \right\}

+ \sum_{j=-M}^{M} \left\{ \int_{s=t_n}^{t_{n+1}} R^e_\varepsilon(s, x_j) ds \int_{s=t_n}^{t_{n+1}} \left( \tilde{\nu}_\varepsilon^M(s, x_j) - \tilde{\nu}_\varepsilon^M(\tau_{\delta_t}(s), x_j) \right) dB_s \right\}

+ \sum_{j=-M}^{M} \left\{ \int_{s=t_n}^{t_{n+1}} \left( \tilde{\nu}_\varepsilon^M(s, x_j) - \tilde{\nu}_\varepsilon^M(\tau_{\delta_t}(s), x_j) \right) dB_s \right\}^2.

By taking expectation and using Ito’s isometry, we have that for sufficiently small $\varepsilon$,

$$\mathbb{E}A_6 \leq \mathbb{E} \left\| \tilde{\xi}_0^n \right\|^2 \sum_{j=-M}^{M} \int_{s=t_n}^{t_{n+1}} \left( \left| \tilde{\nu}_\varepsilon^M(s, x_j) + \tilde{E}(x) \right| ds \right)^2 + \mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R^e_\varepsilon(s, \cdot) \right\|^2_2 ds

+ 3 \sum_{j=-M}^{M} \mathbb{E} \int_{s=t_n}^{t_{n+1}} \left( \tilde{\nu}_\varepsilon^M(s, x_j) - \tilde{\nu}_\varepsilon^M(\tau_{\delta_t}(s), x_j) \right)^2 ds

\leq 4L^2 \delta_t \varepsilon \int_{s=t_n}^{t_{n+1}} \mathbb{E} \left\| \tilde{\nu}_\varepsilon^M(s, \cdot) \right\|^2_2 ds + 7\mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R^e_\varepsilon(s, \cdot) \right\|^2_2 ds + 6\mathbb{E} \left\| \tilde{\xi}_0^n \right\|^2_2 \delta_t

\leq 8L^2 (L^2 + \| \tilde{E} \|^2_2) \delta_t \varepsilon + 7\mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R^e_\varepsilon(s, \cdot) \right\|^2_2 ds + 6\mathbb{E} \left\| \tilde{\xi}_0^n \right\|^2_2 \delta_t.

We again used the fact that $|\tilde{\nu}_\varepsilon^M(t, 0)| \leq L$ and $|\tilde{\nu}_\varepsilon^M(t, 0)| \leq 2 \left\| \tilde{\nu}_\varepsilon^M(t, \cdot) \right\|_{H^2} \leq 2L$ for $t < \tau_L$. When we combine all the things together, there exist constants $K_4 > 0$ and $K_5 > 0$ which only depend on $M, L$ and $\nu$ such that

$$\mathbb{E} \left( \left\| \tilde{\xi}_{n+1} \right\|^2_2 - \left\| \tilde{\xi}_n \right\|^2_2 \right) \leq (\nu + K_4 \delta_t) \mathbb{E} \left( \left\| \tilde{\xi}_{n+1} \right\|^2_2 - \left\| \tilde{\xi}_n \right\|^2_2 \right) - (1 - 2\nu) \left( 1 - \frac{4}{5} (1 - 2\nu) \right) \mathbb{E} \left\| \nabla \tilde{\xi}_n \right\|^2_2 \delta_t

+ K_5 \mathbb{E} \left\| \tilde{\xi}_n \right\|^2_2 \delta_t + 8\mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R^e_\varepsilon(s, \cdot) \right\|^2_2 ds + 8L^2 (L^2 + \| \tilde{E} \|^2_2) \delta_t \varepsilon.

Since we have that $1 - (\nu + K_4 \delta_t) > 0$ and $(1 - 2\nu)(1 - 4(1 - 2\nu)/5) > 0$ for any $0 < \delta_t < \varepsilon < 1$ by (3.4.6),

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there exist constants $K_6 > 0$, $K_7 > 0$, $K_8 > 0$ and $K_9 > 0$ which only depend on $M, L$ and $\nu$ such that

$$
\mathbb{E} \left( \left\| \mathcal{E}^{n+1} \right\|_2^2 - \left\| \mathcal{E}^n \right\|_2^2 \right) + K_6 \mathbb{E} \left\| \nabla_x \mathcal{E}^n \right\|_2^2 \delta_x \leq + K_7 \mathbb{E} \left\| \mathcal{E}^n \right\|_2^2 + K_8 \mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R^\nu_{x,s} \right\|_2^2 ds + K_9 \delta_x^2.
$$

Now let $a_n \overset{\text{def}}{=} \mathbb{E} \left\| \mathcal{E}^n \right\|_2^2$ and $b_n \overset{\text{def}}{=} \{1/(1 + K_7 \delta_x)\}^n$. By the construction of $b_n$ and $1/(1 + K_7 \delta_x) < 1$, we have that

$$
a_{n+1}b_{n+1} - a_nb_n = b_{n+1}(a_{n+1} - a_n) + a_n(b_{n+1} - b_n) \leq -K_6 b_{n+1} \mathbb{E} \left\| \nabla_x \mathcal{E}^n \right\|_2^2 + K_8 \mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R^\nu_{x,s} \right\|_2^2 ds + K_9 \delta_x^2,
$$

which implies that for some constant $K_{10} > 0$ which only depends on $M, L, \nu$ and $T$,

$$
K_6 \sum_{n=0}^{N} b_{n+1} \mathbb{E} \left\| \nabla_x \mathcal{E}^n \right\|_2^2 \delta_x \leq K_8 \mathbb{E} \int_{s=t_0}^{t_N} \left\| R^\nu_{x,s} \right\|_2^2 ds + K_{10} \delta_x^2.
$$

Note that $b_n \geq b_N$ for $0 \leq n \leq N$ and $b_N \geq \exp(-K_7 T)$ for sufficiently small $\varepsilon$ (which implies $\delta_x$ sufficiently small). Therefore we can now use Lemma 3.4.5 and 3.4.6 to complete the proof.

Let’s now consider $\tilde{u}^M_\varepsilon - \hat{u}^M_\varepsilon$. Define $\mathcal{E}_j^n = \mathcal{E}_j^n(t_n, x_j) - \tilde{u}^M_\varepsilon(t_n, x_j)$ for $0 \leq j \leq \tilde{M}$. Similar to the case for $\tilde{u}^M$ and $\hat{u}^M_\varepsilon$, Lemma 3.4.2 (3.4.1) and (3.4.5) imply that

$$
\mathcal{E}^{n+1}_j = \mathcal{E}^n_j + \Delta_x \mathcal{E}^n_x \delta_x + \hat{\alpha} \mathcal{E}^n_x \delta_x + (1 - \hat{\mathcal{E}}^M_\varepsilon(t_n, 0)) \nabla_x \mathcal{E}^n_j \delta_x
$$

$$
+ \int_{s=t_n}^{t_{n+1}} \left( \tilde{u}^M_\varepsilon(s, x_j) - \tilde{u}^M_\varepsilon(\tau_{\delta_x}(s), x_j) \right) dB_s + \int_{s=t_n}^{t_{n+1}} R^\nu_{x,s}(s, x_j) ds, \quad 0 < j < \tilde{M}
$$

$$
\mathcal{E}_0^n = \mathcal{E}^n_M = 0,
$$

where

$$
R^\nu_{x}(t, x_j) \overset{\text{def}}{=} \{ \tilde{u}^M_\varepsilon(t, x_j) - \Delta_x \tilde{u}^M_\varepsilon(\tau_{\delta_x}(t), x_j) \} + \hat{\alpha} \{ \tilde{u}^M_\varepsilon(t, x_j) - \tilde{u}^M_\varepsilon(\tau_{\delta_x}(t), x_j) \}
$$

$$
+ (1 - \hat{\mathcal{E}}^M_\varepsilon(t_n, 0)) \{ \tilde{u}^M_\varepsilon(t, x_j) - \Delta_x \tilde{u}^M_\varepsilon(\tau_{\delta_x}(t), x_j) \}
$$

$$
+ \{ \tilde{u}^M_\varepsilon(t, x_j) + E(x_j) \} \{ \tilde{u}^M_\varepsilon(\tau_{\delta_x}(t), 0) - \tilde{u}^M_\varepsilon(\tau_{\delta_x}(t), 0) \}
$$

$$
+ \{ \tilde{u}^M_\varepsilon(t, x_j) + E(x_j) \} \{ \tilde{u}^M_\varepsilon(0, \tau_{\delta_x}(t), 0) \}.
$$

Lemma 3.4.7. We have

$$
\lim_{\varepsilon \to 0} \int_{s=0}^{t_N} \varepsilon \left\| R^\nu_{x,s} \right\|_2^2 ds = 0.
$$
Proof. Since the proof is very similar to that of Lemma 3.4.5 we may skip some calculations which are redundant. Considering the space discretization, we also have that

\[ \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \left\| \tilde{u}_M^\tau (\tau_{\delta_s}(s), \cdot) - \Delta_{x} \tilde{u}_M^\tau (\tau_{\delta_s}(s), \cdot) \right\|_2^2 \right) ds \leq 2MTL^2 \varepsilon \]

\[ \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \left\| \tilde{u}_M^\tau (\tau_{\delta_s}(s), \cdot) - \nabla_{x} \tilde{u}_M^\tau (\tau_{\delta_s}(s), \cdot) \right\|_2^2 \right) ds \leq 2MTL^2 \varepsilon. \]

Furthermore, as in the proof of Lemma 3.4.5 Assumption 3.3.6 enables us to have that

\[ \lim_{\varepsilon \to 0} \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \left\| \partial_i \tilde{u}_M^\tau (s, \cdot) - \partial_i \tilde{u}_M^\tau (\tau_{\delta_s}(s), \cdot) \right\|_2^2 \right) = 0 \quad \text{and} \]

\[ \lim_{\varepsilon \to 0} \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left( \varepsilon \left\| \tilde{u}_M^\tau (s, 0) - \tilde{u}_M^\tau (\tau_{\delta_s}(s), 0) \right\|_2^2 \right) \left( \varepsilon \left\| \tilde{u}_M^\tau ((s, \cdot) + \bar{E}(\cdot) \right\|_2^2 ds \right) = 0. \]

Let’s control the nonlinear term:

\[ \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \left| \tilde{u}_M^\tau (\tau_{\delta_s}(s), 0) - \tilde{u}_M^\tau (\tau_{\delta_s}(s), 0) \right| \left( \varepsilon \left\| \tilde{u}_M^\tau ((s, \cdot) + \bar{E}(\cdot)^2 \right\|_2^2 ds \right) \leq 2L^2 \sum_{n=0}^{N} \mathbb{E} \left( \varepsilon \left\| \nabla_{x} \left( \tilde{u}_M^\tau (t_n, \cdot) - \tilde{u}_M^\tau (t_n, \cdot) \right) \right\|_2^2 \delta_{\varepsilon}. \]

Using Lemma 3.4.6 and combining all the things finishes the proof. \( \square \)

Finally we can prove Theorem 3.4.4

Proof of Theorem 3.4.4 The proof is very similar to that of Lemma 3.4.6 Thus, we skip some calculations which are shown in Lemma 3.4.6 As in the proof of Lemma 3.4.6 we also let \( \nu \overset{\text{def}}{=} \delta_{\varepsilon}^2 / \varepsilon \) and consider the difference of 2-norms of \( \mathcal{E}^{n+1} \) and \( \mathcal{E}^n \), i.e.,

\[ \left\| \mathcal{E}^{n+1} \right\|_2^2 - \left\| \mathcal{E}^n \right\|_2^2 = \sum_{j=0}^{\tilde{M}} \left( (\mathcal{E}_j^{n+1})^2 - (\mathcal{E}_j^n)^2 \right) = \sum_{i=1}^{5} A_i, \]

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where

\[ A_1 \overset{\text{def}}{=} \sum_{j=0}^{\hat{M}} (E_j^{n+1} - E_j^n) E_j^n + \sum_{j=-\hat{M}}^{\hat{M}} E_j^{n+1} (\Delta_z E_j^n \delta_z) \]
\[ A_2 \overset{\text{def}}{=} \hat{\alpha} \sum_{j=0}^{\hat{M}} (E_j^n E_j^{n+1}) \delta_z \]
\[ A_3 \overset{\text{def}}{=} (1 - \hat{\delta}_z(t_n,0)) \sum_{j=0}^{\hat{M}} (E_j^{n+1} \nabla_z E_j^n) \delta_z \]
\[ A_4 \overset{\text{def}}{=} \sum_{j=0}^{\hat{M}} E_j^{n+1} \int_{s=t_n}^{t_{n+1}} R_z^s(s, x_j) ds \]
\[ A_5 \overset{\text{def}}{=} \sum_{j=0}^{\hat{M}} E_j^{n+1} \int_{s=t_n}^{t_{n+1}} (\hat{u}_z^M(s, x_j) - \hat{u}_z^M(\tau_{\delta_z}(s), x_j)) dB_s. \]

First consider \( A_1 \). There exists constant \( \tilde{K}_1 > 0 \) such that

\[
\mathbb{E} A_1 \leq \mathbb{E} \left\{ \nu \left( \left\| E^{n+1} \right\|_2^2 - \left\| E^n \right\|_2^2 \right) - (1 - 2\nu) \left\| \nabla_z E^n \right\|_2^2 \delta_z + (1 - 2\nu) \hat{\alpha} \left\| E^n \right\|_2^2 \delta_z \\
+ (1 - 2\nu)(1 - \hat{\delta}_z(t_n,0)) \sum_{j=-\hat{M}}^{\hat{M}} (E_j^n \nabla_z E_j^n) \delta_z + (1 - 2\nu) \sum_{j=-\hat{M}}^{\hat{M}} \left( E_j^n \int_{s=t_n}^{t_{n+1}} R_z^s(s, x_j) ds \right) \right\}
\]
\[
\leq \mathbb{E} \left\{ \nu \left( \left\| E^{n+1} \right\|_2^2 - \left\| E^n \right\|_2^2 \right) - (1 - 2\nu) \left( \frac{1 - 2\nu}{2} \right) \left\| \nabla_z E^n \right\|_2^2 \delta_z + \tilde{K}_1 \left\| E^n \right\|_2^2 \delta_z \\
+ \int_{s=t_n}^{t_{n+1}} \left\| R_z^s(s, \cdot) \right\|_2^2 ds \right\}.
\]

(3.4.10)

For \( A_1 \)'s where \( 2 \leq i \leq 5 \), there exist some constant \( \tilde{K}_2 > 0 \) such that

\[
\mathbb{E} A_2 \leq |\hat{\alpha}| \mathbb{E} \left\{ \left\| E^{n+1} \right\|_2^2 + \left\| E^n \right\|_2^2 \right\} \delta_z
\]
\[
\mathbb{E} A_3 \leq \tilde{K}_2 \mathbb{E} \left\{ \left\| E^{n+1} \right\|_2^2 \delta_z + \left( \frac{1 - 2\nu}{2} \right) \left\| \nabla_z E^n \right\|_2^2 \delta_z \right\}
\]
\[
\mathbb{E} A_4 \leq \mathbb{E} \left\{ \left\| E^{n+1} \right\|_2^2 \delta_z + \int_{s=t_n}^{t_{n+1}} \left\| R_z^s(s, \cdot) \right\|_2^2 ds \right\}
\]
\[
\mathbb{E} A_5 \leq 6 \mathbb{E} \left\| E^n \right\|_2^2 \delta_z + 7 \mathbb{E} \int_{s=t_n}^{t_{n+1}} \left\| R_z^s \right\|_2^2 ds.
\]

(3.4.11)

Thus, by combining (3.4.10) and (3.4.11) and using the fact that \(- (1 - 2\nu - (1 - 2\nu)^2) \leq 0 \) (due to (3.4.9),

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there exist constants $\tilde{K}_3 > 0$, $\tilde{K}_4 > 0$, and $\tilde{K}_5 > 0$ such that

$$\mathbb{E} \left\{ \| \mathcal{E}^{n+1} \|_2^2 - \| \mathcal{E}^n \|_2^2 \right\} \leq (\nu + \tilde{K}_3 \delta) \mathbb{E} \left\{ \| \mathcal{E}^{n+1} \|_2^2 - \| \mathcal{E}^n \|_2^2 \right\} + \tilde{K}_4 \mathbb{E} \| \mathcal{E}^n \|_2^2 \delta + \tilde{K}_5 \mathbb{E} \int_{s=t_n}^{t_{n+1}} \| R^\alpha_{\varepsilon} (s, \cdot) \|_2^2 ds.$$  

Again by (3.4.6), we have $\nu + \tilde{K}_3 \delta < 1$ for sufficiently small $\delta$. Therefore there exist some constants $\tilde{K}_6 > 0$ and $\tilde{K}_7 > 0$ such that

$$\mathbb{E} \left\{ \| \mathcal{E}^{n+1} \|_2^2 - \| \mathcal{E}^n \|_2^2 \right\} \leq \tilde{K}_6 \mathbb{E} \| \mathcal{E}^n \|_2^2 \delta + \tilde{K}_7 \mathbb{E} \int_{s=t_n}^{t_{n+1}} \| R^\alpha_{\varepsilon} (s, \cdot) \|_2^2 ds,$$

which implies that for some constant $\tilde{K}_8 > 0$

$$\mathbb{E} \| \mathcal{E}^N \|_2^2 \leq \tilde{K}_8 \mathbb{E} \int_{s=0}^{t_N \wedge \tau_L} \| R^\alpha_{\varepsilon} (s, \cdot) \|_2^2 ds$$

(see the end of the proof of Lemma 3.4.6). By Lemma 3.4.7, the proof is complete. \( \square \)

### 3.5 Simulations

In this section, we will see from numerical simulations where the boundary is. Let us revisit numerical approximations. We first consider $\tilde{v}_\varepsilon$:

$$\tilde{v}_\varepsilon^M(t_{n+1}, x_j) = \tilde{v}_\varepsilon^M(t_n, x_j) + \Delta_t \tilde{v}_\varepsilon^M(t_n, x_j) \delta + \alpha (\tilde{v}_\varepsilon^M(t_n, x_j) - E(x_j) \delta) - E(x_j) \delta + \alpha (\tilde{v}_\varepsilon^M(t_n, x_j) - E(x_j) \delta - E(x_j)) (B_{t_n+1} - B_{t_n}), \quad 0 \leq j < M$$

$$\tilde{v}_\varepsilon^M(t_n, M) = 0,$$

$$\tilde{v}_\varepsilon^M(0, x_j) = \tilde{\omega}_\varepsilon^M(x_j), \quad 0 \leq j \leq M.$$

Note that $\Delta_t \tilde{v}_\varepsilon^M(t_n, x_0) = 2 (\tilde{v}_\varepsilon^M(t_n, x_1) - \tilde{v}_\varepsilon^M(t_n, x_0)) / \delta^2$ and $\nabla_x \tilde{v}_\varepsilon^M(t_n, x_0) = 0$. Here we consider $\tilde{v}_\varepsilon^M$ for $0 \leq x_j \leq M$ since $\tilde{v}_\varepsilon^M$ is even.
Once we obtain \( \hat{u}_\varepsilon^M \), we can get \( \tilde{u}_\varepsilon^M \) according to the following equation:

\[
\tilde{u}_\varepsilon^M(t_{n+1}, x_j) = \hat{u}_\varepsilon^M(t_n, x_j) + \Delta \varepsilon \hat{u}_\varepsilon^M(t_n, x_j) \delta \varepsilon + \alpha (\hat{u}_\varepsilon^M(t_n, x_j) - E(x_j) \delta \varepsilon - E(x_j) \delta \varepsilon \\
+ (1 - \hat{u}_\varepsilon^M(t_n, 0)) \left( \nabla \varepsilon \hat{u}_\varepsilon^M(t_n, x_j) + E(x_j) \right) \delta \varepsilon + (\hat{u}_\varepsilon^M(t_n, x_j) - E(x_j)) \left( B_{t_{n+1}} - B_{t_n} \right), \quad 0 < j < M
\]

\[
\hat{u}_\varepsilon^M(t_n, 0) = 1,
\]

\[
\hat{u}_\varepsilon^M(t_n, M) = 0,
\]

\[
\tilde{u}_\varepsilon^M(0, x_j) = \tilde{u}_\varepsilon^M(x_j), \quad 0 \leq j \leq M.
\]

Note that, for both simulations above, \( 2 \delta \varepsilon \) should be strictly less than \( \varepsilon^2 \) due to a stability issue.

In order to obtain numerical approximations of a solution \( u \) of the stochastic moving boundary problem (3.1.1), we use transformation:

\[
u(t, x) = \begin{cases} 
\hat{u}(t, x - \beta(t)) - \exp \left[ - (x - \beta(t)) \right] & x \geq \beta(t), \ 0 \leq t < \tau \\
0 & x < \beta(t), \ 0 \leq t < \tau.
\end{cases}
\]  (3.5.1)

In Figure 3.1, (a) shows the moving boundary problem without noise, i.e., without the term \( u \circ dB_t \) in (3.1.1) whereas (b) represents one realization of the process \( u(t, x) \) of the stochastic moving boundary problem (3.1.1). Here we use \( \varepsilon = 0.035 \), \( \delta \varepsilon = 0.0005 \), \( \alpha = 0 \), i.e., \( \hat{\alpha} = 1/2 \) and initial condition

\[
u_0(x) = \begin{cases} 
\left( \frac{x^2}{2} + \frac{x^3}{6} \right) \xi(x) & \text{if } x \geq 0 \\
0 & \text{else}
\end{cases}
\]

where \( \xi \) is a \( C^\infty \) monotone decreasing function that \( \xi = 1 \) for \( x < 6 \) and decays exponentially. More precisely, we simulate \( \tilde{u}_\varepsilon^M \) for \( M = 24 \) then shift it according to (3.5.1) and show \( u \) for \(-6 \leq x \leq 6\).
We can clearly see there are two phases separated by the black lines, which are the moving boundaries. As we expect, the variation of \( u \) on the colored region for the stochastic moving boundary problem (see (b)) is much greater due to the noise term. However, Figure 3.1 also shows that the interface is not much rough for the stochastic moving boundary problem. This is simply due to the term \( u \circ dB_t \), i.e. the effect of noise is insignificant when it approaches to the interface (note that \( u = 0 \) at the interface).

Figure 3.2 shows one realization of the process \( u(t, x) \) of the stochastic moving boundary problem (3.1.1) with the same \( \varepsilon, \delta \varepsilon \) and initial condition as above, but a different \( \alpha = 1.5 \) (i.e. \( \hat{\alpha} = 2 \)). Since \( \alpha \) is increased, we may expect that there is more variation in \( u \) and the interface, and this can be seen from Figure 3.2.
References


