ASYMPTOTIC STABILITY AND COMPLETENESS IN 2D NONLINEAR SCHRÖDINGER EQUATION

BY

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DISSERTATION

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Abstract

In this thesis we obtained new results on the asymptotic stability of ground states of the nonlinear Schrödinger equation in space dimension two. Under our hypotheses, the result actually shows asymptotic completeness in the regime of small initial data, i.e. any small initial data evolves into a superposition of a solitary wave (ground state) and a radiative part that decays in time.
To my parents, Dr. Anusorn - Dr. Jaruwan Skulkhu.
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List of Symbols

\[ H = -\Delta + V; \]

\[ L^p = \{ f : \mathbb{R}^2 \rightarrow \mathbb{C} \mid f \text{ measurable and } \int_{\mathbb{R}^2} |f(x)|^p dx < \infty \}, 1 \leq p < \infty, \text{ endowed with the standard norm } \| f \|_{L^p} = \left( \int_{\mathbb{R}^2} |f(x)|^p dx \right)^{1/p}; \]

\[ L^p' \text{ is the dual space of } L^p, \text{ where } L^p' \text{ is such that } \frac{1}{p} + \frac{1}{p'} = 1, \text{ while for } p = \infty, \]

\[ L^\infty = \{ f : \mathbb{R}^2 \rightarrow \mathbb{C} \mid f \text{ measurable and esssup}|f(x)| < \infty \}, \text{ and it is endowed with the norm: } \| f \|_{L^\infty} = \text{essup}|f(x)|; \]

\[ < x > = (1 + |x|^2)^{1/2}; \]

\[ L^p_\sigma \text{ denotes the } L^p, 1 \leq p \leq \infty \text{ space with weight } < x >^\sigma, \sigma \in \mathbb{R} \text{ i.e. the space of functions } f(x) \text{ such that } (< x >^\sigma f(x))^p \text{ are integrable endowed with the norm } \| f(x) \|_{L^p_\sigma} = \| < x >^\sigma f(x) \|_p, \text{ while for } p = \infty, L^\infty_\sigma \text{ denotes the vector space of measurable functions } f(x) \text{ such that } \text{esssup} |< x >^\sigma f(x)| < \infty \text{ endowed with the norm } \| f(x) \|_{L^\infty_\sigma} = \| < x >^\sigma f(x) \|_{L^\infty}; \]

\[ \overline{z} = \text{the complex conjugate of the complex number } z; \]

\[ \langle f, g \rangle = \int_{\mathbb{R}^2} \overline{f(x)}g(x) dx \text{ is the scalar product in } L^2; \]

\[ P_c \text{ is the projection associated to the continuous spectrum of the self adjoint operator } H \text{ on } L^2, \text{ range} P_c = \mathcal{H}_0; \]

\[ H^n \text{ denote the Sobolev spaces of measurable functions having all distributional partial derivatives up to order } n \text{ in } L^2, \| \cdot \|_{H^n} \text{ denotes the standard norm in this spaces.} \]
Chapter 1

Introduction

In this dissertation we study the long time behavior of solutions of the nonlinear Schrödinger equation (NLS) with potential in two space dimensions (2-d):

\[ i\partial_t u(t, x) = [-\Delta_x + V(x)]u + g(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2 \]  \hspace{1cm} (1.1)

\[ u(0, x) = u_0(x) \]  \hspace{1cm} (1.2)

where the potential \( V : \mathbb{R}^2 \to \mathbb{R} \) decays at infinity in a manner which will be made precise in the next chapter, and the local nonlinearity is constructed from the real valued, odd, \( C^2 \) function \( g : \mathbb{R} \to \mathbb{R} \) satisfying

\[ g(0) = g'(0) = 0 \quad \text{and} \quad |g''(s)| \leq C(|s|^{|\alpha_1|^{-1}} + |s|^{|\alpha_2|^{-1}}), \quad s \in \mathbb{R}, \quad \sqrt{2} < \alpha_1 \leq \alpha_2 < \infty \]  \hspace{1cm} (1.3)

which is then extended to a complex function via the gauge symmetry:

\[ g(e^{i\theta}s) = e^{i\theta}g(s), \quad \theta \in \mathbb{R}. \]  \hspace{1cm} (1.4)

The equation has important applications in statistical physics, optics and water waves. For example, it describes certain limiting behavior of Bose-Einstein condensates, see [6, 12], and propagation of time harmonic waves in wave guides, see [13, 16]. In the latter case, \( t \) plays the role of the coordinate along the axis of symmetry of the wave guide and we can infer that beyond a transition region all (small) waves converge to a propagating mode.

The main contribution of this thesis is to allow for a larger range of nonlinearities. In
particular, we are able to control lower power nonlinearities, $\sqrt{2} < \alpha_1 \leq \frac{3}{2}$, compared to the previous methods which required at least $\alpha_1 > \frac{3}{2}$, see [10]. While this improvement might seem technical, it is an important step forward because it allows us to study nonlinearities near the Strauss limit $\alpha_1 = \sqrt{2}$ and brings the two dimensional case on par with the three or higher dimensional cases, see [8] and [9]. Note that there are no results for the one dimensional case with subcritical nonlinearities, $\alpha_1 < 4$, not to mention in the vicinity of the Strauss limit, $\alpha_1 = \frac{1+\sqrt{17}}{2}$, while important applications, such as the propagation of information through dispersion managed optical fibers are even below this regime requiring $\alpha_1 = 2$. We hope that the methods developed in this thesis will help bridge the gap in the one dimensional case and also in other similar dispersive type wave equations. Moreover, in Chapter 5 we show that our result is optimal in the context in which the radiative part of the dynamics is controlled by bulk dispersive estimates. We are not aware of any results which can do a finer analysis on the radiative part and improve upon ours.

We analyze the solutions of (1.1)-(1.2) by decomposing them into a time dependent projection onto solitary waves and a radiative part, see Chapter 2. The first original contribution of this thesis is to prove sharp dispersive estimates for the time dependent linearized dynamics of the radiative part, see Chapter 3. The second original contribution of this thesis is to estimate the radiative part of the solutions using Sobolev spaces in both time and space variables, see Chapter 4. This is different from the Strichartz type estimates in similar Sobolev spaces employed in [14], see also [7], which required critical or supercritical nonlinearities $\alpha_1 \geq 2$. Moreover, these new estimates are the ones that allow us to control the nonlinearity in the regime $\sqrt{2} < \alpha_1 \leq \frac{3}{2}$. 
Chapter 2

Decomposition of the Dynamics. The Center Manifold formed by the Ground States.

This chapter follows closely the decomposition into solitary waves and radiative part introduced in [10]. We include it here for completeness.

The center manifold is formed by the collection of periodic solutions for (1.1):

\[ u_E(t, x) = e^{-iEt} \psi_E(x) \]  

where \( E \in \mathbb{R} \) and \( 0 \not\equiv \psi_E \in H^2(\mathbb{R}^2) \) satisfy the time independent equation:

\[
[-\Delta + V] \psi_E + g(\psi_E) = E \psi_E
\]  

Clearly the function constantly equal to zero is a solution of (2.2) but (iii) in the following hypotheses on the potential \( V \) allows the bifurcation of a nontrivial, one (complex) parameter family of solutions:

(H1) Assume that the following hold.

(i) There exists \( C > 0 \) and \( \rho > 3 \) such that:

(1) \( |V(x)| \leq C < x >^{-\rho} \), for all \( x \in \mathbb{R}^2 \);

(2) the Fourier transform of \( V \) is in \( L^1(\mathbb{R}^2) \).

(ii) 0 is a regular point\(^1\) of the spectrum of the linear operator \( H = -\Delta + V \) acting on \( L^2 \).

\(^1\)see [19, Definition 7] or \( M_\mu = \{0\} \) in relation (3.1) in [15]
(iii) $H$ acting on $L^2$ has exactly one negative eigenvalue $E_0 < 0$ with corresponding normalized eigenvector $\psi_0$. It is well known that $\psi_0(x)$ can be chosen strictly positive and exponentially decaying as $|x| \to \infty$.

Conditions (i)-(ii) guarantee the applicability of dispersive estimates in Murata [15] and Schlag [19] to the Schrödinger group $e^{-iHt}$. These estimates are used to obtain estimates for the linearized dynamics, see Theorems 3.2 and 3.3. In particular (i), combined with (1.3) and (1.4), imply the local well-posedness in $H^1$ of the initial value problem (1.1-1.2), see Proposition 2.8.

By the standard bifurcation argument in Banach spaces [17] for (2.2) at $E = E_0$, condition (iii) guarantees existence of nontrivial solutions. Moreover, these solutions can be organized as a $C^2$ manifold (center manifold), see [8, section 2] or [7]. The proofs of the following results can be found in [8, section 2] or [7]. Note that $\alpha_1$, $\alpha_2$ in the current notation correspond to $\alpha_1 + 1$, respectively, $\alpha_2 + 1$ in [8].

**Proposition 2.1.** There exist $\delta', \delta > 0$, the $C^2$ function

$$h : \{a \in \mathbb{R} \times \mathbb{R} : |a| < \delta\} \mapsto L^2_\sigma \cap H^2, \text{ for all } \sigma \in \mathbb{R}$$

and the $C^1$ function $E : (-\delta, \delta) \mapsto \mathbb{R}$ such that in a $\delta'$ neighborhood of $(E_0, 0) \in \mathbb{R} \times H^2(\mathbb{R}^2)$, i.e. for $|E - E_0| < \delta'$, $\|\psi_E\|_{H^2} < \delta'$, the eigenvalue problem (2.2) has a unique solution up to multiplication with $e^{i\theta}$, $\theta \in [0, 2\pi)$, which can be represented as a center manifold:

$$\psi_E = a\psi_0 + h(a), \quad E = E(|a|), \quad \langle \psi_0, h(a) \rangle = 0, \quad h(e^{i\theta}a) = e^{i\theta}h(a), \quad |a| < \delta. \quad (2.3)$$

Moreover $E(|a|) = O(|a|^\alpha)$, $h(a) = O(|a|^{1+\alpha})$, and for $a \in \mathbb{R}$, $|a| < \delta$, $h(a)$ is a real valued function with $\frac{d^2h}{da^2}(a) = O(|a|^{\alpha_1-1})$ and $\frac{dh}{da}(0) = 0$.

Since $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$ the proposition implies that $\psi_E \in L^2_\sigma$. A regularity argument, see [21], gives a stronger result:
Corollary 2.2. For any $\sigma \in \mathbb{R}$, there exists a finite constant $C_\sigma$ such that:

$$\| < x >^\sigma \psi_E \|_{H^2} \leq C_\sigma \| \psi_E \|_{H^2}.$$ 

Moreover, standard regularity theory, see [4, Theorem 8.1.1], implies that for any $p$, $2 \leq p < \infty$, there exists a constant $C_p > 0$ such that

$$\| \psi_E \|_{W^{2,p}} \leq C_p \| \psi_E \|_{H^1}.$$

We are going to decompose the solution of \((1.1)-(1.2)\) into a projection onto the center manifold and a correction. To ensure that the correction disperses to infinity on long times we require that the correction is always in the invariant subspace of the linearized dynamics at the projection that complements the tangent space to the center manifold. A short description of the decomposition follows, for more details and the proofs see [8].

Consider the linearization of \((1.1)\) at a function on the center manifold $\psi_E = a\psi_0 + h(a)$, $a = a_1 + ia_2 \in \mathbb{C}$, $|a| < \delta$:

$$\frac{\partial w}{\partial t} = -iL_{\psi_E}[w] - iEw \tag{2.4}$$

where

$$L_{\psi_E}[w] = (-\Delta + V - E)w + Dg\psi_E[w] = (-\Delta + V - E)w + \lim_{\varepsilon \to 0} \frac{g(\psi_E + \varepsilon w) - g(\psi_E)}{\varepsilon} \tag{2.5}$$

Remark 2.3. Note that for $a \in \mathbb{R}$ we have $\psi_E = a\psi_0 + h(a)$ is real valued and

$$Dg\psi_E[w] = g'(\psi_E)\Re w + i\frac{g(\psi_E)}{\psi_E} \Im w = \frac{1}{2} \left( g'(\psi_E) + \frac{g(\psi_E)}{\psi_E} \right) w + \frac{1}{2} \left( g'(\psi_E) - \frac{g(\psi_E)}{\psi_E} \right) \bar{w}$$
hence

\[ |Dg_{\psi_E}[w]| \leq |w| \max\left\{ |g'(\psi_E)|, \left| \frac{g(\psi_E)}{\psi_E} \right| \right\} \leq C(|\psi_E|^{1+\alpha_1} + |\psi_E|^{1+\alpha_2})|w| \quad (2.6) \]

where we used (1.3). For \( a = |a|e^{i\theta} \in \mathbb{C} \) we have, using the equivariant symmetry (1.4), \( \psi_E = a\psi_0 + h(a) = e^{i\theta}(|a|\psi_0 + h(|a|)) = e^{i\theta}\psi_{E}^{\text{real}} \), where \( \psi_{E}^{\text{real}} \) is real valued, and \( Dg_{\psi_E}[w] = e^{i\theta}Dg_{\psi_{E}^{\text{real}}}[e^{-i\theta}w] \), hence (2.6) is valid for any \( \psi_E \) on the manifold of ground states.

In Chapter 3 we also need some smoothness for the effective (linear) potential induced by the nonlinearity:

(H2) We assume \( \frac{g'(\psi_E)}{\psi_E}, \frac{g(\psi_E)}{\psi_E} \) are uniformly bounded in \( L^1(\mathbb{R}^2) \). Here \( \hat{f} \) stands for the Fourier transform of the function \( f \).

Note that our hypotheses on \( g \) already imply \( \frac{g(\psi_E)}{\psi_E} \) are uniformly bounded in \( L^1(\mathbb{R}^2) \), provided \( \psi_E \) are solutions of (2.2) uniformly bounded in \( H^1 \). Indeed, via standard regularity theory, see Corollary 2.2, we have \( \psi_E \in H^2 \hookrightarrow L^\infty \) uniformly bounded, which combined with (1.3) and (1.4) imply \( \frac{g'(\psi_E(x))}{\psi_E(x)} \in C^1(\mathbb{R}^2) \) and \( \frac{g(\psi_E(x))}{\psi_E(x)} \in C^1(\mathbb{R}^2) \) if \( \alpha_1 \geq 2 \) or \( \frac{g'(\psi_E(x))}{\psi_E(x)} \in C^{0,\alpha_1-1} \) and \( \frac{g(\psi_E(x))}{\psi_E(x)} \in C^{0,\alpha_1-1} \) if \( \alpha_1 < 2 \). Now, using the uniform bounds for \( \psi_E \in W^{2,p} \hookrightarrow C^{1,\gamma}, \ 2 \leq p < \infty, \ 0 < \gamma < 1 \), we have uniform bounds for \( \xi^{\gamma} \nabla \frac{g(\psi_E(x))}{\psi_E(x)}(\xi) \in L^2 \) if \( \alpha_1 \geq 2 \) or \( \xi^{\alpha_1-1} \nabla \frac{g(\psi_E(x))}{\psi_E(x)} \in L^2 \) if \( \alpha_1 < 2 \). By combining them with \( (1 + |\xi|)^{-1-\min(\gamma,\alpha_1-1)} \in L^2(\mathbb{R}^2) \) we get, via Cauchy-Schwarz inequality, uniform bounds for \( \frac{g(\psi_E)}{\psi_E} \in L^1(\mathbb{R}^2) \).

For a given nonlinearity \( g(s) \), one can verify directly whether \( \frac{g'(\psi_E)}{\psi_E} \in L^1 \), or a similar argument shows that it suffices to assume a local Hölder continuity for the second derivative of \( g \):

**Remark 2.4.** If, in addition to (1.3) and (1.4), there exists \( 0 < \alpha \leq 1 \) such that \( |g''(s_1) - g''(s_2)| \leq C(R)|s_1 - s_2|^\alpha \), for any \( s_1, s_2 \in \mathbb{R} \) with \( |s_1| \leq R, |s_2| \leq R \), then (H2) holds for solutions of (2.2) which are uniformly bounded in \( H^1 \).

**Properties of the linearized operator:**
(1) \( L_{\psi_E} \) is real linear and symmetric with respect to the real scalar product \( \Re\langle \cdot, \cdot \rangle \), on \( L^2(\mathbb{R}^2) \), with domain \( H^2(\mathbb{R}^2) \).

(2) Zero is an e-value for \(-iL_{\psi_E}\) and its generalized eigenspace includes \( \{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \} \).

(3) \( \text{span}_\mathbb{R} \{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \} \) and \( \mathcal{H}_a \) defined by \( \{ -i\frac{\partial \psi_E}{\partial a_2}, i\frac{\partial \psi_E}{\partial a_1} \} \perp \), where orthogonality is with respect to the real scalar product in \( L^2(\mathbb{R}^2) \), are invariant subspaces for \(-iL_{\psi_E}\) and, by possibly choosing \( \delta > 0 \) smaller than the one in Proposition 2.1, we have:

\[
L^2(\mathbb{R}^2) = \text{span}_\mathbb{R} \left\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \right\} \oplus \mathcal{H}_a, \quad \text{for all } a \in \mathbb{C}, \ |a| < \delta.
\]

Note that \( \mathcal{H}_0 \) coincides with the subspace of \( L^2 \) associated to the continuous spectrum of the self-adjoint operator \( H = -\Delta + V \).

(4) the above decomposition can be extended to \( H^{-1}(\mathbb{R}^2) : \)

\[
H^{-1}(\mathbb{R}^2) = \text{span}_\mathbb{R} \left\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \right\} \oplus \mathcal{H}_a, \quad \text{for all } a \in \mathbb{C}, \ |a| < \delta, \quad (2.7)
\]

where

\[
\mathcal{H}_a = \left\{ \phi \in H^{-1} \mid \Re\langle -i\frac{\partial \psi_E}{\partial a_2}, \phi \rangle = 0, \text{ and } \Re\langle i\frac{\partial \psi_E}{\partial a_1}, \phi \rangle = 0 \right\}
\]

Our goal is to decompose the solution of \( (1.1) \) at each time into:

\[
u = \psi_E + \eta = a\psi_0 + h(a) + \eta, \quad \eta \in \mathcal{H}_a
\]

which ensures that \( \eta \) is not in the non-decaying directions of the linearized equation \( (2.4) \) at \( \psi_E \). The fact that this can be done in an unique manner is a consequence of the following lemma:

**Lemma 2.5.** There exists \( \delta_1, \ 0 < \delta_1 < \delta/2 \) such that any \( \phi \in H^{-1}(\mathbb{R}^2) \) satisfying \( \|\phi\|_{H^{-1}} \leq \)
\( \delta_1 \) can be uniquely decomposed:

\[
\phi = \psi_E + \eta = a\psi_0 + h(a) + \eta
\]

where \( a = a_1 + ia_2 \in \mathbb{C}, \ |a| < \delta, \ \eta \in \mathcal{H}_a \). Moreover the maps \( \phi \mapsto a \) and \( \phi \mapsto \eta \) are \( C^1 \) and there exists the constant \( C \) independent on \( \phi \) such that

\[
|a| \leq 2\|\phi\|_{H^{-1}}, \quad \|\eta\|_{H^{-1}} \leq C\|\phi\|_{H^{-1}}
\]

while for \( \phi \in L^2(\mathbb{R}^2) \) we have \( \eta \in L^2(\mathbb{R}^2) \) and:

\[
|a| \leq 2\|\phi\|_{L^2}, \quad \|\eta\|_{L^2} \leq C\|\phi\|_{L^2}.
\]

**Remark 2.6.** The above lemma uses the implicit function theorem applied to

\[
F : \mathbb{R}^2 \times H^{-1}(\mathbb{R}^2) \mapsto \mathbb{R}^2 \quad F(a_1, a_2, \phi) = \begin{bmatrix}
\mathbb{R}\langle \Psi_1, \psi_E - \phi \rangle \\
\mathbb{R}\langle \Psi_2, \psi_E - \phi \rangle
\end{bmatrix}
\]

where \( \psi_E = (a_1 + ia_2)\psi_0 + h(a_1 + ia_2) \) and

\[
\Psi_1(a_1, a_2) = -i\frac{\partial \psi_E}{\partial a_2} \left( \mathbb{R}\langle -i\frac{\partial \psi_E}{\partial a_2}, \frac{\partial \psi_E}{\partial a_1} \rangle \right)^{-1}
\]

\[
\Psi_2(a_1, a_2) = i\frac{\partial \psi_E}{\partial a_1} \left( \mathbb{R}\langle i\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \rangle \right)^{-1}
\]

form the dual basis of \( \left\{ \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \right\} \) with respect to the decomposition (2.7). Note that

\[
\frac{\partial F}{\partial (a_1, a_2)}(a_1, a_2, \phi) = \mathbb{I}_{\mathbb{R}^2} - M(a_1, a_2, \phi)
\]
where the entries of the two by two matrix $M$ are

$$M_{ij} = \Re \langle \frac{\partial \Psi_i}{\partial a_j}, \phi - \psi_E \rangle$$

and, consequently, $M(0,0,0)$ is the zero matrix. Thus the implicit function theorem applies to $F = 0$, in a neighborhood of $(a_1, a_2, \phi) = (0,0,0)$ and the number $\delta_1$ in the above lemma is chosen such that:

$$\left| \Re \langle i \frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2} \rangle \right| \geq \frac{1}{2}, \quad \text{whenever } |(a_1, a_2)| \leq 2\delta_1,$$

and the norm of the matrix $M$ as a linear, bounded operator on $\mathbb{R}^2$ satisfies:

$$\|M_{\phi}\| = \|M(a_1(\phi), a_2(\phi), \phi)\| \leq \frac{1}{2}, \quad \text{whenever } \|\phi\|_{H^{-1}} \leq \delta_1, \quad (2.8)$$

see [8, section 2] for details.

We need one more technical result relating the spaces $\mathcal{H}_a$ and the space corresponding to the continuous spectrum of $-\Delta + V$. Recall that $P_c : L^2 \mapsto \mathcal{H}_0 = \{|\psi, i\psi\}_\perp$ is the projection onto the continuous spectrum of $-\Delta + V$.

**Lemma 2.7.** With $\delta_1$ given by the previous lemma we have that for any $a \in \mathbb{C}$, $|a| \leq 2\delta_1$, the linear map $P_c|_{\mathcal{H}_a} : \mathcal{H}_a \mapsto \mathcal{H}_0$ is invertible, and its inverse $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ satisfies:

$$\|R_a\zeta\|_{L^2_{-\sigma}} \leq C_{-\sigma} \|\zeta\|_{L^2_{-\sigma}}, \quad \sigma \in \mathbb{R} \quad \text{and for all } \zeta \in \mathcal{H}_0 \cap L^2_{-\sigma}$$

$$\|R_a\zeta\|_{L^p} \leq C_p \|\zeta\|_{L^p}, \quad 1 \leq p \leq \infty \quad \text{and for all } \zeta \in \mathcal{H}_0 \cap L^p$$

$$\overline{R_a\zeta} = R_a\overline{\zeta}$$

where the constants $C_{-\sigma}, C_p > 0$ are independent of $a \in \mathbb{C}$, $|a| \leq 2\delta_1$.

We are now ready to decompose arbitrary solutions of $(1.1)$ with small initial data in $H^1$
into an evolution on the center manifold and a correction belonging to the complementary invariant subspace of the linearized dynamics at any given time:

**Proposition 2.8.** If hypothesis (1.3), (1.4), (H1) hold then there exists \( \varepsilon_0 > 0 \) such that for all initial conditions \( u_0(x) \) satisfying

\[
\|u_0\|_{H^1} \leq \varepsilon_0
\]

the initial value problem (1.1)-(1.2) is globally well-posed in \( H^1 \), and the solution decomposes into a projection onto the center manifold of ground states and a radiative part.

More precisely, there exist a \( C^1 \) function \( a : \mathbb{R} \mapsto \mathbb{C} \) such that, for all \( t \in \mathbb{R} \) we have:

\[
u(t, x) = a(t)\psi_0(x) + h(a(t)) + \eta(t, x)
\]

where \( \psi_E(t) \) is on the central manifold (i.e it is a ground state) and \( \eta(t, x) \in \mathcal{H}_{a(t)} \), see Proposition 2.1 and Lemma 2.5. Moreover, \( \tilde{a}(t) = e^{i\theta(t)} a(t), \quad \theta(t) = \int_0^t E(|a(s)|)ds \), and \( \eta(t, x) \) satisfies equations (2.23), respectively, (2.25).

**Proof:** It is well known that under hypotheses (H1)(i), (1.3), and (1.4), the initial value problem (1.1)-(1.2) is locally well-posed in the energy space \( H^1 \) and its \( L^2 \) norm is conserved, see for example \[4\] Corollary 4.3.3. at p. 92. Global well-posedness follows via energy estimates from \( \|u_0\|_{H^1} \) small, see \[4\] Corollary 6.1.5 at p. 165.

We choose \( \varepsilon_0 \leq \delta_1 \) given by Lemma 2.5. Then, for all times, \( \|u(t)\|_{H^{-1}} \leq \|u(t)\|_{L^2} \leq \varepsilon_0 \leq \delta_1 \) and, via Lemma 2.5 we can decompose the solution into a solitary wave and a dispersive component:

\[
u(t) = a(t)\psi_0 + h(a(t)) + \eta(t) = \psi_E(t) + \eta(t), \quad \text{where } |a(t)| = |a_1(t) + ia_2(t)| \leq 2\varepsilon_0 \leq 2\delta_1 \forall t \in \mathbb{R}.
\]

(2.9)

Note that since \( a \mapsto h(a) \) is \( C^2 \), see Proposition 2.1 and \( a \) is uniformly bounded in time we
deduce that there exists the constant $C_H > 0$ such that:

$$\max \left\{ \| \psi_E(t) \|_{H^2}, \| \frac{\partial \psi_E}{\partial a_1}(t) \|_{H^2}, \| \frac{\partial \psi_E}{\partial a_2}(t) \|_{H^2} \right\} \leq C_H \varepsilon_0, \quad \text{for all } t \in \mathbb{R},$$

which combined with Corollary 2.2 implies that for any $\sigma \in \mathbb{R}$ there exists a constant $C_{H,\sigma} > 0$ such that:

$$\max \left\{ \| < x >^\sigma \psi_E(t) \|_{H^2}, \| < x >^\sigma \frac{\partial \psi_E}{\partial a_1}(t) \|_{H^2}, \| < x >^\sigma \frac{\partial \psi_E}{\partial a_2}(t) \|_{H^2} \right\} \leq C_{H,\sigma} \varepsilon_0, \quad \forall \ t \in \mathbb{R}. \quad (2.10)$$

Consequently, using the continuous imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, $2 \leq p \leq \infty$ and $L^2_\sigma(\mathbb{R}^2) \hookrightarrow L^1(\mathbb{R}^2)$, $\sigma > 1$ we have that for all $1 \leq p \leq \infty$ and all $\sigma \in \mathbb{R}$, there exists the constants $C_{p,\sigma}$ such that

$$\sup_{t \in \mathbb{R}} \max \left\{ \| \psi_E(t) \|_{L^p}, \| \frac{\partial \psi_E}{\partial a_1}(t) \|_{L^p}, \| \frac{\partial \psi_E}{\partial a_2}(t) \|_{L^p}, \| \Psi_1(a(t)) \|_{L^p}, \| \Psi_1(a(t)) \|_{L^p} \right\} \leq C_{p,\sigma} \varepsilon_0, \quad \text{see Remark 2.6 for the definitions of } \Psi_j(a), \ j = 1, 2. \quad (2.11)$$

In addition, since

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2)),$$

and $u \mapsto a$ respectively $u \mapsto \eta$ are $C^1$, we get that $a(t)$ is $C^1$ and $\eta \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$.

The solution is now described by the $C^1$ function $a : \mathbb{R} \mapsto \mathbb{C}$ and $\eta(t) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. To obtain estimates for them it is useful to first remove their dominant phase. Consider the $C^2$ function:

$$\theta(t) = \int_0^t E(|a(s)|)ds \quad (2.12)$$

and

$$\tilde{u}(t) = e^{i\theta(t)}u(t), \quad (2.13)$$
then \( \tilde{u}(t) \) satisfies the differential equation:

\[
i\partial \tilde{u}(t) = -E(|a(t)|)\tilde{u}(t) + (-\Delta + V)\tilde{u}(t) + g(\tilde{u}(t)),
\]

(2.14)

see (1.1) and (1.4). Moreover, like \( u(t) \), \( \tilde{u}(t) \) can be decomposed:

\[
\tilde{u}(t) = \tilde{a}(t)\psi_0 + h(\tilde{a}(t)) + \tilde{\eta}(t)
\]

(2.15)

where

\[
\tilde{a}(t) = e^{i\theta(t)}a(t), \quad \tilde{\eta}(t) = e^{i\theta(t)}\eta(t) \in \mathcal{H}_{\tilde{a}(t)}
\]

(2.16)

By plugging in (2.15) into (2.14) we get

\[
\begin{align*}
\frac{i\partial \tilde{\eta}}{\partial t} + iD\tilde{\psi}_E|\tilde{a} \frac{d\tilde{a}}{dt} &= (-\Delta + V - E(|a|))(\tilde{\psi}_E + \tilde{\eta}) + g(\tilde{\psi}_E) + g(\tilde{\psi}_E + \tilde{\eta}) - g(\tilde{\psi}_E) \\
&= L_{\tilde{\psi}_E}\tilde{\eta} + g_2(\tilde{\psi}_E, \tilde{\eta})
\end{align*}
\]

or, equivalently,

\[
\frac{\partial \tilde{\eta}}{\partial t} + \frac{\partial \tilde{\psi}_E}{\partial a_1} \frac{d\tilde{a}_1}{dt} + \frac{\partial \tilde{\psi}_E}{\partial a_2} \frac{d\tilde{a}_2}{dt} = -iL_{\tilde{\psi}_E}\tilde{\eta} - ig_2(\tilde{\psi}_E, \tilde{\eta})
\]

(2.17)

where \( L_{\tilde{\psi}_E} \) is defined by (2.5):

\[
L_{\tilde{\psi}_E}\tilde{\eta} = (-\Delta + V - E(|\tilde{a}|))\tilde{\eta} + \frac{d}{d\varepsilon}g(\tilde{\psi}_E + \varepsilon\tilde{\eta})|_{\varepsilon=0}
\]

and we used \( |a| = |\tilde{a}| \), while \( g_2 \) is defined by:

\[
g_2(\tilde{\psi}_E, \tilde{\eta}) = g(\tilde{\psi}_E + \tilde{\eta}) - g(\tilde{\psi}_E) - \frac{d}{d\varepsilon}g(\tilde{\psi}_E + \varepsilon\tilde{\eta})|_{\varepsilon=0}
\]

(2.18)

and we also used the fact that \( \tilde{\psi}_E \) is a solution of the eigenvalue problem (2.2). Note that
$g_2$ is at least quadratic in the second variable, more precisely:

**Lemma 2.9.** There exists a constant $C > 0$ such that for all $a, z \in \mathbb{C}$ we have:

$$|g_2(a, z)| \leq C(|a|^\alpha_1 + |a|^\alpha_2 + |z|^\alpha_1 + |z|^\alpha_2)|z|^2$$

**Proof:** From the definition \((2.18)\) of $g_2$ we have:

$$g_2(a, z) = g(a+z) - g(a) - Dg_a[z] = \int_0^1 \langle Dg_{a+\tau z} - Dg_a \rangle |z| d\tau = \int_0^1 \int_0^1 D^2g_{a+s\tau z} |\tau z| |z| d\tau ds.$$

Now \((1.3)\) and \((1.4)\) imply that there exists a constant $C_1 > 0$ such that the bilinear form $Dg$ on $\mathbb{C} \times \mathbb{C}$ satisfies:

$$\|D^2g_b\| \leq C_1(|b|^\alpha_1 + |b|^\alpha_2), \quad \forall b \in \mathbb{C}. \quad (2.19)$$

Hence

$$|g_2(a, z)| \leq C_1 ((2 \max(|a|, |z|))^\alpha_1 + (2 \max(|a|, |z|))^\alpha_2) \frac{1}{2}|z|^2,$$

which proves the lemma. □

We now project \((2.17)\) onto the invariant subspaces of $-iL\tilde{\psi}_E$, namely span$_{\mathbb{R}} \{ \partial\tilde{\psi}_E / \partial a_1, \partial\tilde{\psi}_E / \partial a_2 \}$, and $\mathcal{H}_a$. More precisely, we evaluate both the left and right-hand side of \((2.17)\) which are functionals in $H^{-1}(\mathbb{R}^2)$ at $\Psi_j = \Psi_j(\tilde{a}(t)), \; j = 1, 2$, see Remark 2.6 and take the real parts. We obtain:

$$\begin{bmatrix}
    \mathbb{R} \langle \Psi_1, \dot{\tilde{\eta}} \rangle \\
    \mathbb{R} \langle \Psi_2, \dot{\tilde{\eta}} \rangle
\end{bmatrix} + \frac{d}{dt} \begin{bmatrix}
    \tilde{a}_1 \\
    \tilde{a}_2
\end{bmatrix} = \begin{bmatrix}
    g_{21}(\tilde{\psi}_E, \tilde{\eta}) \\
    g_{22}(\tilde{\psi}_E, \tilde{\eta})
\end{bmatrix}$$
where for all $t \in \mathbb{R}$:

$$|g_{2j}(\tilde{\psi}_E(t), \tilde{\eta}(t))| = \Re\langle \Psi_j(\tilde{a}(t)), g_2(\tilde{\psi}_E(t), \tilde{\eta}(t)) \rangle \quad (2.20)$$

$$|g_{2j}(\tilde{\psi}_E(t), \tilde{\eta}(t))| \leq C \int_{\mathbb{R}^2} |\Psi_j(t, x)| \left( |\tilde{\psi}_E(t, x)|^{\alpha_1} + |\tilde{\psi}_E(t, x)|^{\alpha_2} + |\tilde{\eta}(t, x)|^{\alpha_1} + |\tilde{\eta}(t, x)|^{\alpha_2} \right) \times |\tilde{\eta}(t, x)|^2 dx \quad (2.21)$$

$$\leq C \left[ \|\Psi_j(t)\|_{L^0} \left( \|\tilde{\psi}_E(t)\|_{L^\infty}^{\alpha_1} + \|\tilde{\psi}_E(t)\|_{L^\infty}^{\alpha_2} \right) \|\tilde{\eta}(t)\|_{L^p}^2 + \|\Psi_j(t)\|_{L^1} \|\tilde{\eta}(t)\|_{L^p}^{2+\alpha_1} + \|\Psi_j(t)\|_{L^2} \|\tilde{\eta}(t)\|_{L^p}^{2+\alpha_2} \right] ,$$

where $r_0^{-1} + (p_2/2)^{-1} = 1$, $r_j^{-1} + (p_2/(2 + \alpha_j))^{-1} = 1$, $j = 1, 2$. Note that for the quadratic term in $\tilde{\eta}$ we can also do a weighted estimate:

$$|g_{2j}(\tilde{\psi}_E(t), \tilde{\eta}(t))| \leq C \left[ \|\Psi_j(t)\|_{L^\infty} \left( \| < x >^{2\sigma} \tilde{\psi}_E(t)\|_{L^\infty}^{\alpha_1} + \| < x >^{2\sigma} \tilde{\psi}_E(t)\|_{L^\infty}^{\alpha_2} \right) \times \| < x >^{-\sigma} \tilde{\eta}(t)\|_{L^2}^2 + \|\Psi_j(t)\|_{L^1} \|\tilde{\eta}(t)\|_{L^p}^{2+\alpha_1} + \|\Psi_j(t)\|_{L^2} \|\tilde{\eta}(t)\|_{L^p}^{2+\alpha_2} \right] \quad (2.22)$$

To calculate $\Re\langle \Psi_j, \frac{\partial \tilde{\eta}}{\partial t} \rangle$, $j = 1, 2$ we use the fact that $\tilde{\eta}(t) \in \mathcal{H}_{\tilde{a}}$, for all $t \in \mathbb{R}$, i.e.

$$\Re\langle \Psi_j(\tilde{a}(t)), \tilde{\eta}(t) \rangle \equiv 0.$$

Differentiating the latter with respect to $t$ we get:

$$\Re\langle \Psi_j, \frac{\partial \tilde{\eta}}{\partial t} \rangle = -\Re\langle \frac{\partial \Psi_j}{\partial a_1} \frac{d\tilde{a}_1}{dt} + \frac{\partial \Psi_j}{\partial a_2} \frac{d\tilde{a}_2}{dt}, \tilde{\eta} \rangle \quad j = 1, 2$$

which replaced into above leads to:

$$\frac{d}{dt} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = (\mathbb{I}_{\mathbb{R}^2} - M_{\tilde{a}})^{-1} \begin{bmatrix} g_{21}(\tilde{\psi}_E, \tilde{\eta}) \\ g_{22}(\tilde{\psi}_E, \tilde{\eta}) \end{bmatrix} , \quad (2.23)$$
where the two by two matrix $M\tilde{u}$ is defined in Remark 2.6 and $g_{21}$, $g_{22}$ are defined in (2.20), see also (2.18). In particular

$$
\begin{bmatrix}
\Re(\Psi_1, \frac{\partial \tilde{\eta}}{\partial t}) \\
\Re(\Psi_2, \frac{\partial \tilde{\eta}}{\partial t})
\end{bmatrix} = -M\tilde{u}(I_{\mathbb{R}^2} - M\tilde{u})^{-1} \begin{bmatrix} g_{21}(\tilde{\psi}_E, \tilde{\eta}) \\
g_{22}(\tilde{\psi}_E, \tilde{\eta}) \end{bmatrix},
$$

which we use to obtain the component in $\mathcal{H}_{\tilde{a}} = \{\Psi_1(\tilde{a}), \Psi_2(\tilde{a})\}^\perp$ of (2.17):

$$
\frac{\partial \tilde{\eta}}{\partial t} = -iL_{\tilde{\psi}_E} \tilde{\eta} - ig_2(\tilde{\psi}_E, \tilde{\eta}) - (I - M\tilde{u})^{-1} g_3(\tilde{\psi}_E, \tilde{\eta}),
$$

where $g_3$ is the projection of $-ig_2$ onto $\text{span}_{\mathbb{R}}\{\frac{\partial \tilde{\psi}_E}{\partial a_1}, \frac{\partial \tilde{\psi}_E}{\partial a_2}\}$ relative to the decomposition (2.7):

$$
g_3(\tilde{\psi}_E, \tilde{\eta}) = g_{21}(\tilde{\psi}_E, \tilde{\eta}) \frac{\partial \tilde{\psi}_E}{\partial a_1} + g_{22}(\tilde{\psi}_E, \tilde{\eta}) \frac{\partial \tilde{\psi}_E}{\partial a_2},
$$

(2.24)

see (2.20) for the definitions of $g_{2j}$, $j = 1, 2$, and $I - M\tilde{u}$ is the linear operator on the two dimensional real vector space $\text{span}_{\mathbb{R}}\{\frac{\partial \tilde{\psi}_E}{\partial a_1}, \frac{\partial \tilde{\psi}_E}{\partial a_2}\}$ whose matrix representation relative to the basis $\{\frac{\partial \tilde{\psi}_E}{\partial a_1}, \frac{\partial \tilde{\psi}_E}{\partial a_2}\}$ is $I_{\mathbb{R}^2} - M\tilde{u}$. It is easier to switch back to the variable $\eta(t) = e^{-i\theta(t)}\tilde{\eta}(t) \in \mathcal{H}_{\tilde{a}}$:

$$
\frac{\partial \eta}{\partial t} = -i(-\Delta + V)\eta - iDg\psi_E \eta - ig_2(\psi_E, \eta) - (I - M\tilde{u})^{-1} g_3(\psi_E, \eta),
$$

(2.25)

where we used the equivariant symmetry (1.4) and its obvious consequences for the symmetries of $Dg$, $g_2$, $g_3$ and $M$, see Remark 2.3, definitions (2.18), (2.24), (2.20), and Remark 2.6.

The proof of Proposition 2.8 is now complete. □

Since by Lemma 2.7 it is sufficient to get estimates for $\zeta(t) = P_c \eta(t)$, we now project (2.25) onto the continuous spectrum of $-\Delta + V$:

$$
\frac{\partial \zeta}{\partial t} = -i(-\Delta + V)\zeta - iP_c Dg\psi_E R_\alpha \zeta - iP_c g_2(\psi_E, R_\alpha \zeta) - P_c (I - M\tilde{u})^{-1} g_3(\psi_E, R_\alpha \zeta),
$$

(2.26)
where \( R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a \) is the inverse of \( P_c \) restricted to \( \mathcal{H}_a \), see Lemma 2.7.

Consider the initial value problem for the linear part of (2.26):

\[
\frac{\partial z}{\partial t} = -i(-\Delta + V)z - iP_cDg_{\psi_{E}(t)}R_{a(t)}z(t) \tag{2.27}
\]

\[
z(s) = v \in \mathcal{H}_0
\]

and write its solution in terms of a family of operators:

\[
\Omega(t, s) : \mathcal{H}_0 \mapsto \mathcal{H}_0, \quad \Omega(t, s)v = z(t), \quad t, s \in \mathbb{R}. \tag{2.28}
\]

In the next chapter we show that such a family of operators exists, is uniformly bounded in \( t, s \) with respect to the \( L^2 \) norm and it has very similar properties with the unitary group of operators \( e^{-i(-\Delta+V)(t-s)}P_c \) generated by the Schrödinger operator \( -i(-\Delta + V)P_c \).

In particular \( \Omega(t, s) \) satisfies certain dispersive decay estimates in weighted \( L^2 \) spaces and \( L^p, p > 2 \) spaces, see Theorem 3.2 and Theorem 3.3. For all these results to hold we only need to add a Hölder continuity hypothesis for \( g'' \), see Remark 2.4, or assume that (H2) holds for \( \psi_{E} \) on the center manifold given in Proposition 2.1. We also need to choose \( \varepsilon_0 \) small enough such that \( \varepsilon_0 C_{H, A\sigma/3} \leq \varepsilon_1 \), where \( \sigma > 1 \) and \( \varepsilon_1 > 0 \) are fixed in Chapter 3 and the constant \( C_{H, A\sigma/3} \) is the one from (2.10).
Chapter 3

The Linearized Dynamics. Dispersive Estimates.

In this chapter we analyze the linearized dynamics of the radiative part of the solution, namely, equation (2.27). We are going to extend for our two space dimension case the methods introduced in [8] and [9] for three and higher space dimensions.

We start from the linear Schrödinger equation with a potential in two dimensions:

\[ i \frac{\partial u}{\partial t} = (-\Delta + V(x)) u \]

\[ u(0) = u_0. \]

If \( V \) satisfies hypothesis (H1) (i) (1) and (ii) it is known, see [15, Example 7.8], that for \( \sigma > 1 \), there exists a constant \( C_\sigma > 0 \) such that

\[ \|e^{-iHt}P_c u_0\|_{L^2_\sigma} \leq \frac{C_\sigma}{(1 + |t|) \log^2(2 + |t|)} \|u_0\|_{L^2_\sigma} \]  

(3.1)

where \( P_c \) is the projection onto the continuous spectrum of \( H = -\Delta + V \).

In addition, if \( V \) satisfies (H1) (i) (1) and (ii) then for each \( 2 \leq p \leq \infty \), \( 1/p' + 1/p = 1 \) there exists a constant \( C_p > 0 \) such that:

\[ \|e^{-iHt}P_c u_0\|_{L^p} \leq \frac{C_p}{|t|^{(1 - \frac{1}{p})}} \|u_0\|_{L^{p'}} \]  

(3.2)

see for example [19].

We would like to extend these estimates to the linearized dynamics around the time
dependent motion on center manifold. We consider the linear equation with initial data at time \( s \):

\[
\frac{dz}{dt} = Hz(t) + P_c Dg|_{\psi_E(t)}[z(t)] - Dh|_{\psi(t)}\langle \psi_0, Dg|_{\psi_E(t)}[z(t)] \rangle
\]

\[z(s) = v\]

where \( Dg|_{\psi_E}[z] = \frac{d}{d\varepsilon} g(\psi_E + \varepsilon z)|_{\varepsilon=0} = \partial_u g(u)|_{u=\psi_E z} + \partial_{\bar{u}} g(u)|_{u=\psi_E \bar{z}}\). Compared to equation (2.27) which we actually want to analyze, the above equation is missing the (linear) operator \( R_a \). This operator does not affect any of the arguments that follow, due to the estimates in Lemma 2.7, so, to simplify notations, we did not include it.

**Remark 3.1.** Note that the above linear equation is globally well-posed in \( H^1 \), see [4, Corollary 6.1.2, page 164]. In what follows we will obtain dispersive estimates for its solution under milder hypotheses.

By Duhamel’s principle we have:

\[
z(t) = e^{-iH(t-s)}P_c v(s) - i \int_s^t e^{-iH(t-\tau)}P_c Dg|_{\psi_E(\tau)}[z(\tau)] - Dh|_{\psi(\tau)}\langle \psi_0, Dg|_{\psi_E(\tau)}[z(\tau)] \rangle d\tau \quad (3.3)
\]

In the next theorems we will extend estimates of type (3.1)-(3.2) to the operators

\[\Omega(t, s)v = z(t), \quad \text{and} \quad T(t, s) = \Omega(t, s) - e^{-iH(t-s)}P_c,\]

relying on the fact that \( \psi_E(t) \) is small and localized in space. We believe that the arguments can be extended for large \( \psi_E(t) \) provided for a certain fixed solution \( \psi_E \) of (2.2) we have \( \| \psi_E(t) - \psi_E \|_{H^1} \) is small uniformly in \( t \in \mathbb{R} \). We start with weighted estimates. While the approach is similar to the one in three or higher space dimensions, see [8], there are important differences due mainly to the fact that estimate (3.2) is not integrable in time at large \( t \), even for \( p = \infty \) when it blows up logarithmically. This will force us to introduce
logarithmic corrections in the $L^p' \mapsto L^p$ estimates.

**Theorem 3.2.** Fix $\sigma > 1$. There exists $\varepsilon_1 > 0$ such that if $\| x > 4^\sigma \psi_E(t) \|_{L^\infty} < \varepsilon_1$ for all $t \in \mathbb{R}$, then there are constants $C_\sigma, C_p, 2 \leq p \leq \infty$ with the property that for any $t, s \in \mathbb{R}$ the following estimates hold:

(i) $\| \Omega(t, s) \|_{L_2^\sigma \rightarrow L_2^\sigma} \leq \frac{C_\sigma}{(1 + |t - s|) \log^2(2 + |t - s|)}$;

(ii) $\| \Omega(t, s) \|_{L_2^\sigma \rightarrow L_2^p} \leq \frac{C_p}{|t - s|^{1 - \frac{2}{p}}}$ for all $2 \leq p < \infty$;

(iii) $\Omega(t, s) \in L_2^\sigma(\mathbb{R}, L^2 \rightarrow L^2_{-\sigma}) \cap L_2^\infty(\mathbb{R}, L^2 \rightarrow L^2_{-\sigma})$;

(iv) if in addition hypothesis (H2) holds then:

$$\| \Omega(t, s) \|_{L_2^p \rightarrow L_2^\sigma} \leq \frac{C_p}{|t - s|^{1 - \frac{2}{p}}} \text{ for all } 2 \leq p \leq \infty;$$

$$\| T(t, s) \|_{L_2^p \rightarrow L_2^\sigma} \leq \frac{C_p}{(1 + |t - s|)^{1 - \frac{2}{p}}} \text{ for all } 2 \leq p \leq \infty.$$

**Proof of Theorem 3.2** Fix $s \in \mathbb{R}$.

(i) By definition, we have $\Omega(t, s)v = z(t)$ where $z(t)$ satisfies equation (3.3). We are going to prove the estimate by showing that the integral equation (3.3) can be solved via contraction principle argument in an appropriate functional space. To this extent let us consider the functional space

$$X_1 := \{ u \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^N)) \mid \sup_{t \in \mathbb{R}} (1 + |t - s|) \log^2(2 + |t - s|) \| u(t) \|_{L^2_{-\sigma}} < \infty \}$$

endowed with the norm

$$\| u \|_{X_1} := \sup_{t \in \mathbb{R}} ((1 + |t - s|) \log^2(2 + |t - s|) \| u(t) \|_{L^2_{-\sigma}}) < \infty$$
Note that the inhomogeneous term in (3.3) \( z_0 = e^{-iH(t-s)} P_c v \) satisfies \( z_0 \in X_1 \) and

\[ \| z_0 \|_{X_1} \leq C_v \| v \|_{L^2} \]  \hspace{1cm} (3.5) \]

because of (3.1). We collect the \( z \) dependent part of the right-hand side of (3.3) in a linear operator \( L(s) : X_1 \to X_1 \)

\[ [L(s)z](t) = -i \int_s^t e^{-iH(t-\tau)} P_c Dg|\psi_{E(\tau)}[z(\tau)] - Dh|_{a(\tau)} \langle \psi_0, Dg|\psi_{E(\tau)}[z(\tau)] \rangle d\tau \]  \hspace{1cm} (3.6) \]

We will show that \( L \) is a well-defined bounded operator from \( X_1 \) to \( X_1 \) whose operator norm can be made less or equal to \( 1/2 \) by choosing \( \varepsilon_1 \) sufficiently small. Consequently \( Id - L \) is invertible and the solution of the equation (3.3) can be written as \( z = (Id - L)^{-1}z_0 \). In particular

\[ \| z \|_{X_1} \leq (1 - \| L \|)^{-1} \| z_0 \|_{X_1} \leq 2 \| z_0 \|_{X_1} \]

which in combination with the definition of \( \Omega \), the definition of the norm \( X_1 \) and the estimate (3.5), finishes the proof of \( (i) \).

By computing the \( L^2_{-\sigma} \) norm of both the left-hand side and right-hand side of (3.6), we have:

\[
\| L(s)z(t) \|_{L^2_{-\sigma}} \leq \int_s^t \| e^{-iH(t-\tau)} P_c \|_{L^2_{-\sigma} \to L^2_{-\sigma}} \left[ \| Dg|\psi_{E}[z] \|_{L^2} + \| Dh|_{a(\tau)} \|_{c \to L^2} \| \langle \psi_0, Dg|\psi_{E}[z] \rangle \| \right] d\tau \\
\leq \int_s^t \| e^{-iH(t-\tau)} P_c \|_{L^2_{-\sigma} \to L^2_{-\sigma}} \left[ \| Dg|\psi_{E}[z] \|_{L^2} \\
+ \| Dh|_{a(\tau)} \|_{c \to L^2} \| \psi_0 \|_{L^2} \| Dg|\psi_{E}[z] \|_{L^2} \right] d\tau 
\]

On the other hand, from (2.6), and (1.3) we obtain:

\[
\| Dg|\psi_{E}[z] \|_{L^2_{-\sigma}} \leq \| x \|^{2\sigma} (|\psi_{E}|^{\alpha_1} + |\psi_{E}|^{\alpha_2}) \|_{L^\infty} \| z \|_{L^2_{-\sigma}} \leq \varepsilon_1^\alpha_1 \| z \|_{L^2_{-\sigma}} \]  \hspace{1cm} (3.7) \\
\| Dg|\psi_{E}[z] \|_{L^2} \leq \| x \|^{\sigma}(|\psi_{E}|^{\alpha_1} + |\psi_{E}|^{\alpha_2}) \|_{L^\infty} \| z \|_{L^2_{-\sigma}} \leq \varepsilon_1^\alpha_1 \| z \|_{L^2_{-\sigma}} \]  \hspace{1cm} (3.8)
Also
\[ \|Dh|_{a(\tau)}\|_{C \rightarrow L^2_2} \leq C_2, \quad \text{for all } \tau \in \mathbb{R} \]

which follow from \( h \) being \( C^1 \) on \( a \in \mathbb{C} \), \( |a| \leq \delta \), with values in \( L^2_\sigma \), see Proposition 2.1 and \( |a(\tau)| \leq 2\varepsilon_0 \leq \delta \) for all \( \tau \in \mathbb{R} \), see (2.9).

Using the last three relations, as well as the estimate (3.1) and the fact that \( z \in X_1 \) we obtain that
\[
\|L(s)\|_{X_1 \rightarrow X_1} \leq \varepsilon_1^{\alpha_1} \sup_{t > 0} (1 + |t - s|) \log^2 (2 + |t - s|) 
\times \int_s^t \frac{1}{(1 + |t - \tau|) \log^2 (2 + |t - \tau|)} \cdot \frac{1}{(1 + |\tau - s|) \log^2 (2 + |\tau - s|)} d\tau
\leq \varepsilon_1^{\alpha_1} \sup_{t > 0} (1 + |t - s|) \log^2 (2 + |t - s|) \frac{1}{(1 + |\frac{t-s}{2}|) \log^2 (2 + |\frac{t-s}{2}|)} \leq C\varepsilon_1^{\alpha_1}
\]

Now choosing \( \varepsilon_1 \) small enough we get
\[ \|L\|_{X_1 \rightarrow X_1} < \frac{1}{2} \]

Therefore
\[ \|\Omega(t, s)\|_{L^2_\sigma \rightarrow L^2_{-\sigma}} \leq \frac{C}{(1 + |t - s|) \log^2 (2 + |t - s|)} \]

(ii) From part (i) we already know that (3.3) has a unique solution in \( L^2_\sigma \) provided that \( v \in L^2_\sigma \). We are going to show that the right-hand side of (3.3) in \( L^p \). Indeed, using (3.2) and \( L^2_\sigma \hookrightarrow L^{p'} \) we have:
\[
\|e^{-iH(t-s)}P_cv\|_{L^p} \leq \frac{C_p}{|t-s|^{\frac{1}{2}p'}} \|v\|_{L^2_\sigma}. \quad (3.9)
\]
The remaining term satisfies:

\[
\|L(s)z(t)\|_{L^p} \leq \int_s^t \|e^{-iH(t-\tau)}P_c|_{L^p} \left[ \|Dg|_{\psi_E[z]}\|_{L^p} + \|Dh|_{C} \langle \psi_0, Dg|_{\psi_E[z]} \rangle \right] d\tau \\
\leq \int_s^t \frac{C}{|t-\tau|^{1 - \frac{2}{p}}} \left[ \|\langle x \rangle^{\sigma} \psi_E\|_{L^2} + \|Dh\|_{L^2} \|\langle x \rangle^{\sigma} \psi_E\|_{L^\infty} \right] \|z(\tau)\|_{L^2} d\tau \\
\leq \int_s^t \frac{C\|v\|_{L^2}}{|t-\tau|^{1 - \frac{2}{p}} (1 + |t-\tau-s|) \log^2 (2 + |t-\tau-s|)} d\tau \\
\leq \frac{C\|v\|_{L^2}}{|t-s|^{1 - \frac{2}{p}}}, \text{ for all } 2 \leq p < \infty \quad (3.10)
\]

Plugging in (3.9) and (3.10) into (3.3) we get:

\[
\|z\|_{L^p} \leq \frac{C}{|t-s|^{1 - \frac{2}{p}}} \|v\|_{L^2}
\]

which by the definition \(\Omega(t, s)v = z(t)\) finishes the proof of part (ii).

(iii) From equation (3.3) we have

\[
\langle x \rangle^{-\sigma}z(t) = \langle x \rangle^{-\sigma}e^{-iH(t-s)}P_c v(s) + \int_s^t \langle x \rangle^{-\sigma}e^{-iH(t-\tau)}P_c Dg|\psi_E[z(\tau)] + Dh|_{\psi_0} \langle \psi_0, Dg|\psi_E[z(\tau)] \rangle \rangle d\tau
\]

and

\[
\|\langle x \rangle^{-\sigma}z(t)\|_{L^2 L^2} \leq C\|v\|_{L^2} + \left\| \int_s^t \frac{C}{(1 + |t-\tau|) \log^2 (2 + |t-\tau|)} \right\| \langle x \rangle^{-\sigma}z(\tau)\|_{L^2} d\tau \right\|_{L^2} \\
\leq C\|v\|_{L^2} + \varepsilon_1 C\|K\|_{L^1} \|\langle x \rangle^{-\sigma}z\|_{L^2 L^2}
\]

where \(K(t) = (1 + |t|)^{-1} \log^{-2} (2 + |t|)\), and for the term

\[
\langle x \rangle^{\sigma}Dg|_{\psi_E}[e^{-iHt}P_c v] = \langle x \rangle^{\sigma}g_a \langle x \rangle^{\sigma}e^{-iHt}P_c v + \langle x \rangle^{\sigma}g_a \langle x \rangle^{\sigma}e^{iHt}P_c v
\]
we used that \( \| \langle x \rangle^{2\sigma} g_u \|_{L^\infty} \) and \( \| \langle x \rangle^{2\sigma} \bar{g}_u \|_{L^\infty} \) are uniformly bounded in \( t \) since \( |g_u| = |\bar{g}_u| \leq C(|\psi_E|^{\alpha_1} + |\psi_E|^{\alpha_2}) \) and the Kato smoothing estimate \( \| \langle x \rangle^{-\sigma} e^{-iHt} P_c v \|_{L^2_t(\mathbb{R}, L^2_x)} \leq C \| v \|_{L^2_x} \).

Choosing \( \varepsilon_1 < 1/(C \| K \|_{L^1}) \) we get \( \| \langle x \rangle^{-\sigma} W \|_{L^2_t L^2_x} < \infty \). In other words \( \Omega(t,s) \in L^2_t(\mathbb{R}, L^2 \to L^2_{-\sigma}) \). And similarly, using now (3.2) with \( p = 2 \) and \( u_0 = v \), we obtain:

\[
\| \langle x \rangle^{-\sigma} z(t) \|_{L^2_x} \leq C \| v \|_{L^2} + \int_s^t \frac{C}{(1 + |t - \tau|) \log(2 + |t - \tau|)} \times (1 + \| D\hat{h} \|_{L^2} \| \psi_0 \|_{L^1}) (\| \langle x \rangle^{2\sigma} g_u \|_{L^\infty} + \| \langle x \rangle^{2\sigma} \bar{g}_u \|_{L^\infty}) \| \langle x \rangle^{-\sigma} z(\tau) \|_{L^2_x} d\tau \leq C \| v \|_{L^2} + \varepsilon_1 C \| \langle x \rangle^{-\sigma} z \|_{L^2_x}
\]

This finishes the proof of \( \Omega(t,s) \in L^2_t(\mathbb{R}, L^2 \to L^2_{-\sigma}) \cap L^\infty_t(\mathbb{R}, L^2 \to L^2_{-\sigma}) \).

(iv) Denote:

\[
T(t,s) v = W(t)
\]

then, by plugging in (3.3), \( W(t) \) satisfies the following equation:

\[
W(t) = -i \int_s^t e^{-iH(t-\tau)} P_c Dg |\psi_E| [e^{-iH(\tau-s)} P_c \bar{v}] d\tau - i \int_s^t e^{-iH(t-\tau)} P_c D\hat{h} |\alpha(\tau)\langle \psi_0, Dg |\psi_E| [e^{-iH(\tau-s)} P_c \bar{v}] \rangle d\tau + [L(s)W](t)
\]

(3.12)

So, it is sufficient to prove that the solution of (3.12) satisfies

\[
\| W(t) \|_{L^2_{-\sigma}} \leq \frac{C_\infty \| v \|_{L^1}}{1 + |t - s|}
\]

The estimates for \( 2 < p < \infty \) are then obtained by Riesz-Thorin interpolation.
Let us also observe that it suffices to obtain estimates only for the forcing terms in (3.12):

\[
  f(t) = -i \int_s^t e^{-iH(t-\tau)} P_c Dg|_{\psi_E} [e^{-iH(\tau-s)} P_c v] d\tau
\]

\[
  \tilde{f}(t) = -i \int_s^t e^{-iH(t-\tau)} P_c Dh|_{a(\tau)} \langle \psi_0, Dg|_{\psi_E} [e^{-iH(\tau-s)} P_c v] \rangle d\tau
\]

because then we will be able to do the contraction principle in the functional space in which \( f(t), \tilde{f}(t) \) are, and thus obtain the same decay for \( W \) as for \( f(t) \) and \( \tilde{f}(t) \). This time we will consider the functional space

\[
  X_2 = \left\{ u \in C(\mathbb{R}, L^2_{-\sigma}(\mathbb{R}^2)) : \sup_{t \in \mathbb{R}} (1 + |t-s|) \|u(t)\|_{L^2_{-\sigma}} < \infty \right\}
\]

endowed with the norm

\[
  \|u\|_{X_2} = \sup_{t \in \mathbb{R}} (1 + |t-s|) \|u(t)\|_{L^2_{-\sigma}}
\]

Now we will estimate \( f(t) + \tilde{f}(t) \). First we will investigate the short time behavior of these terms. If \( s \leq t \leq s + 1 \):
\[
\|f(t)\|_{L^2_{\sigma}} \leq \|(x)\|_{L^2}^{-\sigma} \int_s^t e^{-iH(t-\tau)} P_c F_1(e^{-iH(\tau-s)} P_c v) d\tau \|_{L^2} \\
\leq \|(x)\|_{L^2}^{-\sigma} \int_s^t \|e^{-iH(t-s)} e^{iH(\tau-s)} P_c g_u e^{-iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
+ \|(x)\|_{L^2}^{-\sigma} \int_s^{s+\frac{t-s}{4}} \|e^{-iH(t+s-2\tau)} e^{-iH(\tau-s)} P_c g_u e^{iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
+ \int_{s+\frac{t-s}{4}}^t \|e^{-iH(t-\tau)} P_c\|_{L_2^\infty \rightarrow L_2^{2,\sigma}} \|(x)^{\sigma} g_u e^{-iH(\tau-s)} P_c v\|_{L^2} d\tau \\
\leq \int_s^t \frac{C}{|t-s|} \sup_{\tau \in [s,t]} \|\hat{g}_u(\tau)\|_{L^1} \|v\|_{L^1} d\tau \\
+ \int_s^{s+\frac{t-s}{4}} \frac{C}{|t+s-2\tau|} \sup_{\tau \in [s,t]} \|\hat{g}_u(\tau)\|_{L^1} \|v\|_{L^1} d\tau \\
+ \int_{s+\frac{t-s}{4}}^t \frac{C}{(1+|t-\tau|)} \|(x)^{\sigma} g_u\|_{L^2} \|e^{-iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
\leq C \sup_{\tau \in [s,t]} (\|\hat{g}_u(\tau)\|_{L^1} + \|\hat{g}_u(\tau)\|_{L^1} + \|(x)^{\sigma} g_u(\tau)\|_{L^2}) \|v\|_{L^1}
\]

where we used the Fourier multiplier type estimate \(\|e^{-iH(\tau-s)} F e^{iH(\tau-s)}\|_{L^p \rightarrow L^p} \leq C \|\hat{F}\|_{L^1}, \ |\tau-s| \leq 1 \) for all \(1 \leq p \leq \infty\), see Appendix in [8, Theorem 5.2].
\[ \left\| \hat{f}(t) \right\|_{L^2_{\rightarrow}} \leq \int_s^t \left\| e^{-iH(t-\tau)} P_c v \right\|_{L^2_{\rightarrow}} \left| \int F_1(\hat{\psi}_0, e^{-iH(\tau-s)} P_c v) \right| \]
\[
\leq \int_s^t \frac{C}{(1 + |t - \tau|) \log^2(2 + |t - \tau|)} \left\| Dh|_a(\tau) \right\|_{L^2_{\rightarrow}} \times \left[ \left| \left\langle \psi_0, g_\tau e^{-iH(\tau-s)} P_c v \right\rangle \right| + \left| \left\langle \psi_0, g_\tau e^{iH(\tau-s)} P_c \bar{v} \right\rangle \right| \right] \]
\[
\leq \int_s^t \frac{C \left\| Dh \right\|}{(1 + |t - \tau|) \log^2(2 + |t - \tau|)} \left| \left\langle e^{iE_\tau(\tau-s)} \psi_0, e^{iH(\tau-s)} g_\tau e^{-iH(\tau-s)} P_c v \right\rangle \right| \]
\[
+ \left| \left\langle e^{-iH(\tau-s)} \psi_0, e^{-iH(\tau-s)} g_\tau e^{iH(\tau-s)} P_c \bar{v} \right\rangle \right| \]
\[
\leq \int_s^t \frac{C \left\| Dh \right\|}{(1 + |t - \tau|) \log^2(2 + |t - \tau|)} \left| e^{iE_\tau(\tau-s)} \psi_0 \right|_{L^\infty} \times \sup_{\tau \in [s, t]} \left( \left\| g_\tau(\tau) \right\|_{L^1} + \left\| g_\bar{\tau}(\tau) \right\|_{L^1} \right) \left\| v \right\|_{L^1} \]
\[
\leq C \left\| Dh \right\| \sup_{\tau \in [s, t]} \left( \left\| g_\tau(\tau) \right\|_{L^1} + \left\| g_\bar{\tau}(\tau) \right\|_{L^1} \right) \left\| \psi_0 \right\|_{L^\infty} \left\| v \right\|_{L^1} \]
\[
\leq C \left\| v \right\|_{L^1} \]

where we also used \( \psi_0 \in H^2 \hookrightarrow L^\infty \).

For \( t > s + 1 \) we will split these two integrals in two parts to be estimated differently:

\[
f(t) = \int_s^{s+\frac{1}{4}} \ldots + \int_{s+\frac{1}{4}}^t \ldots
\]

and

\[
\hat{f}(t) = \int_s^{s+\frac{1}{4}} \ldots + \int_{s+\frac{1}{4}}^t \ldots
\]
Then we have:

\[
\|I_1\|_{L^2_{-\sigma}} \leq \|\langle x \rangle^{-\sigma} \int_s^{s+\frac{1}{4}} e^{-iH(t-s)} P_c F_1 (e^{-iH(\tau-s)} P_c v) d\tau \|_{L^2} \\
\leq \|\langle x \rangle^{-\sigma} \|_2 \int_s^{s+\frac{1}{4}} \|e^{-iH(t-s)} e^{iH(\tau-s)} P_c g_u e^{-iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
+ \|\langle x \rangle^{-\sigma} \|_2 \int_s^{s+\frac{1}{4}} \|e^{-iH(t-s-2\tau)} e^{-iH(\tau-s)} P_c g_u e^{iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
\leq \|\langle x \rangle^{-\sigma} \|_2 \int_s^{s+\frac{1}{4}} \frac{C_\infty}{|t-s|} \|e^{iH(\tau-s)} P_c g_u e^{-iH(\tau-s)} P_c v\|_{L^1} d\tau \\
+ \|\langle x \rangle^{-\sigma} \|_2 \int_s^{s+\frac{1}{4}} \frac{C_\infty}{|t+s-2\tau|} \|e^{-iH(\tau-s)} P_c g_u e^{iH(\tau-s)} P_c v\|_{L^1} d\tau \\
\leq C \|\langle x \rangle^{-\sigma} \|_2 \left(\frac{1}{|t-s|} + \frac{1}{|t-s - \frac{1}{2}|}\right) \int_s^{s+\frac{1}{4}} \sup(\|\hat{g}_u\|_{L^1} + \|\hat{g}_u\|_{L^1}) \|v\|_{L^1} d\tau \\
\leq C \|\langle x \rangle^{-\sigma} \|_2 \sup(\|\hat{g}_u\|_{L^1} + \|\hat{g}_u\|_{L^1}) \left(\frac{1}{|t-s|} + \frac{1}{|t-s - \frac{1}{2}|}\right) \|v\|_{L^1} \\
\leq C \|v\|_{L^1} \left(1 + \frac{1}{|t-s|}\right)
\]

For the second integral we have

\[
\|I_2\|_{L^2_{-\sigma}} \leq \int_{s+\frac{1}{4}}^t \|e^{-iH(t-\tau)} P_c\|_{L^2_{-\sigma}} \|F_1 (e^{-iH(\tau-s)} P_c v)\|_{L^2} d\tau \\
\leq \int_{s+\frac{1}{4}}^t \frac{C}{(1 + |t-\tau|) \log^2(2 + |t-\tau|)} \|\langle x \rangle^{\sigma} \psi E\|_{L^3} \|e^{-iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
\leq C \|v\|_{L^1} \int_{s+\frac{1}{4}}^t \frac{1}{(1 + |t-\tau|) \log^2(2 + |t-\tau|) |\tau-s|} d\tau \\
\leq C \|v\|_{L^1} \left(1 + \frac{1}{|t-s|}\right)
\]
For the second forcing term \( \tilde{f}(t) \), we use again \( \psi_0 \in H^2 \hookrightarrow L^\infty \): 

\[
\| II_1 \|_{L^2_{t,s}} \leq \frac{\int_s^{s+\frac{1}{4}} C}{(1 + |t - \tau|)} \log^2(2 + |t - \tau|) \| Dh \|_{L^2_{\mu}} \langle \psi_0, F_1(e^{-iH(\tau-s)P_c}v) \rangle d\tau 
\]

\[
\leq \frac{C}{(1 + |t - s - \frac{1}{4}|)} \int_s^{s+\frac{1}{4}} \left[ \langle e^{iH(\tau-s)} \psi_0, e^{iH(\tau-s)} g_k e^{-iH(\tau-s)} P_c v \rangle 
\right. 
\]

\[
+ \left. \langle e^{-iH(\tau-s)} \psi_0, e^{-iH(\tau-s)} g_k e^{iH(\tau-s)} P_c \bar{v} \rangle \right] d\tau 
\]

\[
\leq \frac{\sup(\| \hat{g}_k \|_{L^1} + \| \hat{g}_k \|_{L^1}) \| v \|_{L^1}}{(1 + |t - s|)} \int_s^{s+\frac{1}{4}} \| e^{\pm iE_0 (\tau-s)} \psi_0 \|_{L^\infty} d\tau 
\]

\[
\leq \frac{C \| v \|_{L^1}}{(1 + |t - s|)} 
\]

\( II_2 \) is estimated exactly the same way as \( I_2 \). Let us observe that the above estimates are for the case \( t > s + 1 \). Because of that we can replace the \( C/|t - s| \) term by \( C/(1 + |t - s|) \) in the \( I_1, I_2 \) and \( II_2 \) integrals.

Theorem 3.2 is now completely proven. □

The next step is to obtain estimates for \( \Omega(t,s) \) and \( T(t,s) \) in unweighted \( L^p \) spaces.

**Theorem 3.3.** Fix \( \sigma > 1 \). Assume that (H2) holds and \( \| x |^\sigma \psi_E \|_{L^\infty} < \varepsilon_1 \) (where \( \varepsilon_1 \) is the one used in Theorem 3.2). Then there exist constants \( C_2, C'_2 \) and \( C_\infty \) such that for all \( t, s \in \mathbb{R} \) the following estimates hold:

(i) \( \| \Omega(t,s) \|_{L^2 \rightarrow L^2} \leq C_2, \quad \| T(t,s) \|_{L^2 \rightarrow L^2} \leq C_2 \)

(ii) \( \| T(t,s) \|_{L^1 \rightarrow L^\infty} \leq C_\infty \frac{\log(2 + |t - s|)}{|t - s|}. \)

**Remark 3.4.** By Riesz-Thorin interpolation, from (i) and (ii), we get

\[
\| T(t,s) \|_{L^p' \rightarrow L^p} \leq C_p \frac{\log^{1-\frac{2}{p}}(2 + |t - s|)}{|t - s|^{1-\frac{2}{p}}}, \quad \text{for all} \quad 2 \leq p \leq \infty
\]

\[
\| \Omega(t,s) \|_{L^p' \rightarrow L^p} \leq C_p \frac{\log^{1-\frac{2}{p}}(2 + |t - s|)}{|t - s|^{1-\frac{2}{p}}}, \quad \text{for all} \quad 2 \leq p \leq \infty.
\]
Using the well-posedness in $H^1$ of the equation that defines the $\Omega$ propagator, see Remark 3.1, we can remove the singularity at $t = s$ if we assume that the initial data is in $H^1$:

$$
\|\Omega(t, s)\|_{L^p' \to L^p} \leq C_p \frac{\log^{1-\frac{2}{p}}(2 + |t - s|)}{(1 + |t - s|)^{1-\frac{2}{p}}}, \quad \text{for all} \quad 2 \leq p < \infty
$$

Proof of Theorem 3.3 Because of the estimate (3.2) and relation $\Omega = T + e^{-iH(t-s)}P_c$, it suffices to prove the theorem for $T(t, s)$. In what follows, we will use the notation from Theorem (3.2).

(i) To estimate the $L^2$ norm we will use duality argument to make use of cancelations.

$$
\|f(t)\|_{L^2}^2 = \langle f(t), f(t) \rangle \\
= \int_s^t \int_s^t \langle e^{-iH(t-\tau)}P_cF_1(e^{-iH(\tau-s)}P_cv), e^{-iH(t-\tau')}P_cF_1(e^{-iH(\tau'-s)}P_cv) \rangle d\tau' d\tau \\
= \int_s^t \int_s^t \langle F_1(e^{-iH(\tau-s)}P_cv), e^{-iH(\tau-\tau')}P_cF_1(e^{-iH(\tau'-s)}P_cv) \rangle d\tau' d\tau \\
= \int_s^t \int_s^t \langle \langle x \rangle^\sigma F_1(e^{-iH(\tau-s)}P_cv), \langle x \rangle^{-\sigma} e^{-iH(\tau-\tau')}P_cF_1(e^{-iH(\tau'-s)}P_cv) \rangle d\tau' d\tau \\
\leq \int_s^t \int_s^t \|F_1(e^{-iH(\tau-s)}P_cv)\|_{L^2_\alpha} \|e^{-iH(\tau-\tau')}P_cF_1(e^{-iH(\tau'-s)}P_cv)\|_{L^2_\alpha} d\tau' d\tau \\
\leq \int_s^t \|\langle x \rangle^\sigma F_1(e^{-iH(\tau-s)}P_cv)\|_{L^2} \int_s^t \frac{C}{(1 + |\tau - \tau'|) \log^2(2 + |\tau - \tau'|)} \\
\times \|\langle x \rangle^\sigma F_1(e^{-iH(\tau'-s)}P_cv)\|_{L^2} d\tau' d\tau \\
\leq C \|\langle x \rangle^\sigma F_1(e^{-iH(\tau-s)}P_cv)\|_{L^2_\alpha L^2_\alpha} \\
\times \left\| \int_s^t \frac{C}{(1 + |\tau - \tau'|) \log^2(2 + |\tau - \tau'|)} \|\langle x \rangle^\sigma F_1(e^{-iH(\tau'-s)}P_cv)\|_{L^2} d\tau' \right\|_{L^2_\alpha} \\
\leq C \|K\|_{L^1} \|\langle x \rangle^\sigma F_1(e^{-iHt}P_cv)\|_{L^2_\alpha L^2_\alpha}^2 \leq C \|v\|_{L^2}^2 < \infty
$$

At the last line, $K(t) = (1 + |t|)^{-1} \log^{-2}(2 + |\tau - \tau'|)$ and we used convolution estimate. For the term $\langle x \rangle^\sigma F_1(e^{-iHt}P_cv) = \langle x \rangle^\sigma (g_u e^{-iHt}P_cv + g_u e^{iHt}P_cv)$ we used the Kato smoothing
estimate $\|\langle x \rangle^{-\sigma} e^{-iHt} P_c v \|_{L^2_t([s,t])} \leq C \|v\|_{L^2_t}$. Similarly we have,

$$\|\tilde{f}\|_{L^2_t}^2 = \int_{s}^{t} \int_{s}^{t} \langle e^{-iH(t-\tau)} P_c Dh(a(\tau) \langle \psi_0, F_1(1) e^{-iH(\tau-s)} P_c v)\rangle, \rangle$$

$$e^{-iH(t-\tau')} P_c Dh(a(\tau) \langle \psi_0, F_1(1) e^{-iH(\tau'-s)} P_c v)\rangle d\tau' d\tau$$

$$\leq \int_{s}^{t} \int_{s}^{t} \|Dh\|_{L^2_t} \|\psi_0\|_{L^2} \|F_1(1) e^{-iH(\tau-s)} P_c v\|_{L^2}$$

$$\times \frac{C}{(1 + |\tau - \tau'| \log^2(2 + |\tau - \tau'|)} \|Dh\|_{L^2_t} \|\psi_0\|_{L^2} \|F_1(1) e^{-iH(\tau'-s)} P_c v\|_{L^2} d\tau' d\tau$$

$$\leq C \|v\|_{L^2_t}^2 < \infty$$

We will estimate $L^2$ norm of $L$ similar to $f$.

$$\|L(s)W\|_{L^2_t}^2 = \langle L(s)W, L(s)W \rangle$$

$$= \langle \int_{s}^{t} e^{-iH(t-\tau)} P_c (F_1(W(\tau)) - Dh(\psi_0, F_1(W(\tau)))) d\tau, \rangle$$

$$\int_{s}^{t} e^{-iH(t-\tau')} P_c (F_1(W(\tau')) - Dh(\psi_0, F_1(W(\tau'))))) d\tau' \rangle$$

$$\int_{s}^{t} \int_{s}^{t} \|F_1(W(\tau)) - Dh(\psi_0, F_1(W(\tau)))) \|_{L^2_t} \|F_1(W(\tau')) - Dh(\psi_0, F_1(W(\tau'))))\|_{L^2_t} d\tau' d\tau$$

$$\leq \int_{s}^{t} (\|\langle x \rangle^\sigma g_u\|_{L^\infty} + \|\langle x \rangle^\sigma g_u\|_{L^\infty}) (1 + \|Dh\|_{L^2_t} \|\psi_0\|_{L^2}) \|\langle x \rangle^{-\sigma} W\|_{L^2}$$

$$\times \int_{s}^{t} C K(1 + |\tau - \tau'| \log^2(2 + |\tau - \tau'|)) (1 + \|Dh\|_{L^2_t} \|\psi_0\|_{L^2}) \|\langle x \rangle^{-\sigma} W\|_{L^2_t} d\tau' d\tau$$

$$\leq C \|\langle x \rangle^{-\sigma} W\|_{L^2_t}^2 \int_{s}^{t} C K(1 + |\tau - \tau'| \log^2(2 + |\tau - \tau'|)) \|\langle x \rangle^{-\sigma} W\|_{L^2_t} d\tau' d\tau$$

$$\leq C \|K\|_{L^1} \|\langle x \rangle^{-\sigma} W\|_{L^2_t}^2 < \infty$$

By Theorem 3.2 (iii), $\|\langle x \rangle^{-\sigma} W\|_{L^2_t} < \infty$.

Therefore we conclude that $\|T(s,t)\|_{L^2 \rightarrow L^2} \leq C$ and $\|\Omega(s,t)\|_{L^2 \rightarrow L^2} \leq C$

(ii) Let us first investigate the short time behavior of the terms $f(t)$ and $\tilde{f}(t)$. We will assume $s < t < s + 1$: 

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\[ \|f(t)\|_{L^\infty} = \left\| \int_s^t e^{-iH(t-\tau)} P_c F_1 (e^{-iH(\tau-s)} P_c v) d\tau \right\|_{L^\infty} \]

\[ \leq \int_s^t \| e^{-iH(t-s)} \|_{L^1 \rightarrow L^\infty} \| e^{iH(\tau-s)} P_c g_u e^{-iH(\tau-s)} P_c v \|_{L^1} d\tau \]

\[ + \int_s^{s + \frac{t-s}{4}} \| e^{-iH(t+s-2\tau)} \|_{L^1 \rightarrow L^\infty} \| e^{-iH(\tau-s)} P_c g_u e^{iH(\tau-s)} P_c \tilde{v} \|_{L^1} d\tau \]

\[ + \int_s^{s + \frac{t-s}{4}} \| e^{-iH(t-s)} P_c \|_{L^1 \rightarrow L^\infty} \| g_u e^{iH(\tau-s)} P_c \tilde{v} \|_{L^1} d\tau \]

\[ + \int_t^{t - \frac{t-s}{4}} \| e^{-iH(t+s-2\tau)} \|_{L^1 \rightarrow L^\infty} \| e^{-iH(\tau-s)} P_c g_u e^{iH(\tau-s)} P_c \tilde{v} \|_{L^1} d\tau \]

\[ \leq \int_s^t \frac{C}{|t-s|} \sup \| \widehat{g_u} \|_{L^1} \| v \|_{L^1} d\tau + \int_s^{s + \frac{t-s}{4}} \frac{C}{|t+s-2\tau|} \sup \| \widehat{g_u} \|_{L^1} \| v \|_{L^1} d\tau \]

\[ + \int_s^{s + \frac{t-s}{4}} \frac{C}{|t-\tau|} \| g_u \|_{L^1} \left\| \| v \|_{L^1} d\tau + \int_t^{t - \frac{t-s}{4}} \frac{C}{|t+s-2\tau|} \sup \| \widehat{g_u} \|_{L^1} \| v \|_{L^1} d\tau \]

\[ \leq C \left( \| g_u \|_{L^p_p = \infty} + \sup (\| \widehat{g_u} \|_{L^1} + \| \widehat{g_u} \|_{L^1}) \right) \| v \|_{L^1} \]

Using \( H^2 \hookrightarrow L^\infty \) and the fact that \( e^{-iHt} \) is unitary on \( H^2 \) we get:

\[ \| \hat{f}(t) \|_{L^\infty} \leq \int_s^t C \| Dh \|_{C \rightarrow H^2} \| \{ \psi_0, F_1 (e^{-iH(\tau-s)} P_c v) \} \| d\tau \]

\[ \leq \int_s^t \| Dh \| \| \{ e^{iH(\tau-s)} \psi_0, e^{iH(\tau-s)} g_u e^{-iH(\tau-s)} P_c v \} \| + \| \{ e^{-iH(\tau-s)} \psi_0, e^{-iH(\tau-s)} g_u e^{iH(\tau-s)} P_c \tilde{v} \} \| d\tau \]

\[ \leq \int_s^t \| Dh \|_{H^2} \| e^{iH(\tau-s)} \psi_0 \|_{L^\infty} \| e^{iH(\tau-s)} g_u e^{-iH(\tau-s)} P_c v \|_{L^1} d\tau \]

\[ + \int_s^t \| Dh \|_{H^2} \| e^{-iH(\tau-s)} \psi_0 \|_{L^\infty} \| e^{-iH(\tau-s)} g_u e^{iH(\tau-s)} P_c \tilde{v} \|_{L^1} d\tau \]

\[ \leq C \| Dh \| \sup (\| \widehat{g_u} \|_{L^1} + \| \widehat{g_u} \|_{L^1}) \| \psi_0 \|_{H^2} |t-s| \| v \|_{L^1} \]

Now let us investigate the long time bevaviour of the terms \( f(t) \) and \( \hat{f}(t) \). We will assume \( t > s + 1 \) and separate these terms into four parts as follows:
\[ f(t) = \int_{s}^{s+1/4} I_1 + \int_{s+1/4}^{t+1/s} I_2 + \int_{t+1/s}^{t-1/4} I_3 + \int_{t-1/4}^{t} I_4 \]

and

\[ \tilde{f}(t) = \int_{s}^{s+1} I_1 + \int_{s+1}^{t+1/s} I_2 + \int_{t+1/s}^{t-1} I_3 + \int_{t-1}^{t} I_4 \]

\[
\|I_1\|_{L^\infty} \leq \int_{s}^{s+1/4} \left\| e^{-iH(t-s)} P_c e^{iH(\tau-s)} P_c g_\tilde{u} e^{-iH(\tau-s)} P_c \bar{v} \right\|_{L^\infty} d\tau \\
+ \int_{s}^{s+1/4} \left\| e^{-iH(t+s-2\tau)} e^{-iH(\tau-s)} P_c g_\tilde{u} \right\|_{L^\infty} d\tau \\
\leq \int_{s}^{s+1/4} \frac{C}{|t-s|} \left\| e^{iH(\tau-s)} P_c g_\tilde{u} \left( e^{-iH(\tau-s)} P_c \bar{v} \right) \right\|_{L^1} d\tau \\
+ \int_{s}^{s+1/4} \frac{C}{|t+s-2\tau|} \left\| e^{iH(\tau-s)} P_c g_\tilde{u} \left( e^{-iH(\tau-s)} P_c \bar{v} \right) \right\|_{L^1} d\tau \\
\leq \int_{s}^{s+1/4} C \left( \frac{1}{|t-s|} + \frac{1}{|t-s-1|} \right) \sup \left( \|g_\tilde{u}\|_{L^1} + \|\tilde{g}_u\|_{L^1} \right) \|v\|_{L^1} d\tau \\
\leq C \|v\|_{L^1_{|t-s|}}
\]

For the second integral we have

\[
\|I_2\|_{L^\infty} \leq \int_{s+1/4}^{t+1/s} \left\| e^{-iH(t-\tau)} P_c (g_u e^{-iH(\tau-s)} P_c v + g_\bar{u} e^{iH(\tau-s)} P_c \bar{v}) \right\|_{L^\infty} d\tau \\
\leq \int_{s+1/4}^{t+1/s} C \left( \|g_u\|_{L^p_{|t-\tau|}} + \|g_\bar{u}\|_{L^p_{|t-\tau|}} \right) \left\| e^{-iH(\tau-s)} P_c v \right\|_{L^1} d\tau \\
\leq \frac{C}{|t-s|} \int_{s+1/4}^{t+1/s} \left( \|g_u\|_{L^p_{|\tau-s|}} + \|g_\bar{u}\|_{L^p_{|\tau-s|}} \right) \left\| e^{-iH(\tau-s)} P_c v \right\|_{L^\infty} d\tau \\
\leq C \left( \|g_u\|_{L^p_{|\tau-s|}} + \|g_\bar{u}\|_{L^p_{|\tau-s|}} \right) \int_{s+1}^{t+1/s} \frac{\|v\|_{L^1}}{|\tau-s|} d\tau \\
\leq C \left( \|g_u\|_{L^p_{|\tau-s|}} + \|g_\bar{u}\|_{L^p_{|\tau-s|}} \right) \frac{\|v\|_{L^1_{|t-s|}}}{|t-s|} \log(2 + |t-s|)
\]

\( I_3 \) and \( I_4 \) are estimated exactly the same way as \( I_2 \) and \( I_1 \), respectively.

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In a similar way we will estimate $\tilde{f}$:

$$\|II_1\|_{L^\infty} \leq \int_s^{s+1} \|e^{-iH(t-\tau)} P_c Dh|_{a(\tau)}\times \langle \psi_0, (g_u e^{-iH(\tau-s)} P_c v + g_u e^{iH(\tau-s)} P_c \bar{v}) \rangle \|_{L^\infty} d\tau$$

$$\leq \int_s^{s+1} \frac{C}{|t-\tau|} \|Dh\|_{L^1}\left[\langle e^{iH(\tau-s)} \psi_0, e^{iH(\tau-s)} g_u e^{-iH(\tau-s)} P_c v \rangle + \langle e^{-iH(\tau-s)} \psi_0, e^{-iH(\tau-s)} g_u e^{iH(\tau-s)} P_c \bar{v} \rangle \right] d\tau$$

$$\leq \int_s^{s+1} \frac{C}{|t-s|} \|Dh\|_{L^1}\|e^{-iH(\tau-s)} \psi_0\|_{L^p} \times \|e^{-iH(\tau-s)} g_u e^{iH(\tau-s)} P_c \bar{v}\|_{L^1} d\tau$$

$$\leq \frac{C\|v\|_{L^1}}{|t-s|} \|Dh\|_{L^1} \sup(\|\bar{g}_u\|_{L^1} + \|\bar{g}_u\|_{L^1}) \int_s^{s+1} \|e^{-iH(\tau-s)} \psi_0\|_{L^p} d\tau$$

$$\leq \frac{C\|v\|_{L^1}}{|t-s|} \|Dh\|_{L^1} \sup(\|\bar{g}_u\|_{L^1} + \|\bar{g}_u\|_{L^1}) \|\psi_0\|_{H^2} \leq \frac{C\|v\|_{L^1}}{|t-s|}$$

For the second integral we have

$$\|II_2\|_{L^\infty} \leq \int_s^{s+1} \frac{C}{|t-\tau|^{N(1/2 - 1/p)}} \|Dh|_{a(\tau)}\|L^1\|\|\psi_0\|_{L^2}(\|g_u\|_{L^{2^p}}^{2^p} + \|g_u\|_{L^{2^p}}^{2^p})\|e^{-iH(\tau-s)} P_c v\|_{L^p} d\tau$$

$$\leq \int_s^{s+1} \frac{C}{|t-s|^{N(1/2 - 1/p)}} \|Dh|_{a(\tau)}\|L^1\|\|\psi_0\|_{L^2}(\|g_u\|_{L^{2^p}}^{2^p} + \|g_u\|_{L^{2^p}}^{2^p}) \|v\|_{L^1} d\tau$$

$$\leq C\frac{\|v\|_{L^1}}{|t-s|} \|Dh|_{a(\tau)}\|L^1\|\|\psi_0\|_{L^2}(\|g_u\|_{L^{2^p}}^{2^p} + \|g_u\|_{L^{2^p}}^{2^p}) \log(2 + |t-s|)$$

$$\leq C\frac{\|v\|_{L^1}}{|t-s|} \log(2 + |t-s|)$$

$II_3$ term is estimated exactly same way as $II_2$ term. Now we estimate $II_4$ term similar to estimating the short time behavior of $\tilde{f}$:
Now it remains to show that $L(s)W$ is bounded in $L^p$. To estimate this term we will iterate the equation for $W$ once, i.e. we will plug in (3.12) in the integral expression (3.6) for $L(s)$. This will help us get rid of the singularity around $\tau = t$ by using certain cancellations.

\[
\begin{align*}
\|I_{4}\|_{L^\infty} & \leq \int_{t-1}^{t} C \|Dh\|_{C-H^2} \langle \psi_0, F_1(e^{-iH(\tau-s)} P_c v) \rangle d\tau \\
& \leq \int_{t-1}^{t} \|Dh\| \|\langle \psi_0, g_u e^{-iH(\tau-s)} P_c v \rangle\| + \|\langle \psi_0, g_u e^{-iH(\tau-s)} P_c \bar{v} \rangle\| d\tau \\
& \leq \int_{t-1}^{t} \|Dh\|_{H^2} \|\psi_0\|_{L^2} (\|g_u\|_{L^2} + \|g_u\|_{L^2}) \|e^{-iH(\tau-s)} P_c v\|_{L^\infty} d\tau \\
& \quad + \int_{t-1}^{t} \|Dh\|_{H^2} \|\psi_0\|_{L^2} (\|g_u\|_{L^2} + \|g_u\|_{L^2}) \|v\|_{L^1} \frac{d\tau}{|t-s|} \\
& \leq \frac{C \|Dh\| \|\psi_0\|_{L^2} (\|g_u\|_{L^2} + \|g_u\|_{L^2}) \|v\|_{L^1}}{|t-s|} \\
\end{align*}
\]

All the terms will be either of the following forms

\[
\begin{align*}
L_1 & = \int_{s}^{t} e^{-iH(t-\tau)} P_c g_u \int_{s}^{\tau} e^{-iH(\tau'-s)} P_c X(\tau') d\tau' d\tau \\
L_2 & = \int_{s}^{t} e^{-iH(t-\tau)} P_c g_u \int_{s}^{\tau} e^{iH(\tau'-\tau)} P_c \overline{X(\tau')} d\tau' d\tau \\
\end{align*}
\]
\[
\begin{align*}
\tilde{L}_1 &= \int_s^t e^{-iH(t-\tau)}P_e g_u \int_s^\tau e^{-iH(\tau-\tau')}P_e Dh(\psi_0, X(\tau'))d\tau'd\tau \\
\tilde{L}_2 &= \int_s^t e^{-iH(t-\tau)}P_e g_u \int_s^\tau e^{iH(\tau-\tau')}P_e\overline{Dh(\psi_0, X(\tau'))}d\tau'd\tau
\end{align*}
\]
where \(X(\tau') = g_u e^{-iH(\tau'-s)} P_e v, g_u e^{iH(\tau'-s)} P_e \bar{v}, g_u W(\tau'), g_u \overline{W(\tau')}\)

First we will add \(e^{iH(t-\tau)}\) and \(e^{-iH(t-\tau)}\) terms after \(g_u\) and \(g_u\) then we will estimate the terms exactly the same way as we estimated \(f\) and \(\bar{f}\).

\[
\begin{align*}
L_1 &= \int_s^t e^{-iH(t-\tau)}P_e g_u e^{iH(t-\tau)} \int_s^\tau e^{-iH(\tau-\tau')}P_e X(\tau')d\tau'd\tau \quad (3.13) \\
L_2 &= \int_s^t e^{-iH(t-\tau)}P_e g_u e^{iH(t-\tau)} \int_s^\tau e^{-iH(\tau-\tau')}P_e \overline{X(\tau')}d\tau'd\tau \quad (3.14) \\
\tilde{L}_1 &= \int_s^t e^{-iH(t-\tau)}P_e g_u e^{iH(t-\tau)} \int_s^\tau e^{-iH(\tau-\tau')}P_e Dh(\psi_0, X(\tau'))d\tau'd\tau \quad (3.15) \\
\tilde{L}_2 &= \int_s^t e^{-iH(t-\tau)}P_e g_u e^{iH(t-\tau)} \int_s^\tau e^{-iH(\tau-\tau')}P_e\overline{Dh(\psi_0, X(\tau'))}d\tau'd\tau \quad (3.16)
\end{align*}
\]

- For \(X(\tau') = g_u e^{-iH(\tau'-s)} P_e v\) and \(s \leq t \leq s + 1\) we have

\[
\begin{align*}
\|L_1\|_{L^\infty} &\leq \int_s^t \|e^{-iH(t-\tau)}P_e g_u e^{iH(t-\tau)}\|_{L^\infty \rightarrow L^\infty}d\tau' \\
&\quad + \int_s^\tau \|e^{-iH(\tau-s)}P_e\|_{L^1 \rightarrow L^\infty} \|e^{iH(\tau'-s)}P_e g_u e^{-iH(\tau'-s)}P_e v\|_{L^1}d\tau'd\tau \\
&\leq \int_s^t \|\tilde{g}_u\|_{L^1} \int_s^\tau \frac{C}{|t-s|} \|\tilde{g}_u\|_{L^1} \|v\|_{L^1} d\tau'd\tau \\
&\leq C \|v\|_{L^1}|t-s|.
\end{align*}
\]
\[ \|L_2\|_{L^\infty} \leq \int_s^{t-\frac{t-s}{4}} \|e^{-iH(t-\tau)} P_c \|_{L^1 \rightarrow L^\infty} \|g_\bar u \| \int_s^\tau e^{-iH(\tau-s)} e^{iH(\tau'-s)} P_c g_\bar u e^{iH(\tau'-s)} P_c \bar v \|_{L^1} d\tau' d\tau \\
+ \int_{t-\frac{t-s}{4}}^t \|e^{-iH(t-\tau)} P_c g_\bar u e^{iH(\tau'-s)} \|_{L^\infty \rightarrow L^\infty} \times \int_s^\tau \|e^{-iH(t+s-2\tau)} P_c \|_{L^1 \rightarrow L^\infty} \|e^{-iH(\tau'-s)} P_c g_\bar u e^{iH(\tau'-s)} P_c \bar v \|_{L^1} d\tau' d\tau \\
\leq \int_s^{t-\frac{t-s}{4}} \int_s^\tau C \|g_\bar u \|_{L^\infty} \|g_\bar u \|_{L^1} \|v\|_{L^1} d\tau' d\tau \\
+ \int_{t-\frac{t-s}{4}}^t \|g_\bar u \|_{L^1} \int_s^\tau \frac{C}{|t-s|} \|g_\bar u \|_{L^1} \|v\|_{L^1} d\tau' d\tau \leq C \|v\|_{L^1} |t-s| \\
\]

\[ \hat{L}_1 \text{ and } \hat{L}_2 \text{ are estimated in the same way as } II_4, \text{ for example:} \]

\[ \|\hat{L}_1\|_{L^\infty} \leq \int_s^t \|e^{-iH(t-\tau)} P_c g_\bar u e^{iH(\tau'-s)} \|_{L^\infty \rightarrow L^\infty} \int_s^\tau C\|Dh\|_{C \rightarrow H^2} \|\psi_0, g_\bar u e^{-iH(\tau'-s)} P_c \bar v\|_{L^1} d\tau' d\tau \\
\leq \int_s^t \|\hat{g}_u \|_{L^1} \int_s^\tau C\|e^{iH(\tau'-s)} \psi_0, e^{iH(\tau'-s)} g_\bar u e^{-iH(\tau'-s)} P_c \bar v\|_{L^1} d\tau' d\tau \\
\leq \int_s^t \|\hat{g}_u \|_{L^1} \int_s^\tau C\|e^{iH(\tau'-s)} \psi_0\|_{L^\infty} \|e^{iH(\tau'-s)} g_\bar u e^{-iH(\tau'-s)} P_c \bar v\|_{L^1} d\tau' d\tau \\
\leq C|t-s|^2 \|v\|_{L^1}. \]

- For \( X(\tau') = g_\bar u R_a e^{iH(\tau'-s)} P_c \bar v \) and \( s \leq t \leq s+1 \) we first change the order of integration then split and use the Fourier multiplier type estimate:
\[ \|L_1\|_{L^\infty} \leq \int_s^{s+\frac{t-s}{4}} \int_{\tau'}^t \|e^{-iH(t-\tau')} P_c g_\hat{u} e^{iH(t-\tau)} e^{-iH(t+s-2\tau')} P_c e^{-iH(\tau'-s)} P_c g_\hat{u} e^{iH(\tau'-s)} P_c \|_{L^\infty} d\tau d\tau' \]

\[ + \int_s^{s+\frac{t-s}{4}} \int_{\tau'}^t \|e^{-iH(t-s)} P_c g_\hat{u} e^{iH(t-\tau)} \|_{L^\infty \to L^\infty} \]

\[ \|e^{-iH(t-\tau')} \|_{L^1 \to L^\infty} \|g_\hat{u} e^{iH(\tau'-s)} P_c \|_{L^1} d\tau d\tau' \]

\[ \leq \int_s^{s+\frac{t-s}{4}} \int_{\tau'}^t \|\hat{g}_\hat{u}\|_{L^1} \frac{C \|v\|_{L^1}}{t + s - 2\tau'} \|\hat{g}_\hat{u}\|_{L^1} d\tau d\tau' \]

\[ + \int_s^{s+\frac{t-s}{4}} \int_{\tau'}^t \|\hat{g}_\hat{u}\|_{L^1} \frac{C}{t - \tau'} \|g_\hat{u}\|_{L^\infty} \|v\|_{L^1} d\tau d\tau' \]

\[ \leq C \|v\|_{L^1} |t - s| \]
\[ \|L_2\|_{L^\infty} \leq \int_{t-s}^{t} \frac{C}{t-\tau} \left[ \|g_u\|_{L^1} \int_{s}^{s+\tau-s} \frac{C}{2\tau' - \tau - s} \|\tilde{g}_u\|_{L^1} \|v\|_{L^1} \right] d\tau' + \int_{s}^{t} \frac{C}{\|g_u\|_{L^2} \|v\|_{L^1}} \left[ \|\tilde{g}_u\|_{L^1} \|v\|_{L^1} \right] d\tau' + \int_{t-s}^{t} \frac{C}{t-2\tau + 2\tau' - s} \|\tilde{g}_u\|_{L^1} \|v\|_{L^1} \right] d\tau' \]

\[ \leq C \|v\|_{L^1} \]

\( \tilde{L}_1 \) and \( \tilde{L}_2 \) are estimated similar to the previous case.

- For \( X(\tau') = g_u W(\tau') \) or \( g_{\bar{u}} \overline{W(\tau')} \) and \( s \leq t \leq s + 1 \) we will change the order of the integration:
\[ \|L_1\|_{L^\infty} \leq \int_s^t \int_{\tau'}^t \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau')}\|_{L^\infty \to L^\infty} \|e^{-iH(t-\tau')} P_c\|_{L^1 \to L^\infty} \|g_u W(\tau')\|_{L^1} d\tau d\tau' \]

\[ \leq \int_s^t \int_{\tau'}^t \|\tilde{g}_u\|_{L^1} \frac{C}{|t-\tau'|} \|\langle x\rangle^\sigma g_u\|_{L^{2,\sigma}} \|W\|_{L^{2,\sigma}} d\tau d\tau' \]

\[ \leq \int_s^t \|g_u\|_{L^1} C \|\langle x\rangle^\sigma g_u\|_{L^2} \frac{C\|v\|_{L^1}}{1 + |\tau' - \tau|} d\tau' \]

\[ \leq C \|v\|_{L^1} \]

\[ \|L_2\|_{L^\infty} \leq \int_s^t \int_{\tau'}^t \|e^{-iH(t-\tau)} P_c\|_{L^1 \to L^\infty} \|g_u e^{-iH(t-\tau')} P_c g_u W(\tau') \|_{L^1} d\tau d\tau' \]

\[ + \int_s^t \int_{\tau'}^t \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau')}\|_{L^\infty \to L^\infty} \|e^{-iH(t+\tau' - 2\tau)} P_c g_u W(\tau')\|_{L^\infty} d\tau d\tau' \]

\[ \leq \int_s^t \int_{\tau'}^t \|\tilde{g}_u\|_{L^1} \frac{C}{|t-\tau'|} \|g_u e^{-iH(t-\tau')}\|_{L^2 \to L^2} \|g_u W(\tau')\|_{L^2} d\tau d\tau' \]

\[ + \int_s^t \int_{\tau'}^t \|\tilde{g}_u\|_{L^1} \frac{C}{|t+\tau' - 2\tau|} \|g_u W(\tau')\|_{L^2} d\tau d\tau' \]

\[ \leq \int_s^t \|g_u\|_{L^2} \|\langle x\rangle^\sigma g_u\|_{L^\infty} \|W(\tau')\|_{L^{2,\sigma}} d\tau' \]

\[ + \int_s^t \|\tilde{g}_u\|_{L^1} \|\langle x\rangle^\sigma g_u\|_{L^\infty} \|W(\tau')\|_{L^{2,\sigma}} d\tau' \]

\[ \leq C \|v\|_{L^1} \]

\[ \|\tilde{L}_1\|_{L^\infty} \leq \int_s^t \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau')}\|_{L^\infty \to L^\infty} \int_{\tau'}^\sigma \|e^{-iH(t-\tau')} P_c D_h \langle \psi_0, g_u W(\tau') \rangle\|_{L^\infty} d\tau d\tau' \]

\[ \leq \int_s^t \|\tilde{g}_u\|_{L^1} \int_s^\sigma \|D_h\| \|\psi_0\|_{L^2} \|\langle x\rangle^\sigma g_u\|_{L^\infty} \|W(\tau')\|_{L^{2,\sigma}} d\tau' d\tau \]

\[ \leq \int_s^t \|\tilde{g}_u\|_{L^1} \int_s^\sigma \|D_h\| \|\psi_0\|_{L^2} \|\langle x\rangle^\sigma g_u\|_{L^\infty} \frac{C}{1 + |\tau' - \tau|} d\tau' d\tau \]

\[ \leq C \|v\|_{L^1} \]
\( \tilde{L}_2 \) is similar to \( \tilde{L}_1 \).

Now we will investigate the long time behavior of the operator \( L(s) \) for \( t > s + 1 \):

\[
L(s)W(t) = \int_{s}^{t-\frac{1}{16}} \cdots + \int_{t-\frac{1}{16}}^{t} \cdots
\]

\( L_3 \) is estimated similar to the previous case. Again we will concentrate on the \( L_4 \) term.

\[
\int_{t-\frac{1}{16}}^{t} e^{-iH(t-\tau)} P_c F_1(W(\tau)) d\tau
\]

\[
= \int_{t-\frac{1}{16}}^{t} e^{-iH(t-\tau)} P_c g_u \left[ \int_{s}^{\tau} e^{-iH(\tau-\tau')} P_c F_1(e^{-iH(\tau'-s)} P_c v) + Dh(\psi_0, F_1(e^{-iH(\tau'-s)} P_c v)) d\tau' \right. \\
+ \int_{s}^{\tau} e^{-iH(\tau-\tau')} P_c F_1(W(\tau')) + Dh(\psi_0, F_1(W(\tau'))) d\tau' \left. \right] d\tau \\
+ \int_{t-\frac{1}{16}}^{t} \| e^{-iH(t-\tau)} P_c g_u \left[ \int_{s}^{\tau} e^{iH(\tau-\tau')} P_c F_1(e^{-iH(\tau'-s)} P_c v) + Dh(\psi_0, F_1(e^{-iH(\tau'-s)} P_c v)) d\tau' \right. \\
+ \int_{s}^{\tau} e^{iH(\tau-\tau')} P_c F_1(W(\tau')) + Dh(\psi_0, F_1(W(\tau'))) d\tau' \left. \right] d\tau
\]

Again we will add \( e^{iH(t-\tau)} \) and \( e^{-iH(t-\tau)} \) terms after \( g_u \) and \( g_\bar{u} \). Then all the terms will be similar to \( L_1, L_2, (3.13) - (3.14) \) respectively. After separating the the inside integrals into pieces, we will estimate short time step integrals exactly the same way we did short time behavior by using Fourier multiplier estimate, and the other integrals will be estimated using the usual norms.
For $X(\tau') = g_u e^{-iH(\tau'-s)} P_c v$ we have

$$
\|L_1\|_{L^\infty} \leq \int_{t - \frac{1}{16}}^{t} \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)} \|_{L^\infty} \rightarrow L^\infty
$$

$$
\left[ \int_{s}^{s + \frac{1}{4}} \|e^{-iH(t-s)} P_c \|_{L^1 \rightarrow L^\infty} \|e^{iH(\tau'-s)} P_c g_u e^{-iH(\tau'-s)} P_c v \|_{L^1} d\tau' \right]
$$

$$
+ \int_{s + \frac{1}{4}}^{t - \frac{1}{4}} \|e^{-iH(t-\tau')} P_c \|_{L^1 \rightarrow L^\infty} \|g_u e^{-iH(\tau'-s)} P_c v \|_{L^\infty} d\tau'
$$

$$
+ \int_{t - \frac{1}{4}}^{t} \|e^{-iH(t-2\tau')} P_c g_u e^{iH(t-2\tau')} \|_{L^1 \rightarrow L^\infty} \|e^{iH(t+s-2\tau')} P_c v \|_{L^1} d\tau'
$$

$$
\leq C \frac{\|v\|_{L^1}}{|t-s|} \log(2 + |t-s|)
$$

$$
\|L_2\|_{L^\infty} \leq \int_{t - \frac{1}{16}}^{t} \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)} \|_{L^\infty} \rightarrow L^\infty
$$

$$
\times \left[ \int_{s}^{s + \frac{1}{4}} \|e^{-iH(t+s-2\tau)} P_c \|_{L^1 \rightarrow L^\infty} \|e^{-iH(\tau'-s)} P_c g_u e^{iH(\tau'-s)} P_c \bar{v} \|_{L^1} d\tau' \right]
$$

$$
+ \int_{s + \frac{1}{4}}^{t - \frac{1}{4}} \|e^{-iH(t-2\tau'+\tau')} P_c \|_{L^1 \rightarrow L^\infty} \|g_u e^{iH(\tau'-s)} P_c \bar{v} \|_{L^1} d\tau'
$$

$$
+ \int_{t - \frac{1}{4}}^{t} \|e^{-iH(t+\tau'-2\tau)} P_c g_u e^{iH(t+\tau'-2\tau)} \|_{L^1 \rightarrow L^\infty} \|e^{iH(t+s-2\tau)} P_c \bar{v} \|_{L^1} d\tau'
$$

$$
\leq C \frac{\|v\|_{L^1}}{|t-s|} \log(2 + |t-s|)
$$
\[\|L_1\|_{L^\infty} \leq \int_{t-\frac{1}{16}}^{-t} \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)}\|_{L^\infty} d\tau \]

\[
\leq \int_{t-\frac{1}{16}}^{-t} \left[ \int_{s}^{s+\frac{1}{4}} \|e^{-iH(t-\tau')} P_c\|_{L^1} \|\langle \tilde{g}_u, e^{iH(t-s')} P_c \psi_0, e^{iH(t-\tau)} P_c g_u e^{-iH(t-s')} P_c v \rangle\|_{L^\infty} d\tau' \right] d\tau \\
+ \int_{s+\frac{1}{4}}^{t} \|e^{-iH(t-\tau')} P_c\|_{L^2} \|Dh\| \|\langle \psi_0, g_u e^{-iH(t-s')} P_c v \rangle\|_{L^\infty} d\tau' \\
\leq \int_{t-\frac{1}{16}}^{-t} \left[ \int_{s}^{s+\frac{1}{4}} \frac{C}{|t-s|} \|\tilde{g}_u\|_{L^1} \|e^{iH(t-s')} P_c \psi_0\|_{L^\infty} \|e^{iH(t-s')} P_c g_u e^{-iH(t-s')} P_c v\|_{L^1} d\tau' \right] d\tau \\
+ \int_{s+\frac{1}{4}}^{t} \|Dh\| \|\psi_0\|_{L^1} \|g_u e^{-iH(t-s')} P_c v\|_{L^\infty} d\tau' \\
\leq \frac{C}{|t-s|} \log(2 + |t-s|)
\]

\[\hat{L}_2\] is similar to \(L_1\).

- For \(X(t') = g_u e^{iH(t-s')} P_c \tilde{v}\) we have

\[\|L_1\|_{L^\infty} \leq \int_{t-\frac{1}{16}}^{-t} \|e^{-iH(t-\tau)} P_c g_u e^{iH(t-\tau)}\|_{L^\infty} d\tau \]

\[
\leq \int_{t-\frac{1}{16}}^{-t} \left[ \int_{s}^{s+\frac{1}{4}} \|e^{-iH(t+s-2\tau')} P_c\|_{L^1} \|e^{-iH(t-s')} P_c g_u e^{iH(t-s')} P_c \tilde{v}\|_{L^1} d\tau' \right] d\tau \\
+ \int_{s+\frac{1}{4}}^{t} \|e^{-iH(t-\tau')} P_c\|_{L^1} \|g_u e^{iH(t-s')} P_c \tilde{v}\|_{L^1} d\tau' \\
+ \int_{s}^{s+\frac{1}{4}} \|e^{-iH(t-\tau')} P_c g_u e^{iH(t-s')} \|_{L^1} \|e^{-iH(t+s-2\tau')} P_c \tilde{v}\|_{L^1} \|e^{iH(t-s')} P_c v\|_{L^\infty} d\tau' \\
\leq \int_{t-\frac{1}{16}}^{-t} \left[ \int_{s}^{s+\frac{1}{4}} \frac{C}{|t+s-2\tau'|} \|\tilde{g}_u\|_{L^1} d\tau' \right] \frac{C}{|t-s|} \log(2 + |t-s|) d\tau' \\
+ \int_{s+\frac{1}{4}}^{t} \frac{C}{|t-s|} \log(2 + |t-s|) d\tau'
\]
\[ \|L_2\|_{L^\infty} \leq \int_{t-\frac{1}{16}}^{t} \|e^{-iH(t-\tau)} P.cg_\alpha e^{iH(t-\tau)}\|_{L^\infty \rightarrow L^\infty} \times \left[ \int_{s}^{s+\frac{1}{4}} \|e^{-iH(t-2\tau+2\tau'-s)} e^{iH(\tau'-s)} P.cg_\alpha e^{-iH(\tau'-s)} P_cv\|_{L^\infty} d\tau' \right. \\
\left. + \int_{t-\frac{1}{4}}^{t-\frac{1}{16}} \|e^{-iH(t-2\tau+\tau')} P_c\|_{L^1 \rightarrow L^1} \|g_\alpha e^{-iH(\tau'-s)} P_cv\|_{L^1} d\tau' \\
+ \int_{t-\frac{1}{4}}^{t} \|e^{-iH(t+\tau'-2\tau)} P_cg_\alpha e^{iH(t+\tau'-2\tau)} e^{-iH(t-2\tau+2\tau'-s)} P_cv\|_{L^\infty} d\tau' \int_{t-\frac{1}{4}}^{t} d\tau \right] \\
\leq \int_{t-\frac{1}{16}}^{t} \|\hat{g}_\alpha\|_{L^1} \left[ \int_{s}^{s+\frac{1}{4}} \frac{C}{|t-2\tau+2\tau'-s|} \|\hat{g}_\alpha\|_{L^1} \|v\|_{L^1} d\tau' \right. \\
+ \int_{s+\frac{1}{4}}^{\tau} \frac{C\|g_\alpha\|_{L^1}}{|t-2\tau+\tau'|} \|v\|_{L^1} d\tau' + \int_{t-\frac{1}{4}}^{\tau} \frac{C\|\hat{g}_\alpha\|_{L^1} \|v\|_{L^1}}{|t-2\tau+2\tau'-s|} d\tau \\
\leq \frac{C\|v\|_{L^1}}{|t-s|} \log(2 + |t-s|) \]

- \(L_1, L_2\) terms corresponding to \(X(\tau') = g_\alpha W(\tau')\) and \(g_\alpha \overline{W(\tau')}\). For \(L_1\) term we will separate the integral into three parts. For the first part we will use integrability of \(\|W\|_{L^2_{\sigma}}\) on short time intervals. For the last part we will change the order of the integration.
Similar to $L_1$ we will split $L_2$ in three integrals. In the first and last we use Fourier multiplier type estimate and in the last one we change the order of integration.

\[
\|L_2\|_{L^\infty} \leq \int_{t - \frac{1}{16}}^{t} \|e^{-iH(t-\tau)}P_c g\bar{u}e^{iH(t-\tau)}\|_{L^\infty \rightarrow L^\infty} \\
\times \int_{t - \frac{1}{4}}^{t - \frac{1}{16}} \|e^{-iH(t+t' - 2\tau)}P_c\|_{L^1 \rightarrow L^\infty} \|g_{a}W(\tau')\|_{L^1} d\tau' d\tau \\
+ \int_{t - \frac{1}{16}}^{t} \int_{s}^{t} \|e^{-iH(t-\tau)}P_c\|_{L^1 \rightarrow L^\infty} \|g\bar{u}\|_{L^2} \\
\times \|e^{iH(\tau-\tau')}P_c\|_{L^2 \rightarrow L^2} \|(x)^{\sigma} g_u\|_{L^\infty} \|W(\tau')\|_{L^2} d\tau' d\tau \\
+ \int_{t - \frac{1}{4}}^{t} \int_{t - \frac{t' - \tau'}{4}}^{t} \|e^{-iH(t+t' - 2\tau)}P_c g\bar{u}e^{iH(t-\tau)}\|_{L^\infty \rightarrow L^\infty} \\
\times \|e^{-iH(t+t' - 2\tau)}P_c\|_{L^1 \rightarrow L^\infty} \|g_{a}W(\tau')\|_{L^1} d\tau d\tau' \\
\leq \int_{t - \frac{1}{16}}^{t} \int_{s}^{t - \frac{1}{4}} \frac{C\|\tilde{g}_{\bar{u}}\|_{L^1} \|(x)^{\sigma} g_u\|_{L^2} \|v\|_{L^1}}{|t + \tau' - 2\tau| (1 + |\tau' - s|)} d\tau' d\tau \\
+ \int_{t - \frac{1}{16}}^{t} \int_{t - \frac{1}{4}}^{t - \frac{1}{16}} \frac{C\|v\|_{L^1}}{|t - \tau| (1 + |\tau' - s|)} d\tau' d\tau \\
+ \int_{t - \frac{1}{4}}^{t} \int_{t - \frac{t' - \tau'}{4}}^{t - \frac{1}{16}} \frac{C\|v\|_{L^1}}{|t + \tau' - 2\tau| (1 + |\tau' - s|)} d\tau d\tau' \\
\leq \frac{C\|v\|_{L^1}}{|t - s|} \log(2 + |t - s|)
\]

$L_1$ and $L_2$ terms are estimated exactly as $L_1$ and $L_2$, respectively.

Now combining all the above estimates we get

\[
\|W(t)\|_{L^\infty} \leq \frac{C\|v\|_{L^1}}{1 + |t - s|} \log(2 + |t - s|).
\]

This finishes the proof of the Theorem. □
Chapter 4


We are now ready to analyze the nonlinear dynamics and to prove our main result:

**Theorem 4.1.** If hypothesis (1.3), (1.4), (H1) and (H2) hold then there exists $\varepsilon_0 > 0$ such that for all initial conditions $u_0(x)$ satisfying

$$\max\{\|u_0\|_{L^{(2\alpha_2 + 2)'}}, \|u_0\|_{H^1}\} \leq \varepsilon_0$$

the initial value problem (1.1)-(1.2) is globally well-posed in $H^1$, and the solution decomposes into a radiative part and a part that asymptotically converges to a ground state.

More precisely, there exists a $C^1$ function $a : \mathbb{R} \mapsto \mathbb{C}$ such that, for all $t \in \mathbb{R}$ we have:

$$u(t, x) = a(t)\psi_0(x) + h(a(t)) + \eta(t, x)$$

where $\psi_E(t)$ is on the central manifold (i.e it is a ground state) and $\eta(t, x) \in \mathcal{H}_{a(t)}$, see Proposition 2.1 and Lemma 2.5. Moreover, there exist the ground states $\psi_{E_{\pm \infty}}$ and the $C^1$ function $\tilde{\theta} : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{|t| \to \infty} \theta(t) = 0$ and:

$$\lim_{t \to \pm \infty} \|\psi_E(t) - e^{-itE_{\pm \theta(t)}}\psi_{E_{\pm \infty}}\|_{H^2 \cap L^2_\sigma} = 0, \ \forall \sigma \in \mathbb{R} \quad (4.1)$$

while $\eta$ satisfies the following decay estimates.
If $\alpha_1 > 3/2$ and $p \leq 2\alpha_2 + 2$ then there is a constant $C(p) > 0$ such that

\[
\|\eta(t)\|_{L^p} \leq \begin{cases} 
C(p)\frac{\log^{1-2/p}(2+|t|)}{(1+|t|)^{1-2/p}} & \text{if } \alpha_1 \geq 2 \text{ or } \alpha_1 < 2 \text{ and } p < \frac{2}{2-\alpha_1}, \\
C(p)\frac{\log^{\alpha_1}(2+|t|)}{(1+|t|)^{\alpha_1-1}} & \text{if } \alpha_1 < 2 \text{ and } p \geq \frac{2}{2-\alpha_1},
\end{cases}
\] (4.2)

If $\sqrt{2} < \alpha_1 \leq 3/2$ then $\|\eta(t)\|_{L^p} \in L^q_t \cap L^\infty_t$ with

\[
q > \begin{cases} 
\frac{p}{p-2} & \text{if } p < \frac{2}{2-\alpha_1}, \\
\frac{1}{\alpha_1-1} & \text{if } p \geq \frac{2}{2-\alpha_1}.
\end{cases}
\] (4.3)

**Corollary 4.2.** If the hypotheses of Theorem 4.1 hold and, in addition, $u_0 \in L^{p'}$, and $p > 2\alpha_2 + 2$, then $\eta(t) \in L^p_t$ and satisfies

\[
\|\eta(t)\|_{L^p} \leq \begin{cases} 
C(p)\frac{\log^{1-2/p}(2+|t|)}{(1+|t|)^{1-2/p}} & \text{if } \alpha_1 \geq 2 \text{ or } \alpha_1 < 2 \text{ and } p < \frac{2}{2-\alpha_1}, \\
C(p)\frac{\log^{\alpha_1}(2+|t|)}{(1+|t|)^{\alpha_1-1}} & \text{if } \alpha_1 < 2 \text{ and } p \geq \frac{2}{2-\alpha_1},
\end{cases}
\] (4.4)

for some constant $C = C(p)$ and $\lim_{p \to \infty} C(p) = \infty$ provided $\alpha_1 > 3/2$. If $\sqrt{2} < \alpha_1 \leq 3/2$ then $\|\eta(t)\|_{L^p} \in L^q_t \cap L^\infty_t$ with

\[
q > \begin{cases} 
\frac{p}{p-2} & \text{if } p < \frac{2}{2-\alpha_1}, \\
\frac{1}{\alpha_1-1} & \text{if } p \geq \frac{2}{2-\alpha_1}.
\end{cases}
\] (4.5)

**Remark 4.3.** The estimates on $\eta$ show that the radiative component of the solution disperses comparable to the solution of the free Schrödinger equation. In the critical and supercritical regimes, $\alpha_1 \geq 2$, our estimates have only logarithmic correction compared to the free Schrödinger equation. In subcritical regimes, $\frac{3}{2} < \alpha_1 < 2$, unlike in the free Schrödinger equation, the decay rate saturates to $|t|^{-\alpha_1}$ for $L^p$, $p \geq \frac{2}{2-\alpha_1}$, while close to the Strauss limit, $\sqrt{2} < \alpha_1 \leq \frac{3}{2}$ we can not get an actual decay rate.
We now proceed with the proofs:

**Proof of Theorem 4.1.** The decomposition part in the statement of this theorem has already been proven in Chapter 2 Proposition 2.8. We will now focus on showing the estimates for the radiative part of the decomposition by analyzing (2.26). The convergence of the projection onto the center manifold will then follow from analyzing equation (2.23).

In Chapter 3 we showed that the linear part of (2.26) generates a semigroup of operators with similar dispersive properties as the free Schrödinger group. Using Duhamel formula, the solution \( \zeta \in C(\mathbb{R}, H^1 \cap \mathcal{H}_0) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^2) \cap \mathcal{H}_0) \) of (2.26) also satisfies:

\[
\zeta(t) = \Omega(t,0)\zeta(0) - i \int_0^t \Omega(t,s)P_c g_2(\psi_E(s), R_a(s)\zeta(s))ds \\
- \int_0^t \Omega(t,s)P_c(\mathbb{I} - M_u(s))^{-1}g_3(\psi_E(s), R_a(s)\zeta(s))ds.
\]

(4.6)

Note that the right-hand side of (4.6) contains only terms that are quadratic and higher order in \( \zeta \), see Lemma 2.9 and (2.21). As in [11] this is essential in controlling low power nonlinearities and it is one of the main differences between our approach and the existing literature on asymptotic stability of coherent structures for dispersive nonlinear equations, see [11] p. 449 for a more detailed discussion.

To obtain estimates for \( \zeta \) we apply a contraction mapping argument to the fixed point problem (4.6) in the following Banach space.

Case I if \( \alpha_1 \geq 2 \), let:

\[
Y = \left\{ v \in C(\mathbb{R}, L^2 \cap L^{2\alpha_2+2}) : \right. \\
\left. \sup_{t \in \mathbb{R}} \| v(t) \|_{L^2} < \infty, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1-\frac{2}{2\alpha_2+2}}}{|\log(2 + |t|)|^{1-\frac{2}{2\alpha_2+2}}} \| v(t) \|_{L^{2\alpha_2+2}} < \infty \right\}.
\]
endowed with the norm
\[ \|v\|_Y = \max \left\{ \sup_{t \in \mathbb{R}} \|v(t)\|_{L^2}, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{2\alpha_2 + 2}}}{\log(2 + |t|)^{1 - \frac{2}{2\alpha_2 + 2}}} \|v(t)\|_{L^{2\alpha_2 + 2}} \right\} \]

Case II if \( 3/2 < \alpha_1 < 2 \), let:

\[ Y = \left\{ v \in C \left( \mathbb{R}, L^2 \cap L^{\alpha_1 + 2} \cap L^{2\alpha_2 + 2} \right) : \sup_{t \in \mathbb{R}} \|v(t)\|_{L^2} < \infty, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{2\alpha_1 + 2}}}{\log(2 + |t|)^{1 - \frac{2}{2\alpha_1 + 2}}} \|v(t)\|_{L^{\alpha_1 + 2}} < \infty, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{n_i}}{\log(2 + |t|)^{m_i}} \|v(t)\|_{L^{2\alpha_2 + 2}} < \infty \right\} ; \]

where

(1) \( n_1 = \alpha_1 - 1, \ m_1 = \alpha_1 \) and is used for \( 2\alpha_2 + 2 \geq \frac{2}{2 - \alpha_1} \);

(2) \( n_2 = 1 - \frac{2}{2\alpha_2 + 2}, \ m_2 = 1 - \frac{2}{2\alpha_2 + 2} \) and is used for \( 2\alpha_2 + 2 < \frac{2}{2 - \alpha_1} \).

endowed with the norm
\[ \|v\|_Y = \max \left\{ \sup_{t \in \mathbb{R}} \|v(t)\|_{L^2}, \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{1 - \frac{2}{2\alpha_1 + 2}}}{\log(2 + |t|)^{1 - \frac{2}{2\alpha_1 + 2}}} \|v(t)\|_{L^{\alpha_1 + 2}}, \right. \]
\[ \left. \sup_{t \in \mathbb{R}} \frac{(1 + |t|)^{n_i}}{\log(2 + |t|)^{m_i}} \|v(t)\|_{L^{2\alpha_2 + 2}} \right\} ; \]

Case III if \( \sqrt{2} < \alpha_1 \leq \frac{3}{2} \), we fix the numbers \( \sigma, \delta, q_1, q_2 \) such that \( \sigma > 1, \ 0 < \delta \leq \frac{\alpha_1^2 \alpha_2}{\alpha_1 (\alpha_1 + 1)} \),

\( 2 \leq q_1 = \frac{\alpha_1 + 2}{(1 - \delta)\alpha_1} \leq 4, \ q_1 \geq q_2 > \frac{1}{(1 - \delta)(\alpha_1 - 1) - \frac{\alpha_1^2 \alpha_2}{\alpha_1 (\alpha_1 + 1)}} \) and consider the Banach space:

\[ Y = \left\{ v \in C \left( \mathbb{R}, L^2 \cap L^{\alpha_1 + 2} \cap L^{2\alpha_2 + 2} \right) : v \in L^\infty \left( \mathbb{R}, L^2 \right) \cap L^2 \left( \mathbb{R}, L^{2 - \sigma} \right) \right. \]
\[ \left. v \in L^{q_1} \left( \mathbb{R}, L^{\alpha_1 + 2} \right) \cap L^\infty \left( \mathbb{R}, L^{\alpha_1 + 2} \right), \ v \in L^{q_2} \left( \mathbb{R}, L^{2\alpha_2 + 2} \right) \cap L^\infty \left( \mathbb{R}, L^{2\alpha_2 + 2} \right) \right\} \]
endowed with the norm

$$\|v\|_Y = \max \left\{\|v\|_{L^\infty(\mathbb{R},L^2)} \cap L^2(\mathbb{R},L^2_{-\sigma}), \|v\|_{L^1(\mathbb{R},L^{\alpha_1+2})} \cap L^\infty(\mathbb{R},L^{\alpha_1+2}), \|v\|_{L^2(\mathbb{R},L^{2\alpha_2+2})} \cap L^\infty(\mathbb{R},L^{2\alpha_2+2}) \right\}$$

Consider now the nonlinear operator in (4.6):

$$N(v)(t) = -i \int_0^t \Omega(t,s)P_c g_2(\psi_E(s), R_{a(s)}v(s)) ds$$

$$- \int_0^t \Omega(t,s)P_c (I - M_{a(s)})^{-1} g_3(\psi_E(s), R_{a(s)}v(s)) ds. \quad (4.7)$$

We have:

**Lemma 4.4.** In all cases above $N : Y \to Y$ is well-defined and locally Lipschitz, i.e. there exists $\hat{C} > 0$, such that

$$\|N v_1 - N v_2\|_Y \leq \hat{C} (\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_Y^{\alpha_1} + \|v_2\|_Y^{\alpha_2} + \|v_1\|_Y^{\alpha_2} + \|v_2\|_Y^{\alpha_2}) \|v_1 - v_2\|_Y.$$ 

Assuming that the lemma has been proven then we can apply the contraction principle for (4.6) in a closed ball in the Banach space $Y$ in the following way. Let

$$v = \Omega(t, 0)\varsigma(0)$$

then by Remark 3.4 we have

$$\|v\|_Y \leq \max \{C_2, C_{2\alpha_2+2}\} \|\varsigma(0)\|_{H^1 \cap L^{2\alpha_2+2}}$$

where we used the interpolation $\|\varsigma(0)\|_{L^{(\alpha_1+2)}} \leq \|\varsigma(0)\|_{L^2 \cap L^{2\alpha_2+2}}$. Recall that

$$\varsigma(0) = P_c \eta(0) = P_c u_0 - h(a(0)) = u_0 - \langle \psi_0, u_0 \rangle \psi_0 - h(a(0))$$
where $u_0 = u(0)$ is the initial data, see also (2.9). Hence

$$
\| \zeta(0) \|_{H^1 \cap L^{2 \alpha_2 + 2}'} \leq \| u_0 \|_{H^1 \cap L^{2 \alpha_2 + 2}'} + \| u_0 \|_{L^2} \| \psi_0 \|_{H^1 \cap L^{2 \alpha_2 + 2}'} + D_1 \| u_0 \|_{L^2} \leq D \varepsilon_0
$$

where $D_1$, $D > 0$ are constants independent on $u_0$ and the estimate on $h(a(0))$ follows from Proposition 2.1 and $|a(0)| \leq 2 \| u_0 \|_{L^2}$, see Lemma 2.5.

Therefore, we can choose $\varepsilon_0$ small enough such that $R = 2 \| v \|_Y$ satisfies

$$
\text{Lip} \overset{\text{def}}{=} 2 \tilde{C}(R + R^{1+\alpha_1} + R^{1+\alpha_2}) < 1.
$$

In this case the integral operator given by the right-hand side of (4.6):

$$
K(\zeta) = v + N(\zeta)
$$

leaves $B(0, R) = \zeta \in Y : \| \zeta \|_Y \leq R$ invariant and it is a contraction on it with Lipschitz constant $\text{Lip}$ defined above. Consequently, the equation (4.6) has a unique solution in $B(0, R)$ and, because $\zeta(t) \in C(\mathbb{R}, H^1) \hookrightarrow C(\mathbb{R}, L^2, L^{\alpha_1+2}, L^{2\alpha_2+2})$ already verified the equation, we deduce that $\zeta(t)$ is in $B(0, R)$, in particular it satisfies the estimates (4.2) in case I and II respectively, and (4.3) in case III, for $p = 2$, $p = \alpha_1 + 2$, $p = 2\alpha_2 + 2$.

Then $\eta(t) = R_a(t)\zeta(t)$ satisfies the same estimates because of Lemma 2.7. In case I, to obtain the estimates for $2 < p < 2\alpha_2 + 2$ we use interpolation in $L^p$ spaces. In case II, if $2\alpha_2 + 2 < \frac{2}{2-\alpha_1}$ we can still use interpolation for all $2 < p < 2\alpha_2 + 2$. If $2\alpha_2 + 2 \geq \frac{2}{2-\alpha_1}$ then for $2 < p < \alpha_1 + 2$ we use interpolation, while for $\alpha_1 + 2 < p < 2\alpha_2 + 2$ we first obtain the estimates (4.2), and respectively (4.3) for $\zeta(t)$, see the proof of Lemma 4.4 and then transfer them to $\eta(t) = R_a(t)\zeta(t)$ via Lemma 2.7.

Moreover, the system of ODE’s (2.23) is integrable in time on the right-hand side because the matrix has norm bounded by 2, see (2.8), while $g_{2j}$ satisfy (2.21) in cases I and II respectively, and (2.22) in case III, where $p_2 = 2\alpha_2 + 2$ and $\tilde{\eta}(t)$ differs from $\eta(t)$ by only a
phase. Note that the $L^p$, $1 \leq p \leq \infty$ norms of $\Psi_j(t)$, $\psi(t)$ are uniformly bounded in time, see (2.11). Consequently, $\tilde{a}_1(t)$ and $\tilde{a}_2(t)$ converge as $t \to \pm \infty$, and, in cases I and II the rate of convergence can be made explicit:

$$\lim_{t \to \pm \infty} \tilde{a}(t) = \lim_{t \to \pm \infty} \tilde{a}_1(t) + i\tilde{a}_2(t) \overset{\text{def}}{=} a_{\pm \infty}, \quad |\tilde{a}(\pm t) - a_{\pm \infty}| \leq C \frac{\log^{2m_i}(2 + t)}{(1 + t)^{2n_i - 1}}, \text{ for all } t \geq 0,$$

where $n_i = m_i = 1 - \frac{2}{2\alpha_2 + 2}$ in case I and in case II for $2\alpha_2 + 2 < \frac{2}{2-\alpha_1}$, while in case II we have $n_i = \alpha_1 - 1$, $m_i = \alpha_1$ if $2\alpha_2 + 2 \geq \frac{2}{2-\alpha_1}$. We can now define

$$\psi_{E \pm \infty} = a_{\pm \infty} \psi_0 + h(a_{\pm \infty}), \quad (4.8)$$

and we have

$$\lim_{t \to \pm \infty} \|\tilde{\psi}_E(t) - \psi_{E \pm \infty}\|_{H^2 \cap L^2_\sigma} = 0, \text{ for } \sigma \in \mathbb{R} \quad (4.9)$$

where we used (2.15) and the continuity of $h(a)$, see Proposition 2.1. In addition, since $E : [-2\delta_1, 2\delta_1] \mapsto (-\delta, \delta)$ is a $C^1$ function, see Proposition 2.1, the following limits exist

$$\lim_{t \to \pm \infty} E(|\tilde{a}(t)|) = E_{\pm \infty},$$

where again in cases I and II the rate of convergence can be made explicit:

$$|E(|\tilde{a}(\pm t)|) - E_{\pm \infty}| \leq C_1 \frac{\log^{2m_i}(2 + t)}{(1 + t)^{2n_i - 1}} \text{ for all } t \geq 0,$$

with $n_i, m_i$ as before. In any case we can define

$$\tilde{\theta}(t) = \begin{cases} \frac{1}{t} \int_0^t E(|\tilde{a}(s)|) - E_{\pm \infty} ds & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ \frac{1}{t} \int_0^t E(|\tilde{a}(s)|) - E_{\pm \infty} ds & \text{if } t < 0 \end{cases} \quad (4.10)$$

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and we have
\[
\lim_{|t| \to \infty} \tilde{\theta}(t) = 0
\]
where the rate of convergence is again explicit in the cases I and II
\[
|\tilde{\theta}(t)| \leq C_2 \log^{2m_i}(2 + t) \frac{(1 + t)^{2n_i - 1}}{(1 + t)^{2n_i - 1}},
\]
with \(n_i, m_i\) as before.

Consider
\[
\tilde{\theta}(t) = \int_0^t E(|a(s)|) ds = \begin{cases} \frac{t(E_\infty + \tilde{\theta}(t))}{2} & \text{if } t \geq 0 \\ \frac{t(E_\infty + \tilde{\theta}(t))}{2} & \text{if } t < 0 \end{cases}
\]
where we used \(|a(t)| = |\tilde{a}(t)|\), see (2.16).

In conclusion, since \(\psi_{E}(t) = e^{i\theta(t)} \tilde{\psi}_{E}(t)\), see (2.9), (2.15) and (2.16), we get from (4.9) and (4.11) the convergence (4.1).

It remains to prove Lemma 4.4:

**Proof of Lemma 4.4:** It suffices to prove the estimate:
\[
\|Nv_1 - Nv_2\|_Y \leq \tilde{C}(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|^{\alpha_1}_Y + \|v_2\|^{\alpha_1}_Y + \|v_1\|^{\alpha_2}_Y + \|v_2\|^{\alpha_2}_Y)\|v_1 - v_2\|_Y, (4.12)
\]
because plugging in \(v_2 \equiv 0\) and using \(N(0) \equiv 0\), see (4.7), will then imply \(N(v_1) \in Y\) whenever \(v_1 \in Y\).

Note that via interpolation in \(L^p\) spaces we have for all \(v \in Y\), all \(2 \leq p \leq 2\alpha_2 + 2\), and all \(t \in \mathbb{R}\):
\[
\|v(t)\|_{L^p} \leq \begin{cases} \|v\|_Y \log^{1/2/p}(2 + |t|) & \text{if } \alpha_1 \geq 2 \text{ or } 3/2 < \alpha_1 < 2 \text{ and } p \leq \alpha_1 + 2, \\ \|v\|_Y \log^{1/2/p}(2 + |t|) & \text{if } 3/2 < \alpha_1 < 2 \text{ and } p > \alpha_1 + 2, \end{cases}
\]
(4.13)
in cases I and II, while in case III
\[
\|v(t)\|_{L^q \cap L^\infty(\mathbb{R}, L^p)} \leq \|v\|_Y, \quad q = q_1 \frac{1 - \frac{2}{p}}{1 - \frac{2}{\alpha_1}} \quad \text{if } p \leq \alpha_1 + 2, \\
\|v(t)\|_{L^q \cap L^\infty(\mathbb{R}, L^p)} \leq \|v\|_Y, \quad \text{if } p > \alpha_1 + 2.
\] (4.14)

Now, from (2.18), we have for any \(v_1, v_2 \in Y\) :
\[
g_2(\psi_E, R_\alpha v_1) - g_2(\psi_E, R_\alpha v_2) = g(\psi_E + R_\alpha v_1) - g(\psi_E + R_\alpha v_2) - Dg_{\psi_E}[R_\alpha(v_1 - v_2)] \\
= \int_0^1 (Dg_{\psi_E + R_\alpha(\tau v_1 + (1-\tau)v_2)} - Dg_{\psi_E})[R_\alpha(v_1 - v_2)] d\tau \\
= \int_0^1 \int_0^1 D^2g_{\psi_E + sR_\alpha(\tau v_1 + (1-\tau)v_2)} \\
\times [R_\alpha(\tau v_1 + (1 - \tau)v_2)][R_\alpha(v_1 - v_2)] d\tau ds \\
= A_1(\psi_E, v_1, v_2) + A_2(\psi_E, v_1, v_2) + A_3(\psi_E, v_1, v_2),
\] (4.15)

where we consider \(\chi_j(t, x), j = 1, 2\) to be the characteristic function of the set \(S_1 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 : |\psi_E(t, x)| \geq \max(|R_\alpha(t_1)v_1(t, x)|, |R_\alpha(t_2)v_2(t, x)|)\}\), respectively
\(S_2 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 : \max(|R_\alpha(t_1)v_1(t, x)|, |R_\alpha(t_2)v_2(t, x)|) \leq 1\}\) and
\[
A_1(\psi_E, v_1, v_2) = \int_0^1 \int_0^1 \chi_1 D^2g_{\psi_E + sR_\alpha(\tau v_1 + (1-\tau)v_2)}[R_\alpha(\tau v_1 + (1 - \tau)v_2)][R_\alpha(v_1 - v_2)] d\tau ds, \\
A_2(\psi_E, v_1, v_2) = \int_0^1 \int_0^1 (1 - \chi_1)\chi_2 D^2g_{\psi_E + sR_\alpha(\tau v_1 + (1-\tau)v_2)} \\
\times [R_\alpha(\tau v_1 + (1 - \tau)v_2)][R_\alpha(v_1 - v_2)] d\tau ds, \\
A_3(\psi_E, u_1, u_2) = \int_0^1 \int_0^1 (1 - \chi_1)(1 - \chi_2) D^2g_{\psi_E + sR_\alpha(\tau v_1 + (1-\tau)v_2)} \\
\times [R_\alpha(\tau v_1 + (1 - \tau)v_2)][R_\alpha(v_1 - v_2)] d\tau ds.
\]

Note that there exists a constant \(C > 0\) such that for any \(\psi_E, v_1, v_2 \in Y,\) any \(t \in \mathbb{R}\) and
almost all \( x \in \mathbb{R}^2 \) we have the pointwise estimates:

\[
|A_1(\psi_E(t, x), v_1(t, x), v_2(t, x))| \leq C \left( 2^{\alpha_1 - 1} |\psi_E(t, x)|^{\alpha_1 - 1} + 2^{\alpha_2 - 1} |\psi_E(t, x)|^{\alpha_2 - 1} \right) \\
\times (|R_{a(t)}v_1(t, x)| + |R_{a(t)}v_2(t, x)|)|R_{a(t)}(v_1(t) - v_2(t))|
\]

\[
|A_2(\psi_E(t, x), v_1(t, x), v_2(t, x))| \leq 2^{\alpha_1 - 1} C \left( |R_{a(t)}v_1(t, x)|^{\alpha_1} + |R_{a(t)}v_2(t, x)|^{\alpha_1} \right) \\
\times |R_{a(t)}(v_1(t) - v_2(t))|
\]

\[
|A_3(\psi_E(t, x), v_1(t, x), v_2(t, x))| \leq 2^{\alpha_2 - 1} C \left( |R_{a(t)}v_1(t, x)|^{\alpha_2} + |R_{a(t)}v_2(t, x)|^{\alpha_2} \right) \\
\times |R_{a(t)}(v_1(t) - v_2(t))|
\]

where we used (2.19). For the quadratic term, we will use the following estimates valid for any \( t \in \mathbb{R} \) and \( \sigma \in \mathbb{R} \):

\[
\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^\beta} \leq C \|2^{\alpha_1 - 1} |\psi_E(t)|^{\alpha_1 - 1} + 2^{\alpha_2 - 1} |\psi_E(t)|^{\alpha_2 - 1}\|_{L^\beta} \\
\times (\|R_{a(t)}v_1(t)\|_{L^{p_2}} + \|R_{a(t)}v_2(t)\|_{L^{p_2}})\|R_{a(t)}(v_1(t) - v_2(t))\|_{L^{p_2}},
\]

where \( p_2 = 2\alpha_2 + 2, \ \frac{1}{s} + \frac{2}{p_2} = \frac{1}{2} \). In cases I and II we have:

\[
\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^\beta} \leq \frac{C_s \log^{\alpha_1}(2 + |t|)}{1 + |t|^{b_1}} (\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y \quad (4.16)
\]

where

\[
b_1 = \begin{cases} 
2 - \frac{4}{p_2} & \text{if } p_2 = 2\alpha_2 + 2 < \frac{2}{2 - \alpha_1}, \\
2\alpha_1 - 2 & \text{if } p_2 = 2\alpha_2 + 2 \geq \frac{2}{2 - \alpha_1},
\end{cases}
\]

and \( a_1 = b_1 \) when \( 2\alpha_2 + 2 < \frac{2}{2 - \alpha_1}, \ a_1 = b_1 + 2 \) when \( 2\alpha_2 + 2 \geq \frac{2}{2 - \alpha_1} \), where we used the definition of the Banach space \( Y \), the Hölder inequality, the estimate (2.11) and the Lemma 2.7.
In case III, we have:

$$\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^2_2 \cap L^\infty_2(\mathbb{R}, L^2_2)} \leq C(\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y. \tag{4.18}$$

Also, for $\Psi_j, j = 1, 2$ defined in Remark 2.6:

$$|\Re(\Psi_j(a(t)), -iA_1(\psi_E(t), v_1(t), v_2(t)))| \leq \|\Psi_j(a(t))\|_{L^2_2} ||A_1(\psi_E(t), v_1(t), v_2(t))||_{L^2_2},$$

hence, in cases I and II

$$|\Re(\Psi_j(a(t)), -iA_1(\psi_E(t), v_1(t), v_2(t)))| \leq C_{2,-\sigma} \frac{C_\sigma \log^{a_1}(2 + |t|)}{(1 + |t|)^{b_1}} (\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y. \tag{4.19}$$

For case III, we will need an $L^1$ estimate in space:

$$\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^1} \leq C(2^{\alpha_1 - 1}|\psi_E(t)|^{\alpha_1} + 2^{\alpha_2 - 1}|\psi_E(t)|^{\alpha_2})_{L^2_2}$$

$$\times (\|R_{a(t)}v_1(t)\|_{L^2_2} + \|R_{a(t)}v_2(t)\|_{L^2_2}) \|R_{a(t)}(v_1(t) - v_2(t))\|_{L^2_2},$$

and, because $\|R_{a(t)}v(t)\|_{L^2_2} \in L^2_2$ for functions $v \in Y$, we have:

$$\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^1 \cap L^\infty(y, L^1)} \leq C(\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y. \tag{4.20}$$

Also:

$$|\Re(\Psi_j(a(t)), -iA_1(\psi_E(t), v_1(t), v_2(t)))| \leq \|\Psi_j(a(t))\|_{L^\infty} ||A_1(\psi_E(t), v_1(t), v_2(t))||_{L^1},$$

hence,

$$|\Re(\Psi_j(a(t)), -iA_1(\psi_E(t), v_1(t), v_2(t)))| \leq C(\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y. \tag{4.21}$$
Moving to the higher order term $A_2$, for $1 \leq r' \leq 2$ we have $(1 + \alpha_1)r' < 2\alpha_2 + 2$, hence, for any $t \in \mathbb{R}$:

$$
\|A_2(\psi(t), v_1(t), v_2(t))\|_{L^{r'}} \leq 2^{\alpha_1-1}C \| R_{a(t)}v_1(t) |^{\alpha_1} + | R_{a(t)}v_2(t) |^{\alpha_1} \|_{L^{(1+\alpha_1)r'}} \times \| R_{a(t)}(v_1(t) - v_2(t))\|_{L^{(1+\alpha_1)r'}}
$$

which in cases I and II gives:

$$
\|A_2(\psi(t), v_1(t), v_2(t))\|_{L^{r'}} \leq \frac{C_{r'} \log^{\alpha_2(r')}(2 + |t|)}{(1 + |t|)^b_{2(r')}} (\|v_1\|_{Y}^{\alpha_1} + \|v_2\|_{Y}^{\alpha_1}) \|v_1 - v_2\|_{Y},
$$

(4.22)

where

$$
b_2(r') = \alpha_1 - 1 + \frac{2}{r}, \quad a_2(r') = b_2(r'), \quad \text{if } \alpha_1 \geq 2 \text{ or } 3/2 < \alpha_1 < 2 \text{ and } (1 + \alpha_1)r' \leq \alpha_1 + 2,
$$

$$
b_2(r') = (1 + \alpha_1)(1 - \frac{2}{\alpha_1 + 2}), \quad a_2(r') = b_2(r'), \quad \text{if } 3/2 < \alpha_1 < 2 \text{ and } (1 + \alpha_1)r' > \alpha_1 + 2,
$$

(4.23)

with $1/r + 1/r' = 1$, and in case III:

$$
\|A_2(\psi(t), v_1(t), v_2(t))\|_{L^{q'(r')} \cap L^\infty(\mathbb{R},L^{r'})} \leq C(\|v_1\|_{Y}^{\alpha_1} + \|v_2\|_{Y}^{\alpha_1}) \|v_1 - v_2\|_{Y},
$$

(4.24)

where

$$
q(r') = \frac{1}{1-\delta} \cdot \frac{1}{\alpha_1 - 1 + \frac{2}{r}}, \quad \text{if } (1 + \alpha_1)r' \leq \alpha_1 + 2
$$

$$
q(r') = 1, \quad \text{if } (1 + \alpha_1)r' > \alpha_1 + 2
$$

(4.25)

Similarly, for the higher order term $A_3$, and $1 \leq r' \leq 2$ we have $(1 + \alpha_2)r' \leq 2\alpha_2 + 2$ and:

$$
\|A_3(\psi(t), v_1(t), v_2(t))\|_{L^{r'}} \leq 2^{\alpha_2-1}C \| R_{a(t)}v_1(t) |^{\alpha_2}
$$

$$
+ | R_{a(t)}v_2(t) |^{\alpha_2} \|_{L^{(1+\alpha_2)r'}} \| R_{a(t)}(v_1(t) - v_2(t))\|_{L^{(1+\alpha_2)r'}}
$$
which in cases I and II give:

$$\|A_3(\psi_E(t), v_1(t), v_2(t))\|_{L^{r'}} \leq C r' \log^{a_3(r')} (2 + |t|) \left( \|v_1\|_{Y}^{\alpha_2} + \|v_2\|_{Y}^{\alpha_2} \right) \|v_1 - v_2\|_Y, \quad (4.26)$$

where

$$b_3(r') = \alpha_2 - 1 + \frac{2}{r'}, \quad a_3(r') = b_3(r'), \quad \text{if } \alpha_1 \geq 2 \text{ or } 3/2 < \alpha_1 < 2 \text{ and } (1 + \alpha_2)r' \leq \alpha_1 + 2,$$

$$b_3(r') = (1 + \alpha_2)(1 - \frac{2}{\alpha_1 + 2}), \quad a_3(r') = b_3(r'), \quad \text{if } 3/2 < \alpha_1 < 2 \text{ and } (1 + \alpha_2)r' > \alpha_1 + 2,$$

and in case III:

$$\|A_3(\psi_E(t), v_1(t), v_2(t))\|_{L^{r'}(\mathbb{R} \cap L^{\infty}(\mathbb{R}))} \leq C (\|v_1\|_{Y}^{\alpha_2} + \|v_2\|_{Y}^{\alpha_2}) \|v_1 - v_2\|_Y, \quad (4.28)$$

where

$$q(r') = \frac{1}{1 - \frac{1}{\alpha_2 - 1 + \frac{2}{r'}}, \quad \text{if } (\alpha_2 + 1)r' \leq \alpha_1 + 2}$$

$$q(r') = 1, \quad \text{if } (\alpha_2 + 1)r' > \alpha_1 + 2 \quad (4.29)$$

Moreover, using Cauchy-Schwartz inequality and (2.11) we have for both $A_k$, $k = 2, 3$:

$$|\Re(\Psi_j(a(t)), -iA_k(\psi_E(t), v_1(t), v_2(t)))| \leq \|\Psi_j(a(t))\|_{L^2} \|A_k(\psi_E(t), v_1(t), v_2(t))\|_{L^2}$$

which in cases I and II give:

$$|\Re(\Psi_j(a(t)), -iA_k(\psi_E(t), v_1(t), v_2(t)))| \leq C_{2.0} \frac{C_2 \log^{a_k(2)} (2 + |t|)}{(1 + |t|)^{a_k(2)}} \times (\|v_1\|_{Y}^{\alpha_{k-1}} + \|v_2\|_{Y}^{\alpha_{k-1}}) \|v_1 - v_2\|_Y, \quad (4.30)$$

while in case III we have

$$\|\Re(\Psi_j(a(t)), -iA_k(\psi_E(t), v_1(t), v_2(t)))\|_{L^1 \cap L^{\infty}(\mathbb{R})} \leq C (\|v_1\|_{Y}^{\alpha_{k-1}} + \|v_2\|_{Y}^{\alpha_{k-1}}) \|v_1 - v_2\|_Y. \quad (4.31)$$
Now, from (2.24) and (2.21) we have

\[ g_3(\psi_E, R_a v_1) - g_3(\psi_E, R_a v_2) \]

\[ = \Re(\Psi_1(a) - i(g_2(\psi_E, R_a v_1) - g_2(\psi_E, R_a v_2))) \frac{\partial \psi_E}{\partial a_1} \]

\[ + \Re(\Psi_2(a) - i(g_2(\psi_E, R_a v_1) - g_2(\psi_E, R_a v_2))) \frac{\partial \psi_E}{\partial a_2} \]

\[ = \Re(\Psi_1(a) - i(A_1 + A_2 + A_3)(\psi_E, v_1, v_2)) \frac{\partial \psi_E}{\partial a_1} \]

\[ + \Re(\Psi_2(a) - i(A_1 + A_2 + A_3)(\psi_E, v_1, v_2)) \frac{\partial \psi_E}{\partial a_2} \]

Consequently, for

\[ A_4(\psi_E, v_1, v_2) \overset{\text{def}}{=} (\mathbb{I} - M_a)^{-1}(g_3(\psi_E, R_a v_1) - g_3(\psi_E, R_a v_2)) \tag{4.32} \]

we have that for any \( \sigma \in \mathbb{R} \) there exists a constant \( C_\sigma > 0 \) such that:

\[ \| A_4(\psi_E(t), v_1(t), v_2(t)) \|_{L^2} \leq \max \left\{ \frac{\| \partial \psi_E(t) \|_{L^2}, \| \psi_E(t) \|_{L^2}}{2} \right\} \sqrt{2}\| (\mathbb{I} - M_a(t))^{-1}\|_{\mathbb{R}^2 \to \mathbb{R}^2} \]

\[ \times \sqrt{\Re(\Psi_1(a(t)), -i(A_1 + A_2 + A_3)(t))^2 + |\Re(\Psi_2(a(t)), -i(A_1 + A_2 + A_3)(t))|^2}, \]

hence, in cases I and II we have

\[ \| A_4(\psi_E(t), v_1(t), v_2(t)) \|_{L^2} \leq \frac{C_\sigma \log^{a_4(2 + \| t \|)}}{(1 + \| t \|)^{b_4}} \]

\[ \times (\| v_1 \|_Y + \| v_2 \|_Y + \| v_1 \|^{a_1}_Y + \| v_2 \|^{a_1}_Y + \| v_1 \|^{a_2}_Y + \| v_2 \|^{a_2}_Y) \| v_1 - v_2 \|_Y \tag{4.33} \]

where:

\[ b_4 = \min\{b_1, b_2(2), b_3(2)\}, \quad a_4 = \max\{a_1, a_2(2), a_3(2)\}, \tag{4.34} \]
while, in case III we have:

\[
\|A_1(\psi_E(t), v_1(t), v_2(t))\|_{L^1 \cap L^\infty(B, L^p_\alpha)} \\
\leq C(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_{Y^1}^\alpha + \|v_2\|_{Y^1}^\alpha + \|v_1\|_{Y^2}^\alpha + \|v_2\|_{Y^2}^\alpha)\|v_1 - v_2\|_Y
\]

(4.35)

and we used \([2.11], [2.8], [4.19], [4.21], [4.30], \) and \([4.31]\).

We are now ready to prove the Lipschitz estimate for the nonlinear operator \(N, (4.12)\).

We start with cases I and II. From its definition \([4.7]\) and \([4.15], (4.32)\) we have for any \(v_1, v_2 \in Y,\) any \(2 \leq p \leq p_2,\) and a fixed \(\sigma > 1:\)

\[
\|N(v_1)(t) - N(v_2)(t)\|_{L^p} = \left\| \int_0^t \Omega(t, s)P_c(-iA_1 - iA_2 - iA_3 - A_4)(\psi_E(s), v_1(s), v_2(s))ds \right\|_{L^p}
\leq \| \Omega(t, s)\|_{L^2_{\sigma} \rightarrow L^p_{\sigma}} \left( \|A_1(\psi_E(s), v_1(s), v_2(s))\|_{L^2_{\sigma}} + \|A_4(\psi_E(s), v_1(s), v_2(s))\|_{L^2_{\sigma}} \right) ds
\]

where \(1/p' + 1/p = 1.\) From Theorem 3.2 and estimates \([4.16], (4.33)\) we get:

\[
\int_0^t \| \Omega(t, s)\|_{L^2_{\sigma} \rightarrow L^p_{\sigma}} \left( \|A_1(\psi_E(s), v_1(s), v_2(s))\|_{L^2_{\sigma}} + \|A_4(\psi_E(s), v_1(s), v_2(s))\|_{L^2_{\sigma}} \right) ds
\leq \left( \|v_1\|_Y + \|v_2\|_Y + \|v_1\|_{Y^1}^\alpha + \|v_2\|_{Y^1}^\alpha + \|v_1\|_{Y^2}^\alpha + \|v_2\|_{Y^2}^\alpha \right)\|v_1 - v_2\|_Y
\times \int_0^t \frac{C_\sigma \log \alpha^1(2 + |s|)}{(1 + |s|)^{b_1}} + \frac{C_\sigma \log \alpha^2(2 + |s|)}{(1 + |s|)^{b_2}} ds
\]

while from Theorem 3.3 and estimates \([4.22], (4.26)\) we get:

\[
\int_0^t \| \Omega(t, s)\|_{L^2_{\sigma} \rightarrow L^p_{\sigma}} \left( \|A_2(\psi_E(s), v_1(s), v_2(s))\|_{L^2_{\sigma}} \right) ds \leq (\|v_1\|_{Y^1}^\alpha + \|v_2\|_{Y^1}^\alpha)
\times \|v_1 - v_2\|_Y \int_0^t \frac{C_\sigma \log 2 + |t - s|}{|t - s|^{1/2/p}} \cdot \frac{C_\sigma \log \alpha^2(2 + |s|)}{(1 + |s|)^{b_2(p')}} ds
\]

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and
\[
\int_0^t \| \Omega(t, s) \|_{L^{p'} \to L^p} \| A_3(\psi_E(s), v_1(s), v_2(s)) \|_{L^{p'}} ds \leq (\| v_1 \|_{Y^2}^{a_2} + \| v_2 \|_{Y^2}^{a_2})
\]
\[
\times \| v_1 - v_2 \|_Y \int_0^t \frac{C_p \log^{1-2/p}(2 + |t - s|)}{|t - s|^{1-2/p}} \cdot \frac{C_p' \log^a(p') (2 + |s|)}{(1 + |s|)^b_3(p')} ds.
\]

In Case I, i.e. \( \alpha_1 \geq 2 \), since \( \alpha_2 \geq \alpha_1 \) and \( 2 + 2\alpha_2 > 4 \), we have from (4.17), (4.23), (4.27) and (4.34) for \( p' \in \{(2\alpha_2 + 2)', 2\} \) and \( 1/p + 1/p' = 1 \):

\[
b_1 = 2 - \frac{4}{2\alpha_2 + 2} > 1,
\quad b_2(p') = \alpha_1 - 1 + \frac{2}{p} > 1,
\]
\[
b_3(p') = \alpha_2 - 1 + \frac{2}{p} > 1,
\quad b_4 = \min\{b_1, b_2(2), b_3(2)\} > 1.
\]

We now use the following known convolution estimate:

\[
\int_0^{|t|} \frac{\log^a(2 + |t - s|)}{|t - s|^b} \cdot \frac{\log^c(2 + |s|)}{(1 + |s|)^d} ds \leq C(a, b, c, d) \frac{\log^a(2 + |t|)}{(1 + |t|^b)}, \quad \text{for } d > 1, b < 1, (4.36)
\]

to bound the integral terms above and obtain for all \( 2 \leq p \leq 2\alpha_2 + 2 \):

\[
\| N(v_1)(t) - N(v_2)(t) \|_{L^p} \leq C_p' \frac{\log^{1-2/p}(2 + |t|)}{(1 + |t|)^{1-2/p}}
\]
\[
\times (\| v_1 \|_Y + \| v_2 \|_Y + \| v_1 \|_{Y^1}^{a_1} + \| v_2 \|_{Y^1}^{a_1} + \| v_1 \|_{Y^2}^{a_2} + \| v_2 \|_{Y^2}^{a_2}) \| v_1 - v_2 \|_Y (4.37)
\]

which, upon moving the time dependent terms to the left-hand side and taking supremum over \( t \in \mathbb{R} \) when \( p \in \{2, 2\alpha_2 + 2\} \), leads to (4.12) for \( \tilde{C} = \max\{C_2, C_{2\alpha_2 + 2}\} \).

In Case II, i.e. \( 3/2 < \alpha_1 < 2 \) we have from (4.17) \( b_1 > 1 \) because \( \alpha_2 \geq \alpha_1 > 3/2 \). From (4.23), under the restriction \( 2 \leq p < \frac{2}{2 - \alpha_1} \), we have either:

\[
b_2(p') = \alpha_1 - 1 + 2/p > 1,
\]
or
\[ b_2(p') = \frac{\alpha_1^2 + \alpha_1}{\alpha_1 + 2} > 1. \]

Since \( \alpha_2 \geq \alpha_1 \) implies \( b_3(\cdot) \geq b_2(\cdot) \) we deduce that, under the restriction \( 2 \leq p < \frac{2}{2 - \alpha_1} \), we also have

\[ b_3(p') \geq b_2(p') > 1, \]

and

\[ b_4 = \min\{b_1, b_2(2), b_3(2)\} > 1. \]

We can again apply (4.36) to the above integral terms and get for \( 2 \leq p < \frac{2}{2 - \alpha_1} \) the estimate (4.37).

For \( p = \frac{2}{2 - \alpha_1} \) we have \((1 + \alpha_1)p' < \alpha_1 + 2 \) hence \( b_2(p') = \alpha_1 - 1 + \frac{2}{p} = 1 \). We get a similar convolution estimate but with a logarithmic correction:

\[
\int_0^{|t|} \frac{\log^{1 - 2/p}(2 + |t - s|)}{|t - s|^{1 - 2/p}} \cdot \frac{\log^{\alpha_2(p')}(2 + |s|)}{(1 + |s|)^{b_2(p')}} \, ds \leq C(p) \frac{\log^{1 - 2/p + \alpha_2(p')(2 + |t|)}}{(1 + |t|)^{1 - 2/p}} = C(p) \frac{\log^{\alpha_1}(2 + |t|)}{(1 + |t|)^{\alpha_1 - 1}}. \tag{4.38}
\]

For \( p > \frac{2}{2 - \alpha_1} \) we again have \((1 + \alpha_1)p' < \alpha_1 + 2 \) hence \( b_2(p') = \alpha_1 - 1 + 2/p < 1 \) and, in the particular case of \( p = p_2 = 2\alpha_2 + 2 \) we get the convolution estimate:

\[
\int_0^{|t|} \frac{\log^{1 - 2/p_2}(2 + |t - s|)}{|t - s|^{1 - 2/p_2}} \cdot \frac{\log^{\alpha_2(p_2')}(2 + |s|)}{(1 + |s|)^{b_2(p_2')}} \, ds \leq C(p_2) \frac{\log^{1 - 2/p_2 + \alpha_2(p_2')(2 + |t|)}}{(1 + |t|)^{1 - 2/p_2 + b_2(p_2') - 1}} \leq \tilde{C}(p_2) \frac{\log^{\alpha_1}(2 + |t|)}{(1 + |t|)^{\alpha_1 - 1}},
\]

Since \( b_3(p') \geq b_2(p') \) we deduce

\[
\|N(v_1)(t) - N(v_2)(t)\|_{L^p} \leq \tilde{C}_p \frac{\log^{\alpha_1}(2 + |t|)}{(1 + |t|)^{\alpha_1 - 1}} \times \left( \|v_1\|_Y + \|v_2\|_Y + \|v_1\|_{Y^1} + \|v_2\|_{Y^1} + \|v_1\|_{Y^2} + \|v_2\|_{Y^2} \right) \|v_1 - v_2\|_Y \tag{4.39}
\]

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for $p = \frac{2}{2 - \alpha_1}$, and $p = p_2 = 2\alpha_2 + 2$, hence for any $p$, $\frac{2}{2 - \alpha_1} \leq p \leq 2\alpha_2 + 2$, via interpolation.

Combining the $p = p_2$ estimate with (4.37) for $p \in \{2, \alpha_1 + 2\}$, after moving the time dependent terms on the left-hand side and taking supremum over $t \in \mathbb{R}$, gives (4.12) in the Case II with $\tilde{C} = \max\{C_2, C_{\alpha_1 + 2}, \tilde{C}_{p_2}\}$.

This finishes the proof of Lemma 4.4 in cases I and II. For case III, recall:

$$
\|N(v_1)(t) - N(v_2)(t)\|_{L^p} = \left\| \int_0^t \Omega(t, s)P_c(-iA_1 - iA_2 - iA_3 - A_4)(\psi_E(s), v_1(s), v_2(s))ds \right\|_{L^p}
\leq \left\| \int_0^t \Omega(t, s)P_cA_1(\psi_E(s), v_1(s), v_2(s))ds \right\|_{L^p} + \left\| \int_0^t \Omega(t, s)P_cA_2(\psi_E(s), v_1(s), v_2(s))ds \right\|_{L^p}
\quad + \left\| \int_0^t \Omega(t, s)P_cA_3(\psi_E(s), v_1(s), v_2(s))ds \right\|_{L^p} + \left\| \int_0^t \Omega(t, s)P_cA_4(\psi_E(s), v_1(s), v_2(s))ds \right\|_{L^p}.
$$

In all these integrals we are going to split the kernel: $\Omega(t, s) = \Omega_1(t, s) + \Omega_2(t, s)$, where

$$
\begin{align*}
\Omega_1(t, s) &= \chi(t-s)\Omega(t, s) \\
\Omega_2(t, s) &= [1 - \chi(t-s)]\Omega(t, s) \\
\chi(\tau) &= \begin{cases} 1 & |\tau| < 1 \\
0 & |\tau| \geq 1 \end{cases}
\end{align*}
$$

Note that, based on Theorems 3.2 and 3.3 we have for any $2 \leq p < \infty$ and $\sigma > 1$

$$
\|\Omega_1(t, s)\|_{L^{p'} \to L^p} \in L_{t-s}^1 \cap L_{t-s}^q, \quad q < \frac{p}{p-2}
$$

$$
\|\Omega_1(t, s)\|_{L^2 \to L^p} \in L_{t-s}^1 \cap L_{t-s}^q, \quad q < \frac{p}{p-2}
$$

while for the second kernel we have for any $2 \leq p \leq \infty$ and $\sigma > 1$

$$
\|\Omega_2(t, s)\|_{L^{p'} \to L^p} \in L_{t-s}^q \cap L_{t-s}^\infty, \quad q > \frac{p}{p-2}, \quad q > 1 \text{ if } p = \infty
$$

$$
\|\Omega_2(t, s)\|_{L^2 \to L^p} \in L_{t-s}^q \cap L_{t-s}^\infty, \quad q > \frac{p}{p-2}, \quad q > 1 \text{ if } p = \infty
$$

(4.40)
\[ \| \Omega(t, s) \|_{L^2_t \rightarrow L^2_{t-s}} \in L^1_{t-s} \cap L^\infty_{t-s} \]
\[ \| <x> \Omega(t, s) \|_{L^1_t \rightarrow L^2_t} \in L^2_{t-s} \]

(4.41)

The easiest estimate is the term containing \( A_4 \) because we have:

\[ \left\| \int_0^t \Omega(t, s) P_c A_4(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^p} \]
\[ \leq \int_0^t \| \Omega(t, s) \|_{L^2_t \rightarrow L^p} \| A_4(\psi_E(s), v_1(s), v_2(s)) \|_{L^2} ds \]
\[ + \int_0^t \| \Omega_2(t, s) \|_{L^2_t \rightarrow L^p} \| A_4(\psi_E(s), v_1(s), v_2(s)) \|_{L^2} ds \]

and according to estimate (4.35) and Young inequality:

\[ \left\| \int_0^t \Omega(t, s) P_c A_4(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^q \cap L^\infty_{\mathbb{R}, L^2_\sigma}} \leq C(\| v_1 \|_Y + \| v_2 \|_Y + \| v_1 \|_{\alpha}^1 + \| v_2 \|_{\alpha}^1 + \| v_1 \|_{\alpha}^2 + \| v_2 \|_{\alpha}^2) \| v_1 - v_2 \|_Y \]  

(4.42)

for \( q > \frac{p}{p-2} \). Also:

\[ \left\| \int_0^t \Omega(t, s) P_c A_4(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^2_t} \]
\[ \leq \int_0^t \| \Omega(t, s) \|_{L^3_t \rightarrow L^2_{t-s}} \| A_4(\psi_E(s), v_1(s), v_2(s)) \|_{L^2} ds \]

and using the fact that the kernel is in \( L^1_{t-s} \) we get

\[ \left\| \int_0^t \Omega(t, s) P_c A_4(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^1_t \cap L^\infty_{\mathbb{R}, L^3}} \leq C(\| v_1 \|_Y + \| v_2 \|_Y + \| v_1 \|_{\alpha}^1 + \| v_2 \|_{\alpha}^1 + \| v_1 \|_{\alpha}^2 + \| v_2 \|_{\alpha}^2) \| v_1 - v_2 \|_Y. \]

both estimates are better than we need to show that this term is back in \( Y \).
For the weighted estimates of the terms containing $A_k, k = 2, 3$ we will use the Kato
smoothing estimate (4.41):

$$
\left\| \int_0^t \Omega(t, s) P_c A_k(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^2_{-\sigma}} \\
\leq \int_0^t \left\| \Omega(t, s) \right\|_{L^{2 \to L^2_{-\sigma}}_s} \left\| A_k(\psi_E(s), v_1(s), v_2(s)) \right\|_{L^2} ds
$$

and, using the fact that the kernel is in $L^2_{t-s}$ combined with estimates (4.24) and (4.28) for
$r' = 2$ which satisfies $(\alpha_2 + 1)r' \geq (\alpha_1 + 1)r' > \alpha_1 + 2$ we get:

$$
\left\| \int_0^t \Omega(t, s) P_c A_k(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^2(\mathbb{R} \times L^2_{-\sigma})} \\
\leq C(\|v_1\|_Y + \|v_2\|_Y + \|v_1\|_{Y^0} + \|v_2\|_{Y^0} + \|v_1\|_{Y^2} + \|v_2\|_{Y^2}) \|v_1 - v_2\|_Y.
$$

The $L^p$ estimates for these terms are treated similarly to cases I and II except that the
estimates (4.24) and (4.28) are used.

The most difficult term in the $\sqrt{2} < \alpha_1 \leq 3/2$ regime is the quadratic term containing
$A_1$. For the weighted estimates we use the same argument as for $A_4$ and the fact that in the
estimate (4.18) $q_2/2 \leq 2$. For the $L^p$ estimates, after we split the kernel into $\Omega_1, \Omega_2$ parts,
the term containing $\Omega_1$ satisfies

$$
\left\| \int_0^t \Omega_1(t, s) P_c A_1(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^p} \\
\leq \int_0^t \left\| \Omega_1(t, s) \right\|_{L^2 \to L^p} \left\| A_1(\psi_E(s), v_1(s), v_2(s)) \right\|_{L^2} ds
$$

and via Young inequality and estimate (4.18) we get

$$
\left\| \int_0^t \Omega_1(t, s) P_c A_1(\psi_E(s), v_1(s), v_2(s)) ds \right\|_{L^{q_2/2 \cap L^\infty(\mathbb{R}, L^p)}} \\
\leq C(\|v_1\|_Y + \|v_2\|_Y) \|v_1 - v_2\|_Y \quad (4.43)
$$

which suffices since $q_2/2 \leq q_1$ and $q_2/2 \leq q_2$.  

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For the quadratic term with kernel $\Omega_2$ we will interpolate between the $L^2$ in spaces estimate and the $L^\infty$ in space estimate. For the $L^\infty$ estimate we rely on the $L^1$ estimate for $A_1$ which took advantage of the Kato smoothing type property of the functions in $Y$, see (4.20). We have:

\[
\left\| \int_0^t \Omega_2(t,s) \psi E(s), v_1(s), v_2(s) ds \right\|_{L^\infty} \leq \int_0^t \left\| \Omega_2(t,s) \right\|_{L^1 \to L^\infty} \left\| A_1(\psi E(s), v_1(s), v_2(s)) \right\|_{L^1} ds
\]

which, using (4.40), (4.20), and Young inequality, gives for any $q > 1$

\[
\left\| \int_0^t \Omega_2(t,s) \psi E(s), v_1(s), v_2(s) ds \right\|_{L^q \cap L^\infty(\mathbb{R},L^\infty)} \leq C(\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y. \tag{4.44}
\]

For the $L^2$ in space estimate we use a duality argument:

\[
\left\| \int_0^t \Omega_2(t,s) \psi E(s), v_1(s), v_2(s) ds \right\|_{L^2} = \sup_{\|\phi\|_{L^2}=1} \int_0^t \left| \langle x \rangle^{-\sigma} \Omega_2^*(t,x)\phi, <x>^\sigma A_1 \right| ds
\]

\[
\leq \int_0^t \left\| <x>^{-\sigma} \Omega_2^*(t,x) \right\|_{L^2 \to L^2} \left\| A_1 \right\|_{L^2} ds
\]

and via (4.41) which is also valid for the splitted operators and their adjoints, estimate (4.18), and Young inequality:

\[
\left\| \int_0^t \Omega_2(t,s) \psi E(s), v_1(s), v_2(s) ds \right\|_{L^r \cap L^\infty(\mathbb{R},L^2)} \leq C(\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y, \tag{4.45}
\]

where $\frac{1}{r} = \frac{2}{q_2} + \frac{1}{2} - 1 \geq 0$. Interpolating between estimates (4.44), (4.45) we get:

\[
\left\| \int_0^t \Omega_2(t,s) \psi E(s), v_1(s), v_2(s) ds \right\|_{L^{q(p)} \cap L^\infty(\mathbb{R},L^p)} \leq C(\|v_1\|_Y + \|v_2\|_Y)\|v_1 - v_2\|_Y, \tag{4.46}
\]
where
\[
\frac{1}{q(p)} = \frac{\beta}{r} + \frac{2 - \beta}{q}, \quad \beta = \frac{2}{p}.
\]

By choosing
\[1 < q \leq \max\{q_2 \frac{2\alpha_2 + 1}{\alpha_2 + 1}, q_1 \frac{2\alpha_1 + 2}{\alpha_1 + 2}\}\]
we obtain
\[q(\alpha_1 + 2) \leq q_1, \quad q(\alpha_2 + 2) \leq q_2.\]

Lemma 4.4 is now complete and implies Theorem 4.1 with estimate (4.2) and slightly worse estimate than (4.3) for \(p \in \{2, \alpha_1 + 2, 2\alpha_2 + 2\}\). The estimates for \(2 < p < \frac{2}{2 - \alpha_1}\) follow from (4.37) while the ones for \(\frac{2}{2 - \alpha_1} \leq p \leq 2\alpha_2 + 2\) follow from (4.39) in cases I and II. In case III we return to equation (4.6) for which we now know that the solution \(\zeta(t)\) is in the Banach space \(Y\) defined for this case on page 48. To improve the \(L^p(\mathbb{R}^2)\) estimate for \(\zeta\) we apply the previous argument to the right-hand side of (4.6). More precisely, we use the decomposition of the nonlinearity \(N\), see (4.7), into \(A_1, A_2, A_3, A_4\) parts, see page 62, but this time \(v_1 = \zeta, v_2 = 0\). We have already obtained all the required estimates (4.3) for the term containing \(A_4\), see (4.42). For the \(A_1\) term they follow from (4.43) and (4.46), since \(q_2/2 \leq \frac{1}{\alpha_1 - 1}\) and \(q(t) \searrow \frac{p}{p - 2}\) as \(q \searrow 1\) in (4.44). A direct application of Young inequality using estimates (4.24), (4.28) shows that the terms containing \(A_2, A_3\) satisfies (4.3) for \(p = \alpha_1 + 2 < \frac{2}{2 - \alpha_1}\). Consequently, the right-hand side of (4.6) satisfies the same estimate, therefore, \(\zeta\) does too. For \(2 < p < \alpha_1 + 2\) we obtain the required estimate for \(\zeta\) via interpolation in \(L^p\) spaces. Armed with these improved estimates for \(\zeta\) in \(L^p(\mathbb{R}^2)\) spaces, \(2 < p < \alpha_1 + 2\), we can now attack the terms containing \(A_2\) and \(A_3\) on the right-hand side of (4.6). Via Young inequality we get (4.3) first for \(p = \frac{2}{2 - \alpha_1}\), then for \(p = 2\alpha_2 + 2\). Consequently, the left-hand side, i.e. \(\zeta\), satisfies (4.3) for \(p = \frac{2}{2 - \alpha_1}\) and \(p = 2\alpha_2 + 2\). For \(\alpha_1 + 2 < p < \frac{2}{2 - \alpha_1}, \frac{2}{2 - \alpha_1} < p < 2\alpha_2 + 2\), we obtain the required estimate via interpolation. The estimates for \(\zeta\) transfer to estimates for \(\eta\) without any modification via \(\eta = R_\alpha \zeta\) and
Lemma 2.7.

The Theorem 4.1 is now completely proven. □

Proof of Corollary 4.2: The $L^p(\mathbb{R}^2)$ estimate is based on equation (4.6) for which we already know that $z(t)$ is a solution in the Banach space $Y$, see pages 47-48. The hypothesis $u_0 \in L^{p'}$ implies $\zeta(0) \in L^{p'}$, see (2.9), which combined with Remark 3.4 shows that the inhomogeneous term $\Omega(t,0)\zeta(0)$ satisfies the required estimate. The nonlinear term is treated exactly as in the proof of Lemma 4.4 and gives the required estimate in $L^p$ for the right-hand side of (4.6). Hence, $\zeta$ satisfies the same estimate and the estimate can be transferred to $\eta = R_a\zeta$ via Lemma 2.7. □
Chapter 5
Conclusions

There is a rich literature concerning asymptotic stability of coherent structures (solitary waves, kinks, breathers) in dispersive equations, starting with the pioneering work in [21, 20] and further developments in [18, 21, 13, 5, 23, 14, 7, 11, 10, 8, 9] and others. This thesis differentiates from the previous results by giving the sharpest convergence rates to the solitary wave in two space dimensions, i.e. the sharpest decay rate of the radiative part, and by allowing the largest range of nonlinearities, including the subcritical ones close to the Strauss limit, which in dimension N is:

\[ \alpha_0(N) = \frac{2 - N + \sqrt{N^2 + 12N + 4}}{2N}. \]

Recall that this limit, more precisely, \( \gamma(N) = \alpha_0(N) + 1 \), was first introduced in [22] where, in the absence of the potential \( V(x) \), the author proves scattering of solutions with small initial data. Note that the presence of the potential completely changes the dynamics since solutions with small initial data asymptotically converge to a nonlinear ground state. However, if we subtract this limit state, the correction, \( \eta(t) \), has properties similar to the free Schrödinger dynamics, see Remark 4.3. It is an open problem whether the correction scatters, i.e. converges in \( H^1 \) as \( t \to \infty \) to a solution of the free Schrödinger equation as was the case in [22, Theorem 9].

The new results rely on two important contributions. The first one is an extension to two space dimensions of the dispersive estimates for the Schrödinger like, but time dependent linear operator which controls the linearized dynamics around the curve of solitary waves.
shadowing the nonlinear dynamics, see Chapter 3. Such estimates have only recently appear in [8, 9] in dimensions three and higher. Note though that less sharp estimates have been previously obtained in two dimensions in [11, 10]. The main difficulty we faced in two dimensions compared to higher ones is the non-integrability for large times of the free Schrödinger propagator:

$$\| e^{-i\Delta t} \|_{L^1 \to L^\infty} \sim |t|^{-\frac{N}{2}}, \quad \text{in } \mathbb{R}^N,$$

which forces us to add a logarithmic correction. How to extend such estimates in one dimension, and, consequently treat subcritical nonlinearities in one dimension is an open problem.

The second important contribution of this thesis is the new technique of employing $L^q(\mathbb{R}, L^p(\mathbb{R}^N))$ spaces to control nonlinearities close to the Strauss limit. We emphasize that these spaces are not the classical Strichartz spaces which actually have been used previously and restricted the nonlinearity to be critical or supercritical, i.e. far away from the Strauss limit, see [14, 7]. We need this new technique because for subcritical nonlinearities the best rate of decay for the radiative part turns out to be $(1 + |t|)^{-\left(\frac{N\alpha}{2} - 1\right)}$ where $N$ is the space dimension and $\alpha$ is the power that dominates the nonlinearity at small inputs, $\alpha = \alpha_1$ in Chapter 4 for $N = 2$, and $\alpha = \alpha_1 + 1$ in [8, 9] for $N = 3$ or higher. An algebraic calculation shows that for $\alpha > \alpha_0(N)$ but close to $\alpha_0(N)$ the rate of decay is strictly less than 1/2 for $N = 2$ (or $N = 1$) but the rate of decay is greater than 1/2 for $N \geq 3$. Consequently, the decay of the quadratic term in the equation for the radiative part is non-integrable in time for dimension $N = 2$ but is integrable in time for dimensions $N \geq 3$. To overcome the non-integrability, we took advantage of the localization in space of this term and Kato smoothing type estimates in order to control it in $L^q(\mathbb{R}, L^p(\mathbb{R}^2))$, see the proof of Theorem 4.1 in Chapter 4. For dimension 3 and higher or for higher power nonlinearities, i.e. away from the Strauss limit, the quadratic term is integrable in time and standard convolution estimates will control it in $(1 + |t|)^{-\left(\frac{N\alpha}{2} - 1\right)} \cdot \| \|_{L^p(\mathbb{R}^N)}$. 

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We claim that in the context of using bulk dispersive estimates for the radiative part the range of nonlinearities we allowed is optimal. We are not aware of a more refined analysis that could improve it. Indeed, consider the simplified model for the radiative part:

$$r(t, x) = \Omega(t, 0)r(0, x) - i \int_0^t \Omega(t, s)r^{1+\alpha}(s, x)ds$$  \hspace{1cm} (5.1)$$

Where for simplicity we can assume that $\Omega(t, s)$ satisfies the standard (bulk) dispersive estimates for the free Schrödinger propagator:

$$\|\Omega(t, s)\|_{L^{p'} \to L^p} \sim |t - s|^{-N(\frac{1}{2} - \frac{1}{p})}, \hspace{1cm} p \geq 2, \hspace{1cm} \frac{1}{p'} + \frac{1}{p} = 1.$$  

We want to bootstrap an estimate of type $\|r(t)\|_{L^p} \sim \frac{C}{(1 + |t|)^{n(p)}}$, for a certain $2 \leq p < \infty$, where we can choose $n(\cdot)$ continuous since $r \in C(\mathbb{R}, H^1)$ and $H^1(\mathbb{R}^2) \hookrightarrow L^p$, $2 \leq p < \infty$, in dimension $N = 2$. We have $\|\Omega(t, 0)r(0)\|_{L^p} \sim \frac{C}{|t|^{1 - \frac{2}{p}}} \|r(0)\|_{L^{p'}}$ hence the best possible estimate for $\|r(t)\|_{L^p}$ is $n(p) \leq 1 - \frac{2}{p}$.

Consider now the nonlinear term:

$$\| \int_0^t \Omega(t, s)r^{1+\alpha}(s, x)ds \|_{L^p} \sim \int_0^t |\Omega(t, s)|_{L^{p'} \to L^p} \|r(s)\|_{L^{p(\alpha+1)p'}}^\alpha ds$$  \hspace{1cm} (5.2)$

$$\sim \int_0^t \frac{1}{|t - s|^{1 - \frac{2}{p}}} \cdot \frac{1}{(1 + |s|)^{(1+\alpha)n((1+\alpha)p')}} ds \sim \frac{1}{(1 + |t|)^m}$$

$m = 1 - \frac{2}{p}$ if $(1 + \alpha)n((1 + \alpha)p') > 1$ and

$m = 1 - \frac{2}{p} + (1 + \alpha)n((1 + \alpha)p') - 1$ \hspace{1cm} if $(1 + \alpha)n((1 + \alpha)p') < 1$

So, in any case the nonlinear term decay at most with the second rate above. To bootstrap we need

$$n(p) \leq 1 - \frac{2}{p} + (1 + \alpha)n((1 + \alpha)p') - 1 \leq 1 - \frac{2}{p} + (1 + \alpha)(1 - \frac{2}{(1 + \alpha)p'}) - 1 = \alpha - 1 \hspace{1cm} (5.3)$$
We also learn from above calculation that to bootstrap the decay estimate in $L^p$ we need the decay estimate in $L^{(1+\alpha)p'}$. The simplest way to do it is to use $p = \alpha + 2$ for which $(1 + \alpha)p' = \alpha + 2 = p$ in which case we get from (5.3)

$$n(\alpha + 2) \leq 1 - \frac{2}{\alpha + 2} + (1 + \alpha)n(\alpha + 2) - 1,$$

(5.4)

or, equivalently,

$$\frac{2}{\alpha + 2} \leq \alpha n(\alpha + 2).$$

But from (5.3) $n(\alpha + 2) \leq \alpha - 1$ which plugged in the above leads, after some algebraic manipulations to:

$$(\alpha - \sqrt{2})(\alpha^2 + (1 + \sqrt{2})\alpha + 2) \geq 0.$$  

For $\alpha > 0$ this is equivalent to $\alpha \geq \sqrt{2}$. Now, if $\alpha = \sqrt{2}$ then the estimate in (5.2) requires a logarithmic correction because $(1 + \alpha)n((1 + \alpha)p') = 1$ which then prevents us to bootstrap. In conclusion,

$$\alpha > \sqrt{2}.$$  

Of course, instead of using $L^{\alpha+2}$ space one can try to bootstrap in $L^{p_1}$, $p_1 > \alpha + 2$ in which case an estimate for $(1 + \alpha)p'_1 = p_2 < \alpha + 2$ is needed. Inductively, we will get $\{p_k\}_{k \in \mathbb{N}}, \lim_{k \to \infty} p_k = \alpha + 2.$ and at each stage (5.3) translates to

$$n(p_k) \leq 1 - \frac{2}{p_k} + (1 + \alpha)n(p_{k+1}) - 1.$$  

Passing to the limit when $k \to \infty$ and using continuity of $n(\cdot)$, we arrive back at (5.4) hence $\alpha > \sqrt{2}$.

A way around falling back on (5.4) is to use interpolation in $L^p$ spaces. For example, to bootstrap the estimate in $L^{p_1}$, $p_1 > \alpha + 2$ we assume that we already got a uniform in time
bound for $\|r(t)\|_{L^2}$ and use

$$\|r(t)\|_{L^{(1+\alpha)p_1'}} \leq \|r(t)\|_{L^2}^{1-\beta} \|r(t)\|_{L^{p_1}}^{\beta}, \quad \frac{1-\beta}{2} + \frac{\beta}{p_1'} = \frac{1}{(1+\alpha)p_1'}$$

Plugging in (5.3) we get after some algebraic manipulations $\alpha > \sqrt{2}$. 
References


