A COMPLETE CHARACTERIZATION OF NIM

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*This research was supported in part by a grant from the Illinois Business Associates.
Abstract

Every vertex of an abstract graph is characterized in terms of a Nim-type game. A vertex is winning if by choosing it a player can assure himself of a win, it is losing if by choosing it he cannot prevent his opponent from winning, and it is drawing if it is neither winning nor losing. The sets of winning, losing, and drawing sets are identified in terms of a set-valued function on the graph.
Introduction

A Nim game is played by two players I and II on a graph \((V, D)\), where \(V\) is a (finite) set of vertices and \(D : V \times V\) is a set of directed arcs. The set of predecessors of a vertex \(v \in V\) is the set \(P(v) = \{x \in V \mid (x, v) \in D\}\) and the set of predecessors of a subset \(S \subseteq V\) is the set \(P(S) = \bigcup_{v \in S} P(v)\).

For each \(S \subseteq V\) the complement of \(P(S)\) is denoted \(U(S) = V - P(S)\). Thus \(U(S)\) is the set of vertices which do not precede any vertex in \(S\). Denote by \(U^2\) the composition of \(U\) with itself.

The game is played as follows: Player I selects a vertex \(v_1 \in V\). Player II selects any vertex \(v_2\) such that \(v_1 \in P(v_2)\). Player I then selects any vertex \(v_3\) such that \(v_2 \in P(v_3)\), and so forth. If a player selects a vertex \(v\) such that \(v \in U(V)\), i.e. a vertex which precedes no other, then that player wins and his opponent loses. We shall call the set \(U(V)\) the set of terminal vertices.

A set \(S \subseteq V\) such that \(S = U(S)\) is called a kernel of the graph. (Such sets are also called solutions (cf. Von Neumann and Morgenstern). It is well known (cf. Berge p. 320) that if the graph possesses a kernel \(S\), and if a player chooses a vertex in \(S\), then he can play in such a way that he is assured of a win or a draw; i.e. he can make his subsequent choices in such a way that he will not lose.

It is the purpose of this paper to characterize all vertices of the graph. A vertex is winning if by choosing it a player can assure himself of a win, it is losing if by choosing it he cannot prevent his opponent from winning, and it is drawing if it is neither winning nor losing. We will also determine for each winning vertex the minimum number of moves in which a player can assure that he will win, and for each losing vertex the maximum number of moves a player can forestall his loss.
Analysis

Define the following collection of subsets of \( V \). Let \( A_0 = \emptyset \) and for \( n = 1, 2, \ldots \) define \( A_n = U^2(A_{n-1}) \). Then \( A_1 = U(V) \), the set of terminal vertices, and for \( n = 1, 2, \ldots \) the sets \( A_n \) have the following properties.

**Proposition**  
a) \( A_{n-1} \subseteq A_n \);  
b) \( A_n \subseteq U(A_n) \)

**Proof:** Observe that \( A \subseteq B \) implies \( U(B) \subseteq U(A) \) and consequently \( U^2(A) \subseteq U^2(B) \). Proposition (a) and (b) are both true for \( n = 1 \). Suppose that it has been shown for some \( n \) that \( A_{n-1} \subseteq A_n \). Then \( A_n = U^2(A_{n-1}) \subseteq U^2(A_n) = A_{n+1} \), and thus (a) is true for all \( n \). Suppose that it has been shown for some \( n \) that \( A_n \subseteq U(A_n) \). Then \( A_{n+1} = U^2(A_n) \subseteq U^2(U(A_n)) = U(U^2(A_n)) = U(A_{n+1}) \) and so (b) is true for all \( n \), completing the proof.

It follows from Proposition (a) and from the finiteness of \( V \) that for a sufficiently large, \( A_n = A_n^* \). Denote by \( n^* \) the smallest such \( n \); then \( A_n^* = A_{n+1} = U^2(A_n^*) \). Note that Proposition (a) implies \( A_n^* = U A_n \).

**Theorem:** The set of winning vertices is precisely the set \( A_n^* \); the set of losing vertices is \( P(A_n^*) \); the set of drawing vertices is \( U(A_n^*) - A_n^* \).

**Proof:** First we show that every vertex in \( A_n^* \) is winning. Suppose a player picks a vertex \( a_n \in A_n \). If \( a_n \in A_1 = U(V) \), that player has won. Otherwise his opponent must pick a vertex \( b_n \) such that \( a_n \in P(b_n) \).

Since \( A_n = U^2(A_{n-1}) = U(U(A_{n-1}) \) it follows that \( a_n \) precedes no vertex in \( U(A_{n-1}) \) and so, since \( a_n \) precedes \( b_n \), \( b_n \in P(A_{n-1}) \); i.e., \( b_n \) precedes some vertex in \( A_{n-1} \). Thus the original player can pick a vertex \( a_{n-1} \in A_{n-1} \).
such that \( b_n \in P(a_{n-1}) \) and so forth: after making \( n \) choices in this way, the original player will have picked a terminal vertex \( a_1 \in A_1 \), and so have won the game. Thus every vertex in \( \Lambda^*_n \) is winning.

It follows immediately that every vertex in \( P(A_n^*) \) is losing. If a player picks a vertex in \( P(A_n^*) \), he cannot prevent his opponent from choosing a vertex in \( \Lambda_n^* \) and winning. It remains to show that every other vertex in \( V \), i.e. every vertex in \( U(A_n^*) \setminus \Lambda_n^* \), is neither winning nor losing.

Suppose a player chooses a vertex \( v \in U(A_n^*) \setminus \Lambda_n^* \). His opponent must choose a vertex \( w \) such that \( v \in P(w) \). Since \( \Lambda_n^* \) is contained in \( U(U(A_n^*)) \), no vertex in \( U(A_n^*) \) precedes any vertex in \( \Lambda_n^* \); so \( w \notin \Lambda_n^* \). But \( \Lambda_n^* \) contains \( U(U(A_n^*)) \) and so, since \( v \notin \Lambda_n^* \), it follows that \( v \in P(U(A_n^*)) \). Thus there is an \( x \in U(A_n^*) \) such that \( v \in P(x) \). Since \( v \notin U(A_n^*) \), we know that \( x \notin \Lambda_n^* \), and so \( x \in U(A_n^*) \setminus \Lambda_n^* \).

Thus if a player chooses a vertex \( v \in U(A_n^*) \setminus \Lambda_n^* \), his opponent can respond only by choosing a vertex \( w \) which is not in \( \Lambda_n^* \). If \( w \in P(A_n^*) \), then the opponent cannot prevent a loss. But we have shown that for each such vertex \( v \), the opponent can choose another vertex \( x \in U(A_n^*) \setminus \Lambda_n^* \). Thus, once a vertex \( v \) in \( U(A_n^*) \setminus \Lambda_n^* \) is chosen, each player can choose in such a way that he never chooses a vertex in \( P(A_n^*) \), and his opponent never has an opportunity to choose a vertex in \( \Lambda_n^* \). In particular, his opponent never has an opportunity to choose a terminal vertex. So each player can make his choices in such a way as to assure that he will not lose, and his opponent will not win. Hence no vertex in \( U(A_n^*) \setminus \Lambda_n^* \) is either winning or losing.

It is clear from the proof that if a player chooses a winning vertex \( v \), then the minimum number of choices in which he can be sure that he will win is equal to the smallest \( m \) such that \( v \in A_m \). If a player chooses a losing vertex \( v \), then the maximum number of choices which he can be sure of making before
he loses is equal to the smallest \( n \) such that \( v \in P(A_n) \).

**Example:**

![Graph diagram](image)

In this graph, which has no kernal, the sets \( A_1 = \{a\}; A_2 = A^*_n = \{a, c\}; \)
\( P(A^*_n) = \{b, d\}; \) and \( U(A^*_n) - A^* = \{e, f, g\} \).

**Concluding Remarks**

It should be noted that the assumption that \( V \) is a finite set is made only to simplify the presentation. If \( V \) is a set of arbitrary ordinality, we must define whether a vertex is 'winning' if it results in the choice of a terminal vertex after only a finite or after an arbitrary sequence of choices. In either case, generalization of the results presented here is straightforward.

It is not difficult to show that any kernal of a graph must contain every winning vertex, or that if a graph has no circuits every vertex is either winning or losing, since in this case \( A^*_n = U(A^*_n) \) is the unique kernal (cf. Berge p.311). (The set \( A^*_n \) is the smallest fixed-point of the kind defined in Roth [1975], and every kernal is a maximal fixed-point of the same kind.)

Like the kernal, which has the same mathematical structure as a solution of a cooperative game, the set \( A^*_n \) has the same structure as the supercore of a cooperative game (cf. Roth [1976]).
References


Roth, A. E. [1975], "A Lattice Fixed-Point Theorem With Constraints," *Bulletin of the American Mathematical Society*, vol. 81, no. 1, January

