Finding Saddle Points on Polyhedra: Solving Certain Continuous Minimax Problems

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Abstract

This paper reviews procedures for computing saddle points of continuous concave-convex functions defined on polyhedra and investigates how certain parameters and payoff functions influence equilibrium solutions. The discussion centers on two widely-studied applications: missile defense and market share attraction games. In both settings, each player allocates a limited resource, called effort, among a finite number of alternatives. Equilibrium solutions to these two-person games are particularly easy to compute either in closed form or in a finite number of steps. One of the more interesting qualitative properties we establish is the identification of conditions under which the maximizing player ignores the values of the alternatives when making its allocation decisions.
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1 Introduction

Consider the following constant-sum, two-person game. Two competitors, Player X and Player Y, allocate a single resource, called effort, among \( n \) independent alternatives. The value of the \( i \)-th alternative is \( v_i \in \mathbb{R} \equiv (-\infty, +\infty) \) and is assumed to be constant with respect to allocations made by either player. Effort allocations, therefore, determine only the apportionment of \( v_i \) among the competitors. In general, there are restrictions on the allocations that each competitor can make. Here we assume these restrictions can be expressed in terms of linear inequalities. Let \( X = (x_1, \ldots, x_n)^T \) and \( Y = (y_1, \ldots, y_n)^T \), where \( x_i, y_i \in \mathbb{R}_+ \equiv [0, \infty), i = 1, \ldots, n \), denote the allocations made by Player X and Player Y, respectively. (Here, \( T \) denotes matrix transpose.) Let \( \Omega_X \equiv \{X | X = (x_1, \ldots, x_n)^T, AX \leq a, x_i \geq 0, i = 1, \ldots, n\} \) and \( \Omega_Y \equiv \{Y | Y = (y_1, \ldots, y_n)^T, BY \leq b, y_i \geq 0, i = 1, \ldots, n\} \) denote the sets of feasible allocations for Player X and Player Y, respectively. Here \( A \) and \( B \) are \( m_1 \times n \) and \( m_2 \times n \) matrices and \( a = (a_1, \ldots, a_{m_1})^T \) and \( b = (b_1, \ldots, b_{m_2})^T \) are \( m_1 \) and \( m_2 \) vectors, respectively. We assume that both \( \Omega_X \) and \( \Omega_Y \) are bounded. Let \( Z \equiv \Omega_X \times \Omega_Y \).

Suppose that \( F : Z \to \mathbb{R} \) is a continuous, differentiable function that specifies the payoff to Player X when the allocations by the respective players are \( X \) and \( Y \), for \( (X, Y) \in Z \). Under the constant-sum assumption, the payoff to Player Y is the residual, \( \sum_{i=1}^n v_i - F(X, Y) \). Assume that for all \( Y \in \Omega_Y, F(\cdot, Y) \) is concave and that for all \( X \in \Omega_X, F(X, \cdot) \) is convex. Danskin [8] refers to problems that have this structure as finite allocation games.

The saddle point problem is to find a pair, say \((X^*, Y^*) \in Z\), such that, for any \((X, Y) \in Z\), the following saddle point inequality is satisfied:

\[
F(X, Y^*) \leq F(X^*, Y) \leq F(X^*, Y^*). \tag{1}
\]

Any \((X^*, Y^*) \in Z\) satisfying (1) is called a saddle (or Nash equilibrium) point on \( Z \).
Since $F$ is continuous on $Z$, concave on the compact set $\Omega_X$, and convex on the compact set $\Omega_Y$, a pure (non-randomized) equilibrium point on $Z$ exists. For a discussion of the existence of equilibrium points under a variety of conditions (some of which are weaker than those imposed here), see Radzik [17].

There are two primary objectives to this paper. The first is to review procedures for computing saddle points of continuous concave-convex functions defined on polyhedra, such as $Z$. Models of this type are used in a variety of game-theoretic applications. In these applications, a functional relationship between decisions made by competitors and payoffs accruing to each competitor is specified. Furthermore, the effectiveness of effort allocations are typically captured by one or more parameters in the model. The second objective of this paper is to say something about how equilibrium solutions depend upon both the form of the payoff function and the specific values of these effectiveness parameters.

In Section 2, we begin the analysis of continuous saddle point problems by examining an early application of finite allocation games to problems dealing with ballistic missile defense. In this setting, alternatives correspond to targets. The determination of saddle points in these applications are particularly tractable under certain conditions. We show that when the effectiveness of the effort expended on a target by one competitor is proportional to the effectiveness of the opponent's effort allocated to that target, equilibrium points can be computed in a finite number of steps. We also show that the optimal allocation of effort to each target is independent of the proportionality constant and that the allocations made on one of the players is (conditionally) independent of the value of the target. We use the resulting algorithm to recompute the solution to an example given by Croucher [6] that contains several errors.

In Section 3, we present a market share attraction model that is prominent in the operations research, economics, and marketing literature. This model is structurally quite similar to the missile-defense model, with the only difference being the form of the payoff function. When the effectiveness of effort parameter (called the attraction elasticity of effort) is constant across markets, saddle points can be expressed as closed-form functions of the parameters of the model. We again examine the proportional-effectiveness case, see how the proportionality constant influences the equilibrium effort allocations in this setting, and compare these results to their missile-defense counterparts.

The efficient solution procedures in each of these applications depends in a critical manner on the parametric requirement that either the effectiveness of effort is proportional or that
attraction elasticity is constant across markets. When these relationships do not hold, the saddle point problem persists but the special-purpose solution techniques can no longer be applied. Fortunately, more general solution procedures are available. While these procedures are not discussed in detail here, Section 4 provides a guide to the relevant literature.

1.1 Related Literature

The computation of saddle points of continuous concave-convex functions has been widely studied in a variety of settings in the operations research literature. The monograph by Dem'yjanov and Malozemov [11], for example, contains over 50 references to work published (mostly in the Soviet Union) prior to 1972. Much of the work published in the West deals with the solution of mathematical games of warfare—models dealing with the deployment of missiles and the structure of defenses against enemy attacks. See, for example, the early monograph by Danskin [8] and the papers by Bracken and McGill [1,2,3,4] and Croucher [6,7].

In general, the computation of saddle points of concave-convex functions can be cast as a more-or-less standard problem in nonlinear programming. See, for example Dem'yjanov and Pevnyi [9], Dem'yyanov [10], and Dem'yyanov and Malozemov [11]. Techniques such as gradient projection (Rosen [18]) and interior penalty functions (Sasai [19]) can be used to calculate (approximate) equilibrium points.

There is a related literature dealing with the computation of solutions to max-min and min-max problems. A max-min (min-max) problem is a two-person game where the minimizing (maximizing) player gets to move last after observing the strategy of the opponent. In the two-person games we consider,

\[
\max_{X \in \Omega_X} \min_{Y \in \Omega_Y} F(X, Y) = \min_{Y \in \Omega_Y} \max_{X \in \Omega_X} F(X, Y),
\]

so that the order of play is irrelevant and is typically assumed to be simultaneous. (The fact that (2) is necessary and sufficient for the existence of a saddle point on \(Z\) is called the minimax (or saddle point) theorem; see, Dem'yjanov and Malozemov [11, Lemma 6.1, page 222].) When one of the players is allowed to observe the action of the competitor and is permitted a final move, the equality in (2) generally does not hold.

The solution of min-max and max-min problems are complicated by the fact that saddle points may not exist; see Danskin [8] for a discussion of this issue. Pearsall [15] develops a double Lagrangian procedure for computing solutions to min-max problems.
Some saddle point problems possess payoff functions with special structure which, in conjunction with certain forms of restrictions on allocations, enables equilibrium points to be determined exactly, either in closed form (Monahan [13]) or by using finite-step algorithms (Croucher [6,7]).

We begin the analysis of continuous saddle point problems by examining an early application in missile defense.

2 Missile Defense Games

Consider the following two-player game:

$$\max_{x_1, \ldots, x_n} \min_{y_1, \ldots, y_n} \sum_{i=1}^{n} v_i p_i(x_i, y_i)$$  \hspace{1cm} (3)

subject to

$$\sum_{i=1}^{n} x_i = B_X$$
$$\sum_{i=1}^{n} y_i = B_Y$$
$$x_i, y_i \geq 0$$ \hspace{1cm} (4)

where $v_i > 0$ is the value of the $i$th alternative, $i = 1, \ldots, n$. The following forms of $p_i(x_i, y_i)$ appear in the literature:

$$p_i(x_i, y_i) = [1 - \exp(-\beta_i x_i)] \exp(-\alpha_i y_i)$$ \hspace{1cm} (5)
$$p_i(x_i, y_i) = x_i \exp(-\alpha_i y_i)$$ \hspace{1cm} (6)
$$p_i(x_i, y_i) = 1 - \exp(-\beta_i x_i/(1 + \alpha_i y_i)),$$ \hspace{1cm} (7)

The interpretation of $p_i(x_i, y_i)$ is context-specific. Danskin suggests that (6) represents the residual quantity of the $i$th weapon system whose initial size is $x_i$ that is attacked by $y_i$ units of "counterforce" effort. In Croucher [6] and [7], (5) and (7), respectively, represent the probability that the $i$th target is destroyed by an attacker who allocates $x_i$ units of force when it is defended by $y_i$ units of force.

Danskin [8] analyzes the game that uses (6) and proposes the game that uses (5) in end-of-chapter exercises. Croucher [6] analyzes the game that uses (5), employing the techniques suggested by Danskin. Croucher [7] applies the same form of analysis to the game that uses (7), with the added restriction that $B_X < 2/\max_i b_i$. 
Note that the equations representing the total amount of offensive and defensive effort that is available to each player can be written as weak inequalities since the stipulation that \( v_i > 0 \) for all \( i \) guarantees that it is always optimal for each player to expend all of its resources. Therefore, the inequality constraints will always be binding at an equilibrium solution.

The parameters \( \alpha_i > 0 \) and \( \beta_i > 0 \) reflect the effectiveness of effort expended by Player X and Player Y, respectively. It is easy to verify that \( p_i(x_i, y_i) \) increases as \( \beta_i \) increases and decreases as \( \alpha_i \) increases, for fixed values of \( x_i \) and \( y_i \). Therefore, we say that the effort allocated by Player X to alternative \( i \) is more effective at \( \beta_i' \) than at \( \beta_i'' \) if \( \beta_i' > \beta_i'' \). The same reasoning applies to the rival: Player Y’s effectiveness of effort increases with \( \alpha_i \).

The following characteristics of the optimal solution to (3) will not only be algorithmically useful but will lend economic insight into the nature of the equilibrium. These results were first suggested by Danskin [8] and were later reported and used by Croucher [6] and [7]; see Danskin [8, Chapter II] for the development leading to the results summarized below. We show the analysis for the problem that uses (5). The analysis of games the use either (6) or (7) are analogous.

Let \( \lambda \) and \( \mu \) denote the Lagrange multipliers associated with Player X’s and Y’s budget constraint, respectively. Under the condition that all effort is expended by both players, it is easily shown that \( \lambda > 0 \) and \( \mu > 0 \). Let \( X^* = (x_1^*, \ldots, x_n^*)^T \) and \( Y^* = (y_1^*, \ldots, y_n^*)^T \) denote an equilibrium solution to (3).

The first result summarizes Exercises 5-7 in Danskin [8, Page 18] and is restated as Lemma 7 in Croucher [6].

**Proposition 2.1** For \( i = 1, \ldots, n, \)

\[
\begin{align*}
  &a. \quad x_i^* = y_i^* = 0 \text{ if and only if } 0 \leq v_i \leq \lambda / \beta_i \\
  &b. \quad x_i^* > 0 \text{ and } y_i^* = 0 \text{ if and only if } \lambda / \beta_i < v_i \leq \mu / \alpha_i + \lambda / \beta_i \\
  &c. \quad x_i^* > 0 \text{ and } y_i^* > 0 \text{ if and only if } v_i > \mu / \alpha_i + \lambda / \beta_i.
\end{align*}
\]

From Proposition 2.1, we see that \( \mu / \alpha_i + \lambda / \beta_i \) is the “hurdle rate” that determines if Player Y allocates positive effort to alternative \( i \). The hurdle rate for Player Y is larger than \( \lambda / \beta_i \), the hurdle rate for Player X; put differently, if Player Y allocates any effort to a alternative, then Player X also allocates positive effort. The criterion used by Player X
to determine if effort should optimally be allocated to alternative $i$ is $\beta_i v_i > \lambda$ and depends upon the value of the alternative, the effectiveness of effort allocated to that alternative, and the player’s multiplier (i.e., the marginal value generated by the last unit of the budget). The criterion for Player Y depends upon the effectiveness parameters of both players, the value of the alternative, and both multipliers.

The next result gives the value of the optimal allocations in terms of the Lagrange multipliers. This result is Exercise 6 in Chapter II in Danskin [8] and is restated by Croucher [6] as Lemmas 4 and 5.

**Proposition 2.2** For $i = 1, \ldots, n$

a. if $x_i^* > 0$ and $y_i^* > 0$, then
\[
    x_i^* = \frac{1}{\beta_i} \ln[(\lambda \alpha_i + \mu \beta_i) / \lambda \alpha_i] \tag{8}
\]
\[
    y_i^* = \frac{1}{\alpha_i} \ln[\alpha_i \beta_i v_i / (\lambda \alpha_i + \mu \beta_i)]. \tag{9}
\]

b. if $x_i^* > 0$ and $y_i^* = 0$, then
\[
    x_i^* = \frac{1}{\beta_i} \ln(v_i \beta_i / \lambda). \tag{10}
\]

In light of this result, the optimal solution is completely determined once $\lambda$ and $\mu$ are computed.

Let
\[
    I_2 = \{i : x_i^* > 0 \text{ and } y_i^* > 0\} \text{ and } I_1 = \{i : x_i^* > 0 \text{ and } y_i^* = 0\}. \tag{11}
\]

Then, using (8)–(10), the value of the game $V \equiv \sum_{i=1}^n v_i p_i(x_i^*, y_i^*)$, is
\[
    V = \mu \sum_{i \in I_2} \frac{1}{\alpha_i} + \sum_{i \in I_1} \left( v_i - \frac{\lambda}{\beta_i} \right), \tag{12}
\]
which corrects a typographical error in Croucher [6] (equation (8) on page 201, which is unnumbered but is referenced on page 202).

In the next subsection, an algorithm that computes equilibrium strategies in a finite number of steps is given. The properties given in Propositions 2.1 and 2.2 are exploited for the special case where the measures of effectiveness, $\alpha_i$ and $\beta_i$, are proportional.
2.1 Proportional Effectiveness

In this section, assume that \( \alpha_i = k\beta_i, \ i = 1, \ldots, n \), for some fixed number \( k > 0 \). The reasonableness of this restriction depends upon the setting. The numerical examples in Croucher [6] and [7] satisfy this condition with \( k = 1 \). The restriction does permit differences in effectiveness of effort among the two competitors, but these differences are in some sense uniform across alternatives.

With this restriction on the parameters, there is a simple procedure for computing an equilibrium. The algorithm given below is based on a procedure reported in Croucher[6] for \( k = 1 \) (equal effectiveness). Here we formalize that procedure and make it general enough to accommodate any value of \( k \).

When \( \alpha_i = k\beta_i \), the criterion for allocating positive effort by Player X is \( \alpha_i v_i > \lambda k \) and for Player Y is \( \alpha_i v_i > \lambda k + \mu \). For notational convenience, let \( \sigma \equiv \lambda k + \mu \).

It will be convenient to express the index sets \( I_1 \) and \( I_2 \), defined in (11), in terms of the Lagrange multipliers and parameters of the problem:

\[
I_1 = \{ i \mid \lambda k < \alpha_i v_i \leq \sigma, \ i = 1, \ldots, n \} \quad \text{and} \quad I_2 = \{ i \mid \alpha_i v_i > \sigma, \ i = 1, \ldots, n \}.
\]

Player Y only allocates positive effort for alternatives whose indices are in \( I_2 \). From (9) and (8), we have

\[
y_i^* = \frac{\ln(\alpha_i v_i / \sigma)}{\alpha_i} \quad \text{and} \quad x_i^* = \frac{k \ln(\sigma / \lambda k)}{\alpha_i},
\]

for \( i \in I_2 \). Therefore,

\[
\sum_{i \in I_2} \frac{1}{\alpha_i} \ln(\alpha_i v_i / \sigma) = B_Y. \tag{14}
\]

Player X allocates positive effort only for alternatives either in \( I_1 \) or \( I_2 \). From (10),

\[
x_i^* = \frac{k \ln(\alpha_i v_i / \lambda k)}{\alpha_i} \quad \text{and} \quad y_i^* = 0
\]

for \( i \in I_1 \). Therefore,

\[
\sum_{i \in I_1} \frac{k}{\alpha_i} \ln(\alpha_i v_i / \lambda k) + \sum_{i \in I_2} \frac{k}{\alpha_i} \ln(\sigma / \lambda k) = B_X. \tag{16}
\]

It will be convenient in the sequel to solve (14) for \( \sigma \), as follows:

\[
\sigma = \exp \left\{ \left[ \sum_{i \in I_2} \frac{1}{\alpha_i} \ln(\alpha_i v_i) - B_Y \right] / \sum_{i \in I_2} \frac{1}{\alpha_i} \right\}. \tag{17}
\]
Similarly, solving (16) for $\lambda k$ yields
\[
\lambda k = \exp \left\{ \left[ \sum_{i \in I_1} \frac{1}{\alpha_i} \ln(\alpha_i v_i) + \ln(\sigma) \sum_{i \in I_2} \frac{1}{\alpha_i} - B_X / k \right] / \sum_{i \in I_1 \cup I_2} \frac{1}{\alpha_i} \right\}. \tag{18}
\]

A (simple) eleven step procedure for computing $\lambda k$, $\mu$, $x_i^*$, and $y_i^*$ for all $i$ is now given. For convenience and with no loss in generality, label the $n$ alternatives so that $\alpha_1 v_1 \geq \alpha_2 v_2 \geq \cdots \geq \alpha_n v_n$, so that $I_2$ is an initial substring of $(1, \ldots, n)$, followed by $I_1$. The algorithm is based upon two observations:

1. If $i \in I_2$, then $i - 1 \notin I_2$. To see this, suppose that $i - 1 \notin I_2$. Then $\alpha_{i-1} v_{i-1} \leq \sigma < \alpha_i v_i$, which contradicts the ordering of the $\alpha_j v_j$ values. Therefore, $I_2 = \{1, \ldots, j\}$, for some positive integer $j \leq n$. The fact that $j \geq 1$ is necessary since Player Y must allocate positive effort to at least one alternative.

2. If $i \in I_1$, then $i - 1 \notin I_1$ for $j + 2 \leq i \leq n$. To see this, suppose that $i \in I_1$ and $i - 1 \notin I_1$. Then exactly one of two conditions must hold—either $\alpha_{i-1} v_{i-1} > \sigma$, which contradicts the fact that $\alpha_{i-1} v_{i-1} \leq \alpha_{j+1} v_{j+1} < \sigma$ or $\alpha_{i-1} v_{i-1} \leq \lambda k$, which contradicts the fact that $\alpha_{i-1} v_{i-1} \geq \alpha_i v_i > \lambda k$. Therefore, $I_1 = \emptyset$ or $I_1 = \{j+1, \ldots, m\}$, for some positive integer $m \leq n$.

These observations simplify the identification of the subsets $\{1, \ldots, n\}$ that constitute both $I_1$ and $I_2$. Indeed, one way to solve the problem is to enumerate all of the subsets of the integers that might possibly constitute $I_1$ and $I_2$. Even for modest values of $n$, this procedure becomes cumbersome, however. The construction of $I_1$ and $I_2$ suggested in the Proportional Effectiveness Algorithm circumvents the necessity of doing a compete enumeration and is therefore highly efficient.

The algorithm proceeds in the following manner. In Steps 1–4, $\sigma$ is computed and $I_2$ is determined. In Steps 5–10, $\lambda k$ is computed and $I_1$ is determined. The optimal strategies are then directly computable from the formulas in Propositions 2.1 and 2.2.

Let $\sigma_j$ be the value of $\sigma$ when $I_2 = \{1, \ldots, j\}$ and $\lambda_m$ be the value of $\lambda$ when $I_1 = \{j+1, \ldots, m\}$ and $I_2 = \{1, \ldots, j\}$.

**Proportional Effectiveness Algorithm**

1. Set $j = 0$. 
2. Increment \( j \) by 1. If \( j > n \), go to Step 5; otherwise, set \( I_2 = \{1, \ldots, j\} \) and compute \( \sigma = \sigma_j \) using (17).

3. If \( \alpha_j v_j > \sigma_j \), go to Step 2; otherwise, set \( j \) to \( j - 1 \).

4. Set \( \sigma \) to \( \sigma_j \) and \( I_2 = \{1, \ldots, j\} \).

5. Set \( m \) to \( j \). If \( m \leq n - 1 \), go to Step 7.

6. Set \( I_1 = \emptyset \), and let

\[
\lambda k = \exp \left\{ \left[ \ln(\sigma) \sum_{i \in I_2} \frac{1}{\alpha_i} - B_X/k \right] / \sum_{i \in I_2} \frac{1}{\alpha_i} \right\};
\]

go to Step 11.

7. Increment \( m \) by 1. If \( m > n \), go to Step 10; otherwise, set \( I_1 = \{j + 1, \ldots, m\} \) and compute \( \lambda_m k \) using (18).

8. If \( \lambda_m k \geq \alpha_m v_m \) and \( m = j + 1 \), go to Step 6.

9. If \( \lambda_m k < \alpha_m v_m \), go to Step 7.

10. Set \( m \) to \( m - 1 \), set \( I_1 = \{j + 1, \ldots, m\} \), and set \( \lambda k \) to \( \lambda_m k \).

11. Set \( \mu = \sigma - \lambda k \). Stop.

The following notes explain in more detail what is being done in the algorithm.

1. In Steps 2–4, the objective is to find an index \( j \) such that \( \sigma_j < \alpha_j v_j \) and \( \sigma_{j+1} \geq \alpha_{j+1} v_{j+1} \). This index \( j \) defines \( I_2 \) and \( \sigma = \sigma_j \).

2. With \( \sigma \) and \( I_2 \) both determined, the associated \( y_i^* \) are computable, using (13).

3. If the index \( j \), determined in Steps 2–4, is equal to \( n \), then \( I_1 \) is empty. In this case, the formula for \( \lambda k \), given in (18), simplifies to the expression given in Step 6.

4. In Steps 7–10, \( \lambda k \) and \( I_1 \) are determined when \( I_1 \) is not empty. The objective is to find an index \( m \), such that \( \lambda_m k \leq \alpha_m v_m \) and \( \lambda_{m+1} k > \alpha_{m+1} v_{m+1} \). This index \( m \) defines \( I_1 \) and \( \lambda k = \lambda_m k \).
Table 1: A Numerical Example

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_i$</th>
<th>$v_i$</th>
<th>$x_i^*$</th>
<th>$y_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20</td>
<td>70</td>
<td>10.50</td>
<td>4.66</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>80</td>
<td>7.00</td>
<td>4.90</td>
</tr>
<tr>
<td>3</td>
<td>0.20</td>
<td>100</td>
<td>10.50</td>
<td>6.44</td>
</tr>
</tbody>
</table>

With both $\lambda k$ and $\sigma$ determined, $\mu$ is computed in Step 11. The associated $x_i^*$ are determined using (13) and (15).

The algorithm was tested on the numerical example in Croucher [6], which contains a number of typographical and computational errors. A solution to the problem is given in Table 1 and corrects the solution given in Table 1 in Croucher [6]. In this example, $k = 1$, $B_X = 28$, and $B_Y = 16$.

The value of the game $V$, $\lambda$, and $\mu$ are also incorrectly reported in Croucher [6]. The correct values are $V = 64.53$, $\lambda = 0.675$ and $\mu = 4.84$.

If the value of alternative 3 is changed to $v_3 = 250$, the optimal allocations for Player Y are $y_1^* = 2.94$, $y_2^* = 3.76$, and $y_3^* = 9.30$, which are the same as those reported by Croucher [6]. The optimal allocations by Player X are those reported in Table 1 here and differ from those in Croucher [6].

In the next subsection, some qualitative characteristics of “proportional effectiveness” equilibrium strategies are discussed.

2.2 Qualitative Properties of Equilibrium Solutions

The first result describes the dependence of Player X’s and Y’s optimal allocations on the parameter $k$. We wish to determine, for example, how the optimal allocations to alternative $i$ differ if Player X is ten times as effective as Player Y (i.e., $\alpha_i = 0.1\beta_i$) versus the situation where Player Y is ten times as effective as Player X (i.e., $\alpha_i = 10\beta_i$).

First, a preliminary result.

Lemma 2.1 $\sigma$, given in (17), is constant with respect to $k$.

Proof The construction of $I_2$ in Steps 2–4 of the Proportional Effectiveness Algorithm depends only on the alternative-specific values of $\alpha_i$, $v_i$, and is independent of $k$. Therefore,
\(\sigma\) is also independent of \(k\). \(\square\)

Recall that \(\sigma = \lambda k + \mu\) and therefore \(k\) appears in the definition of the set \(I_2\). Lemma 2.1 highlights the fact, however, that \(k\) does not influence the magnitude of \(\sigma\) but only determines its composition. The notation \(\sigma = \lambda k + \mu\) gives a somewhat distorted view of \(\sigma\)'s "dependence" on \(k\), since both \(\lambda\) and \(\mu\) also depend upon \(k\).

Lemma 2.1 leads readily to the first characterization result.

**Proposition 2.3**

a. \(y_i^*\) is independent of \(k\) for all \(i\).

b. If \(y_i^* > 0\) for all \(i\), then \(x_i^*\) is independent of both \(k\) and \(v_i\) for all \(i\).

**Proof**

When \(\alpha_i = k\beta_i\), the formulas for the optimal allocations to alternative \(i\), (8) and (9), reduce to the expressions in (13) and (15).

That \(y_i^*\) is independent of \(k\) follows directly from Lemma 2.1.

We will now show that \(x_i^*\) is also independent of both \(v_i\) and \(k\) when \(y_i^* > 0\) for all \(i\).

Since \(y_i^* > 0\) for all \(i\), we know from Proposition 2.1-c that \(x_i^* > 0\) for all \(i\). Therefore, \(I_1 = \emptyset\).

From (16), we have \(\ln(\sigma/\lambda k) = B_X/(k \sum_{i \in I_1} 1/\alpha_i)\). Substituting this into (13) yields

\[
x_i^* = B_X \frac{1/\alpha_i}{\sum_{j=1}^{n} 1/\alpha_j},
\]

which is independent of both \(k\) and \(v_i\). \(\square\)

Part b more accurately establishes that \(x_i^*\) is conditionally independent of \(v_i\). Player Y's allocations depend upon each of the \(v_i\)'s. If the values of the alternatives are such that Player Y allocates positive effort to each of them, Player X follows suit. Part b establishes the interesting fact that the apportionment of \(B_X\) that is made to the \(i\)th alternative is based solely on the relative size of \(1/\alpha_i\). The least effective alternative receives the highest allocation. Although Player X seeks to maximize \(\sum_{i=1}^{n} v_i p_i(x_i, y_i)\), the optimal strategy is defensive in character: if all of the alternatives are worth fighting for, effort is allocated only on the basis of its relative effectiveness.

The other interesting feature of this result is the fact that if both competitors find it in their best interests to compete over all of the alternatives, the proportionality of effectiveness \(k\) does not influence either player's allocations.

It is straightforward to show by numerical example that when \(y_i^* = 0\) or when \(x_i^* > 0\), \(y_i^* > 0\), and \(I_1 \neq \emptyset\), \(x_i^*\) can depend upon \(k\). Figures 1 and 2 depict optimal allocations to alternative 1 when \(n = 2\), \(B_X = 10\), \(B_Y = 5\), \(\alpha_1 = \alpha_2 = 0.4\), \(v_1 = 100\), and \(v_2 = 500\) when
$k = 10$ and $k = 0.1$, respectively. The optimal allocations for alternative 2 are simply each player’s budget value less their optimal allocation to alternative 1. Note what happens when $\alpha_1$ is low. Suppose, for example, that $\alpha_1 = 0.10$. Then $x_i^* = 0$ when $k = 10$ but is positive when $k = 0.10$. Here is a case where $x_i^* > 0$, $y_i^* > 0$, but $I_1 \neq \emptyset$ (i.e., $y_1^* = 0$) and $x_i^*$ depends on $k$ for $i = 1, 2$. The optimal allocations are identical for both values of $k$ when all of the alternatives have positive allocations. This occurs when $\alpha_1 > 0.27$, approximately.

The next results indicate the impact that changes in the effectiveness parameter of a single alternative has on both competitors’ allocations.

For convenience, let $y_i(\alpha_i)$ and $x_i(\alpha_i)$ denote the optimal allocations made by Players X and Y, respectively, when the level of effectiveness is $\alpha_i$. The next result establishes the fact that if it is optimal for Player Y to allocate effort to the $i$-th alternative for some level of effectiveness, then it is optimal to allocate effort to that alternative for any higher level of effectiveness. Since Player X allocates effort to all alternatives that Player Y allocates effort, the same result holds for Player X.

**Proposition 2.4** Assume that $y_i(\alpha_i^*) > 0$ for some $i$, where $\alpha_i = \alpha_i^*$. Then $y_i(\alpha_i) > 0$ and $x_i(\alpha_i) > 0$ for all $\alpha_i > \alpha_i^*$.

**Proof** Let $I_2(\alpha_i)$ and $\sigma(\alpha_i)$ denote the dependence of the index set $I_2$ and $\sigma$ on $\alpha_i$, respectively. The hypothesis that $y_i(\alpha_i^*) > 0$ ensures that $\alpha_i^* v_i > \sigma(\alpha_i^*)$, so that $i \in I_2(\alpha_i^*)$. Suppose that for some $\alpha_i > \alpha_i^*$, $\alpha_i v_i \leq \sigma(\alpha_i)$, so that $i \notin I_2$. We now show that this cannot occur: $\alpha_i^* v_i > \sigma(\alpha_i^*)$ is equivalent to

$$
\ln(\alpha_i^* v_i) \sum_{j \in I_2(\alpha_i^*)} \frac{1}{\alpha_j} > \sum_{j \in I_2(\alpha_i^*)} \frac{1}{\alpha_j} \ln(\alpha_j v_j) - B_Y.
$$

Subtracting $\ln(\alpha_i^* v_i)/\alpha_i^*$ from both sides and simplifying, yields:

$$
\ln(\alpha_i^* v_i) = \frac{\sum_{j \in I_2(\alpha_i^*)} \frac{1}{\alpha_j} \ln(\alpha_j v_j) - B_Y}{\sum_{j \in I_2(\alpha_i^*)} \frac{1}{\alpha_j}}.
$$

Let $K$ denote the right-hand-side of (20). Note that $K$ is independent of $\alpha_i$. Since $\ln(\alpha_i)$ is an increasing function of $\alpha_i$, $\ln(\alpha_i) > K$ for $\alpha_i > \alpha_i^*$, so that $i$ remains in $I_2$ and $y_i^*(\alpha_i) > 0$.

The next result states that if the Y-player allocates effort to the $i$-th alternative when the level of effectiveness $\alpha_i^*$, then the X-player’s effort is nonincreasing in $\alpha_i$ on $[\alpha_i^*, \infty)$.
Proposition 2.5 Assume that for effectiveness level $\alpha_i^*$, $y_i(\alpha_i^*) > 0$ for all $i$. Then $x_i(\alpha_i)$ is nonincreasing on the interval $[\alpha_i^*, \infty)$; i.e., for any $\alpha_i^* \leq \alpha_i' < \alpha_i''$, $x_i^*(\alpha_i') \geq x_i^*(\alpha_i'')$.

Proof Since $y_i(\alpha_i^*) > 0$, $i \in I_2$ and $y_i(\alpha_i) > 0$ for all $\alpha_i > \alpha_i^*$ by Proposition 2.4. That $x_i(\alpha_i) > 0$ decreases on $[\alpha_i^*, \infty)$ follows directly from (19). □

Note that the condition that $y(\alpha_i^*) > 0$ for all $i$ is necessary. Figure 2 illustrates that $x_i(\alpha_i)$ can increase in $\alpha_i$ if $y_i = 0$.

2.3 Generality of the Procedure

An analogous version of the Proportional Effectiveness Algorithm can be used to compute the saddle point for each of the games when $p_i(x_i, y_i)$ has one the forms (5)-(7). There are two features that these models share that facilitate this type of analysis. First, the form of the payoff function in each of the three cases is such that the first-order conditions associated with allocations made by each player to the $i$th alternative can be solved explicitly so that $x_i^*$ and $y_i^*$ are expressed only in terms of the Lagrange multipliers and parameters of the problem; i.e., the form of the payoff function makes it straightforward to simultaneously solve

$$\frac{\partial (v_i p_i(x_i, y_i))}{\partial x_i} = \lambda$$
$$\frac{\partial (v_i p_i(x_i, y_i))}{\partial y_i} = \mu$$

for $x_i^*$ and $y_i^*$ in terms of $\lambda$, $\mu$, and the parameters of the problem. While the examples given here involve exponential functions, the procedure is in principle generalizable to any payoff function that possesses this separability property. An example of a non-exponential payoff function that has this property is given in the next section.

The second feature of these models is that set of alternatives that receive positive allocations by either one or both players can be partitioned on the basis of the Lagrange multipliers and parameters of the problem. This feature allows a systematic determination of the multipliers and therefore the saddle point.

Another form of finite allocation game is now examined. This game shares the first property discussed above but not the second. Equilibrium solutions can still be determined in closed form for some versions of the problem, however.
3 Market Share Attraction Games

A competitive effort allocation model that appears extensively in the economics, management science, and marketing literature is the market share attraction model. See, for example, Schmalensee [20], Shapley and Shubik [21], Case [5, Chapter 4], Ponsard [16, Chapter 3], Shakun [22], Monahan [13], and Monahan and Sobel [14]. In attraction models, a market of a fixed size is partitioned among a number of competitors on the basis of the relative size of the “effective” effort expended by each competitor. When there are two competitors, the share of the market accruing to one competitor has the form

\[
\text{"my effective effort"} \over \text{"my effective effort"} + \text{"your effective effort"}.
\]

In a marketing context, the \( n \) alternatives of the general model corresponds to \( n \) product markets. The effective effort resulting from an allocation of \( x_i \) units of effort by Player X can be thought of as the effectiveness of an advertising campaign directed at product \( i \) that costs \( s_i \). The result of this effort allocation is an “attraction” of \( \alpha_i x_i^\gamma \). The parameter \( \gamma_i \) is the attraction elasticity of effort, which is a measure of the responsiveness of sales (in this case) to effort and is defined as

\[
\gamma_i = \frac{d}{dx_i} \left( \frac{\alpha_i x_i^\gamma}{\beta_i x_i^\gamma} \right).
\]

(Here \( d(f(x))/dx \) represents the derivative of \( f(\cdot) \) with respect to \( x \).) The effectiveness of effort resulting in \( y_i \) units allocated by Player Y is \( \alpha_i y_i^\gamma \). The proportion of the \( i \)th market (i.e., the “market share”) accruing to Player Y is

\[
p_i(x_i, y_i) = \left( \frac{\alpha_i y_i^\gamma}{\alpha_i y_i^\gamma + \beta_i x_i^\gamma} \right). \quad (21)
\]

The share of the \( i \)th market accruing to Player X is the residual of that going to Player Y, namely \( 1 - p_i(x_i, y_i) \).

As in the missile-defense model, suppose that a single budget constraint is the only restriction on the effort allocations made by each player. The problem then, is the same as (3) and (4), with \( p_i(x_i, y_i) \) given by (21). This is a special case of a model analyzed by Monahan [13]. (Here we assume for simplicity that the parameters \( M_i \) and \( m_i \) in [13] are equal to 1 for all \( i \).) The solution to this problem is computable in closed-form: let \((X^*, Y^*)\) denote the saddle point on \( Z \). Then

\[
x_i^* = B_X \frac{T_i}{\sum_{j=1}^n T_j} \quad \text{and} \quad y_i^* = B_Y \frac{T_i}{\sum_{j=1}^n T_j}, \quad (22)
\]
where

\[ T_i = v_i \gamma_i \alpha_i \beta_i \rho^n / (\beta_i + \alpha_i \rho^n)^2 \]  
\[ (23) \]

and \( \rho = B_Y / B_X \). See Monahan [13] for details regarding the derivation of this solution, as well as extensions of the model that include the determination of optimal budgets.

Monahan and Sobel [14] study a dynamic, stochastic version of the market share attraction game specified by (3) and (4). They formulate a sequential game whose (discounted) equilibrium point corresponds to the equilibrium point of a static (one-period) game that is analogous to (3), where \( p_i \) has the market share attraction form given in (21). Thus, the equilibrium point of the dynamic stochastic game can also be determined in closed-form.

### 3.1 Comparing Qualitative Characteristics

We now make comparisons between certain characteristics of optimal allocations in the market share model and in the missile-defense problem.

Suppose, as we did in Section 2.1, that the effectiveness of effort between Player X and Player Y are proportional; i.e., suppose that \( \alpha_i = k \beta_i \) for all \( i \). Then, the expression for \( T_i \) given in (23) simplifies to:

\[ T_i = \frac{v_i \gamma_i \rho^n}{(1 + k \rho^n)^2}. \]  
\[ (24) \]

When the elasticity parameter is the same across markets, i.e., \( \gamma_1 = \ldots = \gamma_n = \gamma \), (24) simplifies even further to

\[ T_i = \frac{v_i}{\sum_{j=1}^{n} v_j}. \]  
\[ (25) \]

We use these observations, in conjunction with (22), to characterize the dependence of the competitive allocations on the effectiveness of effort parameter.

**Proposition 3.1** Suppose that \( \alpha_i = k \beta_i \) for all \( i \).

a. Competitive effort allocations are independent of market specific effectiveness—only relative effectiveness, which is the same across markets, matters; i.e., \( x_i^* \) and \( y_i^* \) are independent of \( \beta_i \) and \( \alpha_i \) but do depend upon \( k \), which is the same for all \( i \).

b. When \( \gamma_1 = \ldots = \gamma_n = \gamma \), competitive effort allocations are independent of all measures of effort effectiveness; i.e., \( x_i^* \) and \( y_i^* \) are not only independent of \( \alpha_i \) and \( \beta_i \), they are independent of \( k \) for all \( i \).
When the effectiveness parameters are proportional in the market share model, it is not surprising that the solution depends on neither \( \alpha_i \) nor \( \beta_i \). It is somewhat surprising, however, to see that the equilibrium effort allocations do not depend upon \( k \) when the elasticity measures in both markets are the same. (Since there is no counterpart to the attraction elasticity parameter in the missile-defense model, the assumption that \( \gamma_i = \gamma \) for all \( i \) seems reasonable when comparing the solutions to the two models.) Part (b) is, therefore, much in the spirit of Proposition 2.3. When inter-market differences are permitted in Part (a), equilibrium allocations are sensitive to the value of \( k \) and the direction of the reaction to changes in \( k \) depend on the relative size of each competitor’s budget.

The next result indicates how \( k \) influences competitive allocations in the market share model. For simplicity, we only consider the case where there are two markets.

**Proposition 3.2** Suppose that \( n = 2 \). If \( \gamma_i < \gamma_{3-i}, \ i = 1, 2, \) then \( x_i^* \) and \( y_i^* \) are increasing (decreasing) in \( k \) if \( \rho > (\leq) 1 \).

**Proof** Since the dependence of \( x_i^* \) and \( y_i^* \) on \( k \) are identical, we establish the result only for \( x_i^* \). From (22), \( x_i^* = B_X T_i/(T_i + T_{3-i}) \), for \( i = 1, 2 \). Let \( T_i' \) denote the derivative of \( T_i \) with respect to \( k \). Then

\[
T_i' = \frac{d}{dk} \left( \frac{v_i \gamma_i \rho^n}{(1 + k \rho^n)^2} \right)
\]

\[
= \frac{-2v_i \gamma_i \rho^{2\gamma_i}}{(1 + k \rho^n)}
\]

\[
= -2\rho^n T_i/(1 + k \rho^n).
\]

Using (22),

\[
\frac{\partial x_i^*}{\partial k} = B_X \left[ \frac{T_{3-i} T_i' - T_i T_{3-i}'}{T_i + T_{3-i}} \right].
\]

Substituting (26) into this expression yields,

\[
\text{Sign} \left( \frac{\partial x_i^*}{\partial k} \right) = \text{Sign} \left( B_X \left[ -2T_i T_{3-i} [\rho^{\gamma_i}/(1 + k \rho^{\gamma_i}) - \rho^{\gamma_{3-i}}/(1 + k \rho^{\gamma_{3-i}})] \right] \right)
\]

\[
= \text{Sign} \left( -2B_X T_i T_{3-i} \left[ \frac{\rho^{\gamma_i}(1 + k \rho^{\gamma_{3-i}}) - \rho^{\gamma_{3-i}}(1 + k \rho^{\gamma_i})}{(1 + k \rho^n)(1 + k \rho^{\gamma_{3-i}})} \right] \right)
\]

\[
= \text{Sign} (\rho^{\gamma_{3-i}} - \rho^{\gamma_i}),
\]

which, under the hypothesis that \( \gamma_i < \gamma_{3-i} \), is \( < (>) 0 \) if \( \rho < (>) 1 \). \( \square \)

Note also that since \( n = 2 \) and each competitor always uses up its entire budget, if \( x_i^* \) is increasing (decreasing) in \( k \), then \( x_{3-i}^* \) is decreasing (increasing) in \( k \).
4 The General Saddle Point Problem

The determination of a closed-form solution to the market share attraction game depends critically on the requirement that the attraction elasticities in each market ($\gamma_i$) are the same for both competitors. If this condition does not hold, a different solution technique is needed. We conclude with a brief discussion of the general problem of finding saddle points on $Z$.

The saddle point problem can be formulated as the optimization of an extremal-value function. For $X \in \Omega_X$, let

$$\phi(X) = \min_{Y \in \Omega_Y} F(X, Y). \quad (27)$$

The problem of finding a saddle point on $Z$ is equivalent to maximizing the extremal-value function $\phi(X)$ on $\Omega_X$; i.e.,

$$\max_{X \in \Omega_X} \phi(X). \quad (28)$$

Hogan [12] studies the optimization and structure of extremal-value functions, which include (28) as a special case. The saddle point problem can be restated in other equivalent forms. Rosen [18], for example, considers the more general problem of solving $n$-person concave games that also include (28) as a special case.

A gradient projection algorithm proposed by Rosen [18] (who formulated a dynamic, continuous-time version of the model), Dem'yanov [10], and Dem'yanov and Pevnyi [9] can be used to solve the general saddle point problem. The details of the algorithm are beyond the scope of this paper. The procedure is highly computationally intensive and computes solutions that are only guaranteed to be within a prescribed distance of the saddle point. (i.e., the algorithm does not guarantee that it will determine the exact saddle point in a finite number of iterations.)

5 Summary

This paper reviewed several techniques for computing saddle points of continuous, concave-convex functions defined on polyhedra. Fast and efficient algorithms that exploit special properties of the payoff functions in two widely-studied applications were presented. References are given to less efficient procedures for determining (approximate) saddle points for the general problem. Several results relating the dependence of equilibrium allocations on the values of certain parameters in the models were also established.
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References


Figure 1: Equilibrium allocations as a function of $\alpha_1$ when $k = 10$. 
Figure 2: Equilibrium allocations as a function of $\alpha_1$ when $k = 0.1$. 