ENUMERATIVE INVARIANTS FOR LOCAL CALABI-YAU THREEFOLDS

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DISSERTATION
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Abstract

This thesis consists of three parts. In the first part, we compute the topological Euler characteristics of the moduli spaces of stable sheaves of dimension one on the total space of rank 2 bundle on $\mathbb{P}^1$ whose determinant is $\mathcal{O}_{\mathbb{P}^1}(-2)$. We count the torus fixed stable sheaves of low degrees and show the results verify the predictions in physics and the local Gromov-Witten theory studied in [7]. In the second part, we compute the Poincaré polynomial of the moduli space of stable sheaves with Hilbert polynomial $4n + 1$ on $\mathbb{P}^2$. This is done by classifying all torus fixed points in the moduli space and computing the torus representation of their tangent spaces. The result is also in agreement with a computation in physics. In the third part, we propose an algorithm to compute the Euler characteristics of the moduli spaces of stable sheaves of dimension one on $\mathbb{P}^2$ by means of Joyce’s wall crossing formula. The wall crossing takes place over the moduli spaces of $\alpha$-stable pairs as the stability parameter $\alpha$ varies. The results verify a conjecture in the theory of curve counting invariants motivated by physics.
In memory of Youngsoo Cho (1939 - 2008)

to my family

Sumin, Jeongwoo, and Hayeon.
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Chapter 1

Introduction

A Calabi-Yau threefold $X$ is an algebraic variety of complex dimension three with trivial canonical bundle. For a given homology class $\beta \in H_2(X)$, the expected dimension of the space of curves of class $\beta$ is zero. Counting the number of curves in $X$ is a long standing problem in classical enumerative geometry.

One of powerful tools in algebraic geometry to approach such problems is to consider a space of isomorphism classes of objects in question. Geometric invariant theory (GIT) provides with tools to construct such spaces as algebraic varieties. One would like to define curve counting invariants by intersecting the fundamental classes on such algebraic varieties parametrizing the curves on $X$. However, we can not define an intersection theory on the algebraic variety parametrizing curves in $X$ because it is highly singular and not compact.

The recent development of virtual intersection theory enables us to define curve counting invariants which are constant under deformation. A rigorous mathematical definition of the virtual fundamental class is given by Li and Tian [34] and Behrend and Fantechi [2] via a perfect obstruction theory. One necessary condition for the existence of the virtual fundamental class is that the moduli space is compact. So, we need a compactification of the moduli space of curves of class $\beta$ in $X$. There are at least four known compactifications which admit a perfect obstruction theory.

The first compactification is the Hilbert scheme, where one considers a curve $C$ in $X$ as the ideal sheaf $I_C$ of $C$. The Hilbert scheme is a projective scheme formed by the collection
of ideal sheaves with fixed Hilbert polynomial. In his thesis [45], Thomas has shown the existence of the virtual class. So, invariants are defined as the intersection numbers with this class, called the Donaldson-Thomas (DT) invariants. The second compactification is via the moduli space of stable maps defined by Kontsevich. We consider a curve in $X$ as a parametrization, in other words, a map from a curve $C$ to $X$. Li and Tian [34] and Behrend and Fantechi [2] have shown that there is a perfect obstruction theory on this space. The resulting invariants are called the Gromov-Witten (GW) invariants. Another compactification is provided by stable pairs. One can think of a curve $C$ as a sheaf $\mathcal{O}_C$ together with a canonical global section $\mathcal{O}_X \to \mathcal{O}_C$. In [40], Pandharipande and Thomas consider such pairs as objects in the derived category and show there is a perfect obstruction theory. The obtained virtual intersection numbers are called Pandharipande-Thomas (PT) invariants. The DT, GW, and PT invariants are conjectured to be equivalent in terms of the generating functions [37, 40]. The last compactification is by considering a curve $C$ as a sheaf $\mathcal{O}_C$. Simpson [44] has proved the collection of stable sheaves with fixed Hilbert polynomial forms a projective scheme. Thomas [45] has proved there is a perfect obstruction theory if there is no strictly semistable sheaves. So, we obtain a virtual class. Recently, Joyce and Song [24] have extended the definition to the case where there exist strictly semistable sheaves. They have shown that the resulting invariants are constant under the deformation. The invariants are called the generalized Donaldson-Thomas invariants.

In this thesis, we are interested in the last compactification and its relation with the Gromov-Witten invariants. Let $\overline{M}_{g,n}(X, \beta)$ denote the moduli space of stable maps $f: C \to X$ from genus $g$ curves $C$ with $n$ marked points such that $f_*[C] = \beta$. This moduli space admits a virtual cycle $[\overline{M}_{g,n}(X, \beta)]^{vir}$ whose expected dimension is $c_1(X) \cdot \beta + (\dim X - 3)(1 - g) + n$. When $X$ is a Calabi-Yau threefold and $n = 0$, the expected dimension is zero.
and the Gromov-Witten invariant is defined as the degree of the virtual cycle.

\[
N_{g}^{\beta}(X) := \deg[M_{g,0}(X, \beta)]^{\text{vir}}.
\]

By the BPS state counts in M-theory, Gopakumar and Vafa [13] have proposed integer-valued invariants \(n_{g}^{\beta}(X)\) of \(X\), called the BPS invariants, which are related to the Gromov-Witten invariants by the Gopakumar-Vafa formula

\[
\sum_{\beta, g} N_{g}^{\beta}(X) q^{\beta} \lambda^{2g-2} = \sum_{\beta, g, k} n_{g}^{\beta}(X) \frac{1}{k} \left(2 \sin \left(\frac{k \lambda}{2}\right)\right)^{2g-2} q^{k \beta}.
\]

A priori, the BPS invariants defined by above formula are rational numbers because the Gromov-Witten invariants are rational numbers. The integrality conjecture is an assertion that they are integers.

Katz [26] has proposed a mathematical definition for the genus zero BPS invariants. Let \(M_{X}(\beta)\) be the moduli space of semistable sheaves \(F\) on \(X\) such that the support of \(F\) has class \(\beta \in H_{2}(X)\) and the Euler characteristic \(\chi(F)\) is 1. Since \(\chi(F) = 1\), semistable sheaves are necessarily stable. Hence by the work of Thomas [45], if \(X\) is a Calabi-Yau threefold, the moduli space \(M_{X}(\beta)\) admits a symmetric perfect obstruction theory. Thus, we may define Donaldson-Thomas type invariants as follows.

**Definition 1.0.1.** Define the genus zero BPS invariant by the Donaldson-Thomas type invariant

\[
n_{\beta}(X) = \deg[M_{X}(\beta)]^{\text{vir}} \in \mathbb{Z}.
\]

If we take this definition, then the genus zero part of the formula (1.1) is a conjecture.

**Conjecture 1.0.2.** Let \(X\) be a Calabi-Yau threefold. For \(\beta \in H_{2}(X, \mathbb{Z})\), let \(N_{0}^{\beta}(X)\) be the
genus zero Gromov-Witten invariant. The Gopakumar-Vafa formula

\[ N^0_\beta(X) = \sum_{m \mid \beta} \frac{n^0_{\beta/m}(X)}{m^3}. \tag{1.2} \]

holds.

Katz [26] has shown that (1.2) holds for embedded contractible rational curves. Li and Wu [35] have verified (1.2) for K3 fibred local Calabi-Yau threefolds for curve classes \(d\beta_0\), where \(d \leq 5\) and \(\beta_0\) generates the Picard group of the central fiber.

Our main focus is on computing the BPS invariants when the Calabi-Yau threefold \(X\) is local \(\mathbb{P}^1\) or local \(\mathbb{P}^2\). Local \(\mathbb{P}^1\) is the total space of \(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-k) \to \mathbb{P}^1\) for an integer \(k \geq -1\), and local \(\mathbb{P}^2\) is the total space of \(\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2\). In case of local \(\mathbb{P}^1\), as the moduli spaces of stable sheaves are not compact, we need to extend the definition of invariants via the virtual torus localization (Section 2.1), and compare them with the equivariant Gromov-Witten invariants similarly defined. Meanwhile, although local \(\mathbb{P}^2\) itself is not compact, its moduli spaces is compact (Lemma 3.1.5). Hence the above definition is valid.

In either cases, the moduli spaces are smooth. By the general theory of symmetric obstruction theory [1], we have

\[ n_\beta(X) = (-1)^{\dim M_X(\beta)} e_{\text{top}}(M_X(\beta)), \]

where \(e_{\text{top}}(-)\) denote the topological Euler characteristic.

Since local \(\mathbb{P}^1\) and local \(\mathbb{P}^2\) are toric varieties, we have a natural action of torus on \(X\), which in turn induces a torus action on the moduli space \(M_X(\beta)\). For a quasi-projective variety with a torus action, its Euler characteristic is given by the Euler characteristic of the torus fixed locus (Theorem 2.1.4). In Chapter 2, we compute the Euler characteristics of the moduli space \(M_X(\beta)\) when \(X\) is local \(\mathbb{P}^1\) by counting torus fixed points in the moduli spaces.
When $X$ is the total space of $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2 - k)$ and $\beta$ is the class $d[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$, we denote the moduli space $M_X(\beta)$ by $M_d(k)$. In Section 2.6 we show that $M_d(k)$ is smooth variety of dimension $kd^2 + 1$.

The torus fixed sheaves are precisely the equivariant sheaves under the torus action. We use the classification of pure torus equivariant sheaves on toric varieties studied by Kool [30]. If we take two torus invariant affine open sets in local $\mathbb{P}^1$, the torus equivariant sheaves are characterized by its weight space decompositions on each open set. Such weight space decompositions must admit an $\mathcal{O}_X$-module structure and satisfy certain gluing conditions (Theorem 2.2.1). We can graphically represent an equivariant sheaf by a diagram that contains all information of its weight spaces in each open set. In Section 2.3 we show that we can extract many data about the associated sheaf from such a diagram. We then classify stable equivariant sheaves on local $\mathbb{P}^1$ in Section 2.4. For $1 \leq d \leq 3$, the torus fixed locus of $M_d(k)$ consists of isolated points, and when $d = 4$, it contains positive dimensional components isomorphic to $\mathbb{P}^1$. We describe them in Sections 2.5 and 2.5.4. By computing the Euler characteristics of the torus fixed locus, we compute the BPS invariants. We show the following.

**Theorem 1.0.3.** Let $n_d(k)$ denote the equivariant BPS invariant for local $\mathbb{P}^1$. Then,

$$
n_1(k) = (-1)^{k+1},
$$

$$
n_2(k) = \begin{cases} 
-\frac{k(k+2)}{4} & \text{if } k \text{ is even}, \\
-\frac{(k+1)^2}{4} & \text{if } k \text{ is odd},
\end{cases}
$$

$$
n_3(k) = (-1)^{k+1} \frac{k(k+1)^2(k+2)}{6},
$$

$$
n_4(k) = -\frac{k(k+1)^2(k+2)(2k^2 + 4k + 1)}{12}, \text{ for } k \leq 100.
$$
This coincides with the prediction in Gromov-Witten theory and the equivariant version of Conjecture 1.0.2 (Conjecture 2.1.3).

Let \( M_{\mathbb{P}^2}(d, \chi) \) denote the moduli space of stable sheaves on the projective plane \( \mathbb{P}^2 \) with Hilbert polynomial \( dn + \chi \). In Chapter 3 we compute the Poincaré polynomial of \( M_{\mathbb{P}^2}(4, 1) \). We continue to use the classification of torus equivariant sheaves on a toric variety. By the theory of Bialynicki-Birula [3, 4], we show \( M_{\mathbb{P}^2}(4, 1) \) has a decomposition into affine bundles over the connected components of its torus fixed locus. The rank of each affine bundle can be computed by looking at the torus representation of the tangent space at the associated fixed point. The main result in Chapter 3 is the following (Theorem 3.1.10).

**Theorem 1.0.4.** Under the natural action of the torus \( T = (\mathbb{C}^*)^2 \), the fixed point locus of \( M_{\mathbb{P}^2}(4, 1) \) consists of 180 isolated points and 6 one-dimensional components isomorphic to \( \mathbb{P}^1 \). Furthermore, the Poincaré polynomial of \( M_{\mathbb{P}^2}(4, 1) \) is

\[
P(M_{\mathbb{P}^2}(4, 1)) = 1 + 2q + 6q^2 + 10q^3 + 14q^4 + 15q^5 + 16q^6 + 16q^7 + 16q^8 + 16q^9 + 16q^{10} + 16q^{11} + 15q^{12} + 14q^{13} + 10q^{14} + 6q^{15} + 2q^{16} + q^{17}.
\]

In Chapter 4 we use Joyce’s wall crossing formula and propose an algorithm to compute the Euler characteristic of the moduli space \( M_{\mathbb{P}^2}(d, \chi) \). The strategy is as follows. We consider the moduli spaces of \( \alpha \)-semistable pairs on local \( \mathbb{P}^2 \) for a positive rational number \( \alpha \). A pair is a pure dimension one sheaf together with a choice of a global section. Here, \( \alpha \)-semistability is defined analogously as \( \alpha \)-semistability of a coherent system by Le Potier [33]. When \( \alpha \) is sufficiently large, a pair is \( \alpha \)-stable if and only if it is a stable pair in the sense of Pandharipande and Thomas [40]. When \( \alpha \) is sufficiently close to zero, an \( \alpha \)-stable pair is precisely a stable pair of Joyce and Song [24]. Following [40] and [24], we call the former moduli space the PT moduli space and the latter the PI moduli space, respectively.
The Euler characteristics of PT moduli spaces can be computed via torus localization. We apply Joyce’s wall crossing formula to get the Euler characteristics of PI moduli spaces. Sheaves appearing in pairs in PI moduli space are necessarily semistable. Hence, there is a forgetful map from PI moduli space to the moduli space of semistable sheaves. By analyzing this map, we relate the Euler characteristics of PI moduli spaces with those of $M_{\mathbb{P}^2}(d, \chi)$.

Joyce’s wall crossing technique is developed over an abelian category with stability conditions. As the stability condition varies, the moduli stack parametrizing the semistable objects varies. The wall crossing formula tells us how the Euler characteristic of the corresponding moduli space changes as the stability parameter $\alpha$ changes. Although the collection of all pairs forms an abelian category, it does not satisfy one of necessary conditions for Joyce’s wall crossing to be applied (See Remark 4.4.1). The author has not been able to verify the technical conditions so that Joyce’s wall crossing formula applies in our situation (Conjecture 4.4.4). However, the results of the Euler characteristics of $M_{\mathbb{P}^2}(d, 1)$ by this strategy coincide with the prediction in physics and Conjecture 1.0.2 (Corollary 4.6.1). This provides with a strong evidence that Joyce’s wall crossing formula applies in this case.
Chapter 2

The Moduli space of Sheaves on local \( \mathbb{P}^1 \)

2.1 Local BPS invariant

Let \( k \) be an integer with \( k \geq -1 \). Let \( X = \text{Spec}(\text{Sym}(E^*)) \) be the total space of a rank 2 bundle

\[
E = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2 - k)
\]
on \( \mathbb{P}^1 \). As a toric variety, \( X \) contains a torus \( T' = (\mathbb{C}^*)^3 \) and has two \( T' \)-invariant affine open sets isomorphic to \( \mathbb{C}^3 \). The transition map is

\[
(z_1, z_2, z_3) \mapsto (z_1^{-1}, z_1^{-k} z_2, z_1^{2+k} z_3).
\]

Here, the torus \( T' \) acts by

\[
(t_1, t_2, t_3). (z_1, z_2, z_3) = (t_1 z_1, t_2 z_2, t_3 z_3).
\]  

(2.1)

We will consider the action of the subtorus

\[
T = \{(t_1, t_2, t_3) \in T' : t_1 t_2 t_3 = 1\}
\]

which preserves a canonical Calabi-Yau form \[ \text{[37]} \].

Let \( L \) be the pullback of \( \mathcal{O}_{\mathbb{P}^1}(1) \) to \( X \). We construct the moduli space of \( L \)-stable sheaves
\( \mathcal{F} \) such that the support of \( \mathcal{F} \) has class \( d[\mathbb{P}^1] \in H_2(X) \) and \( \chi(\mathcal{F}) = 1 \). Although \( X \) is not projective, we may define Hilbert polynomial and Gieseker semistability for such sheaves.

For a sheaf \( \mathcal{F} \) whose support is in class \( d[\mathbb{P}^1] \in H_2(X) \), we define the \textit{multiplicity} \( r(\mathcal{F}) \) by \( r(\mathcal{F}) = d \) and the Hilbert polynomial by

\[
P_F(n) = r(\mathcal{F})n + \chi(\mathcal{F}).
\] (2.2)

A sheaf \( \mathcal{F} \) is called (Gieseker) semistable with respect to \( L \) if for any proper nonzero subsheaf \( \mathcal{G} \), we have

\[
\frac{\chi(\mathcal{G})}{r(\mathcal{G})} \leq \frac{\chi(\mathcal{F})}{r(\mathcal{F})}.
\]

Stable sheaf is defined with the strict inequality. For details and the construction of the moduli space of semistable sheaves, we refer to [18].

We consider the moduli space of \( L \)-(semi)stable coherent sheaves of pure dimension 1 on \( X \)

\[
M_d(k) = \{ \mathcal{F} : P_\mathcal{F} = dn + 1, \mathcal{F} \text{ is } L \text{-}\text{(semi)stable} \}.
\]

By the condition \( \chi(\mathcal{F}) = 1 \), semistability agrees with stability. So, there exists a perfect obstruction theory on \( M_d(k) \) [15]. Unfortunately, since \( X \) is not compact, the virtual cycle for \( M_d(k) \) is not well-defined.

The \( T \)-action (2.1) on \( X \) induces a \( T \)-action on the moduli space. Thus, we may define an equivariant version of invariant by means of the virtual localization with respect to the torus action (2.1). For this, we need a following lemma.

\textbf{Lemma 2.1.1.} \textit{The fixed point locus of the induced \( T \)-action on \( M_d(k) \) is compact.}

\textit{Proof.} We will see in Section 2.6 that we can embed \( M_d(k) \) into a compact moduli space via an embedding of \( X \) into the Hirzebruch surface. The torus fixed locus supported on \( \mathbb{P}^1 \) is
the same, and hence it is compact. 

Therefore, we can define an invariant by the residue integral on the fixed locus using the virtual localization formula [14].

**Definition 2.1.2.** Let $M^T_i$ be connected component of $T$-fixed locus $M_d(k)^T$. Let $N_i^{\text{vir}}$ be the virtual normal bundle to $M^T_i$ obtained from the moving part of the virtual tangent space. We define the *genus zero equivariant BPS invariant* by

$$n_d(k) = \sum_i \int_{[M^T_i]^{\text{vir}}} \frac{1}{e(N^{\text{vir}}_i)}.$$ 

Here, $e(-)$ is the equivariant Euler class.

The equivariant Gromov-Witten theory of $X$ is studied by Bryan and Pandharipande [7]. They use the natural $(\mathbb{C}^*)^2$-action on $X$ via scalar multiplication on each fiber, and compute residue Gromov-Witten invariants by localization and degeneration methods. After taking anti-diagonal subtorus of $(\mathbb{C}^*)^2$, they get a closed formula for the Gromov-Witten partition function [7, Cor. 7.2].

The torus action (2.1) restricts to the action of their anti-diagonal subtorus. So, we expect the genus zero Gopakumar-Vafa formula (1.2) holds for the total space $X$ of $E$. The following is the equivariant version of Conjecture 1.0.2 for local $\mathbb{P}^1$.

**Conjecture 2.1.3.** For $\beta = d[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$, let $N_d^{\text{GW}}(k)$ be the genus zero local Gromov-Witten invariant computed in [7] and $n_d(k)$ be the equivariant local BPS invariant defined by the residue integral in Definition 2.1.2. Then, the Gopakumar-Vafa formula

$$N_d^{\text{GW}}(k) = \sum_{m \mid d} \frac{n_d/m(k)}{m^3}$$

holds.
In [6], Bryan and Gholampour study the equivariant version of BPS invariant for the resolution of ADE polyhedral singularities $\mathbb{C}^3/G$. They prove the equivariant BPS invariants so defined is in agreement with the prediction of equivariant Gromov-Witten theory via formula (1.2).

In this chapter, we prove Conjecture 2.1.3 for $d = 1, 2$ and $3$ for any $k$ and for $d = 4$ for $k \leq 100$. In Section 2.6, we show the moduli space $M_d(k)$ is smooth of dimension $kd^2 + 1$. So, the BPS invariant in Definition 2.1.2 is given by the signed topological Euler characteristic. The following is standard. See for example [9].

**Theorem 2.1.4.** Let $M$ be a quasi-projective $\mathbb{C}$-scheme of finite type. Let $T$ be an algebraic torus acting regularly on $M$. Then $e_{\text{top}}(M) = e_{\text{top}}(M^T)$.

**Definition 2.1.5.** Let $X$ be a toric variety.

1. A sheaf $\mathcal{F}$ on $X$ called $T$-fixed if $t^* \mathcal{F} \simeq \mathcal{F}$.

2. Let $\sigma : T \times X \to X$ be $T$-action on $X$ and $p : T \times X \to X$ be the projection. A sheaf $\mathcal{F}$ is $T$-equivariant if we have an isomorphism $\Phi : \sigma^* \mathcal{F} \to p^* \mathcal{F}$ satisfying a cocycle condition

\[(\mu \times 1_X)^* \Phi = p_{23}^* \Phi \circ (1_T \times \sigma)^* \Phi,\]

where $\mu : T \times T \to T$ is the multiplication map and $p_{23} : T \times T \times X \to T \times X$ is the projection to the second and the third factors.

**Proposition 2.1.6** ([30] [25]). A stable sheaf on a toric variety supported on a compact subscheme is $T$-fixed if and only if it is $T$-equivariant.

In the following sections, we count the torus fixed sheaves using the classification of equivariant sheaves studied by Kool [30].

We note that the stable sheaves on $X$ are actually supported on a smaller subspace.
Lemma 2.1.7. Denote by $Y$ the total space of $\mathcal{O}_{\mathbb{P}^1}(k)$. If $\mathcal{F} \in M_d(k)$, then the scheme theoretic support of $\mathcal{F}$ is in $Y$.

Proof. The ideal sheaf of $Y$ is $L^{2+k}$. We have an exact sequence

$$\mathcal{F} \otimes L^{2+k} \to \mathcal{F} \to \mathcal{F}|_Y \to 0.$$ 

Since $2 + k$ is a positive number, by the stability of $\mathcal{F}$, the first map is zero, and hence the map $\mathcal{F} \to \mathcal{F}|_Y$ is an isomorphism. 

So, we can consider $\mathcal{F}$ as a sheaf on $Y$. We will also denote by $L$ the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ to $Y$. Then, $M_d(k)$ is the moduli space of $L$-stable sheaves on $Y$.

Note that the zero section of $\mathbb{P}^1$ is the only compact $T$-invariant curve in $Y$. Hence if a sheaf $\mathcal{F}$ is $T$-fixed, its reduced support must be $\mathbb{P}^1$. In the next section, we will describe $T$-fixed sheaves supported on $\mathbb{P}^1$ using toric geometry.

### 2.2 Equivariant Sheaves

As a toric variety, $Y$ contains a two-dimensional torus $(\mathbb{C}^*)^2$ which is isomorphic to $T$ by the isomorphism

$$(\mathbb{C}^*)^2 \ni (t_1, t_2) \mapsto (t_1, t_2, t_1^{-1} t_2^{-1}) \in T.$$ 

The action of this torus is the same as the restriction of $T$-action on $X$ to $Y$. So, by a slight abuse of notation, we also denote this embedded torus by $T$ and consider $T$-equivariant sheaves on $Y$.

In this section, we give a description of pure equivariant sheaves $\mathcal{F}$ on $Y$ following [30]. Let $M$ be the group of characters of $T$ and $N$ be the group of one parameter subgroups.
Then, the fan associated to $Y$ (which lies in $N \otimes \mathbb{R}$) is

$$\{\sigma_1 = \text{Cone}((0, 1), (1, 0)), \sigma_2 = \text{Cone}((0, 1), (-1, -k))\}$$

where Cone($v_1, v_2$) denote the convex cone generated by vectors $v_1$ and $v_2$. The $T$-invariant subvariety associated to the face $(0, 1)$ is the zero section of $\mathbb{P}^1$.

We have two $T$-invariant affine open sets $U_{\sigma_i} = \text{Spec}(k[S_{\sigma_i}]), i = 1, 2,$ where $S_{\sigma_i}$ is the semigroup defined by $\sigma_i$

$$S_{\sigma_i} = \sigma_i^\vee \cap M.$$  

For a notational convenience, we let $M^i$ be the copy of $M$ whose elements are expressed with respect to the semigroup generator of $S_{\sigma_i}$, i.e.,

$$M^1 = \{m_1(1, 0) + m_2(0, 1)\} \text{ and } M^2 = \{m_1(-1, 0) + m_2(-k, 1)\}.$$  

For $m, m' \in M^i$, we say $m' \geq m$ if every component of $m' - m$ is nonnegative. Note that in the standard basis of $M$, this means $m' - m$ is an element of the semigroup $S_{\sigma_i}$.  

Figure 2.1: Toric fan of $Y$
Then, we have a decomposition into weight spaces

\[ \Gamma(U_\sigma, \mathcal{F}) = \bigoplus_{m \in M^i} \Gamma(U_\sigma, \mathcal{F})_m. \]

Denote the weight space \( \Gamma(U_\sigma, \mathcal{F})_m \) by \( F_i^m \), \( m = (m_1, m_2) \in M^i \). Since \( \mathcal{F} \) is \( \mathcal{O}_Y \)-module, each \( \Gamma(U_\sigma, \mathcal{F}) \) is \( M^i \)-graded \( S_{\sigma_i} \)-module. We can reformulate the \( S_{\sigma_i} \)-module structure by the following data: \( k \)-linear maps \( \chi^{i}_{m,m'} : F_i^m \rightarrow F_i^{m'} \) for all \( m, m' \in M^i \) with \( m' \geq m \) such that

\[ \chi^{i}_{m,m} = 1 \text{ and } \chi^{i}_{m,m'} = \chi^{i}_{m',m''} \circ \chi^{i}_{m,m'}. \]  

We have the following \[30\] Chapter 2).

**Proposition 2.2.1.** Let \( \mathcal{F} \) be a pure equivariant sheaf on \( Y \) with support \( \mathbb{P}^1 \). Then,

1. There are integers \( A_1^1, A_1^2 \) and \( A \leq B \) such that \( F^i(m_1, m_2) = 0 \) unless \( A^1_1 \leq m_1 \) and \( A \leq m_2 \leq B \).

2. For each \( A \leq m_2 \leq B \), the maps \( \chi^{i}_{(m_1, m_2), (m_1+1, m_2)} \) are all injective and the direct limit \( \varinjlim_{m_1} F^i(m_1, m_2) \) is a finite dimensional vector space denoted by \( F^i(\infty, m_2) \).

3. For each \( A \leq m_2 \leq B \),

\[ F^1(\infty, m_2) \simeq F^2(\infty, m_2) \]

and under this identification,

\[ \chi^{1}_{(\infty, m_2), (\infty, m_2+1)} = \chi^{2}_{(\infty, m_2), (\infty, m_2+1)} \]

where \( \chi^{i}_{(\infty, m_2), (\infty, m_2+1)} = \varinjlim_{m_1} \chi^{i}_{(m_1, m_2), (m_1, m_2+1)} \cdot \)

Moreover, let \( C \) be the category whose objects are \( \{F^i(m), \chi^{i}_{m,m'}\} \) satisfying above conditions
and morphisms

\[ \phi: \{F^i(m), \chi^i_{m,m'}\} \to \{G^i(m), \lambda^i_{m,m'}\} \]

are collections of linear maps \(\phi^i(m): F^i(m) \to G^i(m)\) satisfying

\[ \phi^i(m') \circ \chi^i_{m,m'} = \lambda^i_{m,m'} \circ \phi^i(m) \text{ and } \phi^i(\infty, m_2) = \phi^2(\infty, m_2). \]

Then, this correspondence is an equivalence between the category of pure equivariant sheaves and equivariant morphisms with the category \(C\).

An object in the category \(C\) is called \(\Delta\)-family [42]. Proposition 2.2.1 is a special case of more general statement about pure equivariant sheaves on a toric variety [30]. We state it for \(Y\) to avoid heavy notation.

Assume \(A\) in condition (1) is maximally chosen. Let \(O_Y(\chi)\) be the structure sheaf of \(Y\) endowed with the equivariant structure induced by a character \(\chi \in M\). Then \(\mathcal{F} \otimes O_Y(\chi)\) is isomorphic to the sheaf \(\mathcal{F}\) with equivariant structure shifted by \(\chi\). Therefore, we may assume \(A = 0\).

By the above proposition, we can illustrate the sheaf \(\mathcal{F}\) by putting a box at the position \((m_1, m_2)\) of \(M^i\) if the corresponding weight space is nonzero. By the condition (3), for each open chart, the asymptotic weight vector spaces are stabilized and identified with each other. So, we place the asymptotic vector spaces in the middle. We will use the following convention.

Con convention 2.2.2. A box in \(M^1\) corresponds to the lattice point of its lower left corner whereas a box in \(M^2\) corresponds to the lattice point of its lower right corner. We use this convention because the horizontal coordinate axes are in different directions.

Example 2.2.3. Let \(C_n\) be the \(n\)-th order thickening of \(\mathbb{P}^1\) in the direction of \(O_{\mathbb{P}^1}(k)\). More precisely, \(C_n\) is \(\text{Spec}(\text{Sym}(O_{\mathbb{P}^1}(-k))/\mathcal{J})\) where \(\mathcal{J}\) is the ideal generated by \(S^n(O_{\mathbb{P}^1}(-k))\).
Then for the sheaf $\mathcal{O}_{C_n}$, we have

$$\Gamma(U_{\sigma_i}, \mathcal{O}_{C_n}(m_1, m_2)) = \begin{cases} 
\mathbb{C} & \text{if } 0 \leq m_2 \leq n - 1 \text{ and } m_1 \geq 0 \\
0 & \text{else}
\end{cases}$$

The sheaf $\mathcal{O}_{C_3}$ can be depicted as in Figure 2.2.

In this particular example, all weight spaces are one-dimensional. We will see other examples that weight spaces have more than one dimension.

From this description, it is clear that the equivariant version of Grothendieck’s theorem holds.

**Theorem 2.2.4.** Let $\mathcal{E}$ be an equivariant vector bundle of rank $r$ on $\mathbb{P}^1$. Then there are integers $a_1, \cdots, a_r$ uniquely determined up to order such that we have an equivariant isomorphism $\mathcal{E} \simeq \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$.

**Proof.** This theorem is originally due to Klyachko [29]. Since the scheme theoretic support is $\mathbb{P}^1$, we must have $A = 0$ and $B = 0$ in the condition (1) of Proposition 2.2.1. Let $\left(\{E^1(m, 0)\}, \{E^2(m, 0)\}\right)$ be the corresponding family. Then, we can pick a basis $\{v_j\}$ of the asymptotic weight space $E^1(\infty, 0) \simeq E^2(\infty, 0)$ in such a way that for any $m$ and $i = 1, 2$, a subset of $\{v_j\}$ forms a basis of $E^i(m, 0)$. Therefore, by taking subfamilies generated by each $v_j$, $\left(\{E^1(m, 0)\}, \{E^2(m, 0)\}\right)$ decomposes into families with one-dimensional weight spaces.
Hence, \( \mathcal{E} \) decomposes equivariantly into equivariant line bundles.

Let \( U_i \) be the intersection of the open set \( U_\sigma \) with \( \mathbb{P}^1 \) for \( i = 1, 2 \). Then \( \{ U_i \} \) be an affine open cover of \( \mathbb{P}^1 \). We fix an \( \mathcal{T} \)-equivariant structure of \( \mathcal{O}_{\mathbb{P}^1}(k) \) by the weight space decomposition on each open set

\[
\Gamma(U_1, \mathcal{O}_{\mathbb{P}^1}(k)) = \bigoplus_{m \geq 0} C_m, \quad \Gamma(U_2, \mathcal{O}_{\mathbb{P}^1}(k)) = \bigoplus_{m \leq k} C_m,
\]

where \( C_m \) is a one-dimensional representation of \( \mathcal{T} \) with character \( \chi(t_1, t_2) = t_1^m, m \in \mathbb{Z} \).

Then, given an equivariant sheaf \( F \) on \( \mathbb{P}^1 \) with

\[
\Gamma(U_i, F)_m = F^i(m),
\]

we have a natural equivariant structure on \( F \otimes \mathcal{O}_{\mathbb{P}^1}(k) \) by

\[
\Gamma(U_1, F \otimes \mathcal{O}_{\mathbb{P}^1}(k))_m = F^1(m) \\
\Gamma(U_2, F \otimes \mathcal{O}_{\mathbb{P}^1}(k))_m = F^2(m - k).
\]

(2.4)

Now, we consider \( j \)-th row of the weight space decompositions. Given an equivariant sheaf \( \mathcal{F} \) on \( Y \), let \( \mathcal{F}_j \) be the sheaf defined by

\[
\Gamma(U_i, \mathcal{F}_j)_{(m_1, m_2)} = \begin{cases} 
\Gamma(U_i, \mathcal{F})_{m_1, m_2} & \text{if } m_2 = j \\
0 & \text{else}
\end{cases}
\]

Then, \( \mathcal{F}_j \) has scheme theoretic support \( \mathbb{P}^1 \) and hence decomposes into equivariant line bundles by Theorem 2.2.4.

**Theorem 2.2.5.** Let \( \mathcal{F} \) be a pure \( \mathcal{T} \)-equivariant sheaf on \( Y \). Then \( \mathcal{F} \) is determined by the
following data: For $0 \leq j \leq B$, $F_j \simeq \bigoplus_{i=1}^{d_j} \mathcal{O}_{\mathbb{P}^1}(a_{ij})$ and equivariant morphisms

$$\phi_j : F_j \to F_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

such that for all $t_1 = (t_1, 1) \in T$, there exist isomorphisms $\alpha_j : t_1^*F_j \to F_j$ such that the diagram

$$t_1^*F_j \xrightarrow{t_1^\ast\phi_j} t_1^*(F_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \quad \xrightarrow{\alpha_j} \xrightarrow{t_1^\ast\alpha_{j+1} \otimes \mu} F_j \xrightarrow{\phi_j} F_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

commutes, where $\mu : t_1^*(\mathcal{O}_{\mathbb{P}^1}(k)) \simeq \mathcal{O}_{\mathbb{P}^1}(k)$ is given by the equivariant structure of $\mathcal{O}_{\mathbb{P}^1}(k)$ fixed in the above discussion.

**Proof.** By Theorem 2.2.3, the horizontal maps $\chi_{(m_1,j),(m_1+1,j)}^i$ will induce the decomposition $F_j \simeq \bigoplus_{i=1}^{d_j} \mathcal{O}_{\mathbb{P}^1}(a_{ij})$. It remains to consider the vertical maps $\chi_{(m_1,j),(m_1+1,j)}^i$.

Recall that we are using different basis of $M$ for $\chi^1$ and $\chi^2$. The $(m_1,j)$ in the subscript means $m_1(1,0) + j(0,1)$ for $\chi^1$ and $m_1(-1,0) + j(-k,1)$ for $\chi^2$. Rewrite in the standard basis of $M$,

$$\chi_{(m_1,j),(m_1+1,j)}^1 : F^1(m_1,j) \to F^1(m_1,j+1)$$

$$\chi_{(m_1,j),(m_1+1,j)}^2 : F^2(-m_1 - k,j,j) \to F^2(-m_1 - k,j,k,j+1).$$

Thus, this will define an equivariant morphism

$$\phi_j : F_j \to F_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

by 2.2.4.

Since $F$ is equivariant, $t_1^*F \simeq F$. Hence, there exist isomorphisms $\alpha_j$'s such that the
Conversely, if we have such isomorphisms $\alpha_j: t^*_1F_j \simeq F_j$, each $F_j$ is equivariant \[25\], so we have an weight space decomposition and horizontal maps $\chi^i_{(m_1,j),(m_1,j+1)}$. The equivariant morphisms $\phi_j$ define $\chi^i_{(m_1,j),(m_1,j+1)}$. By the commutativity of (2.5), they commute with each other. Hence the data $(F_j, \phi_j, \alpha_j)$ determines $F$, by Proposition 2.2.1. \[ \square \]

**Remark 2.2.6.** Let $\pi: Y \to \mathbb{P}^1$ be the natural projection. In the above theorem, it is clear that

$$\pi_*F \simeq \bigoplus_{j=0}^{B} \bigoplus_{i=1}^{d_j} \mathcal{O}_{\mathbb{P}^1}(a_{ij}).$$

$\pi_*$ induces an equivalence between the category of $\mathcal{O}_Y$-modules and the category of $\pi_*\mathcal{O}_Y$-modules on $\mathbb{P}^1$ [13], Ex.II.5.17]. Since $\pi_*\mathcal{O}_Y \simeq \text{Sym}(\mathcal{O}_{\mathbb{P}^1}(-k))$, $\pi_*\mathcal{O}_Y$-modules structure on $\pi_*F$ is given by a map

$$\pi_*F \to \pi_*F \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

The previous theorem shows that if $F$ is a pure equivariant sheaf, this map is given by $\phi_j$’s. In this sense, we will call the collection $\{F_j, \phi_j\}$ associated to a sheaf $F$ a $\pi_*\mathcal{O}_Y$-modules structure of $F$.

**Theorem 2.2.7.** Let $\mathcal{F}$ and $\mathcal{G}$ be $T$-equivariant sheaves on $Y$ whose $\pi_*\mathcal{O}_Y$-modules structure are $\{F_j, \phi_j\}$ and $\{G_j, \psi_j\}$ respectively. Then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic to each other if and only if there exist isomorphisms $\mu_j : F_j \rightarrow G_j$ such that $\mu_{j+1} \circ \phi_j = \phi_{j+1} \circ \mu_j$.

**Proof.** This is a straightforward consequence of the equivalence between the category of $\mathcal{O}_Y$-modules and the category of $\pi_*\mathcal{O}_{\mathbb{P}^1}$-modules on $\mathbb{P}^1$. \[ \square \]

**Theorem 2.2.8.** Let $P_\mathcal{F}(u) = \chi(\mathcal{F} \otimes L^\otimes n) = dn + \chi(\mathcal{F})$. Then

$$d = \sum_{j=0}^{B} d_j \text{ and } \chi(\mathcal{F}) = \sum_{j=0}^{B} \sum_{i=1}^{d_i} (a_{ij} + 1)$$
Proof. The first equation is clear since the support of $\mathcal{F}$ has multiplicity $\sum_{j=0}^{B} d_{i}$ along $\mathbb{P}^{1}$. The second equation follows from $\chi(\mathcal{F}) = \chi(\pi_{*}\mathcal{F})$ since $\pi$ is affine. 

To test the stability, we only need to test for equivariant subsheaves.

**Proposition 2.2.9** ([30 Proposition 3.19]). Suppose $X$ is a projective variety with a torus action. Let $\mathcal{F}$ be a pure equivariant sheaf on $X$. Then $\mathcal{F}$ is (Gieseker) stable if and only if $p_{G} < p_{\mathcal{F}}$ for any proper equivariant subsheaf $\mathcal{G}$.

Therefore, a sheaf $\mathcal{F}$ associated to $\{\mathcal{F}_{j}, \phi_{j}\}$ is stable if and only if for any $\pi_{*}\mathcal{O}_{Y}$-submodule $\mathcal{G} = \{\mathcal{G}_{j}, \psi_{j}\}$, i.e., a collection of equivariant subsheaves $\mathcal{G}_{j} \subset \mathcal{F}_{j}$ compatible with $\phi_{i}$, we have

$$\frac{\chi(\mathcal{G})}{r(\mathcal{G})} < \frac{\chi(\mathcal{F})}{d}.$$  

where $r(\mathcal{G})$ is the multiplicity of $\mathcal{G}$ along $\mathbb{P}^{1}$.

**Definition 2.2.10.** For a pure equivariant sheaf $\mathcal{F}$ as in Theorem 2.2.5, we will call the dimension vector $(d_{0}, d_{1}, \cdots, d_{B})$ the type of $\mathcal{F}$.

**Definition 2.2.11.** Throughout the rest of this chapter, we denote by $x$ and $y$ the projective coordinates of $\mathbb{P}^{1}$ where $x = 0$ defines a fixed point in $U_{\sigma_{1}}$ and $y = 0$ defines a fixed point in $U_{\sigma_{2}}$ and by $z$ the coordinate of $Y$ in $\mathcal{O}_{\mathbb{P}^{1}(k)}$ direction.

### 2.3 Reading the Diagrams

We saw from the previous sections that $T$-equivariant sheaves can be graphically represented as in Example 2.2.3. In this section, we discuss data about the associated sheaf we can get from such diagrams. We assume the purity of the sheaves. For sheaves with torsion, the discussion is very similar.
Throughout this section, we will use a typical example of \(T\)-equivariant sheaf depicted in Figure 2.3. This picture shows the weight space decomposition on two copies of the character group \(M\), denoted by \(M^1\) and \(M^2\) in Section 2.2, associated to the open sets \(U_{\sigma_1}\) and \(U_{\sigma_2}\). The numbers written on boxes are the dimensions of corresponding weight spaces.

By Proposition 2.2.1, \(T\)-equivariant sheaves are described by data of weight spaces together with the maps between the weight spaces. In Figure 2.3 by the comparability condition (2.3), all such maps are determined by the following maps in the notation of Section 2.2.

\[
\begin{align*}
\chi^1_{(\infty, 1), (\infty, 2)} &= \chi^2_{(\infty, 1), (\infty, 2)} : \mathbb{C} \to \mathbb{C}^2, \\
\chi^1_{(\infty, 2), (\infty, 3)} &= \chi^2_{(\infty, 2), (\infty, 3)} : \mathbb{C}^2 \to \mathbb{C}, \\
\chi^1_{(-1, 1), (\infty, 1)} &= \chi^2_{(-1, 1), (\infty, 1)} : \mathbb{C} \to \mathbb{C}^2, \\
\chi^2_{(-2, 1), (\infty, 1)} &= \chi^2_{(-2, 1), (\infty, 1)} : \mathbb{C} \to \mathbb{C}^2.
\end{align*}
\]

Recall that by Convention 2.2.2, a box in \(M^2\) corresponds to the lattice point of its lower right corner, while a box in \(M^1\) corresponds to the lattice point of its lower left corner. Each map is indicated by an arrow in Figure 2.3.

We assume that the composition \(\chi^1_{(\infty, 2), (\infty, 3)} \circ \chi^1_{(\infty, 1), (\infty, 2)}\) is nonzero, because otherwise
the sheaf is decomposable. Furthermore, we assume

\[ \chi_{(-1,1),(\infty,1)}^1(1) = v_1 \quad \text{and} \quad \chi_{(-2,1),(\infty,1)}^2(1) = v_2, \]  

(2.6)

where \( v_1 \) and \( v_2 \) are linearly independent vectors in

\[ F^1(\infty,1) \cong F^2(\infty,1) \cong \mathbb{C}^2. \]

Up to isomorphism, we may assume \( v_1 = (1,0) \) and \( v_2 = (0,1) \). In case that \( v_1 \) and \( v_2 \) are linearly dependent, we will get a different sheaf. We will see more detail in Example 2.5.10.

### 2.3.1 Euler Characteristic

To compute the Euler characteristics of \( T \)-equivariant sheaves, note that the Euler characteristic is additive along exact sequences. Hence, we have

\[ \chi(F) = \sum \chi(F_j). \]

Each \( F_j \) is a torsion free sheaf supported on \( \mathbb{P}^1 \). So, it has a decomposition into equivariant line bundles on \( \mathbb{P}^1 \) by Theorem 2.2.4. We will describe such decomposition and compute the Euler characteristic of each component.

In Figure 2.3, even though we have infinitely many boxes with positive dimensional weight spaces in the middle, their dimensions must be stabilized and the asymptotic weight spaces for each open set must be isomorphic to each other by Proposition 2.2.1.

By identifying the asymptotic weight spaces, we connect the dotted line in the middle of Figure 2.3 and consider a sheaf as a collection of horizontal strips each of which corresponds to a \( T \)-equivariant sheaf on \( \mathbb{P}^1 \). This is possible because each \( F_j \) is a direct sum of line bundles.
Figure 2.4: Decomposition of sheaf in Figure 2.3 on $\mathbb{P}^1$. Figure 2.4 shows how we can think of a sheaf as a collection of horizontal strip. Note that in $\mathcal{F}_1$ in this example, 1-dimensional weight spaces at each end belong to different strips after decomposition, because of the assumption (2.6). If $v_1$ and $v_2$ were linearly dependent, these two 1-dimensional weight spaces must belong to the same strip, because their images in the horizontal asymptotic weight spaces agree up to a constant multiple.

Now, we compute the Euler characteristic of each strip. The diagram associated to $\mathcal{O}_{\mathbb{P}^1}$ is a horizontal strip from the origin of $M^1$ to the origin of $M^2$. As the Euler characteristic of $\mathcal{O}_{\mathbb{P}^1}$ is one and we can compute the Euler characteristic of each strip by comparing with it. For example, in Figure 2.4, $\mathcal{F}_0$ has Euler characteristic 2, because it has one more box than $\mathcal{O}_{\mathbb{P}^1}$.

The number written on each strip in Figure 2.4 is the Euler characteristic. Note that since the vertical coordinate axis of $M^2$ is $(-k, 1)$, strips located on the second row have actually $k$ less boxes than they look. This explains terms $-k$ on $\mathcal{F}_1$ and $-2k$ on $\mathcal{F}_2$.

Therefore, we conclude that the Euler characteristic of the sheaf in this example is

$$2 + (3 - k) + (3 - k) + (5 - 2k) = 13 - 4k.$$
2.3.2 Cohomologies

we have

\[ H^0(\mathcal{F}) \simeq H^0(\pi_*\mathcal{F}), \]

as the projection \( \pi \) is an affine morphism [15, Ex.III.8.2]. Hence, we can compute \( H^0(\mathcal{F}) \) by computing \( H^0 \) of each strip in the figure. For this, we assume \( k = 2 \) in our example.

In the example, as can be easily checked, \( \mathcal{F}_0 \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(1) \) which has two linearly independent global sections. We can also easily identify all strips with line bundles on \( \mathbb{P}^1 \) and count its global sections. However, we introduce more straightforward way to count the global sections of \( \mathcal{F} \).

The \( T \)-equivariant transition map between open sets \( U_1 \) and \( U_2 \) is

\[ (z_1, z_2) \mapsto (z_1^{-1}, z_1^{-k}z_2). \]

Hence, if a global section of \( \mathcal{F} \) restricts to a weight vector in \( \Gamma(U_1, \mathcal{F}) \) with weight \( t_1^a t_2^b \), then it restricts in \( \Gamma(U_2, \mathcal{F}) \) to a weight vector with weight \( t_1^{a-bk} t_2^b \). Conversely, a pair of such weight vectors in each open set defines a global section if and only if their images in \( \Gamma(U_1 \cap U_2, \mathcal{F}) \) agree with each other.

The space of section \( \Gamma(U_1 \cap U_2, \mathcal{F}) \) is nothing but the localization of \( \Gamma(U_1, \mathcal{F}) \) that allows to invert the element \( z_1 \). Hence, it is generated by the asymptotic weight spaces \( \oplus_j \mathcal{F}^i(\infty,j) \) and \( z_1^{\pm 1} \). So the images of weight vectors in \( \Gamma(U_1 \cap U_2, \mathcal{F}) \) agree with each other if and only if their images in the asymptotic weight space agree with each other. Thus, the weight vectors defining a global section must be in the same horizontal strip.

Figure 2.3 shows how to count the number of linearly independent global sections from the diagram when \( k = 2 \). For example, the global section indicated by \( 3 \) in the picture is defined by a pair of weight vectors with weight \( t_1^{-1} t_2 \) in \( \Gamma(U_1, \mathcal{F}) \) and weight \( t_1^{-1} t_2 \) in...
Figure 2.5: Counting the global sections $\Gamma(U_2, \mathcal{F})$.

Since we have computed $\chi(\mathcal{F}) = 5$ when $k = 2$, and we have

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}),$$

we conclude that $h^0(\mathcal{F}) = 5$ and $h^1(\mathcal{F}) = 0$.

### 2.3.3 Restriction to $\mathbb{P}^1$

We can also describe the restriction $\mathcal{F}|_{\mathbb{P}^1}$ to $\mathbb{P}^1$. Let $I$ be the ideal sheaf of $\mathbb{P}^1$ in $Y$. Then, the restriction is given by

$$I \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F}|_{\mathbb{P}^1} \to 0.$$

As $I$ is generated by $z$ and the multiplication by $z$ is given by the vertical maps

$$\chi_i^i(\infty, 1), (\infty, 2) \quad \text{and} \quad \chi_i^i(\infty, 2), (\infty, 3),$$

the image of $I \otimes \mathcal{F} \to \mathcal{F}$ is the subsheaf of $\mathcal{F}$ generated by the images of these maps.

What is left after taking quotient is depicted in Figure 2.6. Note that in this picture,
The sheaf defined by the strip on the middle row is a sheaf supported on $\mathbb{P}^1$ with shifted equivariant structure. Similarly, one weight space on the top row defines a zero dimension sheaf supported on the fixed point of $U_1$ with shifted equivariant structure. We will denote this point by $p$. We conclude by forgetting the equivariant structure that

$$\mathcal{F}|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_p.$$  

### 2.3.4 Expression of the Sheaf in terms of an Exact Sequence

In practice, it is useful to have a description of a certain sheaf as a quotient of a subsheaf of a locally free sheaf. In this subsection, we explain how to get such a description from the diagram. We continue to use our example in Figure 2.6.

In the left side of Figure 2.7 are shown minimal saturated subsheaves of $\mathcal{F}$ containing $\mathcal{F}_0$, $F^1(-1,1)$, and $F^2(2,1)$ from the top.

The sheaf on top left is identified with an ideal sheaf of zero-dimensional subscheme in the triple line. More precisely, the triple line $C_3$ is defined by $\{z^3 = 0\}$ as in Example 2.2.3 and the subscheme $D_1$ is defined by the ideal $(x^2, z^2)(y, z^2)$. Then this sheaf is isomorphic to $I_{D_1,C_3}(4)$. The twist 4 is because the diagram is dilated by 4 from the picture of $\mathcal{O}_{C_3}$.

Similarly, if we let $C_2$ be the double line and $D_2$, and $D_3$ be the subscheme defined
by ideals \((x, z)(y, z)\) and \((x^2, z)\) respectively, the sheaves in the middle and bottom left is isomorphic to \(I_{D_2,C_2}(4-k)\) and \(I_{D_3,C_2}(4-k)\), respectively.

From the diagram, it is clear that the direct sum of these three sheaves surjects onto \(\mathcal{F}\). We can also identify the kernel of this surjection. As the image of \(\chi^2_{(\infty,1),(\infty,2)}\) can be expressed as a linear combination of \(v_1\) and \(v_2\), the subsheaf of direct sum generated by this relation is in the kernel. If we let \(D_4\) be scheme defined by \((x^2, z)(y, z)\), it is isomorphic to \(I_{D_4,C_2}(4-k)\). Finally, we have one more relation on the top row, which gives rise to an another sheaf in the kernel isomorphic to \(\mathcal{O}_{\mathbb{P}^3}(4-2k)\).
In conclusion, our sheaf $\mathcal{F}$ is expressed as the quotient

$$
\begin{align*}
0 & \rightarrow \mathcal{O}_{\mathbb{P}^1}(4 - 2k) \oplus I_{D_3, C_2}(4) \oplus I_{D_2, C_1}(4 - k) \rightarrow I_{D_2, C_1}(4 - k) \oplus I_{D_1, C_0}(4 - k) \rightarrow \mathcal{F} \rightarrow 0.
\end{align*}
$$

### 2.4 Enumeration of Equivariant Sheaves

Using the classification given in Section 2.2, we want to count the (virtual) number of $T$-equivariant sheaves.

**Definition 2.4.1.** Let $M_{(d_0, \ldots, d_B)}(k)$ denote the subscheme of $M_d(k)$ which consists of stable $T$-equivariant sheaves of type $(d_0, \ldots , d_B)$ with $d = \sum_{j=0}^{B} d_j$. We define

$$
N_d(k) = e_{\text{top}}(M_d(k)),
$$

$$
N_{(d_0, \ldots, d_B)}(k) = e_{\text{top}}(M_{(d_0, \ldots, d_B)}^T(k))
$$

where $e_{\text{top}}(-)$ is the topological Euler characteristic.

It is clear from the localization formula that

$$
N_d(k) = \sum_{(d_0, \ldots, d_B) \vdash d} N_{(d_0, \ldots, d_B)}(k),
$$

(2.7)

where the sum runs over the set of all ordered partitions of $d$. 

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2.4.1 Type \((1^d)\)

Let \((1^d)\) denote \((1, 1, \cdots, 1)\) with 1 repeated \(d\) times. Let \(\mathcal{F}\) be a \(T\)-equivariant sheaf of type \((1^d)\) whose \(\pi_*\mathcal{O}_Y\)-module structure is \(\{\mathcal{F}_j, \phi_j\}\). Assume \(\mathcal{F}_j \simeq \mathcal{O}_{\mathbb{P}^1}(a_j)\) for \(0 \leq j \leq d - 1\). Then, since \(\chi(\mathcal{F}) = 1\), we have

\[
\sum_{j=0}^{d-1} (a_j + 1) = 1.
\]

Let \(x\) and \(y\) be homogeneous coordinates of \(\mathbb{P}^1\). By the condition (2.5) in Theorem 2.2.5, the map \(\phi_j\) is given by a monomial in \(x\) and \(y\) of degree \(a_{j+1} - a_j + k\).

Proposition 2.4.2. \(\mathcal{F}\) of type \((1^d)\) is stable if and only if \(\phi_j\)'s are all nonzero and

\[
\sum_{j=0}^{h-1} (a_j + 1) \geq 1
\]

for any \(1 \leq h \leq d\).

Proof. Since \(\mathcal{F}\) is indecomposable, \(\phi_j\)'s are all nonzero. To check the stability, it is enough to check for the subsheaf \(\mathcal{G}\) with

\[
\mathcal{G}_j = \begin{cases} 
\mathcal{F}_j & \text{if } j \geq h \\
0 & \text{else}
\end{cases}
\]

for \(0 \leq h \leq d - 1\). Hence, the stability condition is

\[
\sum_{j=0}^{h-1} (a_j + 1) \geq 1
\]

where the left side is the Euler characteristic of \(\mathcal{F}/\mathcal{G}\). \(\Box\)
Corollary 2.4.3. $N_{(1^d)}$ is equal to

$$\sum_{\lambda_{d-1} \geq \cdots \geq \lambda_0 \geq 0} \prod_{j=0}^{d-2} (\lambda_{j+1} - \lambda_j + 1)$$

where the sum runs over all $\lambda_{d-1} \geq \cdots \geq \lambda_0 \geq 0$ such that

$$\sum_{j=0}^{d-1} \lambda_j = \frac{d(d-1)}{2} k - (d - 1)$$

and for any $1 \leq h \leq d$,

$$\sum_{j=0}^{h-1} \lambda_j \geq \frac{h(h - 1)}{2} k - (h - 1).$$

Proof. Since $\phi_j$ is nonzero, we have $a_j \leq a_{j+1} + k$. We let

$$\lambda_j = a_j + jk$$

so that $\lambda_{d-1} \geq \lambda_{d-2} \geq \cdots \geq \lambda_0 \geq 0$. Then, $\phi_j$ is a monomial of degree

$$a_{j+1} - a_j + k = \lambda_{j+1} - \lambda_j.$$

By Theorem 2.2.7, each coefficient of the monomial $\phi_j$ can be set to be 1 by scaling isomorphisms. So, we have $\lambda_{j+1} - \lambda_j + 1$ choices for $\phi_j$. The condition for $\lambda_j$’s can be easily seen to be equivalent to the condition in $a_j$’s in the previous proposition.

$\square$

2.4.2 Types $(n, 1^d)$ and $(1^d, n)$

We will use the following lemma frequently.
**Convention 2.4.4.** For a monomial $\alpha$ in $x$ and $y$, we set $\gcd(\alpha, 0) = \gcd(0, \alpha) = \alpha$. Hence,

\[
\deg(\gcd(\alpha, 0)) = \deg(\gcd(0, \alpha)) = \deg(\alpha).
\]

**Lemma 2.4.5.** Suppose

\[
\phi = (\alpha_1, \alpha_2) : \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \to \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathcal{O}_{\mathbb{P}^1}(k)
\]

and

\[
\psi = (\beta_1, \beta_2)^I : \mathcal{O}_{\mathbb{P}^1}(c) \to (\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)) \otimes \mathcal{O}_{\mathbb{P}^1}(k)
\]

are nonzero maps between sheaves on $\mathbb{P}^1$ where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are monomials of appropriate degrees in the homogeneous coordinates $x$ and $y$. Let $K$ be the kernel of $\phi$ and $Q$ be such that $Q \otimes \mathcal{O}_{\mathbb{P}^1}(k)$ be the torsion free part of the cokernel of $\psi$. Then

\[
\begin{align*}
\deg K &= a_1 + a_2 - b - k + \deg(\gcd(\alpha_1, \alpha_2)), \\
\deg Q &= d_1 + d_2 - c + k - \deg(\gcd(\beta_1, \beta_2)).
\end{align*}
\]

**Proof.** Let $r = \deg(\gcd(\alpha_1, \alpha_2))$. If either of $\alpha_1$ or $\alpha_2$ is zero, by symmetry, we may assume $\alpha_1$ is zero. Then, $\alpha_2$ is nonzero monomial of degree $b - a_2 + k$. So, $K \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)$ and we have (2.8). Now, suppose $\alpha_1$ and $\alpha_2$ are both nonzero. Since $\alpha_i$ is a monomial of degree $b - a_i + k$, there are monomials $p$ and $q$ of degree $b - a_1 + k - r$ and $b - a_2 + k - r$ respectively, such that

\[
q\alpha_1 = p\alpha_2.
\]

Then the image of the inclusion

\[
\mathcal{O}_{\mathbb{P}^1}(a_1 + a_2 - b - k + r)(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))
\]
is $K$. Therefore, we have (2.8).

The proof of (2.9) is similar. □

**Definition 2.4.6.** Given a map $\psi$ as in the above lemma, we will call $Q$ the *torsion free cokernel* of $\psi$.

We start with types $(n, 1)$ and $(1, n)$. Let $\mathcal{F}$ be a $T$-equivariant sheaf of type $(n, 1)$ which corresponds to the collection $(\{\mathcal{F}_0, \mathcal{F}_1\}, \phi)$. Assume $\mathcal{F}_0 \simeq \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ and $\mathcal{F}_1 \simeq \mathcal{O}_{\mathbb{P}^1}(b)$ and $\phi = (\alpha_1, \ldots, \alpha_n)$, where

$$\alpha_i : \mathcal{O}_{\mathbb{P}^1}(a_i) \to \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

Then, $\chi(\mathcal{F}) = 1$ is equivalent to

$$\sum_{i=1}^n (a_i + 1) + (b + 1) = 1. \quad (2.10)$$

As before, by condition (2.5) in Theorem 2.2.5, $\alpha_i$ is given by a monomial in $x$ and $y$ of degree $b - a_i + k$.

**Proposition 2.4.7.** $\mathcal{F}$ of type $(n, 1)$ is stable if and only if

- $\alpha_i$’s are nonzero,
- $a_i \geq 0$,
- for all $1 \leq i, j \leq n$, $\deg(\gcd(\alpha_i, \alpha_j)) \leq b - a_i - a_j + k - 1$.

*Proof.* As before, for $\mathcal{F}$ to be indecomposable, $\alpha_i$’s are nonzero. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$ whose $\pi_*\mathcal{O}_Y$-module structure is $(\{\mathcal{G}_0, \mathcal{G}_1\}, \psi)$. Since $\mathcal{F}_1$ is of rank 1, we have two cases: $\mathcal{G}_1 = \mathcal{F}_1$ or $\mathcal{G}_1 = 0$. 

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Suppose $G_1 = \mathcal{F}_1$. Let $G_0 \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a'_i)$ where $r$ is the rank of $G_0$. Without loss of generality, we may assume $a_i$’s and $a'_i$’s are nonincreasing. Then for $1 \leq i \leq r$,

$$a'_i \leq a_i$$

because otherwise there does not exist an injective map from $G_0$ to $\mathcal{F}_0$. So, it is enough to check for the cases $a'_i = a_i$, i.e., $G_0 \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some $0 \leq r \leq n$. Then by looking at the quotients, it is easy to see that the stability implies $a_i \geq 0$ for all $1 \leq i \leq n$. Note that if $G_0 = 0$, we have $b \leq -1$ which is a consequence of (2.10) and $a_i \geq 0$.

Now suppose $G_1 = 0$. Then, $G_0$ is a subsheaf of $K = \ker \phi$. Let $K_{ij}$ be the kernel of the restricted map

$$(\alpha_i, \alpha_j): \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1}(a_j) \to \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

When $G_0 = K_{ij}$, by (2.8), the stability implies

$$a_i + a_j - b - k + \deg(\gcd(\alpha_i, \alpha_j)) \leq -1,$$

which is the third condition. For an arbitrary $G_0$, it suffices to show the degree of $G_0$ is negative provided that the degrees of $K_{ij}$ are negative for all $1 \leq i, j \leq n$. Hence we may assume that $G_0$ is a line bundle. By Proposition 2.2.9 we may also assume $G_0$ is equivariant subsheaf of $\mathcal{F}_0$, that is, the inclusion

$$G_0 \to \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \simeq \mathcal{F}_0$$

is given by a matrix with monomial entries. Let $(p_1, \cdots, p_n)^t$ be the inclusion map where
$p_i$’s are monomials. Then, we have

$$\sum_{i=1}^{n} p_i \alpha_i = 0.$$ 

Since all terms are monomials and at least two terms are nonzero, this implies that there exist $j_1, j_2$ such that $p_{j_1} \alpha_{j_1}$ and $p_{j_2} \alpha_{j_2}$ are nonzero and proportional. Then $\deg(p_{j_1} \alpha_{j_1}) \geq \deg(\text{lcm}(\alpha_{j_1}, \alpha_{j_2}))$, and

$$\deg \mathcal{G}_0 = a_{j_1} - \deg(p_{j_1}) \leq a_{j_1} + \deg(\alpha_{j_1}) - \deg(\text{lcm}(\alpha_{j_1}, \alpha_{j_2}))$$

$$= a_{j_1} - \deg(\alpha_{j_2}) + \deg(\text{gcd}(\alpha_{j_1}, \alpha_{j_2}))$$

$$= a_{j_1} + a_{j_2} - b + k + \deg(\text{gcd}(\alpha_{j_1}, \alpha_{j_2}))$$

$$= \deg K_{j_1, j_2} \leq -1.$$ 

Hence it is enough to check for subsheaves $K_{ij}$. □

The type $(1, n)$ is dual to the type $(n, 1)$. Now, assume $F_0 \simeq \mathcal{O}_{\mathbb{P}^1}(c)$ and $F_1 \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i)$ and $\phi = (\beta_1, \ldots, \beta_n)^t$, where

$$\beta_i : \mathcal{O}_{\mathbb{P}^1}(c) \to \mathcal{O}_{\mathbb{P}^1}(d_i) \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

is given by a monomial in $x$ and $y$ of degree $d_i - c + k$. Then, $\chi(F) = 1$ is equivalent to

$$(c + 1) + \sum_{i=1}^{n} (d_i + 1) = 1. \quad (2.11)$$

**Proposition 2.4.8.** $F$ of type $(1, n)$ is stable if and only if

- $\beta_i$’s are nonzero,
- $d_i \leq -1$, 

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• for all $1 \leq i, j \leq n$, $\deg(\gcd(\beta_i, \beta_j)) \leq d_i + d_j - c + k$.

Proof. The proof is dual to the proof of the previous proposition. Indecomposability implies $\beta_i$’s are nonzero. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$ whose $\pi_*\mathcal{O}_Y$-module structure is $(\{\mathcal{G}_0, \mathcal{G}_1\}, \psi)$. If $\mathcal{G}_0 = 0$, it is enough to check for $\mathcal{G}_1 \simeq \mathcal{O}_{\mathbb{P}^1}(d_i)$. So, we have $d_i \leq -1$.

Suppose $\mathcal{G}_0 = \mathcal{O}_{\mathbb{P}^1}(c)$. Let $Q_{ij}$ be the torsion free cokernel of the map

$$\beta_{ij} = (\beta_i, \beta_j)^t: \mathcal{O}_{\mathbb{P}^1}(c) \to (\mathcal{O}_{\mathbb{P}^1}(d_i) \oplus \mathcal{O}_{\mathbb{P}^1}(d_j)) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

If $\mathcal{G}_1$ is the saturation of $\bigoplus_{i \neq j} \mathcal{O}_{\mathbb{P}^1}(d_i) \oplus \text{im} \beta_{ij}$, then by (2.9),

$$\deg Q_{ij} = d_i + d_j - c + k - \deg(\gcd(\beta_i, \beta_j)) \geq 0,$$

which is the third condition. Now, let $\mathcal{G}_1$ be an arbitrary subsheaf of $\mathcal{F}_1$ containing the image of $\phi$. We may assume $\mathcal{G}_1$ is an equivariant saturated subsheaf of rank $n - 1$. Let $(q_1, \cdots, q_n)$ be the natural projection map from $\mathcal{F}_1$ to the quotient $\mathcal{F}_1/\mathcal{G}_1$ where $q_i$’s are monomials. Then

$$\sum_{i=1}^n \beta_i q_i = 0.$$

As in the previous proposition, we can find $j_1, j_2$ such that $\beta_{j_1} q_{j_1}$ and $\beta_{j_2} q_{j_2}$ are nonzero and proportional. Thus,

$$\deg \frac{\mathcal{F}_1}{\mathcal{G}_1} = d_{j_1} + \deg(q_{j_1}) \geq d_{j_1} + \deg(\beta_{j_2}) - \deg(\gcd(\beta_{j_1}, \beta_{j_2}))$$

$$= d_{j_1} + d_{j_1} - c + k - \deg(\gcd(\beta_{j_1}, \beta_{j_2}))$$

$$= \deg Q_{j_1, j_2} \geq 0.$$

So, it is enough to check for $Q_{ij}$.
Propositions 2.4.7 and 2.4.8 have straightforward generalizations to types \((n, 1^d)\) and \((1^d, n)\).

**Proposition 2.4.9.** For a sheaf \(\mathcal{F}\) of type \((n, 1^d)\), let \(\mathcal{F}_0 \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)\) and \(\mathcal{F}_j \simeq \mathcal{O}_{\mathbb{P}^1}(b_j)\) for \(1 \leq j \leq d\). By \(\chi(\mathcal{F}) = 1\), we have

\[
\sum_{i=1}^{n} (a_i + 1) + \sum_{j=1}^{d} (b_j + 1) = 1. \tag{2.12}
\]

Then, \(\mathcal{F}\) of type \((n, 1^d)\) is stable if and only if

- all maps \(\phi_j, 0 \leq j \leq d\) have nonzero monomial entries,
- \(a_i \geq 0, \sum_{j=s}^{d} (b_j + 1) \leq 0\) for \(1 \leq s \leq d\),
- \(\deg(\gcd(\alpha_i, \alpha_j)) \leq b_i - a_i - a_j + k - 1\),

where \(\phi_0 = (\alpha_1, \ldots, \alpha_n)\).

**Proposition 2.4.10.** For a sheaf \(\mathcal{F}\) of type \((1^d, n)\), let \(\mathcal{F}_j \simeq \mathcal{O}_{\mathbb{P}^1}(c_j)\) for \(0 \leq j \leq d - 1\) and \(\mathcal{F}_d \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i)\). By \(\chi(\mathcal{F}) = 1\), we have

\[
\sum_{j=0}^{d-1} (c_j + 1) + \sum_{i=1}^{n} (d_i + 1) = 1. \tag{2.13}
\]

Then, \(\mathcal{F}\) of type \((1^d, n)\) is stable if and only if

- all maps \(\phi_j, 0 \leq j \leq d\) have nonzero monomial entries,
- \(d_i \leq -1, \sum_{j=0}^{s} (d_j + 1) \geq 1\) for \(0 \leq s \leq d - 1\),
- \(\deg(\gcd(\beta_i, \beta_j)) \leq d_i + d_j - c_{d-1} + k\),

where \(\phi_{d-1} = (\beta_1, \ldots, \beta_n)^t\).
Corollary 2.4.11. All stable equivariant sheaves of type \((1^d), (n, 1^d)\) or \((1^d, n)\) are isolated points in \(M_d(k)^T\).

Proof. By scaling automorphisms in each case, we can set the coefficients of monomials to be 1. So, equivariant sheaves of these types are isolated. \(\square\)

Corollary 2.4.12. For any \(k \geq -1\),

\[ N_{(1,n)}(k) = N_{(n,1)}(k + n - 1) \]

Proof. For a given \(c\) and \(d_j\)'s as in Proposition 2.4.8, we let \(a_j = -1 - d_j\) and \(b = -n - c\) and \(\alpha_j = \beta_j\). Note that \(\text{deg}(\beta_j) = d_j - c + k = c - a_j + (k + n - 1)\) as required. Moreover, \(b - a_i - a_j + ((k + n - 1) - 1) = d_i + d_j - c + k\), and the formula (2.10) for the Euler characteristic is equivalent to (2.11). So, \(a_j, b, \alpha_j\) so defined will determine a stable sheaf in \(M_{(n,1)}^T(k + n - 1)\). Hence, this gives a bijection between \(M_{(1,n)}^T(k)\) and \(M_{(n,1)}^T(k + n - 1)\). \(\square\)

2.5 The Calculation of BPS invariants

In this section, we compute the local BPS invariants when \(d = 1, 2, 3, \) or 4. In Section 2.6, we will show

\[ n_d(k) = (-1)^{kd^2 + 1} e_{\text{top}}(M_d(k)). \quad (2.14) \]
Hence it is enough to compute \( N_d(k) = e_{\text{top}}(M_d(k)) \). Conjecture 2.1.3 combined with the Gromov-Witten theory [7] predicts that

\[
    n_1(k) = (-1)^{k+1},
\]
\[
    n_2(k) = \begin{cases} 
    -\frac{k(k+2)}{4} & \text{if } k \text{ is even,} \\
    -\frac{(k+1)^2}{4} & \text{if } k \text{ is odd,} 
    \end{cases}
\]
\[
    n_3(k) = (-1)^{k+1} \frac{k(k+1)^2(k+2)}{6}.
\]
\[
    n_4(k) = -\frac{k(k+1)^2(k+2)(2k^2 + 4k + 1)}{12}.
\]

By (2.14), signs are correct. By Corollary 2.4.11 and the localization formula (2.7), we compute \( N_d(k) \) by counting \( T \)-equivariant sheaves.

2.5.1 \( d = 1 \)

By Corollary 2.4.3 it is easy to see that \( N_1(k) = 1 \). We can see this more directly. Let \( \mathcal{F} \) be a stable sheaf with Hilbert polynomial \( n + 1 \) whose support is \( \mathbb{P}^1 \). Then \( \mathcal{F} \) has a section, or a nonzero morphism \( \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F} \). Since \( \mathcal{O}_{\mathbb{P}^1} \) is stable with the Hilbert polynomial \( n + 1 \), this morphism is an isomorphism. Hence

\[
    M^T_1(k) = \{ \mathcal{O}_{\mathbb{P}^1} \}.
\]

Hence, we have

\[
    N_1(k) = 1.
\]
2.5.2 \( d = 2 \)

Only sheaves of type \((1,1)\) appear. By Corollary 2.4.3,

\[
N_2(k) = \sum_{\lambda_1 \geq \lambda_0} (\lambda_1 - \lambda_0 + 1)
\]

where the sum is over all partitions \(\lambda_1 + \lambda_0 = k - 1\). Therefore,

\[
N_2(k) = \sum_{\lambda_0=0}^{\lfloor \frac{k+1}{2} \rfloor} (k - 2\lambda_0) = \begin{cases} \frac{k(k+2)}{4} & \text{if } k \text{ is even.} \\ \frac{(k+1)^2}{4} & \text{if } k \text{ is odd.} \end{cases}
\]

2.5.3 \( d = 3 \)

In this case, sheaves of type \((1,1,1)\), \((2,1)\) and \((1,2)\) appear. By Corollary 2.4.12,

\[
N_{(1,2)}(k) = N_{(2,1)}(k+1).
\] (2.19)

We start with the type \((2,1)\).

To count the \(T\)-equivariant sheaves of type \((2,1)\), we let

\[
S_{(2,1)}(k) = \left\{ (a_1, a_2, b) \in \mathbb{Z}^3 : \begin{array}{l} a_1 + a_2 + b = -2, \\ 0 \leq a_1, a_2 \leq b + k, \ b \leq -1 \end{array} \right\}.
\]

For \((a_1, a_2, b) \in S_{(2,1)}(k)\), we count pairs \((\alpha_1, \alpha_2)\) of nonzero monomials with no common factor of degree greater than \(b - a_1 - a_2 + k - 1 = 2b + k + 1\).

**Definition 2.5.1.** For \(r < \min(n, m)\),
\[ P_{(n,m,r)} = \left\{ (v, w) : v, w \text{ monomials in } x \text{ and } y \right\} \]

\[ \deg v = n, \deg w = m, \deg(\gcd(v, w)) \leq r \]

**Lemma 2.5.2.** \(|P_{(n,m,r)}| = \begin{cases} (r + 1)(r + 2) & \text{if } 0 \leq r < \min(n, m) \\
0 & \text{if } r < 0 \end{cases} \]

**Proof.** For \((v, w) \in P_{(n,m,r)}\), let \(g = \gcd(v, w)\) and \(d\) be its degree. Then \((v, w)\) is either \((x^{n-d}g, y^{m-d}g)\) or \((y^{n-d}g, x^{m-d}g)\). Since there are \(d + 1\) choices for \(g\), \(|P_{(n,m,r)}| = 2 \sum_{d=0}^{r}(d + 1) = (r + 1)(r + 2)\). \(\square\)

Note that if \(a_1 = a_2 = a\), switching two factors of \(\mathcal{F}_0 = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a)\) gives an isomorphism between two sheaves determined by \((\alpha_1, \alpha_2)\) and \((\alpha_2, \alpha_1)\). So, we must count half of such pairs \((\alpha_1, \alpha_2)\) if the degree of \(\alpha_1\) and \(\alpha_2\) are the same.

We let

\[ f(n, m, r) = \begin{cases} |P_{(n,m,r)}| & \text{if } n \neq m \\
\frac{1}{2}|P_{(n,m,r)}| & \text{if } n = m \end{cases} \]

Then, the total number of \(T\)-fixed sheaves of type \((2, 1)\) is

\[ N_{(2,1)}(k) = \sum_{(a_1, a_2, b) \in S_{(2,1)}(k)} f(b - a_1 + k, b - a_2 + k, 2b + k + 1). \tag{2.20} \]

**Lemma 2.5.3.** If \(k \geq 1\),

\[ N_{(2,1)}(k) = \sum_{b = \left\lceil \frac{k - 3}{4} \right\rceil}^{k - 1} \left\lfloor \frac{b + 1}{2} \right\rfloor (k + 2b + 2)(k + 2b + 3) \]

\[ + \frac{1}{2} \sum_{a = 0}^{k - 4} (k - 4a - 2)(k - 4a - 1) \tag{2.21} \]
Proof. Each sum corresponds to the case $a_1 > a_2$ and $a_1 = a_2$ respectively. Note that in (2.20), $f$ has $\frac{1}{2}$ factor if and only if $a_1 = a_2$.

First, we count the case $a_1 > a_2$. From (2.20), since $r = 2b + k + 1 \geq 0$, $-\frac{k+1}{2} \leq b \leq -1$. For each $b$, we can check there are $\left\lfloor \frac{k+1}{2} \right\rfloor$ pairs of $(a_1, a_2)$ with $a_1 > a_2$ satisfying all the required conditions. By the Lemma 2.5.2, this verifies the first sum.

If $a_1 = a_2 = a$, then $b = -2 - 2a \geq -\frac{k+1}{2}$. So, $0 \leq a \leq \frac{k-3}{4}$. Thus by (2.20) and Lemma 2.5.2, we get the second sum.

Now, to count sheaves of type $(1, 1, 1)$, let

$$S_{(1,1,1)}(k) = \left\{(\lambda_0, \lambda_1, \lambda_2) : \sum_{i=1}^{3} \lambda_i = 3k - 2, 0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq 2k - 1 \right\}$$

Then by Corollary 2.4.3

$$N_{(1,1,1)}(k) = \sum_{(\lambda_0, \lambda_1, \lambda_2) \in S_{(1,1,1)}(k)} (\lambda_2 - \lambda_1 + 1)(\lambda_1 - \lambda_0 + 1). \quad (2.22)$$

Theorem 2.5.4.

$$N_3(k) = \frac{k(k+1)^2(k+2)}{6} \quad (2.23)$$

Proof. We compute $N_3(k) - N_3(k-1)$ and prove (2.23) by induction. It remains to count type $(1,1,1)$ sheaves.

The map

$$(\lambda_0, \lambda_1, \lambda_2) \mapsto (\lambda_0 + 1, \lambda_1 + 1, \lambda_2 + 1)$$

gives an injection from $S_{(1,1,1)}(k-1)$ to $S_{(1,1,1)}(k)$. Since the summand in (2.22) does not change under this map, the corresponding terms cancel each other in $N_3(k) - N_3(k-1)$.
The remaining terms in $N_3(k)$ are for $\lambda_0 = 0$ or $\lambda_2 = 2k - 1$. We claim that

$$N_{(1,1,1)}(k) - N_{(1,1,1)}(k - 1)$$

$$= \sum_{\lambda_1 = k - 1}^{\lfloor \frac{k-2}{2} \rfloor} (3k - 2\lambda_1 - 1)(\lambda_1 + 1) + \sum_{\lambda_0 = 1}^{\lfloor \frac{k-1}{2} \rfloor} (\lambda_0 + k + 1)(k - 2\lambda_0).$$

If $\lambda_0 = 0$, then we must have $\lambda_1 + \lambda_2 = 3k - 2$, $\lambda_2 \leq 2k - 1$. So, $\lambda_2 = 3k - 2 - \lambda_1$ and $k - 1 \leq \lambda_1 \leq \frac{3k-2}{2}$. Hence we have the first term.

If $\lambda_0 \neq 0$ and $\lambda_2 = 2k - 1$, we must have $\lambda_0 + \lambda_1 = k - 1$, and $\lambda_0 > 0$. So, $1 \leq \lambda_0 \leq \frac{k-1}{2}$, which verifies the second term.

Now, using the Lemma 2.5.3 and (2.19), we can check case by case ($k \mod 4$) that

$$N_3(k) - N_3(k - 1) = \frac{k(k+1)(2k+1)}{3}.$$  

Since it is easy to verify (2.23) for small values of $k$, this proves the theorem.

\[\square\]

Corollary 2.5.5. Conjecture 2.1.3 holds for $d = 1, 2$ and 3.

2.5.4 $d = 4$

Types $(1,1,1,1)$, $(3,1)$, $(1,3)$, $(2,1,1)$ and $(1,1,2)$ are treated in Section 2.4. The remaining types are $(1,2,1)$ and $(2,2)$. In these types, positive dimensional torus fixed loci can occur.

Example 2.5.6. We give an example of a positive dimensional $T$-fixed locus in degree 4 of type $(1,2,1)$ when $k = 2$.

Let $\mathcal{F}_0 = \mathcal{O}_{\mathbb{P}^1}$, $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^1}(-1)$. The $\pi_*, \mathcal{O}_\mathcal{Y}$-module structure is

$$\phi_0 = \begin{pmatrix} x \\ y \end{pmatrix}: \mathcal{O}_{\mathbb{P}^1} \to (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(2),$$

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\( \phi_1 = \begin{pmatrix} c_1 y^2 & c_2 xy \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(2), \)

where \( c_1 \) and \( c_2 \) are in \( \mathbb{C} \). It is easy to see that this satisfies the condition \( (2.5) \) for an equivariant sheaf. As only scaling isomorphisms are allowed, we cannot set all coefficients to be 1 using isomorphisms.

Let \( \mathcal{F}(c_1, c_2) \) be such a sheaf. One can see that \((c_1, c_2)\) can not be \((0, 0)\) and that \( \mathcal{F}(c_1, c_2) \simeq \mathcal{F}(\lambda c_1, \lambda c_2) \) for \( \lambda \in \mathbb{C}^* \). So, this \( T \)-fixed locus is isomorphic to \( \mathbb{P}^1 \).

Let the \( \pi_* \mathcal{O}_Y \)-module structure of a sheaf of type \((1, 2, 1)\) be

\[
\phi_0 = (\alpha_1, \alpha_2)^t : \mathcal{O}_{\mathbb{P}^1}(a) \to (\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \\
\phi_1 = (\beta_1, \beta_2) : \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2) \to \mathcal{O}_{\mathbb{P}^1}(c) \otimes \mathcal{O}_{\mathbb{P}^1}(k),
\]

where \( \alpha_i \) and \( \beta_i \) are monomials with coefficient 1.

\[
\chi(\mathcal{F}) = 1 \text{ is equivalent to } a + b_1 + b_2 + c = -3. \tag{2.24}
\]

Without loss of generality, we assume \( b_1 \geq b_2 \). Suppose that all entries \( \alpha_1, \beta_i \) are nonzero. Then by condition \((2.5)\),

\[
\text{wt}(\alpha_1) - \text{wt}(\alpha_2) = \text{wt}(\beta_2) - \text{wt}(\beta_1), \tag{2.25}
\]

where \( \text{wt}(\cdot) \) denotes the \( T \)-weight of a monomial.

**Proposition 2.5.7.** Suppose \( \mathcal{F} \) is a sheaf of type \((1, 2, 1)\) as above. Assume \( b_1 \geq b_2 \). Then \( \mathcal{F} \) is stable if and only if

1. No more than one of \( \alpha_1, \alpha_2, \beta_1 \) or \( \beta_2 \) is zero.

2. \( c \leq -1, \ a \geq 0, \ b_1 + c \leq -2. \)
3. \( \deg(\gcd(\alpha_1, \alpha_2)) \leq b_1 + b_2 - a + k, \)
\( \deg(\gcd(\beta_1, \beta_2)) \leq c + k - b_1 - b_2 - 1. \)

4. If \( \alpha_1 \beta_1 + \alpha_2 \beta_2 = 0 \), then \( \deg(\gcd(\beta_1, \beta_2)) \leq c + k - b_1 - b_2 - a - 2. \)

**Proof.** If at least two of \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are zero, \( \mathcal{F} \) is decomposable.

Suppose \( \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2) \) is a \( \pi_* \mathcal{O}_Y \)-submodule, where \( \mathcal{G}_0 \subset \mathcal{O}_{\pi_1}(a), \mathcal{G}_1 \subset \mathcal{O}_{\pi_1}(b_1) \oplus \mathcal{O}_{\pi_1}(b_2) \) and \( \mathcal{G}_2 \subset \mathcal{O}_{\pi_1}(c). \) Let

\[ \text{rank}(\mathcal{G}) = (\text{rank}(\mathcal{G}_0), \text{rank}(\mathcal{G}_1), \text{rank}(\mathcal{G}_2)). \]

For each possible choice of the rank of \( \mathcal{G} \), we examine the stability condition.

1. \( \text{rank}(\mathcal{G}) = (0, 0, 1) : c \leq -1. \)

2. \( \text{rank}(\mathcal{G}) = (0, 2, 1) : b_1 + b_2 + c \leq -3 \) or \( a \geq 0 \) by (2.24).

3. \( \text{rank}(\mathcal{G}) = (0, 1, 1) : \) Since the degree of \( \mathcal{G}_1 \) is no more than \( b_1 \) as \( b_1 \geq b_2 \), we have \( b_1 + c \leq -2. \)

4. \( \text{rank}(\mathcal{G}) = (1, 1, 1) : \) We can reduce to the case when \( \mathcal{F}/\mathcal{G} \) is the torsion free cokernel of \( \phi_0. \) So, by Lemma 2.4.5, stability condition is

\[ b_1 + b_2 - a + k - \deg(\gcd(\alpha_1, \alpha_2)) \geq 0. \]

5. \( \text{rank}(\mathcal{G}) = (0, 1, 0) : \) The kernel of \( \phi_1 \) has degree \( b_1 + b_2 - c - k + \deg(\gcd(\beta_1, \beta_2)) \). So,

\[ \deg(\gcd(\beta_1, \beta_2)) \leq c + k - b_1 - b_2 - 1. \]
6. rank(\(G\)) = (1, 1, 0) : A subsheaf of this type exists only if the image of \(\phi_0\) is in the kernel of \(\phi_1\), i.e., if \(\alpha_1\beta_1 + \alpha_2\beta_2 = 0\). In such a case, we take \(G_0 = \mathcal{O}_{\mathbb{P}^1}(a)\) and \(G_1 = \ker \phi_1\). So,

\[ a + (b_1 + b_2 - c - k + \deg(\gcd(\beta_1, \beta_2))) \leq -2, \]

which is the condition (4).

\[ \square \]

Let the \(\pi_*\mathcal{O}_Y\)-module structure of a sheaf \(F\) of type (2, 2) be

\[
\phi = \begin{pmatrix}
\phi_{11} & \phi_{12} \\
\phi_{21} & \phi_{22}
\end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \to (\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)) \otimes \mathcal{O}_{\mathbb{P}^1}(k).
\]

\(\chi(F) = 1\) is equivalent to

\[ a_1 + a_2 + b_1 + b_2 = -3. \]  \hfill (2.26)

Suppose that all entries \(\phi_{ij}\) are nonzero. Then by condition (2.5),

\[ \wt(\phi_{11}) - \wt(\phi_{21}) = \wt(\phi_{12}) - \wt(\phi_{22}), \]  \hfill (2.27)

which means \(\phi_{11}\phi_{22}\) and \(\phi_{12}\phi_{21}\) are proportional.

**Proposition 2.5.8.** Suppose \(F\) is a sheaf of type (2, 2) as above. Assume \(a_1 \geq a_2\) and \(b_1 \geq b_2\). Then \(F\) is stable if and only if

1. \(\phi_{21}\) is nonzero. No more than one of \(\phi_{ij}\) is zero.

2. \(a_1 \geq a_2 \geq 0\) and \(b_2 \leq b_1 \leq -1\).

3. \(\deg(\gcd(\phi_{11}, \phi_{21})) \leq a_2 + b_1 + b_2 - a_1 + k + 1\),

\[ \deg(\gcd(\phi_{21}, \phi_{22})) \leq b_2 - b_1 - a_1 - a_2 + k - 2. \]
4. If $\phi_{11}\phi_{22} = \phi_{12}\phi_{21}$, then

$$\deg(\text{gcd}(\phi_{11}, \phi_{21})) \leq b_1 + b_2 - a_1 + k, \text{ and}$$

$$\deg(\text{gcd}(\phi_{11}, \phi_{12})) \leq b_1 + k - a_1 - a_2 - 1.$$ 

Proof. If at least two of $\phi_{ij}$ are zero, $\mathcal{F}$ is decomposable.

Let

$$r_1 = \deg(\text{gcd}(\phi_{11}, \phi_{12})), \quad r_2 = \deg(\text{gcd}(\phi_{21}, \phi_{22})),$$

$$s_1 = \deg(\text{gcd}(\phi_{11}, \phi_{21})), \quad s_2 = \deg(\text{gcd}(\phi_{12}, \phi_{22})).$$

Then by (2.27),

$$r_2 = r_1 + b_2 - b_1 \text{ and } s_2 = s_1 + a_1 - a_2,$$  \hspace{1cm} (2.28)

provided that $\phi_{ij}$ are all nonzero.

Suppose $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is a $\pi_*\mathcal{O}_Y$-submodule. For each possible choice of the rank of $\mathcal{G}$, we examine the stability conditions.

1. $\text{rank}(\mathcal{G}) = (0, 1) : b_1 \leq -1$.

2. $\text{rank}(\mathcal{G}) = (0, 2) : b_1 + b_2 \leq -2$ which is implied by the above condition (1).

3. $\text{rank}(\mathcal{G}) = (1, 2) : a_2 \geq 0$.

4. $\text{rank}(\mathcal{G}) = (1, 1)$: Let $\mathcal{G}_0 = \mathcal{O}_{P^1}(m)$ and $\mathcal{G}_1 = \mathcal{O}_{P^1}(n)$. If $a_2 < m \leq a_1$, $\mathcal{G}_0$ is a subsheaf of $\mathcal{O}_{P^1}(a_1)$. Hence, we can replace $\mathcal{G}_0$ by $\mathcal{O}_{P^1}(a_1)$ and take $\mathcal{G}_1$ to be the saturation of the image of $\mathcal{O}_{P^1}(a_1)$ under $\phi$. The quotient is $(\mathcal{O}_{P^1}(a_2)$, the torsion free cokernel of $\phi|_{\mathcal{O}_{P^1}(a_1)}$). So, for $\mathcal{F}$ to be stable, we must have

$$a_2 + b_1 + b_2 - a_1 + k - s_1 \geq -1,$$
by Lemma 2.4.5. Note that if \( \phi_{21} \) is zero, \( s_1 = b_1 - a_1 + k \). Then the quotient has degree \( a_2 + b_2 \leq -2 \) by (2.26) contradicting the stability.

Now suppose \( m \leq a_2 \). If \( n \leq b_2 \), since \( a_2 + b_2 \leq -2 \), there is nothing to check. If \( b_2 \leq n \leq b_1 \), we can replace \( G_1 \) by \( \mathcal{O}_{\mathbb{P}^1}(b_1) \) and take \( G_0 \) to be the inverse image of \( \mathcal{O}_{\mathbb{P}^1}(b_1) \), i.e., the kernel of the map

\[
(\phi_{21}, \phi_{22}) : \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \to \mathcal{O}_{\mathbb{P}^1}(b_2) \otimes \mathcal{O}_{\mathbb{P}^1}(k).
\]

Then the condition is

\[
b_1 + a_1 + a_2 - b_2 - k + r_2 \leq -2.
\]

5. rank(\( G \)) = (1, 0) or (2, 1) : A subsheaf of these types exists only if the image of \( \phi \) has rank 1, in other words, if \( \phi_{11} \phi_{22} = \phi_{12} \phi_{21} \). Then, the torsion free cokernel of \( \phi \) has degree

\[
b_1 + b_2 - a_1 + k - s_1 = b_1 + b_2 - a_2 + k - s_2,
\]

and the kernel of \( \phi \) has degree

\[
a_1 + a_2 - b_1 - k + r_1 = a_1 + a_2 - b_2 - k + r_2,
\]

by (2.28). Hence the conditions are

\[
s_1 \leq b_1 + b_2 - a_1 + k \text{ and } r_1 \leq b_1 + k - a_1 - a_2 - 1.
\]

Remark 2.5.9. As we will see in the next example, all positive dimensional loci of type (1,2,1) can be expressed as a GIT quotient of \( (\mathbb{P}^1)^4 \) by the action of \( SL_2(\mathbb{C}) \). While the
linearization may be different, the quotient is always isomorphic to \( \mathbb{P}^1 \). Similar argument for type \((2, 2)\) holds. So, we can see that all \( T \)-fixed loci in degree 4 are either isolated points or \( \mathbb{P}^1 \).

**Example 2.5.10.** In Example 2.5.6, \( F_0, F_1 \) and \( F_2 \) are unchanged along the one dimensional torus fixed locus. The condition (4) in Propositions 2.5.7 and 2.5.8 suggests this is not true in general.

Assume \( k = 3 \) and let \( F_0 = \mathcal{O}_{\mathbb{P}^1}(1), F_1 = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \) and \( F_2 = \mathcal{O}_{\mathbb{P}^1}(-2) \). The \( \pi_* \mathcal{O}_Y \)-module structure is

\[
\phi_0 = \begin{pmatrix} x \\ y \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(1) \to (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(3),
\]

\[
\phi_1 = \begin{pmatrix} c_1 xy & c_2 x^2 \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(3),
\]

where \( c_1 \) and \( c_2 \) are in \( \mathbb{C} \). By Proposition 2.5.7, we can check the corresponding sheaf \( \mathcal{F}(c_1, c_2) \) is stable unless \( c_1 = -c_2 \).

As the \( T \)-fixed locus \( M_d(k) \) is compact, the limit of above family at \( c_1 = -c_2 \) exists in \( M_d(k)^T \). To see what the limit is, we need to examine \( \Delta \)-family described in Proposition 2.2.1.

Assume the fixed point in the open set \( U_{\sigma_1} \) is given by \( x = 0 \) and the fixed point in \( U_{\sigma_2} \) by \( y = 0 \). Then the above \( \pi_* \mathcal{O}_Y \)-module structure has weight space decomposition as Figure 2.8.

In Figure 2.8, \( A, B, C \) and \( Q \) are one dimensional. By Proposition 2.2.1 \( T \)-fixed sheaves with such weight space decomposition are determined by inclusions of \( A, B \) and \( C \) into \( \mathbb{C}^2 \) and a surjection \( \mathbb{C}^2 \to Q \). The \( SL_2(\mathbb{C}) \) action on \( \mathbb{C}^2 \) via change of basis encodes isomorphism between sheaves. See [30, Chapter 3] for a detailed discussion.
We identify \( C^2 \to Q \) with its kernel \( K \) so that \( A, B, C \) and \( K \) are in \( Gr(1, C^2) \simeq \mathbb{P}^1 \). We want to relate Gieseker stability to GIT stability condition for the action of \( SL_2(\mathbb{C}) \) on \((\mathbb{P}^1)^4\). It can be checked that the associated sheaf is Gieseker stable unless

\[
A = B \text{ or } A = C \text{ or } A = K \text{ or } B = C = K. \tag{2.29}
\]

Meanwhile, a point \((p_1, p_2, p_3, p_4) \in (\mathbb{P}^1)^4\) is GIT stable with respect to a line bundle \( O(k_1, k_2, k_3, k_4) \) if and only if for any point \( p \in \mathbb{P}^1 \)

\[
\sum_{p = p_i} k_i < \frac{1}{2} \sum_{i=1}^4 k_i. \tag{2.30}
\]

See [10, Theorem 11.2], [38, Section 4.4]. If we take \( k_1 = 2, k_2 = k_3 = k_4 = 1 \), these two conditions agree with each other when we let \((p_1, p_2, p_3, p_4) = (A, B, C, K)\). This is an example of matching GIT stability and Gieseker stability discussed in [30, Chapter 3].

Therefore, the \( T \)-fixed locus is

\[
(\mathbb{P}^1)^4 \sslash SL_2(\mathbb{C}) \simeq \mathbb{P}^1.
\]

All positive dimensional fixed loci can be analyzed similarly.

The condition \( c_1 = -c_2 \) is equivalent to \( A = K \). It is easy to check that at the limit in
$(\mathbb{P}^1)^4 / SL_2(\mathbb{C})$, we have $B = C$ and $A, B, K$ are distinct. By reading equivariant vector bundles in each rows, we can see the limit has $\pi_*\mathcal{O}_Y$-module structure

$$\phi_0 = \begin{pmatrix} xy \\ 1 \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(1) \to (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \mathcal{O}_{\mathbb{P}^1}(3),$$

$$\phi_1 = \begin{pmatrix} x & x^2y \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(3).$$

Note that since $xy$ is a multiple of 1, or $x^2y$ is a multiple of $x$, we can set all the coefficients of monomials to be 1 up to isomorphism.

**Remark 2.5.11.** Based on the classification of $T$-equivariant stable sheaves studied above and in Section 2.4, we can compute $N_4(k)$. The author has verified that the result is consistent with (2.18) when $k \leq 100$ using a computer program. However, we don’t have a proof for general $k$.

### 2.6 Equivariant Residue

In this section, we compute the virtual tangent space of $M_d(k)$ and verify the sign in the BPS invariants. The virtual tangent space at $\mathcal{F} \in M_d(k)$ is

$$\text{Ext}^1_X(\mathcal{F}, \mathcal{F}) - \text{Ext}^2_X(\mathcal{F}, \mathcal{F}).$$

Since $T$ preserves a canonical Calabi-Yau form, the canonical bundle on $X$ is trivial with trivial weight. By equivariant Serre duality,

$$\text{Ext}^1_X(\mathcal{F}, \mathcal{F}) \simeq \text{Ext}^2_X(\mathcal{F}, \mathcal{F})^*$$
as $T$-representations. So, the dual weights of the moving parts will be canceled and we just count signs.

Let $\mathcal{H}_k$ be the Hirzebruch surface whose toric fan has ray generators $u_1 = (-1, k)$, $u_2 = (0, 1)$, $u_3 = (1, 0)$, $u_4 = (0, -1)$. Denote the corresponding divisors by $D_1$, $D_2$, $D_3$, $D_4$. The total space $Y$ of $\mathcal{O}_{\mathbb{P}^1}(k)$ can be described as a toric variety by the fan $\{\text{Cone}(u_3, u_4), \text{Cone}(u_4, u_1)\}$. Hence, $Y$ is a subvariety of $\mathcal{H}_k$ and the zero section of $Y$ is the divisor $D_4$. Let $i: Y \to \mathcal{H}_k$ be the inclusion.

By the equivalence $D_1 \sim D_3$ and $D_2 \sim D_4 - kD_1$, any divisor on $\mathcal{H}_k$ can be expressed as $aD_3 + bD_4$ for integers $a$ and $b$. We have

$$aD_3 + bD_4 \text{ is ample if and only if } a, b > 0.$$ 

We fix an ample line bundle $D = 2D_3 + D_4$. Then, we have a well defined moduli space

$$M_{\mathcal{H}_k}(d) = \{ \mathcal{F} \text{ sheaf on } \mathcal{H}_k: c_1(\mathcal{F}) = dD_4, \chi(\mathcal{F}) = 1, D-(semi)stable \}.$$ 

Let $\mathcal{F}$ be a sheaf on $Y$ supported on a curve of class $d[\mathbb{P}^1]$. Then, $i_*\mathcal{F}$ is supported on a curve of class $dD_4$. Then, since $D_4 \cdot D = k + 2$, we have

$$\chi(i_*\mathcal{F} \otimes \mathcal{O}(nD)) = d(k + 2)n + \chi(\mathcal{F}) = P_{\mathcal{F}}((k + 2)n),$$

where $P_{\mathcal{F}}(n)$ is the Hilbert polynomial defined in [2.2]. Thus, since $k + 2 > 0$, $i_*\mathcal{F}$ is $D$-semistable if and only if $\mathcal{F}$ is semistable. Hence, $i_*$ induce an injective morphism from $M_d(k)$ to $M_{\mathcal{H}_k}(d)$.

**Proposition 2.6.1.** $M_{\mathcal{H}_k}(d)$ is a smooth projective variety of dimension $kd^2 + 1$. 

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Proof. By Serre duality, \( \text{Ext}^2(F, F) = \text{Hom}(F, F \otimes K)^* \). Since

\[
c_1(F) \cdot K = dD_4 \cdot K = -(k + 2) < 0,
\]

we have \( \chi(F \otimes K) < \chi(F) \) by Riemann-Roch theorem. Hence, by stability of \( F \), we have \( \text{Hom}(F, F \otimes K) = 0 \). Therefore, there is no obstruction and \( M_{H_k}(d) \) is a smooth projective variety.

We compute the dimension of \( \text{Ext}^1(F, F) \) using Riemann-Roch Theorem.

\[
\chi(F, F) = 1 - \dim \text{Ext}^1(F, F) = \int_{H_k} \text{ch}^\vee(F) \text{ch}(F) \text{td}(H_k)
\]

Since the rank of \( F \) is zero and \( c_1(F) = dD_4 \), the degree 2 term of right side is \( -d^2D_4^2 = -kd^2 \).

Therefore,

\[
\dim \text{Ext}^1(F, F) = 1 - \chi(F, F) = k^d^2 + 1.
\]

Thus \( \dim M_{H_k}(d) = k^d^2 + 1. \)

Corollary 2.6.2.

\[
n_d(k) = (-1)^{kd^2+1} e_{top}(M_d(k))
\]

Proof. \( M_d(k) \) is open subscheme of \( M_{H_k}(d) \), hence smooth of dimension \( kd^2 + 1 \). Then, this is a consequence of general properties of Donaldson-Thomas type invariant with symmetric obstruction theory \[1\].

Chapter 3

The Moduli Space of Sheaves on $\mathbb{P}^2$

In this chapter, we compute the Poincaré polynomial of the moduli space of stable sheaves with Hilbert polynomial $4n + 1$ on the projective plane. We use the natural torus action on $\mathbb{P}^2$ and apply the techniques of Białyńcki-Birula.

3.1 Moduli Spaces of Sheaves of Dimension one on $\mathbb{P}^2$

We fix a very ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$.

Definition 3.1.1. We define the Hilbert polynomial of a sheaf $\mathcal{F}$ by

$$P_{\mathcal{F}}(n) = \chi(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^2}(n)).$$

Let $[\mathbb{P}^1] \in H_2(\mathbb{P}^2, \mathbb{Z})$ be the class of line in $\mathbb{P}^2$. We are interested in sheaves whose supports have homology class $d[\mathbb{P}^1]$. For such a sheaf, it is easy to check that

$$P_{\mathcal{F}}(n) = dn + \chi(\mathcal{F}),$$

by Riemann-Roch theorem, since $c_1(\mathcal{F}) = d[\mathbb{P}^1]$. Here, we slightly abuse notation to denote by $[\mathbb{P}^1]$ the class of line in the Chow group $A^1(\mathbb{P}^2)$ or the cohomology group $H^1(\mathbb{P}^2)$.

Definition 3.1.2. Let $r(\mathcal{F})$ be the leading coefficient of the Hilbert polynomial of $\mathcal{F}$, called
the multiplicity. The reduced Hilbert polynomial of $\mathcal{F}$ is defined by

$$p_{\mathcal{F}}(n) = \frac{P_{\mathcal{F}}(n)}{r(\mathcal{F})}. $$

The sheaf $\mathcal{F}$ is semistable (resp. stable) if for any nonzero proper coherent subsheaf $\mathcal{G}$ of $\mathcal{F}$, $p_{\mathcal{G}}(n) \leq p_{\mathcal{F}}(n)$ (resp. $p_{\mathcal{G}}(n) < p_{\mathcal{F}}(n)$), for sufficiently large $n$.

**Definition 3.1.3.** We denote by $M_{\mathbb{P}^2}(d, \chi)$ the moduli space of semistable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $dn + \chi$.

A general construction of the moduli spaces of semistable sheaves is done by Simpson (See [18]) by a GIT quotient of a certain Quot scheme by a reductive group. The geometry of the moduli space $M_{\mathbb{P}^2}(d, \chi)$ was studied by Le Potier [32].

**Theorem 3.1.4 (Le Potier [32]).**

1. If $d$ and $\chi$ are coprime, $M_{\mathbb{P}^2}(d, \chi)$ is a smooth projective scheme of dimension $d^2 + 1$.

2. $M_{\mathbb{P}^2}(d, \chi) \cong M_{\mathbb{P}^2}(d, d + \chi)$

**Proof.** Given $\mathcal{F} \in M_{\mathbb{P}^2}(d, \chi)$, we have by Serre duality,

$$\text{Ext}^2(\mathcal{F}, \mathcal{F}) = \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^2}(-3))^* = 0.$$ 

Moreover, we can see the dimension of $\text{Ext}^1(\mathcal{F}, \mathcal{F})$ is $d^2 + 1$, by Riemann-Roch. The computation is similar to that of Proposition 2.6.1. This proves part 1.

For part 2, sending $\mathcal{F}$ to $\mathcal{F}(1)$ induces an isomorphism between these moduli spaces. □

Consider the total space of the canonical bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$ on $\mathbb{P}^2$, which we call local $\mathbb{P}^2$. We take $L$ to be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Then, for a sheaf whose support has class $d[\mathbb{P}^1] \in H_2(\mathbb{P}^2, \mathbb{Z})$, we may define Hilbert polynomial and stability with respect to $L$. 

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Table 3.1: Genus zero BPS invariants for local $\mathbb{P}^2$

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_d(X)$</td>
<td>3</td>
<td>-6</td>
<td>27</td>
<td>-192</td>
<td>1695</td>
</tr>
<tr>
<td>$d$</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$n_d(X)$</td>
<td>-17064</td>
<td>188454</td>
<td>-2228160</td>
<td>27748899</td>
<td>-360012150</td>
</tr>
</tbody>
</table>

**Lemma 3.1.5.** If a sheaf $\mathcal{F}$ on local $\mathbb{P}^2$ is semistable with respect to $L$, $\mathcal{F}$ is supported scheme-theoretically on $\mathbb{P}^2$.

**Proof.** The ideal sheaf of $\mathbb{P}^2$ is $L^3$. We have an exact sequence

$$
\mathcal{F} \otimes L^3 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}|_{\mathbb{P}^2} \longrightarrow 0.
$$

By the $L$-semistability of $\mathcal{F}$, the first map is zero, and hence the map $\mathcal{F} \rightarrow \mathcal{F}|_{\mathbb{P}^2}$ is an isomorphism. 

**Theorem 3.1.6.** Let $X$ be local $\mathbb{P}^2$. Let $n_d(X)$ denote the genus zero BPS invariant in Definition 1.0.1 when $\beta = d[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$. Then,

$$
n_d(X) = (-1)^{d^2+1}e_{\text{top}}(M_{\mathbb{P}^2}(d, 1)).
$$

**Proof.** By Lemma 3.1.5, a stable sheaf on $X$ is necessarily supported on $\mathbb{P}^2$. Hence, the moduli space $M_X(d[\mathbb{P}^1])$ of sheaves on local $\mathbb{P}^2$ in Definition 1.0.1 is isomorphic to $M_{\mathbb{P}^2}(d, 1)$. By Theorem 3.1.4, this moduli space is smooth of dimension $d^2 + 1$. Hence, $n_d(X) = (-1)^{d^2+1}e_{\text{top}}(M_{\mathbb{P}^2}(d, 1))$. 

Table 3.1 shows a prediction in physics [27] of the genus zero BPS invariants for local $\mathbb{P}^2$ for low degrees.

Le Potier [32] has studied the moduli spaces up to degree 3.
**Theorem 3.1.7** (Le Potier).

1. \( \text{M}_{\mathbb{P}^2}(1, 1) \cong \mathbb{P}^2 \).

2. \( \text{M}_{\mathbb{P}^2}(2, 1) \cong \mathbb{P}^5 \).

3. \( \text{M}_{\mathbb{P}^2}(3, 1) \) is the universal cubic curve in \( \mathbb{P}^2 \), which is isomorphic to a \( \mathbb{P}^8 \)-bundle over \( \mathbb{P}^2 \).

By Theorem 3.1.4 and the duality result Proposition 4.2.7, this determines the moduli spaces \( \text{M}_{\mathbb{P}^2}(d, \chi) \) for \( 1 \leq d \leq 3 \) when \( d \) and \( \chi \) are coprime.

We can easily compute the Poincaré polynomials.

**Definition 3.1.8.** For a variety \( X \), let

\[
P(X) = \sum_{j \geq 0} \dim_{\mathbb{Q}} H_j(X, \mathbb{Q}) q^{j/2}
\]

be the Poincaré polynomial.

Definition 3.1.8 defines a polynomial only if all the odd cohomologies vanish. However, it can be shown that all odd cohomologies of the moduli spaces of stable sheaves on \( \mathbb{P}^2 \) vanish. (cf. [12]) So, all Poincaré polynomials in this thesis are actual polynomials.

**Corollary 3.1.9.** We have

1. \( P(\text{M}_{\mathbb{P}^2}(1, 1)) = 1 + q + q^2 \).

2. \( P(\text{M}_{\mathbb{P}^2}(2, 1)) = 1 + q + q^2 + q^3 + q^4 + q^5 \).

3. \( P(\text{M}_{\mathbb{P}^2}(3, 1)) = (1 + q + q^2)(1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8) \).
By substituting \( q^{1/2} = -1 \), we get the topological Euler characteristics of the moduli spaces. The results are consistent with Table 3.1 for \( d = 1, 2, \) and \( 3 \). The aim of this chapter is to compute the Poincaré polynomial of \( \text{M}_{\mathbb{P}^2}(4, 1) \). We will prove the following theorem.

**Theorem 3.1.10.** Under the natural action of the torus \( T = (\mathbb{C}^*)^2 \), the fixed point locus of \( \text{M}_{\mathbb{P}^2}(4, 1) \) consists of 180 isolated points and 6 one-dimensional components isomorphic to \( \mathbb{P}^1 \). Furthermore, the Poincaré polynomial of \( \text{M}_{\mathbb{P}^2}(4, 1) \) is

\[
P(\text{M}_{\mathbb{P}^2}(4, 1)) = 1 + 2q + 6q^2 + 10q^3 + 14q^4 + 15q^5 + 16q^6 + 16q^7 + 16q^8 + 16q^9 + 16q^{10} + 16q^{11} + 15q^{12} + 14q^{13} + 10q^{14} + 6q^{15} + 2q^{16} + q^{17}.
\]

(3.1)

This shows that the Euler characteristic of \( \text{M}_{\mathbb{P}^2}(4, 1) \) is 192, which agrees with Table 3.1. We note that Sahin [43] has computed the Euler characteristic using a different method. Recently, Huang, Kashani-Poor, and Klemm [17] have computed (3.1) in physics.

### 3.2 Torus Action and Cohomology

In this section, we review the theory of Bialynicki-Birula [3, 4, 5]. Let \( X \) be a smooth projective variety with an action of a torus \( T \). As usual, we denote by \( M \) the group of characters of \( T \), and by \( N \) the group of one-parameter subgroups of \( T \). Any one-parameter subgroup \( \lambda \in N \) determines a \( \mathbb{C}^* \)-action on \( X \) via \((t, x) \mapsto \lambda(t) \cdot x\). Using this \( \mathbb{C}^* \) action, we can decompose \( X \) into plus cells or minus cells as follows.

Let \( X^\lambda_1, \cdots, X^\lambda_r \) denote irreducible components of \( \mathbb{C}^* \)-fixed point locus \( X^\lambda \). We define

\[
X^\lambda_+ = \{ x \in X : \lim_{t \to 0} \lambda(t) \cdot x \in X^\lambda_+ \},
\]

\[
X^\lambda_- = \{ x \in X : \lim_{t \to \infty} \lambda(t) \cdot x \in X^\lambda_- \}.
\]
These are called plus cell and minus cell respectively. As $X$ is a projective variety, we have decompositions of $X$ into plus cells and minus cells, which are called plus and minus decompositions.

The following is a theorem of Białynicki-Birula [3, 5].

**Theorem 3.2.1** ([3, Theorem II.4.2]). Let $X$ be a smooth projective variety with a $\mathbb{C}^*$-action $\lambda$ with fixed points. Then the plus and minus decompositions have the following properties.

1. every component $X^\lambda_i$ of fixed point locus is nonsingular.

2. all the cells $X^\lambda_i^+$ and $X^\lambda_i^-$ are locally closed and we have the natural maps $p^+_i : X^\lambda_i^+ \to X^\lambda_i$ and $p^-_i : X^\lambda_i^- \to X^\lambda_i$.

3. if we decompose

$$T(X)|_{X^\lambda_i} \simeq T^0_i \oplus T^+_i \oplus T^-_i,$$

where $T^0_i$, $T^+_i$, $T^-_i$ are respectively subbundles of $T(X)|_{X^\lambda_i}$ where $\mathbb{C}^*$ acts with zero, positive, negative weights, then we have $\mathbb{C}^*$ isomorphisms

$$T^+_i \simeq X^\lambda_i^+ \text{ and } T^-_i \simeq X^\lambda_i^-$$

which lift the $p^+_i$.

An immediate consequence of this is a formula for the Poincaré polynomial.

**Theorem 3.2.2.** For $x \in X^\lambda_i$, let $p(i) = \dim T^+_{i,x}$, and $m(i) = \dim T^-_{i,x}$. Then,

$$P(X) = \sum_i P(X^\lambda_i)q^{p(i)} = \sum_i P(X^\lambda_i)q^{m(i)}.$$

This theorem is originally proved by Białynicki-Birula [3]. The formula is also valid for a compact Kähler manifold [5].
By the following lemma, we can compute the Poincaré polynomial by classifying $T$-fixed loci and computing $T$-representations of the tangent spaces.

**Lemma 3.2.3.** Suppose $X^T$ is finite set. Then, for general one-parameter subgroup $\lambda \in T$, we have $X^T = X^\lambda$.

*Proof.* [11, Remark 1.7] Let $x_i$ be a point in $X^T$. Then, $T$-action on $X$ induces a linear action on the tangent space $T_{x_i,X}$. By complete reducibility theorem [12, Theorem 2.30], $T_{x_i,X}$ is decomposed into one-dimensional representations, corresponding to characters of $T$. Let $\{\chi_1, \cdots, \chi_r\}$ be the set of all characters occurring in the $T_{x_i,X}$. Then, $X^T = X^\lambda$ if and only if $\chi_j \circ \lambda$ are non-trivial characters of $\mathbb{C}^*$ for all $j = 1, \cdots, r$. The latter property is true for a general one-parameter subgroup.  

**Remark 3.2.4.** In Section 3.3, we will see that the torus fixed locus of the moduli space $M_{\mathbb{P}^2}(4, 1)$ is not a finite set, but it contains positive dimensional components isomorphic to $\mathbb{P}^1$. However, we will show that the $T$-representations of the tangent spaces are unchanged along each connected component of torus fixed locus. Then, the same argument as in the above proof holds and we can pick $\lambda$ so that $X^T = X^\lambda$.

### 3.3 Torus Fixed Locus of Moduli space of Sheaves

As we noted in Chapter 2, a sheaf on $\mathbb{P}^2$ is $T$-fixed if and only if it is $T$-equivariant. In this section, we use a classification of $T$-equivariant sheaves on a toric variety to study the torus fixed locus of the moduli space of sheaves on $\mathbb{P}^2$.

We use the following torus invariant affine open cover of $\mathbb{P}^2$. For $\alpha = 0, 1, 2$, let $U_\alpha$ be the affine subset of $\mathbb{P}^2$

$$U_\alpha = \{(x_0, x_1, x_2) \in \mathbb{P}^2 : x_\alpha \neq 0\}.$$
The torus $T \simeq (\mathbb{C}^*)^2$ acts on $U_\alpha$ by

$$(t_1, t_2) \cdot (x_0, x_1, x_2) = (x_0, t_1^{-1}x_1, t_2^{-1}x_2).$$

Then,

$$U_0 = \text{Spec } \mathbb{C}[x, y]$$
$$U_1 = \text{Spec } \mathbb{C}[x^{-1}, x^{-1}y]$$
$$U_2 = \text{Spec } \mathbb{C}[y^{-1}, xy^{-1}],$$

where the induced $T$-action is given by

$$(t_1, t_2) \cdot (x, y) = (t_1x, t_2y).$$

Let $R_\alpha$ denote the coordinate ring $\Gamma(U_\alpha)$, and $p_\alpha, q_\alpha$ denote the $T$-characters for the generators of $R_\alpha$, in other words,

$$(p_0, q_0) = (t_1, t_2),$$
$$(p_1, q_1) = (t_1^{-1}, t_1^{-1}t_2)$$
$$(p_2, q_2) = (t_2^{-1}, t_1t_2^{-1}).$$

As in Section [2.2] we let $M^\alpha$ be a copy of the character group $M = \text{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^2$ whose elements are expressed in terms of $p_\alpha$ and $q_\alpha$. For $m, m' \in M^\alpha$, we say $m' \geq m$ if every component of $m' - m$ is nonnegative.

As in Section [2.3] we draw $M_i$ so that we can encode the gluing conditions easily as follows.
Let $\mathcal{F}$ be a pure $T$-equivariant sheaf on $\mathbb{P}^2$. We have a decomposition into weight spaces

$$\Gamma(U_\alpha, \mathcal{F}) = \bigoplus_{m \in M^\alpha} \Gamma(U_\alpha, \mathcal{F})_m.$$ 

Denote the weight space $\Gamma(U_\alpha, \mathcal{F})_m$ by $F^{\alpha}(m)$. Since $\mathcal{F}$ is $\mathcal{O}_{\mathbb{P}^2}$-module, each $\Gamma(U_\alpha, \mathcal{F})$ is $M^\alpha$-graded $R^\alpha$-module. We can reformulate the $R^\alpha$-module structure by the following data: linear maps $\chi^{\alpha}_{m,m'}: F^{\alpha}(m) \to F^{\alpha}(m')$ for all $m, m' \in M^i$ with $m' \geq m$ such that

$$\chi^{i}_{m,m} = 1 \quad \text{and} \quad \chi^{i}_{m,m'} = \chi^{i}_{m',m} \circ \chi^{i}_{m,m'}.$$  

(3.2)

The pure one dimensional $T$-equivariant sheaf $\mathcal{F}$ is supported on the union of three torus fixed lines in $\mathbb{P}^2$. They are in one to one correspondence with the collection of weight spaces and linear maps.

In the following theorem, for each $\alpha \in \{1, 2, 3\}$, let $\beta_1, \beta_2 \in \{1, 2, 3\} \setminus \{\alpha\}$ be such that $p_\alpha^{-1}$ is among the $T$-characters of $R_{\beta_1}$, and $q_\alpha^{-1}$ is among the $T$-characters of $R_{\beta_2}$. For example, if $\alpha = 0$, then $\beta_1 = 1$ and $\beta_2 = 2$.

**Theorem 3.3.1** ([30 Chapter 2]). The category of pure one dimensional equivariant sheaf $\mathcal{F}$ on $\mathbb{P}^2$ is equivalent to the category $\mathcal{C}$ that can be described as follows. An object of $\mathcal{C}$ is a
collection of weight spaces and linear maps between weight spaces

\[ \{F^\alpha(m), \chi_{m,m'}^\alpha: m \in M^\alpha, \alpha = 1, 2, 3\}, \]

as described above which satisfies the following conditions.

1. For \( i = 1, 2 \), there are integers \( A_{\beta_1}, A_{\alpha\beta_i}, \) and \( B_{\alpha\beta_i} \) such that \( F^\alpha(m) = 0 \) unless

\[ m \in [A_{\beta_1}, \infty) \times [A_{\alpha\beta_1}, B_{\alpha\beta_1}] \cup [A_{\alpha\beta_2}, B_{\alpha\beta_2}] \times [A_{\beta_2}, \infty). \]

We assume \( A_{\beta_1} \leq A_{\alpha\beta_2} \) and \( A_{\beta_2} \leq A_{\alpha\beta_1} \), and \( A_{\alpha\beta_i} \)'s are maximally chosen, and \( B_{\alpha\beta_i} \)'s are minimally chosen. It is possible that \((A_{\beta_1}, A_{\alpha\beta_1}, B_{\alpha\beta_1}) = (\infty, \infty, \infty)\) or \((A_{\beta_2}, A_{\alpha\beta_2}, B_{\alpha\beta_2}) = (\infty, \infty, \infty)\), which means that the set \([A_{\beta_1}, \infty) \times [A_{\alpha\beta_1}, B_{\alpha\beta_1}] \) or \([A_{\alpha\beta_2}, B_{\alpha\beta_2}] \times [A_{\beta_2}, \infty)\) respectively is an empty set.

2. Assume \((m_1, m_2) \notin [A_{\alpha\beta_2}, B_{\alpha\beta_2}] \times [A_{\alpha\beta_1}, B_{\alpha\beta_1}]\). If \((m_1, m_2) \in [A_{\beta_1}, \infty) \times [A_{\alpha\beta_1}, B_{\alpha\beta_1}]\), then \( \chi_{(m_1, m_2), (m_1 + 1, m_2)}^\alpha \) is injective. Similarly, if \((m_1, m_2) \in [A_{\alpha\beta_2}, B_{\alpha\beta_2}] \times [A_{\beta_2}, \infty)\), then \( \chi_{(m_1, m_2), (m_1, m_2 + 1)}^\alpha \) is injective. Hence, the direct limits \( F_{\alpha\beta_1}(m_2) := \lim_{m_1 \to m_2} F^\alpha(m_1, m_2) \) and \( F_{\alpha\beta_2}(m_1) := \lim_{m_2 \to m_1} F^\alpha(m_1, m_2) \) are well-defined. They are required to be finite dimensional vector spaces.

3. For \( m \in [A_{\alpha\beta_2}, B_{\alpha\beta_2}] \times [A_{\alpha\beta_1}, B_{\alpha\beta_1}] \), the map

\[ \chi_{m, (B_{\alpha\beta_1} + 1)} \oplus \chi_{m, (B_{\alpha\beta_2} + 1, m_2)}^\alpha: F^\alpha(m) \to F^\alpha(m_1, B_{\alpha\beta_1} + 1) \oplus F^\alpha(B_{\alpha\beta_2} + 1, m_2) \]

is injective.

4. For \( m_2 \in [A_{\alpha\beta_1}, B_{\alpha\beta_1}] \),

\[ F_{\alpha\beta_1}(m_2) \simeq F_{\beta_1\alpha}(m_2) \]

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and under this identification,

\[ \chi^\alpha_{\langle \infty, m_2 \rangle, \langle \infty, m_2+1 \rangle} = \chi^\beta_1_{\langle \infty, m_2 \rangle, \langle \infty, m_2+1 \rangle}, \]

where \( \chi^\alpha_{\langle \infty, m_2 \rangle, \langle \infty, m_2+1 \rangle} = \lim_{m_1 \to \infty} \chi^\alpha_{\langle m_1, m_2 \rangle, \langle m_1, m_2+1 \rangle}. \)

An analogous statement holds for \( \beta_2. \)

A morphism

\[ \phi: \{ F^\alpha(m), \chi^\alpha_{m,m'} \} \to \{ G^\alpha(m), \lambda^\alpha_{m,m'} \} \]

in \( C \) is a collection of linear maps \( \phi^\alpha(m): F^\alpha(m) \to G^\alpha(m) \) which commute with \( \chi^\alpha \) and \( \lambda^\alpha \) such that

\[ \phi^{\alpha_1}_1(m_2) = \phi^{\beta_1}_1(m_2) \text{ and } \phi^{\alpha_2}_2(m_1) = \phi^{\beta_2}_2(m_1), \]

with obvious notations.

By Theorem 3.3.1 the dimensions of weight spaces of pure equivariant sheaf must satisfy the following condition.

1. The dimension of weight space at position \( w_1 p_\alpha + w_2 q_\alpha \) is at least the dimensions of weight spaces at positions \( (w_1 - 1) p_\alpha + w_2 q_\alpha \) and \( w_1 p_\alpha + (w_2 - 1) q_\alpha. \)

2. Moreover, if \( (w_1, w_2) \in [A_{\alpha_2}, B_{\alpha_2}] \times [A_{\alpha_1}, B_{\alpha_1}], \) the dimension of weight space at the position \( w_1 p_\alpha + w_2 q_\alpha \) is at most the sum of dimensions of weight spaces at positions \( (B_{\alpha_2} + 1) p_\alpha + w_2 q_\alpha \) and \( w_1 p_\alpha + (B_{\alpha_1} + 1) q_\alpha. \)

We will refer this condition as condition \((*)\).

As in Chapter 2 we illustrate an equivariant sheaf by putting boxes on \( M^\alpha \) labeled by
the dimension of the corresponding weight spaces. As shown in Section 2.3, by identifying the asymptotic weight spaces, we can consider a sheaf as a collection of strips.

We continue to use our convention that a box in $M^\alpha$ corresponds to the lattice point of its corner towards the origin of $M^\alpha$.

**Convention 3.3.2.** A box in $M^0$, $M^1$, $M^2$ respectively corresponds to the lattice point of its lower left corner, lower right corner, and upper left corner respectively.

The torus fixed stable sheaves of degree 1, 2, and 3 are described in [31, Section 2.4]. In what follows, we will describe stable $T$-equivariant sheaves with Hilbert polynomial $4n + 1$.

### 3.3.1 Case 1: Sheaves Supported on One Line

If the sheaf is supported on one line, the problem is the same as the problem on local $\mathbb{P}^1$ with $k = 1$ studied in Chapter 2. By the discussion in Section 2.5.4 and (2.18), we have $N_4(1) = 7$ equivariant sheaves supported on one line. Since there are three $T$-invariant lines, the contribution from sheaves of this type is 21. Examples of $T$-equivariant sheaves supported on one line is depicted as follows.

In (a), since the sheaf is stable, its quotient has Euler characteristic greater than 1. Recall that $C_4$ denotes fourth order thickening of $\mathbb{P}^1$ in its normal direction. By Section 2.5.11, the Euler characteristic of strips in the sheaf $\mathcal{O}_{C_4}$ are $1, 0, -1, -2$. Thus, one more box must be
added to the third row of the sheaf $\mathcal{O}_{C_4}$ and this forces another box on the fourth row by condition (*). There are two ways to add these two boxes, either as shown in the picture or on the opposite side. Since the Euler characteristic of each strip is now 1,0,0, and -1, we have one more box to be added. The boxes with diagonal lines show three possible ways to add the last box. Therefore, we got six $T$-equivariant sheaves of these type.

The sheaves of type (1,1,2) as in (b) is also possible. Since the asymptotic weight space of the third row is two dimensional, we need to specify the images of one-dimensional weight spaces at each end. By stability, their images must be linearly independent, hence we may assume they are (1,0) and (0,1). It is easy to see this sheaf is stable and is of Euler characteristic 1.

The sheaves of other types can be easily seen decomposable. Hence, these are all stable sheaves supported on one irreducible line.

### 3.3.2 Case 2 : Sheaves Supported on the Union of Two Lines

First we consider the case where all asymptotic weight spaces are one-dimensional. The scheme theoretic support of the sheaf is either a union of a triple line and another line or a union of two double lines. We analyze the former case as shown in the following picture.
As before, since the sheaf is stable, the third row must contain one more box than the structure sheaf. There are three possible ways (c), (d), and (e) as shown in the above picture.

The Euler characteristics of (c) and (d) are $-1$, and that of (e) is zero. So we need to add two more boxes to (c) and (d) and one more box to (e). The possible places for the additional boxes are shown as labeled boxes with diagonal lines. Adding a box to an existing one-dimensional weight space means the increase of its dimension to two. The resulting weight space configurations must satisfy the condition ($\ast$) and the stability condition. Here
are the lists of all possible ways to add two more boxes for each case:

\[
\begin{align*}
(c): & \quad \{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{4, 5\}, \\
(d): & \quad \{1, 2\}, \{1, 3\}, \{3, 5\}, \{2, 4\}, \{3, 4\}, \\
(e): & \quad \{1\}, \{2\}.
\end{align*}
\]

For example, in (d), \{1, 4\} or \{2, 3\} is not allowed by stability, because the subsheaf generated by them has Euler characteristic 1.

Therefore, we have \(6 \times (5 + 5 + 2) = 72\) equivariant sheaves of this type.

Next, we consider the case where the scheme theoretic support is a union of two double lines.

If we remove one box from the structure sheaf as in (g), the stability forces boxes on the other sides. By condition (\(\ast\)), the sheaf in (g) is the only possible one. In (f), we need to add three more boxes to the structure sheaf. As before, possible places are shown by labeled boxes with diagonal lines. There are 12 possible ways to add these three boxes without
violating the stability as follows:

\[
(f): \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,4,5\}, \{1,4,6\}, \{4,5,6\}, \\
\{1,3,7\}, \{1,4,7\}, \{4,6,7\}, \{1,7,9\}, \{3,7,8\}, \{7,8,9\}.
\]

Similarly as in the previous case, examples of adding three boxes on the same line such as \{1,2,7\} and \{1,7,8\} are not allowed by stability. So, there are \(3 \times (12 + 1) = 39\) equivariant sheaves of this type.

There are two more equivariant sheaves which have two-dimensional asymptotic weight spaces as shown below, which leads to the contribution \(6 \times 2 = 12\). We can check no other weight space configurations are possible.

In conclusion, there are 123 torus fixed sheaves supported on the union of two lines.

### 3.3.3 Case 3 : Sheaves Supported on the Union of Three Lines

In case all the asymptotic weight spaces are one-dimensional, the support of the sheaf is a union of two line and one double line, which we denote by \(C\). The sheaf in this case is obtained by either adding three boxes to \(\mathcal{O}_C\) or removing one box from \(\mathcal{O}_C(1)\). We start with the former case.
As before, the possible places for three added boxes are shown by boxes with diagonal lines. By the stability, three boxes added cannot be on the same line, otherwise the subsheaf generated by these three boxes has Euler characteristic 1. By counting, we can check there are 10 sheaves of this type:

\begin{align*}
(j): \{1, 2, 3\}, \{1, 2, 7\}, \{1, 3, 4\}, \{1, 4, 7\}, \\
\{4, 5, 6\}, \{1, 4, 5\}, \{1, 6, 7\}, \\
\{7, 8, 9\}, \{1, 7, 8\}, \{4, 7, 9\},
\end{align*}

Thus, total contribution is $3 \times 10 = 30$.

Now, we consider the latter case of removing one box from $\mathcal{O}_C(1)$.
In all sheaves described so far, by changing bases of the weight spaces and using Theorem 3.3.1 we can assume the $\chi^\alpha$ maps are either identities, or projections, or inclusions, depending on the dimensions of the weight spaces involved. However, we cannot do so for the sheaf shown in (k) above. If we start fixing bases from the weight spaces in lower right corner of the diagram so that the $\chi^\alpha$ maps are all identities, we have a contradiction at the box in upper left corner. So, this weight space configuration will determine a one-dimensional torus fixed locus. In Example 3.3.5 we will show that this one-dimensional locus is isomorphic to $\mathbb{P}^1$. Hence, there are 6 one-dimensional components isomorphic to $\mathbb{P}^1$ in the torus fixed locus.

It is clear that the diagram (l) defines a stable sheaf and there are 3 of them.

The final example is where we have a two-dimensional asymptotic weight space. If the kernels of two $\chi^\alpha$ maps shown by arrows are distinct, the sheaf is not decomposable, and hence, stable. There are 3 equivariant sheaves of this kind.

In conclusion, we have the first part of Theorem 3.1.10.

**Theorem 3.3.3.** The $(\mathbb{C}^*)^2$-fixed point locus of $\mathcal{M}_{\mathbb{P}^2}(4,1)$ consists of 180 isolated points and 6 one-dimensional components isomorphic to $\mathbb{P}^1$’s.

**Corollary 3.3.4.** The topological Euler characteristic of $\mathcal{M}_{\mathbb{P}^2}(4,1)$ is 192.
Example 3.3.5 (Positive dimensional fixed locus).

Let \( L_1, L_2, L_3 \) be three torus fixed lines in \( \mathbb{P}^2 \). Let \( C = 2L_1 \cup L_2 \).

By the same technique as in Section 2.3, if we read the diagram of one-dimensional fixed locus above, we get the stable sheaf defined by the following short exact sequence.

\[
0 \longrightarrow \mathcal{F} \longrightarrow I_{p,C}(1) \oplus \mathcal{O}_{L_3} \overset{\begin{pmatrix} -a \cdot \text{rest}_p & b \cdot \text{rest}_p \\ -c \cdot \text{rest}_r & d \cdot \text{rest}_r \end{pmatrix}}{\longrightarrow} \mathcal{O}_p \oplus \mathcal{O}_r \longrightarrow 0
\]

where \( a, b, c, d \) are any complex numbers and \( \text{rest}_p \) and \( \text{rest}_r \) are the restriction maps to the corresponding points. Then, by stability, \( a \) and \( c \) cannot be zero. Indeed, if we suppose \( a \) is zero, \( \mathcal{F} \) is the kernel of the map \( I_{p,C}(1) \oplus I_{p,L_3} \rightarrow \mathcal{O}_r \). In particular, \( I_{(p,r),C}(1) \) is a subsheaf of \( \mathcal{F} \). Similarly, if \( c \) is zero, \( I_{2p,C}(1) \) is a subsheaf of \( \mathcal{F} \). These subsheaves have Hilbert polynomials \( 3n + 1 \), destabilizing \( \mathcal{F} \).

Now, by using an automorphism of \( \mathcal{O}_p \oplus \mathcal{O}_r \), we may assume \( a = c = 1 \) up to isomorphism. Denote by \( \mathcal{F}(b,d) \) the sheaf corresponding to \( (b,d) \in \mathbb{C}^2 \). To classify such stable sheaves, note that \( I_{(2p,r),C}(1) \simeq \mathcal{O}_C \) is a subsheaf of \( \mathcal{F}(b,d) \). Since the quotient is \( \mathcal{O}_{L_3} \), \( \mathcal{F}(b,d) \) fits in the short exact sequence

\[
0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F}(b,d) \rightarrow \mathcal{O}_{L_3} \rightarrow 0.
\]

This exact sequence splits if and only if \( \mathcal{O}_{L_3} \) is a subsheaf of \( \mathcal{F}(b,d) \), which is if and only if \( (b,d) = (0,0) \). In other words, \( \mathcal{F}(b,d) \) is stable if and only if \( (b,d) \) is in \( \mathbb{C}^2 - \{(0,0)\} \).
By the automorphism of $O_{L_3}$ given by a multiplication of $k \in \mathbb{C}^*$, we see that $\mathcal{F}(b, d) \simeq \mathcal{F}(kb, kd)$. So, $\{[\mathcal{F}(b, d)]\}$ forms a $T$-fixed locus isomorphic to $\mathbb{P}^1$.

### 3.4 $T$-representation of Tangent Space

The tangent space of the moduli space of semistable sheaves at a point corresponding to a sheaf $\mathcal{F}$ is given by $\text{Ext}^1(\mathcal{F}, \mathcal{F})$.

Consider
\[
\chi(\mathcal{F}, \mathcal{F}) = \sum_{i=0}^{2} (-1)^i \text{Ext}^i(\mathcal{F}, \mathcal{F}).
\]

For a stable sheaf $\mathcal{F}$ in $M_{\mathbb{P}^2}(d, 1)$, we have $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$, and the $T$-action on $\text{Hom}(\mathcal{F}, \mathcal{F}) \simeq \mathbb{C}$ is trivial. Hence, in the representation ring of the torus $T$, we have
\[
\text{Ext}^1(\mathcal{F}, \mathcal{F}) = 1 - \chi(\mathcal{F}, \mathcal{F}).
\]

So, it is enough to compute the representation of $\chi(\mathcal{F}, \mathcal{F})$. We use the technique of \[37\] to compute it. Using the local-to-global spectral sequence, we have
\[
\chi(\mathcal{F}, \mathcal{F}) = \sum_{i,j=0}^{2} (-1)^{i+j} H^i(\text{Ext}^j(\mathcal{F}, \mathcal{F})).
\]

For $\alpha = 0, 1, 2$, let $U_\alpha$ be the affine open subset of $\mathbb{P}^2$ defined in Section 3.3. Let $U_{\alpha \beta}$ be the intersection of $U_\alpha$ and $U_\beta$. We replace the cohomology with the Čech complex $\mathfrak{C}^i(\text{Ext}^j(\mathcal{F}, \mathcal{F}))$ with respect to open cover $\{U_\alpha\}$. A $T$-fixed sheaf $\mathcal{F}$ is necessarily supported on $T$-invariant lines. Since no $T$-invariant line intersects with the intersection of three open
sets, we only have to consider $C^0$ and $C^1$ terms. So,

$$\chi(\mathcal{F}, \mathcal{F}) = \bigoplus_{\alpha=0}^{2} \sum_j (-1)^j \Gamma(U_\alpha, \mathcal{E}xt^j(\mathcal{F}, \mathcal{F}))$$

$$- \bigoplus_{\alpha, \beta} \sum_j (-1)^j \Gamma(U_{\alpha\beta}, \mathcal{E}xt^j(\mathcal{F}, \mathcal{F})).$$

Let $Q_\alpha$ be the $T$-character of $\Gamma(U_\alpha, \mathcal{F})$. Define

$$Q_\alpha(t_1, t_2) = Q_\alpha(t_1^{-1}, t_2^{-1}).$$

Recall that $R_\alpha$ is the coordinate ring $\Gamma(U_\alpha)$ and $p_\alpha, q_\alpha$, are $T$-characters for the generators of $R_\alpha$.

Consider a $T$-equivariant free resolution of $F_\alpha = \Gamma(U_\alpha, \mathcal{F})$.

$$0 \to F_s \to \cdots \to F_2 \to F_1 \to F_\alpha \to 0. \quad (3.3)$$

Each term in (3.3) is of the form

$$F_i = \bigoplus_j R_\alpha(d_{ij}), \quad d_{ij} \in \mathbb{Z}^2.$$

Let

$$P_\alpha(t_1, t_2) = \sum_{i,j} (-1)^i t^{d_{ij}}.$$

Then, from the exact sequence (3.3),

$$Q_\alpha(t_1, t_2) = \frac{P_\alpha(t_1, t_2)}{(1 - p_\alpha)(1 - q_\alpha)}.$$
The representation \( \chi(F_\alpha, F_\alpha) \) is given by the alternating sum

\[
\chi(F_\alpha, F_\alpha) = \sum_{i,j,k,l} (-1)^{i+k} \text{Hom}(R_\alpha(d_{ij}), R_\alpha(d_{kl})) \\
= \sum_{i,j,k,l} (-1)^{i+k} R_\alpha(d_{kl} - d_{ij}) \\
= \frac{P_\alpha(t_1, t_2) P_\alpha(t_1^{-1}, t_2^{-1})}{(1 - p_\alpha)(1 - q_\alpha)} \\
= Q_\alpha Q_\alpha (1 - p_\alpha^{-1})(1 - q_\alpha^{-1})
\]

Similarly for intersection \( U_{\alpha\beta} \), let \( R_{\alpha\beta} \) denote the coordinate ring \( \Gamma(U_{\alpha\beta}) \). Let \( p_{\alpha\beta}, q_{\alpha\beta} \) be \( T \)-characters for the generators of \( R_\alpha \) where \( p_{\alpha\beta}^{-1} \) is in \( R_{\alpha\beta} \). For example, since \( R_0 = \mathbb{C}[x, y] \) and \( R_{01} = \mathbb{C}[x, x^{-1}, y] \), we take \( (p_{01}, q_{01}) = (t_1, t_2) \). Similarly, \( (p_{12}, q_{12}) = (t_1^{-1} t_2, t_2^{-1}) \), etc.

Then, the \( T \)-character of \( F_{\alpha\beta} = \Gamma(U_{\alpha\beta}, \mathcal{F}) \) has an overall factor

\[
\delta(p_{\alpha\beta}) = \sum_{n=-\infty}^{\infty} p_{\alpha\beta}^n = \frac{1}{1 - p_{\alpha\beta}} + \frac{p_{\alpha\beta}^{-1}}{1 - p_{\alpha\beta}^{-1}}.
\]

Let \( Q_{\alpha\beta} \) be such that the \( T \)-character of \( F_{\alpha\beta} \) is

\[
\delta(p_{\alpha\beta}) Q_{\alpha\beta}.
\]

By the same computation as before, we get

\[
\chi(F_{\alpha\beta}, F_{\alpha\beta}) = \delta(p_{\alpha\beta}) Q_{\alpha\beta} Q_{\alpha\beta} (1 - q_{\alpha\beta}^{-1}).
\]

Therefore, we get the following.
Proposition 3.4.1.

\[ \chi(\mathcal{F}, \mathcal{F}) = \sum_{\alpha=0}^{2} Q_{\alpha} Q_{\alpha}(1 - p_{\alpha}^{-1})(1 - q_{\alpha}^{-1}) - \sum_{\alpha, \beta} \delta(p_{\alpha \beta}) Q_{\alpha \beta} Q_{\alpha \beta}(1 - q_{\alpha \beta}^{-1}). \]

Although each term in the summation is infinite dimensional, the total sum is necessarily finite.

Our proof of Theorem 3.1.10 is a case by case computation using the classification in Section 3.3 and Proposition 3.4.1. In Example 3.4.2, we carry out the computation for the last example in Section 3.3. The computation for the other sheaf is similar and equally complicated.

By Lemma 3.2.3 and Remark 3.2.4, once the \( T \)-representation is computed, we can take generic one-parameter subgroup of \( T \) to compute \( p(i) = \dim T_{i,x}^{+} \) in Theorem 3.2.2. For this, we will use one-parameter subgroup \( \lambda: \mathbb{C}^{\ast} \to T \) defined by

\[ \lambda(t) = (t, t^{l}), \]  

for a sufficiently large \( l \in \mathbb{Z} \). Hence, the number \( p(i) \) is given by the number of terms in the \( T \)-representation of the form \( t_{1}^{a} t_{2}^{b} \) with

\[ b > 0 \text{ or } (a > 0 \text{ and } b = 0). \]

Example 3.4.2. As an example, we compute \( T \)-representation of the tangent space at the sheaf of the last example (m) in Section 3.3. We include the diagram again in Figure 3.1.
Figure 3.1: The example (m) in Section 3.3

From the diagram, we can see that

\[ Q_0 = 2(1 + t_1 + t_1^2 + \cdots) + (t_2 + t_2^2 + \cdots) = \frac{2}{1 - t_1} + \frac{t_2}{1 - t_2}, \]
\[ Q_1 = 2(1 + t_1^{-1} + t_1^{-2} + \cdots) + (t_1^{-1}t_2 + t_1^{-2}t_2^2 + \cdots) = \frac{2}{1 - t_1^{-1}} + \frac{t_1^{-1}t_2}{1 - t_1^{-1}t_2}, \]
\[ Q_2 = (1 + t_2^{-1} + t_2^{-2} + \cdots) + (t_1^{-1}t_2^{-1} + t_1^{-1}t_2^{-2} + \cdots) = \frac{1}{1 - t_2^{-1}} + \frac{t_1^{-1}t_2^{-1}}{1 - t_1^{-1}t_2^{-1}}, \]
\[ Q_{01} = 2, \]
\[ Q_{12} = 1, \]
\[ Q_{20} = 1. \]

By applying Proposition 3.4.1, we get the T-character

\[ \text{Ext}^1(\mathcal{F}, \mathcal{F}) = t_1^{-1}t_2^2 + t_2 + t_1^{-1}t_2 + t_1 + t_1^{-1} + 2t_1^2t_2^{-1} + 4t_1t_2^{-1} + 4t_2^{-1} + 2t_1^{-1}t_2^{-1}. \]

Hence, with respect to the one-parameter subgroup (3.4), the dimension of plus cell attached to this fixed point is 4. The computation for other fixed point are similar. By Theorem 3.2.2, we get Theorem 3.1.10.
Chapter 4
Stable Pair Wall Crossing

In this chapter, we propose an algorithm to compute the topological Euler characteristics of the moduli space $M_{\mathbb{P}^2}(d,\chi)$ by means of Joyce’s wall crossing method. Wall crossing occurs on the moduli spaces of $\alpha$-stable pairs as the stability parameter $\alpha$ varies. When $\alpha$ is sufficiently large, the $\alpha$-stable pairs are precisely the stable pairs of Pandharipande and Thomas [40]. When $\alpha$ is sufficiently small but positive, the $\alpha$-stable pairs are the stable pairs studied by Joyce and Song [24]. On $\mathbb{P}^2$, we can relate the moduli space of the latter stable pairs with the moduli space $M_{\mathbb{P}^2}(d,\chi)$ of stable sheaves. The Euler characteristics of the moduli spaces of stable pairs of Pandharipande and Thomas are computed via torus localization [39].

4.1 Moduli Space of Stable Pairs on $\mathbb{P}^2$

Let $X$ be a smooth projective scheme over $\mathbb{C}$. In this section, we introduce the moduli space of $\alpha$-semistable pairs on $X$. The $\alpha$-semistable pairs are examples of the $\alpha$-semistable coherent systems studied by Le Potier [33].

**Definition 4.1.1.** A coherent system on $X$ is a pair $(\Gamma,F)$, where $F$ is a coherent sheaf on $X$ and $\Gamma$ is a finite dimensional subspace of $H^0(F)$. A morphism of coherent systems $\varphi: (\Gamma,F) \rightarrow (\Gamma',F')$ is a morphism of sheaves $\varphi: F \rightarrow F'$ such that $H^0(\varphi)(\Gamma) \subseteq \Gamma'$.

A coherent system with $\dim \Gamma = 1$ is simply called a pair.
Definition 4.1.2. A pair on $X$ is a pair $(\mathcal{F}, s)$ where $\mathcal{F}$ is a coherent sheaf on $X$ and $s \in H^0(\mathcal{F})$ is a nonzero section. A morphism of pairs $\varphi : (\mathcal{F}', s') \to (\mathcal{F}, s)$ is a morphism $\varphi : \mathcal{F}' \to \mathcal{F}$ of coherent sheaves such that $\lambda s = H^0(\varphi)s'$ for some $\lambda \in \mathbb{C}$.

Fix an ample line bundle $\mathcal{O}_X(1)$ on $X$. Let $P_{\mathcal{F}}(n)$ be the Hilbert polynomial of $\mathcal{F}$

$$P_{\mathcal{F}}(n) = \chi(\mathcal{F}(n)).$$

In this chapter, we only consider one dimensional sheaves, whose Hilbert polynomials are linear. We define the multiplicity $r(\mathcal{F})$ of $\mathcal{F}$ by the linear coefficient of $P_{\mathcal{F}}(n)$. Let $\alpha$ be a positive rational number. For a pair $F = (\mathcal{F}, s)$, the reduced Hilbert polynomial $p^\alpha_F$ relative to $\alpha$ is defined by

$$p^\alpha_F(n) = \frac{P_{\mathcal{F}}(n) + \alpha r(\mathcal{F})}{r(\mathcal{F})}.$$

We define the semistability of a pair with respect to $\alpha$.

Definition 4.1.3. A pair $F = (\mathcal{F}, s)$ is called $\alpha$-semistable if

1. $\mathcal{F}$ is a pure sheaf

2. For all proper nonzero subsheaf $\mathcal{F}'$ of $\mathcal{F}$, we have

$$\frac{\chi(\mathcal{F}') + \epsilon(s, \mathcal{F}')\alpha}{r(\mathcal{F}')} \leq \frac{\chi(\mathcal{F}) + \alpha}{r(\mathcal{F})},$$

(4.1)

where $\epsilon(s, \mathcal{F}') = 1$ if $s$ factors through $\mathcal{F}'$ and zero otherwise.

If strict inequality in the second condition holds, $F$ is called $\alpha$-stable.

We define the $\alpha$-slope $\mu_\alpha(F)$ by

$$\mu_\alpha(F) = \frac{\chi(\mathcal{F}) + \alpha}{r(\mathcal{F})}.$$
When $\alpha$ is zero, the $\alpha$-slope is equal to the usual slope of the sheaf $\mathcal{F}$, which we will denote by $\mu(\mathcal{F})$.

The moduli functor of $\alpha$-semistable pairs

$$\mathcal{M}_X^\alpha(P) : \text{Sch} \to \text{Sets}$$

is defined to be the contravariant functor from a category of schemes to a category of sets that takes a scheme $S$ to the set of isomorphism class of $S$-families of $\alpha$-semistable pairs $(\mathcal{F}, s)$ with $P_F = P$.

Since a pair is a special case of a coherent system, as in the case of sheaves, each $\alpha$-semistable pair has a finite Jordan-Hölder filtration.

**Definition 4.1.4.** Let $F$ be a $\alpha$-semistable pair with the reduced Hilbert polynomial $p^\alpha$. A Jordan-Hölder filtration of $F$ is a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = F,$$

such that each factor $gr_i(F) = F_i/F_{i-1}$ is a stable sheaf or a $\alpha$-stable pair with reduced Hilbert polynomial $p^\alpha$. Let $gr(F) := \oplus gr_i(F)$ be the associated grading of $F$. We say two pairs $F_1$ and $F_2$ are $S$-equivalent if there is an isomorphism $gr(F_1) \simeq gr(F_2)$.

**Proposition 4.1.5** (Le Potier [33]). Suppose $X$ is a smooth projective scheme. For the functor $\mathcal{M}_X^\alpha(P)$, there exists a coarse moduli space $\mathcal{M}_X^\alpha(P)$, which is a projective algebraic scheme whose closed points are $S$-equivalence classes of $\alpha$-semistable pairs $(\mathcal{F}, s)$ such that $P_F = P$.

For noncompact $X$, the moduli space $\mathcal{M}_X^\alpha(P)$ still exists, but it needs not be proper. However, we can still calculate its Euler characteristic [24, Section 6.7].
When $\alpha$ is sufficiently small, $\alpha$-semistable pairs are semistable pairs of Joyce and Song [24].

**Lemma 4.1.6.** When $\alpha$ is sufficiently small, $\alpha$-semistability and $\alpha$-stability coincide. A pair $(\mathcal{F}, s)$ is $\alpha$-stable if and only if

1. $\mathcal{F}$ is a pure sheaf,
2. For all proper nonzero subsheaf $\mathcal{G}$ of $\mathcal{F}$, we have $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$,
3. If $s$ factors through a subsheaf $\mathcal{G}$, then $\mu(\mathcal{G}) < \mu(\mathcal{F})$.

In this case, we denote $\alpha = 0^+$. 

**Proof.** For a sufficiently small $\alpha$, the inequality (4.1) is always strictly satisfied when it holds. Hence, $\alpha$-semistability and $\alpha$-stability coincide.

Let $r$ be the multiplicity of $\mathcal{F}$. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$. By $\alpha$-stability condition, we have

$$\frac{\mathcal{G}}{r(\mathcal{G})} \leq \frac{\chi(\mathcal{F}) + \alpha}{r}.$$ 

Since $\alpha$ is sufficiently small and the coefficients of Hilbert polynomial are rational numbers, this implies the inequality $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$.

If $s$ factors through a subsheaf $\mathcal{G}$, we have

$$\mu(\mathcal{G}) + \frac{\alpha}{r(\mathcal{G})} \leq \mu(\mathcal{F}) + \frac{\alpha}{r}.$$ 

Since $r(\mathcal{G}) < r$, we have $\mu(\mathcal{G}) < \mu(\mathcal{F})$.

The converse is obvious. \(\square\)

Therefore, if a pair $F = (\mathcal{F}, s)$ is $0^+$-semistable, then $\mathcal{F}$ is Gieseker semistable.
Remark 4.1.7. In [24], Joyce and Song assume the vanishing of higher cohomologies of the sheaf \( \mathcal{F} \) by twisting it with a sufficiently large tensor power of an ample line bundle. This property is essential in proving the existence of a symmetric obstruction theory [24, Chapter 12]. We do not assume this vanishing property here, because our main focus is on computing the topological Euler characteristic of the moduli space of stable sheaves. Therefore, we do not have a symmetric obstruction theory and our invariant defined below by taking Euler characteristic is not the virtual invariant of Joyce and Song.

When \( \alpha \) is sufficiently large, \( \alpha \)-semistable pairs are semistable pairs of Pandharipande and Thomas [40].

Lemma 4.1.8. Suppose \( \alpha \) is sufficiently large. Then \( \alpha \)-semistability and \( \alpha \)-stability coincide. A pair \((\mathcal{F}, s)\) is \( \alpha \)-stable if and only if

1. \( \mathcal{F} \) is a pure sheaf
2. \( s: \mathcal{O}_X \to \mathcal{F} \) has a zero dimensional cokernel.

In this case, we denote \( \alpha = \infty \).

Proof. See [40, Lemma 1.3].

4.2 Euler Characteristics of \( \mathcal{M}_{\mathbb{P}^2}(d, \chi) \)

From now on, let \( X \) be the projective plane \( \mathbb{P}^2 \). Denote by \( \mathcal{M}_{\mathbb{P}^2}^0(d, \chi) \) the moduli space of \( \alpha \)-semistable pairs \((\mathcal{F}, s)\) on \( \mathbb{P}^2 \) with \( P_X(n) = dn + \chi \). By Lemma 4.1.6 there exist a forgetting morphism (cf. He [16])

\[ \zeta: \mathcal{M}_{\mathbb{P}^2}^0(d, \chi) \to \mathcal{M}_{\mathbb{P}^2}(d, \chi). \]
Let \( \mathbb{P}^2(d, \chi)_k \) be a subscheme of \( \mathbb{P}^2(d, \chi) \) defined by

\[
\mathbb{P}^2(d, \chi)_k = \{ \mathcal{F} \in \mathbb{P}^2(d, \chi) : h^0(\mathcal{F}) = k \}.
\]

Assume \( d \) and \( \chi \) are coprime. Then, \( \mathbb{P}^2(d, \chi) \) is a smooth projective variety and admits a universal family \( \mathcal{U} \). Then, by the semicontinuity theorem, we have immediately

**Lemma 4.2.1.** \( \{ \mathbb{P}^2(d, \chi)_k \}_{k \geq 0} \) is a finite locally closed stratification of \( \mathbb{P}^2(d, \chi) \).

Denote by \( \mathbb{P}^2(d, \chi)_k \) the preimage of \( \mathbb{P}^2(d, \chi)_k \) under the projection \( \zeta \). The following is an analogue of [28, Lemma 5.113].

**Lemma 4.2.2.** Suppose \( d \) and \( \chi \) are coprime. Then, the restriction \( \mathbb{P}^2(d, \chi)_k \to \mathbb{P}^2(d, \chi)_k \) of \( \zeta \) is a Zariski locally trivial fibration with fiber \( \mathbb{P}^{k-1} \). So, we have

\[
\epsilon_{\text{top}}(\mathbb{P}^2(d, \chi)) = \sum_{k \geq 0} k \cdot \epsilon_{\text{top}}(\mathbb{P}^2(d, \chi)_k).
\]

**Proof.** It is enough to prove the first statement. Note that if \( d \) and \( \chi \) are coprime, all semistable sheaves in \( \mathbb{P}^2(d, \chi) \) are stable, and hence any nonzero global section gives a \( 0^+ \)-stable pair by Lemma 4.1.6. The map \( \zeta \) can be explicitly constructed as a projective bundle.

Let \( \mathcal{U} \) be a universal family on \( \mathbb{P}^2 \times \mathbb{P}^2(d, \chi) \) and let \( p : \mathbb{P}^2 \times \mathbb{P}^2(d, \chi) \to \mathbb{P}^2(d, \chi) \) be the projection. For points \( [\mathcal{F}] \in \mathbb{P}^2(d, \chi)_k \), the groups \( H^0(\mathcal{F}) \) and \( H^1(\mathcal{F}) \) has constant dimensions \( k \) and \( k - \chi \) respectively. Since \( Rp_* \mathcal{U} \) is supported in cohomological degree 0 and 1, the restrictions of \( R^1p_* \mathcal{U} \) to \( \mathbb{P}^2(d, \chi)_k \) are locally free of rank \( k - \chi \). By cohomology and base change [15, Corollary III.12.11], \( p_* \mathcal{U} \) is locally free of rank \( k \) on \( \mathbb{P}^2(d, \chi)_k \). Then, \( \mathbb{P}^2(d, \chi)_k \) is isomorphic to the projective bundle \( \mathbb{P}(p_* \mathcal{U})^* \), whose fiber is \( \mathbb{P}^{k-1} \).

**Remark 4.2.3.** The condition that \( d \) and \( \chi \) are coprime is essential in the previous lemma.

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In particular, not every section of a strictly semistable sheaf defines a $0^+\!$-stable pair. For example, consider the sheaf $\mathcal{O}_L \oplus \mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$. A pair $s: \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_L \oplus \mathcal{O}_L$ may be regarded as a pair $(s_1, s_2)$ of sections in $H^0(\mathcal{O}_L)$. Such a pair is $0^+\!$-stable only if $s_1$ and $s_2$ are linearly independent, which is not possible because $H^0(\mathcal{O}_L)$ is one dimensional. See [24, Example 6.1] for more detail.

**Lemma 4.2.4.** Let $\mathcal{F}$ be a semistable sheaf on $\mathbb{P}^2$ with Hilbert polynomial $dn + \chi$. Then $H^1(\mathcal{F}) = 0$ if $\chi \geq \frac{(d-1)(d-2)}{2}$.

**Proof.** We know that $\mathcal{F}$ is supported on some degree $e$ Cohen-Macaulay curve $C$ in $\mathbb{P}^2$ where $1 \leq e \leq d$. By adjunction formula, we have $\omega_C \simeq \mathcal{O}_C(e-3)$. By applying Serre duality on $C$, we have $H^1(\mathcal{F})^* \simeq \text{Hom}(\mathcal{F}, \mathcal{O}_C(e-3))$. Suppose there is a nonzero map: $\mathcal{F} \to \mathcal{O}_C(e-3)$. Then, by semistability of $\mathcal{F}$ and $\mathcal{O}_C$, we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{O}_C(e-3)).$$  \hspace{1cm} (4.3)

Since $\chi(\mathcal{O}_C(e-3)) = \frac{e(3-e)}{2} + e(e-3) = \frac{e(e-3)}{2}$, we have $\mu(\mathcal{O}_C(e-3)) = \frac{e-3}{2}$.

Then by (4.3),

$$\frac{\chi}{d} \leq \frac{e-3}{2} \leq \frac{d-3}{2}.$$  

Therefore, if $\chi \geq \frac{d(d-3)}{2} + 1 = \frac{(d-1)(d-2)}{2}$, then $H^1(\mathcal{F}) = 0$.

**Corollary 4.2.5.** Suppose $d$ and $\chi$ are coprime. If $\chi \geq \frac{(d-1)(d-2)}{2}$, then

$$e_{\text{top}}(M_{\mathbb{P}^2}(d, \chi)) = (1/\chi)e_{\text{top}}(M_{\mathbb{P}^2}^0(d, \chi)).$$

**Proof.** By Lemma 4.2.4, we have $M_{\mathbb{P}^2}(d, \chi) = M_{\mathbb{P}^2}(d, \chi)$. So, it is straightforward from (4.2).
Hence, if $d$ and $\chi$ are coprime, by using isomorphisms in Theorem 3.1.4, the Euler characteristics of $\mathcal{M}_{\mathbb{P}^2}(d, \chi)$ is immediately calculated from the Euler characteristics of $\mathcal{M}_{\mathbb{P}^2}^0(d, \chi)$.

However, in practice, the latter is hard to be computed for large $\chi$, as the wall crossing gets complicated. In what follows, we prove another formula for the Euler characteristics of $\mathcal{M}_{\mathbb{P}^2}(d, 1)$, which is the case we are interested in.

**Definition 4.2.6.** Let $\mathcal{F}$ be a coherent sheaf on codimension $c$ on a smooth projective variety $X$. Then the dual sheaf is defined as $\mathcal{F}^D = \mathcal{E}xt_c^X(\mathcal{F}, \omega_X)$.

**Proposition 4.2.7.** The association $\mathcal{F} \mapsto \mathcal{F}^D$ gives an isomorphism between the moduli spaces $\mathcal{M}_{\mathbb{P}^2}(d, \chi)$ and $\mathcal{M}_{\mathbb{P}^2}(d, -\chi)$.

**Proof.** See [36, Theorem 13]. □

**Proposition 4.2.8.** Let $\mathcal{F}$ be a pure coherent sheaf on $\mathbb{P}^2$ with Hilbert polynomial $dn + \chi$. Then

$$h^0(\mathcal{F}^D) = h^0(\mathcal{F}) - \chi.$$

**Proof.** Since $\mathcal{F}$ is a pure sheaf with one-dimensional support, we have $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}^2}) = 0$ unless $i = 1$ [18, Proposition 1.1.6]. Hence, the local-to-global spectral sequence

$$E_2^{pq} = H^p(\mathbb{P}^2, \mathcal{E}xt^q(\mathcal{F}, \omega_{\mathbb{P}^2}))$$

degenerate at level two. So,

$$H^0(\mathcal{F}^D) = H^0(\mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}^2})) \simeq \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}^2}) \simeq \mathcal{E}xt^1(\mathcal{F}, \omega_X) \simeq H^1(\mathcal{F})^*,$$

by Serre duality. Therefore, $\chi = h^0(\mathcal{F}) - h^1(\mathcal{F}) = h^0(\mathcal{F}) - h^0(\mathcal{F}^D)$. □
Hence, by the isomorphism $M_{\mathbb{P}^2}(d, \chi)$ and $M_{\mathbb{P}^2}(d, -\chi)$ in Proposition 4.2.7, we have

$$M_{\mathbb{P}^2}(d, \chi)_k \simeq M_{\mathbb{P}^2}(d, -\chi)_{k-\chi}. \quad (4.4)$$

Combining (4.2) and (4.4), we get the following result.

**Proposition 4.2.9.**

$$e_{\top}(M_{\mathbb{P}^2}(d, 1)) = e_{\top}(M_{\mathbb{P}^2}^{0+}(d, 1)) - e_{\top}(M_{\mathbb{P}^2}^{0+}(d, -1)). \quad (4.5)$$

**Proof.**

$$e_{\top}(M_{\mathbb{P}^2}^{0+}(d, 1)) - e_{\top}(M_{\mathbb{P}^2}^{0+}(d, -1))) = \sum_{k \geq 0} k \cdot e_{\top}(M_{\mathbb{P}^2}(d, 1)_k) - \sum_{k \geq 0} k \cdot e_{\top}(M_{\mathbb{P}^2}(d, -1)_k)$$

$$= \sum_{k \geq 0} k \cdot e_{\top}(M_{\mathbb{P}^2}(d, 1)_k) - \sum_{k \geq 0} k \cdot e_{\top}(M_{\mathbb{P}^2}(d, 1)_{k+1})$$

$$= \sum_{k \geq 0} e_{\top}(M_{\mathbb{P}^2}(d, 1)_k)$$

$$= \chi(M_{\mathbb{P}^2}(d, 1)).$$

We compute $e_{\top}(M_{\mathbb{P}^2}^{0+}(d, \chi))$ via wall crossing method of Joyce. To use his machinery, the base scheme needs to be a Calabi-Yau threefold. So, from now on, we assume $X$ is local $\mathbb{P}^2$, the total space of the bundle $O_{\mathbb{P}^2}(-3)$ on $\mathbb{P}^2$. By Lemma 3.1.5 and Lemma 4.1.6, for $0^+$-stable pair $F = (\mathcal{F}, s)$, $\mathcal{F}$ is necessarily supported on $\mathbb{P}^2$. Therefore, if $d$ and $\chi$ are coprime, the moduli spaces $M_{\mathbb{P}^2}^{0+}(d, \chi)$ and $M_{\mathbb{P}^2}(d, \chi)$ do not change after we replace $\mathbb{P}^2$ by local $\mathbb{P}^2$, and hence Corollary 4.2.5 and Proposition 4.2.9 hold.
4.3 Review of Joyce’s Wall Crossing Formula

In this section, we review the general theory of Joyce on wall crossing [19, 20, 21, 23, 24]. The wall crossing occurs for stability conditions on an abelian category. Most of the materials in this section are taken from [47].

4.3.1 Preliminaries

Definition 4.3.1. Let $\mathcal{A}$ be an abelian category. The Grothendieck group $K(\mathcal{A})$ is an abelian group generated by isomorphism classes $[E]$ of objects $E$ in $\mathcal{A}$, with the relation $[F] = [E] + [G]$ for each short exact sequence $0 \to E \to F \to G \to 0$ in $\mathcal{A}$. We assume that $\text{Ext}^i(E, F)$ is a finite dimensional vector space for any $E, F \in \mathcal{A}$ and integer $i$. Define the Euler form by

$$\chi([E], [F]) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(E, F).$$  (4.6)

The numerical Grothendieck group $N(\mathcal{A})$ is the quotient $K(\mathcal{A})/I$, where

$$I = \{[[E] \in K(\mathcal{A}) : \chi([E], [F]) = \chi([F], [E]) = 0 \text{ for all } [F] \in K(\mathcal{A})\}.$$

Then $\chi$ descends to a biadditive form $N(\mathcal{A}) \times N(\mathcal{A}) \to \mathbb{Z}$. Define the positive cone $C(\mathcal{A})$ in $N(\mathcal{A})$ to be $C(\mathcal{A}) = \{[[E] \in N(\mathcal{A}) : 0 \not\cong E \in \mathcal{A}\}.

We define a stability condition on an abelian category $\mathcal{A}$. For a notational convenience, we will simply write $E$ for an isomorphism class $[E]$ in $N(\mathcal{A})$ or $C(\mathcal{A})$.

Definition 4.3.2. Let $(T, \geq)$ be a totally ordered set. Let $\mu : C(\mathcal{A}) \to T$ be a map. We call $\mu$ a stability function if for $E, F, G \in C(\mathcal{A})$ with $F = E + G$, we have either $\mu(E) < \mu(F) < \mu(G)$, or $\mu(E) > \mu(F) > \mu(G)$, or $\mu(E) = \mu(F) = \mu(G)$. We call $\mu$ a weak stability function if for $E, F, G \in C(\mathcal{A})$ with $F = E + G$, we have either $\mu(E) \leq \mu(F) \leq \mu(G)$, or
\[ \mu(E) \geq \mu(F) \geq \mu(G). \]

**Definition 4.3.3.** Let \( \mu : C(\mathcal{A}) \to T \) be a weak stability function. A nonzero object \( E \) in \( \mathcal{A} \) is called \( \mu \)-stable (resp. \( \mu \)-semistable) if for any proper nonzero subobject \( F \subset E \), we have

\[ \mu(F) < \mu(E/F). \quad (\text{resp. } \mu(F) \leq \mu(E/F)). \]

**Definition 4.3.4.** The Harder-Narasimhan filtration of \( E \in C(\mathcal{A}) \) is a filtration

\[ 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E, \]

such that each subquotient \( F_i = E_i / E_{i-1} \) is \( \mu \)-semistable and \( \mu(F_1) > \mu(F_2) > \cdots \mu(F_n) \).

If all objects in \( \mathcal{A} \) admit a Harder-Narasimhan filtration, the (weak) stability function is called the (weak) stability condition.

Examples of stability conditions include the Gieseker stability and \( \mu \)-stability condition on the category of coherent sheaves on a variety.

**Definition 4.3.5.** An abelian category \( \mathcal{A} \) is called noetherian if for any object \( E \in \mathcal{A} \), all ascending chains of subobjects \( E_1 \subset E_2 \subset \cdots \subset E \) stabilize, artinian if all descending chains of subobjects \( E \supset E_1 \supset E_2 \supset \cdots \) stabilize. We call \( \mathcal{A} \) \( \mu \)-artinian if there is no infinite chains of subobjects \( E \supset E_1 \supset E_2 \supset \cdots \) with \( E_{i+1} \neq E_i \) and \( \mu(E_{i+1}) \geq \mu(E_i / E_{i+1}) \) for all \( i \).

To apply Joyce’s theory, we need the following assumption.

**Assumption 4.3.6.**

1. \( \mathcal{A} \) is noetherian and \( \mu \)-artinian.

2. Objects in \( \mathcal{A} \) is parametrized by an Artin stack \( \mathcal{M}_\mathcal{A} \) of locally finite type.

3. For \( v \in C(\mathcal{A}) \), the substack \( \mathcal{M}_v^\text{ss}(\mu) \) of \( \mu \)-semistable objects is an open substack of \( \mathcal{M}_\mathcal{A} \), and it is of finite type.
4.3.2 Ringel-Hall Algebra and Definition of Invariants

We introduce a Ringel-Hall algebra $H(A)$ associated to an abelian category $A$. First, we introduce the notion of the Grothendieck ring of Artin stacks.

**Definition 4.3.7 ([22]).** For an Artin $C$-stack $\mathcal{F}$ and a 1-morphism $x: \text{Spec} C \to \mathcal{F}$, a stabilizer group $\text{Iso}_\mathcal{F}(x)$ is the group of 2-morphism $x \to x$. We say that an Artin $C$-stack $\mathcal{F}$ has affine geometric stabilizers if $\text{Iso}_\mathcal{F}(x)$ is an affine algebraic group over $C$ for all 1-morphisms $x: \text{Spec} C \to \mathcal{F}$.

Fix an Artin stack $\mathcal{F}$ of the locally finite type over $C$. Consider pairs $(\mathcal{R}, \rho)$ where $\mathcal{R}$ is an Artin stack of finite type with affine geometric stabilizers, and $\rho: \mathcal{R} \to \mathcal{F}$ is a 1-morphism. Two pairs $(\mathcal{R}, \rho)$ and $(\mathcal{R}', \rho')$ are equivalent if there exist 1-isomorphism $\iota: \mathcal{R} \to \mathcal{R}'$ such that $\rho' \circ \iota$ and $\rho$ are 2-isomorphic. Write $[(\mathcal{R}, \rho)]$ for the equivalence class of $(\mathcal{R}, \rho)$. We define $H(A)$ to be the $\mathbb{Q}$-vector space

$$K(\text{St}/\mathcal{F}) := \bigoplus \mathbb{Q}[(\mathcal{R}, \rho)]/\sim,$$

where the relations $\sim$ are given by

$$[(\mathcal{R}, \rho)] \sim [(\mathcal{G}, \rho|_{\mathcal{G}})] + [(\mathcal{R}\backslash \mathcal{G}, \rho|_{\mathcal{R}\backslash \mathcal{G}})],$$

for closed substacks $\mathcal{G}$.

The multiplication $\cdot$ on $K(\text{St}/\mathcal{F})$ is given by the fiber product

$$[(\mathcal{R}, \rho)] \cdot [(\mathcal{R}', \rho')] = [(\mathcal{R} \times_{\mathcal{F}} \mathcal{R}', \rho \circ \pi_{\mathcal{R}})].$$

**Remark 4.3.8.** There are obvious notions of the pullback $\phi^*: K(\text{St}/\mathcal{G}) \to K(\text{St}/\mathcal{F})$ and the pushforward $\phi_*: K(\text{St}/\mathcal{F}) \to K(\text{St}/\mathcal{G})$ for a 1-morphism $\phi: \mathcal{F} \to \mathcal{G}$, and tensor product.
\(\otimes: K(\text{St}/\mathcal{F}) \times K(\text{St}/\mathcal{G}) \to K(\text{St}/\mathcal{F} \times \mathcal{G})\). We omit the detail here and refer to [24, Section 2.2]. The ring \(K(\text{St}/\mathcal{F})\) is denoted by \(\mathbb{S}(\mathcal{F})\) in [24], and its elements are called the stack functions.

**Definition 4.3.9.** Let \(\mathcal{M}_A\) be the locally finite type Artin stack parametrizing objects in the abelian category \(A\). We define \(\mathcal{H}(A)\) by

\[
\mathcal{H}(A) := K(\text{St}/\mathcal{M}_A).
\]

There is an associative multiplication \(*\) on \(\mathcal{H}(A)\). For this, we let \(\mathfrak{E}\text{xt}_A\) be the moduli stack of short exact sequences \(0 \to E_1 \to E_2 \to E_3 \to 0\) in \(A\). It is an Artin \(\mathbb{C}\)-stack of locally finite type [24, Section 3.2]. Let \(\pi_i: \mathfrak{E}\text{xt}_A \to \mathcal{M}_A\) be 1-morphism of Artin stacks projecting \(0 \to E_1 \to E_2 \to E_3 \to 0\) to \(E_i\). Take \(f = [(\mathcal{R}, \rho)]\) and \(f' = [(\mathcal{R}', \rho')]\). We have the following diagram.

\[
\begin{array}{ccc}
\mathfrak{E}\text{xt}_A & \xrightarrow{u} & \mathcal{M}_A \\
\downarrow & & \downarrow \pi_2 \\
\mathcal{R} \times \mathcal{R}' & \xrightarrow{\rho \times \rho'} & \mathcal{M}_A \times \mathcal{M}_A
\end{array}
\]

**Definition 4.3.10.** We define the \(*\) product by

\[
f * f' = [(\pi_1 \times \pi_3)^*(\mathcal{R} \times \mathcal{R}'), \pi_2 \circ u].
\]

If we use the notion of pushforwards, pullbacks, tensor products (Remark 4.3.8), we can write it as \(f * f' = (\pi_2)_*((\pi_1 \times \pi_3)^*(f \otimes f'))\).

It is shown in [20, Theorem 5.2] that \(*\) is associative. Thus, \(\mathcal{H}(A)\) is a \(\mathbb{Q}\)-algebra with identity \([0 \to \mathcal{M}_A]\), where 0 is the zero object in \(\mathcal{M}_A\). We call \(\mathcal{H}(A)\) the Ringel-Hall algebra of \(A\). We may view \(\mathcal{H}(A)\) as a Lie algebra with Lie bracket \([f, g] = f * g - g * f\).
Definition 4.3.11. Let $\mu: C(A) \to T$ be a weak stability condition satisfying Assumption 4.3.6. We define

$$\delta^v(\mu) = [\mathcal{M}_{\mu}^v(\mu) \hookrightarrow \mathcal{M}_A] \in \mathcal{H}(A),$$

$$\epsilon^v(\mu) = \sum_{n \geq 1, v_1, \ldots, v_n \in C(A)} \frac{(-1)^{n-1}}{n} \delta^{v_1}(\mu) \ast \delta^{v_2}(\mu) \ast \cdots \ast \delta^{v_n}(\mu) \in \mathcal{H}(A).$$

It is shown in [21, Proposition 4.9] that the sum (4.8) has only finitely many nonzero terms. From Definition 4.3.11, it is clear that when stability and semistability coincide, we have $\epsilon^v(\mu) = \delta^v(\mu)$. We will define counting invariants by “integrating” the elements $\epsilon^v(\mu)$.

Let $K(\text{Var}/\text{Spec} \mathbb{C})$ be the Grothendieck ring of varieties over $\mathbb{C}$, i.e., $K(\text{Var}/\text{Spec} \mathbb{C})$ is the abelian group generated by isomorphism classes of quasiprojective varieties over $\mathbb{C}$ with the relation $[X] = [X\backslash C] + [C]$ for a variety $X$ and its closed subvariety $C$. We consider a ring homomorphism $\Upsilon: K(\text{Var}/\text{Spec} \mathbb{C}) \to \mathbb{Q}(t)$ such that

$$\Upsilon([X]) = \sum_i (-1)^i \dim H^i(X, \mathbb{C}) t^i,$$

for a smooth projective variety $X$. In [22, Section 4.1], it is shown that $\Upsilon$ can be extended to $K(\text{St}/\text{Spec} \mathbb{C})$ so that if $X$ is a quasiprojective variety and $G$ is a special algebraic group, we have $\Upsilon([X/G]) = \Upsilon([X])/\Upsilon([G])$. Here, an algebraic group is called special if every principal $G$-bundle is a locally trivial Zariski fibration.

Consider the composition

$$\Pi: \mathcal{H}(A) \xrightarrow{p_*} K(\text{St}/\text{Spec} \mathbb{C}) \xrightarrow{\Upsilon} \mathbb{Q}(t),$$

where $p: \mathcal{M}_A \to \text{Spec} \mathbb{C}$ is the structure morphism and $p_*$ sends $[(\mathcal{R}, \rho)]$ to $[(\mathcal{R}, p \circ \rho)]$. Joyce
shows that the rational function $\Pi(\epsilon^v(\mu))$ has a pole at $t = 1$ at most order one.

**Remark 4.3.12.** Joyce constructs a certain Lie subalgebra of $\mathcal{H}(\mathcal{A})$ denoted by $\text{SF}_{\text{ind}}^\text{al}(\mathcal{M}_\mathcal{A})$ in his papers [19, 20, 21, 23]. We can think of elements in $\text{SF}_{\text{ind}}^\text{al}(\mathcal{M}_\mathcal{A})$ as stack functions supported on *virtual indecomposables* in $\mathcal{M}_\mathcal{A}$. We refer to [24, Chapter 2] for a summary. It is shown in [21, Theorem 8.7] that $\epsilon^v(\mu)$ lies in $\text{SF}_{\text{ind}}^\text{al}(\mathcal{M}_\mathcal{A})$. In [23, Section 6.2], it is shown that for $\epsilon \in \text{SF}_{\text{ind}}^\text{al}(\mathcal{M}_\mathcal{A})$, the image $\Pi(\epsilon)$ has a pole at $t = 1$ at most order one.

Therefore, we may define an invariant as follows.

**Definition 4.3.13** ([23 Definition 6.7], [47 Definition 2.21]). For $\epsilon \in \text{SF}_{\text{ind}}^\text{al}(\mathcal{M}_\mathcal{A})$, define

$$\Theta(\epsilon) = (t^2 - 1)\Pi(\epsilon)|_{t=1} \in \mathbb{Q}.$$  

We define the invariant $J^v(\mu) \in \mathbb{Q}$ by

$$J^v(\mu) = \Theta(\epsilon^v(\mu)).$$

**Remark 4.3.14.** The factor $(t^2 - 1)$ is from $\Upsilon(\mathbb{C}^*) = (t^2 - 1)$. When semistability agrees with stability, and the moduli stack $\mathcal{M}_{\text{ss}}^v(\mu)$ is the quotient stack $[M_{\text{ss}}^v(\mu)/\mathbb{C}^*]$, where $M_{\text{ss}}^v(\mu)$ is a scheme and $\mathbb{C}^*$ is the stabilizer group of stable objects, then $J^v(\mu)$ is the Euler characteristic of the moduli space $M_{\text{ss}}^v(\mu)$.

**Definition 4.3.15.** Let $X$ be a Calabi-Yau threefold. We define $DT^v_\beta := J^\beta(\mu)$ where $\mathcal{A}$ is the category $\text{Coh}(X)$ of coherent sheaves on $X$, $\beta$ is a class in $C(\text{Coh}(X))$, and $\mu$ is Gieseker stability on $\text{Coh}(X)$.

**Remark 4.3.16.** The invariant $DT^v_\beta$ is the Euler characteristic version of the generalized Donaldson-Thomas invariants defined in [24]. In [24], the invariant $DT^v_\beta$ is denoted by
$J^\beta(\tau)$, where $\tau$ denote Gieseker stability of coherent sheaves. By Remark 4.3.14 if there is no strictly semistable sheaves in class $\beta$, $DT^u_\beta$ is equal to the Euler characteristic of the moduli space of sheaves in class $\beta$.

Assume that $\chi: N(A) \times N(A) \to \mathbb{Z}$ is anti-symmetric, we define the Lie algebra $L(A)$ to be the $\mathbb{Q}$-vector space

$$L(A) = \bigoplus_{v \in N(A)} \mathbb{Q}\lambda^v,$$

with Lie bracket $[\lambda^v, \lambda^w] = \chi(v, w)\lambda^{v+w}$.

We define a map $\Phi: SF_{\text{ind}}(\mathfrak{M}_A) \to L(A)$ as follows. An element $\epsilon$ in the Lie algebra $SF_{\text{ind}}(\mathfrak{M}_A)$ can be uniquely written as $\epsilon = \sum_{v \in C(A)} c^v \epsilon^v$, for some $c^v \in \mathbb{Q}$ and $\epsilon^v \in K(\text{St}/\mathfrak{M}_A)$, where $\mathfrak{M}_A$ is an Artin stack parametrizing objects in class $v$. We define

$$\Phi(\epsilon) = \sum_{v \in C(A)} c^v \Theta(\epsilon^v)\lambda^v.$$

From Definition 4.3.13 it is clear that

$$\Phi(\epsilon^v(\mu)) = J^v(\mu)\lambda^v. \quad (4.9)$$

Then, we have the following theorem.

**Theorem 4.3.17** ([20, Theorem 6.12], [47, Theorem 2.23]). Under the assumption that $\chi: N(A) \times N(A) \to \mathbb{Z}$ is anti-symmetric, the map $\Phi$ is a Lie algebra homomorphism.

### 4.3.3 Wall Crossing Formula

In this section we describe transformation laws for $\epsilon^v(\mu)$ under change of stability conditions [23]. By applying the Lie algebra homomorphism $\Phi$, we will get a wall crossing formula for $J(\epsilon^v(\mu))$. We follow the notations of [24].
**Definition 4.3.18** ([24 Definition 3.12]). Let \( \mu, \mu' : C(A) \to T \) be weak stability conditions on \( A \). We say \( \mu' \) dominates \( \mu \) if \( \mu(v) \leq \mu(w) \) implies \( \mu'(v) \leq \mu'(w) \) for all \( v, w \in C(A) \).

Let \( v_1, \ldots, v_l \in C(A) \). Suppose that for each \( i = 1, \ldots, l - 1 \), we have either

\[
\mu(v_i) \leq \mu(v_{i+1}) \text{ and } \mu'(v_i + \cdots + v_l) > \mu'(v_{i+1} + \cdots + v_l) \quad \text{ or } \quad (4.10)
\]

\[
\mu(v_i) > \mu(v_{i+1}) \text{ and } \mu'(v_i + \cdots + v_l) \leq \mu'(v_{i+1} + \cdots + v_l). \quad (4.11)
\]

Then, define \( S(v_1, \ldots, v_l; \mu, \mu') = (-1)^r \), where \( r \) is the number of \( i = 1, \ldots, l - 1 \) satisfying (4.10). Otherwise define \( S(v_1, \ldots, v_l; \mu, \mu') = 0 \).

We need another combinatorial coefficient used in wall crossing formulae [24 Definition 3.12]. Define a nondecreasing surjective map \( \psi : \{1, \ldots, l\} \to \{1, \ldots, m\} \) for some \( m \leq l \) such that \( \psi(i) = \psi(j) \) only if \( \mu(v_i) = \mu(v_j) \). This map \( \psi \) defines a combining of indices \( i = 1, \ldots, l \) so that two indices \( i, j \) can be combined together if \( \mu(v_i) = \mu(v_j) \).

Now define another nondecreasing surjective map \( \xi : \{1, \ldots, m\} \to \{1, \ldots, m'\} \) for some \( m' \leq m \) such that \( \mu'\left(\sum_{\xi(k) = i} v_k\right) = \mu'(v_1 + \cdots + v_l) \) for all \( i \in \{1, \ldots, m'\} \). This map \( \xi \) gives another level of combining so that after combination, the value of \( \mu' \) are constant.

Given \( \psi, \xi \) as above, and \( i \in \{1, \ldots, m'\} \), we let \( \xi^{-1}(i) = \{i_1, \ldots, i_k\} \) with \( i_1 \leq \cdots \leq i_k \) and let \( w_{i_t} = \sum_{j \in \psi^{-1}(i_t)} v_j \).

Now we define

\[
U(v_1, \ldots, v_l; \mu, \mu') = \sum_{1 \leq m' \leq m \leq l, \psi, \xi \text{ as above}} \frac{(-1)^{m'-1}}{m'} \cdot \prod_{i=1}^{m'} S(w_{i_1}, \ldots, w_{i_k}; \mu, \mu') \prod_{b=1}^{m} \frac{1}{|\psi^{-1}(b)|!} \quad (4.12)
\]

The following wall crossing formula is derived in [23 Section 5].

**Theorem 4.3.19.** For two weak stability conditions \( \mu \) and \( \mu' \), assume that there are weak stability conditions \( \mu = \mu_0, \mu_1, \ldots, \mu_l = \mu' \) and \( \omega_1, \ldots, \omega_n \) such that \( \omega_i \) dominates \( \mu_{i-1} \) and
\[ \delta^v(\mu') = \sum_{v_1 + \cdots + v_l = v} S(v_1, \cdots, v_l; \mu, \mu') \delta^{v_1}(\mu) * \delta^{v_2}(\mu) * \cdots \delta^{v_l}(\mu) \] (4.13)

\[ e^v(\mu') = \sum_{v_1 + \cdots + v_l = v} U(v_1, \cdots, v_l; \mu, \mu') e^{v_1}(\mu) * e^{v_2}(\mu) * \cdots e^{v_l}(\mu) \] (4.14)

provided that the sums on the right sides have only finitely many nonzero terms.

**Remark 4.3.20.** There is the notion of convergence of an infinite sum, and under a certain finiteness condition on the stability conditions, the equations (4.13) and (4.14) hold without the assumption that the sum has only finitely many nonzero term. However, we omit the detail, as in our case of interest, the sum is finite. For detail, see [23, Section 5].

In [23, Theorem 5.4], it is proven that we can rewrite (4.14) as a \( \mathbb{Q} \)-linear combination of multiple commutators of \( e^v(\mu) \). Hence, the equality (4.14) is in the Lie subalgebra \( \mathfrak{sp}^{\text{ind}}_{\text{all}}(\mathfrak{M}_A) \) of \( \mathcal{H}(A) \). So we may apply the Lie algebra homomorphism \( \Phi \) to (4.14), and we get the following wall crossing formula for \( J^v(\mu) \).

**Theorem 4.3.21** ([23, Theorem 6.28, Equation (130)]). For two weak stability condition \( \mu \) and \( \mu' \) satisfying the conditions in Theorem 4.3.19, we have

\[ J^v(\mu') = \sum_{v_1 + \cdots + v_l = v} \sum_{\text{connected, simply connected digraphs } \Gamma \text{ with vertex } \{1, \cdots, l\}, \bullet^i \to \bullet^j \text{ implies } i < j} \frac{1}{2^{l-1}} U(v_1, \cdots, v_l; \mu, \mu') \prod_{\bullet^i \to \bullet^j \text{ in } \Gamma} \chi(v_i, v_j) \prod_{i=1}^l J^{v_i}(\mu) \] (4.15)

Toda [47] extends this result by relaxing Assumption 4.3.6. We briefly explain his results.
For a stability condition \( \mu \) on an abelian category \( \mathcal{A} \) and \( t \in T \), we let

\[
\mathcal{A}_{\mu \geq t} = \langle E : E \text{ is } \mu\text{-semistable with } \mu(E) \geq t \rangle,
\]
\[
\mathcal{A}_{\mu < t} = \langle E : E \text{ is } \mu\text{-semistable with } \mu(E) < t \rangle.
\]

Here, for a set \( S \) of objects in \( \mathcal{A} \), the notation \( \langle S \rangle \) means the smallest extension closed subcategory of \( \mathcal{A} \). For objects \( E, F \) in \( \mathcal{A}_{\mu < t} \), we call a morphism \( f : E \to F \) in \( \mathcal{A} \) strict if its kernel and cokernel are in the subcategory \( \mathcal{A}_{\mu < t} \).

Fix \( v \in C(\mathcal{A}_{\mu < t}) \), and define

\[
C_{\leq v}(\mathcal{A}_{\mu < t}) = \{ v' \in C(\mathcal{A}_{\mu < t}) : \text{there is } v'' \in C(\mathcal{A}_{\mu < t}) \text{ with } v' + v'' = v \}.
\]

We modify the definition of dominant stability condition in Definition 4.3.18.

**Definition 4.3.22.** Let \( \mu, \mu' : C(\mathcal{A}) \to T \) be weak stability conditions on \( \mathcal{A} \). For \( t \in T \) and \( v \in C(\mathcal{A}_{\mu < t}) \), we say \( \mu' \) dominates \( \mu \) with respect to \( (v, t) \) if \( \mu(v) \leq \mu(w) \) implies \( \mu'(v) \leq \mu'(w) \) for all \( v, w \in C_{\leq v}(\mathcal{A}_{\mu < t}) \), and we have \( \mathcal{A}_{\mu < t} = \mathcal{A}_{\mu' < t} \) and \( \mathcal{A}_{\mu \geq t} = \mathcal{A}_{\mu' \geq t} \).

Toda [47, Theorem 2.28] modifies Assumption 4.3.6 and generalizes Theorem 4.3.21 as follows.

**Assumption 4.3.23.**

1. \( \mathcal{A}_{\mu < t} \) is noetherian and artinian with respect to strict monomorphisms.

2. Objects in \( \mathcal{A} \) is parametrized by an Artin stack \( \mathcal{M}_\mathcal{A} \) of locally finite type.

3. For \( v' \in C_{\leq v}(\mathcal{A}_{\mu < t}) \), the substack of \( \mathcal{M}_{\text{ss}}(\mu) \) of \( \mu \)-semistable objects is an open substack of \( \mathcal{M}_\mathcal{A} \), and it is of finite type.
Theorem 4.3.24. Given \( t \in T \) and \( v \in C(A_{\mu < t}) \), suppose that Assumption 4.3.23 is satisfied and that two weak stability conditions \( \mu \) and \( \mu' \) on \( A \) satisfy the condition in Theorem 4.3.19 with dominance of stability conditions replaced by dominance with respect to \((v, t)\). Then, (4.14) holds with \( v_i \in C(A_{\mu < t}) \). If there are only finitely many terms in (4.14), then (4.15) holds with \( v_i \in C(A_{\mu < t}) \).

4.4 Stable Pairs and Wall Crossing on Local \( \mathbb{P}^2 \)

4.4.1 Stable Pair Invariants

Let \( X \) be local \( \mathbb{P}^2 \) and Coh\((X)\) be the category of coherent sheaves on \( X \). Let \( \mathcal{C} \) be the category of objects of the form \((\mathcal{F}, s)\), where \( \mathcal{F} \) is a sheaf whose reduced support is a curve in \( \mathbb{P}^2 \) and \( s: \mathcal{O}_X \to \mathcal{F} \) is a morphism for some finite integer \( r \). Given objects \((\mathcal{F}, s)\) and \((\mathcal{F}', s')\) in \( \mathcal{C} \), a morphism \((g, f): (\mathcal{F}, s) \to (\mathcal{F}', s')\) in \( \mathcal{C} \) is a pair \((g, f)\), where \( g: \mathcal{O}_X \to \mathcal{O}'_X \) and \( f: \mathcal{F} \to \mathcal{F}' \) are morphisms satisfying \( f \circ s = s' \circ g \).

Remark 4.4.1. Although the category \( \mathcal{C} \) itself is an abelian category, there is no notion of Serre duality in \( \mathcal{C} \). So, the Euler form is not anti-symmetric. Joyce and Song [24, Section 13.1] use a weakened version of Euler form defined by the equation

\[
\chi(A, B) = \dim \text{Hom}(A, B) - \dim \text{Ext}^1(A, B) + \dim \text{Ext}^1(B, A) - \dim \text{Hom}(B, A) \quad (4.16)
\]

for objects \( A \) and \( B \) in \( \mathcal{C} \). This Euler form is enough to prove wall crossing formula [24, Proposition 13.4].

However, if we use the Euler form (4.16), the vanishing of higher cohomology is essential for Joyce’s wall crossing formula to be applied. It is because otherwise, the Euler form above is not given by the Chern characters of \( A \) and \( B \). To see this, let \( A = (0 \to \mathcal{F}) \) and...
\( B = (\mathcal{O}_X \to 0) \) for some one dimensional sheaf \( \mathcal{F} \). Then we can see

\[
\begin{align*}
\text{Hom}(A, B) &= 0, \\
\text{Ext}^1(A, B) &= 0, \\
\text{Hom}(B, A) &= 0, \\
\text{Ext}^1(B, A) &= H^0(\mathcal{F}).
\end{align*}
\]

Note that Hom and Ext groups are taken in the category \( \mathcal{C} \) of pairs. So, \( \chi(A, B) \) is not constant under deformation of \( \mathcal{F} \) unless \( H^1(\mathcal{F}) = 0 \). This example is pointed out to the author by Yukinobu Toda.

To avoid such problem, we work on a subcategory of the derived category \( D(X) := D(\text{Coh}(X)) \).

**Definition 4.4.2.** We define

\[
\begin{align*}
\text{Coh}_{\leq 1}(X) &= \{ E \in \text{Coh}(X) : \dim \text{Supp}(E) \leq 1 \}, \\
\mathcal{A}_X &= \langle \mathcal{O}_X[1], \text{Coh}_{\leq 1}(X) \rangle.
\end{align*}
\]

In [46, Lemma 3.5], it is proved that \( \mathcal{A}_X \) is an abelian category. We consider \( \mathcal{C} \) as a subcategory of \( \mathcal{A}_X \) by associating \( (\mathcal{F}, s) \) with two term complex \( (\mathcal{O}_X^s \to \mathcal{F}) \) with \( \mathcal{F} \) located in degree 0.

Let \( \text{Coh}_{\leq 1}^c(X) \) be the subcategory of \( \text{Coh}_{\leq 1}(X) \) whose objects are sheaves \( E \in \text{Coh}_{\leq 1}(X) \) with \( (\text{Supp}(E))_{\text{red}} \subset \mathbb{P}^2 \). Let \( \text{N}(\text{Coh}_{\leq 1}^c(X)) \) be the image of \( K(\text{Coh}_{\leq 1}^c(X)) \) in \( N(\text{Coh}(X)) \). Then, we may write \( N(\mathcal{C}) = \mathbb{Z} \oplus N(\text{Coh}_{\leq 1}^c(X)) \). For an object \( (\mathcal{O}_X^s \xrightarrow{s} \mathcal{F}) \), write its numerical class in \( N(\mathcal{C}) \) as \((r, [\mathcal{F}])\). Since the reduced support of \( \mathcal{F} \) is on \( \mathbb{P}^2 \), the class \([\mathcal{F}]\) is completely determined by its Hilbert polynomial. For a class \( \beta = [\mathcal{F}] \), we write \( P(\beta) \) for its Hilbert polynomial, and \( \chi(\beta) \) for its Euler characteristic.

Following [24, Definition 13.3], we define the Euler form as follows.
Definition 4.4.3. \( \chi((r_1, \beta_1), (r_2, \beta_2)) = r_2 \chi(\beta_1) - r_1 \chi(\beta_2). \)

This is the natural Euler form on \( N(A) \) induced by the embedding \( C \to A_X \). Since in \( N(A_X) \)

\[ [\mathcal{O}_X] \to [\mathcal{F}] = r[\mathcal{O}_X[1]] + [\mathcal{F}] = -r[\mathcal{O}_X] + [\mathcal{F}], \]

we have

\[ \chi((r_1, \beta_1), (r_2, \beta_2)) = r_1 r_2 \chi([\mathcal{O}_X], [\mathcal{O}_X]) - r_1 \chi([\mathcal{O}_X], \beta_2) - r_2 \chi(\beta_1, [\mathcal{O}_X]) + \chi(\beta_1, \beta_2) \]

\[ = -r_1 \chi(\beta_2) + r_2 \chi(\beta_1). \]

Note that \( \chi([\mathcal{O}_X], [\mathcal{O}_X]) = 0 \) and \( \chi(\beta_1, \beta_2) = 0 \) by Riemann-Roch theorem and the condition that \( \beta_i \)'s are classes of one dimensional sheaves.

Conjecture 4.4.4. Let \( v = (1, \beta) \) with \( P(\beta) = dn + \chi \). Joyce’s wall crossing formula \( \text{[1.15]} \) holds where \( J^{(1, \beta)}(\mu) = e_{\text{top}}(M^{(1)}_{d, \chi} (d, \chi)), J^{(1, \beta)}(\mu') = e_{\text{top}}(M^{(1)}_{d, \chi} (d, \chi)), \) and \( J^{(0, \beta)}(\mu_\alpha) = DT^{\text{eu}}_{dn+\chi} \) (cf. Definition \( \text{[4.3.15]} \)).

We write

\[ PI^{\text{eu}}_{dn+\chi} = e_{\text{top}}(M^{(1)}_{d, \chi} (d, \chi)) \quad \text{and} \quad PT^{\text{eu}}_{dn+\chi} = e_{\text{top}}(M^{(1)}_{d, \chi} (d, \chi)). \] (4.17)

\( PI^{\text{eu}}_{dn+\chi} \) is the Euler characteristic version of \( PI_{dn+\chi} \) of Joyce and Song \( \text{[24]} \) and \( PT^{\text{eu}}_{dn+\chi} \) is the Euler characteristic version of \( PT_{dn+\chi} \) of Pandharipande and Thomas \( \text{[40]} \).

One way to prove Conjecture \( \text{[4.4.4]} \) is to construct an abelian subcategory \( B \) of \( D(X) \) containing \( C \) and a weak stability condition \( \mu_\alpha \) on \( B \) for all positive rational number \( \alpha \) satisfying the following conditions.

1. Given a class \( \delta \in N(A) \) of the form \( (1, \beta) \) or \( (0, \beta) \), there exist a class \( \gamma \in N(B) \) such that for \( \alpha = \infty \) or \( 0^+ \), the moduli stack of \( \mu_\alpha \)-semistable objects of class \( \gamma \) in \( B \) is isomorphic to the moduli stack of \( \alpha \)-semistable pairs of class \( \delta \).
2. For a class $\gamma \in N(B)$ corresponding to a class $(1, \beta) \in N(A)$, $B$ and $\mu_0$ satisfies all assumptions so that Joyce’s wall crossing formula \(^{(4.15)}\) holds for $\mu' = \mu_\infty$ and $\mu = \mu_0^+$. 

As in Section \(^{4.1}\) we write $\alpha = \infty$ (respectively $0^+$) for a sufficiently large (respectively small) $\alpha$.

A natural candidate for $B$ and $\mu_\alpha$ is the category $A_X$ and the stability condition similarly defined as $\alpha$-stability condition of pairs. It can be shown that $A_X$ satisfies condition 2. However, we do not know whether $A_X$ satisfies condition 1, because a general object in $A_X$ can not be expressed as a pair of the form $(\mathcal{O}_X \to \mathcal{F})$.

Suppose that such subcategory $B$ and stability condition $\mu_\alpha$ exist. By abuse of notation, we write $(1, \beta)$ and $(0, \beta)$ for the corresponding classes in $N(B)$. Assume that $P(\beta) = dn + \chi$. Then, by Remark \(^{4.3.14}\) we have (cf. Definition \(^{4.3.13}\))

$$J^{(1, \beta)}(\mu_0^+) = PT^{eu}_{dn+\chi} = e_{top}(M^0_\beta(d, \chi)),$$

$$J^{(1, \beta)}(\mu_\infty) = PT^{eu}_{dn+\chi} = e_{top}(M_\infty^0(d, \chi)).$$

as semistability agrees with stability for these spaces. Since objects of class $(0, \beta)$ are sheaves of class $\beta$ and $\alpha$-stability in this case is Gieseker stability for sheaves, $J^{(0, \beta)}(\mu_\alpha)$ is by definition $DT^{eu}_{dn+\chi}$. Hence the existence of the category $B$ and $\mu_\alpha$ proves Conjecture \(^{4.4.4}\).

We assume Conjecture \(^{4.4.4}\) holds for the rest of this chapter.

\subsection*{4.4.2 Wall Crossing Formula for Local $\mathbb{P}^2$}

In this section, we apply \(^{(4.15)}\) to get a wall crossing formula between $PT^{eu}_\beta$ and $PI^{eu}_\beta$ invariants.
Lemma 4.4.5. If

\[ U((0, \beta_1), \cdots, (0, \beta_k), (1, \beta_0), (0, \beta_{k+1}), \cdots, (0, \beta_l); 0^+, \infty) \]

is nonzero, then we must have

\[ \mu(\beta_1) \geq \cdots \geq \mu(\beta_k) > \mu(\beta_0) \text{ and } \]
\[ \mu(\beta_0) < \mu(\beta_{k+1}) \leq \cdots \leq \mu(\beta_l). \]

In this situation, define surjective maps

\[ \psi_1: \{1, \cdots, k\} \to \{1, \cdots, m_1\} \text{ and } \]
\[ \psi_2: \{k+1, \cdots, l\} \to \{1, \cdots, m_2\} \]

so that

\[ \psi_1(i) = \psi_1(j) \text{ if and only if } \mu((0, \beta_i)) = \mu((0, \beta_j)) \text{ for } i, j \leq k, \]
\[ \psi_2(i) = \psi_2(j) \text{ if and only if } \mu((0, \beta_i)) = \mu((0, \beta_j)) \text{ for } i, j > k. \]

Then,

\[ U((0, \beta_1), \cdots, (0, \beta_k), (1, \beta_0), (0, \beta_{k+1}), \cdots, (0, \beta_l); 0^+, \infty) \]
\[ = (-1)^{l-k} \prod_{b_1=1}^{m_1} \frac{1}{|\psi_1^{-1}(b_1)|!} \prod_{b_2=1}^{m_2} \frac{1}{|\psi_2^{-1}(b_2)|!} \]
Proof. Note that

\[ S((0, \beta_1), \cdots, (0, \beta_k), (1, \beta_0), (0, \beta_{k+1}), \cdots, (0, \beta_l); 0^+, \infty) \]

is \((-1)^{l-k}\) if

\[ \mu_0^+((0, \beta_1)) > \cdots > \mu_0^+((0, \beta_k)) > \mu_0^+((1, \beta_0)) \]

and zero otherwise.

Since \(\mu_0^+((0, \beta_i)) = \mu(\beta_i)\), and \(\mu_0^+((1, \beta_0)) \leq \mu_0^+((0, \beta_{k+1}))\) implies \(\mu(\beta_0) < \mu(\beta_{k+1})\), the condition in the statement must be satisfied.

Then,

\[ U((0, \beta_1), \cdots, (0, \beta_k), (1, \beta_0), (0, \beta_{k+1}), \cdots, (0, \beta_l); 0^+, \infty) = \prod_{b_1=1}^{m_1} \frac{1}{\psi_1^{-1}(b_1)!} \prod_{b_2=1}^{m_2} \left( \sum_{d_1 + \cdots + d_i = |\psi_2^{-1}(b_2)|} \frac{(-1)^i}{d_1! \cdots d_i!} \right) \]

\[ = \prod_{b_1=1}^{m_1} \frac{1}{\psi_1^{-1}(b_1)!} \prod_{b_2=1}^{m_2} \frac{(-1)^{|\psi_2^{-1}(b_2)|}}{|\psi_2^{-1}(b_2)|!} \]

\[ = (-1)^{l-k} \prod_{b_1=1}^{m_1} \frac{1}{\psi_1^{-1}(b_1)!} \prod_{b_2=1}^{m_2} \frac{1}{|\psi_2^{-1}(b_2)|!} \]

The second equality is due to [23, Lemma 13.9].

\[ \square \]

Theorem 4.4.6. Assume that Conjecture 4.4.4 holds. Then,

\[ PT_{\beta^u} = \sum_{\beta_0 + \beta_1 + \cdots + \beta_l = \beta \atop \mu(\beta_i) > \mu(\beta_0) \text{ for all } i} \frac{1}{l!} PI_{\beta_0}^u \prod_{i=1}^{l} (\chi(\beta_i) DT_{\beta_i}^u) \quad (4.18) \]
Proof. We decompose

\[(1, \beta) = (1, \beta_0) + (0, \beta_1) + \cdots + (0, \beta_l)\].

We rewrite this as

\[(1, \beta) = (1, \beta_0) + d_1(0, \beta_1) + \cdots + d_k(0, \beta_k),\]

where \(\beta_i\)'s are distinct.

By definition,

\[\chi((0, \beta_i), (0, \beta_j)) = 0\] and \[\chi((0, \beta_i), (1, \beta_0)) = \chi(\beta_i).\] (4.19)

Thus, by Lemma 4.4.5 and (4.19), a term in the sum (4.15) is non-zero only if the graph \(\Gamma\) is of the following form,

\[
\begin{array}{c}
\bullet (0, \beta_1) \\
\vdots \\
\bullet (0, \beta_k) \\
\end{array}
\begin{array}{c}
\rightarrow (1, \beta_0) \\
\end{array}
\begin{array}{c}
\vdots \\
\bullet (0, \beta_1) \\
\vdots \\
\end{array}
\begin{array}{c}
\rightarrow (0, \beta_k) \\
\end{array}
\]

and \(\mu(\beta_0) < \mu(\beta_i)\) for all \(i\), and the slopes are nonincreasing on the left side and nondecreasing on the right side of the above graph.

Let \(n_i\) be the number of \(\beta_i\)’s appearing on the left side of the above diagram. Then, clearly \(d_i - n_i\) of \(\beta_i\)’s are appearing on the right side. Let \(\sum n_i = n\) and \(\sum d_i = d\).

Define \(\psi: \{1, \cdots, k\} \rightarrow \{1, \cdots, m\}\) so that

\[\psi(i) = \psi(j)\] if and only if \(\mu(\beta_i) = \mu(\beta_j)\) for \(1 \leq i, j \leq k\).
Then,

\[
U((0, \beta_1), \cdots, (0, \beta_k), (1, \beta_0), (0, \beta_1), \cdots, (0, \beta_k); 0^+, \infty)
\]

corresponding to the above directed graph is

\[
(-1)^{d-n} \prod_{b=1}^{m} \frac{1}{(\sum_{\psi(i)=b} n_i)!} \prod_{b=1}^{m} \frac{1}{(\sum_{\psi(i)=b} d_i - n_i)!}
\]

The product of \(\chi(v_i, v_j)\) in (4.15) is

\[
\prod_{i=1}^{l} \left( \prod_{j=1}^{n_i} \chi((0, \beta_i), (1, \beta_0)) \prod_{j=1}^{n_i} \chi((1, \beta_0), (0, \beta_i)) \right) = (-1)^{d-n} \prod_{i=1}^{k} \chi(\beta_i)^{d_i}.
\]

There are \(m \prod_{b=1}^{\psi(i)=b} n_i)!\) and \(m \prod_{b=1}^{\psi(i)=b} d_i - n_i)!\) possible arrangements for vertices on each side respectively. Therefore the wall crossing formula is

\[
PT^{eu}_{\beta} = \sum_{\beta_0 + d_1 \beta_1 + \cdots + d_k \beta_k = \beta} \frac{1}{2d} \sum_{n_i \leq d_i} PI^{eu}_{\beta_0} \prod_{i=1}^{k} \frac{1}{n_i!(d_i - n_i)!} \chi(\beta_i)^{d_i}(DT^{eu}_{\beta_i})^{d_i}
\]

\[
= \sum_{\beta_0 + d_1 \beta_1 + \cdots + d_k \beta_k = \beta} \frac{1}{2d} \sum_{n_i = 0}^{d_i} PI^{eu}_{\beta_0} \prod_{i=1}^{k} \frac{1}{n_i!(d_i - n_i)!} \chi(\beta_i)^{d_i}(DT^{eu}_{\beta_i})^{d_i}
\]

\[
= \sum_{\beta_0 + d_1 \beta_1 + \cdots + d_k \beta_k = \beta} PI^{eu}_{\beta_0} \prod_{i=1}^{k} \left( \frac{1}{d_i!} \chi(\beta_i)^{d_i}(DT^{eu}_{\beta_i})^{d_i} \right), \tag{4.20}
\]

because \(\sum_{n_i=0}^{d_i} \frac{d_i!}{n_i!(d_i - n_i)!} = 2^{d_i}\). The expression (4.20) is equivalent to (4.18).

Therefore, by (4.18), we can recursively compute \(PI^{eu}\) invariants from \(PT^{eu}\) and \(DT^{eu}\) invariants. The \(DT^{eu}\) invariants can also be computed from \(PT^{eu}\) invariant recursively by
Theorem 4.4.7 ([24, Theorem 5.30]). For $\beta \in C(\text{coh}(X))$ with $\chi(\beta) \geq \frac{(d - 1)(d - 2)}{2}$, we have

$$\text{PI}_{\beta}^{cu} = \sum_{\substack{\beta_1 + \cdots + \beta_l = \beta \\ \mu(\beta_i) = \mu(\beta)}} \frac{1}{l!} \prod_{i=1}^{l} \chi(\beta_i) \text{DT}_{\beta_i}^{cu}$$

(4.21)

with only finitely many nonzero terms in the sum.

Proof. In [24, Theorem 5.30], the condition $n \gg 0$ is needed only for the vanishing of higher cohomology $H^1(\mathcal{F}) = 0$ for $\mathcal{F}$ in the class $\beta$. So, by Lemma 4.2.4, the condition $\chi(\beta) \geq \frac{(d - 1)(d - 2)}{2}$ is sufficient. By taking $n = 0$ with this condition in [24, Theorem 5.30], we get (4.21). $\square$

For $\beta$ with small $\chi(\beta)$, the wall crossing formula (4.21) does not hold. However, by the following proposition, we are able to compute $\text{DT}^{cu}$ invariants.

Proposition 4.4.8. $\text{DT}_{dn + \chi}^{cu} = \text{DT}_{dn + (\chi + d)}^{cu}$.

Proof. Tensoring with $\mathcal{O}_{\mathbb{P}^2}(1)$ induces an 1-isomorphism between the moduli stacks of semi-stable sheaves of Hilbert polynomial $dn + \chi$ and $dn + (\chi + d)$. Thus, $\text{DT}_{dn + \chi}^{cu} = \text{DT}_{dn + (\chi + d)}^{cu}$. $\square$

### 4.5 Euler Characteristics of PT Moduli Spaces via Torus Localization

To complete our algorithm for computing the topological Euler characteristics of $M_{\mathbb{P}^2}(d, 1)$, it remains to compute $\text{PT}_{dn + \chi}^{cu}$ in the wall crossing formula (4.21). These invariant is the Euler characteristic of the moduli space of stable pairs by Pandharipande and Thomas. Its torus fixed locus is described in [39]. In the case of local $\mathbb{P}^2$, the torus fixed locus consists
of isolated points. Thus, we get the Euler characteristic by counting them. In this section, we review the classification of torus fixed stable pairs via box configurations studied in [39]. The stable pair in this section refers to the stable pair in the sense of Pandharipande and Thomas [40], which is the $\infty$-stable pair in Section 4.1.

Let $s: \mathcal{O}_X \to \mathcal{F}$ be a torus fixed stable pair. Consider the exact sequence associated to it

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{Q} \to 0.$$

Then, the curve $C$ is a torus invariant curve in $X$ and the zero-dimensional cokernel $\mathcal{Q}$ of $s$ is supported on torus fixed points. Pandharipande and Thomas [40] shows the following.

**Proposition 4.5.1.** Let $m \subset \mathcal{O}_C$ be the ideal of a zero dimensional subscheme of a curve $C$. A stable pair $(\mathcal{F}, s)$ with support $C$ satisfying

$$\text{Support}^{\text{red}}(\mathcal{Q}) \subset \text{Support}(\mathcal{O}_C/m)$$

is equivalent to a subsheaf of $\mathcal{H}om(m^r, \mathcal{O}_C)/\mathcal{O}_C$, for $r \gg 0$.

We have inclusions

$$\mathcal{H}om(m^r, \mathcal{O}_C) \to \mathcal{H}om(m^{r+1}, \mathcal{O}_C)$$

by the purity of $\mathcal{O}_C$. Hence, by Proposition 4.5.1 we may consider a stable pair as a subsheaf of the limit

$$\lim_{r \to \infty} \mathcal{H}om(m^r, \mathcal{O}_C)/\mathcal{O}_C.$$

Then, under the equivalence, the subsheaf $\lim_{r \to \infty} \mathcal{H}om(m^r, \mathcal{O}_C)/\mathcal{O}_C$ corresponds to the cokernel $\mathcal{Q}$.

As $\mathcal{H}om(m^r, \mathcal{O}_C)/\mathcal{O}_C$ is supported on each fixed point, we may restrict our attention to affine torus invariant open sets containing a unique fixed points. For example in local $\mathbb{P}^2$,
we have three fixed points and three affine open sets containing each of these fixed points.

Let $x_1, x_2, x_3$ be the coordinates on an affine torus invariant open set $U$ such that the torus $T \simeq (\mathbb{C}^*)^3$ acts by

$$(t_1, t_2, t_3) \cdot x_i = t_i x_i.$$ 

By Proposition 4.5.1, the pair $(F|_U, s|_U)$ corresponds to a subsheaf of

$$\lim_{r \to \infty} \mathcal{H}om(m^r, \mathcal{O}_C)/\mathcal{O}_C,$$

where $C$ is a $T$-invariant line in $U$ and $m$ is the ideal sheaf of the origin. By purity of $F$, $C$ is a Cohen-Macaulay curve, i.e., a pure dimension one curve with no embedded point. Therefore, $C$ is defined by a monomial ideal $I$ of the polynomial ring $R = \mathbb{C}[x_1, x_2, x_3]$ such that $R/I$ is of pure dimension one. Such a monomial ideal is in correspondence with a pair of three outgoing partitions as follows.

A monomial ideal $I$ in $R$ is associated to a three dimensional partition $\pi$ by considering a union of boxes corresponding to the weights of $R/I$ in the group of characters isomorphic to $\mathbb{Z}^3$. The localizations

$$(I)_{x_i} \subset \mathbb{C}[x_1, x_2, x_3]_{x_i},$$

for $i = 1, 2, 3$ are all $T$-fixed, and hence each corresponds to a two-dimensional partition $\pi^i$. One can think of $\pi^i$ as a cross-section of three dimensional partition $\pi$ by a plane $x_i = c$ for a large integer $c$. We will call $\pi^i$ the outgoing partition of $\pi$. Conversely, given a triple $(\pi^1, \pi^2, \pi^3)$ of outgoing partitions, the monomial ideal $I$ is defined by a unique minimal three dimensional partition with outgoing partition $(\pi^1, \pi^2, \pi^3)$. The minimality assumption is due to the Cohen-Macaulay property of the curve $C$. We denote the curve corresponding to the outgoing partition $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$ by $C_{\vec{\pi}}$. 

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Let $\pi^1[x_2, x_3]$ be the monomial ideal of $\mathbb{C}[x_2, x_3]$ defined by the partition $\pi^1$, and let

$$M_1 = \mathbb{C}[x_1, x_1^{-1}] \otimes \left( \mathbb{C}[x_2, x_3]/\pi^1[x_2, x_3] \right).$$

We define $M_2$ and $M_3$ similarly. Hence, $M_i$ may be viewed in the space of $T$-characters as an infinite cylinder $\text{Cyl}_i \in \mathbb{Z}^3$ along the $x_i$ axis with cross-section $\pi^i$.

Then we have

$$\lim_{r \to \infty} \text{Hom}(m^r, O_C) \simeq \bigoplus_{i=1}^3 M_i.$$

Let $M = \bigoplus_{i=1}^3 M_i$. Then, by the above isomorphism, the submodule $O_C$ is identified with the submodule of $M$ generated by $(1, 1, 1)$. Hence, the $T$-fixed stable pair $(\mathcal{F}|_U, s|_U)$ corresponds to a finite dimensional $T$-invariant submodule of $M/\langle (1, 1, 1) \rangle$. In [39], this submodule is described by box configurations as follows.

There are three types of $T$-weights of $M/\langle (1, 1, 1) \rangle$:

(i) weights which are contained in exactly one cylinder $\text{Cyl}_i$ and have negative $i$-th coordinate. The set of all weights of these type is denoted by $I^-$.  

(ii) weights which are contained in exactly two and three cylinders. The sets of weights of these types are denoted by $\mathbb{I}$ and $\mathbb{II}$ respectively.

Let $\mathbb{C}_w$ be the one dimensional weight space with the weight $w$. Then, we have by definition

$$M/\langle (1, 1, 1) \rangle = \bigoplus_{w \in I^- \cup \mathbb{I}} \mathbb{C}_w \oplus \bigoplus_{w \in \mathbb{II}} (\mathbb{C}_w)^2. \quad (4.22)$$

The straightforward $R$-module structure on $M/\langle (1, 1, 1) \rangle$ is that the multiplication of $x_i$ increases $i$-th coordinate of weight by one. Therefore, the $T$-fixed pair on $U$ can be described by a subspace of (4.22) that respects $R$-module structure.
If \( M/\langle (1,1,1) \rangle \) contains the type III weights, additional data is needed for each type III box to specify a \( T \)-fixed component, and \( T \)-fixed locus may contain positive dimensional components. In case of local \( \mathbb{P}^2 \), however, we do not have type III weights since \( T \)-fixed curves are on \( \mathbb{P}^2 \). So, in what follows, we only consider the case with no type III weights. We refer to [39] for a description of the \( T \)-fixed locus in case there are type III weights.

By the above discussion, if there is no type III weights, the submodule of \( M/\langle (1,1,1) \rangle \) corresponds to box configurations supported on the set of weights in \( \mathcal{I}^- \) and \( \mathcal{I} \) satisfying the following rule.

\( ^\dagger \) For \( w = (w_1, w_2, w_3) \in \mathcal{I}^- \cup \mathcal{I} \), if any of

\[
(w_1 - 1, w_2, w_3), (w_1, w_2 - 1, w_3), (w_1, w_2, w_3 - 1)
\]

support a box then \( w \) must support a box.

The length of the box configuration is defined by the dimension of corresponding submodule of \( M/\langle (1,1,1) \rangle \) as a vector space, which in our case is the same as the number of boxes.

Local \( \mathbb{P}^2 \) has three \( T \)-fixed points, denoted by \( p_0, p_1, \) and \( p_2 \). Let \( L_{ij} \) be the \( T \)-invariant line connecting \( p_i \) and \( p_j \). Then, we get the following conclusion.

**Proposition 4.5.2.** The invariant \( PT_{\text{cu}}^{\text{eu} + \chi} \) is the number of tuples \((B_0, B_1, B_2)\) of three box configurations satisfying the rule \((\dagger)\) such that for some triple \( \vec{\lambda} = (\lambda^{01}, \lambda^{02}, \lambda^{12}) \) of partitions with \( |\lambda^{01}| + |\lambda^{02}| + |\lambda^{12}| = d \), the outgoing partitions of \( B_0, B_1, B_2 \) are \((\lambda^{01}, \lambda^{02}, \emptyset), (\lambda^{12}, \lambda^{01}, \emptyset)\), and \((\lambda^{02}, \lambda^{12}, \emptyset)\) respectively, and the sum of length of \( B_0, B_1, B_2 \) is equal to \( \chi - \chi(\mathcal{O}_{C(\vec{\lambda})}) \), where \( C(\vec{\lambda}) \) is the torus fixed curve on \( \mathbb{P}^2 \) defined by the partition \( \lambda^{ij} \) along \( T \)-invariant line \( L_{ij} \).
Proof. A $T$-fixed stable pair $(\mathcal{F}, s)$ with Hilbert polynomial $dn + \chi$ is supported on a $T$-invariant curve of degree $d$, which is given by $C(\vec{\lambda})$ for some triple $\vec{\lambda}$ of partitions. By the exact sequence

$$0 \to \mathcal{O}_{C(\vec{\lambda})} \to \mathcal{F} \to Q \to 0,$$

we have

$$\chi = \chi(\mathcal{F}) = \chi(Q) + \chi(\mathcal{O}_{C(\vec{\lambda})}).$$

Moreover, $Q$ must be supported on fixed points, and by the above discussion, at each fixed point $Q$ corresponds to a box configuration satisfying the rule (†). Since all $T$-fixed stable pairs are isolated points, by the torus localization Theorem 2.1.4, we can compute the topological Euler characteristic by counting them.

For a triple $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$ of partitions, define the renormalized volume $|\vec{\pi}|$ by

$$|\vec{\pi}| = \# \left\{ \pi \cap [0, \ldots, N]^3 \right\} - (N + 1) \sum_{i=1}^{3} |\pi^i|, \quad N \gg 0,$$

where $\pi$ is the three dimensional partition corresponding to the curve $C_{\vec{\pi}}$.

**Lemma 4.5.3.** In Proposition 4.5.2,

$$\chi(\mathcal{O}_{C(\vec{\lambda})}) = \sum_{\lambda \in \{\lambda^{01}, \lambda^{02}, \lambda^{12}\}} \left( \sum_j \frac{\lambda_j(3 - \lambda_j)}{2} + 3j\lambda_j \right)$$

$$+ |(\lambda^{01}, \lambda^{02}, \emptyset)| + |(\lambda^{12}, \lambda^{01}, \emptyset)| + |(\lambda^{02}, \lambda^{12}, \emptyset)|,$$

where $\lambda_j$ is the $j$-th part of the partition $\lambda$.

**Proof.** Each torus fixed line $L_{ij}$ is isomorphic to $\mathbb{P}^1$, and has normal bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. This follows from an elementary computation by applying [37] Lemma 4].

\[109\]
Therefore, we can compute $PT_{d^n + \chi}^{eu}$ by counting. The results for $d \leq 10$ and $\chi \leq 5$ are shown in Table 4.1.

**Remark 4.5.4.** The invariant shown in Table 4.1 is the Euler characteristics of the moduli space of stable pairs. In [39], by calculating the virtual tangent space, the contribution of each fixed point to the virtual invariant (the Pandharipande-Thomas invariant) is computed. The contribution is either $+1$ or $-1$, depending only on the length of $Q$ and the outgoing partitions. When applying their result, we can see that the contribution in our case is $(-1)^{d + \chi + 1}$. Hence, by the sign change in Table 4.1 we get the Pandharipande-Thomas invariants for local $\mathbb{P}^2$. 
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Table 4.1: $PT_{d+n+\chi}^{eu}$ for local $\mathbb{P}^2$
4.6 Results

By the wall crossing formulae (4.18) and (4.21), $PI_{dn+\chi}$ is recursively determined by $PT^{cu}$ invariants of lower degrees. The results for $d \leq 10$ and $\chi \leq 5$ are shown in Table 4.2.

By (4.5), we get the topological Euler characteristics of $M_{P^2}(d,1)$, and hence, the BPS invariants $n_d(X)$ by Theorem 3.1.6. The results agree with the prediction in physics given in Table 3.1.

Corollary 4.6.1. Assume that Conjecture 4.4.4 holds. Then, Conjecture 1.0.2 is true for local $\mathbb{P}^2$ up to degree 10.

Note that when $d$ is small, Table 4.2 shows that the results using Corollary 4.2.5 are also valid. This gives a strong evidence for Conjecture 4.4.4 which is an assertion that we may apply Joyce’s wall crossing formula to stable pairs on local $\mathbb{P}^2$.

In the next example, we recalculate the Poincaré polynomial of $M_{P^2}(4,1)$ in Theorem 3.1.10 via the wall crossing method.

Example 4.6.2 (Poincaré polynomial for $d = 4$). We computed the Poincaré polynomial of $M_{P^2}(4,1)$ in Chapter 3. We can also compute it by wall crossing method described in this chapter. This example is taught to the author by Kiryong Chung (private communication).

The Poincaré polynomial version of (4.2) is

$$P(M_{P^2}^{0+}(d,\chi)) = \sum_{k \geq 0} P(\mathbb{P}^{k-1}) P(M_{P^2}(d,\chi)_k). \tag{4.23}$$

When $d = 4$ and $\chi = 1$, we have the following.

Lemma 4.6.3 (43). Let $\mathcal{F}$ be a stable sheaf of Hilbert polynomial $4n + 1$. Then $h^0(\mathcal{F})$ is either 1 or 2. Moreover, if $h^0(\mathcal{F}) = 2$, $\mathcal{F} \simeq I_{p,C}(1)$ for a quartic curve $C$ and a point $p \in C$.  

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Table 4.2: $P_{n+\chi}^{eu}$ for local $\mathbb{P}^2$
Proof. Let \( C \) be the support of \( \mathcal{F} \). Then \( C \) is a quartic curve in \( \mathbb{P}^2 \). By the Serre duality on \( C \), we have an isomorphism \( \text{Hom}(\mathcal{F}, \mathcal{O}_C(1)) \simeq H^1(\mathcal{F})^* \). Hence, if \( h^0(\mathcal{F}) \geq 2 \), then we have a nonzero map \( \mathcal{F} \to \mathcal{O}_C(1) \). By stability, this map must be injective. The cokernel must be of length 1. Hence, \( \mathcal{F} \) is isomorphic to \( I_{p,C}(1) \) for some point \( p \) on \( C \), in which case, \( h^0(\mathcal{F}) = 2 \). \( \square \)

Hence, by (4.23),

\[
P(M_{\mathbb{P}^2}(4, 1)) = P(M_{\mathbb{P}^2}(4, 1)_1) + P(\mathbb{P}^1)P(M_{\mathbb{P}^2}(4, 1)_2)
= P(M_{\mathbb{P}^2}(4, 1)_1) + (1 + q)P(M_{\mathbb{P}^2}(4, 1)_2)
= P(M_{\mathbb{P}^2}(4, 1)) + qP(M_{\mathbb{P}^2}(4, 1)_2)
\]

Let \( B^{(d,r)} \) denote the space parametrizing pairs \((C, Z)\) where \( C \) is a degree \( d \) curves on \( \mathbb{P}^2 \) and \( Z \) is a subscheme of \( C \) of length \( r \). When \( r < d + 2 \), it is known that the natural morphism

\[B^{(d,r)} \to \text{Hilb}^r(\mathbb{P}^2)\]

to the Hilbert scheme of \( r \) points is \( \mathbb{P}^{\frac{d(d+3)}{2} - r} \)-bundle.

By Lemma 4.6.3, \( M_{\mathbb{P}^2}(d, \chi)_2 \) is isomorphic to \( B^{(4,1)} \). So we have

\[
P(M_{\mathbb{P}^2}(d, \chi)_2) = P(\mathbb{P}^{13})P(\text{Hilb}^1(\mathbb{P}^2)) = \frac{1 - q^{14} - q^3}{1 - q - 1 - q}.
\]

The moduli space \( M_{\mathbb{P}^2}^\infty(4, 1) \) is isomorphic to \( B^{(4,3)} \) by [33, Corollary 5.12] (cf. [41, Proposition B.8]). Roughly, a pair \( s: \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F} \) in \( M_{\mathbb{P}^2}^\infty(4, 1) \) is associated to the pair \((C, Q)\) in
\[ B^{(4,3)}, \text{ where } C \text{ is the support of } F \text{ and } Q \text{ is the cokernel of } s. \text{ Thus,} \]
\[
P(M_{\mathbb{P}^2}(4, 1)) = P(\mathbb{P}^{11})P(\text{Hilb}^3(\mathbb{P}^2))
\]
\[
= \frac{1 - q^{12}}{1 - q}(1 + 2q + 5q^2 + 6q^3 + 5q^4 + 2q^5 + q^6).
\]

From this, we compute \( P(M_{\mathbb{P}^2}(4, 1)) \) by wall crossing. The PT moduli space for Hilbert polynomial \( dn + \chi \) is empty if \( \chi \leq \frac{d(3-d)}{2} \). So, in Theorem 4.4.6 the only nontrivial wall crossing occurs when \( \beta_0 = 3n \) and \( \beta_1 = n + 1 \).

Note that \( M_{\mathbb{P}^2}(3, 0) \) is isomorphic to \( B^{(3,0)} \simeq \mathbb{P}^9 \) for any \( \alpha \), and \( M_{\mathbb{P}^2}(1, 1) \) is isomorphic to \( \mathbb{P}^2 \). Let \( F_{n+1} \) and \( F_{3n} \) be sheaves of Hilbert polynomial \( n + 1 \) and \( 3n \) respectively. By Riemann-Roch theorem and [16, Corollary 1.6], we can compute the extension group defined on the category of pairs.

\[
\text{Ext}^1(F_{n+1}, O_{\mathbb{P}^2} \to F_{3n}) \simeq \mathbb{C}^3 \tag{4.24}
\]
\[
\text{Ext}^1(O_{\mathbb{P}^2} \to F_{3n}, F_{n+1}) \simeq \mathbb{C}^4 \tag{4.25}
\]

The extension given by an element in (4.25) is stable when \( \alpha = \infty \) and becomes unstable when \( \alpha = 0^+ \). The extension given by an element in (4.24) behaves in exact the other way. So, at the wall as we cross from \( \alpha = \infty \) to \( \alpha = 0^+ \), the \( \mathbb{P}^3 \)-bundle on \( \mathbb{P}^9 \times \mathbb{P}^2 \) is replaced by \( \mathbb{P}^2 \)-bundle on the same space \( \mathbb{P}^9 \times \mathbb{P}^2 \). This is exactly what happens at the walls in the wall crossing formula in Theorem 4.4.6.

Hence, we have

\[
P(M_{\mathbb{P}^2}^{(4,1)}(4, 1)) = P(M_{\mathbb{P}^2}(4, 1)) + (P(\mathbb{P}^2) - P(\mathbb{P}^3))P(\mathbb{P}^9)P(\mathbb{P}^2).
\]
In conclusion,

\[ P(M_{\mathbb{P}^2}(4, 1)) = P(M_{\mathbb{P}^2}^0(4, 1)) - qP(M_{\mathbb{P}^2}(4, 1)) \]

\[ = P(M_{\mathbb{P}^2}(4, 1) + (P(\mathbb{P}^2) - P(\mathbb{P}^3))P(\mathbb{P}^9)P(\mathbb{P}^2) - qP(M_{\mathbb{P}^2}(4, 1)) \]

\[ = \frac{1 - q^{12}}{1 - q} (1 + 2q + 5q^2 + 6q^3 + 5q^4 + 2q^5 + q^6) \]

\[ - q^3 \frac{1 - q^{10}}{1 - q} \frac{1 - q^3}{1 - q} - q \frac{1 - q^{14}}{1 - q} \frac{1 - q^3}{1 - q} \]

\[ = 1 + 2q + 6q^2 + 10q^3 + 14q^4 + 15q^5 + 16q^6 + 16q^7 + 16q^8 + 16q^9 + 16q^{10} + 16q^{11} + 15q^{12} + 14q^{13} + 10q^{14} + 6q^{15} + 2q^{16} + q^{17}. \]

The result agrees with Theorem 3.1.10. Kiryong Chung has also carried out the computation for \( M_{\mathbb{P}^2}(5, 1) \) using the same technique. There, we have three walls, each of which can be analyzed as above. We include his result omitting the details.

**Proposition 4.6.4.** The Poincaré polynomial of \( P(M_{\mathbb{P}^2}(5, 1)) \) is

\[ P(M_{\mathbb{P}^2}(5, 1)) = 1 + 2q + 6q^2 + 13q^3 + 26q^4 + 45q^5 + 68q^6 + 87q^7 + 100q^8 + 107q^9 + 111q^{10} + 112q^{11} + 113q^{12} + 113q^{13} + 113q^{14} + 112q^{15} + 111q^{16} + 107q^{17} + 100q^{18} + 87q^{19} + 68q^{20} + 45q^{21} + 26q^{22} + 13q^{23} + 6q^{24} + 2q^{25} + q^{26}. \]

This result is also consistent with the recent result in physics [17] and Table 3.1.
References


