TRANSFORMATION FORMULAS ASSOCIATED WITH INTEGRALS INVOLVING THE RIEMANN Ξ-FUNCTION AND RANK-CRANK TYPE PDES IN PARTITION THEORY

BY

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DISSERTATION

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Abstract

There are two parts to this thesis. The first part deals with obtaining "modular"-type transformation formulas involving various special functions such as the digamma function, Hurwitz zeta function and modified Bessel functions, through integrals involving the Riemann Ξ-function. Our motivation is a beautiful formula involving the digamma and the Riemann Ξ-functions found on page 220 in the volume "Ramanujan's lost notebook and other unpublished papers". In Chapter 2, we give two different proofs of this result, the second one being implicitly sought by A.P. Guinand while rediscovering this result. However, Guinand only partially rediscovered it in the sense that he did not see the connection with the integral involving the Riemann Ξ-function. A simple and cute, yet very useful, trick based upon the invariance of this integral under a certain map is used to provide the proof of the complete identity of Ramanujan.

From Chapter 3 onwards, we consider general integrals involving the Riemann Ξ-function and evaluate these using Mellin transforms and residue calculus. Then we use the above-mentioned trick to obtain modular transformation formulas. This provides a very convenient way of generating these formulas without using Poisson summation or Fourier transforms. Chapter 3 is devoted to transformation formulas of the type $F(\alpha) = F(\beta)$, where $\alpha\beta = 1$. Here, we unify many well-known formulas in the literature such as Koshliakov's formula, Hardy's formula, Ferrar's formula and, of course, Ramanujan's above transformation formula, in the sense that all of them can be generated in similar ways starting from the general integral involving the Riemann Ξ-function. This also leads us to extended versions of Koshliakov's formula and Ferrar's formula.

In Chapter 4, we study one-parameter generalizations of the extended versions of Koshliakov's and Ramanujan's formulas from Chapter 3, i.e., we obtain formulas of the type $F(z,\alpha) = F(z,\beta)$, where $\alpha\beta = 1$. In this case, we need to consider integrals involving a product of two Riemann Ξ-functions, the prototypical example of which was given by Ramanujan in his paper "New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$". Here we obtain a new proof as well as an extended version of the Ramanujan-Guinand formula. Also, Ramanujan's formula is generalized to a new one involving the Hurwitz zeta function. New analogues of this generalization are also obtained. Furthermore, we end this chapter by discussing proofs of two of the three transformation formulas involving the Hurwitz zeta function through special functions and without using contour integration and Mellin transforms. The proofs require certain identitites of Ramanujan from the above-referred paper. However, two identities in the paper are incorrect. Here, we derive correct versions of the same.

Chapter 5 is devoted to analogues of some of the results from Chapters 3 and 4 for primitive Dirichlet characters. These are of the form $F(z,\alpha,\chi) = F(-z,\beta,\overline{\chi}) = F(-z,\alpha,\overline{\chi}) = F(z,\beta,\chi)$, where $\alpha\beta = 1$. Character analogues of the Hurwitz zeta function and digamma function make their appearance here. Chapter 6 contains material from joint work with Bruce C. Berndt and Jaebum Sohn and involves a different character analogue of the Ramanujan-Guinand formula. We also give character analogues of the results found on pages 253–254 in Ramanujan's lost notebook. However, here we give results for only even primitive characters. The ones for even as well as odd primitive characters can be found in [18].

A new topic which fills the remainder of this thesis is discussed in Chapter 7. This is a topic in Partition Theory and it is concerned with obtaining Rank-Crank type PDE's through certain bilateral basic hypergeometric series identities due to M. Jackson and S.H. Chan. The motivation comes from the fact that the original Rank-Crank PDE due to A.O.L. Atkin and F.G. Garvan was obtained through an identity of Atkin and H.P.F. Swinnerton-Dyer, which is a special case of Chan's identity. We give proof of only the fourth order PDE of

Garvan and not of the general PDE. This contains part of the material from joint work with Chan and Garvan [26]. The proof given here is different from the one given there.

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Finally, I would like to thank Ramanujan for the wonderful mathematics that he created in spite of so many hurdles that he faced in life. This thesis would not have existed in the first place had Ramanujan not found his amazing transformation formula involving the Gamma and the Riemann zeta functions. His story and his mathematics will continue to inspire me and several other mathematicians all through our lives.

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Chapter 1

Introduction

Modular transformations are ubiquitous in the works of Ramanujan, namely his Notebooks [75], his published papers [76], and the Lost Notebook and other unpublished papers [78]. Ramanujan usually expressed them in a symmetric form involving α and β, and in this form they are valid under the conditions $\alpha\beta = k$, where k is some constant. Page 220 in the volume [78] including Ramanujan's lost notebook, being a part of one of the manuscripts of S. Ramanujan in the handwriting of G.N. Watson, contains a beautiful identity which is in the same spirit of these above-mentioned modular transformation formulas in one respect, and special in another (for reasons to be clear shortly). Many different proofs of this identity, its generalization, analogues, and unification of a vast number of transformation formulas of this nature in the sense that all can be derived through similar algorithms, will constitute a major portion of this thesis.

Pages 219–227 of [78] are devoted to material "Copied from the Loose Papers." These pages are *not* part of the original lost notebook found by George E. Andrews at Trinity College Library, Cambridge in the spring of 1976. These "loose papers", in the handwriting of G.N. Watson, are found in the Oxford University Library; the original manuscripts are in the library at Trinity College and have not been photocopied for publication. Most of these nine pages, which are divided into three rough, partial manuscripts, are connected with material in Ramanujan's published papers. However, there is much that is new in these fragments. See [3] for more details. One claim on pages 219–220 is the most interesting theorem in the first manuscript, and provides a beautiful series transformation involving the logarithmic derivative of the gamma function and the Riemann zeta function. To state

Ramanujan's claim, it will be convenient to use the familiar notation [43, p. 952, formulas 8.360, 8.362, no. 1]

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{k+x} - \frac{1}{k+1}\right), \tag{1.0.1}$$

where γ denotes Euler's constant. We also need to recall some basic properties of the Riemann zeta function $\zeta(s)$. This function is defined for Re $s > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}. \tag{1.0.2}$$

It can be analytically continued, first to $0 <$ Re $s < 1$ by an elementary argument, and then to the whole complex plane, except for a simple pole at $s = 1$, by means of the following functional equation [81, p. 22, eqn. (2.6.4)]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.0.3}$$

which can also be written in the following asymmetric form [81, p. 13, eqn. (2.1.1)]

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\tfrac{1}{2}\pi s\right). \tag{1.0.4}$$

Define the Riemann ξ-function by

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s), \tag{1.0.5}$$

Riemann's Ξ-function can then be defined by

$$\Xi(t) := \xi(\tfrac{1}{2} + it). \tag{1.0.6}$$

We also review some properties of the gamma function $\Gamma(s)$, which is defined for Re $s > 0$

by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx. \tag{1.0.7}$$

The reflection formula for the gamma function [80, p. 46] is given by

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{1.0.8}$$

Also, the duplication formula for the gamma function [80, p. 46] is given by

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \pi^{1/2} 2^{1-2s}\Gamma(2s). \tag{1.0.9}$$

Theorem 1.0.1 (Ramanujan)**.** *Define*

$$\lambda(x) := \psi(x) + \frac{1}{2x} - \log x. \tag{1.0.10}$$

If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\lambda(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\lambda(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt, \tag{1.0.11}$$

where γ denotes Euler's constant and $\Xi(x)$ denotes Riemann's Ξ-function.

Ramanujan claims this relation as 'curious' [78, p. 220]. Indeed, the first identity in (1.0.11) is beautiful in its elegant symmetry and surprising as well, because why would subtracting the two leading terms in the asymptotic expansion of the logarithmic derivative of the Gamma function, in order to gain convergence of the infinite series on the left side, yield a "modular relation" for the resulting function? The second identity in (1.0.11) is also surprising, for why would the first identity foreshadow a connection with the Riemann zeta function in the second?

Although Ramanujan does not provide a proof of (1.0.11), he does indicate that (1.0.11) "can be deduced from"

$$\int_0^\infty (\psi(1+x) - \log x)\cos(2\pi nx)\,dx = \frac{1}{2}\left(\psi(1+n) - \log n\right). \tag{1.0.12}$$

This latter result was rediscovered by A.P. Guinand [45] in 1947, and he later found a simpler proof of this result in [46]. In a footnote at the end of his paper [46], Guinand remarks that T.A. Brown had told him that he himself had proved the self-reciprocality of $\psi(1+x)-\log x$ some years ago, and that when he (Brown) communicated the result to G.H. Hardy, Hardy told him that the result was also given by Ramanujan in a progress report to the University of Madras, but was not published elsewhere. However, we cannot find this result in any of the three *Quarterly Reports* that Ramanujan submitted to the University of Madras [12], [13]. Therefore, Hardy's memory was perhaps imperfect; it would appear that he saw (1.0.12) in the aforementioned manuscript that Watson had copied. On the other hand, the only copy of Ramanujan's *Quarterly Reports* that exists is in Watson's handwriting! It could be that the manuscript on pages 219–220 of [78], which is also in Watson's handwriting, was somehow separated from the original *Quarterly Reports*, and therefore that Hardy was indeed correct in his assertion!

The first equality in (1.0.11) was rediscovered by Guinand in [45] and appears in a footnote on the last page of his paper [45, p. 18]. It is interesting that Guinand remarks, "This formula also seems to have been overlooked." Guinand's version of (1.0.11) is as follows.

Theorem 1.0.2. *For any complex z such that $|\arg z| < \pi$, we have*

$$\sum_{n=1}^\infty \left(\frac{\Gamma'}{\Gamma}(nz) - \log nz + \frac{1}{2nz}\right) + \frac{1}{2z}(\gamma - \log 2\pi z)$$
$$= \frac{1}{z}\sum_{n=1}^\infty \left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right) - \log\frac{n}{z} + \frac{z}{2n}\right) + \frac{1}{2}\left(\gamma - \log\frac{2\pi}{z}\right). \tag{1.0.13}$$

The first equality in (1.0.11) can be easily obtained from Guinand's version by multiplying both sides of (1.0.13) by $\sqrt{z}$ and then letting $z = \alpha$ and $1/z = \beta$. Although not offering a proof of (1.0.13) in [45], Guinand did remark that it can be obtained by using an appropriate form of Poisson's summation formula, namely the form given in Theorem 1 in [44]. Later Guinand gave another proof of Theorem 1.0.2 in [46], while also giving extensions of (1.0.13) involving derivatives of the ψ-function. He also established a finite version of (1.0.13) in [48]. However, Guinand apparently did not discover the connection of his work with Ramanujan's integral involving Riemann's Ξ-function.

Although the Riemann zeta function appears at various instances throughout Ramanujan's notebooks [75] and lost notebook [78], he wrote only one paper in which the zeta function plays the leading role [74], [76, pp. 72–77]. In fact, a result proved by Ramanujan in [74], namely equation (2.3.7) in Section 2.3, is a key to proving (1.0.11). About the integral involving Riemann's Ξ-function in this result, Hardy [50] comments that "*the properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in the Acta Mathematica, to prove that*

$$\int_{-T}^{T} \left| \zeta\left(\frac{1}{2} + ti\right) \right|^2 dt \sim 2T \log T. \text{"} \tag{1.0.14}$$

In [74], Ramanujan also offers identities for a more general integral whose special case is the integral in (2.3.7). We will be working with this integral in Chapter 4. At this stage, it is appropriate to mention here the remarks made by Hardy on Ramanujan's formulas in [74] in his article in the Journal of Indian Mathematical Society [51] describing Ramanujan's mathematical work in England. Hardy says, "*It is difficult at present to estimate the importance of these results. The unsolved problems concerning the zeros of $\zeta(s)$ or of $\Xi(t)$ are among the most obscure and difficult on the whole range of Pure Mathematics. Any new formulae involving $\zeta(s)$ or $\Xi(t)$ are of very great interest, because of the possibility that they may throw light on some of these outstanding questions. It is, as I have shown in a short note attached to Mr. Ramanujan's paper, certainly possible to apply his formulae in this direction; but the* *results which can be deduced from them do not at present go beyond those obtained already by Mr. Littlewood and myself in other ways. But I should not be at all surprised if still more important applications were to be made of Mr. Ramanujan's formulae in the future*".

The short note that Hardy refers in the above paragraph is [50]. It is also interesting that on a page in the original lost notebook [78, p. 195], Ramanujan defines

$$\phi(x) := \psi(x) + \frac{1}{2x} - \gamma - \log x \tag{1.0.15}$$

and then concludes that (1.0.11) is valid. However, with the definition (1.0.15) of $\phi(x)$, the series $\sum_{n=1}^{\infty} \phi(n\alpha)$ and $\sum_{n=1}^{\infty} \phi(n\beta)$ do not converge.

Having taken a look at the complete history, we turn our attention back to Theorem 1.0.1. Observe that if we have proved the equality of the first and the last expression in (1.0.11), the complete identity in (1.0.11) can be obtained by just replacing α by β in the proved one, and by noticing that the integral there is invariant under the transformation $\alpha \to \beta$ since $\alpha\beta = 1$. Not only is this trick elegant and apparently simple but also very powerful. This is because, it provides a way of generating these modular-type transformation formulas without using advanced machinery of Fourier Transforms and other related topics. This nice idea was first utilized in [17] and later in [30, 31, 32, 33, 34]. Of course, Ramanujan seems to have been acquainted with it.

The second part of this thesis deals with a topic in the Theory of Partitions, namely, on obtaining Rank-Crank type PDE's through certain bilateral basic hypergeometric identities, as opposed to the method motivated from the theory of Jacobi forms utilized by S. Zwegers [87]. This contains part of the material from joint work with Song Heng Chan and Frank G. Garvan [26]. Before we state the Rank-Crank PDE or its analogue or generalization, we wish to provide certain background of this topic.

F.J. Dyson [35], [36, p. 52] defined the rank of a partition as the largest part minus the number of parts. Dyson conjectured that the residue of the rank mod 5 divides the partitions of $5n+4$ into 5 equal clases thereby providing a combinatorial interpretation of Ramanujan's famous partition congruence $p(5n+4) \equiv 0 \pmod 5$. He also conjectured that the rank mod 7 likewise gives Ramanujan's partition congruence $p(7n+5) \equiv 0 \pmod 7$. Dyson's rank conjectures were proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [8]. The following was the crucial identity that Atkin and Swinnerton-Dyer needed for the proof of the Dyson rank conjectures. It was first proved by G.N. Watson [83].

$$\begin{aligned}
&\zeta \frac{[\zeta^2]_\infty}{[\zeta]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n} + \frac{[\zeta]_\infty [\zeta^2]_\infty (q)_\infty^2}{[z/\zeta]_\infty [z]_\infty [\zeta z]_\infty} \\
&\quad = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n+1)/2} \left(\frac{\zeta^{-3n}}{1-zq^n/\zeta} + \frac{\zeta^{3n+3}}{1-z\zeta q^n} \right).
\end{aligned} \tag{1.0.16}$$

Throughout we use the following standard q-notation, where $q = e^{2\pi i \tau}$ with $\operatorname{Im} \tau > 0$:

$$\begin{aligned}
(x)_0 &:= (x;q)_0 := 1, \\
(x)_n &:= (x;q)_n := \prod_{m=0}^{n-1} (1 - xq^m), \\
(x_1, \ldots, x_m)_n &:= (x_1, \ldots, x_m; q)_n := (x_1;q)_n \ldots (x_m;q)_n, \\
[x]_n &:= (x;q)_n (q/x;q)_n, \\
[x_1, \ldots, x_m]_n &:= [x_1, \ldots, x_m; q]_n := [x_1]_n \ldots [x_m]_n.
\end{aligned}$$

when n is a nonnegative integer. Assuming $|q| < 1$, we also use this notation when $n = \infty$ by interpreting its meaning as the limit as $n \to \infty$. Later M. Jackson [54] proved an analogue of the above identity,

$$\frac{\zeta^2[\zeta^2]_\infty[x\zeta]_\infty[x/\zeta]_\infty}{[\zeta]_\infty[x]^2_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n}+\frac{[\zeta]_\infty[\zeta^2]_\infty[x\zeta]_\infty[\zeta/x]_\infty(q)^2_\infty}{[z/x]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[zx]_\infty}$$
$$+\frac{\zeta}{x}\frac{[\zeta]_\infty[\zeta^2]_\infty}{[x]_\infty[x^2]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{5n(n+1)/2}\left(\frac{x^{-5n}}{1-zq^n/x}+\frac{x^{5n+5}}{1-zxq^n}\right) \tag{1.0.17}$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{5n(n+1)/2}\left(\frac{\zeta^{-5n}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{5n+5}}{1-z\zeta q^n}\right).$$

Recently, Chan [25] found a generalization of the above two identities, namely,

$$\frac{x_1^m[x_2/x_1,\cdots,x_m/x_1,x_1x_m,\cdots,x_1x_2,x_1^2]_\infty}{[x_1]_\infty[x_2,\cdots,x_m]^2_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-zq^n}$$
$$+\frac{[x_1/x_2,\cdots,x_1/x_m,x_1,x_1x_m,\cdots,x_1x_2,x_1^2]_\infty(q)^2_\infty}{[z/x_1,z/x_2,\cdots,z/x_m,z,zx_m,\cdots,zx_1]_\infty}$$
$$+\Bigg\{\frac{x_1}{x_2}\frac{[x_1/x_3,\cdots,x_1/x_m,x_1,x_1x_m,\cdots,x_1x_3,x_1^2]_\infty}{[x_2/x_3,\cdots,x_2/x_m,x_2,x_2x_m,\cdots,x_2^2]_\infty}$$
$$\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{x_2^{-(2m+1)n}}{1-zq^n/x_2}+\frac{x_2^{(2m+1)(n+1)}}{1-zx_2q^n}\right)+\mathrm{idem}(x_2;x_3,\cdots,x_m)\Bigg\}$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{x_1^{-(2m+1)n}}{1-zq^n/x_1}+\frac{x_1^{(2m+1)(n+1)}}{1-zx_1q^n}\right), \tag{1.0.18}$$

where $g(a_1,a_2,\cdots,a_m)+$ idem$(a_1;a_2,\cdots,a_n)$ denotes the sum $\sum_{i=1}^n g(a_i,a_2,\cdots,a_{i-1},a_1,a_{i+1},\cdots,a_m)$, i.e., the i-th term of the sum is obtained from the first by interchanging a_1 and a_i.

Chan proved (1.0.18) using partial fractions. Indeed, the $m=1$ case of (1.0.18) is equivalent to (1.0.16), while the $m=2$ case is equivalent to (1.0.17). The fact that the right-hand side of (1.0.17) is independent of x, and that the right-hand side of (1.0.18) is independent of $x_2,x_3,\cdots,x_m$ seems to be intriguing at first. Indeed, in Section 7.1 of Chapter 7, we show that the left-hand sides of (1.0.17) and (1.0.18) are really elliptic functions of order less than 2, in fact entire functions as we show, in the respective variables (x for (1.0.17) and x_2 for (1.0.18) while holding $x_3,\cdots,x_m$ fixed) and therefore that they must be constants which are nothing but the right-hand sides of (1.0.17) and (1.0.18) respectively. Since (1.0.17) follows from (1.0.18), we show this only for (1.0.18).

Let $N(m,n)$ denote the number of partitions of n with rank m. Then the rank generating function $R(z,q)$ is given by

$$R(z,q)=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}N(m,n)z^mq^n=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}. \tag{1.0.19}$$

In [4], Andrews and Garvan defined the crank of a partition, a partition statistic hypothesized by Dyson in [35]. It is the largest part if the partition contains no ones, and otherwise is the number of parts larger than the number of ones minus the number of ones. For $n>1$, we let $M(m,n)$ denote the number of partitions of n with crank m. If we amend the definition of $M(m,n)$ for $n=1$, then the generating function can be given as an infinite product. Accordingly, we assume

$$M(0,1)=-1,\ M(-1,1)=M(1,1)=1,\ \text{and } M(m,1)=0 \text{ otherwise.}$$

Then the crank generating function $C(z,q)$ is given by

$$C(z,q)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}M(m,n)z^mq^n=\frac{(q)_\infty}{(zq)_\infty(z^{-1}q)_\infty}. \tag{1.0.20}$$

Atkin and Garvan [7] found a partial differential equation (PDE) relating $R(z,q)$ and $C(z,q)$, which they called the Rank-Crank PDE. To state this PDE, first define the differential operators

$$\delta_z=z\frac{\partial}{\partial z},\qquad \delta_q=q\frac{\partial}{\partial q}. \tag{1.0.21}$$

Then the Rank-Crank PDE can be written as

$$z(q)_\infty^2[C^*(z,q)]^3=\left(3\delta_q+\frac{1}{2}\delta_z+\frac{1}{2}\delta_z^2\right)R^*(z,q), \tag{1.0.22}$$

where

$$R^*(z,q) := \frac{R(z,q)}{1-z},$$
$$C^*(z,q) := \frac{C(z,q)}{1-z}. \tag{1.0.23}$$

In [7], it was shown how the Rank-Crank PDE and certain results for the derivatives of Eisenstein series lead to exact relations between rank and crank moments. As in [41], define $N_k(m,n)$ by

$$\sum_{n=0}^{\infty} N_k(m,n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2+|m|n}(1-q^n), \tag{1.0.24}$$

of any positive integer k. When $k=1$ this is the generating function for $M(m,n)$, and when $k=2$ it is the generating function for $N(m,n)$. In [41, Equation (1.11)], Garvan defined a more general partition statistic called k-rank, for $k \geq 2$, which is defined as follows. For a partition π, let $n_1(\pi), n_2(\pi)\cdots$ be the sizes of the successive Durfee squares of π. Then the k-rank of the partition π is the 'number of columns in the Ferrers graph of π which lie to the right of the first Durfee square and whose length $\leq n_{k-1}(\pi)$ minus the number of parts of π that lie below the $(k-1)$-th Durfee square'. When $k \geq 2$, $N_k(m,n)$ can be combinatorially interpreted as the number of partitions of n into at least $k-1$ successive Durfee squares with k-rank equal to m. We define

$$R_k(z,q) := \sum_{n=0}^{\infty}\sum_{m=-n}^{n} N_k(m,n)z^m q^n. \tag{1.0.25}$$

From [41, Equation (4.5)], this generating function can be written as

$$R_k(z,q) = \sum_{n_{k-1}\geq n_{k-2}\geq\cdots\geq n_1\geq 1} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}}, \tag{1.0.26}$$

when $k \geq 2$.

In an unpublished work, Garvan found a 4^{th} order PDE, which is an analogue of the Rank-Crank PDE. To state this PDE, we first need to give some definitions and notations. Define for $k \geq 1$,

$$\Sigma^{(k)}(z,q) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{kn(n+1)/2}}{1 - zq^n}. \tag{1.0.27}$$

Now if we define

$$G^{(5)}(z,q) := \frac{\Sigma^{(5)}(z,q)}{(q)_\infty^3}, \tag{1.0.28}$$

and

$$\Phi_3(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \tag{1.0.29}$$

then using (1.0.21) and (1.0.23), the analogue of the Rank-Crank PDE can be written as

$$\begin{aligned} &24(q)_\infty^2 [C^*(z,q)]^5 \\ &= 24(1 - 10\Phi_3(q)) G^{(5)}(z,q) \\ &\quad + \left(100\delta_q + 50\delta_z + 100\delta_q\delta_z + 35\delta_z^2 + 20\delta_q\delta_z^2 + 100\delta_q^2 + 10\delta_z^3 + \delta_z^4\right) G^{(5)}(z,q). \end{aligned} \tag{1.0.30}$$

Recently, S. Zwegers [87] found a general PDE which gives the Rank-Crank PDE and Garvan's 4^{th} order analogue of the Rank-Crank PDE as special cases. Before we state Zwegers' PDE, we note that for $l \in \mathbb{Z}_{>0}$, the level l Appell function is defined by

$$A_l(u,v) := A_l(u,v;\tau) := z^{l/2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{ln} q^{ln(n+1)/2} w^n}{1 - zq^n}, \tag{1.0.31}$$

where $z = e^{2\pi i u}$, $w = e^{2\pi i v}$, $u, v \in \mathbb{C}$. Define the modified rank and crank generating functions by

$$\mathcal{R} := \mathcal{R}(u;\tau) := \frac{z^{1/2}q^{-1/24}}{1-z}R(z,q),$$
$$\mathcal{C} := \mathcal{C}(u;\tau) := \frac{z^{1/2}q^{-1/24}}{1-z}C(z,q). \tag{1.0.32}$$

Then the following theorem due to Zwegers, gives for odd $l \geq 3$, a generalization of the Rank-Crank PDE.

Theorem 1.0.3. *Let $l \geq 3$ be an odd integer. Define*

$$\mathcal{H}_k := \frac{l}{\pi i}\frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial u^2} - \frac{l(2k-1)}{12}E_2,$$
$$\mathcal{H}^k := \mathcal{H}_{2k-1}\mathcal{H}_{2k-3}\cdots\mathcal{H}_3\mathcal{H}_1, \tag{1.0.33}$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ is the usual Eisenstein series of weight 2 with $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then there exist holomorphic modular forms f_j (which can be constructed explicitly), with $j = 4, 6, 8, \ldots, l-1$, on $SL_2(\mathbb{Z})$ of weight j, such that

$$\left(\mathcal{H}^{(l-1)/2} + \sum_{k=0}^{(l-5)/2} f_{l-2k-1}\mathcal{H}^k\right)A_l(u,0) = (l-1)!\eta^l\mathcal{C}^l, \tag{1.0.34}$$

where η is the Dedekind η-function, given by $\eta(\tau) = q^{1/24}(q)_\infty$.

As mentioned before, Zwegers proved this theorem using the formulas and methods motivated from the theory of Jacobi forms. In contrast to this, the proof of the Rank-Crank PDE by Atkin and Garvan, which corresponds to the $l = 3$ case of Zwegers' PDE, depends upon simply taking the second derivative with respect to ζ of both sides of (1.0.16). In Chapter 7 Section 7.3, we show that the 4^{th} order PDE (1.0.30) follows from a special case of (1.0.17) in a similar fashion. In [26], an equivalent version of Zwegers' general Rank-Crank PDE is obtained. Of course, this gives an equivalent version of (1.0.30) as a special case. However, the proofs of both the versions are different.

In the light of (1.0.31), it should be observed that the identities (1.0.16) and (1.0.17) are really the identities involving certain combinations of level 3 and level 5 Appell functions, respectively, while (1.0.18) is an identity involving a combination of level $(2m + 1)$ Appell functions.

Chapter 2

A transformation formula in Ramanujan's lost notebook

2.1 A claim in the lost notebook

This section is devoted to a proof of a beautiful transformation formula involving the Gamma and Riemann zeta functions given by Ramanujan on page 220 in [78]. The proof discussed here is published in [17].

Theorem 2.1.1. *Define*

$$\lambda(x) := \psi(x) + \frac{1}{2x} - \log x, \tag{2.1.1}$$

where

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=0}^{\infty}\left(\frac{1}{m+x} - \frac{1}{m+1}\right) \tag{2.1.2}$$

is the logarithmic derivative of the Gamma function. Let Riemann's ξ*- and* Ξ*-functions be defined in* (1.0.5) *and* (1.0.6) *respectively. If* α *and* β *are positive numbers such that* $\alpha\beta = 1$*, then*

$$\begin{aligned}\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{k=1}^{\infty}\lambda(k\alpha)\right\} &= \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{k=1}^{\infty}\lambda(k\beta)\right\}\\ &= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt,\end{aligned} \tag{2.1.3}$$

where γ *denotes Euler's constant.*

2.2 Preliminary results

We first collect several well-known theorems that we use in our proof. First, from [28, p. 191], for $t \neq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{t^2 + 4n^2\pi^2} = \frac{1}{2t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right). \tag{2.2.1}$$

Second, from [85, p. 251], we find that, for $\operatorname{Re} z > 0$,

$$\lambda(z) = -2\int_0^{\infty} \frac{t\,dt}{(t^2 + z^2)(e^{2\pi t} - 1)}. \tag{2.2.2}$$

Third, we require Binet's integral for $\log\Gamma(z)$, i.e., for $\operatorname{Re} z > 0$ [85, p. 249], [43, p. 377, formula 3.427, no. 4],

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^{\infty}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-zt}}{t}\,dt. \tag{2.2.3}$$

Fourth, from [43, p. 377, formula 3.427, no. 2], we find that

$$\int_0^{\infty}\left(\frac{1}{1 - e^{-x}} - \frac{1}{x}\right)e^{-x}\,dx = \gamma, \tag{2.2.4}$$

where γ denotes Euler's constant. Lastly, we state Frullani's integral theorem [71, p. 612, Equation (1)].

Theorem 2.2.1. *Let $f(x)$ be a Lebesgue integrable function over any interval $0 < A \leq x \leq B < \infty$. Assuming that the limits exist, write $f(0) = \lim_{x\to 0^+} f(x)$ and $f(\infty) = \lim_{x\to\infty} f(x)$. Then, for $a, b > 0$,*

$$\int_0^{\infty} \frac{f(at) - f(bt)}{t}dt = (f(\infty) - f(0))\log\left(\frac{a}{b}\right). \tag{2.2.5}$$

As a corollary, we obtain the following integral evaluation [43, p. 378, formula 3.434, no. 2]:

$$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x}\, dx = \log \frac{\nu}{\mu}, \qquad \mu, \nu > 0. \tag{2.2.6}$$

2.3 Proof of Theorem 2.1.1

Proof. Our first goal is to establish an integral representation for the far left side of (2.1.3). Replacing z by $n\alpha$ in (2.2.2) and summing on n, $1 \le n < \infty$, we find, by absolute convergence, that

$$\begin{aligned}\sum_{n=1}^\infty \lambda(n\alpha) &= -2 \sum_{n=1}^\infty \int_0^\infty \frac{t\, dt}{(t^2 + n^2\alpha^2)(e^{2\pi t} - 1)} \\ &= \frac{-2}{\alpha^2} \int_0^\infty \frac{t\, dt}{(e^{2\pi t} - 1)} \sum_{n=1}^\infty \frac{1}{(t/\alpha)^2 + n^2}.\end{aligned} \tag{2.3.1}$$

Invoking (2.2.1) in (2.3.1), we see that

$$\sum_{n=1}^\infty \lambda(n\alpha) = -\frac{2\pi}{\alpha} \int_0^\infty \frac{1}{(e^{2\pi t} - 1)} \left(\frac{1}{e^{2\pi t/\alpha} - 1} - \frac{\alpha}{2\pi t} + \frac{1}{2} \right) dt. \tag{2.3.2}$$

Next, setting $x = 2\pi t$ in (2.2.4), we readily find that

$$\gamma = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t} - 1} - \frac{e^{-2\pi t}}{t} \right) dt. \tag{2.3.3}$$

By Frullani's integral (2.2.6),

$$\int_0^\infty \frac{e^{-t/\alpha} - e^{-2\pi t}}{t}\, dt = \log\left(\frac{2\pi}{1/\alpha} \right) = \log(2\pi\alpha). \tag{2.3.4}$$

Combining (2.3.3) and (2.3.4), we arrive at

$$\gamma - \log(2\pi\alpha) = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t} - 1} - \frac{e^{-t/\alpha}}{t} \right) dt. \tag{2.3.5}$$

Hence, from (2.3.2) and (2.3.5), we deduce that

$$\sqrt{\alpha}\left(\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^{\infty}\lambda(n\alpha)\right) \tag{2.3.6}$$
$$=\frac{1}{2\sqrt{\alpha}}\int_0^{\infty}\left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-t/\alpha}}{t}\right)dt$$
$$-\frac{2\pi}{\sqrt{\alpha}}\int_0^{\infty}\frac{1}{(e^{2\pi t}-1)}\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}+\frac{1}{2}\right)dt$$
$$=\int_0^{\infty}\left(\frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)}-\frac{2\pi}{\sqrt{\alpha}(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)}-\frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt.$$

Now from [74, p. 260, eqn. (22)] or [76, p. 77], for n real,

$$\int_0^{\infty}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\left(\Xi\left(\frac{1}{2}t\right)\right)^2\frac{\cos nt}{1+t^2}\,dt$$
$$=\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos nt}{1+t^2}\,dt$$
$$=\pi^{3/2}\int_0^{\infty}\left(\frac{1}{e^{xe^{n}}-1}-\frac{1}{xe^{n}}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx. \tag{2.3.7}$$

Letting $n=\frac{1}{2}\log\alpha$ and $x=2\pi t/\sqrt{\alpha}$ in (2.3.7), we deduce that

$$-\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos(\frac{1}{2}t\log\alpha)}{1+t^2}\,dt \tag{2.3.8}$$
$$=-\frac{2\pi}{\sqrt{\alpha}}\int_0^{\infty}\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}\right)dt$$
$$=\int_0^{\infty}\left(\frac{-2\pi/\sqrt{\alpha}}{(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)}+\frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)}+\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)}-\frac{\sqrt{\alpha}}{2\pi t^2}\right)dt.$$

Hence, combining (2.3.6) and (2.3.8), in order to prove that the far left side of (2.1.3) equals the far right side of (2.1.3), we see that it suffices to show that

$$\int_0^{\infty}\left(\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)}-\frac{\sqrt{\alpha}}{2\pi t^2}+\frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt$$
$$=\frac{1}{\sqrt{\alpha}}\int_0^{\infty}\left(\frac{1}{u(e^{u}-1)}-\frac{1}{u^2}+\frac{e^{-u/(2\pi)}}{2u}\right)du=0, \tag{2.3.9}$$

where we made the change of variable $u = 2\pi t/\alpha$. In fact, more generally, we show that

$$\int_0^\infty \left(\frac{1}{u(e^u-1)} - \frac{1}{u^2} + \frac{e^{-ua}}{2u} \right) du = -\frac{1}{2}\log(2\pi a), \tag{2.3.10}$$

so that if we set $a = 1/(2\pi)$ in (2.3.10), we deduce (2.3.9).

Consider the integral, for $t > 0$,

$$\begin{aligned} F(a,t) :&= \int_0^\infty \left\{ \left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2} \right) \frac{e^{-tu}}{u} + \frac{e^{-ua} - e^{-tu}}{2u} \right\} du \\ &= \log\Gamma(t) - \left(t - \frac{1}{2}\right)\log t + t - \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\frac{t}{a}, \end{aligned} \tag{2.3.11}$$

where we applied (2.2.3) and (2.2.6). Upon the integration of (2.1.2), it is easily gleaned that, as $t \to 0$,

$$\log\Gamma(t) \sim -\log t - \gamma t,$$

where γ denotes Euler's constant. Using this in (2.3.11), we find, upon simplification, that, as $t \to 0$,

$$F(a,t) \sim -\gamma t - t\log t + t - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log a.$$

Hence,

$$\lim_{t\to 0} F(a,t) = -\frac{1}{2}\log(2\pi a). \tag{2.3.12}$$

Letting t approach 0 in (2.3.11), taking the limit under the integral sign on the right-hand side using Lebesgue's dominated convergence theorem, and employing (2.3.12), we immediately deduce (2.3.10). As previously discussed, this is sufficient to prove the equality of the first and third expressions in (2.1.3), namely,

$$\sqrt{\alpha}\left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^\infty \lambda(n\alpha) \right\} = -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt. \tag{2.3.13}$$

Lastly, using (2.3.13) with α replaced by β and employing the relation $\alpha\beta = 1$, we conclude that

$$\begin{aligned}
&\sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\lambda(n\beta)\right\} \\
&= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\beta\right)}{1+t^2}\,dt \\
&= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log(1/\alpha)\right)}{1+t^2}\,dt \\
&= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt.
\end{aligned}$$

Hence, the equality of the second and third expressions in (2.1.3) has been demonstrated, and so the proof is complete. □

2.3.1 Another proof of (2.3.9)

Here, we give another proof of (2.3.9) which utilizes Ramanujan's generalization of Frullani's integral theorem [13, p. 313] given below.

Theorem 2.3.1. *Let f and g be continuous functions on $[0,\infty)$ such that*

$$f(t) = \sum_{k=0}^{\infty}\frac{u(k)(-t)^k}{k!}$$
$$g(t) = \sum_{k=0}^{\infty}\frac{v(k)(-t)^k}{k!}.$$

Assume that $\dfrac{u(s)}{\Gamma(s+1)}$ and $\dfrac{u(s)}{\Gamma(s+1)}$ satisfy the hypotheses of Hardy's theorem [13, p. 299]. *Furthermore, assume that $f(0) = g(0)$ and $f(\infty) = g(\infty)$, where $f(0) = \lim_{t\to 0} f(t)$, $f(\infty) =$*

$\lim_{t\to\infty} f(t)$, $g(0) = \lim_{t\to 0} g(t)$ *and* $g(\infty) = \lim_{t\to\infty} g(t)$. *Then if* $a, b > 0$,

$$\lim_{n\to 0^+} I_n \equiv \lim_{n\to 0^+} \int_0^\infty x^{n-1}(f(ax) - g(bx))dx$$
$$= (f(0) - f(\infty)) \left(Log\left(\frac{b}{a}\right) + \frac{d}{ds}\left(Log\left(\frac{v(s)}{u(s)}\right)\right)\Big|_{s=0}\right). \tag{2.3.14}$$

Let $f(t) = \left(\frac{1}{e^{2\pi t/\alpha} - 1} - \frac{\alpha}{2\pi t}\right)$ and $g(t) = \frac{-e^{-t/\alpha}}{2}$. We first show that $f(0) = g(0)$ and $f(\infty) = g(\infty)$.

Now

$$f(0) = \lim_{t\to 0}\left(\frac{1}{e^{2\pi t/\alpha} - 1} - \frac{\alpha}{2\pi t}\right) = \lim_{t\to 0} \frac{\alpha + 2\pi t - \alpha e^{2\pi t/\alpha}}{2\pi t(e^{2\pi t/\alpha} - 1)}$$
$$= -\alpha \lim_{t\to 0} \frac{\sum_{n=2}^{\infty}(2\pi t/\alpha)^n/n!}{2\pi t(e^{2\pi t/\alpha} - 1)} = -\lim_{t\to 0} \frac{\sum_{n=2}^{\infty}(2\pi t/\alpha)^{n-1}/n!}{(e^{2\pi t/\alpha} - 1)}$$
$$= \lim_{t\to 0} \frac{-\frac{\pi t}{\alpha} - \sum_{n=3}^{\infty}(2\pi t/\alpha)^{n-1}/n!}{(e^{2\pi t/\alpha} - 1)}$$
$$= -\frac{1}{2}, \tag{2.3.15}$$

where in the last step we have employed L'Hopital's rule.

Also,

$$f(\infty) = \lim_{t\to\infty}\left(\frac{1}{e^{2\pi t/\alpha} - 1} - \frac{\alpha}{2\pi t}\right) = 0. \tag{2.3.16}$$

Similarly, one can easily show that $g(0) = -\frac{1}{2}$ and $g(\infty) = 0$. Thus we have $f(0) = g(0)$ and $f(\infty) = g(\infty)$.

We now expand $f(t)$ and $g(t)$ as Maclaurin series in the form as required in Theorem 2.3.1.

For $|t/\alpha| < 1$, we have

$$\begin{aligned} f(t) = \left(\frac{1}{e^{2\pi t/\alpha}-1} - \frac{\alpha}{2\pi t}\right) &= \frac{\alpha}{2\pi t}\left(\frac{2\pi t/\alpha}{e^{2\pi t/\alpha}-1} - 1\right) \\ &= \frac{\alpha}{2\pi t}\sum_{k=1}^{\infty}\frac{B_k\,(2\pi t/\alpha)^k}{k!} = \sum_{k=0}^{\infty}\frac{B_{k+1}\,(2\pi t/\alpha)^k}{(k+1)!} \\ &= -\sum_{k=0}^{\infty}\frac{(-1)^{k+1}B_{k+1}\,(2\pi/\alpha)^k\,(-t)^k}{(k+1)!} \\ &= -\sum_{k=0}^{\infty}\frac{B_{k+1}(1)\,(2\pi/\alpha)^k\,(-t)^k}{(k+1)!} \\ &= -\sum_{k=0}^{\infty}\frac{\zeta(-k)\,(2\pi/\alpha)^k\,(-t)^k}{k!}. \end{aligned} \tag{2.3.17}$$

Also,

$$g(t) = \frac{-e^{-t/\alpha}}{2} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-t)^k}{\alpha^k k!}. \tag{2.3.18}$$

From (2.3.17) and (2.3.18), we find that

$$\begin{aligned} u(s) &= -\zeta(-s)\left(\frac{2\pi}{\alpha}\right)^s, \\ v(s) &= -\frac{1}{2\alpha^s}. \end{aligned} \tag{2.3.19}$$

Thus from (2.3.15), (2.3.16), (2.3.19) and Theorem 2.3.1, we have

$$\begin{aligned} &\int_0^{\infty}\left(\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)} - \frac{\sqrt{\alpha}}{2\pi t^2} + \frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt \\ &= \frac{-1}{2}\left(\log(1) + \frac{d}{ds}\log\left(\frac{-\frac{1}{2\alpha^s}}{-\zeta(-s)\left(\frac{2\pi}{\alpha}\right)^s}\right)\Bigg|_{s=0}\right) \\ &= \frac{-1}{2}\frac{d}{ds}\left(\log(1/2) - \log\zeta(-s) - s\log(2\pi)\right)\Big|_{s=0} \\ &= \frac{-1}{2}\left(\frac{\zeta'(-s)}{\zeta(-s)} - \log(2\pi)\right)\Bigg|_{s=0} = \frac{-1}{2}\left(\frac{\zeta'(0)}{\zeta(0)} - \log(2\pi)\right) \\ &= \frac{-1}{2}\left(\frac{(-1/2)\log(2\pi)}{-1/2} - \log(2\pi)\right) = 0. \end{aligned} \tag{2.3.20}$$

2.4 Guinand's proof of the first equality in Theorem 2.1.1

As remarked in the introduction, A.P. Guinand rediscovered the first equality in (2.1.3). Guinand's version is as follows [45].

Theorem 2.4.1. *For any complex z such that $|\arg z| < \pi$, we have*

$$\sum_{n=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}(nz) - \log nz + \frac{1}{2nz}\right) + \frac{1}{2z}(\gamma - \log 2\pi z)$$
$$= \frac{1}{z}\sum_{n=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right) - \log\frac{n}{z} + \frac{z}{2n}\right) + \frac{1}{2}\left(\gamma - \log\frac{2\pi}{z}\right). \quad (2.4.1)$$

In this section, we prove (2.4.1) using Guinand's generalization of Poisson's summation formula [44] given below.

Theorem 2.4.2. *If $f(x)$ is an integral, $f(x)$ tends to zero as $x \to \infty$, and $xf'(x)$ belongs to $L^p(0,\infty)$, for some p, $1 < p \leq 2$, then*

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} f(n) - \int_0^N f(t)\,dt\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} g(n) - \int_0^N g(t)\,dt\right), \quad (2.4.2)$$

where

$$g(x) = 2\int_0^{\to\infty} f(t)\cos(2\pi xt)\,dt. \quad (2.4.3)$$

We also need the following lemma. We are indebted to M. L. Glasser for the following slick proof of this lemma. Another proof, somewhat longer, will be discussed after giving the proof of (2.4.1).

Lemma 2.4.3. *If $\psi(x)$ is defined by (2.1.2), then*

$$\int_0^{\infty}\left(\psi(t+1) - \frac{1}{2(t+1)} - \log t\right)dt = \frac{1}{2}\log 2\pi. \quad (2.4.4)$$

Proof. Let I denote the integral on the left-hand side of (2.4.4). Then,

$$\begin{aligned} I &= \int_0^\infty \frac{d}{dt}\left(\log \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}\right) dt \\ &= \lim_{t\to\infty} \log \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \lim_{t\to 0} \log \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \\ &= \log \lim_{t\to\infty} \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \log\left(\lim_{t\to 0} e^t\Gamma(t+1)\right) - \lim_{t\to 0} t\log t - \lim_{t\to 0}\frac{1}{2}\log(t+1) \\ &= \log \lim_{t\to\infty} \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}. \end{aligned} \tag{2.4.5}$$

Next, Stirling's formula [43, p. 945, formula 8.327] tells us that

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-1/2}e^{-z}, \tag{2.4.6}$$

as $|z| \to \infty$ for $|\arg z| \le \pi - \delta$, where $0 < \delta < \pi$. Hence, employing (2.4.6), we find that

$$\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \sim \frac{\sqrt{2\pi}}{e}\left(1+\frac{1}{t}\right)^t, \tag{2.4.7}$$

so that

$$\lim_{t\to\infty} \frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} = \sqrt{2\pi}. \tag{2.4.8}$$

Thus, from (2.4.5) and (2.4.8), we conclude that

$$I = \frac{1}{2}\log 2\pi. \tag{2.4.9}$$

□

Now we are ready to give the proof of Theorem 2.4.1. We first prove it for a fixed z such that Re $z > 0$. Let

$$f(x) := \psi(xz+1) - \log xz$$
$$= \psi(xz) + \frac{1}{xz} - \log xz, \tag{2.4.10}$$

since from [1, p. 258, formula 6.3.5], we have

$$\psi(s+1) = \psi(s) + \frac{1}{s}. \tag{2.4.11}$$

We show that $f(x)$ satisfies the hypotheses of Theorem 2.4.2. From (1.0.12), we see that $f(x)$ is an integral. Next, we need two formulas for the psi function. First, from [1, p. 259, formula 6.3.18], for $|\arg s| < \pi$, as $s \to \infty$,

$$\psi(s) \sim \log s - \frac{1}{2s} - \frac{1}{12s^2} + \frac{1}{120s^4} - \frac{1}{252s^6} + \cdots . \tag{2.4.12}$$

Second, from [85, p. 250],

$$\psi'(s) = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}. \tag{2.4.13}$$

From (2.4.10) and (2.4.12) it follows that

$$f(x) \sim \frac{1}{2xz} - \frac{1}{12x^2z^2} + \frac{1}{120x^4z^4} - \frac{1}{252x^6z^6} + \cdots , \tag{2.4.14}$$

so that

$$\lim_{x\to\infty} f(x) = 0. \tag{2.4.15}$$

Next, we show that $xf'(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$. Using

(2.4.14), we find that, as $x \to \infty$,

$$xf'(x) \sim -\frac{1}{2xz}, \tag{2.4.16}$$

so that $|xf'(x)|^p \sim (2x|z|)^{-p}$. Thus, for $p > 1$, we see that $xf'(x)$ is locally integrable near ∞. Also, using (2.4.10) and (2.4.13), we have

$$\begin{aligned}
\lim_{x\to 0} xf'(x) &= \lim_{x\to 0}\left(xz\sum_{n=0}^{\infty}\frac{1}{(xz+n)^2} - \frac{1}{xz} - 1\right)\\
&= \lim_{x\to 0}\left(xz\sum_{n=1}^{\infty}\frac{1}{(xz+n)^2} - 1\right)\\
&= -1.
\end{aligned} \tag{2.4.17}$$

This proves that $xf'(x)$ is locally integrable near 0. Hence, we have shown that $xf'(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$.

Now from (2.4.3) and (2.4.10), we find that

$$g(x) = 2\int_0^{\infty}(\psi(tz+1) - \log tz)\cos(2\pi xt)\,dt.$$

Employing the change of variable $y = tz$ and using (1.0.12), we find that

$$\begin{aligned}
g(x) &= \frac{2}{z}\int_0^{\infty}(\psi(y+1) - \log y)\cos(2\pi xy/z)\,dy\\
&= \frac{1}{z}\left(\psi\left(\frac{x}{z}+1\right) - \log\left(\frac{x}{z}\right)\right).
\end{aligned} \tag{2.4.18}$$

Substituting the expressions for $f(x)$ and $g(x)$ from (2.4.10) and (2.4.18), respectively, in

(2.4.2), we find that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}(\psi(nz+1)-\log nz)-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\psi\left(\frac{n}{z}+1\right)-\log\frac{n}{z}\right)-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)\,dt\right)\right]. \tag{2.4.19}$$

Thus,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}(nz)+\frac{1}{2nz}-\log nz\right)+\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right)+\frac{z}{2n}-\log\frac{n}{z}\right)+\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)\,dt\right)\right]. \tag{2.4.20}$$

Now if we can show that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)=\frac{\gamma-\log 2\pi z}{2z}, \tag{2.4.21}$$

then replacing z by $1/z$ in (2.4.21) will give us

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)\,dt\right)=\frac{z(\gamma-\log(2\pi/z))}{2}. \tag{2.4.22}$$

Then substituting (2.4.21) and (2.4.22) in (2.4.20) will complete the proof of the theorem. To that end,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\lim_{N\to\infty}\left(\frac{1}{2z}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)+\frac{\log N}{2z}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{\gamma}{2z}+\lim_{N\to\infty}\left(-\frac{\log z}{2z}+\frac{\log Nz}{2z}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$

$$
\begin{aligned}
&= \frac{\gamma}{2z} - \frac{\log z}{2z} + \lim_{N\to\infty} \left(\frac{\log(Nz+1)}{2z} - \frac{1}{z}\int_0^{Nz} (\psi(t+1) - \log t)\, dt - \frac{1}{2z}\log\left(1 + \frac{1}{Nz}\right)\right) \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} + \frac{1}{z}\lim_{N\to\infty} \left(\frac{\log(Nz+1)}{2} - \int_0^{Nz} (\psi(t+1) - \log t)\, dt \right) \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} + \frac{1}{z}\lim_{N\to\infty} \left(\frac{1}{2}\int_0^{Nz} \frac{1}{t+1}\, dt - \int_0^{Nz} (\psi(t+1) - \log t)\, dt \right) \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z}\lim_{N\to\infty} \int_0^{Nz} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z}\int_0^{\infty} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt \\
&= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{\log 2\pi}{2z} \\
&= \frac{\gamma - \log 2\pi z}{2z}, \qquad (2.4.23)
\end{aligned}
$$

where in the antepenultimate line we have made use of Lemma 2.4.3. This completes the proof of (2.4.21) and hence the proof of Theorem 2.4.1 for Re $z > 0$. But both sides of (2.4.1) are analytic for $|\arg z| < \pi$. Hence, by analytic continuation, the theorem is true for all complex z such that $|\arg z| < \pi$.

2.4.1 Another proof of Lemma 2.4.3

Let I denote the integral on the left-hand side of (2.4.4). Then,

$$
I = \int_0^1 \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt + \int_1^{\infty} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt. \quad (2.4.24)
$$

In the second integral, let $t \to 1/t$. Then after some simplification, we find

$$
\begin{aligned}
I &= \int_0^1 \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t + \frac{\psi(1+\frac{1}{t})}{t^2} - \frac{1}{2t(t+1)} + \frac{\log t}{t^2} \right) dt \\
&= \int_0^1 \psi(t+1) dt + \int_0^1 \left(\frac{\psi(1+\frac{1}{t})}{t^2} - \frac{1}{2t} - \left(1 - \frac{1}{t^2}\right) \log t \right) dt. \qquad (2.4.25)
\end{aligned}
$$

Now from [43, p. 691, formula 6.462], we know that for $\alpha > 0$,

$$\int_0^1 \psi(\alpha + x)dx = \log \alpha. \tag{2.4.26}$$

Thus from (2.4.26) with $\alpha = 1$ and (2.4.24), we obtain after some simplification,

$$\begin{aligned} I &= \int_0^1 \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t + \frac{\psi(1+\frac{1}{t})}{t^2} - \frac{1}{2t(t+1)} + \frac{\log t}{t^2} \right) dt \\ &= \int_0^1 \left(\frac{\psi(\frac{1}{t})}{t^2} + \frac{1}{2t} - \left(1 - \frac{1}{t^2}\right) \log t \right) dt, \end{aligned} \tag{2.4.27}$$

where, we employed (2.4.11) on the right-hand side.

After making a change of variable $t = 1/x$ in the latter integral above, we find

$$\begin{aligned} I &= \int_1^\infty \left(x^2\psi(x) + \frac{x}{2} + (1 - x^2) \log x \right) \frac{dx}{x^2} \\ &= \int_1^\infty \left(\psi(x) + \frac{1}{2x} - \log x \right) dx + \int_1^\infty \frac{\log x}{x^2} dx \\ &= I_1 + I_2. \end{aligned} \tag{2.4.28}$$

The integral I_2 is quite easy to evaluate. After making a change of variable $x = 1/t$, we readily find that

$$\begin{aligned} I_2 &= -\int_0^1 \log t \, dt \\ &= -\left(t \log t - t\right)\Big|_0^1 \\ &= 1. \end{aligned} \tag{2.4.29}$$

Now we proceed to evaluate I_1. First, from [85, p. 251], we find that for $\text{Re } z > 0$,

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{2z} - \log z = -2 \int_0^\infty \frac{t \, dt}{(t^2 + z^2)(e^{2\pi t} - 1)}. \tag{2.4.30}$$

Hence using (2.4.30), we find that

$$\begin{aligned}
I_1 &= -2\int_1^{\infty}\int_0^{\infty}\frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)}dx\\
&= -2\int_0^{\infty}\frac{t\,dt}{e^{2\pi t}-1}\int_1^{\infty}\frac{dx}{(t^2+x^2)}\\
&= -2\int_0^{\infty}\frac{t\,dt}{e^{2\pi t}-1}\left(\frac{\tan^{-1}(x/t)}{t}\right)\Bigg|_1^{\infty}\\
&= -2\int_0^{\infty}\frac{\tan^{-1}(t)}{e^{2\pi t}-1}dt, \qquad (2.4.31)
\end{aligned}$$

where in the last step we have made use of the fact that $\tan^{-1}(t)+\tan^{-1}(1/t)=\frac{\pi}{2}$. The interchange of two integrals in the above calculation can be easily justified. Now by Binet's second expression for $\log\Gamma(z)$ [85, p. 251], we have for $\text{Re } z>0$

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z - z+\frac{1}{2}\log(2\pi)+2\int_0^{\infty}\frac{\tan^{-1}(t/z)}{e^{2\pi t}-1}dt. \qquad (2.4.32)$$

Letting $z=1$ in (2.4.32) and simplifying, we find that

$$2\int_0^{\infty}\frac{\tan^{-1}(t)}{e^{2\pi t}-1}dt = 1-\frac{1}{2}\log 2\pi. \qquad (2.4.33)$$

Thus from (2.4.28), (2.4.29), (2.4.31) and (2.4.33) we have

$$\begin{aligned}
I &= 1-\left(1-\frac{1}{2}\log 2\pi\right)\\
&= \frac{1}{2}\log 2\pi. \qquad (2.4.34)
\end{aligned}$$

This completes the proof of the lemma.

Chapter 3

Transformation formulas of the form $F(\alpha) = F(\beta)$ where $\alpha\beta = 1$

After Ramanujan, N.S. Koshliakov was another mathematician to do significant research in this area. Besides using contour integration, Mellin transforms and several summation formulas that he developed, he frequently used a method similar to that developed by Ramanujan in [74], [76, pp. 72–77] to obtain old and new transformation formulas of the form $F(\alpha) = F(\beta)$, where $\alpha\beta = k$ for some constant k. He also obtained deep generalizations of many well-known formulas of Ramanujan and of G.H. Hardy (such as [50, Equation (2)]), some of them being analogues in rational and number fields. See [58, 59, 60, 62, 63, 64, 65]. In [61, 57], Koshliakov used Fourier's integral theorem to obtain expressions for the Riemann Ξ-function, a method also enunciated by Ramanujan [74]. Around the same time, W.L. Ferrar [38] also worked on transformation formulas of the above kind.

3.1 General strategy for deriving transformation formulas from integrals involving the Riemann Ξ-function

In Chapter 2, we gave a new proof of Theorem 2.1.1 and then another proof of the first equality in Theorem 2.1.1 through the approach suggested by Guinand. In this chapter, we show that the first equality in Theorem 2.1.1 can be thought of as being generated from the integral involving the Riemann Ξ-function that is present in the theorem. In other words, if we evaluate this integral to be one of the other two expressions occurring in the theorem, then the other expression can be obtained just by replacing α by β in the proved identity. This idea allows us to obtain more transformation formulas by considering other integrals involving the Riemann Ξ-function or their generalizations, and shows that Ramanujan's theorem (Theorem 2.1.1) is not an isolated example but rather a prototype of this general phenomenon.

In this chapter, we study an integral involving the Riemann Ξ-function of a general type whose one special case is the integral present in Theorem 2.1.1. The basic strategy is to convert this integral into an equivalent line integral and then employ residue calculus and the theory of Mellin transforms. This general integral and the approach of converting it into a complex integral is given in [81, p. 35]. Using this approach, we show that various well-known formulas in the literature can be unified in the sense that all can be derived by evaluating this general integral. The contents of this chapter were published in [30].

3.2 Unification of formulas of S. Ramanujan, N.S. Koshliakov, W.L. Ferrar and G.H. Hardy

In the year 1929, Koshliakov [56] discovered a result now remembered as Koshliakov's formula. To state his theorem, let $K_\nu(z)$ denote the modified Bessel function of order ν, and let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if γ denotes Euler's constant and $a > 0$,

$$\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi a n) = \frac{1}{a}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \quad (3.2.1)$$

Koshliakov's formula can be considered as an analogue of the familiar transformation formula for the classical theta function [20, p. 122]. Later in 1936, W.L. Ferrar [38] showed that Koshliakov's formula is equivalent to the functional equation for $\zeta^2(s)$. Ferrar rephrased Koshliakov's formula in the form $F(\alpha) = F(\beta)$, where $\alpha\beta = 1$, given below.

Theorem 3.2.1. *If $K_\nu(z)$, $d(n)$ and γ are defined as before and if α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\beta)\right). \tag{3.2.2}$$

Actually Ramanujan had discovered Koshliakov's formula before Koshliakov, as can be seen from page 253 of Ramanujan's Lost Notebook [78]. See [21] for details.

Ferrar [38] demonstrated some other solutions of the general equation $F(\alpha) = F(\beta)$, where $\alpha\beta = 1$. For example,

Theorem 3.2.2. *If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned}&\sqrt{\alpha}\left(-\gamma+\log 16\pi-2\log\alpha+2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\alpha^2n^2}{2}}K_0\left(\frac{\pi\alpha^2n^2}{2}\right)-\frac{1}{n\alpha}\right)\right)\\&=\sqrt{\beta}\left(-\gamma+\log 16\pi-2\log\beta+2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\beta^2n^2}{2}}K_0\left(\frac{\pi\beta^2n^2}{2}\right)-\frac{1}{n\beta}\right)\right).\end{aligned} \tag{3.2.3}$$

Ferrar's method in [38] is general in the sense that it applies to any Dirichlet series having a functional equation. Here, we give extended versions of Koshliakov's and Ferrar's formulas. By an 'extended version', we mean that the identity known before is linked to an integral involving the Riemann Ξ function. These extended versions are as follows.

Theorem 3.2.3 (Extended version of Koshliakov's formula)**.** *If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned}&\sqrt{\alpha}\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta}-4\sum_{n=1}^{\infty}d(n)K_0(2\pi n\beta)\right)\\&=-\frac{32}{\pi}\int_0^{\infty}\frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2\cos\left(\frac{1}{2}t\log\alpha\right)\,dt}{(1+t^2)^2}.\end{aligned} \tag{3.2.4}$$

Theorem 3.2.4 (Extended version of Ferrar's formula)**.** *If α and β are positive numbers* *such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(\frac{-\gamma+\log 16\pi+2\log\alpha}{\alpha}-2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\alpha^2n^2}{2}}K_0\left(\frac{\pi\alpha^2n^2}{2}\right)-\frac{1}{n\alpha}\right)\right)$$
$$=\sqrt{\beta}\left(\frac{-\gamma+\log 16\pi+2\log\beta}{\beta}-2\sum_{n=1}^{\infty}\left(e^{\frac{\pi\beta^2n^2}{2}}K_0\left(\frac{\pi\beta^2n^2}{2}\right)-\frac{1}{n\beta}\right)\right)$$
$$=4\pi^{-\frac{3}{2}}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \tag{3.2.5}$$

Two further examples of a transformation formula and an integral involving the Riemann Ξ-function associated with it, namely equations (3.2.7) and (3.2.10), can be easily derived from Ramanujan's formula (Equation (3.2.6) below) and Hardy's formula (Equation (3.2.9) below) respectively. In [74], Ramanujan derives the identity for real n,

$$e^{-n}-4\pi e^{-3n}\int_0^{\infty}\frac{xe^{-\pi x^2e^{-4n}}}{e^{2\pi x}-1}\,dx=\frac{1}{4\pi\sqrt{\pi}}\int_0^{\infty}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos nt\,dt. \tag{3.2.6}$$

Letting $n = \frac{1}{2}\log\alpha$ in (3.2.6) and noting that the integral on the right-hand side is invariant under the map $\alpha \to \beta$, where $\alpha\beta = 1$, we deduce the following result.

Theorem 3.2.5. *If α and β are two positive numbers such that $\alpha\beta = 1$, then*

$$\alpha^{-\frac{1}{2}}-4\pi\alpha^{-\frac{3}{2}}\int_0^{\infty}\frac{xe^{-\frac{\pi x^2}{\alpha^2}}}{e^{2\pi x}-1}\,dx=\beta^{-\frac{1}{2}}-4\pi\beta^{-\frac{3}{2}}\int_0^{\infty}\frac{xe^{-\frac{\pi x^2}{\beta^2}}}{e^{2\pi x}-1}\,dx$$
$$=\frac{1}{4\pi\sqrt{\pi}}\int_0^{\infty}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt. \tag{3.2.7}$$

The first equality in the above formula can be easily seen to be equivalent to the following well-known identity of Ramanujan.

Theorem 3.2.6. *If α and β be any two positive numbers such that $\alpha\beta = \pi^2$, then*

$$\alpha^{-\frac{1}{4}}\left(1+4\alpha\int_0^{\infty}\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right)=\beta^{-\frac{1}{4}}\left(1+4\beta\int_0^{\infty}\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right). \tag{3.2.8}$$

Ramanujan discussed (3.2.8) in [74], just after proving (3.2.6). Another proof of this identity can be seen in a paper of Ramanujan [77]. It also appears in Ramanujan's first letter to Hardy [76, p. xxvi]. It is also in a problem Ramanujan submitted to the Journal of Indian Mathematical Society. Further, this result was also established by C. T. Preece [73]. See [14, p. 291] for more details.

Another example of such a function F can be easily derived from an identity found in a 1915 paper of G.H. Hardy [50] (see (3.2.9) below) in the Quarterly Journal of Mathematics, immediately following Ramanujan's paper [74]. Interestingly, this short note is not reproduced in any of the seven volumes of the Collected Papers of G.H. Hardy (see [49, pp. 691–692] for example). In this note, Hardy says that the integral on the right-hand side in Ramanujan's formula (3.2.6) can be used to prove Hardy's result that there are infinitely many zeros of $\zeta(s)$ on the critical line Re $s = \frac{1}{2}$, and then he concludes the note by giving (3.2.9) below, which he says is not unlike (3.2.6). It turns out that there is a small error in the original formula given by Hardy which was first observed by Koshliakov (see [59] for example), namely, the sign of $\dfrac{\gamma}{2}$ in it should be $+$ and not $-$. Hardy did not give a proof of his formula. Later, Koshliakov proved it in several of his papers [59, Equations 18, 19], [61, Equation 20], [64, Equation 30.5], [65, Equation 34.10][1]. He also gave two different generalizations of Hardy's formula; one in [64, Equation 30.4] and [65, Equation 34.1], and another in [62, Equation 27]. We will sketch a proof of this formula in Section 5.

Theorem 3.2.7 (Correct version of Hardy's claim)**.** *For n real, we have*

$$\int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt = \frac{1}{4}e^{-n}\left(2n+\frac{1}{2}\gamma+\frac{1}{2}\log\pi+\log 2\right)+\tfrac{1}{2}e^{n}\int_0^\infty \psi(x+1)e^{-\pi x^2e^{4n}}\,dx, \tag{3.2.9}$$

where $\psi(x)$ is defined in (2.1.2)*.*

Now letting $n = \frac{1}{2}\log\alpha$ in (3.2.9) and noting that the integral on the right-hand side is invariant under the map $\alpha \to \beta$, where $\alpha\beta = 1$, we have another example of a function F

[1] The formula here contains a typo, as the factor $\frac{1}{2}\log 2\pi$ should be $\frac{1}{2}\log\pi$.

satisfying $F(\alpha)=F(\beta)$, where $\alpha\beta=1$.

Theorem 3.2.8. *If α and β are two positive numbers such that $\alpha\beta=1$, then*

$$\begin{aligned}
&\frac{\sqrt{\alpha}}{2}\int_0^\infty \psi(x+1)e^{-\pi\alpha^2x^2}\,dx+\frac{1}{4\sqrt{\alpha}}(\log\alpha+\tfrac{1}{2}\gamma+\tfrac{1}{2}\log\pi+\log 2)\\
&=\frac{\sqrt{\beta}}{2}\int_0^\infty \psi(x+1)e^{-\pi\beta^2x^2}\,dx+\frac{1}{4\sqrt{\beta}}(\log\beta+\tfrac{1}{2}\gamma+\tfrac{1}{2}\log\pi+\log 2)\\
&=\int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{\cosh\frac{1}{2}\pi t}\,dt. \qquad (3.2.10)
\end{aligned}$$

3.3 General integral involving the Riemann Ξ-function

The general integral involving the Riemann Ξ-function that we discussed in the beginning of this chapter is $\int_0^\infty f(t)\Xi(t)\cos nt\,dt$, where n is real and $f(t)$ is defined by

$$f(t)=|\phi(it)|^2=\phi(it)\phi(-it), \qquad (3.3.1)$$

where ϕ is analytic in t. For self-containedness, we prove a formula found in [81, p. 35] which transforms the above integral into an equivalent line integral which then allows us to use residue calculus and the theory of Mellin transforms for its evaluation. Let $y=e^n$ for n real. Then,

$$\begin{aligned}
\int_0^\infty f(t)\Xi(t)\cos nt\,dt&=\frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it}\,dt\\
&=\frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\xi\left(\tfrac{1}{2}+it\right)y^{it}\,dt\\
&=\frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi(s-\tfrac{1}{2})\phi(\tfrac{1}{2}-s)\xi(s)y^s\,ds. \qquad (3.3.2)
\end{aligned}$$

Actually we use (3.3.2) in a slightly different form. Replacing t by $t/2$ on the left-hand side of (3.3.2) and then replacing n by $2n$ in (3.3.2) yields

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos nt\,dt = \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi(s-\tfrac{1}{2})\phi(\tfrac{1}{2}-s)\xi(s)y^s\,ds, \tag{3.3.3}$$

where $y = e^{2n}$. It is (3.3.3) that we will use in the subsequent sections of this chapter.

We now review some basic properties of Mellin transforms. Let $F(z)$ denote the Mellin transform of $f(x)$, i.e.,

$$F(z) = \int_0^\infty x^{z-1}f(x)\,dx. \tag{3.3.4}$$

The inverse Mellin transform is given by [81, p. 33]

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)x^{-z}\,dz, \tag{3.3.5}$$

where c lies in the fundamental strip (or the strip of analyticity) for which $F(z)$ is defined. The Mellin convolution theorem [72, p. 83] states that if $F(z)$ and $G(z)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively, then

$$\int_0^\infty x^{z-1}f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(z-s)\,ds, \tag{3.3.6}$$

c again being in the associated fundamental strip.

Now let $F(z)$ be related to $f(x)$ by (3.3.4) and (3.3.5), where $f(x)$ is locally integrable on $(0,\infty)$, is $O(x^{-a})$ as $x \to 0^+$ and $O(x^{-b})$, where $b > 1$, as $x \to \infty$, and $a < c < b$. Then for $\max\{1, a\} < c < b$, we have

$$\sum_{k=1}^\infty f(kx) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\zeta(s)x^{-s}\,ds, \tag{3.3.7}$$

where $\zeta(s)$ denotes the Riemann zeta function (see [72, p. 117]).

We will also need Stirling's formula on a vertical strip which states that if $s = \sigma + it$, then for $\alpha \leq \sigma \leq \beta$ and $|t| \geq 1$,

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}} |t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right). \tag{3.3.8}$$

3.4 Proofs of formulas of Ferrar, Hardy, Koshliakov and Ramanujan

3.4.1 Ferrar's formula

Here, we give a proof of Theorem 3.2.4. Let

$$f(t) := \frac{2}{\frac{1}{4} + t^2} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \Gamma\left(\frac{1}{4} - \frac{it}{2}\right). \tag{3.4.1}$$

Then using (3.3.1), we see that

$$\phi(s) = \frac{\sqrt{2}}{\left(\frac{1}{2} - s\right)} \Gamma\left(\frac{1}{4} + \frac{s}{2}\right). \tag{3.4.2}$$

Hence from (3.3.3) with $y = e^{2n}$, we find that

$$\begin{aligned} &8 \int_0^\infty \Gamma\left(\frac{1+it}{4}\right) \Gamma\left(\frac{1-it}{4}\right) \Xi\left(\frac{t}{2}\right) \frac{\cos nt}{1+t^2}\, dt \\ &= -\frac{1}{i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \pi^{-\frac{s}{2}} \Gamma^2\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \zeta(s) y^s\, ds. \end{aligned} \tag{3.4.3}$$

While examining the integral in the last expression in (3.4.3), we shift the line of integration from Re $s = \frac{1}{2}$ to Re $s = 1 + \delta$, for some $\delta \in (0, 2)$, so that we can use (1.0.2). But while doing that, we need to take care of the pole of order 2 at $s = 1$ of the integrand in the last expression in (3.4.3).

Let $T > 0$ denote a real number. Then by the residue theorem, we know that

$$\begin{aligned}
&\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds \\
&= \left[\int_{\frac{1}{2}-iT}^{1+\delta-iT} + \int_{1+\delta-iT}^{1+\delta+iT} + \int_{1+\delta+iT}^{\frac{1}{2}+iT}\right] \pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds \\
&- 2\pi i \lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\right). \qquad (3.4.4)
\end{aligned}$$

Using the product rule for differentiation and simplifying, we have

$$\begin{aligned}
&\frac{d}{ds}\left((s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)\right) \\
&= (s-1)\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(2\zeta(s) - \frac{1}{2}(s-1)\psi\left(\frac{1-s}{2}\right)\zeta(s) + (s-1)\zeta'(s)\right) \\
&\quad + (s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\log\left(\frac{y}{\sqrt{\pi}}\right)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s) \\
&\quad + (s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^s\Gamma^2\left(\frac{s}{2}\right)\psi\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s) \\
&= f_1(s) + f_2(s) + f_3(s), \qquad (3.4.5)
\end{aligned}$$

say.

Now we need the well-known Laurent expansion [43, p. 944, formula 8.321, no. 1]

$$\Gamma(s) = \frac{1}{s} - \gamma + \cdots, \qquad (3.4.6)$$

so that

$$\psi\left(\frac{1-s}{2}\right) = \frac{2}{s-1} - \gamma + \cdots. \qquad (3.4.7)$$

From [81, p. 20], we have

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots. \qquad (3.4.8)$$

Then from (3.4.8) and (3.4.7), we have

$$\lim_{s\to 1}\left(2\zeta(s)-\frac{1}{2}(s-1)\psi\left(\frac{1-s}{2}\right)\zeta(s)+(s-1)\zeta^{'}(s)\right)=\frac{3\gamma}{2}, \qquad (3.4.9)$$

so that from (3.4.5) and (3.4.9), we obtain

$$\begin{aligned}\lim_{s\to 1} f_1(s) &= \lim_{s\to 1}(s-1)\left(\frac{y}{\sqrt{\pi}}\right)^{s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\\ &\quad\times\left(2\zeta(s)-\frac{1}{2}(s-1)\psi\left(\frac{1-s}{2}\right)\zeta(s)+(s-1)\zeta^{'}(s)\right)\\ &= -3y\gamma\sqrt{\pi}. \qquad (3.4.10)\end{aligned}$$

Also,

$$\lim_{s\to 1} f_2(s)=\lim_{s\to 1}(s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^{s}\log\left(\frac{y}{\sqrt{\pi}}\right)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)=-2y\sqrt{\pi}\log\left(\frac{y}{\sqrt{\pi}}\right), \qquad (3.4.11)$$

and using $\frac{\Gamma^{'}}{\Gamma}\left(\frac{1}{2}\right)=-\gamma-2\log 2$ [43, p. 895, formula 8.366, no. 2], we find that

$$\lim_{s\to 1} f_3(s)=\lim_{s\to 1}(s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^{s}\Gamma^2\left(\frac{s}{2}\right)\psi\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)=-2y\sqrt{\pi}\left(-\gamma-2\log 2\right). \qquad (3.4.12)$$

Finally from (3.4.10), (3.4.11) and (3.4.12), we deduce that

$$\begin{aligned}&\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\left(\frac{y}{\sqrt{\pi}}\right)^{-s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)\right)\\ &= -3y\gamma\sqrt{\pi}-2y\sqrt{\pi}\log\left(\frac{y}{\sqrt{\pi}}\right)-2y\sqrt{\pi}\left(-\gamma-2\log 2\right)\\ &= y\sqrt{\pi}\left(\log 16\pi-2\log y-\gamma\right). \qquad (3.4.13)\end{aligned}$$

Let $T\to\infty$ in (3.4.4). Then the integrals along the horizontal segments $[\frac{1}{2}-iT, 1+\delta-iT]$ and $[1+\delta+iT, \frac{1}{2}+iT]$ tend to 0 as can be seen by an application of Stirling's formula. Next using (1.0.2), we have

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty} \pi^{-\frac{s}{2}}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)y^s\,ds = \sum_{m=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds, \tag{3.4.14}$$

where we have interchanged the order of summation and integration because of absolute convergence. Now from [70, p. 115, formula 11.4], we know that if $\pm$ Re $\nu < c =$ Re $s < 1/2$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\pi^{-\frac{1}{2}}(2a)^{-s}\cos(\pi\nu)\Gamma\left(\tfrac{1}{2}-s\right)\Gamma(s+\nu)\Gamma(s-\nu)x^{-s}\,ds = e^{ax}K_{\nu}(ax). \tag{3.4.15}$$

Letting $\nu = 0$, and replacing s by $s/2$ and a by $a/2$ in (3.4.15), we find that for $0 < c =$ Re $s < 1$,

$$\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} a^{-\frac{s}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)x^{-\frac{s}{2}}\,ds = \pi^{\frac{1}{2}}e^{\frac{1}{2}ax}K_0\left(\frac{1}{2}ax\right). \tag{3.4.16}$$

Now let $a = 1$ and $x = \pi m^2/y^2$ in (3.4.16). Then for $0 < c =$ Re $s < 1$, we have

$$\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds = 4\pi^{\frac{3}{2}}ie^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right). \tag{3.4.17}$$

But since (3.4.17) is valid only for $0 <$ Re $s < 1$, in order to simplify the integral on the right-hand side of (3.4.14), we need to shift the line of integration from Re $s = 1+\delta$, where $\delta \in (0, 2)$ to Re $s = c$, where $c \in (0, 1)$ and then use the residue theorem. While doing that, we encounter a pole of order 1 at $s = 1$ of the integrand on the right-hand side of (3.4.14). Thus by another application of the residue theorem, we see that

$$\begin{aligned}&\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds\\ &=\int_{c-i\infty}^{c+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds + 2\pi i\lim_{s\to 1}(s-1)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}.\end{aligned} \tag{3.4.18}$$

From Stirling's formula and the reflection formula for the gamma function,

$$\lim_{s\to 1}(s-1)\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}=-\frac{2\sqrt{\pi}}{m/y}. \tag{3.4.19}$$

Hence from (3.4.17), (3.4.18) and (3.4.19), we find that

$$\int_{1+\delta-i\infty}^{1+\delta+i\infty}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{m\sqrt{\pi}}{y}\right)^{-s}ds=2\pi i\left(2\sqrt{\pi}\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right)-\frac{1}{m/y}\right)\right). \tag{3.4.20}$$

Then from (3.4.3), (3.4.4), (3.4.13) and (3.4.20), we see that

$$\begin{aligned}
&8\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos nt}{1+t^2}\,dt\\
&=\frac{-1}{i\sqrt{y}}\left(2\pi i\left(2\sqrt{\pi}\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right)-\frac{1}{m/y}\right)-y\sqrt{\pi}\left(\log 16\pi-2\log y-\gamma\right)\right)\right)\\
&=\frac{2\pi^{3/2}}{\sqrt{y}}\left(y\left(\log 16\pi-2\log y-\gamma\right)-2\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2}{2y^2}}K_0\left(\frac{\pi m^2}{2y^2}\right)-\frac{1}{m/y}\right)\right).
\end{aligned} \tag{3.4.21}$$

Now letting $n=\frac{1}{2}\log\alpha$ in (3.4.21), noting that $y=e^{2n}$ and $\alpha\beta=1$ and then simplifying, we observe that

$$\begin{aligned}
&4\pi^{-3/2}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt\\
&=\sqrt{\beta}\left(\frac{\log 16\pi+2\log\beta-\gamma}{\beta}-2\sum_{m=1}^{\infty}\left(e^{\frac{\pi m^2\beta^2}{2}}K_0\left(\frac{\pi m^2\beta^2}{2}\right)-\frac{1}{m\beta}\right)\right).
\end{aligned} \tag{3.4.22}$$

Now switching α and β in (3.4.22) and combining with (3.4.22), we arrive at (3.2.5), since the left-hand side of (3.4.22) is invariant under the map $\alpha\to\beta$.

3.4.2 Hardy's formula

Here we give a brief sketch of a proof of Hardy's formula (3.2.9). Let

$$f(t) := \frac{1}{32\pi^2}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right). \tag{3.4.23}$$

Using (3.3.1), we observe that

$$\phi(s) = \frac{1}{4\sqrt{2}\pi}\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{s}{2}\right), \tag{3.4.24}$$

and using (1.0.8), we find that

$$\begin{aligned} f\left(\frac{t}{2}\right) &= \frac{1}{32\pi^2}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\\ &= \frac{1}{1+t^2}\frac{1}{2\sin\left(\pi\left(\frac{1+it}{4}\right)\right)\sin\left(\pi\left(\frac{1-it}{4}\right)\right)}\\ &= \frac{1}{1+t^2}\frac{1}{\cos\left(\frac{i\pi t}{2}\right)-\cos\left(\frac{\pi}{2}\right)}\\ &= \frac{1}{(1+t^2)\cosh\frac{1}{2}\pi t}. \end{aligned} \tag{3.4.25}$$

Hence from (3.3.3) with $y = e^{2n}$, (1.0.9) and (1.0.5), we see that

$$\begin{aligned} \int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt &= \frac{1}{32\pi^2 i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{-s}{2}\right)\xi(s)y^s\,ds\\ &= \frac{1}{4\pi i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds, \end{aligned} \tag{3.4.26}$$

To examine the integral in the last expression in (3.4.26), we wish to move the line of integration from $\text{Re } s = \frac{1}{2}$ to $\text{Re } s = 1+\delta$, for some $\delta \in (0,1)$, so that we can use (1.0.2). In this process, we encounter the pole of order 2 of the integrand at $s = 1$ in the last expression in (3.4.26).

Let $T > 0$ denote a real number. Then by the residue theorem, we know that

$$\begin{aligned}
&\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds \\
&= \left[\int_{\frac{1}{2}-iT}^{1+\delta-iT} + \int_{1+\delta-iT}^{1+\delta+iT} + \int_{1+\delta+iT}^{\frac{1}{2}+iT}\right] \Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds \\
&\quad - 2\pi i \lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\right). \qquad (3.4.27)
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\right) \\
&= \left(\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\right)\right)\xi(s)y^s + \left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi'(s)y^s\right) \\
&\quad + (s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\log y. \qquad (3.4.28)
\end{aligned}$$

Using (1.0.8) to simplify the first expression on the right-hand side of (3.4.28) and then using L'Hopital's rule twice, we easily see that

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\right) = -1. \qquad (3.4.29)$$

Now $\Gamma(s)$ has residue $\frac{(-1)^n}{n!}$ at $s = -n$ where n is a positive integer [43, p. 883, formula 8.310, no. 2]. Hence we see that

$$\lim_{s\to 1}(s-1)\Gamma(-s) = 1. \qquad (3.4.30)$$

Also from [29, pp. 80–81], we know that

$$\xi(1) = \frac{1}{2}, \qquad (3.4.31)$$

and

$$-\frac{\xi'(1)}{\xi(1)} = -\frac{\gamma}{2} - 1 + \frac{1}{2}\log 4\pi, \tag{3.4.32}$$

so that

$$\xi'(1) = \frac{\gamma}{4} + \frac{1}{2} - \frac{1}{4}\log 4\pi. \tag{3.4.33}$$

Thus from (3.4.28), (3.4.29), (3.4.30), (3.4.31) and (3.4.33), we deduce that

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\Gamma(s-1)\Gamma(-s)\xi(s)y^s\right) = \frac{y}{4}\left(\gamma - \log 4\pi + 2\log y\right). \tag{3.4.34}$$

Now let $T \to \infty$ in (3.4.27). From Stirling's formula, we see that the integrals along the horizontal segments $[\frac{1}{2} - iT, 1+\delta - iT]$ and $[1+\delta+iT, \frac{1}{2}+iT]$ tend to 0. Finally,

$$\begin{aligned}
&\int_{1+\delta-i\infty}^{1+\delta+i\infty} \Gamma(s-1)\Gamma(-s)(s-1)\Gamma\left(1+\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds \\
&= \int_{1+\delta-i\infty}^{1+\delta+i\infty} -\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)y^s\,ds \\
&= \int_{1+\delta-i\infty}^{1+\delta+i\infty} -\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\sum_{k=1}^{\infty}\frac{1}{k^s}y^s\,ds \\
&= \sum_{k=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty} -\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s} ds,
\end{aligned} \tag{3.4.35}$$

where in the last step, we have interchanged the order of summation and integration because of absolute convergence. Now employing a change of variable $s \to s+1$, we see that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty} -\frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s} ds \\
&= \sum_{k=1}^{\infty}\frac{y}{k\sqrt{\pi}}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\pi}{2\sin\pi s}\Gamma\left(\frac{s+1}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s} ds.
\end{aligned} \tag{3.4.36}$$

Next, let $\mathcal{M}(f(x);s) := F(s)$ denote the Mellin transform of $f(x)$ and let $\mathcal{M}^{-1}(F(s);x) =$

$f(x)$ be the inverse Mellin transform of $F(x)$.

$$\mathcal{M}^{-1}(F(s);x) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}\,ds, \tag{3.4.37}$$

where $c = \text{Re } s$ lies in the fundamental strip (or the strip of analyticity) for which $F(s)$ is defined. Since [72, p. 404]

$$\mathcal{M}^{-1}\left(F(s+a);x\right) = x^a f(x), \tag{3.4.38}$$

and [72, p. 406]

$$\mathcal{M}^{-1}\left(\frac{1}{2}\Gamma\left(\frac{s}{2}\right);x\right) = e^{-x^2}, \tag{3.4.39}$$

for Re $s > 0$, we see that

$$\mathcal{M}^{-1}\left(\frac{1}{2}\Gamma\left(\frac{s+1}{2}\right);x\right) = xe^{-x^2}. \tag{3.4.40}$$

Also for $0 < \text{Re } s < 1$, we have [72, p. 91, eqn. (3.3.10)]

$$\mathcal{M}^{-1}\left(\frac{\pi}{\sin \pi s};x\right) = \frac{1}{1+x}. \tag{3.4.41}$$

But from [72, p.83, eqn. (3.1.13)], we observe that

$$\mathcal{M}^{-1}\left(F(s)G(s);w\right) = \int_0^\infty f(x)g\left(\frac{w}{x}\right)\frac{dx}{x}, \tag{3.4.42}$$

where $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively.

Thus from (2.1.2), (3.4.40), (3.4.41) and (3.4.42), we see that for $0 < \delta < 1$,

$$\begin{aligned}
&\sum_{k=1}^\infty \frac{y}{k\sqrt{\pi}}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\pi}{2\sin \pi s}\Gamma\left(\frac{s+1}{2}\right)\left(\frac{k\sqrt{\pi}}{y}\right)^{-s}\,ds\\
&= 2\pi i\sum_{k=1}^\infty \frac{y}{k\sqrt{\pi}}\int_0^\infty \frac{xe^{-x^2}}{1+\frac{k\sqrt{\pi}}{xy}}\frac{dx}{x}
\end{aligned}$$

$$= 2\pi i \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} \frac{x e^{-\pi x^2 y^{-2}}}{x+k} \, dx$$
$$= 2\pi i \int_0^{\infty} e^{-\pi x^2 y^{-2}} \sum_{k=1}^{\infty} \frac{x}{k(x+k)} \, dx$$
$$= 2\pi i \int_0^{\infty} e^{-\pi x^2 y^{-2}} \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{x+1+k} \right) dx$$
$$= 2\pi i \int_0^{\infty} e^{-\pi x^2 y^{-2}} \left(\psi(x+1) + \gamma \right) dx,$$
$$= 2\pi i \left(\frac{\gamma y}{2} + \int_0^{\infty} \psi(x+1) e^{-\pi x^2 y^{-2}} \, dx \right), \tag{3.4.43}$$

since $\int_0^{\infty} e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$. Here again the interchange of the order of integration and summation is justified by absolute convergence. Thus from (3.4.26), (3.4.27), (3.4.34), (3.4.35), (3.4.36) and (3.4.43), we deduce that

$$\int_0^{\infty} \frac{\Xi(\frac{1}{2}t)}{1+t^2} \frac{\cos nt}{\cosh \frac{1}{2}\pi t} \, dt$$
$$= \frac{1}{4\pi i \sqrt{y}} \left(2\pi i \left(\frac{\gamma y}{2} + \int_0^{\infty} \psi(x+1) e^{-\pi x^2 y^{-2}} \, dx \right) - \frac{2\pi i y}{4} \left(\gamma - \log 4\pi + 2 \log y \right) \right)$$
$$= \frac{e^n}{4} \left(-2n + \frac{1}{2}\gamma + \frac{1}{2} \log \pi + \log 2 \right) + \frac{e^{-n}}{2} \int_0^{\infty} \psi(x+1) e^{-\pi x^2 e^{-4n}} \, dx. \tag{3.4.44}$$

Finally, since the left-hand side of (3.4.44) is an even function of n, replacing n by $-n$ in (3.4.44), we obtain (3.2.9). This completes the proof.

3.4.3 Koshliakov's formula

We can prove Theorem 3.2.3 by substituting $f(t) = 4\Xi(t)(\frac{1}{4}+t^2)^{-2}$ in (3.3.3), but we defer its proof till the next chapter since there we obtain it as a special case of a more general theorem, namely, the extended version of the Ramanujan-Guinand formula. However, we present a corollary of Theorem 3.2.3.

Corollary 3.4.1. *Let* $G(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. *Then,*

$$4\sum_{n=1}^{\infty} d(n)K_0(2\pi n) = \gamma - \log(4\pi) + \int_0^{\infty} \left(G(x^2) - 1 - \frac{1}{x} \right)^2 dx. \tag{3.4.45}$$

Proof. Letting $\alpha = 1$ in (3.2.4), we see that

$$\gamma - \log(4\pi) - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n) = -\frac{32}{\pi}\int_0^{\infty} \frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2}{(1+t^2)^2}\, dt. \tag{3.4.46}$$

Now from Theorem 4 in [53], $0 < \sigma < 1$,

$$\int_{-\infty}^{\infty} \left|\zeta(\sigma + it)\Gamma(\tfrac{1}{2}\sigma + \tfrac{1}{2}it)\right|^2 dt = 2\pi^{1+\sigma}\int_0^{\infty} (uG(u^2) - 1 - u)^2 u^{2\sigma - 3}\, du. \tag{3.4.47}$$

Substituting $\sigma = 1/2$ in (3.4.47) and writing the left-hand side in terms of the Riemann Ξ-function, we see that

$$\int_0^{\infty} \frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2}{(1+t^2)^2}\, dt = \frac{\pi}{32}\int_0^{\infty} \left(G(x^2) - 1 - \frac{1}{x} \right)^2 dx. \tag{3.4.48}$$

Combining (3.4.46) and (3.4.48), we obtain (3.4.45). □

Remark. Equation (3.4.45) is not unlike the first equality of a formula on page 254 in Ramanujan's Lost Notebook [78], given below in a different version in [21], i.e.,

$$\int_0^{\infty} \frac{dx}{x(e^{2\pi x} - 1)(e^{2\pi a/x} - 1)} = 2\sum_{n=1}^{\infty} d(n)K_0(4\pi\sqrt{an}) \tag{3.4.49}$$

$$= \frac{a}{\pi^2}\sum_{n=1}^{\infty} \frac{d(n)\log(a/n)}{a^2 - n^2} - \frac{1}{2}\gamma - \left(\frac{1}{4} + \frac{1}{4\pi^2 a}\right)\log a - \frac{\log(2\pi)}{2\pi^2 a}. \tag{3.4.50}$$

3.4.4 A third proof of Theorem 2.1.1

Again, we can prove Theorem by substituting $f(t) := \dfrac{\Xi(t)}{\frac{1}{4} + t^2}\Gamma\left(-\frac{1}{4} + \frac{it}{2}\right)\Gamma\left(-\frac{1}{4} - \frac{it}{2}\right)$ in (3.3.3). However, we defer it till the next chapter and obtain it as a special case of a general theorem.

Chapter 4

Transformation formulas of the form $F(z,\alpha) = F(z,\beta)$ where $\alpha\beta = 1$

In this chapter, we give a one-parameter generalization of the general integral involving the Riemann Ξ-function considered in Chapter 3. The general form of such an integral is

$$\int_0^\infty f\left(z, \frac{t}{2}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \cos \mu t \, dt. \tag{4.0.1}$$

The invariance of the above integral under the transformation $\alpha \to \beta$, where $\alpha\beta = 1$, gives rise to transformation formulas of the form $F(z,\alpha) = F(z,\beta)$. Again, a prototypical example of such an integral was given by Ramanujan in [74], where he considered the following integral for n real,

$$\int_0^\infty \Gamma\left(\frac{s-1+it}{4}\right) \Gamma\left(\frac{s-1-it}{4}\right) \Xi\left(\frac{t+is}{2}\right) \Xi\left(\frac{t-is}{2}\right) \frac{\cos nt}{(s+1)^2+t^2} \, dt, \tag{4.0.2}$$

and evaluated it in terms of another integral for various values of s, for example, in Re $s > 1$, $-1 < \text{Re } s < 1$, $-3 < \text{Re } s < -1$ and so on. We will discuss more on Ramanujan's evaluations in the last section of this chapter. The results in this chapter are published in [31, 32].

4.1 General form of an integral generating formulas of the type $F(z,\alpha) = F(z,\beta)$

The following theorem generalizes (3.3.2) from Chapter 3.

Theorem 4.1.1. *Let*

$$f(z,t) = \phi(z,it)\phi(z,-it), \tag{4.1.1}$$

where ϕ is analytic in t as a function of a real variable and analytic in z in some complex domain. Assume that the integral in (4.0.1) converges. Also let $y = e^{\mu}$ for μ real. Then,

$$\begin{aligned}&\int_0^\infty f\left(z,t\right)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)\cos\mu t\,dt\\&=\frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^{s}\,ds.\end{aligned} \tag{4.1.2}$$

Proof. Let

$$I(z,\mu) := \int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)\cos\mu t\,dt. \tag{4.1.3}$$

Then,

$$\begin{aligned}I(z,\mu) = \frac{1}{2}\bigg[&\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)y^{it}\,dt\\&+\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)y^{-it}\,dt\bigg].\end{aligned} \tag{4.1.4}$$

Now since $\xi(s) = \xi(1-s)$, where ξ is defined in (1.0.5), we have

$$\begin{aligned}&\Xi\left(\pm t+\frac{iz}{2}\right)=\xi\left(\frac{1}{2}\pm it-\frac{z}{2}\right)=\xi\left(1-\left(\frac{z+1}{2}\mp it\right)\right)=\xi\left(\frac{z+1}{2}\mp it\right),\\&\Xi\left(\pm t-\frac{iz}{2}\right)=\xi\left(\frac{1}{2}\pm it+\frac{z}{2}\right)=\xi\left(\frac{z+1}{2}\pm it\right).\end{aligned} \tag{4.1.5}$$

Hence,

$$\Xi\left(-t\pm\frac{iz}{2}\right)=\Xi\left(t\mp\frac{iz}{2}\right). \tag{4.1.6}$$

From (4.1.4), change of variable $t\to -t$ in the second integral in (4.1.4), (4.1.6) and from the fact that f is an even function of t, we deduce that

$$I(z,\mu) = \frac{1}{2}\int_{-\infty}^{\infty} f(z,t)\Xi\left(t+\frac{iz}{2}\right)\Xi\left(t-\frac{iz}{2}\right)y^{it}\,dt$$
$$= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^{s}\,ds, \tag{4.1.7}$$

where in the penultimate line, we made the change of variable $s = \frac{1}{2} + it$. □

For our purpose here, we replace μ by 2μ in (4.1.2) and then t by $t/2$ on the left-hand side of (4.1.2). Thus with $y = e^{2\mu}$, we find that

$$\int_{0}^{\infty} f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\mu t\,dt$$
$$= \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^{s}\,ds. \tag{4.1.8}$$

It is this equation with which we will be working throughout this chapter.

4.2 Modular transformation formulas involving the Hurwitz zeta function through Ramanujan's integral (4.0.2)

In this section, we derive three new formulas of the form $F(z,\alpha) = F(z,\beta)$, where $\alpha\beta = 1$. One of the three is a generalization of Ramanujan's transformation formula, i.e., Theorem 2.1.1.

4.3 Generalization of Theorem 2.1.1

The Hurwitz zeta function is defined for Re $z > 1$ by

$$\zeta(z, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^z}. \tag{4.3.1}$$

It can also be analytically continued to the entire z-plane except for a simple pole at $z = 1$. The asymptotic expansion of $\zeta(z, x)$ [69, p. 25] for large $|x|$ and $|\arg x| < \pi$ is given by

$$\zeta(z, x) = \frac{1}{\Gamma(z)} \left(x^{1-z}\Gamma(z-1) + \frac{1}{2}\Gamma(z)x^{-z} + \sum_{k=1}^{m-1} \frac{B_{2k}}{(2k)!}\Gamma(z+2k-1)x^{-2k-z+1} \right) + O(x^{-2m-z-1}), \tag{4.3.2}$$

where B_{2k} is the $2k^{\text{th}}$ Bernoulli number.

Theorem 4.3.1. *Let* $-1 < \text{Re } z < 1$. *Define* $\varphi(z, x)$ *by*

$$\varphi(z, x) = \zeta(z+1, x) - \frac{x^{-z}}{z} - \frac{1}{2}x^{-z-1}, \tag{4.3.3}$$

where $\zeta(z, x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta = 1$,

$$\alpha^{\frac{z+1}{2}} \left(\sum_{n=1}^{\infty} \varphi(z, n\alpha) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z} \right) = \beta^{\frac{z+1}{2}} \left(\sum_{n=1}^{\infty} \varphi(z, n\beta) - \frac{\zeta(z+1)}{2\beta^{z+1}} - \frac{\zeta(z)}{\beta z} \right)$$
$$= \frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)} \int_0^{\infty} \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\, dt, \tag{4.3.4}$$

where $\Xi(t)$ *is defined in* (1.0.6).

As can be easily gleaned from (4.3.3), in the definition of $\varphi(z, x)$, we have subtracted from

$\zeta(z+1,x)$ the first two terms in its asymptotic expansion. In the sequel, we use

$$R_a = R_a(g) \tag{4.3.5}$$

to denote the residue of a function g at a. We choose to write just R_a instead of $R_a(g)$ when the function in consideration is understood and there is not any ambiguity.

Proof. Let

$$f(z,t) = \frac{1}{(4t^2+(z+1)^2)}\Gamma\left(\frac{z-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{z-1}{4}-\frac{it}{2}\right), \tag{4.3.6}$$

so that $\phi(z,s) = \dfrac{1}{(2s+z+1)}\Gamma\left(\dfrac{z-1}{4}+\dfrac{s}{2}\right)$. Then by an application of (4.1.8) with the above f and ϕ, we observe that

$$\begin{aligned}
&\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(z+1)^2+t^2}\,dt\\
&= \frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{1}{\left(\frac{z}{2}+s\right)}\Gamma\left(\frac{z-2}{4}+\frac{s}{2}\right)\frac{1}{\left(\frac{z+2}{2}-s\right)}\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds\\
&= -\frac{1}{16i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(z,s)\left(\frac{\pi}{y}\right)^{-s}\,ds,
\end{aligned} \tag{4.3.7}$$

where

$$\begin{aligned}
H(z,s) :=& \left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\\
&\times\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right),
\end{aligned} \tag{4.3.8}$$

and we have used (1.0.5) in the penultimate line.

To evaluate the integral in the last step in (4.3.7), we want to use the series representation for $\zeta\left(s+\frac{z}{2}\right)$, valid for Re $\left(s+\frac{z}{2}\right)>1$, i.e., Re $s>$ Re $\left(\frac{2-z}{2}\right)$. But since $-1<$ Re $z<1$, we have $0<$ Re $\left(s+\frac{z}{2}\right)<1$. Thus we move the line of integration from Re $s=\frac{1}{2}$ to Re $s=\frac{3}{2}$. In this process, we encounter a pole of order 1 at $s=\frac{z+2}{2}$ (due to $\zeta(s-\frac{z}{2})$) and a pole of order 1 at $s = \frac{2-z}{2}$ (due to $\Gamma(\frac{s}{2} + \frac{z-2}{4})$). Note that the pole of $\zeta\left(s + \frac{z}{2}\right)$ at $s = \frac{2-z}{2}$ is cancelled by the zero of $\left(s + \frac{z-2}{2}\right)$ at $s = \frac{2-z}{2}$, since

$$\lim_{s \to \frac{2-z}{2}} \left(s + \frac{z-2}{2}\right) \zeta\left(s + \frac{z}{2}\right) = 1. \tag{4.3.9}$$

Let $T > 0$ denote a real number. Then by the residue theorem, and the notation in (4.3.5), we know that

$$\begin{aligned}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds = &\left[\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}\right] H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds \\ &- 2\pi i\left(R_{\frac{z+2}{2}} + R_{\frac{2-z}{2}}\right).\end{aligned} \tag{4.3.10}$$

We evaluate the first residue above by using the functional equation of the Gamma function $\Gamma(z+1) = z\Gamma(z)$, (1.0.9) and the facts that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$\lim_{s \to \frac{z+2}{2}} \left(s - \frac{z+2}{2}\right) \zeta\left(s - \frac{z}{2}\right) = 1. \tag{4.3.11}$$

Thus,

$$\begin{aligned}R_{\frac{z+2}{2}} &= 4 \lim_{s \to \frac{z+2}{2}} \left(s - \frac{z+2}{2}\right)\Gamma\left(\frac{s}{2} + \frac{z-2}{4} + 1\right)\Gamma\left(\frac{z}{4} - \frac{s}{2}\right) \\ &\qquad \times \Gamma\left(\frac{s}{2} - \frac{z}{4} + 1\right)\Gamma\left(\frac{s}{2} + \frac{z}{4}\right)\zeta\left(s - \frac{z}{2}\right)\zeta\left(s + \frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s} \\ &= 4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{z}{2} + 1\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{z+1}{2}\right)\zeta(z+1)\left(\frac{\pi}{y}\right)^{-\frac{z}{2}-1} \\ &= -2^{2-z}\pi^{\frac{1-z}{2}}y^{1+\frac{z}{2}}\Gamma(z+1)\zeta(z+1).\end{aligned} \tag{4.3.12}$$

For the second limit, using (1.0.4) and (1.0.8), we have

$$R_{\frac{2-z}{2}} = 4 \lim_{s \to \frac{2-z}{2}} \left(s - \frac{2-z}{2}\right)\Gamma\left(\frac{s}{2} + \frac{z-2}{4} + 1\right)\Gamma\left(\frac{z}{4} - \frac{s}{2}\right)$$

$$\times \Gamma\left(\frac{s}{2}-\frac{z}{4}+1\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s}$$

$$= 4\Gamma\left(\frac{3-z}{2}\right)\Gamma\left(\frac{z-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\zeta(1-z)\pi^{\frac{z}{2}-1}\left(\frac{\pi}{y}\right)^{\frac{z}{2}-1}$$

$$- 4\frac{\pi}{\cos\left(\frac{1}{2}\pi z\right)}\sqrt{\pi}2^{1-z}\pi^{-z}\Gamma(z)\zeta(z)\sin\left(\frac{1}{2}\pi(1-z)\right)\pi^{\frac{z}{2}-1}y^{1-\frac{z}{2}}$$

$$= -2^{3-z}\pi^{\frac{1-z}{2}}y^{1-\frac{z}{2}}\Gamma(z)\zeta(z). \tag{4.3.13}$$

It can easily be seen, by the use of Stirling's formula (3.3.8), that as $T \to \infty$, the integrals along the horizontal segments $[\frac{1}{2}-iT, \frac{3}{2}-iT]$ and $[\frac{3}{2}+iT, \frac{1}{2}+iT]$ tend to zero, i.e.,

$$\lim_{T\to\infty}\int_{\frac{1}{2}\pm iT}^{\frac{3}{2}\pm iT} H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds = 0. \tag{4.3.14}$$

It remains to evaluate

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds. \tag{4.3.15}$$

We first simplify the integrand using (1.0.9) and (1.0.8). Thus,

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds = 4\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac{1}{2}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(1+\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)$$

$$\times \zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)\left(\frac{\pi}{y}\right)^{-s} ds$$

$$= \frac{4\pi^{\frac{3}{2}}}{2^{\frac{z}{2}-1}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\Gamma\left(s+\frac{z}{2}\right)\zeta\left(s-\frac{z}{2}\right)}{2^s\sin\left(\pi\left(\frac{z}{4}-\frac{s}{2}\right)\right)}\left(\frac{\pi m}{y}\right)^{-s} ds, \tag{4.3.16}$$

where we have used (1.0.2) in the penultimate step and then interchanged the order of summation and integration because of absolute convergence. Using (1.0.4) with s replaced by $s-\frac{z}{2}$, we find that

$$\begin{aligned}
&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds\\
&=\frac{4\pi^{\frac{3}{2}}}{2^{\frac{z}{2}-1}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{2^{s-\frac{z}{2}}\pi^{s-\frac{z}{2}-1}\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\sin\left(\frac{\pi}{2}\left(s-\frac{z}{2}\right)\right)\Gamma\left(s+\frac{z}{2}\right)}{2^s\sin\left(\pi\left(\frac{z}{4}-\frac{s}{2}\right)\right)}\\
&\qquad\qquad\times\left(\frac{\pi m}{y}\right)^{-s} ds\\
&=-2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds. \quad (4.3.17)
\end{aligned}$$

Now from [70, p. 203, formula 5.84], we see that for $0<c=\text{Re } s<\text{Re } \nu-1$ and $\text{Re } \alpha>0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} a^{-s}\Gamma(s)\Gamma(\nu-s)\zeta(\nu-s,\alpha)x^{-s}\, ds=\Gamma(\nu)\zeta(\nu,\alpha+ax). \quad (4.3.18)$$

Let $\nu=z+1$, $a=1$, $\alpha=1$ and $x=m/y$. Then noting that $\zeta(s,1)=\zeta(s)$, for $0<c=\text{Re}$ $s<\text{Re } z$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(z+1-s)\zeta(z+1-s)\left(\frac{m}{y}\right)^{-s} ds=\Gamma(z+1)\zeta\left(z+1,1+\frac{m}{y}\right). \quad (4.3.19)$$

Next, replacing s by $s+\frac{z}{2}$, we see that, for $-\text{ Re}\left(\frac{z}{2}\right)<\text{Re } s<\text{Re}\left(\frac{z}{2}\right)$,

$$\begin{aligned}
&\int_{c-\text{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\text{Re}\left(\frac{z}{2}\right)+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds\\
&=2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1,1+\frac{m}{y}\right). \quad (4.3.20)
\end{aligned}$$

Hence in order to use (4.3.20) to simplify the integral in the last expression in (4.3.17), we shift the line of integration from $\text{Re } s=c-\text{Re}\left(\frac{z}{2}\right)$ to $\text{Re } s=\frac{3}{2}$. In doing that, we encounter a pole of order 1 at $s=\frac{z}{2}$ (due to $\zeta\left(1-s+\frac{z}{2}\right)$) and a pole of order 1 at $s=1+\frac{z}{2}$ (due to $\Gamma\left(1-s+\frac{z}{2}\right)$). Let $T>0$ denote a real number. Thus by another application of the residue theorem with the observation that the integrals along the horizontal segments tend to zero as $T \to \infty$, we see that

$$\begin{aligned}
&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds \\
&= \int_{c-\mathrm{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\mathrm{Re}\left(\frac{z}{2}\right)+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds \\
&\quad + 2\pi i\left(R_{\frac{z}{2}} + R_{1+\frac{z}{2}}\right) \\
&= \int_{c-\mathrm{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\mathrm{Re}\left(\frac{z}{2}\right)+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds \\
&\quad + 2\pi i\left(-\left(\frac{m}{y}\right)^{-\frac{z}{2}}\Gamma(z) + \frac{1}{2}\left(\frac{m}{y}\right)^{-1-\frac{z}{2}}\Gamma(1+z)\right). \qquad (4.3.21)
\end{aligned}$$

Now we make use of the simple fact that

$$\zeta(s, a+1) = \zeta(s, a) - a^{-s}, \qquad (4.3.22)$$

which is easily seen first for Re $s > 1$, and then for all s by analytic continuation. Then from (4.3.20), (4.3.21) and (4.3.22), we deduce that

$$\begin{aligned}
&\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds \\
&= 2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\left(\zeta\left(z+1, \frac{m}{y}\right) - \frac{1}{(m/y)^{z+1}} + \frac{1}{2\,(m/y)^{z+1}} - \frac{(m/y)^{-z}}{z}\right) \\
&= 2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\varphi\left(z, \frac{m}{y}\right), \qquad (4.3.23)
\end{aligned}$$

where $\varphi(z, x)$ is defined in (4.3.3). Thus from (4.3.17) and (4.3.23), we see that

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} H(z, s)\left(\frac{\pi}{y}\right)^{-s} ds = -2\pi i 2^{3-z}\pi^{\frac{1-z}{2}}y^{-\frac{z}{2}}\Gamma(z+1)\sum_{m=1}^{\infty}\varphi\left(z, \frac{m}{y}\right). \qquad (4.3.24)$$

From (4.3.7), (4.3.10), (4.3.12), (4.3.13), (4.3.14) and (4.3.24), we observe that

$$\int_0^{\infty} \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(z+1)^2+t^2}\,dt$$
$$= -\frac{1}{16i\sqrt{y}}\Bigg(-2\pi i 2^{3-z}\pi^{\frac{1-z}{2}}y^{-\frac{z}{2}}\Gamma(z+1)\sum_{m=1}^{\infty}\varphi\left(z,\frac{m}{y}\right)$$
$$-2\pi i\Big(-2^{2-z}\pi^{\frac{1-z}{2}}y^{1+\frac{z}{2}}\Gamma(z+1)\zeta(z+1) - 2^{3-z}\pi^{\frac{1-z}{2}}y^{1-\frac{z}{2}}\Gamma(z)\zeta(z)\Big)\Bigg)$$
$$= 2^{-z}\pi^{\frac{3-z}{2}}y^{-\frac{(z+1)}{2}}\Gamma(z+1)\left(\sum_{m=1}^{\infty}\varphi\left(z,\frac{m}{y}\right) - \frac{y^{z+1}\zeta(z+1)}{2} - \frac{y\zeta(z)}{z}\right). \tag{4.3.25}$$

Now let $\mu = \frac{1}{2}\log\alpha$ in (4.3.25) so that $y = \alpha$. Since $\alpha\beta = 1$, (4.3.25) becomes

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty} \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$
$$= \beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\varphi(z,n\beta) - \frac{\zeta(z+1)}{2\beta^{z+1}} - \frac{\zeta(z)}{\beta z}\right). \tag{4.3.26}$$

Finally, switching the roles of α and β in (4.3.26) and then combining the result with (4.3.26), we arrive at (4.3.4), since with $\alpha\beta = 1$, the left-hand side of (4.3.26) is invariant under the map $\alpha \to \beta$. This completes the proof. □

In the following corollary, we derive Theorem 2.1.1 as the limiting case when $z \to 0$ of the above theorem.

Corollary 4.3.2. *If*

$$\lambda(x) := \psi(x) + \frac{1}{2x} - \log x, \tag{4.3.27}$$

and α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{k=1}^{\infty}\lambda(k\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{k=1}^{\infty}\lambda(k\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \quad (4.3.28)$$

Proof. Let $z \to 0$ in (4.3.4). Then Lebesgue's dominated convergence theorem gives

$$\lim_{z\to 0}\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\varphi(z, n\alpha) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z}\right)$$
$$= \lim_{z\to 0}\beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\varphi(z, n\beta) - \frac{\zeta(z+1)}{2\beta^{z+1}} - \frac{\zeta(z)}{\beta z}\right)$$
$$= \frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \quad (4.3.29)$$

Formula (4.3.2) shows that $\sum_{k=1}^{\infty}\varphi(z, k\alpha)$ and $\sum_{k=1}^{\infty}\varphi(z, k\beta)$ converge absolutely and uniformly in a neighborhood of $z = 0$. Therefore,

$$\lim_{z\to 0}\alpha^{\frac{z+1}{2}}\left(\sum_{k=1}^{\infty}\varphi(z, k\alpha) - \frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{\zeta(z)}{\alpha z}\right)$$
$$= \left(\lim_{z\to 0}\alpha^{\frac{z+1}{2}}\right)\left(\sum_{k=1}^{\infty}\left(\lim_{z\to 0}\varphi(z, k\alpha)\right) - \lim_{z\to 0}\left(\frac{\zeta(z+1)}{2\alpha^{z+1}} + \frac{\zeta(z)}{\alpha z}\right)\right)$$
$$= \left(\lim_{z\to 0}\alpha^{\frac{z+1}{2}}\right)\left(\sum_{k=1}^{\infty}\left(\lim_{z\to 0}\varphi(z, k\alpha)\right) - \lim_{z\to 0}\left(\frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{1}{2z\alpha^{z+1}}\right) - \lim_{z\to 0}\left(\frac{1}{2z\alpha^{z+1}} + \frac{\zeta(z)}{\alpha z}\right)\right).$$
$$(4.3.30)$$

It is known [81, p. 16] that

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma. \quad (4.3.31)$$

Hence

$$\lim_{z\to 0}\left(\frac{\zeta(z+1)}{2\alpha^{z+1}} - \frac{1}{2z\alpha^{z+1}}\right) = \frac{\gamma}{2\alpha}. \quad (4.3.32)$$

Then,

$$\begin{aligned}
\lim_{z\to 0}\left(\frac{1}{2z\alpha^{z+1}}+\frac{\zeta(z)}{\alpha z}\right) &= \lim_{z\to 0}\frac{1}{2\alpha^{z+1}}\cdot\lim_{z\to 0}\frac{1+2\alpha^z\zeta(z)}{z}\\
&= \lim_{z\to 0}\frac{1}{2\alpha^{z+1}}\cdot\lim_{z\to 0}\left(2\alpha^z\zeta'(z)+2\zeta(z)\alpha^z\log\alpha\right)\\
&= \frac{1}{2\alpha}\left(2\zeta'(0)+2\zeta(0)\log\alpha\right)\\
&= -\frac{\log(2\pi\alpha)}{2\alpha}, \qquad (4.3.33)
\end{aligned}$$

since $\zeta(0)=-\frac{1}{2}$ and $\zeta'(0)=-\frac{1}{2}\log(2\pi)$ [81, pp. 19–20, eqns. (2.4.3), (2.4.5)]. Since [69, p. 23]

$$\lim_{z\to 0}\left(\zeta(z+1,a)-\frac{1}{z}\right)=-\psi(a), \qquad (4.3.34)$$

it follows that

$$\begin{aligned}
\lim_{z\to 0}\varphi(z,k\alpha) &= \lim_{z\to 0}\left(\zeta(z+1,k\alpha)-\frac{1}{2}(k\alpha)^{-(z+1)}+\frac{(k\alpha)^{-z}}{-z}\right)\\
&= \lim_{z\to 0}\left[\left(\zeta(z+1,k\alpha)-\frac{1}{z}\right)-\frac{(k\alpha)^{-(z+1)}}{2}+\frac{(k\alpha)^{-z}-1}{-z}\right]\\
&= -\psi(k\alpha)-\frac{1}{2k\alpha}+\lim_{z\to 0}\frac{-(k\alpha)^{-z}\log k\alpha}{-1}\\
&= -\psi(k\alpha)-\frac{1}{2k\alpha}+\log(k\alpha)\\
&= -\lambda(k\alpha), \qquad (4.3.35)
\end{aligned}$$

where $\phi(x)$ is defined in (4.3.27).

Hence from (4.3.30), (4.3.32), (4.3.33) and (4.3.35), we find that

$$\lim_{z\to 0}\alpha^{\frac{z+1}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\alpha)-\frac{\zeta(z+1)}{2\alpha^{z+1}}-\frac{\zeta(z)}{\alpha z}\right)=-\sqrt{\alpha}\left(\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{k=1}^{\infty}\lambda(k\alpha)\right). \qquad (4.3.36)$$

Thus from (4.3.29), (4.3.36) and (4.3.36) with α replaced by β, we obtain

$$-\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{k=1}^{\infty}\lambda(k\alpha)\right\}=-\sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{k=1}^{\infty}\lambda(k\beta)\right\}$$
$$=\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt. \quad (4.3.37)$$

This produces (4.3.28). □

4.4 Further modular transformations involving the Hurwitz zeta function

We derive further transformation formulas by evaluating the same integral as in (4.3.4) for values of z other than $-1 < \operatorname{Re} z < 1$.

Theorem 4.4.1. *Let* Re $z > 1$ *and let* $\zeta(z,a)$ *be the Hurwitz zeta function defined in* (4.3.1). *If* α *and* β *are positive numbers such that* $\alpha\beta = 1$, *then*

$$\alpha^{-\frac{(z+1)}{2}}\sum_{k=1}^{\infty}\zeta\left(z+1,1+\frac{k}{\alpha}\right)=\beta^{-\frac{(z+1)}{2}}\sum_{k=1}^{\infty}\zeta\left(z+1,1+\frac{k}{\beta}\right)$$
$$=\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt, \quad (4.4.1)$$

where Ξ *is defined as in* (1.0.6).

A finite version of the first equality in (4.4.1) appears in [24] and is given by

$$n^z\sum_{k=1}^{m}\zeta\left(z,nx+\frac{n(k-1)}{m}\right)=m^z\sum_{k=1}^{n}\zeta\left(z,mx+\frac{m(k-1)}{n}\right), \quad (4.4.2)$$

where Re $z > 1$, m and n are positive integers and x is real. In (4.4.2), let m and n be replaced by mN and nN respectively. Then,

$$\sum_{k=1}^{mN} \zeta\left(z, nNx + \frac{n(k-1)}{m}\right) = \left(\frac{m}{n}\right)^z \sum_{k=1}^{nN} \zeta\left(z, mNx + \frac{m(k-1)}{n}\right) \tag{4.4.3}$$

Next, let $m/n = \alpha$, $n/m = \beta$ and $x = \frac{1}{N}\left(\frac{1}{m} + \frac{1}{n}\right)$, then replace z by $z+1$ and assuming Re $z > 1$, let $N \to \infty$. This yields the first equality in (4.4.1). We now prove Theorem 4.4.1.

Proof. Since the right-hand side of (4.4.1) is exactly the same as that in (4.3.4), the proof as well as many of the calculations in Theorem 4.4.1 are similar, in fact simpler than those of Theorem 4.3.1, and so we will be brief. We use (4.3.7) again. Since Re $z > 1$, we see that Re $\left(s + \frac{z}{2}\right) > 1$ and thus we can directly use (1.0.2). Hence,

$$\begin{aligned}
&\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(z+1)^2+t^2}\,dt \\
&= -\frac{1}{16i\sqrt{y}}\sum_{m=1}^\infty \frac{1}{m^{\frac{z}{2}}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right) \\
&\qquad \times \Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\left(\frac{\pi m}{y}\right)^{-s} ds,
\end{aligned} \tag{4.4.4}$$

where we have interchanged the order of summation and integration because of absolute convergence. Then, using the exact same method as in (4.3.16) and (4.3.17) to simplify the integrand in the last step in (4.4.4), we see that

$$\begin{aligned}
&\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(s-\frac{z}{2}\right)\left(s+\frac{z-2}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z-2}{4}\right)\Gamma\left(\frac{z}{4}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right) \\
&\qquad \times \Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\left(\frac{\pi m}{y}\right)^{-s} ds \\
&= -2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^\infty \frac{1}{m^{\frac{z}{2}}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds.
\end{aligned} \tag{4.4.5}$$

Next, we want to use (4.3.20) to simplify the inner integral on the right-hand side of (4.4.4). Thus we need to shift the line of integration from Re $s = c-$Re $\left(\frac{z}{2}\right)$ to the line Re $s = \frac{1}{2}$.

But unlike in the previous case where $-1 < \text{Re } z < 1$, here we do not encounter any poles of the integrand on the left-hand side of (4.3.20) in the shifting process. Thus for $c = \text{Re } s$,

$$\begin{aligned}
&\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds \\
&= \int_{c-\text{Re}\left(\frac{z}{2}\right)-i\infty}^{c-\text{Re}\left(\frac{z}{2}\right)+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds \\
&= 2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1, 1+\frac{m}{y}\right). \qquad (4.4.6)
\end{aligned}$$

Then from (4.4.4), (4.4.5) and (4.4.6), we find that

$$\begin{aligned}
&\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(z+1)^2+t^2}\,dt \\
&= -\frac{1}{16i\sqrt{y}}\left(-2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^{\infty}\frac{2\pi i}{m^{\frac{z}{2}}}\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1,1+\frac{m}{y}\right)\right) \\
&= \frac{1}{8}(4\pi)^{-\frac{(z-3)}{2}}y^{-\frac{(z+1)}{2}}\Gamma(z+1)\sum_{m=1}^{\infty}\zeta\left(z+1,1+\frac{m}{y}\right). \qquad (4.4.7)
\end{aligned}$$

Now letting $\mu = \frac{1}{2}\log\alpha$ in (4.4.7) so that $y = \alpha$, we observe that

$$\begin{aligned}
&\frac{8(4\pi)^{\frac{(z-3)}{2}}}{\Gamma(z+1)}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt \\
&= \alpha^{-\frac{(z+1)}{2}}\sum_{m=1}^{\infty}\zeta\left(z+1,1+\frac{m}{\alpha}\right). \qquad (4.4.8)
\end{aligned}$$

Now replacing α by β in (4.4.8) and then combining the result with (4.4.8), we arrive at (4.4.1) since the left-hand side of (4.4.8) is invariant under the map $\alpha \to \beta$. This completes the proof. □

When $-3 < \text{Re } z < -1$, we get a third transformation formula as follows:

Theorem 4.4.2. *Let* $-3 < \text{Re } z < -1$. *Define* $\Lambda(z,x)$ *by*

$$\Lambda(z,x) = \zeta(z+1,x) - \frac{x^{-z}}{z} - \frac{1}{2}x^{-z-1} - \frac{(z+1)x^{-z-2}}{12}, \tag{4.4.9}$$

where $\zeta(z,x)$ *denotes the Hurwitz zeta function. Then if* α *and* β *are any positive numbers such that* $\alpha\beta = 1$,

$$\begin{aligned}
&\alpha^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(z,n\alpha) - \frac{\zeta(z)}{\alpha z} + \frac{\zeta(z+1)}{2} + \frac{(z+1)\zeta(z+2)}{12\alpha^{z+2}}\right)\\
&= \beta^{\frac{z+1}{2}}\left(\sum_{n=1}^{\infty}\Lambda(z,n\beta) - \frac{\zeta(z)}{\beta z} + \frac{\zeta(z+1)}{2} + \frac{(z+1)\zeta(z+2)}{12\beta^{z+2}}\right)\\
&= \frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt,
\end{aligned} \tag{4.4.10}$$

where $\Xi(t)$ *is defined in* (1.0.6).

Proof. Here, in the definition of $\Lambda(z,x)$, we have subtracted from $\zeta(z+1,x)$, the first three terms in its asymptotic expansion. The proof is along the lines similar to those in the proof of Theorem 4.3.1, and so we will be brief.

We want to evaluate the integral in the last step in (4.3.7) by making use of (1.0.2), valid for $\text{Re }\left(s+\frac{z}{2}\right) > 1$, i.e., $\text{Re } s > \text{Re }\left(\frac{2-z}{2}\right)$. But since $-3 < \text{Re } z < -1$, we have $-1 < \text{Re }\left(s+\frac{z}{2}\right) < 0$. While shifting the line of integration from $\text{Re } s = \frac{1}{2}$ to $\text{Re } s = \frac{5}{2}$, we encounter poles of order 1 at $s = \frac{2-z}{2}$, $s = \frac{z+4}{2}$ and $s = -\frac{z}{2}$. Let $T > 0$ denote a real number. We apply the residue theorem by considering the rectangle with segments $[\frac{1}{2}-iT, \frac{5}{2}-iT]$, $[\frac{5}{2}-iT, \frac{5}{2}+iT]$, $[\frac{5}{2}+iT, \frac{1}{2}+iT]$ and $[\frac{1}{2}+iT, \frac{1}{2}-iT]$. The residues at the above poles can be calculated similarly as before and so we suppress the calculations. Thus using the notation in (4.3.5), we find that

$$
\begin{aligned}
R_{\frac{2-z}{2}} &= -2^{3-z}\pi^{\frac{1-z}{2}}y^{1-\frac{z}{2}}\Gamma(z)\zeta(z),\\
R_{\frac{z+4}{2}} &= \frac{1}{3}2^{1-z}\pi^{\frac{1-z}{2}}y^{2+\frac{z}{2}}\Gamma(z+2)\zeta(z+2),\\
R_{\frac{-z}{2}} &= 2^{2-z}\pi^{\frac{1-z}{2}}y^{-\frac{z}{2}}\Gamma(1+z)\zeta(1+z). \qquad (4.4.11)
\end{aligned}
$$

As $T \to \infty$, the integrals along the horizontal segments $[\frac{1}{2} - iT, \frac{5}{2} - iT]$ and $[\frac{5}{2} + iT, \frac{1}{2} + iT]$ tend to zero. Next, doing the exact same calculations as in (4.3.16) and (4.3.17), we have

$$
\begin{aligned}
&\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty} H(z,s)\left(\frac{\pi}{y}\right)^{-s} ds\\
&= -2^{3-z}\pi^{\frac{1-z}{2}}\sum_{m=1}^{\infty}\frac{1}{m^{\frac{z}{2}}}\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds, \quad (4.4.12)
\end{aligned}
$$

where $H(z, s)$ is defined in (4.3.8). In order to use (4.3.20) to simplify the integral in the last expression in (4.4.12), we shift the line of integration from Re $s = c - \text{Re}\left(\frac{z}{2}\right)$ to Re $s = \frac{5}{2}$ and then from another application of the residue theorem, we see after simplification that

$$
\begin{aligned}
&\int_{\frac{5}{2}-i\infty}^{\frac{5}{2}+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)\zeta\left(1-s+\frac{z}{2}\right)\left(\frac{m}{y}\right)^{-s} ds\\
&= 2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\zeta\left(z+1, 1+\frac{m}{y}\right) + 2\pi i\left(R_{\frac{z}{2}} + R_{1+\frac{z}{2}} + R_{2+\frac{z}{2}} + R_{3+\frac{z}{2}}\right)\\
&= 2\pi i\left(\frac{m}{y}\right)^{\frac{z}{2}}\Gamma(z+1)\Lambda\left(z, \frac{m}{y}\right). \qquad (4.4.13)
\end{aligned}
$$

Finally from (4.3.7), (4.4.11), (4.4.12) and (4.4.13), we find after simplification that

$$
\begin{aligned}
&\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^{\infty}\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(z+1)^2+t^2}\,dt\\
&= y^{-\frac{(z+1)}{2}}\left(\sum_{m=1}^{\infty}\Lambda\left(z, \frac{m}{y}\right) - \frac{y\zeta(z)}{z} + \frac{\zeta(z+1)}{2} + \frac{(z+1)y^{z+2}\zeta(z+2)}{12}\right). \qquad (4.4.14)
\end{aligned}
$$

Now letting $\mu = \frac{1}{2}\log\alpha$ in (4.4.14) so that $y = \alpha$, we observe that since $\alpha\beta = 1$,

$$\frac{8(4\pi)^{\frac{z-3}{2}}}{\Gamma(z+1)}\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(z+1)^2+t^2}\,dt$$
$$= \beta^{\frac{z+1}{2}}\left(\sum_{n=1}^\infty \Lambda(z,n\beta) - \frac{\zeta(z)}{\beta z} + \frac{\zeta(z+1)}{2} + \frac{(z+1)\zeta(z+2)}{12\beta^{z+2}}\right). \tag{4.4.15}$$

Finally, switching the roles of α by β in (4.4.15) and then combining the result with (4.4.15), we arrive at (4.4.10), since with $\alpha\beta = 1$, the left-hand side of (4.4.15) is invariant under the map $\alpha \to \beta$. This completes the proof. □

As can be seen from the above transformation formulas, one can construct a general transformation formula in the strip $-(2k+1) < \text{Re } z < -(2k-1)$, where $k \geq 0$, by subtracting from $\zeta(z+1,x)$, the requisite number of terms from the asymptotic expansion (4.3.2) for that particular strip and evaluating the integral

$$\int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(z+1)^2+t^2}\,dt$$

. However, the formulas soon start to become complicated and so we do not continue our study in that direction.

4.5 Extended version of the Ramanujan-Guinand formula

Another special case of (4.0.1) that we study here extends a formula found by Guinand [47] which, in fact, was discovered by Ramanujan several years earlier (see [78, p. 253]). We thus call this the Ramanujan-Guinand formula. This formula is associated with the Fourier expansions of Eisenstein series and Epstein zeta functions (see [21]). Ramanujan's version of this formula is given below.

Theorem 4.5.1 (Ramanujan-Guinand formula)**.** *Let $K_\nu(z)$ denote the modified Bessel function of order ν, let $\sigma_k(n) = \sum_{d|n} d^k$ and let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}(2n\beta)$$
$$= \frac{1}{4}\Gamma\left(\frac{z}{2}\right)\zeta(z)\{\beta^{(1-z)/2} - \alpha^{(1-z)/2}\} + \frac{1}{4}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\{\beta^{(1+z)/2} - \alpha^{(1+z)/2}\}. \quad (4.5.1)$$

The series involved in this identity are remindful of the Fourier expansion of non-holomorphic Eisenstein series on $SL(2,\mathbb{Z})$, or Maass wave forms. See [21] for details. For elaborate discussions on the Fourier expansion of Maass forms and related topics, we refer the reader to [23], [42]. The proof of this identity in [21] as well as in [47] makes use of a theorem of G. N. Watson [82] given below.

Theorem 4.5.2. *Let $K_\nu(z)$ be defined as before. If $x > 0$ and $\mathrm{Re}\ \nu > 0$, then*

$$\frac{1}{4}(\pi x)^{-\nu}\Gamma(\nu) + \sum_{n=1}^{\infty} n^{\nu}K_\nu(2\pi n x)$$
$$= \frac{1}{4}\sqrt{\pi}(\pi x)^{-\nu-1}\Gamma(\nu+\tfrac{1}{2}) + \frac{\sqrt{\pi}}{2x}\left(\frac{x}{\pi}\right)^{\nu+1}\Gamma(\nu+\tfrac{1}{2})\sum_{n=1}^{\infty}(n^2+x^2)^{-\nu-1/2}. \quad (4.5.2)$$

Our extended version of the Ramanujan-Guinand formula given below not only gives a new proof of (4.5.1), which does not make use of Watson's theorem, but also shows a surprising connection between the Fourier expansion of Maass waveforms and the Riemann Ξ-function.

Theorem 4.5.3 (Extended version of the Ramanujan-Guinand formula)**.** *Let $K_\nu(s), \sigma_k(n)$, and $\Xi(t)$ be defined as before and let $-1 < \mathrm{Re}\ z < 1$. Then if α and β are positive numbers* *such that $\alpha\beta = 1$, we have*

$$\sqrt{\alpha}\left(\alpha^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\alpha\right)\right)$$
$$=\sqrt{\beta}\left(\beta^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{\frac{z}{2}}\left(2n\pi\beta\right)\right)$$
$$=-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt. \tag{4.5.3}$$

Proof. Let

$$f(z,t)=\frac{4}{\left(t^2+\frac{(z-1)^2}{4}\right)\left(t^2+\frac{(z+1)^2}{4}\right)}. \tag{4.5.4}$$

Substituting this representation of f in the integral in (4.0.1) gives the integral in (4.5.3). Using (3.3.8), one can easily show that this integral converges.

From (4.5.4) and (4.1.1), it can be easily seen that $\phi(z,s)=\frac{2}{\left(\frac{z-1}{2}-s\right)\left(\frac{z+1}{2}-s\right)}$. Thus using (4.1.8) with these f and ϕ, we find that

$$64\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt$$
$$=\frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{4}{\left(\frac{z}{2}-s\right)\left(\frac{z+2}{2}-s\right)\left(\frac{z-2}{2}+s\right)\left(\frac{z}{2}+s\right)}\xi\left(s-\frac{z}{2}\right)\xi\left(s+\frac{z}{2}\right)y^s\,ds$$
$$=\frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}G(z,s)\left(\frac{\pi}{y}\right)^{-s}\,ds, \tag{4.5.5}$$

where

$$G(z,s):=\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right), \tag{4.5.6}$$

and in the penultimate line, we have used (1.0.5). To evaluate the integral in the last step in (4.5.5), we want to use the series representation for $\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)$, namely,

$$\zeta\left(s-\frac{z}{2}\right)\zeta\left(s+\frac{z}{2}\right)=\sum_{m=1}^{\infty}\frac{\sigma_{-z}(m)}{m^{s-\frac{z}{2}}}, \tag{4.5.7}$$

valid for Re $\left(s-\frac{z}{2}\right)>1$ and Re $\left(s+\frac{z}{2}\right)>1$, i.e., Re $s>$ Re $\left(\frac{z+2}{2}\right)$ and Re $s>$ Re $\left(\frac{2-z}{2}\right)$ (see [81, p. 8, eqn. 1.3.1]). But since $-1<\text{Re } z<1$, we have $0<\text{Re}\left(s-\frac{z}{2}\right)<1$ and $0<\text{Re}\left(s+\frac{z}{2}\right)<1$. Thus we move the line of integration from Re $s=\frac{1}{2}$ to Re $s=\frac{3}{2}$. In this process, we encounter in the integrand a pole of order 1 at $s=\frac{z+2}{2}$ (due to $\zeta(s-\frac{z}{2})$) and a pole of order 1 at $s=\frac{2-z}{2}$ (due to $\zeta(s+\frac{z}{2})$).

Let $T>0$ denote a real number. Then using the residue theorem and (4.3.5), we know that

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} G(z,s)\left(\frac{\pi}{y}\right)^{-s} ds=\left[\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}\right] G(z,s)\left(\frac{\pi}{y}\right)^{-s} ds$$
$$-2\pi i\left(R_{\frac{z+2}{2}}+R_{\frac{2-z}{2}}\right). \tag{4.5.8}$$

Also, it is easily observed that

$$\lim_{s\to\frac{z+2}{2}}\left(s-\frac{z+2}{2}\right)\zeta\left(s-\frac{z}{2}\right)=1. \tag{4.5.9}$$

Thus using (1.0.3) and (4.5.9), we see that

$$\begin{aligned} R_{\frac{z+2}{2}} &= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{z+1}{2}\right)\zeta(z+1)\left(\frac{\pi}{y}\right)^{\left(-\frac{z+2}{2}\right)} \\ &= y^{1+\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z). \end{aligned} \tag{4.5.10}$$

Also,

$$\lim_{s\to\frac{2-z}{2}}\left(s-\frac{2-z}{2}\right)\zeta\left(s+\frac{z}{2}\right)=1. \tag{4.5.11}$$

Hence using (1.0.3) and (4.5.11), we deduce that

$$\begin{aligned} R_{\frac{2-z}{2}} &= \Gamma\left(\frac{1-z}{2}\right)\Gamma\left(\frac{1}{2}\right)\zeta(1-z)\left(\frac{\pi}{y}\right)^{\left(-\frac{2-z}{2}\right)} \\ &= y^{1-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z). \end{aligned} \tag{4.5.12}$$

Next, we show that as $T \to \infty$, the integrals along the horizontal segments $[\frac{1}{2} - iT, \frac{3}{2} - iT]$ and $[\frac{3}{2} + iT, \frac{1}{2} + iT]$ tend to zero. We use the following estimate for $\zeta(s)$ valid for $\sigma \geq -\delta$ [81, p. 95, eqn. 5.1.1],

$$\zeta(s) = O(t^{\frac{3}{2}+\delta}). \tag{4.5.13}$$

Since $\frac{1}{2} < \text{Re } s < \frac{3}{2}$ and $-1 < \text{Re } z < 1$, we readily see that $0 < \text{Re } \left(s - \frac{z}{2}\right) < 2$ and $0 < \text{Re } \left(s + \frac{z}{2}\right) < 2$. So if $s = \sigma + it$, $z = x + iy$ and δ is any positive number, then using (3.3.8) and (4.5.13), we find that

$$\begin{aligned}
&\left| \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} G(z,s) \left(\frac{\pi}{y}\right)^{-s} ds \right| \\
&\leq C(2\pi) \left| \frac{T}{2} - \frac{y}{4} \right|^{\frac{\sigma-1}{2} - \frac{x}{4}} \left| \frac{T}{2} + \frac{y}{4} \right|^{\frac{\sigma-1}{2} + \frac{x}{4}} e^{-\frac{\pi}{2}\left(\left|\frac{T}{2} - \frac{y}{4}\right| + \left|\frac{T}{2} + \frac{y}{4}\right|\right)} \left(T - \frac{y}{2}\right)^{\frac{3}{2}+\delta} \\
&\quad \times \left(T + \frac{y}{2}\right)^{\frac{3}{2}+\delta} \left(1 + O\left(\frac{1}{\left|\frac{T}{2} - \frac{y}{4}\right|}\right)\right) \left(1 + O\left(\frac{1}{\left|\frac{T}{2} + \frac{y}{4}\right|}\right)\right) \\
&= o(1),
\end{aligned} \tag{4.5.14}$$

as $T \to \infty$, where C is some constant. Thus,

$$\lim_{T\to\infty} \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} G(z,s) \left(\frac{\pi}{y}\right)^{-s} ds = 0. \tag{4.5.15}$$

Similarly we can show that

$$\lim_{T\to\infty} \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} G(z,s) \left(\frac{\pi}{y}\right)^{-s} ds = 0. \tag{4.5.16}$$

Now it remains to evaluate

$$\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} G(z,s) \left(\frac{\pi}{y}\right)^{-s} ds. \tag{4.5.17}$$

From [70, p. 115, formula 11.1], for $c = \text{Re } s > \pm \text{ Re } \nu$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s-2}a^{-s}\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)x^{-s}\,ds = K_{\nu}(ax). \tag{4.5.18}$$

Using (4.5.7) in (4.5.17), interchanging the order of summation and integration because of absolute convergence and then using (4.5.18) with $\nu = \frac{z}{2}$, $a = 2$ and $x = \pi m/y$, we see that since $\text{Re } s = \frac{3}{2} > \text{Re } \frac{z}{2}$,

$$\begin{aligned}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} G(z,s)\left(\frac{\pi}{y}\right)^{-s} ds &= \sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}\right)\left(\frac{\pi m}{y}\right)^{-s} ds\\ &= 8\pi i\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}K_{\frac{z}{2}}\left(\frac{2\pi m}{y}\right). \end{aligned}\tag{4.5.19}$$

Thus from (4.5.5), (4.5.8), (4.5.10), (4.5.12), (4.5.15), (4.5.16) and (4.5.19), we see that, with $y = e^{2\mu}$,

$$\begin{aligned}&64\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\mu t}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt\\ &= -\frac{2\pi}{\sqrt{y}}\left(y^{1-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+y^{1+\frac{z}{2}}\pi^{\frac{z}{2}}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)-4\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{\frac{z}{2}}K_{\frac{z}{2}}\left(\frac{2\pi m}{y}\right)\right). \end{aligned}\tag{4.5.20}$$

Now let $\mu = \frac{1}{2}\log\alpha$. Then $y = e^{2\mu}$ implies that $y = \alpha$. Since $\alpha\beta = 1$, (4.5.20) becomes

$$\begin{aligned}&-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt\\ &= \sqrt{\beta}\left(\beta^{\frac{z}{2}-1}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-\frac{z}{2}-1}\pi^{\frac{z}{2}}\Gamma\left(\frac{-z}{2}\right)\zeta(-z)-4\sum_{m=1}^{\infty}\sigma_{-z}(m)m^{z/2}K_{\frac{z}{2}}\left(2m\pi\beta\right)\right). \end{aligned}\tag{4.5.21}$$

Finally, switching the roles of α and β in (4.5.21) and then combining the result with (4.5.21), we arrive at (4.5.3), since with $\alpha\beta = 1$, the left-hand side of (4.5.21) is invariant under the map $\alpha \to \beta$. This completes the proof. $\square$

Remarks. 1. Since $K_z(x)$ is an even function of z, it can readily be seen from (4.5.3) that the identity is invariant if we replace z by $-z$.
2. The first equality in (4.5.3) can be easily simplified to obtain (4.5.1), where $\alpha\beta = \pi^2$.

As a corollary of Theorem 4.5.3, we obtain Theorem 3.2.3.

Corollary 4.5.4 (Extended version of Koshliakov's formula)**.** *Let $d(n)$ denote the number of positive divisors of n and let $K_s(n)$ be defined as before. If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\sqrt{\alpha}\left(\frac{\gamma - \log(4\pi\alpha)}{\alpha} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\alpha)\right) = \sqrt{\beta}\left(\frac{\gamma - \log(4\pi\beta)}{\beta} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\beta)\right)$$
$$= -\frac{32}{\pi}\int_0^{\infty} \frac{\left(\Xi\left(\frac{t}{2}\right)\right)^2 \cos\left(\frac{1}{2}t\log\alpha\right)\, dt}{(1+t^2)^2}. \tag{4.5.22}$$

Proof. Let $z \to 0$ in Theorem 4.5.3. Using Lebesgue's dominated convergence theorem, we can interchange the limit and the integral sign which readily gives the integral in (4.5.22). For obtaining the first equality in (4.5.22), we follow along similar lines as in the proof of Corollary 3.4 in [21], i.e., by using the following series expansions [43, p. 944, formula 8.321, no. 1]

$$\Gamma(z) = \frac{1}{z} - \gamma + \cdots, \tag{4.5.23}$$

and [81, pp. 19–20, eqns. (2.4.3), (2.4.5)]

$$\zeta(z) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)z + \cdots, \tag{4.5.24}$$

along with

$$\alpha^{z/2} = e^{\frac{z}{2}\log\alpha} = 1 + \frac{z}{2}\log\alpha + \frac{z^2}{4.2!}(\log\alpha)^2 + \cdots, \tag{4.5.25}$$

and noting that $\lim_{z\to 0}\sigma_{-z}(n) = d(n)$. □

4.6 Ramanujan's evaluation of (4.0.2)

Let us define the function $\Omega(s,t)$ by

$$\Omega(s,t) := \Gamma\left(\frac{s-1+it}{4}\right)\Gamma\left(\frac{s-1-it}{4}\right)\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right), \tag{4.6.1}$$

where s is a complex number and t is real, both assuming those values for which the right-hand side of (4.6.1) exists. As remarked in the beginning of this chapter, Ramanujan studied the integral $\int_0^\infty \frac{\Omega(s,t)\cos nt}{(s+1)^2+t^2}\,dt$ for n real and represented it in terms of another integral for various values of s, for example, in Re $s>1$, $-1<$ Re $s<1$ and $-3<$ Re $s<-1$ (see equations (17)-(19) in [74]). These can be stated in the following theorem.

Theorem 4.6.1. *Let $\Omega(s,t)$ be defined in* (4.6.1) *and let n be a real number. Then,*

$$\int_0^\infty \frac{\Omega(s,t)\cos nt}{(s+1)^2+t^2}\,dt$$
$$=\begin{cases} \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\left(\int_0^\infty \frac{x^s}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\,dx - 2\Gamma(s)\zeta(s)\cosh n(1-s)\right), \ \textit{if}\ \text{Re } s>1 \ \text{(i)}, \\ \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx, \ \textit{if}\ -1<\text{Re } s<1 \ \text{(ii)}, \\ \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\left(\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}+\frac{1}{2}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}+\frac{1}{2}\right)dx \right. \\ \qquad \left. -\Gamma(1+s)\zeta(1+s)\cosh n(1+s)\right), \ \textit{if}\ -3<\text{Re } s<-1 \ \text{(iii)}. \end{cases} \tag{4.6.2}$$

After giving this theorem, Ramanujan says 'and so on', by which he means that one can evaluate this integral in the vertical strips $-5<$ Re $s<-3$, $-7<$ Re $s<-5$ etc. However, identities (4.6.2)(i) and (4.6.2)(iii) in the above theorem, i.e., the ones in the half-plane Re $s>1$ and in the vertical strip $-3<$ Re $s<-1$ respectively, are incorrect. The second term on the right-hand side of (4.6.2)(i), namely, $-\frac{1}{4}(4\pi)^{\frac{(s-3)}{2}}\Gamma(s)\zeta(s)\cosh n(1-s)$ should not be present. Similarly, the second term on the right-hand side of (4.6.2)(iii), namely, $-\frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\Gamma(1+s)\zeta(1+s)\cosh n(1+s)$, should not be present. The corrected form of Theorem 4.6.1 is as follows.

Theorem 4.6.2. *Let* $\Omega(s,t)$ *be defined in* (4.6.1) *and let* n *be a real number. Then,*

$$\int_0^\infty \frac{\Omega(s,t)\cos nt}{(s+1)^2+t^2}\,dt$$
$$=\begin{cases} \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty \frac{x^s}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\,dx, \textit{if } \operatorname{Re} s>1 \text{ (i)},\\ \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx, \textit{if } -1<\operatorname{Re} s<1 \text{ (ii)},\\ \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty x^s\left(\frac{1}{e^{xe^n}-1}-\frac{1}{xe^n}+\frac{1}{2}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}+\frac{1}{2}\right)dx, \textit{if}\\ -3<\operatorname{Re} s<-1 \text{ (iii)}.\end{cases} \tag{4.6.3}$$

4.6.1 Proof of (4.6.3)(i)

Here we show that the second term on the right-hand side of (4.6.2(i)), namely $-\dfrac{1}{4}(4\pi)^{\frac{(s-3)}{2}}\Gamma(s)\zeta(s)\cosh n(1-s)$, is not present and thus indeed that (4.6.3)(i) is actually the correct version of (4.6.2(i)). Since the exposition in Sections 4 and 5 of [74] is quite terse, we will derive (4.6.3)(i) giving all the details.

First, identity (15) in [74] states that for $\operatorname{Re} s>-1$ and $\alpha\beta=4\pi^2$,

$$\begin{aligned} G(s;\alpha) &:= \frac{\zeta(1-s)}{4\cos(\pi s/2)}\alpha^{(s-1)/2}+\frac{\zeta(-s)}{8\sin(\pi s/2)}\alpha^{(s+1)/2}\\ &\quad+\alpha^{(s+1)/2}\int_0^\infty\int_0^\infty \frac{x^s\sin(\alpha xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy\\ &=\frac{\zeta(1-s)}{4\cos(\pi s/2)}\beta^{(s-1)/2}+\frac{\zeta(-s)}{8\sin(\pi s/2)}\beta^{(s+1)/2}\\ &\quad+\beta^{(s+1)/2}\int_0^\infty\int_0^\infty \frac{x^s\sin(\beta xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy. \end{aligned} \tag{4.6.4}$$

This relation can be proved from the integral representations for $\dfrac{\zeta(1-s)}{4\cos(\pi s/2)}$ and $\dfrac{\zeta(-s)}{8\sin(\pi s/2)}$ and by using the identity (see [74, p. 253])

$$\int_0^\infty \frac{\sin(\alpha xy)}{e^{2\pi y}-1}\,dy = \frac{1}{2}\left(\frac{1}{e^{\alpha x}-1}-\frac{1}{\alpha x}+\frac{1}{2}\right). \tag{4.6.5}$$

Since we are concerned with the case Re $s>1$, as far as identity (19) in [74] is concerned, we prove (4.6.4) for Re $s>1$ only. Other cases can be similarly proved.

For Re $s>1$, the integral representation of $\zeta(s)$ [81, p. 18, eqn. (2.4.1)] gives

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt. \tag{4.6.6}$$

From (1.0.4) and (4.6.6), we have

$$\frac{\zeta(1-s)}{4\cos(\frac{1}{2}\pi s)} = \frac{1}{2}\int_0^\infty \frac{x^{s-1}dx}{e^{2\pi x}-1}, \tag{4.6.7}$$

and

$$\frac{\zeta(-s)}{8\sin(\frac{1}{2}\pi s)} = \frac{-1}{4}\int_0^\infty \frac{x^s dx}{e^{2\pi x}-1} \tag{4.6.8}$$

for Re $s>1$. Hence using (4.6.5), (4.6.7) and (4.6.8) in (4.6.4), we see that

$$G(s;\alpha) = \frac{\alpha^{(s+1)/2}}{2}\int_0^\infty \frac{x^s}{(e^{2\pi x}-1)(e^{\alpha x}-1)}\,dx. \tag{4.6.9}$$

Thus (4.6.4) follows from the identity

$$\frac{\alpha^{(s+1)/2}}{2}\int_0^\infty \frac{x^s}{(e^{2\pi x}-1)(e^{\alpha x}-1)}\,dx = \frac{\beta^{(s+1)/2}}{2}\int_0^\infty \frac{x^s}{(e^{2\pi x}-1)(e^{\beta x}-1)}\,dx, \tag{4.6.10}$$

which follows readily from the change of variable $x = 2\pi y/\alpha$ on the left-hand side of (4.6.10) and from the fact that $\alpha\beta = 4\pi^2$.

Next, identity (17) in [74] states that when $\alpha\beta = 4\pi^2$ and Re $s > -1$, we have

$$\frac{\zeta(1-s)}{4\cos(\pi s/2)}\frac{s-1}{(s-1)^2+t^2}\left(\alpha^{\frac{(s-1)}{2}}+\beta^{\frac{(s-1)}{2}}\right)+\frac{\zeta(-s)}{8\sin(\pi s/2)}\frac{s+1}{(s+1)^2+t^2}\left(\alpha^{\frac{(s+1)}{2}}+\beta^{\frac{(s+1)}{2}}\right)$$
$$+\alpha^{\frac{(s+1)}{2}}\int_0^\infty\int_0^\infty\left(\frac{\alpha xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\alpha xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right)\frac{x^s dx dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$
$$+\beta^{\frac{(s+1)}{2}}\int_0^\infty\int_0^\infty\left(\frac{\beta xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(\beta xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right)\frac{x^s dx dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$
$$=\frac{2^{\frac{(s-3)}{2}}}{\pi}\frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right)\cos\left(\frac{1}{4}t\log\frac{\alpha}{\beta}\right). \tag{4.6.11}$$

Letting $\alpha = \beta = 2\pi$ in (4.6.11), we find that

$$\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}\frac{s-1}{(s-1)^2+t^2}+\frac{\zeta(-s)}{\sin(\pi s/2)}\frac{s+1}{(s+1)^2+t^2}$$
$$+8\int_0^\infty\int_0^\infty\left(\frac{2\pi xy}{1!}\frac{s+3}{(s+3)^2+t^2}-\frac{(2\pi xy)^3}{3!}\frac{s+7}{(s+7)^2+t^2}+\cdots\right)\frac{x^s dx dy}{(e^{2\pi x}-1)(e^{2\pi y}-1)}$$
$$=\frac{1}{\pi^{(s+3)/2}}\frac{\Gamma\left(\frac{1}{4}(s-1+it)\right)\Gamma\left(\frac{1}{4}(s-1-it)\right)}{(s+1)^2+t^2}\Xi\left(\frac{t+is}{2}\right)\Xi\left(\frac{t-is}{2}\right). \tag{4.6.12}$$

Now for Re $a > 0$,

$$\int_0^\infty e^{-au}\cos bu\, du = \frac{a}{a^2+b^2}. \tag{4.6.13}$$

This is used to replace the fractions of the form $\dfrac{s+j}{(s+j)^2+t^2}$ in (4.6.12) by integrals. Since Re $s > 1$, for $j \geq -1$, we have

$$\frac{s+j}{(s+j)^2+t^2} = \int_0^\infty e^{-(s+j)u}\cos tu\, du. \tag{4.6.14}$$

Hence using (4.6.14) in (4.6.12) and inverting the order of integration gives

$$\frac{1}{\pi^{(s+3)/2}}\frac{\Omega(s,t)}{(s+1)^2+t^2}$$

$$
\begin{aligned}
&= \frac{\zeta(1-s)}{\pi\cos(\pi s/2)} \int_0^\infty e^{-(s-1)u}\cos tu\,du + \frac{\zeta(-s)}{\sin(\pi s/2)} \int_0^\infty e^{-(s+1)u}\cos tu\,du \\
&\quad + 8\int_0^\infty \int_0^\infty \left(\frac{2\pi xy}{1!} \int_0^\infty e^{-(s+3)u}\cos tu\,du - \frac{(2\pi xy)^3}{3!} \int_0^\infty e^{-(s+7)u}\cos tu\,du + \cdots \right) \\
&\qquad \times \frac{x^s dxdy}{(e^{2\pi x}-1)(e^{2\pi y}-1)} \\
&= \int_0^\infty \Bigg[\frac{\zeta(1-s)}{\pi\cos(\pi s/2)} e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)} e^{-(s+1)u} \\
&\quad + 8e^{-(s+1)u} \int_0^\infty \int_0^\infty \left(\frac{2\pi xye^{-2u}}{1!} - \frac{(2\pi xye^{-2u})^3}{3!} + \cdots \right) \frac{x^s dxdy}{(e^{2\pi x}-1)(e^{2\pi y}-1)} \Bigg] \cos tu\,du \\
&= \int_0^\infty \Bigg[\frac{\zeta(1-s)}{\pi\cos(\pi s/2)} e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)} e^{-(s+1)u} \\
&\qquad + 8e^{-(s+1)u} \int_0^\infty \int_0^\infty \frac{x^s \sin\left(2\pi xye^{-2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\, dxdy \Bigg] \cos tu\,du. \qquad (4.6.15)
\end{aligned}
$$

Define

$$
\begin{aligned}
f(u,s) := &\frac{\zeta(1-s)}{\pi\cos(\pi s/2)} e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)} e^{-(s+1)u} \\
&+ 8e^{-(s+1)u} \int_0^\infty \int_0^\infty \frac{x^s \sin\left(2\pi xye^{-2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\, dxdy, \qquad (4.6.16)
\end{aligned}
$$

and

$$
\widehat{f}(t,s) := \frac{1}{\pi^{(s+3)/2}} \frac{\Omega(s,t)}{(s+1)^2+t^2}. \qquad (4.6.17)
$$

Then from (4.6.15), (4.6.16) and (4.6.17), the relation

$$
\widehat{f}(t,s) = \int_0^\infty f(u,s)\cos tu\,du \qquad (4.6.18)
$$

holds. Next, it is shown that f is an even function in u.

Let $\alpha = 2\pi e^{-2u}$ and $\beta = 2\pi e^{2u}$ in (4.6.4) to produce

$$\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)u} + 8e^{-(s+1)u}\int_0^\infty\int_0^\infty \frac{x^s\sin\left(2\pi xye^{-2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy$$
$$= \frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{(s-1)u} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{(s+1)u} + 8e^{(s+1)u}\int_0^\infty\int_0^\infty \frac{x^s\sin\left(2\pi xye^{2u}\right)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}\,dxdy. \tag{4.6.19}$$

This proves that f is an even function in u. Also using the fact that $\Xi(-t) = \Xi(t)$, it follows that $\widehat{f}$ is an even function in t.

The Fourier integral theorem and (4.6.18) give

$$f(n,s) = \frac{2}{\pi}\int_0^\infty \widehat{f}(t,s)\cos nt\,dt \tag{4.6.20}$$

for n real. Now define

$$F(n,s) := \frac{\pi^{\frac{s+5}{2}}}{2}f(n,s). \tag{4.6.21}$$

Then (4.6.16), (4.6.17) and (4.6.20) give

$$F(n,s) = \int_0^\infty \frac{\Omega(s,t)}{(s+1)^2+t^2}\cos nt\,dt$$
$$= \frac{\pi^{\frac{s+5}{2}}}{2}\left(\frac{\zeta(1-s)}{\pi\cos(\pi s/2)}e^{-(s-1)n} + \frac{\zeta(-s)}{\sin(\pi s/2)}e^{-(s+1)n}\right.$$
$$\left. + 8e^{-(s+1)n}\int_0^\infty\int_0^\infty \frac{t^s\sin\left(2\pi tye^{-2n}\right)}{(e^{2\pi t}-1)(e^{2\pi y}-1)}\,dtdy\right). \tag{4.6.22}$$

Substituting (4.6.5), (4.6.7) and (4.6.8) in (4.6.22), we find that

$$F(n,s) = \frac{\pi^{\frac{s+5}{2}}}{2}\left[\frac{2e^{-(s-1)n}}{\pi}\int_0^\infty \frac{t^{s-1}dt}{e^{2\pi t}-1} - 2e^{-(s+1)n}\int_0^\infty \frac{t^s dt}{e^{2\pi t}-1}\right.$$
$$\left. + 4e^{-(s+1)n}\int_0^\infty \frac{t^s dt}{e^{2\pi t}-1}\left(\frac{1}{e^{2\pi te^{-2n}}-1} - \frac{1}{2\pi te^{-2n}} + \frac{1}{2}\right)\right]. \tag{4.6.23}$$

Let $t = \dfrac{e^n x}{2\pi}$ in (4.6.23) to obtain

$$\begin{aligned} F(n,s) &= \frac{\pi^{\frac{s+5}{2}}}{2}\Bigg[\frac{e^n}{2^{s-1}\pi^{s+1}}\int_0^\infty \frac{x^{s-1}\,dx}{(e^{xe^n}-1)} - \frac{1}{2^s\pi^{s+1}}\int_0^\infty \frac{x^s\,dx}{(e^{xe^n}-1)} \\ &\quad + \frac{1}{2^{s-1}\pi^{s+1}}\int_0^\infty \frac{x^s dx}{(e^{xe^n}-1)}\left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}} + \frac{1}{2}\right)\Bigg] \\ &= \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\Bigg[e^n\int_0^\infty \frac{x^{s-1}\,dx}{(e^{xe^n}-1)} - \frac{1}{2}\int_0^\infty \frac{x^s\,dx}{(e^{xe^n}-1)} \\ &\quad + \int_0^\infty \frac{x^s dx}{(e^{xe^n}-1)}\left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}} + \frac{1}{2}\right)\Bigg] \\ &= \frac{1}{8}(4\pi)^{-\frac{(s-3)}{2}}\int_0^\infty \frac{x^s\,dx}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}. \end{aligned} \tag{4.6.24}$$

The identity (4.6.3)(i) now follows from (4.6.22) and (4.6.24).

4.6.2 Proof of (4.6.3)(iii)

In this section, it is shown that (4.6.3)(iii) is actually the correct version of (4.6.2(iii)). Our exposition will be brief. Note that the identity (4.6.3)(iii) cannot be proved using Ramanujan's method delineated in Section 5 of [74], since (4.6.11), which was crucially employed there, is true only when Re $s > -1$. Instead, a reverse route is used in the sense that (4.6.3)(iii) is obtained by means of Theorem 4.4.2. Since Theorem 4.4.2 is proved in Section 4.4 using contour integration and Mellin transforms, there is no circular reasoning involved here. To establish (4.6.3)(iii), the full strength of (4.4.10) in Theorem 4.4.2 is not required, rather just the equality of the first and the third expressions. The result is first established for $-2 <$ Re $s < -1$.

We note the following formula [69, p. 23]

$$\zeta(z,a) = \frac{a^{-z}}{2} - \frac{a^{1-z}}{1-z} + \frac{1}{\Gamma(z)}\int_0^\infty e^{-ax}x^{z-1}\left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right)dx, \tag{4.6.25}$$

First of all, it is easy to adapt (4.6.25) to see that, for Re $s > -2$,

$$\Gamma(s+1)\Lambda(s,n\alpha) = \int_0^\infty e^{-ax}x^s\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}-\frac{x}{12}\right)dx, \tag{4.6.26}$$

where $\Lambda(z,x)$ is defined in (4.4.9). Hence,

$$\alpha^{(s+1)/2}\Gamma(s+1)\sum_{n=1}^\infty \Lambda(s,n\alpha) = \alpha^{-(s+1)/2}\int_0^\infty \frac{t^s}{(e^t-1)}\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}+\frac{1}{2}-\frac{t/\alpha}{12}\right)dt. \tag{4.6.27}$$

Since $-2 < \text{Re } s < -1$, we also see that

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \int_0^\infty t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right)dt,\\
\Gamma(s+1)\zeta(s+1) &= \int_0^\infty t^{s}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt,\\
\Gamma(s+2)\zeta(s+2) &= \int_0^\infty t^{s+1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt.
\end{aligned} \tag{4.6.28}$$

Then from the equality of the first and the third expressions in (4.4.10), and from (4.6.27) and (4.6.28), we see that

$$\begin{aligned}
&8(4\pi)^{\frac{s-3}{2}}\int_0^\infty \frac{\Omega(s,t)\cos\left(\frac{1}{2}t\log\alpha\right)}{(s+1)^2+t^2}dt\\
&=\alpha^{-\frac{(s+1)}{2}}\int_0^\infty t^s\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}+\frac{1}{2}\right)dt\\
&=\int_0^\infty x^s\left(\frac{1}{e^{x\sqrt{\alpha}}-1}-\frac{1}{x\sqrt{\alpha}}+\frac{1}{2}\right)\left(\frac{1}{e^{x/\sqrt{\alpha}}-1}-\frac{1}{x/\sqrt{\alpha}}+\frac{1}{2}\right)dx,
\end{aligned} \tag{4.6.29}$$

where in the ultimate step, we made a change of variable $t = x\sqrt{\alpha}$. Finally, let $\alpha = e^{2n}$ in the last integral in (4.6.29) to obtain (4.6.3)(iii) for $-2 < \text{Re } s < -1$. It is known [81, p. 95] that

$$\zeta(s) = O(u^{\frac{3}{2}+\delta}),\ \ \sigma \geq -\delta \text{ and } \delta > 0. \tag{4.6.30}$$

. From (3.3.8) and (4.6.30), it can be shown that the left-hand side of (4.6.3)(iii) is absolutely and uniformly convergent for $-3 < \text{Re } s < -1$ and hence analytic in that strip. Similarly, it is easily seen from the behavior of the integrand at 0 and at ∞ that the right-hand side of (4.6.3)(iii) is analytic for $-3 < \text{Re } s < -1$. Thus by analytic continuation, (4.6.3)(iii) holds for $-3 < \text{Re } s < -1$.

Remarks. 1. Identity (4.6.3)(ii), for $-1 < \text{Re } s < 1$, is derived in a very similar manner as the derivation of (4.6.3)(i) in Section 5, except that since $\text{Re } s < 1$, the first expression on the left-hand side of (4.6.12) is written as $-\dfrac{\zeta(1-s)}{\pi\cos(\pi s/2)}\dfrac{1-s}{(1-s)^2+t^2}$ and then we use the evaluation,

$$\frac{1-s}{(1-s)^2+t^2} = \int_0^\infty e^{(s-1)u}\cos tu\, du. \tag{4.6.31}$$

This along with an analysis similar to that in Section 5 gives (4.6.3)(ii). Now it turns out that if we use (4.6.31) instead of (4.6.14) with $j = -1$ when $\text{Re } s > 1$, then we *do* get the second term on the right-hand side of (4.6.2)(i), i.e., $-\frac{1}{4}(4\pi)^{\frac{(s-3)}{2}}\Gamma(s)\zeta(s)\cosh n(1-s)$, as given by Ramanujan. This explains how Ramanujan was erroneously led to his identity.

4.7 Two more proofs of Theorem 4.4.1

This section is devoted to two alternative proofs of Theorem 4.4.1, both of which make use of (4.6.3(i)). However, we rephrase this theorem in a slightly different way below so as to clearly view the right-hand side of (4.6.3(i)) as the Mellin transform of the product of $\dfrac{1}{e^{xe^n}-1}$ and $\dfrac{1}{e^{xe^{-n}}-1}$.

Theorem 4.7.1. *Let $\Omega(s,t)$ be defined in (4.6.1). Let $\text{Re } z > 2$ and $1 < c < \text{Re } z - 1$. If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned}\alpha^{-\frac{z}{2}}\sum_{k=1}^\infty \zeta\left(z, 1+\frac{k}{\alpha}\right) &= \beta^{-\frac{z}{2}}\sum_{k=1}^\infty \zeta\left(z, 1+\frac{k}{\beta}\right)\\ &= \frac{\alpha^{\frac{z}{2}}}{2\pi i\Gamma(z)}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\, ds\end{aligned}$$

$$= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)} \int_0^\infty \frac{\Omega(z-1,t)}{z^2+t^2} \cos\left(\tfrac{1}{2}t\log\alpha\right)\, dt. \tag{4.7.1}$$

Proof. Replace s by $z-1$ in (4.6.3)(i) and then multiply the resulting two sides by $8(4\pi)^{\frac{z-4}{2}}e^{-nz}$. Then for Re $z>2$ and n real, we have

$$8(4\pi)^{\frac{z-4}{2}}e^{-nz}\int_0^\infty \frac{\Omega(z-1,t)}{z^2+t^2}\cos nt\, dt = e^{-nz}\int_0^\infty \frac{x^{z-1}}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\, dx. \tag{4.7.2}$$

The change of variable $t=xe^n$ in (4.6.6) gives

$$e^{-nz}\Gamma(z)\zeta(z) = \int_0^\infty \frac{x^{z-1}}{e^{xe^n}-1}\, dx. \tag{4.7.3}$$

The change of variable $t=xe^{-n}$ in (4.6.6) yields

$$e^{nz}\Gamma(z)\zeta(z) = \int_0^\infty \frac{x^{z-1}}{e^{xe^{-n}}-1}\, dx. \tag{4.7.4}$$

Thus from (3.3.6), (4.7.3) and (4.7.4), it can be seen that

$$\int_0^\infty \frac{x^{z-1}}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\, dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-ns}\Gamma(s)\zeta(s)e^{n(z-s)}\Gamma(z-s)\zeta(z-s)\, ds \tag{4.7.5}$$

for $1<c<\text{Re } z-1$, and hence

$$e^{-nz}\int_0^\infty \frac{x^{z-1}}{(e^{xe^n}-1)(e^{xe^{-n}}-1)}\, dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-2ns}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\, ds. \tag{4.7.6}$$

Define α by $n=\frac{1}{2}\log\alpha$ in (4.7.2) and (4.7.6) and combine them to obtain

$$8(4\pi)^{\frac{z-4}{2}}\alpha^{-\frac{z}{2}}\int_0^\infty \frac{\Omega(z-1,t)}{z^2+t^2}\cos\left(\tfrac{1}{2}t\log\alpha\right)\, dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\, ds. \tag{4.7.7}$$

Upon simplification, this gives the last equality in (4.7.1). Now since $1<c<\text{Re } z-1$, Re

$(z-s) > 1$ if s is on the line Re $s = c$. Therefore the representation

$$\zeta(z-s) = \sum_{k=1}^{\infty} \frac{1}{k^{z-s}}. \tag{4.7.8}$$

is valid there. Using (4.7.8) on the right-hand side of (4.7.7), by absolute convergence, it can be seen that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds \\
&= \sum_{k=1}^{\infty} k^{-z}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\left(\frac{\alpha}{k}\right)^{-s}\,ds \\
&= \Gamma(z)\sum_{k=1}^{\infty} k^{-z}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)}\zeta(s)\left(\frac{\alpha}{k}\right)^{-s}\,ds.
\end{aligned} \tag{4.7.9}$$

The Euler beta function $B(s, z-s)$ is given by

$$B(s, z-s) = \int_0^{\infty} \frac{x^{s-1}}{(1+x)^z}\,dx = \frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)}, \;\; 0 < \text{ Re}\, s < \text{ Re}\, z. \tag{4.7.10}$$

In other words, $B(s, z-s)$ is the Mellin transform of $(1+x)^{-z}$. For Re $z > 2$, $f(x) := (1+x)^{-z}$ is locally integrable on $(0,\infty)$. Also, as $x \to 0^+$, $f(x) = O(1)$ and as $x \to \infty$, $f(x) \sim x^{-z} = O\left(x^{-\text{Re}(z)}\right)$. In particular, using (3.3.7) and (4.7.10), we find that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(s, z-s)\zeta(s)x^{-s}\,ds = \sum_{m=1}^{\infty}(1+xm)^{-z} \tag{4.7.11}$$

for $1 < c <$ Re $z - 1$. The identities (4.7.9), (4.7.10) and (4.7.11) produce

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds &= \Gamma(z)\sum_{k=1}^{\infty} k^{-z}\sum_{m=1}^{\infty}\left(1+\frac{\alpha m}{k}\right)^{-z} \\
&= \Gamma(z)\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}(k+\alpha m)^{-z}
\end{aligned}$$

$$= \alpha^{-z}\Gamma(z) \sum_{k=1}^{\infty} \zeta\left(z, 1 + \frac{k}{\alpha}\right). \tag{4.7.12}$$

Invoking (4.7.12) in (4.7.7), we observe that

$$\alpha^{-\frac{z}{2}} \sum_{k=1}^{\infty} \zeta\left(z, 1 + \frac{k}{\alpha}\right) = \frac{8(4\pi)^{\frac{(z-4)}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{\Omega(z-1,t)}{z^2+t^2} \cos\left(\tfrac{1}{2}t \log \alpha\right) \, dt. \tag{4.7.13}$$

This shows that the extreme left and right-hand sides in Theorem 4.7.1 are equal. Now replacing α by β in (4.7.13), and making use of the fact that $\alpha\beta = 1$ and that $\cos\theta$ is an even function of θ, we obtain the equality of second and fourth expressions in Theorem 4.7.1 as well. This completes the first proof of Theorem 4.7.1. □

Remark: An alternative way to proceed from (4.7.7) is to use the series definition of $\zeta(s)$, interchange the order of summation and integration, and then use, with $x = 1$, the formula from [70, p. 202, formula 5.78]

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(z-s)\zeta(z-s)a^{-s}x^{-s} \, ds = \Gamma(z)\zeta(z, 1+ax), \ \ 0 < c = \text{Re}\, s < \text{Re}\, z - 1. \tag{4.7.14}$$

Now we give a short and simpler proof of Theorem 4.7.1.

Proof. Let $n = \frac{1}{2}\log\alpha$ in (4.7.2) and multiply both sides by $1/\Gamma(z)$. Then,

$$\begin{aligned} \frac{8(4\pi)^{\frac{(z-4)}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{\Omega(z-1,t)}{z^2+t^2} \cos\left(\tfrac{1}{2}t\log\alpha\right) \, dt &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{x^{z-1}}{(e^{x\sqrt{\alpha}}-1)(e^{x/\sqrt{\alpha}}-1)} \, dx \\ &= \frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{(e^t-1)(e^{t/\alpha}-1)} \, dt. \end{aligned} \tag{4.7.15}$$

Also,

$$\frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{(e^t-1)(e^{t/\alpha}-1)} \, dt = \frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}e^{-t}}{1-e^{-t}} \sum_{k=1}^{\infty} e^{-kt/\alpha} \, dt$$

$$= \frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)} \sum_{k=1}^{\infty} \int_0^{\infty} \frac{t^{z-1} e^{-(1+k/\alpha)t}}{1-e^{-t}}\, dt, \tag{4.7.16}$$

where the order of summation and integration can be interchanged because of absolute convergence. Now [81, p. 37, eqn. (2.17.1)] states that

$$\zeta(z,a) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{x^{z-1} e^{-ax}}{1-e^{-x}}\, dx. \tag{4.7.17}$$

Using (4.7.17) in (4.7.16), we deduce that

$$\frac{\alpha^{-\frac{z}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{(e^t-1)(e^{t/\alpha}-1)}\, dt = \alpha^{-\frac{z}{2}} \sum_{k=1}^{\infty} \zeta\left(z, 1+\frac{k}{\alpha}\right) \tag{4.7.18}$$

for Re $z > 1$. Thus from (4.7.15) and (4.7.18), we derive (4.7.13). Then following the same argument as in the first proof, we obtain the equality of second and fourth expressions in (4.7.1) as well. This finishes the second proof. □

Corollary 4.7.2. *For* Re $z > 2$, *we have*

$$\sum_{k=1}^{\infty} \zeta\left(z, 1+k\right) = \zeta(z-1) - \zeta(z) = \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)} \int_0^{\infty} \frac{\Omega(z-1,t)}{z^2+t^2}\, dt. \tag{4.7.19}$$

Proof. Set $\alpha = 1$ in (4.7.1) and note that from [81, p. 35], for $1 < c <$ Re $z-1$, we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\, ds = \Gamma(z)\left(\zeta(z-1)-\zeta(z)\right). \tag{4.7.20}$$

□

4.7.1 Guinand's formula as a special case of (4.7.1)

Let $\psi^{(j)}(x)$ denote the j^{th} derivative of the digamma function $\psi(x)$ defined in (2.1.2), also known as the polygamma function of order j. In [46], Guinand gave the following formula

$$\sum_{k=1}^{\infty} \psi^{(j)}(1+kx) = x^{-j-1} \sum_{k=1}^{\infty} \psi^{(j)} \left(1+\frac{k}{x}\right), \tag{4.7.21}$$

for $j \geq 2$. This formula is shown to be a special case of (4.7.1). Let $z \in \mathbb{N}, z > 2$. Successive differentiations of (2.1.2) give

$$\psi^{(z-1)}(x) = (-1)^z (z-1)! \sum_{m=1}^{\infty} \frac{1}{(m-1+x)^z}. \tag{4.7.22}$$

Thus,

$$\begin{aligned} \alpha^{\frac{z}{2}} \sum_{k=1}^{\infty} \psi^{(z-1)}(1+k\alpha) &= (-1)^z (z-1)! \alpha^{\frac{z}{2}} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+k\alpha)^z} \\ &= (-1)^z (z-1)! \alpha^{\frac{z}{2}} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(m+k\alpha)^z} \\ &= (-1)^z (z-1)! \alpha^{\frac{-z}{2}} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(k+m/\alpha)^z} \\ &= (-1)^z (z-1)! \alpha^{\frac{-z}{2}} \sum_{m=1}^{\infty} \zeta\left(z, 1+\frac{m}{\alpha}\right), \end{aligned} \tag{4.7.23}$$

where the change in the order of summation in the second equality is justified by absolute convergence.

An alternative version of (4.7.1) follows from (4.7.23) and (4.7.1) in the case $z > 2$. Thus,

$$\begin{aligned} \alpha^{\frac{z}{2}} \sum_{k=1}^{\infty} \psi^{(z-1)}(1+k\alpha) &= \beta^{\frac{z}{2}} \sum_{k=1}^{\infty} \psi^{(z-1)}(1+k\beta) \\ &= 8(-1)^z (4\pi)^{\frac{z-4}{2}} \int_0^{\infty} \frac{\Omega(z-1,t)}{z^2+t^2} \cos\left(\tfrac{1}{2} t \log \alpha\right) dt. \end{aligned} \tag{4.7.24}$$

To obtain (4.7.21), we simply replace $z-1$ by j, α by x and β by $1/x$ in the first equality of (4.7.24).

4.8 Another proof of Theorem 4.3.1

We give another proof of Theorem 4.3.1 that uses the theory of special functions and does not involve contour integration. However, we rephrase this theorem in the following way so that it is easier to interpret the right-hand side of (4.3.4) as a Mellin transform.

Theorem 4.8.1. *Let* $0 < \mathrm{Re}\ z < 2$. *Define* $\varphi(z,x)$ *as*

$$\varphi(z,x) = \zeta(z,x) - \frac{1}{2}x^{-z} + \frac{x^{1-z}}{1-z}, \tag{4.8.1}$$

where $\zeta(z,x)$ *denotes the Hurwitz zeta function. Let* $\Omega(s,t)$ *be defined in* (4.6.1). *If* α *and* β *are any positive numbers such that* $\alpha\beta = 1$, *then*

$$\begin{aligned}\alpha^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\alpha) - \frac{\zeta(z)}{2\alpha^{z}} - \frac{\zeta(z-1)}{\alpha(z-1)}\right) &= \beta^{\frac{z}{2}}\left(\sum_{k=1}^{\infty}\varphi(z,k\beta) - \frac{\zeta(z)}{2\beta^{z}} - \frac{\zeta(z-1)}{\beta(z-1)}\right)\\ &= \frac{8(4\pi)^{\frac{z-4}{2}}}{\Gamma(z)}\int_0^{\infty}\frac{\Omega(z-1,t)}{z^2+t^2}\cos\left(\tfrac{1}{2}t\log\alpha\right)\,dt.\end{aligned} \tag{4.8.2}$$

Proof. The fact that $\sum_{k=1}^{\infty}\varphi(z,k\alpha)$ as well as $\sum_{k=1}^{\infty}\varphi(z,k\beta)$ are analytic functions for $0 < \mathrm{Re}\ z < 2$ follows from (4.3.2). We first prove the result for $1 < \mathrm{Re}\ z < 2$ and later extend it to $0 < \mathrm{Re}\ z < 2$ using analytic continuation. Replacing s by $z-1$ in (4.6.3)(ii), we find that for $0 < \mathrm{Re}\ z < 2$,

$$\int_0^{\infty}\frac{\Omega(z-1,t)}{z^2+t^2}\cos nt dt = \frac{1}{8}(4\pi)^{\frac{-(z-4)}{2}}\int_0^{\infty}x^{z-1}\left(\frac{1}{e^{xe^{n}}-1} - \frac{1}{xe^{n}}\right)\left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}}\right)dx. \tag{4.8.3}$$

Multiplying both sides of (4.8.3) by $8(4\pi)^{\frac{(z-4)}{2}}$ and then letting $n = \frac{1}{2}\log\alpha$, we see that

$$\begin{aligned} &8(4\pi)^{\frac{(z-4)}{2}} \int_0^\infty \frac{\Omega(z-1,t)}{z^2+t^2}\cos\left(\tfrac{1}{2}t\log\alpha\right)\, dt \\ &= \int_0^\infty x^{z-1}\left(\frac{1}{e^{x\sqrt{\alpha}}-1}-\frac{1}{x\sqrt{\alpha}}\right)\left(\frac{1}{e^{x/\sqrt{\alpha}}-1}-\frac{1}{x/\sqrt{\alpha}}\right) dx. \qquad (4.8.4) \end{aligned}$$

The change of variable $x = t/\sqrt{\alpha}$ in the integral on the right-hand side of (4.8.4) gives

$$\begin{aligned} &8(4\pi)^{\frac{(z-4)}{2}} \int_0^\infty \Omega(z-1,t)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\, dt \\ &= \alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\left(\frac{1}{e^{t/\alpha}-1}-\frac{1}{t/\alpha}\right) dt \\ &= \alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{(e^{t/\alpha}-1)(e^t-1)}-\frac{\alpha}{t(e^t-1)}-\frac{1}{t(e^{t/\alpha}-1)}+\frac{\alpha}{t^2}\right) dt, \\ &= I_1(z,\alpha)+I_2(z,\alpha), \qquad (4.8.5) \end{aligned}$$

where

$$I_1(z,\alpha) = \alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{1}{(e^{t/\alpha}-1)(e^t-1)}-\frac{\alpha}{t(e^t-1)}+\frac{e^{-t/\alpha}}{2t}\right) dt, \qquad (4.8.6)$$

and

$$I_2(z,\alpha) = \alpha^{-z/2}\int_0^\infty t^{z-1}\left(\frac{-1}{t(e^{t/\alpha}-1)}+\frac{\alpha}{t^2}-\frac{e^{-t/\alpha}}{2t}\right) dt. \qquad (4.8.7)$$

First,

$$\begin{aligned} I_1(z,\alpha) &= \alpha^{-\frac{z}{2}}\int_0^\infty t^{z-1}\left(\frac{1}{(e^{t/\alpha}-1)(e^t-1)}-\frac{\alpha}{t(e^t-1)}+\frac{1}{2(e^t-1)}-\frac{1}{2(e^t-1)}+\frac{e^{-t/\alpha}}{2t}\right) dt \\ &= \alpha^{-\frac{z}{2}}\int_0^\infty \frac{t^{z-1}}{(e^t-1)}\left(\frac{1}{(e^{t/\alpha}-1)}-\frac{1}{t/\alpha}+\frac{1}{2}\right) dt \\ &\quad -\frac{\alpha^{-\frac{z}{2}}}{2}\int_0^\infty t^{z-1}\left(\frac{1}{e^t-1}-\frac{e^{-t/\alpha}}{t}\right) dt \\ &= I_3(z,\alpha)+I_4(z,\alpha), \qquad (4.8.8) \end{aligned}$$

where

$$I_3(z,\alpha) = \alpha^{-z/2} \int_0^\infty \frac{t^{z-1}}{e^t-1} \left(\frac{1}{e^{t/\alpha}-1} - \frac{1}{t/\alpha} + \frac{1}{2} \right) dt, \tag{4.8.9}$$

and

$$I_4(z,\alpha) = -\frac{\alpha^{-z/2}}{2} \int_0^\infty t^{z-1} \left(\frac{1}{e^t-1} - \frac{e^{-t/\alpha}}{t} \right) dt. \tag{4.8.10}$$

To evaluate $I_3(z,\alpha)$, employ (4.6.25), which is valid for Re $z > -1$ and Re $a > 0$. Recognizing $1/(e^t-1)$ as the sum of a geometric series and interchanging the summation and integration because of absolute convergence yields

$$\begin{aligned} I_3(z,\alpha) &= \alpha^{-z/2} \int_0^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}} \left(\frac{1}{e^{t/\alpha}-1} - \frac{1}{t/\alpha} + \frac{1}{2} \right) dt \\ &= \alpha^{-z/2} \sum_{k=1}^\infty \int_0^\infty t^{z-1}e^{-kt} \left(\frac{1}{e^{t/\alpha}-1} - \frac{1}{t/\alpha} + \frac{1}{2} \right) dt \\ &= \alpha^{z/2} \sum_{k=1}^\infty \Gamma(z) \left(\zeta(z,k\alpha) - \frac{(k\alpha)^{-z}}{2} + \frac{(k\alpha)^{1-z}}{1-z} \right) \\ &= \alpha^{z/2}\Gamma(z) \sum_{k=1}^\infty \varphi(z,k\alpha), \end{aligned} \tag{4.8.11}$$

where in the penultimate step, the change of variable $t = \alpha x$ and then (4.6.25) were used. Next we evaluate $I_4(z,\alpha)$. Since Re $z > 1$, using (4.6.6) and the integral representation for $\Gamma(z-1)$, we find that

$$\begin{aligned} I_4(z,\alpha) &= -\frac{\alpha^{-z/2}}{2} \left(\int_0^\infty \frac{t^{z-1}}{e^t-1}\, dt - \int_0^\infty t^{z-2}e^{-t/\alpha}\, dt \right) \\ &= -\frac{\alpha^{-z/2}}{2} \left(\Gamma(z)\zeta(z) - \alpha^{z-1}\Gamma(z-1) \right) \\ &= -\frac{\alpha^{-\frac{z}{2}}}{2}\Gamma(z)\zeta(z) + \frac{\alpha^{\frac{z}{2}-1}}{2}\Gamma(z-1). \end{aligned} \tag{4.8.12}$$

Hence from (4.8.8), (4.8.11) and (4.8.12), it is seen that

$$I_1(z,\alpha) = \alpha^{z/2}\Gamma(z) \sum_{k=1}^\infty \varphi(z,k\alpha) - \frac{\alpha^{-\frac{z}{2}}}{2}\Gamma(z)\zeta(z) + \frac{\alpha^{\frac{z}{2}-1}}{2}\Gamma(z-1). \tag{4.8.13}$$

It remains to evaluate $I_2(z,\alpha)$. Now from [81, p. 23, eqn. (2.7.1)] for $0 < \text{Re } z < 1$, we have

$$\Gamma(z)\zeta(z) = \int_0^\infty t^{z-1}\left(\frac{1}{e^t-1} - \frac{1}{t}\right)\, dt. \tag{4.8.14}$$

The change of variable $u = t/\alpha$ in (4.8.7) and then an application of (4.8.14) give

$$\begin{aligned} I_2(z,\alpha) &= -\alpha^{\frac{z}{2}-1}\int_0^\infty u^{z-1}\left(\frac{1}{u(e^u-1)} - \frac{1}{u^2} + \frac{e^{-u}}{2u}\right) du \\ &= -\alpha^{\frac{z}{2}-1}\left(\int_0^\infty u^{z-2}\left(\frac{1}{e^u-1} - \frac{1}{u}\right) du + \frac{1}{2}\int_0^\infty e^{-u}u^{z-2}\, du\right) \\ &= -\alpha^{\frac{z}{2}-1}\left(\Gamma(z-1)\zeta(z-1) + \frac{1}{2}\Gamma(z-1)\right). \end{aligned} \tag{4.8.15}$$

Finally (4.8.5), (4.8.13) and (4.8.15) give

$$\frac{8(4\pi)^{\frac{(z-4)}{2}}}{\Gamma(z)}\int_0^\infty \frac{\Omega(z-1,t)}{z^2+t^2}\cos\left(\tfrac{1}{2}t\log\alpha\right)\, dt = \alpha^{\frac{z}{2}}\left(\sum_{k=1}^\infty \varphi(z,k\alpha) - \frac{\zeta(z)}{2\alpha^z} - \frac{\zeta(z-1)}{\alpha(z-1)}\right). \tag{4.8.16}$$

Now replace α by β in (4.8.16) and then combine the result with (4.8.16) to obtain (4.8.2), since with $\alpha\beta = 1$, the left-hand side of (4.8.16) is invariant under the map $\alpha \to \beta$. This implies (4.8.2) for $1 < \text{Re } z < 2$. Next, (3.3.8) and (4.6.30) readily yield absolute and uniform convergence of the extreme right-hand side of (4.8.2) for $0 < \text{Re } z < 2$, and hence it is analytic in that strip. Also the first two expressions in (4.8.2) are analytic for $0 < \text{Re } z < 2$, except for a possible pole at $z = 1$. But it is easily seen that $z = 1$ is a removable singularity because the residue of $\zeta(z)$ at $z = 1$ is equal to 1 and because $\zeta(0) = -\frac{1}{2}$. Hence by analytic continuation, (4.8.2) holds for $0 < \text{Re } z < 2$. □

Chapter 5

Character analogues of the general integral involving the Riemann Ξ-function

In this chapter, a new class of integrals involving the product of Ξ-functions associated with primitive Dirichlet characters is considered. These integrals give rise to transformation formulas of the type $F(z, \alpha, \chi) = F(-z, \beta, \overline{\chi}) = F(-z, \alpha, \overline{\chi}) = F(z, \beta, \chi)$, where $\alpha\beta = 1$. We then give new character analogues of the Ramanujan-Guinand formula and Koshliakov's formula (Theorems 4.5.3 and 3.2.3 respectively) as well as those of Ramanujan's transformation formula and its generalization (Theorems 2.1.1 and 4.3.1) as particular examples. Finally, character analogues of a conjecture of Ramanujan, Hardy and Littlewood involving infinite series of Möbius functions are derived. See [33].

5.1 Preliminary results

We review some basic properties of Dirichlet characters, L-functions and Gauss sums. Throughout this article, we will be concerned with the principal branch of the logarithm. Since we frequently use the functional equation for L-functions (see (5.1.6) below), we work only with a primitive, non-principal Dirichlet character χ modulo q, where q is the period of the character; see [5, Theorem 1]. It is easy to see that its conjugate character $\overline{\chi}$ is also a primitive, non-principal character modulo q and $\overline{\chi}$ is even (odd) if and only if χ is even (resp. odd). Let $L(s, \chi)$ denote the Dirichlet L-function defined by $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ for Re $s > 1$. This series converges conditionally for $0 <$ Re $s < 1$. Also, it can be analytically continued to an entire function of s. Let $G(\chi) := G(1, \chi)$, where $G(n, \chi)$ is the Gauss sum defined by

$$G(n,\chi) := \sum_{m=1}^{q} \chi(m) e^{2\pi i m n/q}.$$

We know that [6, p. 168]

$$|G(\chi)|^2 = q \tag{5.1.1}$$

and it is easy to see that

$$\overline{G(\chi)} = \begin{cases} G(\overline{\chi}), & \text{for } \chi \text{ even}, \\ -G(\overline{\chi}), & \text{for } \chi \text{ odd}. \end{cases} \tag{5.1.2}$$

Define b as follows:

$$b = \begin{cases} 0, & \chi(-1) = 1, \\ 1, & \chi(-1) = -1. \end{cases} \tag{5.1.3}$$

Then the function $\xi(s,\chi)$ is defined by

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-(s+b)/2} \Gamma\left(\frac{s+b}{2}\right) L(s,\chi), \tag{5.1.4}$$

and the analogue of the Riemann Ξ-function for Dirichlet characters is then defined as

$$\Xi(t,\chi) := \xi\left(\frac{1}{2} + it, \chi\right). \tag{5.1.5}$$

L-functions satisfy the functional equation [6, p. 263]

$$L(1-s,\chi) = \frac{q^{s-1}\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2} + \chi(-1) e^{\pi i s/2}\right) G(\chi) L(s,\overline{\chi}), \tag{5.1.6}$$

which can be rephrased in terms of $\xi(s,\chi)$ as [29]

$$\xi(1-s,\overline{\chi}) = \epsilon(\chi)\xi(s,\chi), \tag{5.1.7}$$

where $\epsilon(\chi) = i^b q^{1/2}/G(\chi)$. By (5.1.1), $|\epsilon(\chi)| = 1$. Now using (5.1.6) and from the fact [29, p. 82] that $|L(s,\chi)| = O(q|t|)$ for Re $s \geq 1/2$, we can easily see that for Re $s \geq -\delta$, $\delta > 0$, we

have

$$L(s,\chi) = O\left(q^{\frac{3}{2}+\delta}|t|^{\frac{3}{2}+\delta}\right). \tag{5.1.8}$$

We will subsequently use this result.

5.2 General form of an integral generating formulas $F(z, \alpha, \chi) = F(-z, \beta, \overline{\chi}) = F(-z, \alpha, \overline{\chi}) = F(z, \beta, \chi)$ where $\alpha\beta = 1$

Here, we derive a character analogue of the integral in (4.0.1). Its general form is

$$\int_0^\infty f\left(z, \frac{t}{2}\right) \Xi\left(\frac{t+iz}{2}, \overline{\chi}\right) \Xi\left(\frac{t-iz}{2}, \chi\right) \cos\left(\frac{1}{2}t\log\alpha\right) dt, \tag{5.2.1}$$

where f is an even function of both the variables z and t. We now give a formal way of transforming the above integral into an equivalent complex integral which allows us to use residue calculus and Mellin transform techniques for its evaluation.

Theorem 5.2.1. *Let*

$$f(z,t) = \frac{\phi(z,it)\phi(z,-it) + \phi(-z,it)\phi(-z,-it)}{2}, \tag{5.2.2}$$

where ϕ is analytic in t as a function of a real variable and analytic in z in some complex domain. Let $y = e^\mu$ with μ real. Then, under the assumption that the integral on the left *side below converges,*

$$\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right)\cos\mu t\,dt$$
$$=\frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)+\phi\left(-z,s-\frac{1}{2}\right)\phi\left(-z,\frac{1}{2}-s\right)\right)$$
$$\times\,\xi\left(s-\frac{z}{2},\overline{\chi}\right)\xi\left(s+\frac{z}{2},\chi\right)y^s\,ds. \tag{5.2.3}$$

Proof. Let

$$I(z,\mu,\chi):=\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right)\cos\mu t\,dt.$$

Then

$$\begin{aligned}I(z,\mu,\chi)&=\frac{1}{2}\Bigg(\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right)y^{it}\,dt\\&\qquad+\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right)y^{-it}\,dt\Bigg)\\&=\frac{1}{2}\Bigg(\int_0^\infty f(z,t)\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right)y^{it}\,dt\\&\qquad+\int_{-\infty}^0 f(z,-t)\Xi\left(-t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(-t-\frac{iz}{2},\chi\right)y^{it}\,dt\Bigg).\end{aligned} \tag{5.2.4}$$

However, using (5.1.7), we readily see that

$$\Xi\left(-t+\frac{iz}{2},\overline{\chi}\right)=\xi\left(\frac{1}{2}-it-\frac{z}{2},\overline{\chi}\right)=\epsilon(\chi)\xi\left(\frac{1}{2}+it+\frac{z}{2},\chi\right)=\epsilon(\chi)\Xi\left(t-\frac{iz}{2},\chi\right),$$
$$\Xi\left(-t-\frac{iz}{2},\chi\right)=\xi\left(\frac{1}{2}-it+\frac{z}{2},\chi\right)=(\epsilon(\chi))^{-1}\xi\left(\frac{1}{2}+it-\frac{z}{2},\overline{\chi}\right)=\epsilon(\chi)^{-1}\Xi\left(t+\frac{iz}{2},\overline{\chi}\right),$$

so that

$$\Xi\left(-t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(-t-\frac{iz}{2},\chi\right)=\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right). \tag{5.2.5}$$

Thus from (5.2.4), (5.2.5) and the fact that f is an even function of t, we obtain

$$
\begin{aligned}
I(z,\mu,\chi) &= \frac{1}{2}\int_{-\infty}^{\infty} f(z,t)\Xi\left(t+\frac{iz}{2},\overline{\chi}\right)\Xi\left(t-\frac{iz}{2},\chi\right)y^{it}\,dt\\
&= \frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)+\phi\left(-z,s-\frac{1}{2}\right)\phi\left(-z,\frac{1}{2}-s\right)\right)\\
&\qquad\times\xi\left(s-\frac{z}{2},\overline{\chi}\right)\xi\left(s+\frac{z}{2},\chi\right)y^{s}\,ds,
\end{aligned}
$$

where in the penultimate line, we made the change of variable $s=\frac{1}{2}+it$. □

For our purpose here, we replace μ by 2μ in (5.2.3) and then t by $t/2$ on the left-hand side of (5.2.3). Thus with $y=e^{2\mu}$, we find that

$$
\begin{aligned}
&\int_{0}^{\infty} f\left(z,\frac{t}{2}\right)\Xi\left(\frac{t+iz}{2},\overline{\chi}\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\mu t\,dt\\
&= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(\phi\left(z,s-\frac{1}{2}\right)\phi\left(z,\frac{1}{2}-s\right)+\phi\left(-z,s-\frac{1}{2}\right)\phi\left(-z,\frac{1}{2}-s\right)\right)\\
&\qquad\times\xi\left(s-\frac{z}{2},\overline{\chi}\right)\xi\left(s+\frac{z}{2},\chi\right)y^{s}\,ds. \qquad (5.2.6)
\end{aligned}
$$

It is this equation with which we will be working throughout this chapter.

Transformation formulas involving Dirichlet characters of the form

$$\sum_{n=1}^{\infty}\chi(n)f(n)=\sum_{n=1}^{\infty}\overline{\chi}(n)g(n),$$

where

$$
g(x)=\begin{cases}\dfrac{2G(\chi)}{q}\displaystyle\int_{0}^{\to\infty}\cos\left(\frac{2\pi xt}{q}\right)f(t)\,dt, & \text{for } \chi(-1)=1,\\[2ex] \dfrac{-2iG(\chi)}{q}\displaystyle\int_{0}^{\to\infty}\sin\left(\frac{2\pi xt}{q}\right)f(t)\,dt, & \text{for } \chi(-1)=-1,\end{cases}
$$

were considered by Guinand [44, Theorems 4–5], though he did not give any particular examples. In this chapter, by means of (5.2.6), we generate character analogues of some of the theorems from previous chapters.

5.3 Character analogue of the extended version of the Ramanujan-Guinand formula

The character analogue of Theorem 4.5.3 for even and odd primitive characters are stated in the theorem below.

Theorem 5.3.1. *Let* $-1 < Re\ z < 1$ *and let* χ *denote a primitive, non-principal character modulo* q*. Let the number* b *be defined as in* (5.1.3)*. Let* $K_\nu(z), d(n)$ *and* γ *be defined as before and let* α *and* β *be positive numbers such that* $\alpha\beta = 1$*. If*

$$F(z,\alpha,\chi) := \alpha^{b+\frac{1}{2}} \sum_{n=1}^{\infty} \chi(n) n^{-\frac{z}{2}+b} \left(\sum_{d|n} \overline{\chi}^2(d) d^z \right) K_{-\frac{z}{2}} \left(\frac{2\pi n \alpha}{q} \right),$$

then

$$\begin{aligned} F(z,\alpha,\chi) &= F(-z,\beta,\overline{\chi}) = F(-z,\alpha,\overline{\chi}) = F(z,\beta,\chi) \\ &= \frac{1}{8\pi} \int_0^\infty \Xi\left(\frac{t+iz}{2},\overline{\chi}\right) \Xi\left(\frac{t-iz}{2},\chi\right) \cos\left(\frac{1}{2} t \log\alpha\right)\, dt. \end{aligned} \quad (5.3.1)$$

We will require a lemma which gives a Dirichlet series representation for a product of two Dirichlet L-functions.

Lemma 5.3.2. *For* Re $s > 1$ *and* Re $(s-\eta) > 1$*,*

$$L(s,\overline{\chi}) L(s-\eta,\chi) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^s} \sum_{d|n} \chi^2(d) d^\eta. \quad (5.3.2)$$

Proof. Since the Dirichlet series for both the L-functions converge absolutely under the given hypotheses, using [6, Theorem 11.5] and the fact that $\chi(k)\overline{\chi}(k) = 1$, we see that

$$L(s,\overline{\chi}) L(s-\eta,\chi) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\overline{\chi}(n)}{n^s} \sum_{\substack{k=1 \\ (k,q)=1}}^{\infty} \frac{\chi(k)}{k^{s-\eta}}$$

$$
\begin{aligned}
&= \sum_{\substack{j=1\\(j,q)=1}}^{\infty} \frac{1}{j^s} \sum_{\substack{nk=j\\(k,q)=1}} \overline{\chi}(n)\chi(k)k^{\eta} \\
&= \sum_{\substack{j=1\\(j,q)=1}}^{\infty} \frac{\overline{\chi}(j)}{j^s} \sum_{\substack{nk=j\\(k,q)=1}} \frac{\chi^2(k)}{\chi(k)\overline{\chi}(k)} k^{\eta}, \\
&= \sum_{j=1}^{\infty} \frac{\overline{\chi}(j)}{j^s} \sum_{nk=j} \chi^2(k)k^{\eta},
\end{aligned}
$$

where in the last step, we have made use of the fact that $\chi(k)\overline{\chi}(k) = 1$ for $(k,q) = 1$. □

Now we first give a proof of Theorem 5.3.1 when χ is even and then give a brief sketch of the proof for χ odd, since the details are similar.

Proof. Let $\phi(z,s) \equiv 1$. Then from (5.2.2), we see that $f(z,t) \equiv 1$. Using (5.1.5), (5.1.4), (3.3.8) and (5.1.8), we find that the integral

$$
M(z,\mu,\chi) := \int_0^{\infty} \Xi\left(\frac{t+iz}{2}, \overline{\chi}\right) \Xi\left(\frac{t-iz}{2}, \chi\right) \cos \mu t \, dt
$$

does converge. Using (5.2.6), we observe that

$$
\begin{aligned}
M(z,\mu,\chi) &= \frac{1}{i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \xi\left(s-\frac{z}{2}, \overline{\chi}\right) \xi\left(s+\frac{z}{2}, \chi\right) y^s \, ds \\
&= \frac{1}{i\sqrt{y}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{s}{2}-\frac{z}{4}\right) \Gamma\left(\frac{s}{2}+\frac{z}{4}\right) L\left(s-\frac{z}{2}, \overline{\chi}\right) L\left(s+\frac{z}{2}, \chi\right) \left(\frac{\pi}{qy}\right)^{-s} ds.
\end{aligned} \tag{5.3.3}
$$

Since Re $s = 1/2$ and $-1 < \text{Re } z < 1$, we have $0 < \text{Re}\left(s - \frac{z}{2}\right) < 1$ and $0 < \text{Re}\left(s + \frac{z}{2}\right) < 1$. Now replace s by $s - \frac{z}{2}$ and let $\eta = -z$ in Lemma 5.3.2. Then, for Re $\left(s - \frac{z}{2}\right) > 1$ and Re $\left(s + \frac{z}{2}\right) > 1$,

$$
L\left(s-\frac{z}{2}, \overline{\chi}\right) L\left(s+\frac{z}{2}, \chi\right) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^{s-\frac{z}{2}}} \sum_{d|n} \chi^2(d) d^{-z}. \tag{5.3.4}
$$

We wish to shift the line of integration from Re $s = 1/2$ to Re $s = 3/2$ in order to be able to use (5.3.4) in (5.3.3). Consider a positively oriented rectangular contour formed by $[\frac{1}{2}+iT, \frac{1}{2}-iT], [\frac{1}{2}-iT, \frac{3}{2}-iT], [\frac{3}{2}-iT, \frac{3}{2}+iT]$ and $[\frac{3}{2}+iT, \frac{1}{2}+iT]$, where T is any positive real number. The integrand on the extreme right-hand side of (5.3.3) does not have any pole inside the contour. Also as $T \to \infty$, the integrals along the horizontal segments $[\frac{1}{2}-iT, \frac{3}{2}-iT]$ and $[\frac{3}{2}+iT, \frac{1}{2}+iT]$ tend to zero, which can be seen by using (3.3.8). Hence, employing the residue theorem, letting $T \to \infty$, using (5.3.4) in (5.3.3), and interchanging the order of summation and integration because of absolute convergence, we observe that

$$M(z,\mu,\chi) = \frac{1}{i\sqrt{y}} \sum_{n=1}^{\infty} \overline{\chi}(n) n^{z/2} \Bigg(\sum_{d|n} \chi^2(d) d^{-z} \Bigg) \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(\frac{s}{2}-\frac{z}{4}\right) \Gamma\left(\frac{s}{2}+\frac{z}{4}\right) \left(\frac{n\pi}{qy}\right)^{-s} ds. \tag{5.3.5}$$

But from [70, p. 115, formula 11.1], for $c = \text{Re } s > \pm \text{ Re } \nu$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s-2} w^{-s} \Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right) \Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right) x^{-s}\, ds = K_\nu(wx). \tag{5.3.6}$$

Hence using (5.3.6) with $c = 3/2$, $\nu = z/2$, $w = 2$ and $x = n\pi/qy$ in (5.3.5), we find that

$$M(z,\mu,\chi) = \frac{8\pi}{\sqrt{y}} \sum_{n=1}^{\infty} \overline{\chi}(n) n^{z/2} \Bigg(\sum_{d|n} \chi^2(d) d^{-z} \Bigg) K_{\frac{z}{2}}\left(\frac{2\pi n}{qy}\right). \tag{5.3.7}$$

Now let $\mu = \frac{1}{2}\log\alpha$ in (5.3.7) so that $y = e^{2\mu}$ implies that $y = \alpha$. Then using the fact that $\alpha\beta = 1$, we deduce that

$$\begin{aligned} &\frac{1}{8\pi} \int_0^{\infty} \Xi\left(\frac{t+iz}{2}, \overline{\chi}\right) \Xi\left(\frac{t-iz}{2}, \chi\right) \cos\left(\frac{1}{2} t \log\alpha\right) dt \\ &= \sqrt{\beta} \sum_{n=1}^{\infty} \overline{\chi}(n) n^{z/2} \Bigg(\sum_{d|n} \chi^2(d) d^{-z} \Bigg) K_{\frac{z}{2}}\left(\frac{2\pi n\beta}{q}\right). \end{aligned} \tag{5.3.8}$$

Next, observing that replacing α by β and/or replacing simultaneously χ by $\overline{\chi}$ and z by $-z$

in (5.3.8) leaves the integral on the left-hand side invariant, we obtain (5.3.1).

Now consider the case when χ is odd. Again the convergence of the integral $M(z,\mu,\chi)$ can be seen from (3.3.8) and (5.1.8). Following similar steps above as in the case of even χ, and using the definition of $\xi(s,\chi)$ from (5.1.4) for χ odd, we find that

$$M(z,\mu,\chi)=\frac{q}{i\pi\sqrt{y}}\sum_{n=1}^{\infty}\overline{\chi}(n)n^{z/2}\sum_{d|n}\chi^2(d)d^{-z}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma\left(\frac{s}{2}-\frac{z}{4}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{z}{4}+\frac{1}{2}\right)\times\left(\frac{n\pi}{qy}\right)^{-s}ds. \tag{5.3.9}$$

Now replacing s by $s+1$ in (5.3.6), we find that for $c=\text{Re } s>\pm\text{ Re }\nu-1$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-1}w^{-s-1}\Gamma\left(\frac{s+1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{s+1}{2}+\frac{\nu}{2}\right)x^{-s}\,ds=xK_{\nu}(wx). \tag{5.3.10}$$

Then using (5.3.10) with $c=3/2$, $\nu=0$, $w=2$ and $x=n\pi/qy$ in (5.3.9), we see that

$$M(z,\mu,\chi)=\frac{8\pi}{y^{3/2}}\sum_{n=1}^{\infty}\overline{\chi}(n)n^{\frac{z}{2}+1}\left(\sum_{d|n}\chi^2(d)d^{-z}\right)K_{z/2}\left(\frac{2\pi n}{qy}\right). \tag{5.3.11}$$

Now let $\mu=\frac{1}{2}\log\alpha$ in (5.3.11) so that $y=e^{2\mu}$ implies that $y=\alpha$. Then using the fact that $\alpha\beta=1$, we deduce that

$$\frac{1}{8\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2},\overline{\chi}\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt$$
$$=\beta^{3/2}\sum_{n=1}^{\infty}\overline{\chi}(n)n^{\frac{z}{2}+1}\left(\sum_{d|n}\chi^2(d)d^{-z}\right)K_{z/2}\left(\frac{2\pi n\beta}{q}\right). \tag{5.3.12}$$

Next, observing that replacing α by β and/or replacing simultaneously χ by $\overline{\chi}$ and z by $-z$ in (5.3.12) leaves the integral on the left-hand side invariant, we obtain (5.3.1). □

Remark. Letting $z\to 0$ in Theorem 5.3.1 gives a new character analogue of the extended version of Koshliakov's formula, that is, Theorem 3.2.3.

When χ is real, Theorem 5.3.1 reduces to the following corollary.

Corollary 5.3.3. *Let $-1 < \text{Re } z < 1$ and let χ denote a real, primitive, non-principal character modulo q. Let the number b be defined as in* (5.1.3). *If*

$$F(z,\alpha,\chi)=\alpha^{b+\frac{1}{2}}\sum_{n=1}^{\infty}\chi(n)n^{-\frac{z}{2}+b}\sigma_z(n)K_{-\frac{z}{2}}\left(\frac{2\pi n\alpha}{q}\right),$$

then

$$\begin{aligned}F(z,\alpha,\chi)&=F(-z,\beta,\chi)=F(-z,\alpha,\chi)=F(z,\beta,\chi)\\&=\frac{1}{8\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2},\chi\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt.\end{aligned}$$

The above corollary (without the integral) is equivalent to the special cases, when χ is real, of the character analogues of the Ramanujan-Guinand formula established in [18] (see Theorems 3.1 and 4.1).

5.4 Character analogues of Ramanujan's transformation formula and its generalization

Here, we give character analogues of Theorems 2.1.1 and 4.3.1. Define $\psi(a,\chi)$ by

$$\psi(a,\chi)=-\sum_{n=1}^{\infty}\frac{\chi(n)}{n+a},\qquad(5.4.1)$$

where $a\in\mathbb{C}\backslash\mathbb{Z}_{<0}$. For a real character χ, this agrees with the character analogue of the psi function obtained by the logarithmic differentiation of the following Weierstrass product form of the character analogue of the gamma function for real characters derived by B.C. Berndt [10]:

$$\Gamma(a,\chi)=e^{-aL(1,\chi)}\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)^{-\chi(n)}e^{a\chi(n)/n}.$$

The character analogue of the Hurwitz zeta function $\zeta(z,a)$ is given by [9, Ex. 3.2]

$$L(z,a,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{(n+a)^z}, \tag{5.4.2}$$

valid for Re $z>0$, and provided $a\in\mathbb{C}$ is a non-negative integer. The above character analogue of the Hurwitz zeta function can also be obtained as the special case when $x=0$ of the function $L(z,x,a,\chi)$ defined by [10]

$$L(z,x,a,\chi):={\sum_{n=0}^{\infty}}' e^{2\pi inx/q}\chi(n)(n+a)^{-z},$$

where the prime indicates that the term corresponding to $n=-a$ is omitted if a is a negative integer and $\chi(a)\neq 0$. As shown in [10], $L(z,x,a,\chi)$ converges for Re $z>0$ if x is not an integer, or if x is an integer and $\gcd(x,q)>1$. If x is an integer and $\gcd(x,q)=1$, the series converges for Re $z>1$. For mean value properties of $L(z,a,\chi)$ and asymptotic formulas, see the recent paper [68]. The character analogues of Theorem 4.3.1 are given below.

Theorem 5.4.1. *Let χ denote an even, primitive, non-principal character modulo q. Let $-1<$ Re $z<1$, and let $L(z,a,\chi)$ be defined as in* (5.4.2). *Define $T(z,\alpha,\chi)$ by*

$$T(z,\alpha,\chi):=\frac{\alpha^{z/2}q^{z/2}\Gamma(z+1)}{2^z\pi^{z/2}G(\chi)}, \tag{5.4.3}$$

and $\Delta(z,t)$ by

$$\begin{aligned}\Delta(z,t):=((z+1)^2+t^2)\Gamma\left(\frac{-z-1+it}{4}\right)\Gamma\left(\frac{-z-1-it}{4}\right)\\ +((z-1)^2+t^2)\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right).\end{aligned} \tag{5.4.4}$$

Then if α and β are positive numbers such that $\alpha\beta = 1$,

$$\begin{aligned}
&\sqrt{\alpha}\left[T(z,\alpha,\chi)\sum_{n=1}^{\infty}\chi(n)L\bigg(z+1,n\alpha,\chi\bigg)+T(-z,\alpha,\overline{\chi})\sum_{n=1}^{\infty}\overline{\chi}(n)L\bigg(-z+1,n\alpha,\overline{\chi}\bigg)\right]\\
&=\sqrt{\beta}\left[T(-z,\beta,\overline{\chi})\sum_{n=1}^{\infty}\overline{\chi}(n)L\bigg(-z+1,n\beta,\overline{\chi}\bigg)+T(z,\beta,\chi)\sum_{n=1}^{\infty}\chi(n)L\bigg(z+1,n\beta,\chi\bigg)\right]\\
&=\frac{1}{64\pi^{3/2}q}\int_0^{\infty}\Delta(z,t)\Xi\left(\frac{t+iz}{2},\overline{\chi}\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt. \qquad (5.4.5)
\end{aligned}$$

Theorem 5.4.2. *Let χ denote an odd, primitive, non-principal character modulo q. Let $-1 < \text{Re } z < 1$ and let $L(z,a,\chi)$ be defined as in* (5.4.2) *and let $T(z,\alpha,\chi)$ be defined as in* (5.4.3)*. Define $\Lambda(z,t)$ by*

$$\Lambda(z,t) := \Gamma\bigg(\frac{z+1+it}{4}\bigg)\Gamma\bigg(\frac{z+1-it}{4}\bigg)+\Gamma\bigg(\frac{-z+1+it}{4}\bigg)\Gamma\bigg(\frac{-z+1-it}{4}\bigg). \qquad (5.4.6)$$

Then if α and β are positive numbers such that $\alpha\beta = 1$,

$$\begin{aligned}
&\sqrt{\alpha}\left[T(z,\alpha,\chi)\sum_{n=1}^{\infty}\chi(n)L\bigg(z+1,n\alpha,\chi\bigg)+T(-z,\alpha,\overline{\chi})\sum_{n=1}^{\infty}\overline{\chi}(n)L\bigg(-z+1,n\alpha,\overline{\chi}\bigg)\right]\\
&=\sqrt{\beta}\left[T(-z,\beta,\overline{\chi})\sum_{n=1}^{\infty}\overline{\chi}(n)L\bigg(-z+1,n\beta,\overline{\chi}\bigg)+T(z,\beta,\chi)\sum_{n=1}^{\infty}\chi(n)L\bigg(z+1,n\beta,\chi\bigg)\right]\\
&=\frac{1}{4\pi^{1/2}iq^2}\int_0^{\infty}\Lambda(z,t)\Xi\left(\frac{t+iz}{2},\overline{\chi}\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt. \qquad (5.4.7)
\end{aligned}$$

5.4.1 Inverse Mellin transforms and asymptotic expansions of certain functions

In this section, we evaluate inverse Mellin transforms of some functions and asymptotic expansions of certain other functions all of which are subsequently used in the later sections.

Lemma 5.4.3. *Let $z \in \mathbb{C}$ be fixed such that $-1 < \text{Re } z < 1$. For a primitive, non-principal character χ mod q, let $L(z,a,\chi)$ be defined as in* (5.4.2)*. Then for $-\text{Re }\frac{z}{2} < c = \text{Re } s < 1+$*

Re $\frac{z}{2}$ *and* $x \in \mathbb{R}\backslash\mathbb{Z}_{<0}$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)x^{-s}\,ds = x^{z/2}\Gamma(z+1)L(z+1,x,\chi). \tag{5.4.8}$$

Proof. We prove the result only for even characters. The case for odd characters can be proved similarly. We first assume $|x|<1$ and later extend it to any real $x \in \mathbb{R}\backslash\mathbb{Z}_{<0}$ by analytic continuation. Let $-\text{Re }\frac{z}{2} < c = \text{Re } s < 1+ \text{ Re }\frac{z}{2}$. Consider a positively oriented rectangular contour formed by $[c-iT, c+iT], [c+iT, -M+iT], [-M+iT, -M-iT]$ and $[-M-iT, c-iT]$, where T is some positive real number and $M = n-1/2$, where n is a positive integer. Let $s = \sigma+it$. Among the poles of the function $\Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}$, the only ones that contribute are the poles at $s = -z/2-m, m \geq 0$. Let $R_f(a)$ denote the residue of the function $f(s) := \Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}$ at a. Then for $m \geq 0$,

$$\begin{aligned}R_f\left(-\frac{z}{2}-m\right) &= \lim_{s\to-\frac{z}{2}-m}\left(s+\frac{z}{2}+m\right)\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)x^{-s}\\ &= \frac{(-1)^m}{m!}\Gamma\left(1+z+m\right)L\left(1+z+m,\chi\right)x^{m+z/2}.\end{aligned} \tag{5.4.9}$$

From (5.4.9) and the residue theorem, we have

$$\begin{aligned}&\left[\int_{c-iT}^{c+iT}+\int_{c+iT}^{-M+iT}+\int_{-M+iT}^{-M-iT}+\int_{-M-iT}^{c-iT}\right]\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)x^{-s}\,ds\\ &= 2\pi i x^{z/2}\sum_{0\leq m<M}\frac{(-1)^m}{m!}\Gamma\left(1+z+m\right)L\left(1+z+m,\chi\right)x^{m}.\end{aligned} \tag{5.4.10}$$

We now estimate the integral along the upper horizontal segment. Using (5.1.8), we find that for $-M \leq \sigma \leq c$,

$$L(1-\sigma \pm iT,\chi) = O\left(q^{c+1/2}T^{c+1/2}\right). \tag{5.4.11}$$

Then (5.4.11) implies that for $-M \leq \sigma \leq c$, i.e., $-M-$ Re $\frac{z}{2} \leq \sigma-$ Re $\frac{z}{2} \leq c-$ Re $\frac{z}{2}$, we have

$$L\left(1-\left(\sigma-\mathrm{Re}\frac{z}{2}\right)-i\left(T-\mathrm{Im}\frac{z}{2}\right),\chi\right)=O\left(q^{c-\mathrm{Re}\frac{z}{2}+\frac{1}{2}}\left(T-\mathrm{Im}\frac{z}{2}\right)^{c-\mathrm{Re}\frac{z}{2}+\frac{1}{2}}\right). \quad (5.4.12)$$

By (3.3.8), we observe that

$$\left|\Gamma\left(s+\frac{z}{2}\right)\right|\sim\sqrt{2\pi}e^{-\frac{\pi}{2}\left|T+\mathrm{Im}\frac{z}{2}\right|}\cdot\left|T+\mathrm{Im}\frac{z}{2}\right|^{\sigma+\mathrm{Re}\frac{z}{2}-\frac{1}{2}}, \quad (5.4.13)$$

and

$$\left|\Gamma\left(1-s+\frac{z}{2}\right)\right|\sim\sqrt{2\pi}e^{-\frac{\pi}{2}\left|T-\mathrm{Im}\frac{z}{2}\right|}\cdot\left|T-\mathrm{Im}\frac{z}{2}\right|^{-\sigma+\mathrm{Re}\frac{z}{2}+\frac{1}{2}}. \quad (5.4.14)$$

Since $|x|<1$, from (5.4.12), (5.4.13) and (5.4.14), we deduce that

$$\left|\int_{c+iT}^{-M+iT}\Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}\,ds\right|$$
$$\leq 2\pi K_1(c+M)|x|^{-c}q^{c-\mathrm{Re}\frac{z}{2}+\frac{1}{2}}e^{-\frac{\pi}{2}\left(\left|T+\mathrm{Im}\frac{z}{2}\right|+\left|T-\mathrm{Im}\frac{z}{2}\right|\right)}\left|T+\mathrm{Im}\frac{z}{2}\right|^{\sigma+\mathrm{Re}\frac{z}{2}-\frac{1}{2}}\left|T-\mathrm{Im}\frac{z}{2}\right|^{c-\sigma+1},$$

where K_1 is some absolute constant. Hence,

$$\int_{c+i\infty}^{-M+i\infty}\Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}\,ds=0. \quad (5.4.15)$$

Similarly for the integral along the lower horizontal segment, using (5.4.13), (5.4.14) and the fact that

$$L\left(1-\left(\sigma-\mathrm{Re}\frac{z}{2}\right)+i\left(T+\mathrm{Im}\frac{z}{2}\right),\chi\right)=O\left(q^{c-\mathrm{Re}\frac{z}{2}+\frac{1}{2}}\left(T+\mathrm{Im}\frac{z}{2}\right)^{c-\mathrm{Re}\frac{z}{2}+\frac{1}{2}}\right),$$

we observe that

$$\int_{-M-i\infty}^{c-i\infty}\Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}\,ds=0. \quad (5.4.16)$$

Hence from (5.4.10), (5.4.15) and (5.4.16), it is clear that

$$\left[\int_{c-i\infty}^{c+i\infty}+\int_{-M+i\infty}^{-M-i\infty}\right]\Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}\,ds$$
$$=2\pi i x^{z/2}\sum_{0\le m<M}\frac{(-1)^m}{m!}\Gamma\left(1+z+m\right)L\left(1+z+m,\chi\right)x^m. \tag{5.4.17}$$

It remains to evaluate $\int_{-M+i\infty}^{-M-i\infty}\Gamma(s+z/2)\Gamma(1-s+z/2)L(1-s+z/2,\chi)x^{-s}\,ds$. Using (3.3.8) and the reflection formula for the gamma function [80, p. 46, Equation (3.5)], we find that as $|t|\to\infty$,

$$\Gamma\left(-M+it\right)=O\left(|t|^{-M-1/2}e^{-\pi|t|/2}\right).$$

Hence as $|t|\to\infty$,

$$\Gamma\left(-M+it+\frac{z}{2}\right)=O\left(\left|t+\mathrm{Im}\frac{z}{2}\right|^{-M+\mathrm{Re}\frac{z}{2}-\frac{1}{2}}e^{-\frac{\pi}{2}|t+\mathrm{Im}\frac{z}{2}|}\right). \tag{5.4.18}$$

Again by (3.3.8), as $|t|\to\infty$,

$$\left|\Gamma\left(1+M-it+\frac{z}{2}\right)\right|=\sqrt{2\pi}e^{-\frac{\pi}{2}|t-\mathrm{Im}\frac{z}{2}|}\cdot\left|t-\mathrm{Im}\frac{z}{2}\right|^{M+\mathrm{Re}\frac{z}{2}+\frac{1}{2}}\left(1+O\left(\frac{1}{\left|t-\mathrm{Im}\frac{z}{2}\right|}\right)\right). \tag{5.4.19}$$

Also, $L\left(1+M-it+\frac{z}{2},\chi\right)$ is bounded as $\mathrm{Re}\left(1+M-it+\frac{z}{2}\right)>1$. Hence,

$$\left|\int_{-M+i\infty}^{-M-i\infty}\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)x^{-s}\,ds\right|$$
$$=\left|i\int_{-\infty}^{\infty}\Gamma\left(-M+it+\frac{z}{2}\right)\Gamma\left(1+M-it+\frac{z}{2}\right)L\left(1+M-it+\frac{z}{2},\chi\right)x^{M-it}\,dt\right|$$
$$=|x|^M\int_{-1}^{1}O(1)\,dt$$
$$+|x|^M\int_{1}^{\pm\infty}O\left(\left|t+\mathrm{Im}\frac{z}{2}\right|^{-M+\mathrm{Re}\frac{z}{2}-\frac{1}{2}}\left|t-\mathrm{Im}\frac{z}{2}\right|^{M+\mathrm{Re}\frac{z}{2}+\frac{1}{2}}e^{-\frac{\pi}{2}\left(\left|t+\mathrm{Im}\frac{z}{2}\right|+\left|t-\mathrm{Im}\frac{z}{2}\right|\right)}\right)dt$$
$$=O\left(|x|^M\right).$$

Since $|x| < 1$,

$$\lim_{M\to\infty} \int_{-M+i\infty}^{-M-i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)x^{-s}\,ds = 0. \tag{5.4.20}$$

From (5.4.17) and (5.4.20), we finally deduce that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)x^{-s}\,ds \\
&= x^{z/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\Gamma\left(1+z+m\right)L\left(1+z+m,\chi\right)x^{m+z/2} \\
&= x^{z/2}\Gamma(z+1)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\Gamma\left(1+z+m\right)}{\Gamma(1+z)}\sum_{k=1}^{\infty}\frac{\chi(k)}{k^{z+m+1}}x^m \\
&= x^{z/2}\Gamma(z+1)\sum_{k=1}^{\infty}\frac{\chi(k)}{k^{z+1}}\sum_{m=0}^{\infty}\frac{\Gamma(1+z+m)}{m!\Gamma(1+z)}\left(\frac{-x}{k}\right)^m \\
&= x^{z/2}\Gamma(z+1)\sum_{k=1}^{\infty}\frac{\chi(k)}{k^{z+1}}\left(1+\frac{x}{k}\right)^{-z-1} \\
&= x^{z/2}\Gamma(z+1)\sum_{k=1}^{\infty}\frac{\chi(k)}{(k+x)^{z+1}} \\
&= x^{z/2}\Gamma(z+1)L(z+1,x,\chi),
\end{aligned}$$

where in the fourth step above, we have utilized the binomial theorem since $|x| < 1$. Since both sides of (5.4.8) are analytic for any $x \in \mathbb{R}\backslash\mathbb{Z}_{<0}$, the result follows by analytic continuation. □

When $z = 0$, we get the following corollary.

Corollary 5.4.4. *For a primitive, non-principal character χ mod q, let $\psi(a,\chi)$ be defined as in* (5.4.1)*. Then for $0 < c = \text{Re } s < 1$ and $x \in \mathbb{R}\backslash\mathbb{Z}_{<0}$,*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{L(1-s,\chi)}{\sin \pi s}x^{-s}\,ds = -\frac{1}{\pi}\psi(x,\chi). \tag{5.4.21}$$

For $j \geq 1$, the generalized Bernoulli numbers $B_j(\chi)$ are given by [10, p. 426]

$$B_{2j}(\chi) = \frac{2(-1)^{j-1}G(\overline{\chi})(2j)!}{q(2\pi/q)^{2j}}L(2j,\chi),$$

for χ even, and by

$$B_{2j-1}(\chi) = \frac{2(-1)^{j-1}iG(\overline{\chi})(2j-1)!}{q(2\pi/q)^{2j-1}}L(2j-1,\chi), \tag{5.4.22}$$

for χ odd. Also it is known [10, p. 423, Corollary 3.4] that $B_{2j-1}(\chi) = 0$ when χ is even and $B_{2j}(\chi) = 0$ when χ is odd. The asymptotic expansion of $L(z, a, \chi)$ as $|a| \to \infty$ is given below.

Lemma 5.4.5. *For* Re $z > 0$ *and* $-\pi < \arg a < \pi$, *as* $|a| \to \infty$,

$$L(z,a,\chi) \sim \chi(-1)\sum_{j=1}^{\infty}\frac{B_j(\overline{\chi})\prod_{m=0}^{j-2}(z+m)}{j!a^{z+j-1}}.$$

Proof. One takes (4.3) and (4.4) in [10, p. 424] valid for χ even and odd respectively, substitutes $A = 0, B = N, r = 1$ and $f(u) = (u + a)^{-z}$, lets $N \to \infty$ and performs repeated integration by parts on the prevalent integral. □

This gives, as a special case, the following asymptotic expansion of $\psi(a, \chi)$ as $|a| \to \infty$.

Corollary 5.4.6. *For* $-\pi < \arg a < \pi$, *as* $|a| \to \infty$,

$$\psi(a,\chi) \sim -\frac{L(0,\chi)}{a} - \chi(-1)\sum_{j=2}^{\infty}\frac{B_j(\overline{\chi})}{ja^j}. \tag{5.4.23}$$

Proof. Specialize $z = 1$ in Lemma 5.4.5. Observe that $L(1, a, \chi) = -\psi(a, \chi)$. For χ even, we have $B_1(\overline{\chi}) = 0$. But from [6, p. 268], $L(0, \chi) = 0$. This yields (5.4.23) for χ even. For χ

odd, we observe from (5.4.22) that

$$B_1(\overline{\chi}) = \frac{i}{\pi} G(\chi) L(1, \overline{\chi}), \tag{5.4.24}$$

and from (5.1.6), it is easy to see that

$$L(1, \overline{\chi}) = \frac{i\pi}{G(\chi)} L(0, \chi). \tag{5.4.25}$$

Now (5.4.23) follows from (5.4.24) and (5.4.25). □

5.4.2 Proofs of Theorems 5.4.1 and 5.4.2

We begin with the proof of Theorem 5.4.1.

Proof. Using Lemma 5.4.5, one sees that the series involving the functions $L(z, a, \chi)$ in the theorem are convergent. Let $\phi(z, s) = (z + 1 + 2s)\Gamma\left(\frac{-z-1}{4} + \frac{s}{2}\right)$. Then from (4.1.1) and (4.6.1), we find that $f\left(z, \frac{t}{2}\right) = \frac{1}{2}\Delta(z, t)$. From (5.2.6), we have

$$\int_0^\infty \Delta(z, t) \Xi\left(\frac{t + iz}{2}, \overline{\chi}\right) \Xi\left(\frac{t - iz}{2}, \chi\right) \cos \mu t \, dt = \frac{1}{i\sqrt{y}} \left(J(z, y, \chi) + J(-z, y, \overline{\chi})\right), \tag{5.4.26}$$

where

$$J(z, y, \chi) := \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} U(z, s, y, \chi) \, ds \tag{5.4.27}$$

with

$$U(z, s, y, \chi) := (-z+2s)(-z+2-2s)\Gamma\left(\frac{z}{4} + \frac{s}{2} - \frac{1}{2}\right) \Gamma\left(\frac{z}{4} - \frac{s}{2}\right) \xi\left(s - \frac{z}{2}, \overline{\chi}\right) \xi\left(s + \frac{z}{2}, \chi\right) y^s.$$

Using (3.3.8) and (5.1.8), one sees that indeed the integral on the left side of (5.4.26) converges. We first simplify the integrand in (5.4.27). Using (5.1.4) with $b = 0$, (1.0.9) and

(1.0.8) in the second equality below, we have

$$\begin{aligned} U(z,s,y,\chi) &= 16\left(\frac{\pi}{qy}\right)^{-s}\left\{\Gamma\left(\frac{z}{4}+\frac{s+1}{2}\right)\Gamma\left(\frac{z}{4}+\frac{s}{2}\right)\right\}\left\{\Gamma\left(\frac{z}{4}-\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}-\frac{z}{4}\right)\right\} \\ &\quad\times L\left(s-\frac{z}{2},\overline{\chi}\right)L\left(s+\frac{z}{2},\chi\right) \\ &= 16\left(\frac{\pi}{qy}\right)^{-s}\cdot\frac{\sqrt{\pi}}{2^{s+\frac{z}{2}-1}}\Gamma\left(s+\frac{z}{2}\right)\cdot\frac{\pi}{\sin\left(\pi\left(\frac{s}{2}-\frac{z}{4}\right)\right)}\cdot L\left(s-\frac{z}{2},\overline{\chi}\right)L\left(s+\frac{z}{2},\chi\right). \end{aligned} \tag{5.4.28}$$

Substituting (5.1.6) in the form

$$L\left(s-\frac{z}{2},\overline{\chi}\right)=\frac{(2\pi)^{s-\frac{z}{2}}L(1-s+\frac{z}{2},\chi)}{2q^{s-\frac{z}{2}-1}G(\chi)\Gamma\left(s-\frac{z}{2}\right)\cos\left(\frac{\pi}{2}\left(s-\frac{z}{2}\right)\right)}$$

in (5.4.28) and then simplifying, we find that

$$U(z,s,y,\chi)=\frac{32y^s2^{-z}\pi^{(1-z)/2}}{q^{-\frac{z}{2}-1}G(\chi)}\Gamma\left(1-s+\frac{z}{2}\right)\Gamma\left(s+\frac{z}{2}\right)L\left(1-s+\frac{z}{2},\chi\right)L\left(s+\frac{z}{2},\chi\right). \tag{5.4.29}$$

We wish to shift the line of integration from Re $s=1/2$ to Re $s=3/2$ in order to evaluate (5.4.27), since then $-1<$ Re $z<1$ implies that Re $(s+z/2)>1$, which allows us to use the series representation of $L\left(s+\frac{z}{2},\chi\right)$. Consider a positively oriented rectangular contour formed by $[\frac{1}{2}+iT,\frac{1}{2}-iT]$, $[\frac{1}{2}-iT,\frac{3}{2}-iT]$, $[\frac{3}{2}-iT,\frac{3}{2}+iT]$ and $[\frac{3}{2}+iT,\frac{1}{2}+iT]$, where T is any positive real number. The integrand in (5.4.27) does not have any pole inside the contour since the pole of $\Gamma\left(1-s+\frac{z}{2}\right)$ at $s=1+z/2$ is cancelled by the zero of $L\left(1-s+\frac{z}{2},\chi\right)$ there. Also as $T\to\infty$, the integrals along the horizontal segments $[\frac{1}{2}-iT,\frac{3}{2}-iT]$ and $[\frac{3}{2}+iT,\frac{1}{2}+iT]$ tend to zero, which can be seen using (3.3.8). Employing the residue theorem, letting $T\to\infty$ and using (5.4.29), we find that

$$\begin{aligned} J(z,y,\chi) = \frac{32\cdot2^{-z}\pi^{(1-z)/2}}{q^{-\frac{z}{2}-1}G(\chi)}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}&\Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right) \\ &\times L\left(1-s+\frac{z}{2},\chi\right)L\left(s+\frac{z}{2},\chi\right)y^s\,ds \end{aligned}$$

$$= \frac{32 \cdot 2^{-z}\pi^{(1-z)/2}}{q^{-\frac{z}{2}-1}G(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{z/2}} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma\left(s+\frac{z}{2}\right)\Gamma\left(1-s+\frac{z}{2}\right) \times L\left(1-s+\frac{z}{2},\chi\right)\left(\frac{n}{y}\right)^{-s} ds. \tag{5.4.30}$$

Now, in order to use Lemma 5.4.3 to evaluate the integral in (5.4.30), we again have to shift the line of integration from Re $s > 3/2$ to Re $s = d$, where $-\frac{1}{2}$ Re $z < d < \frac{1}{2}$ Re z. Again, we do not encounter any pole in this process. Hence,

$$J(z,y,\chi) = \frac{64i2^{-z}y^{-z/2}\pi^{(3-z)/2}\Gamma(z+1)}{q^{-\frac{z}{2}-1}G(\chi)} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(n)\chi(k)}{(k+n/y)^{z+1}}. \tag{5.4.31}$$

Since $-1 < \mathrm{Re}(z) < 1$, the other integral, namely $J(-z,y,\overline{\chi})$, can be evaluated by simply replacing z by $-z$ and χ by $\overline{\chi}$ in (5.4.31). Now (5.4.26), (5.4.31), (5.4.3) and the discussion in the previous line give

$$\int_0^{\infty} \Delta(z,t)\Xi\left(\frac{t+iz}{2},\overline{\chi}\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\mu t\, dt$$
$$= \frac{64\pi^{3/2}q}{\sqrt{y}}\left(T(z,y^{-1},\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{(k+n/y)^{z+1}} + T(-z,y^{-1},\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{(k+n/y)^{-z+1}}\right), \tag{5.4.32}$$

where it is easy to see from the fact that $-1 < \mathrm{Re}\, z < 1$, from the discussion just preceding the statement of Theorem 5.4.1 and from Lemma 5.4.5 that both the double series on the right-hand side of (5.4.32) converge.

Now let $\mu = \frac{1}{2}\log\alpha$ in (5.4.32) so that $y = e^{2\mu}$ implies that $y = \alpha$. Then using the fact that $\alpha\beta = 1$ and using (5.4.2) in the second equality below, we deduce that

$$\int_0^{\infty} \Delta(z,t)\Xi\left(\frac{t+iz}{2},\overline{\chi}\right)\Xi\left(\frac{t-iz}{2},\chi\right)\cos\left(\tfrac{1}{2}t\log\alpha\right)\, dt$$
$$= 64\pi^{3/2}q\sqrt{\beta}\left(T(z,\beta,\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{(k+n\beta)^{z+1}} + T(-z,\beta,\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{(k+n\beta)^{-z+1}}\right)$$

$$= 64\pi^{3/2} q\sqrt{\beta}\left(T(z,\beta,\chi)\sum_{n=1}^{\infty}\chi(n)L(z+1,n\beta,\chi)+T(-z,\beta,\overline{\chi})\sum_{n=1}^{\infty}\overline{\chi}(n)L(-z+1,n\beta,\overline{\chi})\right).$$

The integral on the extreme left-hand side above is invariant under the transformation $\alpha \to \beta$ or under the simultaneous application of the transformations $\alpha \to \beta, \chi \to \overline{\chi}$ and $z \to -z$. Thus we obtain (5.4.5). □

Next we give an analogue of Ramanujan's transformation formula (Theorem 2.1.1) for even characters.

Corollary 5.4.7. *For an even, primitive, and nonprincipal character χ modulo q, define $P(\alpha, \chi)$ by*

$$P(\alpha,\chi) := \sqrt{\alpha}\,\mathrm{Re}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha}\right) = -\sqrt{\alpha}\,\mathrm{Re}\left(G(\chi)\sum_{n=1}^{\infty}\overline{\chi}(n)\psi\left(n\alpha,\overline{\chi}\right)\right),$$

where $\psi(a, \chi)$ is defined in (5.4.1)*. Then we have*

$$\begin{aligned}
&P(\alpha,\chi) = P(\beta,\overline{\chi}) = P(\alpha,\overline{\chi}) = P(\beta,\chi)\\
&= \frac{1}{64\pi^{3/2}}\int_0^{\infty}(1+t^2)\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2},\overline{\chi}\right)\Xi\left(\frac{t}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt.
\end{aligned} \tag{5.4.33}$$

Proof. Using Corollary 5.4.6, we readily see that the double series in the definition of $P(\alpha, \chi)$ converges. Let $z \to 0$ in (5.4.5). Then multiplying both sides by q and using (5.1.2), we have

$$\begin{aligned}
&\sqrt{\alpha}\left(G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\alpha}+G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha}\right)\\
&= \sqrt{\beta}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\beta}+G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\beta}\right)
\end{aligned}$$

$$= \frac{1}{32\pi^{3/2}} \int_0^\infty (1+t^2)\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2},\overline{\chi}\right)\Xi\left(\frac{t}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt. \tag{5.4.34}$$

Each of the first two expressions in (5.4.34) can be written in two different ways as real parts of a double series. Thus,

$$\begin{aligned}
&\sqrt{\alpha}\,\mathrm{Re}\left(G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\alpha}\right) = \sqrt{\alpha}\,\mathrm{Re}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha}\right)\\
&= \sqrt{\beta}\,\mathrm{Re}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\beta}\right) = \sqrt{\beta}\,\mathrm{Re}\left(G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\beta}\right)\\
&= \frac{1}{64\pi^{3/2}} \int_0^\infty (1+t^2)\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2},\overline{\chi}\right)\Xi\left(\frac{t}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt.
\end{aligned}$$

This implies (5.4.33). □

Moreover, if we start with the integral in Corollary 5.4.7, evaluate it using (5.2.6) with $z = 0$ and make use of Corollary 5.4.4 when χ is even, we obtain the same result as in Corollary 5.4.7, except that the function $P(\alpha, \chi)$ is replaced by the function $F(\alpha, \chi)$ defined by

$$F(\alpha,\chi) := \sqrt{\alpha}G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha} = -\sqrt{\alpha}G(\chi)\sum_{n=1}^{\infty}\overline{\chi}(n)\psi\left(n\alpha,\overline{\chi}\right). \tag{5.4.35}$$

It is then trivial to see that $F(\alpha, \chi) = P(\alpha, \chi)$. Theorem 5.4.2 can be analogously proved using Lemma 5.4.3 for χ odd. We just note that there we have to take care of the pole of $\Gamma\left(1 - s + \frac{1}{2}z\right)$ in the integrands of two separate integrals. However, in the calculations that follow later, the two residues turn out to be additive inverses of each other and hence do not contribute anything.

The following is an analogue of Ramanujan's transformation formula (Theorem 2.1.1) for odd characters.

Corollary 5.4.8. *For an odd, primitive, and nonprincipal character χ modulo q, define*

$Q(\alpha,\chi)$ by

$$Q(\alpha,\chi) := \sqrt{\alpha}\,\mathrm{Im}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha}\right) = -\sqrt{\alpha}\,\mathrm{Im}\left(G(\chi)\sum_{n=1}^{\infty}\overline{\chi}(n)\psi\left(n\alpha,\overline{\chi}\right)\right),$$

where $\psi\left(a,\chi\right)$ is defined in (5.4.1)*. Then we have*

$$\begin{aligned}&Q(\alpha,\chi) = Q(\beta,\overline{\chi}) = Q(\alpha,\overline{\chi}) = Q(\beta,\chi)\\ &= \frac{1}{4\pi^{1/2}q}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2},\overline{\chi}\right)\Xi\left(\frac{t}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt.\end{aligned} \tag{5.4.36}$$

Proof. Using Corollary 5.4.6, we find that the double series in the definition of $Q(\alpha,\chi)$ converges. Let $z\to 0$ in Theorem 5.4.2. Multiplying both sides by $-q$ and using (5.1.1) and (5.1.2), we observe that

$$\begin{aligned}&\sqrt{\alpha}\left(G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\alpha}+G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha}\right)\\ &= \sqrt{\beta}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\beta}+G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\beta}\right)\\ &= \frac{i}{2\pi^{1/2}q}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2},\overline{\chi}\right)\Xi\left(\frac{t}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt.\end{aligned} \tag{5.4.37}$$

Now using (5.1.2) for odd characters to simplify (5.4.37), we see that

$$\begin{aligned}&2i\sqrt{\alpha}\,\mathrm{Im}\left(G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\alpha}\right) = 2i\sqrt{\alpha}\,\mathrm{Im}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\alpha}\right)\\ &= 2i\sqrt{\beta}\,\mathrm{Im}\left(G(\chi)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\overline{\chi}(n)\overline{\chi}(k)}{k+n\beta}\right) = 2i\sqrt{\beta}\,\mathrm{Im}\left(G(\overline{\chi})\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(n)\chi(k)}{k+n\beta}\right)\\ &= \frac{i}{2\pi^{1/2}q}\int_0^{\infty}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2},\overline{\chi}\right)\Xi\left(\frac{t}{2},\chi\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt.\end{aligned} \tag{5.4.38}$$

This implies (5.4.36). ☐

If we now start with the integral in Corollary 5.4.8, evaluate it using (5.2.6) with $z=0$

and make use of Corollary 5.4.4 when χ is odd, we obtain the same result as in Corollary 5.4.8, except that the function $Q(\alpha, \chi)$ is replaced by $-iF(\alpha, \chi)$, where $F(\alpha, \chi)$ is defined in (5.4.35). It is then trivial to see that $F(\alpha, \chi) = iQ(\alpha, \chi)$.

We separately record the following corollary resulting from the discussion on the previous line and the one succeeding Corollary 5.4.7.

Corollary 5.4.9. *The sum $F(\alpha, \chi)$ defined in* (5.4.35) *is real if χ is even and purely imaginary if χ is odd.*

5.5 Character analogues of the Ramanujan-Hardy-Littlewood conjecture

In [52, p. 156, Section 2.5], Hardy and Littlewood discuss the following interesting conjecture suggested by the work of Ramanujan.

Theorem 5.5.1. *Let $\mu(n)$ denote the Möbius function. Let α and β be two positive numbers such that $\alpha\beta = 1$. Assume that the series $\sum_\rho \left(\Gamma\left(\frac{1-\rho}{2}\right)/\zeta'(\rho)\right) a^\rho$ converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then*

$$\begin{aligned}
&\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi\alpha^2/n^2} - \frac{1}{4\sqrt{\pi}\sqrt{\alpha}}\sum_\rho \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\pi^{\frac{\rho}{2}}\alpha^\rho \\
&= \sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi\beta^2/n^2} - \frac{1}{4\sqrt{\pi}\sqrt{\beta}}\sum_\rho \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\pi^{\frac{\rho}{2}}\beta^\rho. \qquad (5.5.1)
\end{aligned}$$

The original formulation of the above identity is slightly different in [52] but can readily be seen to be equivalent to (5.5.1). See also [15, p. 470], [72, p. 143] and [81, p. 219, Section 9.8] for discussions on this identity. Based on certain assumptions, the character analogues of (5.5.1) for even and odd primitive Dirichlet characters, which furnish two examples of transformation formulas of the form $F(\alpha,\chi) = F(\beta,\overline{\chi})$, are derived here. A one-variable generalization of (5.5.1) was recently obtained in [34].

We require Lemma 3.1 from [2] which states that if χ is a primitive character of conductor N and $k \geq 2$ is an integer such that $\chi(-1) = (-1)^k$, then

$$\frac{(k-2)!N^{k-2}G(\chi)}{2^{k-1}\pi^{k-2}i^{k-2}}L(k-1,\overline{\chi}) = L'(2-k,\chi). \tag{5.5.2}$$

We first begin with the analogue for odd characters.

Theorem 5.5.2. *Let χ be an odd, primitive character modulo q, and let α and β be two positive numbers such that $\alpha\beta = 1$. Assume that the series $\sum_{\rho} \frac{\pi^{\rho/2}\alpha^{\rho}\Gamma((2-\rho)/2)}{q^{\rho/2}L'(\rho,\chi)}$ and $\sum_{\rho} \frac{\pi^{\rho/2}\beta^{\rho}\Gamma((2-\rho)/2)}{q^{\rho/2}L'(\rho,\overline{\chi})}$ converge, where ρ runs through the non-trivial zeros of $L(s,\chi)$ and $L(s,\overline{\chi})$ respectively, and that the non-trivial zeros of the associated Dirichlet L-functions are simple. Then*

$$\begin{aligned}
&\alpha\sqrt{\alpha}\sqrt{G(\chi)}\left(\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}e^{-\frac{\pi\alpha^2}{qn^2}} - \frac{q}{4\pi\alpha^2}\sum_{\rho}\frac{\Gamma\left(\frac{2-\rho}{2}\right)}{L'(\rho,\chi)}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\alpha^{\rho}\right)\\
&= \beta\sqrt{\beta}\sqrt{G(\overline{\chi})}\left(\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n^2}e^{-\frac{\pi\beta^2}{qn^2}} - \frac{q}{4\pi\beta^2}\sum_{\rho}\frac{\Gamma\left(\frac{2-\rho}{2}\right)}{L'(\rho,\overline{\chi})}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\beta^{\rho}\right).
\end{aligned} \tag{5.5.3}$$

Proof. From [66], we have for $\text{Re } s > 1$,

$$\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^s} = \frac{1}{L(s,\chi)}. \tag{5.5.4}$$

Also, since for $-1 < c = \text{Re } s < 0$,

$$(1-e^{-x}) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}\,ds, \tag{5.5.5}$$

replacing s by $s+1$, we find that for $-2 < c < -1$,

$$(1-e^{-x}) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s+1)x^{-s-1}\,ds. \tag{5.5.6}$$

Using (5.5.4) and (5.5.6), we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}e^{-\frac{\pi\alpha^2}{n^2 q}} &= \frac{1}{L(2,\chi)} - \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}(1-e^{-\frac{\pi\alpha^2}{n^2 q}})\\ &= \frac{1}{L(2,\chi)} + \frac{q}{2\pi^2 i\alpha^2}\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{-2s}}\Gamma(s+1)\left(\frac{\pi\alpha^2}{q}\right)^{-s} ds\\ &= \frac{1}{L(2,\chi)} + \frac{q}{2\pi^2 i\alpha^2}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s+1)}{L(-2s,\chi)}\left(\frac{\pi\alpha^2}{q}\right)^{-s} ds, \end{aligned} \tag{5.5.7}$$

where in the second step above, we interchanged the order of summation and integration, which is valid because of absolute convergence. For χ odd, (5.1.6) can be put in the form

$$\left(\frac{\pi}{q}\right)^{-(2-s)/2}\Gamma\left(\frac{2-s}{2}\right)L(1-s,\overline{\chi}) = \frac{iq^{1/2}}{G(\chi)}\left(\frac{\pi}{q}\right)^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi).$$

Hence,

$$\frac{\Gamma(s+1)}{L(-2s,\chi)} = \frac{G(\overline{\chi})}{iq^{1/2}}\left(\frac{\pi}{q}\right)^{2s+\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}. \tag{5.5.8}$$

Substituting (5.5.8) in (5.5.7), we observe that

$$\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}e^{-\frac{\pi\alpha^2}{n^2 q}} = \frac{1}{L(2,\chi)} - \frac{G(\overline{\chi})}{2\pi^{3/2}\alpha^2}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\frac{q\alpha^2}{\pi}\right)^{-s} ds. \tag{5.5.9}$$

We wish to shift the line of integration from Re $s=c$, $-2<c<-1$, to Re $s=\lambda$, where $\frac{1}{2}<\lambda<\frac{3}{2}$. Consider a positively oriented rectangular contour formed by $[c-iT,\lambda-iT]$, $[\lambda-iT,\lambda+iT]$, $[\lambda+iT,c+iT]$ and $[c+iT,c-iT]$, where T is any positive real number. Let $\rho=\delta+i\gamma$ denote a non-trivial zero of $L(s,\overline{\chi})$. Let $T\to\infty$ through values such that $|T-\gamma| > \exp(-A_1\gamma/\log\gamma)$ for every ordinate γ of a zero of $L(s,\overline{\chi})$. It is known [29, p. 102] that for t not coinciding with the ordinate γ of a zero, and $-1 \leq \sigma \leq 2$,

$$\frac{L'(s,\overline{\chi})}{L(s,\overline{\chi})} = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O\left(\log\left(q(|t|+2)\right)\right),$$

from which we can conclude that

$$\log L(s,\overline{\chi}) = \sum_{|t-\gamma|\leq 1} \log(s-\rho) + O\left(\log\left(q(|t|+2)\right)\right). \tag{5.5.10}$$

Taking real parts in (5.5.10) gives

$$\begin{aligned}\log|L(s,\overline{\chi})| &= \sum_{|t-\gamma|\leq 1} \log|s-\rho| + O\left(\log\left(q(|t|+2)\right)\right) \\ &\geq \sum_{|t-\gamma|\leq 1} \log|t-\gamma| + O\left(\log\left(q(|t|+2)\right)\right).\end{aligned} \tag{5.5.11}$$

Hence from (5.5.11), we have

$$\begin{aligned}\log|L(\sigma+iT,\overline{\chi})| &\geq -\sum_{|T-\gamma|\leq 1} A_1\gamma/\log\gamma + O\left(\log\left(q(|T|+2)\right)\right) \\ &> -A_2 T,\end{aligned} \tag{5.5.12}$$

where $A_2 < \pi/4$ if A_1 is small enough and $T > T_0$ for some fixed T_0. From (5.5.12), we see that

$$\left|\frac{1}{L(2s+1,\overline{\chi})}\right| < e^{A_3 T}, \tag{5.5.13}$$

where $A_3 < \pi/2$. Using (3.3.8) and (5.5.13), we observe that as $T \to \infty$ through the above values, the integrals along the horizontal segments tend to zero. Now let $\frac{\rho-1}{2} := \delta + i\gamma$ denote a non-trivial zero of $L(2s+1,\overline{\chi})$. Using the notation in (4.3.5), we denote the residue at a of the function $f(s) := \dfrac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\dfrac{q\alpha^2}{\pi}\right)^{-s}$ by $R_a(f)$. The non-trivial zeros of $L(2s+1,\overline{\chi})$ lie in the critical strip $-\frac{1}{2} < \text{Re } s < 0$, whereas the trivial zeros are at $-1,-2,-3,\cdots$. Also, $\Gamma\left(\frac{1}{2}-s\right)$ has poles at $\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots$. Then the residue theorem yields

$$\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\frac{q\alpha^2}{\pi}\right)^{-s}ds = \int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\frac{q\alpha^2}{\pi}\right)^{-s}ds$$
$$-2\pi i\left(R_{-1}(f)+\sum_{\rho}R_{\frac{\rho-1}{2}}(f)+R_{\frac{1}{2}}(f)\right), \qquad (5.5.14)$$

where

$$R_{-1}(f) = \lim_{s\to -1}(s+1)\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\frac{q\alpha^2}{\pi}\right)^{-s} = \frac{\alpha^2 q}{4\sqrt{\pi}L'(-1,\overline{\chi})},$$
$$R_{\frac{\rho-1}{2}}(f) = \lim_{s\to\frac{\rho-1}{2}}\left(s-\frac{\rho-1}{2}\right)\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\frac{q\alpha^2}{\pi}\right)^{-s} = \frac{\Gamma\left(\frac{2-\rho}{2}\right)}{2L'(\rho,\overline{\chi})}\left(\frac{\pi}{q\alpha^2}\right)^{\frac{\rho-1}{2}},$$
$$R_{\frac{1}{2}}(f) = -\frac{\sqrt{\pi}}{\alpha\sqrt{q}L(2,\overline{\chi})}. \qquad (5.5.15)$$

Of course, here we have assumed that the non-trivial zeros of $L(2s+1,\overline{\chi})$ are all simple and that $\sum_\rho R_{\frac{\rho-1}{2}}(f)$ converges, since the afore-mentioned discussion regarding the integrals along the horizontal segments tending to zero as $T\to\infty$ through the chosen sequence does not imply the convergence of $\sum_\rho R_{\frac{\rho-1}{2}}(f)$ in the ordinary sense. It only means that the series converges only when we bracket the terms in such a way that the two terms for which

$$|\gamma-\gamma'| < \exp\left(-A_1|\gamma|/\log(|\gamma|+2)\right)+\exp\left(-A_1|\gamma'|/\log(|\gamma'|+2)\right)$$

are included in the same bracket. Using (5.5.4) and interchanging the order of summation and integration, which is valid because of absolute convergence, we obtain

$$\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})}\left(\frac{q\alpha^2}{\pi}\right)^{-s}ds = \sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma\left(\frac{1}{2}-s\right)\left(\frac{q\alpha^2n^2}{\pi}\right)^{-s}ds$$

$$= \frac{\sqrt{\pi}}{\alpha\sqrt{q}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\mu(n)}{n^2} \int_{d-i\infty}^{d+i\infty} \Gamma(s) \left(\frac{\pi}{\alpha^2 n^2 q}\right)^{-s} ds, \tag{5.5.16}$$

where in the penultimate line, we have made the change of variable $s \to \frac{1}{2} - s$ so that $-1 < d < 0$. Thus, (5.5.14), (5.5.15), (5.5.16) and (5.5.5) imply

$$\begin{aligned}
&\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{1}{2}-s\right)}{L(2s+1,\overline{\chi})} \left(\frac{q\alpha^2}{\pi}\right)^{-s} ds \\
&= -\frac{2\pi^{3/2} i}{\alpha\sqrt{q}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\mu(n)}{n^2} \left(1 - e^{-\frac{\pi}{\alpha^2 n^2 q}}\right) \\
&\quad - 2\pi i \left(\frac{\alpha^2 q}{4\sqrt{\pi} L'(-1,\overline{\chi})} + \sum_{\rho} \frac{\Gamma\left(\frac{2-\rho}{2}\right)}{2L'(\rho,\overline{\chi})} \left(\frac{\pi}{q\alpha^2}\right)^{\frac{\rho-1}{2}} - \frac{\sqrt{\pi}}{\alpha\sqrt{q} L(2,\overline{\chi})} \right).
\end{aligned} \tag{5.5.17}$$

From (5.5.9), (5.5.17) and the facts that $\alpha\beta = 1$ and $\sqrt{G(\chi)G(\overline{\chi})} = i\sqrt{q}$, we find that

$$\begin{aligned}
&\alpha\sqrt{\alpha}\sqrt{G(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^2} e^{-\frac{\pi\alpha^2}{n^2 q}} \\
&= \frac{\alpha\sqrt{\alpha}\sqrt{G(\chi)}}{L(2,\chi)} - \frac{\beta\sqrt{\beta}\sqrt{G(\overline{\chi})}}{L(2,\overline{\chi})} + \beta\sqrt{\beta}\sqrt{G(\overline{\chi})} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\mu(n)}{n^2} e^{-\frac{\pi\beta^2}{n^2 q}} \\
&\quad - \frac{\alpha\sqrt{\alpha} q^{3/2} \sqrt{G(\overline{\chi})}}{4\pi L'(-1,\overline{\chi})} - \frac{q\sqrt{G(\overline{\chi})}}{2\pi\sqrt{\beta}} \sum_{\rho} \frac{\Gamma\left(\frac{2-\rho}{2}\right)}{L'(\rho,\overline{\chi})} \left(\frac{\pi}{q}\right)^{\rho/2} \beta^{\rho} + \frac{\beta\sqrt{\beta}\sqrt{G(\overline{\chi})}}{L(2,\overline{\chi})}.
\end{aligned} \tag{5.5.18}$$

Applying (5.5.2) with $N = q$ and $k = 3$ and replacing χ by $\overline{\chi}$ gives

$$\frac{1}{L'(-1,\overline{\chi})} = \frac{4\pi i}{qG(\overline{\chi})L(2,\chi)}. \tag{5.5.19}$$

Thus (5.5.18) and (5.5.19) yield

$$\begin{aligned}
&\alpha\sqrt{\alpha}\sqrt{G(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^2} e^{-\frac{\pi\alpha^2}{n^2 q}} - \beta\sqrt{\beta}\sqrt{G(\overline{\chi})} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\mu(n)}{n^2} e^{-\frac{\pi\beta^2}{n^2 q}} \\
&= -\frac{q\sqrt{G(\overline{\chi})}}{2\pi\sqrt{\beta}} \sum_{\rho} \frac{\Gamma\left(\frac{2-\rho}{2}\right)}{L'(\rho,\overline{\chi})} \left(\frac{\pi}{q}\right)^{\rho/2} \beta^{\rho}.
\end{aligned} \tag{5.5.20}$$

Switching the roles of α and β and those of χ and $\overline{\chi}$ gives

$$\frac{q\sqrt{G(\chi)}}{2\pi\sqrt{\alpha}}\sum_{\rho}\frac{\Gamma\left(\frac{2-\rho}{2}\right)}{L'(\rho,\chi)}\left(\frac{\pi}{q}\right)^{\rho/2}\alpha^{\rho}+\frac{q\sqrt{G(\overline{\chi})}}{2\pi\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{2-\rho}{2}\right)}{L'(\rho,\overline{\chi})}\left(\frac{\pi}{q}\right)^{\rho/2}\beta^{\rho}=0. \tag{5.5.21}$$

Finally (5.5.20) and (5.5.21) give (5.5.3) upon simplification. □

Remark. The approach used above for proving that the integrals along the horizontal segments tend to zero as $T\to\infty$ through the chosen sequence is adapted from [81, p. 219].

Theorem 5.5.3. *Let χ be an even, primitive character modulo q, and let α and β be two positive numbers such that $\alpha\beta = 1$. Assume that the series $\sum_{\rho}\frac{\pi^{\rho/2}\alpha^{\rho}\Gamma((2-\rho)/2)}{q^{\rho/2}L'(\rho,\chi)}$ and $\sum_{\rho}\frac{\pi^{\rho/2}\beta^{\rho}\Gamma((2-\rho)/2)}{q^{\rho/2}L'(\rho,\overline{\chi})}$ converge, where ρ runs through the non-trivial zeros of $L(s,\chi)$ and $L(s,\overline{\chi})$ respectively, and that the non-trivial zeros of the associated Dirichlet L-functions are simple. Then*

$$\sqrt{\alpha}\sqrt{G(\chi)}\left(\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}e^{-\frac{\pi\alpha^2}{qn^2}}-\frac{\sqrt{q}}{4\sqrt{\pi}\alpha}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{L'(\rho,\chi)}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\alpha^{\rho}\right)$$
$$=\sqrt{\beta}\sqrt{G(\overline{\chi})}\left(\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n}e^{-\frac{\pi\beta^2}{qn^2}}-\frac{\sqrt{q}}{4\sqrt{\pi}\beta}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{L'(\rho,\overline{\chi})}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\beta^{\rho}\right). \tag{5.5.22}$$

To prove Theorem 5.5.3, we require the following lemma.

Lemma 5.5.4.

$$\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}=\frac{1}{L(1,\chi)}.$$

Proof. Dividing n into its residue classes mod q by letting $n = qr + b$, $0 \le r < \infty$, $0 \le b \le q-1$, we find that since χ has period q,

$$\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}=\sum_{r=0}^{\infty}\sum_{b=0}^{q-1}\frac{\chi(b)\mu(qr+b)}{qr+b}=\sum_{b=0}^{q-1}\chi(b)\sum_{r=0}^{\infty}\frac{\mu(qr+b)}{qr+b}. \tag{5.5.23}$$

The series $\sum_{r=0}^{\infty}\mu(qr+b)/(qr+b)$ was first studied by J.C. Kluyver [55] and its convergence was proved by E. Landau [66]. In fact, Landau gave an explicit representation for this series in terms of a finite sum consisting of L-functions. Thus (5.5.23) implies convergence of $\sum_{n=1}^{\infty} \chi(n)\mu(n)/n$. Then using (5.5.4) and an analogue of Abel's theorem for power series, we see that

$$\sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} = \lim_{s\to 1} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^s} = \lim_{s\to 1} \frac{1}{L(s,\chi)} = \frac{1}{L(1,\chi)}.$$

□

Proof. The proof is very similar to that of Theorem 5.5.2 and hence we omit the details. However we note that Lemma 5.5.4, (5.1.6) in the form [29, p. 69]

$$\pi^{-(1-s)/2} q^{(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) L(1-s,\overline{\chi}) = \frac{q^{1/2}}{G(\chi)} \pi^{-s/2} q^{s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi),$$

and (5.5.2) with $N = q$ and $k = 2$ are used in the proof. □

Chapter 6

Character analogues of theorems of Ramanujan, Koshliakov and Guinand

In this chapter, we establish even character analogues of some results involving the modified Bessel function $K_\nu(z)$ found in Ramanujan's Lost Notebook [78] and recently proved by B.C. Berndt, Y. Lee, and J. Sohn [21]. These include formulas of N.S. Koshliakov [56] and A.P. Guinand [47], i.e., (3.2.1) and Theorem 4.5.1 respectively. Some more results similar to these type of identities can be found in [27]. Our starting point in this paper is a result (Theorem 6.1.1) arising from a character analogue of a theorem of G.N. Watson [82] (see (4.5.2)) established by Berndt in [11] using periodic or character analogues of the Poisson summation formula [10, 11, 22].

In this chapter, we prove an analogue (Theorem 6.2.1) of the Ramanujan-Guinand formula for even primitive characters. We further establish an even character analogue (Theorem 6.2.3) of Koshliakov's formula. Several corollaries and special cases are also given. We could have generalized our results to arbitrary even periodic sequences; but with the restriction that the sequences be completely multiplicative, such sequences must be Dirichlet characters [6, p. 145, Exercise 17(b)]. The results in this chapter are published in [18], where corresponding results for odd characters are also given.

6.1 Preliminary results

We use the well-known fact [43, p. 978, formula 8.469, no. 3]

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{6.1.1}$$

We require the simple asymptotic formula [84, p. 202]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad z \to \infty,$$

to confirm the convergence of series and integrals and to also justify the interchange of integration and summation several times in the sequel. We need several integrals of Bessel functions beginning with [43, p. 705, formula 6.544, no. 8]

$$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu(bx) \frac{dx}{x^2} = \frac{\pi}{a} K_{2\nu}(2\sqrt{ab}), \qquad \text{Re } a > 0, \text{Re } b > 0. \tag{6.1.2}$$

Also,

$$\int_0^\infty K_0(a/x) K_0(yx) dx = \frac{\pi}{y} K_0(2\sqrt{ay}) \tag{6.1.3}$$

and

$$\int_0^\infty x^{\nu-1} e^{-\beta/x - \gamma x} dx = 2\left(\beta/\gamma\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}), \tag{6.1.4}$$

for $s \in \mathbb{C}$, Re $\beta > 0$ and Re $\gamma > 0$. We need the related pair [43, p. 697, formula 6.521, no. 3]

$$\int_0^\infty x K_\nu(ax) K_\nu(bx) dx = \frac{\pi (ab)^{-\nu}(a^{2\nu} - b^{2\nu})}{2\sin(\pi\nu)(a^2 - b^2)}, \qquad |\text{Re } \nu| < 1, \ \text{Re } (a+b) > 0, \tag{6.1.5}$$

and

$$\int_0^\infty x K_0(ax) K_0(bx) dx = \frac{\log(a/b)}{a^2 - b^2}, \qquad a, b > 0, \tag{6.1.6}$$

which can be obtained by letting $\nu \to 0$ in (6.1.5). Also, we need the evaluation [43, p. 708, formula 6.561, no. 16], for Re $a > 0$ and Re$(\mu + 1 \pm \nu) > 0$,

$$\int_0^\infty x^\mu K_\nu(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{6.1.7}$$

We now state a theorem of Berndt specialized to characters [11, p. 171], which generalizes a theorem of Watson [82], and which will be used in the proof of Theorem 6.2.1.

Theorem 6.1.1. *Let $x > 0$. If χ is even with period k and* Re $\nu > 0$*, then*

$$\sum_{n=1}^{\infty} \chi(n) n^{\nu} K_{\nu}(2\pi n x/k) = \frac{\pi^{\frac{1}{2}}}{2xG(\overline{\chi})} \left(\frac{kx}{\pi}\right)^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right) . \sum_{n=1}^{\infty} \overline{\chi}(n)(n^2 + x^2)^{-\nu-\frac{1}{2}}. \quad (6.1.8)$$

6.2 Character analogues of theorems of Ramanujan, Koshliakov and Guinand for even primitive characters

Throughout this section, $\chi(n)$ denotes an even primitive character of modulus q.

Theorem 6.2.1. *If α and β are positive numbers such that $\alpha\beta = \pi^2$ and s is any complex number, then*

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \chi(n) \sigma_{-s}(n) \left(\frac{n}{q}\right)^{s/2} K_{s/2}(2n\alpha/q) = \frac{q\sqrt{\beta}}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) \sigma_{-s}(n) \left(\frac{n}{q}\right)^{s/2} K_{s/2}(2n\beta/q). \quad (6.2.1)$$

Proof. Invoking (6.1.8) in the third equality below, we find that, for $\sigma = \text{Re } s > 1$,

$$\begin{aligned}
&\sqrt{\alpha} \sum_{n=1}^{\infty} \chi(n) \sigma_{-s}(n) \left(\frac{n}{q}\right)^{s/2} K_{s/2}(2n\alpha/q) \\
&= \sqrt{\alpha} \sum_{n=1}^{\infty} \chi(n) \sum_{d|n} d^{-s} \left(\frac{n}{q}\right)^{s/2} K_{s/2}(2n\alpha/q) \\
&= \sqrt{\alpha} \sum_{d=1}^{\infty} \chi(d) \sum_{m=1}^{\infty} \chi(m) \left(\frac{m}{dq}\right)^{s/2} K_{s/2}(2dm\alpha/q) \\
&= \sqrt{\alpha} q^{-s/2} \sum_{d=1}^{\infty} \chi(d) d^{-s/2} \left(\frac{\pi^{3/2}}{2d\alpha} \left(\frac{qd\alpha}{\pi^2}\right)^{s/2+1} \Gamma\left(\frac{s+1}{2}\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{G(\overline{\chi})(n^2 + (d\alpha/\pi)^2)^{(s+1)/2}} \right) \\
&= \frac{q\sqrt{\pi}}{2G(\overline{\chi})} \alpha^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(d)\overline{\chi}(n)}{(n^2\pi^2 + d^2\alpha^2)^{(s+1)/2}}. \quad (6.2.2)
\end{aligned}$$

By symmetry, for $\sigma > 1$,

$$\begin{aligned}
&\sqrt{\beta}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{q}\right)^{s/2}K_{s/2}(2n\beta/q)\\
&=\frac{\sqrt{\pi}}{2}\beta^{(s+1)/2}G(\overline{\chi})\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(d)\chi(n)}{(n^2\pi^2+d^2\beta^2)^{(s+1)/2}}\\
&=\frac{\sqrt{\pi}}{2}\beta^{-(s+1)/2}G(\overline{\chi})\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(d)\chi(n)}{(n^2\pi^2/\beta^2+d^2)^{(s+1)/2}}\\
&=\frac{\sqrt{\pi}}{2}\alpha^{(s+1)/2}G(\overline{\chi})\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\chi(d)}{(d^2\alpha^2+\pi^2n^2)^{(s+1)/2}}.
\end{aligned}$$

Thus, for $\sigma > 1$,

$$\begin{aligned}
&\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\chi(d)}{(d^2\alpha^2+\pi^2n^2)^{(s+1)/2}}\\
&\qquad=\frac{2\sqrt{\beta}}{\sqrt{\pi}\alpha^{(s+1)/2}G(\overline{\chi})\Gamma((s+1)/2)}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-s}(n)\left(\frac{n}{q}\right)^{s/2}K_{s/2}(2n\beta/q). \quad (6.2.3)
\end{aligned}$$

Substituting (6.2.3) in (6.2.2), we deduce (6.2.1) for $\sigma > 1$. By analytic continuation, we complete the proof. □

For example, let $\chi(n) = \left(\frac{n}{p}\right)$, the Legendre symbol modulo p, where p is a prime such that $p \equiv 1 \pmod 4$. Then from a well-known theorem of Gauss [19], we know that for any prime $p \equiv 1 \pmod 4$, $G(\chi) = \sqrt{p}$, where $\chi(n) = \left(\frac{n}{p}\right)$. Thus, substituting $\left(\frac{n}{p}\right)$ for $\chi(n)$ in (6.2.1) and simplifying, we find that

$$\begin{aligned}
&\sqrt{\alpha}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-s}(n)\left(\frac{n}{p}\right)^{s/2}K_{s/2}(2n\alpha/p)\\
&\qquad=\sqrt{\beta}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-s}(n)\left(\frac{n}{p}\right)^{s/2}K_{s/2}(2n\beta/p).
\end{aligned} \quad (6.2.4)$$

Now let $s = 1$ in (6.2.4). Then using (6.1.1), we deduce the following corollary.

Corollary 6.2.2. *If α and β are positive numbers, $\alpha\beta = \pi^2$, and $\left(\dfrac{n}{p}\right)$ is the Legendre symbol, where p is a prime with $p \equiv 1 \pmod 4$, then*

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(n) e^{-2n\alpha/p} = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(n) e^{-2n\beta/p}. \tag{6.2.5}$$

For more identities similar to the above one, see [67]. Next we state and prove a character analogue of Koshliakov's formula.

Theorem 6.2.3. *If α and β are positive numbers, $\alpha\beta = \pi^2$, and s is any complex number, then*

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \chi(n) d(n) K_0(2n\alpha/q) = \frac{q\sqrt{\beta}}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) d(n) K_0(2n\beta/q). \tag{6.2.6}$$

Proof. Setting $s = 0$ in (6.2.1) and noting that $\sigma_0(n) = d(n)$, the number of divisors of n, we arrive at (6.2.6). $\square$

Letting $\chi(n) = \left(\frac{n}{p}\right)$ in (6.2.6), we find that

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) d(n) K_0(2n\alpha/p) = \sqrt{\beta} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) d(n) K_0(2n\beta/p). \tag{6.2.7}$$

Next we give analogues of theorems on page 254 of Ramanujan's Lost Notebook [21, pp. 30–34], [78].

Theorem 6.2.4. *If $a > 0$, then*

$$\begin{aligned}
&\int_0^{\infty} \left(\sum_{m=1}^{\infty} \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi m x} \right) \left(\sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi n a/x} \right) \frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2} \int_0^{\infty} \left(\frac{\sum_{s=0}^{q-1} \overline{\chi}(s) e^{-2\pi s x}}{1 - e^{-2\pi q x}} \right) \left(\frac{\sum_{v=0}^{q-1} \overline{\chi}(v) e^{-2\pi a v/x}}{1 - e^{-2\pi a q/x}} \right) \frac{dx}{x} \\
&= \frac{2}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n) d(n) K_0(4\pi\sqrt{an}) \\
&= \frac{aq^2}{\pi^2} \sum_{n=1}^{\infty} \frac{\chi(n) d(n) \log(n/aq^2)}{n^2 - a^2 q^4}.
\end{aligned} \tag{6.2.8}$$

Proof. Writing $m = qr + s$, $0 \le r < \infty$, $0 \le s \le q-1$, and $n = qu + v$, $0 \le u < \infty$, $0 \le v \le q-1$, we find that, since χ has period q,

$$\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi m x}\right)\left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi n a/x}\right)\frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\sum_{r=0}^\infty \sum_{s=0}^{q-1} \overline{\chi}(qr+s)e^{-2(qr+s)\pi x}\right)\left(\sum_{u=0}^\infty \sum_{v=0}^{q-1} \overline{\chi}(qu+v)e^{-2(qu+v)\pi a/x}\right)\frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\sum_{r=0}^\infty e^{-2\pi q r x}\sum_{s=0}^{q-1} \overline{\chi}(s)e^{-2\pi s x}\right)\left(\sum_{u=0}^\infty e^{-2\pi a k u/x}\sum_{v=0}^{q-1} \overline{\chi}(v)e^{-2\pi a v/x}\right)\frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2}\int_0^\infty \left(\frac{\sum_{s=0}^{q-1}\overline{\chi}(s)e^{-2\pi s x}}{1-e^{-2\pi q x}}\right)\left(\frac{\sum_{v=0}^{q-1}\overline{\chi}(v)e^{-2\pi a v/x}}{1-e^{-2\pi a q/x}}\right)\frac{dx}{x},
\end{aligned} \tag{6.2.9}$$

which proves the first equality.

Interchanging the order of summation and integration by absolute convergence, we find that

$$\begin{aligned}
&\int_0^\infty \left(\sum_{m=1}^\infty \frac{\overline{\chi}(m)}{G(\overline{\chi})} e^{-2\pi m x}\right)\left(\sum_{n=1}^\infty \frac{\overline{\chi}(n)}{G(\overline{\chi})} e^{-2\pi n a/x}\right)\frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \overline{\chi}(mn)\int_0^\infty e^{-2\pi(mx+an/x)}\frac{dx}{x} \\
&= \frac{1}{G(\overline{\chi})^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \overline{\chi}(mn)\int_0^\infty e^{-2\pi(u+amn/u)}\frac{du}{u} \\
&= \frac{2}{G(\overline{\chi})^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \overline{\chi}(mn)K_0(4\pi\sqrt{amn}) \\
&= \frac{2}{G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(4\pi\sqrt{an}),
\end{aligned} \tag{6.2.10}$$

which proves the second equality.

Now letting $\alpha = \pi a$ and $\beta = \pi/a$ for $a > 0$ in (6.2.6) and simplifying, we deduce that

$$\sum_{n=1}^\infty \chi(n)d(n)K_0(2n\pi a/q) = \frac{q}{aG(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(2n\pi/aq). \tag{6.2.11}$$

Multiplying both sides by $aK_0(2\pi ayq)$ and integrating with respect to a from 0 to ∞, we have

$$\int_0^\infty a\sum_{n=1}^\infty \chi(n)d(n)K_0(2n\pi a/q)K_0(2\pi ayq)da$$
$$= \frac{q}{G(\overline{\chi})^2}\int_0^\infty \sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(2n\pi/aq)K_0(2\pi ayq)da. \qquad (6.2.12)$$

Interchanging the order of summation and integration and using (6.1.3) and (6.1.6), we find that

$$\frac{q^2}{4\pi^2}\sum_{n=1}^\infty \frac{\chi(n)d(n)\log(n/yq^2)}{n^2-y^2q^4} = \frac{1}{2yG(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(4\pi\sqrt{ny}). \qquad (6.2.13)$$

Multiplying both sides by $4y$ and then letting $y=a$, we find that

$$\frac{aq^2}{\pi^2}\sum_{n=1}^\infty \frac{\chi(n)d(n)\log(n/aq^2)}{n^2-a^2q^4} = \frac{2}{G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)d(n)K_0(4\pi\sqrt{na}). \qquad (6.2.14)$$

This proves the last equality in Theorem 3.4 and completes the proof. □

Theorem 6.2.5. *If $a>0$, then*

$$\sum_{n=1}^\infty \chi(n)\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}} = \frac{aq^{7/2}}{\pi G(\overline{\chi})^2}\sum_{n=1}^\infty \frac{\overline{\chi}(n)\sigma_{-1/2}(n)}{(n+aq^2)(\sqrt{n}+q\sqrt{a})}. \qquad (6.2.15)$$

Proof. Let $s=1/2$ in (6.2.1) and set $\alpha=x$ and $\beta=\pi^2/x$. Then

$$\sqrt{x}\sum_{n=1}^\infty \chi(n)\sigma_{-1/2}(n)\left(\frac{n}{q}\right)^{1/4}K_{1/4}(2nx/q)$$
$$= \frac{q\pi}{\sqrt{x}G(\overline{\chi})^2}\sum_{n=1}^\infty \overline{\chi}(n)\sigma_{-1/2}(n)\left(\frac{n}{q}\right)^{1/4}K_{1/4}(2n\pi^2/xq). \qquad (6.2.16)$$

Now multiplying both sides by $x^{-5/2}K_{1/4}(2a\pi^2q/x)$, integrating with respect to x from 0 to ∞, and interchanging the order of summation and integration by absolute convergence, we find that

$$q^{-1/4}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1/2}(n)n^{1/4}\int_0^{\infty}\frac{1}{x^2}K_{1/4}(2nx/q)K_{1/4}(2a\pi^2q/x)dx$$
$$=\frac{q^{3/4}\pi}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1/2}(n)n^{1/4}\int_0^{\infty}\frac{1}{x^3}K_{1/4}(2n\pi^2/xq)K_{1/4}(2a\pi^2q/x)dx. \quad (6.2.17)$$

Now using (6.1.1), (6.1.2), and (6.1.5) and simplifying, we see that

$$\frac{1}{4\sqrt{2}q^{5/4}a^{5/4}\pi}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}}=\frac{\sqrt{2}\pi q^{9/4}a^{-1/4}}{8\pi^3G(\overline{\chi})^2}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sigma_{-1/2}(n)}{(n+aq^2)(\sqrt{n}+q\sqrt{a})}, \quad (6.2.18)$$

which upon further simplification yields (6.2.15). □

Finally we give a character analogue of the last theorem on page 254 of Ramanujan's Lost Notebook [78].

Theorem 6.2.6. *For $a>0$,*

$$2\sqrt{a}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})=\frac{aq^3}{2\pi G(\overline{\chi})^2}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sigma_{-1}(n)}{(n+aq^2)}=\frac{-aq^2}{2\pi G(\overline{\chi})^2}S(a,q), \quad (6.2.19)$$

where

$$S(a,q)=\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^2}\sum_{m=0}^{q-1}\overline{\chi}(m)\sum_{t=1}^{\infty}\frac{(aq^2-mn)}{t(tnq+aq^2-mn)}. \quad (6.2.20)$$

Proof. Letting $s=1$ in (6.2.1) and setting $\alpha=x$ and $\beta=\pi^2/x$, we arrive at

$$\sqrt{x}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\left(\frac{n}{q}\right)^{1/2}K_{1/2}(2nx/q)$$
$$=\frac{q\pi}{\sqrt{x}G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)\left(\frac{n}{q}\right)^{1/2}K_{1/2}(2n\pi^2/xq). \quad (6.2.21)$$

Multiplying both sides of (6.2.21) by $x^{-5/2}K_{1/2}(2a\pi^2q/x)$, integrating with respect to x from 0 to ∞, and interchanging the order of summation and integration by absolute convergence, we find that

$$\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\left(\frac{n}{q}\right)^{1/2}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx/q)K_{1/2}(2a\pi^2q/x)dx$$
$$=\frac{q\pi}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)\left(\frac{n}{q}\right)^{1/2}\int_0^{\infty}\frac{1}{x^3}K_{1/2}(2n\pi^2/qx)K_{1/2}(2a\pi^2q/x)dx. \quad (6.2.22)$$

Making the substitution $x=1/u$ in the integral on the right-hand side of (6.2.22) and then using (6.1.5) for this integral, and (6.1.2) for the integral on the left side, we find after simplifying that

$$\frac{1}{2a\pi q}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})=\frac{q^2}{8\pi^2\sqrt{a}G(\overline{\chi})^2}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sigma_{-1}(n)}{(n+aq^2)}, \quad (6.2.23)$$

which after simplification yields the first equality.

Next, to show that the first and third expressions of (6.2.19) are equal, we use (6.1.1) on the right-hand side of (6.2.21). After simplification, (6.2.21) becomes

$$\frac{2}{\sqrt{q\pi}}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{nx}K_{1/2}(2nx/q)=\frac{q}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/qx}. \quad (6.2.24)$$

Now multiplying both sides of (6.2.24) by $x^{-5/2}K_{1/2}(2a\pi^2q/x)$, integrating with respect to x from 0 to ∞, interchanging the order of summation and integration by absolute convergence, and using (6.1.2), we find that

$$\frac{2}{\sqrt{q\pi}}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx/q)K_{1/2}(2a\pi^2q/x)dx$$
$$=\frac{1}{aq^{3/2}\pi^{3/2}}\sum_{n=1}^{\infty}\chi(n)\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})$$
$$=\int_0^{\infty}\left(\frac{q}{G(\overline{\chi})^2}\sum_{n=1}^{\infty}\overline{\chi}(n)\sigma_{-1}(n)e^{-2n\pi^2/qx}\right)x^{-5/2}K_{1/2}(2a\pi^2q/x)dx. \quad (6.2.25)$$

Since

$$\sum_{m=1}^{\infty} \overline{\chi}(m) e^{-2md\pi^2/qx} = \frac{1}{(1-e^{-2d\pi^2/x})} \sum_{s=0}^{q-1} \overline{\chi}(s) e^{-2ds\pi^2/qx}, \tag{6.2.26}$$

with d replaced by n, we deduce that

$$\begin{aligned}\frac{q}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n)\sigma_{-1}(n) e^{-2n\pi^2/qx} &= q \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} \frac{\overline{\chi}(m)\overline{\chi}(d)}{dG(\overline{\chi})^2} e^{-2md\pi^2/qx} \\ &= q \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{nG(\overline{\chi})} \left(\frac{\sum_{s=0}^{q-1} \overline{\chi}(s) e^{-2ns\pi^2/qx}}{G(\overline{\chi})(1-e^{-2n\pi^2/x})} \right). \end{aligned} \tag{6.2.27}$$

Hence

$$\begin{aligned}&\int_0^{\infty} \left(\frac{q}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \overline{\chi}(n)\sigma_{-1}(n) e^{-2n\pi^2/qx} \right) x^{-5/2} K_{1/2}(2a\pi^2 q/x) dx \\ &= \frac{\sqrt{q}}{2\sqrt{a\pi}} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{nG(\overline{\chi})} \left(\frac{\sum_{s=0}^{q-1} \overline{\chi}(s) e^{-2ns\pi^2/qx}}{G(\overline{\chi})(1-e^{-2n\pi^2/x})} \right) e^{-2a\pi^2 q/x} \frac{dx}{x^2} \\ &= \frac{\sqrt{q}}{4\sqrt{a}\pi^{5/2}} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \left(\frac{\sum_{s=0}^{q-1} \overline{\chi}(s) e^{-us/q}}{G(\overline{\chi})(1-e^{-u})} \right) e^{-aqu/n} du. \end{aligned} \tag{6.2.28}$$

Now

$$\begin{aligned}\frac{\sum_{s=0}^{q-1} \overline{\chi}(s) e^{-us/q}}{G(\overline{\chi})(1-e^{-u})} &= \frac{\sum_{s=0}^{q-1} \overline{\chi}(s) e^{u(q-s)/q}}{G(\overline{\chi})(e^u-1)} \\ &= \frac{\sum_{m=1}^{q-1} \overline{\chi}(m) e^{um/q}}{G(\overline{\chi})(e^u-1)} - \frac{\sum_{m=0}^{q-1} \overline{\chi}(m)}{uG(\overline{\chi})} \\ &= \sum_{m=0}^{q-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \left(\frac{e^{um/q}}{e^u-1} - \frac{1}{u} \right), \end{aligned} \tag{6.2.29}$$

for $\sum_{m=0}^{q-1} \overline{\chi}(m) = 0$, since χ is non-principal. Thus, for $0 \le m \le q-1$, we need to evaluate

$$\begin{aligned}&\int_0^{\infty} \left(\frac{e^{um/q}}{e^u-1} - \frac{1}{u} \right) e^{-aqu/n} du = -\int_0^{\infty} \left(\frac{e^{-aqu/n}}{u} - \frac{e^{-(aq/n-m/q+1)u}}{1-e^{-u}} \right) du \\ &= -\left(\int_0^{\infty} \frac{e^{-aqu/n} - e^{-(aq/n-m/q+1)u}}{u} du + \int_0^{\infty} \left(\frac{1}{u} - \frac{1}{1-e^{-u}} \right) e^{-(aq/n-m/q+1)u} du \right). \end{aligned} \tag{6.2.30}$$

The first integral in (6.2.30) can be evaluated with the help of Theorem 2.2.1 with $f(u) = e^{-u}$. Since $aq/n > 0$,

$$\int_0^\infty \frac{e^{-aqu/n} - e^{-(aq/n-m/q+1)u}}{u} du = \log\left(\frac{aq/n - m/q + 1}{aq/n}\right). \tag{6.2.31}$$

This integral can also be evaluated by differentiation under the integral sign.

To evaluate the second integral, we use the following representation for $\psi(z)$, the logarithmic derivative of the gamma function $\Gamma(z)$ [43, p. 952, formula 8.361, no. 8]:

$$\psi(z) = \log(z) + \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) e^{-zt} dt, \tag{6.2.32}$$

for Re $(z) > 0$.

Thus from (6.2.30), (6.2.32), and (6.2.31),

$$\begin{aligned}
&\int_0^\infty \left(\frac{e^{um/q}}{e^u - 1} - \frac{1}{u}\right) e^{-aqu/n} du \\
&= -\log\left(\frac{aq}{n} - \frac{m}{q} + 1\right) + \log\left(\frac{aq}{n}\right) - \psi\left(\frac{aq}{n} - \frac{m}{q} + 1\right) + \log\left(\frac{aq}{n} - \frac{m}{q} + 1\right) \\
&= \log\left(\frac{aq}{n}\right) + \gamma - \sum_{t=1}^\infty \frac{(aq/n - m/q)}{t\,(aq/n - m/q + t)} \\
&= \log\left(\frac{aq}{n}\right) + \gamma - \sum_{t=1}^\infty \frac{(aq^2 - mn)}{t(tnq + aq^2 - mn)},
\end{aligned} \tag{6.2.33}$$

since

$$\psi(z) = -\gamma + \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{z+n-1}\right) = -\gamma + \sum_{n=1}^\infty \frac{z-1}{n(z+n-1)}. \tag{6.2.34}$$

Hence by absolute convergence, we find that

$$\begin{aligned}
&\int_0^\infty \left(\frac{q}{G(\overline{\chi})^2} \sum_{n=1}^\infty \overline{\chi}(n)\sigma_{-1}(n) e^{-2n\pi^2/qx}\right) x^{-5/2} K_{1/2}(2a\pi^2 q/x) dx \\
&= \frac{\sqrt{q}}{4\sqrt{a}\pi^{5/2}} \int_0^\infty \sum_{n=1}^\infty \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \left(\sum_{m=0}^{q-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \left(\frac{e^{um/q}}{e^u - 1} - \frac{1}{u}\right)\right) e^{-aqu/n} du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{q}}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \left(\sum_{m=0}^{q-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \int_0^{\infty} \left(\frac{e^{um/q}}{e^u - 1} - \frac{1}{u} \right) e^{-aqu/n} du \right) \\
&= \frac{\sqrt{q}}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^2 G(\overline{\chi})} \sum_{m=0}^{q-1} \frac{\overline{\chi}(m)}{G(\overline{\chi})} \left(\log\left(\frac{aq}{n}\right) + \gamma - \sum_{t=1}^{\infty} \frac{(aq^2 - mn)}{t(tnq + aq^2 - mn)} \right) \\
&= \frac{-\sqrt{q}}{4\sqrt{a}\pi^{5/2}} \left(\frac{1}{G(\overline{\chi})^2} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^2} \sum_{m=0}^{q-1} \overline{\chi}(m) \sum_{t=1}^{\infty} \frac{(aq^2 - mn)}{t(tnq + aq^2 - mn)} \right) \\
&= \frac{-\sqrt{q}}{4\sqrt{a}\pi^{5/2} G(\overline{\chi})^2} S(a, q), \qquad (6.2.35)
\end{aligned}
$$

where $S(a,q)$ is defined in (6.2.20) and where we made use of the fact that $\sum_{m=0}^{q-1} \overline{\chi}(m) = 0$ in the penultimate step.

Thus from (6.2.25), we deduce that

$$
\frac{1}{aq^{3/2}\pi^{3/2}} \sum_{n=1}^{\infty} \chi(n)\sigma_{-1}(n)\sqrt{n} K_1(4\pi\sqrt{an}) = -\frac{\sqrt{q}}{4\sqrt{a}\pi^{5/2} G(\overline{\chi})^2} S(a, q), \qquad (6.2.36)
$$

which upon trivial simplification shows the equality of the first and third expressions in (6.2.19). □

Corollary 6.2.7. *If $a > 0$ and $\left(\frac{n}{p}\right)$ is the Legendre symbol, where p is a prime with $p \equiv 1 \pmod 4$, then*

$$
2\sqrt{a} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(n)\sqrt{n} K_1(4\pi\sqrt{an}) = \frac{p^2 a}{2\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right) \sigma_{-1}(n)}{n + p^2 a}. \qquad (6.2.37)
$$

Proof. Set $\chi(n) = \left(\frac{n}{p}\right)$ in (6.2.19). Then after simplification, the first equality in (6.2.19) yields (6.2.37). □

Remark. The middle expression in Theorem 6.2.6 converges extremely slowly. For example, let $a = 1$ and $\chi(n) = \left(\frac{n}{5}\right)$. The value of the leftmost expression of Theorem 6.2.6 is $2.51273028 \cdots \times 10^{-6}$, which is correct up to the decimals listed. However, the middle expression

$$\frac{25}{2\pi}\sum_{n=1}^{\infty}\frac{\left(\frac{n}{5}\right)\sigma_{-1}(n)}{n+25}, \tag{6.2.38}$$

as we indicated above, converges very slowly. To illustrate, the correct value for 49 million terms of (6.2.38) is $2.59028733\cdots\times 10^{-6}$, while the correct value of (6.2.38) for 50 million terms is $2.4317244512\cdots\times 10^{-6}$. Thus, the convergence of (6.2.38) is slow to stabilize, and the partial sums oscillate about the correct value.

Chapter 7

Rank-Crank type PDE's in the Theory of Partitions

7.1 Chan's identity through the elliptic function theory

As remarked in the introduction, in this section, we prove (1.0.18) using the theory of elliptic functions. Let

$$z = e^{2\pi iu}, x_1 = e^{2\pi iv}, x_2 = e^{2\pi iw}, \tag{7.1.1}$$

and let

$$x_j = e^{2\pi ia_j}, \quad j = 3, \cdots, m, \tag{7.1.2}$$

where $u, v, w, a_3, \ldots, a_m$ are all complex numbers.

Now define

$$J(a) := J(a, q) := [b]_\infty, \text{ where } b = e^{2\pi ia}, a \in \mathbb{C}. \tag{7.1.3}$$

Then using the Jacobi triple product identity [16, p. 10, Theorem 1.3.3] in the form

$$[\zeta]_\infty = \frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k-1)/2} \zeta^k, \tag{7.1.4}$$

we easily find that,

$$J(a, q) = \frac{ie^{\pi ia} q^{-1/8}}{(q)_\infty} \theta(a), \tag{7.1.5}$$

where

$$\theta(z) = \theta(z;\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i\left(n+\frac{1}{2}\right)^2\tau+2\pi i\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}. \tag{7.1.6}$$

Comparing this with the classical definition of $\theta_1(z)$ [37, p. 355, Section 13.19, Equation 10], we find that upon replacing q by $q^{1/2}$ in this classical definition, $\theta(z) = -\theta_1(z)$. From [86, p. 8], we see that

$$\theta(z+1) = -\theta(z), \tag{7.1.7}$$

$$\theta(z+\tau) = -e^{-\pi i\tau-2\pi iz}\theta(z), \tag{7.1.8}$$

$$\theta(-z) = -\theta(z), \tag{7.1.9}$$

$$\theta'(0;\tau) = -2\pi q^{1/8}(q)_\infty^3. \tag{7.1.10}$$

Using (7.1.7) and (7.1.8), we have

$$J(a+1,q) = J(a,q), \tag{7.1.11}$$

$$J(a+\tau,q) = -e^{-2\pi ia}J(a,q), \tag{7.1.12}$$

$$J(a-\tau,q) = -q^{-1}e^{2\pi ia}J(a,q), \tag{7.1.13}$$

$$J(-a,q) = -e^{-2\pi ia}J(a,q). \tag{7.1.14}$$

Using (7.1.3), we rephrase (1.0.18) as follows:

$$\begin{aligned}
&\frac{e^{2\pi imv}J(w-v)J(w+v)J(2v)}{J(v)(J(w))^2}\prod_{3\le j\le m}\frac{J(a_j-v)J(a_j+v)}{(J(a_j))^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi iu}q^n}\\
&+\frac{J(v-w)J(v)J(v+w)J(2v)(q)_\infty^2}{J(u-v)J(u-w)J(u)J(u+w)J(u+v)}\prod_{3\le j\le m}\frac{J(v-a_j)J(v+a_j)}{J(u-a_j)J(u+a_j)}\\
&+\left\{e^{2\pi i(v-w)}\frac{J(v)J(2v)}{J(w)J(2w)}\prod_{3\le j\le m}\frac{J(v-a_j)J(v+a_j)}{J(w-a_j)J(w+a_j)}\right.\\
&\quad\left.\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\right)+\mathrm{idem}(w;a_3,\cdots,a_m)\right\}
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i v(2m+1)n}}{1 - e^{2\pi i(u-v)} q^n} + \frac{e^{2\pi i v(2m+1)(n+1)}}{1 - e^{2\pi i(u+v)} q^n} \right). \tag{7.1.15}$$

Fix $a_3, \cdots, a_m$ and consider the left-hand side of (7.1.15) as a function of w only and denote it by $g(w)$. Let $f_1(w)$ denote the expression in line 1 of (7.1.15), $f_2(w)$ the expression in line 2 of (7.1.15) and $f_3(w)$ the expression in lines 3 and 4 of (7.1.15). Then, using (7.1.11) and (7.1.12), we readily see that

$$f_1(w+1) = f_1(w), \tag{7.1.16}$$

$$f_1(w+\tau) = f_1(w). \tag{7.1.17}$$

Similarly from (7.1.11), one can easily obtain

$$f_2(w+1) = f_2(w). \tag{7.1.18}$$

Now from (7.1.12) and (7.1.13),

$$\begin{aligned} f_2(w+\tau) &= \frac{J(v-w-\tau)J(v)J(v+w+\tau)J(2v)(q)_\infty^2}{J(u-v)J(u-w-\tau)J(u)J(u+w+\tau)J(u+v)} \prod_{3\le j\le m} \frac{J(v-a_j)J(v+a_j)}{J(u-a_j)J(u+a_j)} \\ &= \frac{\left(-q^{-1}e^{2\pi i(v-w)}J(v-w)\right) J(v) \left(-e^{-2\pi i(v+w)}J(v+w)\right) J(2v)(q)_\infty^2}{J(u-v)\left(-q^{-1}e^{2\pi i(u-w)}J(u-w)\right) J(u) \left(-e^{-2\pi i(u+w)}J(u+w)\right) J(u+v)} \\ &\quad \times \prod_{3\le j\le m} \frac{J(v-a_j)J(v+a_j)}{J(u-a_j)J(u+a_j)} \\ &= f_2(w). \end{aligned} \tag{7.1.19}$$

Using (7.1.11), we find that

$$f_3(w+1) = f_3(w). \tag{7.1.20}$$

Another application of (7.1.12) and (7.1.13) gives

$f_3(w+\tau)$

$$
\begin{aligned}
&= \frac{e^{2\pi i(v-w-\tau)}J(v)J(2v)}{J(w+\tau)J(2w+2\tau)} \prod_{3\le j\le m} \frac{J(v-a_j)J(v+a_j)}{J(w+\tau-a_j)J(w+\tau+a_j)} \\
&\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i(w+\tau)(2m+1)n}}{1-e^{2\pi i(u-w-\tau)}q^n} + \frac{e^{2\pi i(w+\tau)(2m+1)(n+1)}}{1-e^{2\pi i(u+w+\tau)}q^n} \right) \\
&\quad + \sum_{k=3}^{m} e^{2\pi i(v-a_k)} \frac{J(v)J(2v)J(v-w-\tau)J(v+w+\tau)}{J(a_k)J(2a_k)J(a_k-w-\tau)J(a_k+w+\tau)} \prod_{\substack{2<j<m+1\\ j\neq k}} \frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)} \\
&\qquad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i a_k(2m+1)n}}{1-e^{2\pi i(u-a_k)}q^n} + \frac{e^{2\pi i a_k(2m+1)(n+1)}}{1-e^{2\pi i(u+a_k)}q^n} \right) \\
&= -e^{2\pi i v+8\pi i w+4\pi i w(m-2)} \frac{J(v)J(2v)}{J(w)J(2w)} \prod_{3\le j\le m} \frac{J(v-a_j)J(v+a_j)}{J(w-a_j)J(w+a_j)} \\
&\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i w(2m+1)n}q^{-(2m+1)n}}{1-e^{2\pi i(u-w)}q^{n-1}} + \frac{e^{2\pi i w(2m+1)(n+1)}q^{(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^{n+1}} \right) \\
&\quad + \sum_{k=3}^{m} e^{2\pi i(v-a_k)} \frac{J(v)J(2v)J(v-w)J(v+w)}{J(a_k)J(2a_k)J(a_k-w)J(a_k+w)} \prod_{\substack{2<j<m+1\\ j\neq k}} \frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)} \\
&\qquad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i a_k(2m+1)n}}{1-e^{2\pi i(u-a_k)}q^n} + \frac{e^{2\pi i a_k(2m+1)(n+1)}}{1-e^{2\pi i(u+a_k)}q^n} \right) \\
&= e^{2\pi i v+4\pi i m w} \frac{J(v)J(2v)}{J(w)J(2w)} \prod_{3\le j\le m} \frac{J(v-a_j)J(v+a_j)}{J(w-a_j)J(w+a_j)} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)(n+1)(n+2)/2} \right. \\
&\quad \left. \times \frac{e^{-2\pi i w(2m+1)(n+1)}q^{-(2m+1)(n+1)}}{1-e^{2\pi i(u-w)}q^n} + \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n-1)/2} \frac{e^{2\pi i w(2m+1)n}q^{(2m+1)n}}{1-e^{2\pi i(u+w)}q^n} \right) \\
&\quad + \sum_{k=3}^{m} e^{2\pi i(v-a_k)} \frac{J(v)J(2v)J(v-w)J(v+w)}{J(a_k)J(2a_k)J(a_k-w)J(a_k+w)} \prod_{\substack{2<j<m+1\\ j\neq k}} \frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)} \\
&\qquad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i a_k(2m+1)n}}{1-e^{2\pi i(u-a_k)}q^n} + \frac{e^{2\pi i a_k(2m+1)(n+1)}}{1-e^{2\pi i(u+a_k)}q^n} \right) \\
&= f_3(w). \qquad (7.1.21)
\end{aligned}
$$

Thus from (7.1.16)–(7.1.21), we deduce that g is a doubly periodic function in w with periods 1 and τ. Our next task is to show that g is an entire function of w and hence a constant (with respect to w). We show that the poles of g at $w = u$ and $w = -u$ are actually removable singularities by proving that $\lim_{w\to\pm u}(w \mp u)\left(f_2(w) + f_3(w)\right) = 0$ which readily implies that $\lim_{w\to\pm u}(w\mp u)g(w)=0$. Let

$$A:=A(v,a_3,\cdots,a_m;q):=J(v)J(2v)\prod_{3\le j\le m}J(v-a_j)J(v+a_j). \tag{7.1.22}$$

Next, applying (7.1.5), (7.1.14) and (7.1.10), we see that

$$\begin{aligned}
&\lim_{w\to u}(w-u)\left(f_2(w)+f_3(w)\right)\\
&=A\lim_{w\to u}(w-u)\Bigg\{\frac{J(v-w)J(v+w)(q)_\infty^2}{J(u-w)J(u+w)J(u-v)J(u)J(u+v)}\prod_{3\le j\le m}\frac{1}{J(u-a_j)J(u+a_j)}\\
&\quad+\frac{e^{2\pi i(v-w)}}{J(w)J(2w)}\prod_{3\le j\le m}\frac{1}{J(w-a_j)J(w+a_j)}\Bigg(\frac{1}{1-e^{2\pi i(u-w)}}+\sum_{\substack{n=-\infty\\n\ne0}}^{\infty}(-1)^nq^{(2m+1)n(n+1)/2}\\
&\quad\times\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\sum_{n=-\infty}^{\infty}(-1)^nq^{(2m+1)n(n+1)/2}\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\Bigg)\Bigg\}\\
&=A\Bigg(-iq^{1/8}(q)_\infty^3\frac{J(v-u)J(v+u)}{\left(-e^{-2\pi i(v-u)}J(v-u)\right)J(u)J(2u)J(u+v)}\prod_{3\le j\le m}\frac{1}{J(u-a_j)J(u+a_j)}\\
&\quad\times\lim_{w\to u}\frac{w-u}{e^{\pi i(u-w)}\theta(u-w)}+\frac{1}{J(u)J(2u)}\prod_{3\le j\le m}\frac{1}{J(u-a_j)J(u+a_j)}\lim_{w\to u}\frac{(w-u)e^{2\pi i(v-w)}}{1-e^{2\pi i(u-w)}}\Bigg)\\
&=A\frac{e^{2\pi i(v-u)}}{J(u)J(2u)}\prod_{3\le j\le m}\frac{1}{J(u-a_j)J(u+a_j)}\left(\frac{-iq^{1/8}(q)_\infty^3}{\theta'(0)}+\frac{1}{2\pi i}\right)\\
&=0.
\end{aligned} \tag{7.1.23}$$

Similarly, $\lim_{w\to -u}(w+u)\left(f_2(w)+f_3(w)\right)=0$. Now the only other possibility of a pole of g is at 0, which arises from f_1 and f_3 each having a pole at 0. Again, to show that this is a removable singularity, it suffices to show that $\lim_{w\to0}w\left(f_1(w)+f_3(w)\right)=0$. To show this, we need Jacobi's duplication formula for theta functions [85, p. 488, Ex. 5]

$$\theta(2w)\theta_2\theta_3\theta_4=2\theta(w)\theta_2(w)\theta_3(w)\theta_4(w). \tag{7.1.24}$$

Let

$$B := B(u, v, a_3, \cdots, a_m; q) := e^{2\pi imv} \frac{J(2v)}{J(v)} \prod_{3\le j\le m} \frac{J(a_j - v)J(a_j + v)}{(J(a_j))^2} \times \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1 - e^{2\pi iu} q^n}. \tag{7.1.25}$$

Then from (7.1.15) and (7.1.25),

$$\begin{aligned}
&\lim_{w\to 0} w\left(f_1(w) + f_3(w)\right) \\
&= \lim_{w\to 0} w \Bigg\{ B \frac{J(w-v)J(w+v)}{(J(w))^2} + A \frac{e^{2\pi i(v-w)}}{J(w)J(2w)} \prod_{3\le j\le m} \frac{1}{J(w-a_j)J(w+a_j)} \\
&\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi iw(2m+1)n}}{1 - e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi iw(2m+1)(n+1)}}{1 - e^{2\pi i(u+w)}q^n} \right) \Bigg\} \\
&\quad + \lim_{w\to 0} w \sum_{k=3}^{m} e^{2\pi i(v-a_k)} \frac{J(v)J(2v)J(v-w)J(v+w)}{J(a_k)J(2a_k)J(a_k-w)J(a_k+w)} \prod_{\substack{2<j<m+1 \\ j\neq k}} \frac{J(v-a_j)J(v+a_j)}{J(a_k-a_j)J(a_k+a_j)} \\
&= \lim_{w\to 0} \frac{w}{\theta(w)} \Bigg\{ B \frac{\theta(w-v)\theta(w+v)}{\theta(w)} - A e^{2\pi iv} q^{1/4} (q)_\infty^2 \frac{e^{-5\pi iw}}{\theta(2w)} \prod_{3\le j\le m} \frac{1}{J(w-a_j)J(w+a_j)} \\
&\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi iw(2m+1)n}}{1 - e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi iw(2m+1)(n+1)}}{1 - e^{2\pi i(u+w)}q^n} \right) \Bigg\} \\
&= \frac{1}{\theta'(0)} \lim_{w\to 0} \frac{D(w)}{\theta(w)},
\end{aligned} \tag{7.1.26}$$

where

$$\begin{aligned}
D(w) &:= D(w, v, a_3, \cdots, a_m; q) := B\theta(w-v)\theta(w+v) - A e^{2\pi iv} q^{1/4} (q)_\infty^2 E(w) \\
E(w) &:= E(w; u, a_3 \cdots, a_m; q) := \frac{e^{-5\pi iw}\theta(w)}{\theta(2w)} \prod_{3\le j\le m} \frac{1}{J(w-a_j)J(w+a_j)} \\
&\quad \times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi iw(2m+1)n}}{1 - e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi iw(2m+1)(n+1)}}{1 - e^{2\pi i(u+w)}q^n} \right).
\end{aligned}$$

$$(7.1.27)$$

Now using (7.1.24), (7.1.5) and (7.1.14), we find that as $w \to 0$,

$$D(w) \to -B\theta^2(v) - \frac{e^{2\pi iv}Aq^{1/4}(q)_\infty^2}{(-1)^{m-2}e^{-2\pi i(a_3+\cdots+a_m)}(J(a_3)\cdots J(a_m))^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi iu}q^n}$$
$$= 0, \tag{7.1.28}$$

which is observed by putting the expressions for A and B back in the first expression on the right side in (7.1.28). Thus,

$$\lim_{w\to 0} w\left(f_1(w)+f_3(w)\right) = \frac{D^{'}(0)}{\theta^{'}(0)^2}. \tag{7.1.29}$$

Now we calculate $D^{'}(0)$.

$$D^{'}(w) = B\left(\theta^{'}(w-v)\theta(w+v)+\theta(w-v)\theta^{'}(w+v)\right) - e^{2\pi iv}Aq^{1/4}(q)_\infty^2 E^{'}(w). \tag{7.1.30}$$

Using (7.1.5) and (7.1.24), we have

$$E(w) = (-1)^{m-2}q^{\frac{m-2}{4}}(q)_\infty^{2(m-2)}\frac{e^{-\pi iw(2m+1)}\theta_2\theta_3\theta_4}{2\theta_2(w)\theta_3(w)\theta_4(w)}\prod_{3\le j\le m}\frac{1}{\theta(w-a_j)\theta(w+a_j)}$$
$$\times\sum_{n=-\infty}^{\infty}(-1)^n q^{(2m+1)n(n+1)/2}\left(\frac{e^{-2\pi iw(2m+1)n}}{1-e^{2\pi i(u-w)}q^n}+\frac{e^{2\pi iw(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n}\right). \tag{7.1.31}$$

Differentiating both sides with respect to w and simplifying, we obtain

$$E^{'}(w) = \frac{1}{2}(-1)^{m-2}q^{\frac{m-2}{4}}(q)_\infty^{2(m-2)}e^{-\pi iw(2m+1)}\theta_2\theta_3\theta_4$$
$$\times\left\{-\frac{\pi i(2m+1)+\frac{\theta_2^{'}(w)}{\theta_2(w)}+\frac{\theta_3^{'}(w)}{\theta_3(w)}+\frac{\theta_3^{'}(w)}{\theta_3(w)}+\sum_{3\le j\le m}\left(\frac{\theta^{'}(w-a_j)}{\theta(w-a_j)}+\frac{\theta^{'}(w+a_j)}{\theta(w+a_j)}\right)}{\theta_2(w)\theta_3(w)\theta_4(w)\prod_{3\le j\le m}\theta(w-a_j)\theta(w+a_j)}\right.$$

$$\times \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i w(2m+1)n}}{1-e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi i w(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n} \right)$$

$$+ \frac{1}{\theta_2(w)\theta_3(w)\theta_4(w)} \prod_{3\le j\le m} \frac{1}{\theta(w-a_j)\theta(w+a_j)} F^{'}(w) \Bigg\}, \tag{7.1.32}$$

where

$$F(w) := F(w,u,m;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i w(2m+1)n}}{1-e^{2\pi i(u-w)}q^n} + \frac{e^{2\pi i w(2m+1)(n+1)}}{1-e^{2\pi i(u+w)}q^n} \right). \tag{7.1.33}$$

It is straightforward to see that

$$F^{'}(0) = 2\pi i(2m+1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2m+1)n(n+1)/2}}{1-e^{2\pi i u}q^n}. \tag{7.1.34}$$

From (7.1.9), we have

$$\theta^{'}(-z) = \theta^{'}(z). \tag{7.1.35}$$

Then letting $w \to 0$ in (7.1.32), and using (7.1.9), (7.1.35), (7.1.34) and the fact that $\theta_k^{'}(0) = 0$ for $2 \le k \le 4$, we find that

$$E^{'}(0) = 0. \tag{7.1.36}$$

Using (7.1.35) and (7.1.36) in (7.1.30), we finally deduce that $D^{'}(0) = 0$.

With the help of (7.1.29), this then implies that $\lim_{w\to 0} w\,(f_1(w) + f_3(w)) = 0$ and thus $\lim_{w\to 0} wg(w) = 0$. Thus $w = 0$ is also a removable singularity, which implies that g is an doubly periodic entire function and hence a constant, say K (which may very well depend on v). Finally, since $J(0) = 0$, we have

$$K = g(v) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(2m+1)n(n+1)/2} \left(\frac{e^{-2\pi i v(2m+1)n}}{1-e^{2\pi i(u-v)}q^n} + \frac{e^{2\pi i v(2m+1)(n+1)}}{1-e^{2\pi i(u+v)}q^n} \right).$$

This completes the proof.

7.2 A useful lemma

In the following lemma, we relate $R_k(z,q)$ to $\Sigma^{(2k-1)}(z,q)$, a special case of the level $(2k-1)$ Appell function. This lemma generalizes Lemma 7.9 in [40] which gives a relation between the rank generating function $R(z,q)$ and a level 3 Appell function. The $k=1$ case of the lemma gives the familiar partial fraction expansion for the reciprocal of Jacobi's theta product $(z)_\infty(z^{-1}q)_\infty$ [78, p. 1], [79, p. 136], since $\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}=0$.

Lemma 7.2.1. *For $k \geq 1$,*

$$R_k(z,q) = \frac{1}{(q)_\infty}\left(z^{k-1}(1-z)\Sigma^{(2k-1)}(z,q) - z\theta_{1,2k-1}(q) + z(1-z)\sum_{m=0}^{k-3} z^m \theta_{2m+3,2k-1}(q)\right), \quad (7.2.1)$$

where

$$\theta_{j,2k-1}(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n((2k-1)n+j)/2}, \quad (7.2.2)$$

for $j = 1,\ 3,\ \ldots,\ 2k-3$.

Proof. From [41, Eq.(4.3)], we see that

$$\begin{aligned} R_k(z,q) &= \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2}(1-q^n)\left(\frac{1}{1-zq^n}+\frac{z^{-1}q^n}{1-z^{-1}q^n}\right) \\ &= \frac{z^{-1}}{(q)_\infty}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}(-1)^{n-1}q^{n((2k-1)n+1)/2}\frac{1-q^n}{1-z^{-1}q^n}. \end{aligned} \quad (7.2.3)$$

Replacing z by z^{-1} in (1.0.26) and (7.2.3), we see that

$$R_k(z,q) = \frac{z}{(q)_\infty}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}(-1)^{n-1}q^{n((2k-1)n+1)/2}\frac{1-q^n}{1-zq^n}$$

$$
\begin{aligned}
&= \frac{z}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^{n-1} q^{n((2k-1)n+1)/2} \left(1 - \frac{(1-z)q^n}{1-zq^n}\right) \\
&= \frac{z}{(q)_\infty} \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^{n-1} q^{n((2k-1)n+1)/2} + (1-z) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n((2k-1)n+3)/2}}{1-zq^n} \right) \\
&= \frac{z}{(q)_\infty} \left(1 - \sum_{n=-\infty}^{\infty} (-1)^n q^{n((2k-1)n+1)/2} + (1-z) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n((2k-1)n+3)/2}}{1-zq^n} \right) \\
&= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty} + \frac{z}{(q)_\infty} \left(1 + (1-z) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n((2k-1)n+3)/2}}{1-zq^n} \right) \\
&= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty} + \frac{z}{(q)_\infty} \Bigg(1 + (1-z) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{n((2k-1)n+3)/2} \\
&\qquad\qquad \times \left(\frac{z^{k-2} q^{(k-2)n}}{1-zq^n} + \frac{1-(zq^n)^{k-2}}{1-zq^n} \right) \Bigg) \\
&= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty} + \frac{z}{(q)_\infty} \Bigg(1 + (1-z) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{n((2k-1)n+3)/2} \\
&\qquad\qquad \times \left(\frac{z^{k-2} q^{(k-2)n}}{1-zq^n} + \sum_{m=0}^{k-3} z^m q^{mn} \right) \Bigg) \\
&= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty} + \frac{z}{(q)_\infty} \Bigg(1 + z^{k-2}(1-z) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{(2k-1)n(n+1)/2}}{1-zq^n} \\
&\qquad\qquad + (1-z) \sum_{m=0}^{k-3} z^m \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{n((2k-1)n+2m+3)/2} \Bigg) \\
&= \frac{-z\theta_{1,2k-1}(q)}{(q)_\infty} + \frac{z}{(q)_\infty} \Bigg(z^{k-2}(1-z) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2k-1)n(n+1)/2}}{1-zq^n} \\
&\qquad\qquad + (1-z) \sum_{m=0}^{k-3} z^m \sum_{n=-\infty}^{\infty} (-1)^n q^{n((2k-1)n+2m+3)/2} \Bigg) \\
&= \frac{1}{(q)_\infty} \left(-z\theta_{1,2k-1}(q) + z^{k-1}(1-z)\Sigma^{(2k-1)}(z,q) + z(1-z) \sum_{m=0}^{k-3} z^m \theta_{2m+3,2k-1}(q) \right).
\end{aligned}
$$

This completes the proof of Lemma (7.2.1). □

7.3 Proof of (1.0.30) through Jackson's identity

Here the role of (1.0.16) is played by the following special case of Jackson's identity (1.0.17). Let $x = \zeta^2$ in (1.0.17). Then,

$$\begin{aligned}
&\frac{\zeta^2[\zeta^3]_\infty}{[\zeta^2]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n}+\frac{[\zeta]_\infty[\zeta^2]_\infty[\zeta^3]_\infty[\zeta^{-1}]_\infty(q)_\infty^2}{[z/\zeta^2]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[z\zeta^2]_\infty}\\
&+\zeta^{-1}\frac{[\zeta]_\infty}{[\zeta^4]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{5n(n+1)/2}\left(\frac{\zeta^{-10n}}{1-z\zeta^{-2}q^n}+\frac{\zeta^{10n+10}}{1-z\zeta^2q^n}\right)\\
&=\sum_{n=-\infty}^{\infty}(-1)^n q^{5n(n+1)/2}\left(\frac{\zeta^{-5n}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{5n+5}}{1-z\zeta q^n}\right). \qquad (7.3.1)
\end{aligned}$$

We consider both sides of (7.3.1) as functions of ζ only. The basic idea is to take the fourth derivative of both sides of (7.3.1) with respect to ζ. Denote the three expressions on the left-hand side of (7.3.1) by $g_1(\zeta)$, $g_2(\zeta)$ and $g_3(\zeta)$ respectively, and let $g_4(\zeta)$ denote the right-hand side of (7.3.1). It is clear that $g_2(\zeta)$ has a zero of order 4 at $\zeta = 1$. Now

$$\begin{aligned}
g_2(\zeta) &= \frac{[\zeta]_\infty[\zeta^2]_\infty[\zeta^3]_\infty[\zeta^{-1}]_\infty(q)_\infty^2}{[z/\zeta^2]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[z\zeta^2]_\infty}\\
&= (1-\zeta)(1-\zeta^2)(1-\zeta^3)(1-\zeta^{-1})\frac{(\zeta q,\zeta^{-1}q,\zeta^2q,\zeta^{-2}q,\zeta^3q,\zeta^{-3}q,\zeta^{-1}q,\zeta q)_\infty(q)_\infty^2}{[z/\zeta^2]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[z\zeta^2]_\infty}\\
&= (1-\zeta)^4\left(-\frac{1}{\zeta}(1+\zeta)\left(1+\zeta+\zeta^2\right)\frac{(\zeta q,\zeta^{-1}q,\zeta^2q,\zeta^{-2}q,\zeta^3q,\zeta^{-3}q,\zeta^{-1}q,\zeta q)_\infty(q)_\infty^2}{[z/\zeta^2]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[z\zeta^2]_\infty}\right)\\
&=: g_{21}(\zeta)g_{22}(\zeta), \qquad (7.3.2)
\end{aligned}$$

where

$$\begin{aligned}
g_{21}(\zeta) &:= (1-\zeta)^4\\
g_{22}(\zeta) &:= -\frac{1}{\zeta}(1+\zeta)\left(1+\zeta+\zeta^2\right)\frac{(\zeta q,\zeta^{-1}q,\zeta^2q,\zeta^{-2}q,\zeta^3q,\zeta^{-3}q,\zeta^{-1}q,\zeta q)_\infty(q)_\infty^2}{[z/\zeta^2]_\infty[z/\zeta]_\infty[z]_\infty[z\zeta]_\infty[z\zeta^2]_\infty}. \qquad (7.3.3)
\end{aligned}$$

Now taking the fourth derivative of the extreme sides of (7.3.2) with respect to ζ and using Leibnitz's rule for successive differentiation, one observes that

$$g_2^{(4)}(\zeta) = g_{21}^{(4)}(\zeta)g_{22}(\zeta) + 4g_{21}^{(3)}(\zeta)g_{22}^{(1)}(\zeta) + 6g_{21}^{(2)}(\zeta)g_{22}^{(2)}(\zeta) + 4g_{21}^{(1)}(\zeta)g_{22}^{(3)}(\zeta) + g_{21}(\zeta)g_{22}^{(4)}(\zeta). \tag{7.3.4}$$

Evaluating above at $\zeta = 1$ annihilates all the terms on the right-hand side except the first one so that

$$g_2^{(4)}(1) = g_{21}^{(4)}(1)g_{22}(1) = (-1)^4 4!(-6)\frac{(q)_\infty^{10}}{[z]_\infty^5} = -144(q)_\infty^5[C^*(z,q)]^5, \tag{7.3.5}$$

where $C^*(z,q)$ is defined in (1.0.23).

Next, we show that

$$g_1^{(4)}(1) = \left(\frac{3}{4} - \frac{15}{2}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} - 225\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2} - \frac{105}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n(n+1)/2}}{1-zq^n}. \tag{7.3.6}$$

From [39, p. 20, Equation (18.6)],

$$\frac{[ab]_\infty(q)_\infty^2}{[a]_\infty[b]_\infty} = \frac{1-ab}{(1-a)(1-b)} + \sum_{m,n=1}^{\infty}\left(a^mb^n - a^{-m}b^{-n}\right)q^{mn}. \tag{7.3.7}$$

Let $a = \zeta$ and $b = \zeta^2$ in (7.3.7). Then,

$$\frac{[\zeta^3]_\infty}{[\zeta^2]_\infty} = \frac{[\zeta]_\infty}{(q)_\infty^2}\left(\frac{1-\zeta^3}{(1-\zeta)(1-\zeta^2)} + \sum_{m,n=1}^{\infty}\left(\zeta^{m+2n} - \zeta^{-m-2n}\right)q^{mn}\right). \tag{7.3.8}$$

Using (7.3.8) and the Jacobi triple product identity in the form

$$[\zeta]_\infty = \frac{1}{(q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{k(k-1)/2}\zeta^k, \tag{7.3.9}$$

we have

$$\begin{aligned}
\zeta^2\frac{[\zeta^3]_\infty}{[\zeta^2]_\infty} &= \frac{1}{(q)_\infty^3}\Bigg(\frac{\zeta^2(1-\zeta^3)}{1-\zeta^2}\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{k(k-1)/2}\zeta^k}{1-\zeta} \\
&\qquad - \sum_{\substack{m,n=1\\k=-\infty}}^{\infty}(-1)^k q^{k(k-1)/2}\zeta^{2+k-m-2n}\left(1-\zeta^{2m+4n}\right)q^{mn}\Bigg) \\
&= \frac{1}{(q)_\infty^3}\Bigg(\frac{\zeta^2(1+\zeta+\zeta^2)}{1+\zeta}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}\sum_{j=0}^{2k}\zeta^{-k+j} \\
&\qquad - \sum_{\substack{m,n=1\\k=0}}^{\infty}(-1)^k q^{k(k+1)/2}\zeta^{2-k-m-2n}(1-\zeta^{2k+1})\left(1-\zeta^{2m+4n}\right)q^{mn}\Bigg) \\
&=: k_1(\zeta)-k_2(\zeta).
\end{aligned} \tag{7.3.10}$$

Now

$$\begin{aligned}
k_1^{(4)}(1) &= \frac{1}{(q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{4}+\frac{22n}{5}+9n^2+7n^3+\frac{3n^4}{2}+\frac{3n^5}{5}\right)q^{n(n+1)/2} \\
&= \frac{1}{20(q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n(2n+1)(6n^4+12n^3+64n^2+58n+15)q^{n(n+1)/2} \\
&= \frac{1}{20(q)_\infty^3}\Bigg(4q\sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{n(n+1)}{2}(3n^2+3n+29)q^{n(n+1)/2-1} \\
&\qquad + 15\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\Bigg) \\
&= \frac{1}{20(q)_\infty^3}\Bigg(4q\frac{d}{dq}\Bigg(6q\sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{n(n+1)}{2}q^{n(n+1)/2-1} \\
&\qquad + 29\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\Bigg) + 15\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\Bigg) \\
&= \frac{1}{20(q)_\infty^3}\left(4q\frac{d}{dq}\left(6q\frac{d}{dq}(q)_\infty^3+29(q)_\infty^3\right)+15(q)_\infty^3\right) \\
&= \frac{1}{20}\left(216\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)^2-72\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2}-348\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+15\right),
\end{aligned} \tag{7.3.11}$$

where in the penultimate step, we have made use of Jacobi's identity [16, Theorem 1.3.9]

$$(q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \tag{7.3.12}$$

and in the last step we have used the fact that $q\dfrac{d}{dq}(q)_\infty^3 = -3(q)_\infty^3\sum_{n=1}^{\infty}\dfrac{nq^n}{1-q^n}$. Next,

$$\begin{aligned}
k_2^{(4)}(1) &= \frac{8}{(q)_\infty^3}\sum_{\substack{m,n=1\\k=0}}^{\infty}(-1)^k(2k+1)(m+2n)\left((m+2n)^2+k^2+k+2\right)q^{k(k+1)/2+mn}\\
&= \frac{8}{(q)_\infty^3}\Bigg[\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}\right)\left(\sum_{m,n=1}^{\infty}(m+2n)^3q^{mn}\right)\\
&\qquad+\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)(k^2+k+2)q^{k(k+1)/2}\right)\left(\sum_{m,n=1}^{\infty}(m+2n)q^{mn}\right)\Bigg]\\
&= 8\sum_{m,n=1}^{\infty}(m+2n)^3q^{mn}+\frac{8}{(q)_\infty^3}\left(2q\frac{d}{dq}(q)_\infty^3+2(q)_\infty^3\right)\sum_{m,n=1}^{\infty}(m+2n)q^{mn} \qquad (7.3.13)
\end{aligned}$$

Since

$$\sum_{m,n=1}^{\infty}(m+2n)^3q^{mn} = 9\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}+18\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2}, \tag{7.3.14}$$

and

$$\sum_{m,n=1}^{\infty}(m+2n)q^{mn} = 3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{7.3.15}$$

we see that

$$k_2^{(4)}(1) = 72\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}+144\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2}-144\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)^2+48\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \tag{7.3.16}$$

Finally from (7.3.10), (7.3.11) and (7.3.16), we find that

$$g_1^{(4)}(1) = \left(k_1^{(4)}(1)-k_2^{(4)}(1)\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n(n+1)/2}}{1-zq^n}$$

$$= \left[\frac{774}{5}\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)^2 - \frac{738}{5}\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2} - \frac{327}{5}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{3}{4} - 72\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\right]$$
$$\times \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n}. \tag{7.3.17}$$

Using Ramanujan's differential equation [16, Equation 4.2.20]

$$q\frac{dP}{dq} = \frac{P^2-Q}{12}, \tag{7.3.18}$$

where

$$P := P(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} \tag{7.3.19}$$

and

$$Q := Q(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{7.3.20}$$

we easily see that

$$\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)^2 = \frac{5}{12}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2}. \tag{7.3.21}$$

Substituting (7.3.21) in (7.3.17) and simplifying produces (7.3.6). Next, by routine algebraic simplification, it can easily be observed that

$$g_4^{(4)}(1) = \sum_{n=-\infty}^{\infty}(-1)^n q^{5n(n+1)/2}\left(\frac{2(625n^4+750n^3+425n^2+120n+12)}{1-zq^n}\right.$$
$$\left. + \frac{8(125n^3+75n^2+25n+3)}{(1-zq^n)^2} + \frac{24(25n^2+5n+1)}{(1-zq^n)^3} + \frac{240n}{(1-zq^n)^4} + \frac{48}{(1-zq^n)^5}\right). \tag{7.3.22}$$

It remains to evaluate $g_3^{(4)}(1)$. We define $g_3(\zeta) := g_{31}(\zeta)g_{32}(\zeta)$ where

$$g_{31}(\zeta) = \zeta^{-1}\frac{[\zeta]_\infty}{[\zeta^4]_\infty} = -\frac{[\zeta^{-1}]_\infty}{[\zeta^4]_\infty},$$

$$g_{32}(\zeta) = \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \left(\frac{\zeta^{-10n}}{1 - z\zeta^{-2}q^n} + \frac{\zeta^{10n+10}}{1 - z\zeta^2 q^n} \right). \tag{7.3.23}$$

By Leibnitz rule for successive differentiation,

$$g_3^{(4)}(1) = g_{31}^{(4)}(1)g_{32}(1) + 4g_{31}^{(3)}(1)g_{32}^{(1)}(1) + 6g_{31}^{(2)}(1)g_{32}^{(2)}(1) + 4g_{31}^{(1)}(1)g_{32}^{(3)}(1) + g_{31}(1)g_{32}^{(4)}(1). \tag{7.3.24}$$

The function $g_{31}(\zeta)$ and its derivatives are evaluated below at $\zeta = 1$:

$$g_{31}(1) = \zeta^{-1}\frac{[\zeta]_\infty}{[\zeta^4]_\infty}\bigg|_{\zeta=1} = \frac{\zeta^{-1}\,(1-\zeta)\,(\zeta q)_\infty(\zeta^{-1}q)_\infty}{(1-\zeta^4)\,(\zeta^4 q)_\infty(\zeta^{-4}q)_\infty}\bigg|_{\zeta=1} = \frac{1}{4}.$$

$$g_{31}^{(1)}(\zeta) = -\frac{[\zeta^{-1}]_\infty}{[\zeta^4]_\infty}\left(\sum_{n=0}^{\infty}\frac{q^n/\zeta^2}{1-q^n/\zeta} - \sum_{n=1}^{\infty}\frac{q^n}{1-\zeta q^n} + \sum_{n=0}^{\infty}\frac{4\zeta^3 q^n}{1-\zeta^4 q^n} - \sum_{n=1}^{\infty}\frac{4q^n/\zeta^5}{1-q^n/\zeta^4}\right).$$

Thus

$$g_{31}^{(1)}(1) = \lim_{\zeta\to 1}\frac{1}{4}\left(\frac{1/\zeta^2}{1-1/\zeta} + \frac{4\zeta^3}{1-\zeta^4}\right) = \frac{1}{4}\left(-\frac{4\zeta^3+3\zeta^2+2\zeta+1}{\zeta(\zeta^3+\zeta^2+\zeta+1)}\right)_{\zeta=1} = -\frac{5}{8}.$$

For $|q| < 1$, $|x| < 1$ and $|y| < 1$, the following transformation is valid.

$$\sum_{n=0}^{\infty}\frac{x^n}{1-yq^n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x^n y^m q^{mn} = \sum_{n=0}^{\infty}\frac{y^n}{1-xq^n}. \tag{7.3.25}$$

This transformation allows us to bring any powers of ζ from denominators of the summands of the series in $g_{31}^{(1)}(\zeta)$ to numerators. This then enables us to obtain Eisenstein-type series upon differentiation. Hence,

$$g_{31}^{(1)}(\zeta) = -\frac{[\zeta^{-1}]_\infty}{[\zeta^4]_\infty}\Bigg(\frac{1/\zeta^2}{1-1/\zeta} + \frac{4\zeta^3}{1-\zeta^4} + \frac{q}{\zeta^2}\sum_{n=0}^{\infty}\frac{q^n}{1-q^n(q/\zeta)} - q\sum_{n=0}^{\infty}\frac{q^n}{1-(\zeta q)q^n}$$
$$+ 4q\zeta^3\sum_{n=0}^{\infty}\frac{q^n}{1-(\zeta^4 q)q^n} - \frac{4q}{\zeta^5}\sum_{n=0}^{\infty}\frac{q^n}{1-q^n(q/\zeta^4)}\Bigg)$$

$$
= -\frac{[\zeta^{-1}]_\infty}{[\zeta^4]_\infty}\Bigg(- \frac{4\zeta^3+3\zeta^2+2\zeta+1}{\zeta(\zeta^3+\zeta^2+\zeta+1)} + \sum_{n=0}^{\infty}\frac{\zeta^{-n-2}q^{n+1}}{1-q^{n+1}} - \sum_{n=0}^{\infty}\frac{\zeta^{n}q^{n+1}}{1-q^{n+1}} + \sum_{n=0}^{\infty}\frac{4\zeta^{4n+3}q^{n+1}}{1-q^{n+1}} - \sum_{n=0}^{\infty}\frac{4\zeta^{-4n-5}q^{n+1}}{1-q^{n+1}}\Bigg).
$$

Therefore

$$
\begin{aligned}
g_{31}^{(2)}(\zeta) &= -\frac{[\zeta^{-1}]_\infty}{[\zeta^4]_\infty}\left(\sum_{n=0}^{\infty}\frac{q^n/\zeta^2}{1-q^n/\zeta} - \sum_{n=1}^{\infty}\frac{q^n}{1-\zeta q^n} + \sum_{n=0}^{\infty}\frac{4\zeta^3 q^n}{1-\zeta^4 q^n} - \sum_{n=1}^{\infty}\frac{4q^n/\zeta^5}{1-q^n/\zeta^4}\right)^2 \\
&\quad - \frac{[\zeta^{-1}]_\infty}{[\zeta^4]_\infty}\Bigg(\frac{4\zeta^6+6\zeta^5+5\zeta^4+2\zeta^2+2\zeta+1}{\zeta^2(\zeta^3+\zeta^2+\zeta+1)^2} + \sum_{n=0}^{\infty}\frac{(-n-2)\zeta^{-n-3}q^{n+1}}{1-q^{n+1}} \\
&\quad - \sum_{n=0}^{\infty}\frac{n\zeta^{n-1}q^{n+1}}{1-q^{n+1}} + \sum_{n=0}^{\infty}\frac{4(4n+3)\zeta^{4n+2}q^{n+1}}{1-q^{n+1}} - \sum_{n=0}^{\infty}\frac{4(-4n-5)q^{n+1}\zeta^{-4n-6}}{1-q^{n+1}}\Bigg) \\
g_{31}^{(2)}(1) &= \frac{1}{4}\left(-\frac{5}{2}\right)^2 + \frac{1}{4}\Bigg(\frac{5}{4} + \sum_{n=0}^{\infty}\frac{(-n-2)q^{n+1}}{1-q^{n+1}} - \sum_{n=0}^{\infty}\frac{nq^{n+1}}{1-q^{n+1}} + \sum_{n=0}^{\infty}\frac{4(4n+3)q^{n+1}}{1-q^{n+1}} \\
&\quad - \sum_{n=0}^{\infty}\frac{4(-4n-5)q^{n+1}}{1-q^{n+1}}\Bigg) \\
&= \frac{15}{8} + \frac{15}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
g_{31}^{(3)}(1) &= -\frac{105}{16} - \frac{315}{4}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \\
g_{31}^{(4)}(1) &= \frac{213}{8} + 645\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 675\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)^2 + \frac{255}{2}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \\
&= \frac{213}{8} + \frac{2805}{4}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1635}{4}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} - \frac{675}{2}\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2},
\end{aligned}
$$

where we employed (7.3.18) in the last line. Also,

$$
g_{32}(1) = 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n},
$$

$$g_{32}^{(1)}(1) = 10 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n},$$

$$g_{32}^{(2)}(1) = \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \left(\frac{16}{(1-zq^n)^3} + \frac{16(5n+1)}{(1-zq^n)^2} + \frac{2(100n^2+60n+29)}{1-zq^n} \right),$$

$$g_{32}^{(3)}(1) = \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \left(\frac{192}{(1-zq^n)^3} + \frac{192(5n+1)}{(1-zq^n)^2} + \frac{48(50n^2+30n+7)}{1-zq^n} \right),$$

$$g_{32}^{(4)}(1) = \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \Bigg(\frac{768}{(1-zq^n)^5} + \frac{3840n}{(1-zq^n)^4} + \frac{48(200n^2+40n+27)}{(1-zq^n)^3} + \frac{16(1000n^3+600n^2+485n+81)}{(1-zq^n)^2} + \frac{40(500n^4+600n^3+625n^2+267n+42)}{1-zq^n} \Bigg).$$

The above calculations for the derivatives of $g_{31}(\zeta)$ and $g_{32}(\zeta)$ together with (7.3.24) imply

$$\begin{aligned}
g_3^{(4)}(1) &= \left(\frac{213}{4} + \frac{2805}{2} \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} + \frac{1635}{2} \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k} - 675 \sum_{k=1}^{\infty} \frac{k^2 q^k}{(1-q^k)^2} \right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n} \\
&- \left(\frac{525}{2} + 3150 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} \right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n} + \left(\frac{45}{2} + 90 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} \right) \\
&\times \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \left(\frac{8}{(1-zq^n)^3} + \frac{8(5n+1)}{(1-zq^n)^2} + \frac{100n^2+60n+29}{1-zq^n} \right) \\
&- 120 \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \left(\frac{4}{(1-zq^n)^3} + \frac{4(5n+1)}{(1-zq^n)^2} + \frac{50n^2+30n+7}{1-zq^n} \right) \\
&+ \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)/2} \Bigg(\frac{192}{(1-zq^n)^5} + \frac{960n}{(1-zq^n)^4} + \frac{12(200n^2+40n+27)}{(1-zq^n)^3} \\
&+ \frac{4(1000n^3+600n^2+485n+81)}{(1-zq^n)^2} + \frac{10(500n^4+600n^3+625n^2+267n+42)}{1-zq^n} \Bigg).
\end{aligned} \tag{7.3.26}$$

Now (7.3.1) implies that $g_2^{(4)}(1) = g_4^{(4)}(1) - g_1^{(4)}(1) - g_3^{(4)}(1)$. Hence from (7.3.5), (7.3.6), (7.3.22) and (7.3.26), we see that

$$
\begin{aligned}
&- 144(q)_\infty^5[C^*(z,q)]^5 \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)/2}}{1-zq^n}\Bigg(- 30n(125n^3+150n^2+55n+6) - 90(10n+3)^2 \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} \\
&\quad - 810\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k} + 900\sum_{k=1}^{\infty}\frac{k^2q^k}{(1-q^k)^2}\Bigg) \\
&\quad + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{(1-zq^n)^2}\Bigg(- 120n(25n^2+15n+2) - 720(5n+1)\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\Bigg) \\
&\quad + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{(1-zq^n)^3}\Bigg(-360n(5n+1) - 720\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\Bigg) \\
&\quad - 720\sum_{n=-\infty}^{\infty}\frac{(-1)^n n q^{5n(n+1)/2}}{(1-zq^n)^4} - 144\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)/2}}{(1-zq^n)^5}. \qquad (7.3.27)
\end{aligned}
$$

Now divide both sides of the above equation by $-6(q)_\infty^3$ and then observe that, after a routine (though laborious) algebraic simplification, the right-hand side turns out to be

$$
\begin{aligned}
&24(1-10\Phi_3(q))G^{(5)}(z,q) \\
&\quad + \left(100\delta_q + 50\delta_z + 100\delta_q\delta_z + 35\delta_z^2 + 20\delta_q\delta_z^2 + 100\delta_q^2 + 10\delta_z^3 + \delta_z^4\right) G^{(5)}(z,q), \qquad (7.3.28)
\end{aligned}
$$

where $G^{(5)}(z,q)$ and $\Phi_3(q)$ are defined in (1.0.28) and (1.0.29) respectively. This completes the proof of (1.0.30).

7.4 Future possible developments

In [26], an equivalent version of Zwegers' general Rank-Crank PDE (1.0.34) is obtained by applying the operator D_{2m}, where $D_\ell := \left(\zeta\frac{\partial}{\partial\zeta}\right)^\ell\Big|_{\zeta=1} = \delta_\zeta^\ell\Big|_{\zeta=1}$, to both sides of the following special case of (1.0.18) [26, Equation (4.1)] obtained by letting $x_i = \zeta^i, 1 \le i \le m$, in

(1.0.18):

$$Y_m(\zeta,z,q)\,(q)_\infty^2 = S_{2m+1}(\zeta,z,q) + \sum_{j=1}^{m-1} F_{j,m}(\zeta,q)\,S_{2m+1}(\zeta^{j+1},z,q) - F_{0,m}(\zeta,q)\,\Sigma^{(2m+1)}(z,q), \tag{7.4.1}$$

where

$$S_k(\zeta,z,q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{kn(n+1)/2}\left(\frac{\zeta^{-kn}}{1-z\zeta^{-1}q^n} + \frac{\zeta^{k(n+1)}}{1-z\zeta q^n}\right), \tag{7.4.2}$$

for k odd and

$$F_{0,m}(\zeta,q) := \zeta^m \frac{[\zeta^{m+1}]_\infty}{[\zeta^m]_\infty}, \tag{7.4.3}$$

$$F_{j,m}(\zeta,q) := \frac{\left[\zeta^{-(m-1)},\zeta^{-(m-2)},\cdots,\zeta^{-(m-j)}\right]_\infty}{\left[\zeta^{m+2},\cdots,\zeta^{m+j+1}\right]_\infty} \quad (\text{for } 1 \le j \le m-1), \tag{7.4.4}$$

$$Y_m(\zeta,z,q) := \frac{\left[\zeta^{-(m-1)},\zeta^{-(m-2)},\cdots,\zeta^{-2},\zeta^{-1},\zeta,\zeta^2,\cdots,\zeta^m,\zeta^{m+1}\right]_\infty}{\left[z\zeta^{-m},z\zeta^{-(m-1)},\cdots,z\zeta^{-2},z\zeta^{-1},z,z\zeta,z\zeta^2,\cdots,z\zeta^{m-1},z\zeta^m\right]_\infty}. \tag{7.4.5}$$

This equivalent version is as follows [26, Theorem 4.4]:

Theorem 7.4.1. *Define the operator*

$$\mathcal{H}_k^* := k\delta_z + 2k\delta_q + \delta_z^2. \tag{7.4.6}$$

Let $R^*(z,q)$ *and* $C^*(z,q)$ *be defined as in* (1.0.23) *and let* $\Sigma^{(k)}(z,q)$ *be defined as in* (1.0.27). *Then,*

$$\begin{aligned}&(-1)^{m+1}(2m)!\,(m+1)!\,(m-1)![C^*(z,q)]^{2m+1}(q)_\infty^{2m+1}\\ &=\Bigg(P_{2m+1,2m}(\mathcal{H}_{2m+1}^*) - D_{2m}(F_{0,m}(\zeta,q))\\ &\quad+\sum_{j=1}^{m-1}\sum_{a=0}^{2m}(j+1)^{2m-a}\binom{2m}{a}D_a(F_{j,m}(\zeta,q))P_{2m+1,2m-a}(\mathcal{H}_{2m+1}^*)\Bigg)\Sigma^{(2m+1)}(z,q),\end{aligned} \tag{7.4.7}$$

where

$$P_{k,\ell}(x) := \sum_{m=0}^{\lfloor \ell/2 \rfloor} \frac{\ell(\ell-m-1)!}{(\ell-2m)!m!} x^m k^{\ell-2m} \tag{7.4.8}$$

and the coefficient functions $D_a(F_{j,m}(\zeta,q))$ $(0 \leq j \leq m-1)$ *are given recursively by*

$$\begin{aligned} D_a(F_{0,m}(\zeta,q)) &= (m+\tfrac{1}{2})D_{a-1}(F_{0,m}(\zeta,q)) \\ &\quad + \sum_{i=1}^{\lfloor a/2 \rfloor} 2\binom{a-1}{2i-1}(m^{2i}-(m+1)^{2i})G_{2i}(q)D_{a-2i}(F_{0,m}(\zeta,q)) \\ D_a(F_{j,m}(\zeta,q)) &= -j(m+\tfrac{1}{2})D_{a-1}(F_{j,m}(\zeta,q)) \\ &\quad + \sum_{i=1}^{\lfloor a/2 \rfloor}\sum_{k=1}^{j} 2\binom{a-1}{2i-1}((m+k+1)^{2i}-(m-k)^{2i})G_{2i}(q)D_{a-2i}(F_{j,m}(\zeta,q)), \end{aligned} \tag{7.4.9}$$

and their initial values

$$\begin{aligned} D_0(F_{0,m}(\zeta,q)) &= F_{0,m}(1,q) = \tfrac{m+1}{m} \\ D_0(F_{j,m}(\zeta,q)) &= F_{j,m}(1,q) = (-1)^j \prod_{i=1}^{j} \frac{m-i}{m+i+1}, \end{aligned} \tag{7.4.10}$$

with G_{2k} *being a normalized Eisenstein series given by*

$$G_{2k} := G_{2k}(q) := \frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n} = -\frac{B_{2k}}{4k} + \Phi_{2k-1}(q), \tag{7.4.11}$$

where B_{2n} *is the* $(2n)$*-th Bernoulli number, and*

$$\Phi_{2k-1} := \Phi_{2k-1}(q) := \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n. \tag{7.4.12}$$

In this equivalent version, the coefficients are quasimodular forms unlike holomorphic modular forms as in (1.0.34). The quasimodular function E_2 occurs in Zwegers's result

(Theorem 1.0.3) as part of the definition of his operator $\mathcal{H}_k$. The coefficient functions in (7.4.7) are given recursively. It was remarked in [26] that it would be interesting to find explicit expressions for the coefficients. Two examples where we have explicit versions of the PDEs, i.e., without coefficient functions expressed recursively, say, are the Rank-Crank PDE (1.0.22) and Garvan's 4^{th} order PDE (1.0.30). The proof of (1.0.22) in [7], and of (1.0.30) that we have given in Section 7.3 here, both involve direct differentiation unlike the use of the operator D_ℓ as in [26]. Based on this, we conjecture that in order to obtain the explicit PDEs, we need to differentiate (7.4.1) $2m$-times with respect to ζ and then let $\zeta = 1$. Garvan found the PDE in (1.0.30) using MAPLE and what we have done here is to merely verify it. One may be able to obtain an explicit higher order Rank-Crank type PDE for a specific value of m using MAPLE or other similar software. However, doing this in general seems to be quite difficult because of the complexity associated with higher-order differentiation of the terms in (7.4.1) with respect to ζ.

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