

Sharp-interface nematic-isotropic phase transitions with flow

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Abstract

We present a sharp-interface theory for phase transformations between the isotropic and uniaxial nematic phases of a flowing liquid crystal. These equations include a supplemental interface condition which expresses the balance of configurational momentum. As an application, we study the behavior a spherical isotropic drop surrounded by a radially-oriented nematic phase. For this problem, the bulk and interfacial equations collapse to a single nonlinear second-order ordinary differential equation for the radius of the droplet. We determine and study the stability of the corresponding equilibria. In addition, we find that motion of the interface generates a backflow in the nematic phase. Our analysis indicates that a backflow measurement can be used to determine independently the density difference between the isotropic and uniaxial nematic phases.

1 Introduction

In a recent paper, CERMELLI, FRIED & GURTIN [5] generalized the classical curvature elasticity theory of OSEEN [19], ZÖCHER [25], and FRANK [13] to

- allow for phase transitions,
- model a nematic-isotropic interface as a sharp surface across which bulk fields may suffer discontinuities,
- account for localized interactions between phases by endowing the interface with excess properties.

Here, we extend the theory of CERMELLI, FRIED & GURTIN [5] to account for flow and thereby take another step toward providing a generalization of the Ericksen–Leslie theory (ERICKSEN [10, 11, 12]; LESLIE [16, 17]) appropriate to describing transformations phase transformations between the isotropic and uniaxial nematic phases of a flowing liquid crystal.

Aside from developing governing equations for the bulk phases and the interface, we revisit a simple application studied by CERMELLI, FRIED & GURTIN [5]. That application involves the growth and equilibrium of a spherical isotropic drop in a nematic ocean in which the director is radially oriented. When flow is accounted for, the solution of this problem involves determining the velocity and pressure fields in the phases. As in the case when flow is neglected, we find that the bulk and interfacial equations collapse to a single nonlinear second-order ordinary differential equation—enforcing the balance of configurational momentum—for the radius of the droplet. On writing R for the radius of the droplet, this equation has the form

$$\frac{[[v]]^2}{v^+v^-} \left\{ R\ddot{R} + \frac{3\dot{R}^2}{2} + \frac{4v^-(\mu_1 + \mu_4)\dot{R}}{R} + \frac{2v^-(\lambda_1 + \lambda_5)\dot{R}}{R^2} \right\} + \frac{\lambda_9\dot{R}}{v^+} = v^-\Psi_0 - [[v]]p_\infty - v^+ \left\{ \frac{2\gamma_0}{R} - \frac{\kappa}{R^2} \right\}, \quad (1.1)$$

where v^+ and v^- denote the specific volumes of the isotropic and nematic phases, $[[v]] = v^+ - v^-$, $\mu_1 + \mu_4 > 0$ (with μ_1 and μ_4 determined by the Leslie coefficients α_1 , α_4 , α_5 , and α_6 via $\mu_1 = (\alpha_1 + \alpha_5 + \alpha_6)/3$ and $\mu_4 = \alpha_4/2$), $\lambda_1 + \lambda_2 > 0$ is the dilational viscosity of the interface, λ_9 is the reciprocal mobility of the interface, Ψ_0 is the free-energy density of the nematic phase, in a state of uniform alignment, measured relative to the isotropic phase, p_∞ is the pressure in the far-field, γ_0 is the isotropic contribution to the

interfacial free-energy density, and $\kappa = 2k_1 - (k_2 + k_4) \geq 0$, (with k_1 , k_2 , and k_4 being the classical Frank moduli for splay, twist, bend, and saddle-splay). When the densities of the phases are equal, so that $[[v]] = 0$, (1.1) reduces to an equation with structure identical to that of the equation derived and studied by Cermelli, Fried & Gurtin [5] for the droplet evolution problem without flow. We determine and study the stability of the equilibria of (1.1). Generally, the number and nature of the equilibria is essentially unchanged from the case without flow. With flow taken into account, however, the detailed expressions for the equilibrium states are influenced by the ambient pressure. Additionally, the dynamics of the droplet are evidently influenced by inertia, the viscosity of the nematic phase, and the dilatational viscosity of the interface.

Within the isotropic phase, the velocity is trivial and the pressure is spatially uniform and determined in terms of the radius of the droplet and relevant material parameters. Within the nematic phase, the velocity and pressure are determined in terms of the radius of the droplet, with the velocity being radial. Whereas the velocity decays is inversely proportional to the square of the distance from the origin, in the far-field the pressure tends toward a uniform ambient value.

Most importantly, when flow is taken into account, we find that motion of the interface generates a backflow in the nematic phase. Writing \mathbf{u}^- for the interfacial limit of the velocity of the nematic phase, \mathbf{m} for the interfacial unit normal (directed into the isotropic phase), we find in particular that

$$\frac{[[v]]}{v^+} = -\frac{\mathbf{u}^- \cdot \mathbf{m}}{R}.$$

Our analysis therefore shows that a backflow measurement provides an independent means to measure the density difference between the phases.

2 Theory for the nematic phase

The theory for the nematic phase is a specialization of the Ericksen–Leslie theory (ERICKSEN [10, 11, 12]; LESLIE [16, 17]) to a setting in which thermal transport is neglected.

The basic kinematical descriptors of the theory are the *velocity* \mathbf{u} and the *director* \mathbf{n} . We write

$$\mathbf{L} = \text{grad } \mathbf{u} \quad \text{and} \quad \mathbf{G} = \text{grad } \mathbf{n}$$

for the *velocity gradient* and the *director gradient* and

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^\top) \quad \text{and} \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^\top)$$

for the *rate of stretch* and the *rate of spin*. As is standard, we assume that the nematic phase is incompressible, so that

$$\text{div } \mathbf{u} = \text{tr } \mathbf{L} = 0. \tag{2.1}$$

Further, since the director describes the axis of the uniaxial nematic order tensor, we assume that

$$|\mathbf{n}| = 1 \tag{2.2}$$

and that \mathbf{n} and $-\mathbf{n}$ are indistinguishable.

We use a superposed dot to denote *material time-differentiation*; e.g., for a scalar field Φ , $\dot{\Phi} = \partial\Phi/\partial t + (\text{grad } \Phi) \cdot \mathbf{u}$. Aside from the constraints (2.1) and (2.2), the governing equations in the nematic phase are then

$$\left. \begin{aligned} \rho \dot{\mathbf{u}} &= -\text{grad } p - \text{div} \left\{ \mathbf{G}^\top \frac{\partial \Psi}{\partial \mathbf{G}} \right\} + \text{div } \mathbf{T}_{\text{dis}}, \\ \sigma (\ddot{\mathbf{n}} + |\dot{\mathbf{n}}|^2 \mathbf{n}) &= \text{div} \left\{ \frac{\partial \Psi}{\partial \mathbf{G}} \right\} + \left\{ \mathbf{G} : \frac{\partial \Psi}{\partial \mathbf{G}} \right\} \mathbf{n} - \frac{\partial \Psi}{\partial \mathbf{n}} + \mathbf{g}_{\text{dis}}, \end{aligned} \right\} \tag{2.3}$$

where ρ is the (constant) mass density, σ is the (constant) director mass density (i.e., the peculiar mass density of a mesogen), p is the pressure required to maintain the constraint (2.1), Ψ is the *free-energy*

(density), as given FRANK's [13] quadratic expression¹

$$\Psi = \Psi_0 + \frac{1}{2}k_1(\operatorname{div} \mathbf{n})^2 + \frac{1}{2}k_2(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2 + \frac{1}{2}k_3|\mathbf{n} \times \operatorname{curl} \mathbf{n}|^2 + \frac{1}{2}(k_2 + k_4)(\operatorname{tr}((\operatorname{grad} \mathbf{n})^2) - (\operatorname{div} \mathbf{n})^2), \quad (2.4)$$

and \mathbf{T}_{dis} and \mathbf{g}_{dis} are given by (ERICKSEN [11]; LESLIE [16])²

$$\left. \begin{aligned} \mathbf{T}_{\text{dis}} &= \mu_1(\mathbf{n} \cdot \mathbf{D}\mathbf{n})(3\mathbf{n} \otimes \mathbf{n} - \mathbf{1}) + \mu_2\boldsymbol{\omega} \otimes \mathbf{n} + \mu_3\mathbf{n} \otimes \boldsymbol{\omega} + 2\mu_4\mathbf{D} \\ &\quad + 2\mu_5(\mathbf{1} - \mathbf{n} \otimes \mathbf{n})\mathbf{D}\mathbf{n} \otimes \mathbf{n} + 2\mu_6\mathbf{n} \otimes (\mathbf{1} - \mathbf{n} \otimes \mathbf{n})\mathbf{D}\mathbf{n}, \\ \mathbf{g}_{\text{dis}} &= -(\mu_3 - \mu_2)\boldsymbol{\omega} - 2(\mu_6 - \mu_5)(\mathbf{1} - \mathbf{n} \otimes \mathbf{n})\mathbf{D}\mathbf{n}, \end{aligned} \right\} \quad (2.5)$$

where $\boldsymbol{\omega} = \dot{\mathbf{n}} - \mathbf{W}\mathbf{n}$ and the viscosities $\mu_1, \mu_2, \mu_3, \mu_4,$ and μ_5 obey

$$\left. \begin{aligned} \mu_1 + \mu_4 &> 0, & \mu_3 &> \mu_2, & \mu_4 &> 0, & 2\mu_4 + \mu_5 + \mu_6 &> 0, \\ (\mu_3 - \mu_2)(2\mu_4 + \mu_5 + \mu_6) &> 2(\mu_2 + \mu_3 + 2\mu_6 - 2\mu_5)^2. \end{aligned} \right\} \quad (2.6)$$

Granted (2.4) and (2.5), the equations (2.3) may be recast in the compact form³

$$\left. \begin{aligned} \dot{\mathbf{p}} &= \operatorname{div} \mathbf{T}, \\ \dot{\mathbf{r}} &= \operatorname{div} \mathbf{S} + \mathbf{g}, \end{aligned} \right\} \quad (2.7)$$

where $\mathbf{p} = \rho\mathbf{u}$ is *linear momentum*, \mathbf{T} is the *Cauchy stress*, $\mathbf{r} = \sigma\dot{\mathbf{n}}$ is *director momentum*, \mathbf{S} is the *director stress*, and \mathbf{g} is the *director body force (density)*, given by the constitutive relations⁴

$$\left. \begin{aligned} \mathbf{T} &= -p\mathbf{1} - \mathbf{G}^\top \mathbf{S} + \mathbf{T}_{\text{dis}}, \\ \mathbf{S} &= \mathbf{n} \otimes \boldsymbol{\alpha} + \frac{\partial \Psi}{\partial \mathbf{G}}, \\ \mathbf{g} &= \beta\mathbf{n} - \mathbf{G}\boldsymbol{\alpha} - \frac{\partial \Psi}{\partial \mathbf{n}} + \mathbf{g}_{\text{dis}}, \end{aligned} \right\} \quad (2.8)$$

with $\boldsymbol{\alpha}$ and β being constitutively indeterminate fields that arise in response to the constraint (2.2).

Granted (2.4), (2.5), and (2.8), it can be verified that

$$\mathbf{T} + \mathbf{S}\mathbf{G}^\top + \mathbf{n} \otimes \mathbf{g} = (\mathbf{T} + \mathbf{S}\mathbf{G}^\top + \mathbf{n} \otimes \mathbf{g})^\top, \quad (2.9)$$

which is equivalent to the balance of angular momentum, and that

$$\dot{\Psi} - (\mathbf{T} + \mathbf{G}^\top \mathbf{S}) : \mathbf{L} - \mathbf{S} : \dot{\mathbf{G}} + \mathbf{g} \cdot \dot{\mathbf{n}} \leq 0 \quad (2.10)$$

which is the dissipation inequality expressing the first and second laws of thermodynamics under isothermal conditions.

¹The quantity Ψ_0 contains information about the energy of the nematic phase, in a state of uniform alignment, relative to that of the isotropic phase. For fixed compositions and in the absence of external electromagnetic fields, we might expect Ψ_0 to be negative at sufficiently low temperatures, positive at sufficiently high temperatures, and zero at some intermediate temperature.

²The expressions (2.5) for the dissipative contributions \mathbf{T}_{dis} and \mathbf{g}_{dis} to \mathbf{T} and \mathbf{g} are defined consistent with the geometric restrictions $\operatorname{tr} \mathbf{T}_{\text{dis}} = 0$ and $\mathbf{g}_{\text{dis}} \cdot \mathbf{n} = 0$ arising from the constraints (2.1) and (2.2) and therefore differ slightly from the standard expressions given by ERICKSEN [11] and LESLIE [16]. Consistency with the standard expressions involving the Leslie coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5,$ and α_6 is obtained by using the relations $\mu_1 = (\alpha_1 + \alpha_5 + \alpha_6)/3$, $\mu_2 = \alpha_2$, $\mu_3 = \alpha_3$, $\mu_4 = \alpha_4/2$, $\mu_5 = \alpha_5/2$, and $\mu_6 = \alpha_6/2$ and absorbing the terms $\mu_1(\mathbf{n} \cdot \mathbf{D}\mathbf{n})\mathbf{1}$ and $2(\mu_6 - \mu_5)(\mathbf{n} \cdot \mathbf{D}\mathbf{n})\mathbf{n}$ into the reactions $-p\mathbf{1}$ and $\beta\mathbf{n}$.

³More precisely, granted that \mathbf{S} and \mathbf{g} are as determined by (2.4) and (2.8)_{2,3}, (2.3)₂ is equivalent to the projection of (2.7)₂ onto the plane orthogonal to \mathbf{n} .

⁴The derivatives in (2.8) are defined consistent with the constraint (2.1)₂ and thus obey $(\partial \Psi / \partial \mathbf{n}) \cdot \mathbf{n} = 0$ and $(\partial \Psi / \partial \mathbf{G})^\top \mathbf{n} = \mathbf{0}$. We avoid a detailed discussion of constraints and associated multiplier fields. A modern geometrical treatment of constraints in a material with nematic microstructure is provided by ANDERSON, CARLSON & FRIED [2] and CHEN & FRIED [7].

3 Configurational momentum and stress in the nematic phase

For homogeneous bulk phases, configurational forces do not describe new physics. When a phase interface is present, however, configurational forces are of essential importance. Moreover, a knowledge of the structure of the bulk configurational forces is central to the understanding of their localized effects at the interface. With this in mind, we now summarize the results of CERMELLI & FRIED [4] concerning the forms of the bulk configurational momentum and stress in nematic liquid-crystals.

We consider a configurational momentum balance involving three fields: a *configurational momentum (density)* \mathbf{q} , a *configurational stress* \mathbf{C} , and a *configurational body force (density)* \mathbf{f} . The local form of the balance of configurational momentum is

$$\dot{\mathbf{q}} = \operatorname{div} \mathbf{C} + \mathbf{f}. \quad (3.1)$$

Since no changes of material structure occur in bulk, we require that the local configurational balance (3.1) be identically satisfied away from the interface: the bulk force \mathbf{f} is therefore indeterminate, and *defined* by the relation $\mathbf{f} = \dot{\mathbf{q}} - \operatorname{div} \mathbf{C}$.

Insight regarding the structure of \mathbf{p} and \mathbf{C} is best obtained using an integral formulation involving control volumes $\mathcal{R}(t)$ that migrate relative to the liquid (GURTIN [14, 15]). One of the advantages of such an approach is that a migrating control volume can be chosen to evolve following the interface. This observation leads to the notion of a *superficial pillbox* introduced later in Section 6.1.

Let $\mathbf{v}_{\partial\mathcal{R}}$ be a velocity field describing the motion of $\partial\mathcal{R}(t)$, and let $V_{\partial\mathcal{R}}$ the (scalar) normal velocity of $\partial\mathcal{R}$ in the direction of the outward unit normal $\mathbf{m}_{\partial\mathcal{R}}$. To characterize the motion of $\mathcal{R}(t)$ relative to the liquid we use the relative velocities

$$\mathbf{v}_{\partial\mathcal{R}}^{\text{rel}} = \mathbf{v}_{\partial\mathcal{R}} - \mathbf{u} \quad \text{and} \quad V_{\partial\mathcal{R}}^{\text{rel}} = V_{\partial\mathcal{R}} - \mathbf{u} \cdot \mathbf{m}_{\partial\mathcal{R}}.$$

Given a migrating control volume $\mathcal{R}(t)$, the field $\partial\mathbf{n}/\partial t + \mathbf{G}\mathbf{v}_{\partial\mathcal{R}}$ represents the *rate of change of the director following the observed motion of $\partial\mathcal{R}$* . Notice that we may write

$$\frac{\partial\mathbf{n}}{\partial t} + \mathbf{G}\mathbf{v}_{\partial\mathcal{R}} = \dot{\mathbf{n}} + \mathbf{G}\mathbf{v}_{\partial\mathcal{R}}^{\text{rel}}.$$

Moreover, since $\operatorname{div} \mathbf{u} = 0$, we conclude that for any scalar field Φ

$$\frac{d}{dt} \int_{\mathcal{R}(t)} \Phi \, dv = \int_{\mathcal{R}(t)} \dot{\Phi} \, dv + \int_{\partial\mathcal{R}(t)} \Phi V_{\partial\mathcal{R}}^{\text{rel}} \, da. \quad (3.2)$$

By integrating the local laws (2.7), (2.10), and (3.1) over $\mathcal{R}(t)$ and using (3.2) and the divergence theorem, we obtain equivalent expressions for a migrating control volume. To begin with, the balance laws for linear, director, and configurational momentum become

$$\left. \begin{aligned} \frac{d}{dt} \int_{\mathcal{R}(t)} \mathbf{p} \, dv &= \int_{\partial\mathcal{R}(t)} \left\{ \mathbf{T}\mathbf{m}_{\partial\mathcal{R}} + \mathbf{p}V_{\partial\mathcal{R}}^{\text{rel}} \right\} \, da, \\ \frac{d}{dt} \int_{\mathcal{R}(t)} \mathbf{r} \, dv &= \int_{\partial\mathcal{R}(t)} \left\{ \mathbf{S}\mathbf{m}_{\partial\mathcal{R}} + \mathbf{r}V_{\partial\mathcal{R}}^{\text{rel}} \right\} \, da + \int_{\mathcal{R}(t)} \mathbf{g} \, dv, \\ \frac{d}{dt} \int_{\mathcal{R}(t)} \mathbf{q} \, dv &= \int_{\partial\mathcal{R}(t)} \left\{ \mathbf{C}\mathbf{m}_{\partial\mathcal{R}} + \mathbf{q}V_{\partial\mathcal{R}}^{\text{rel}} \right\} \, da + \int_{\mathcal{R}(t)} \mathbf{f} \, dv. \end{aligned} \right\} \quad (3.3)$$

In the form (3.3), the balances suggest a view of $\mathbf{T}\mathbf{m}_{\partial\mathcal{R}} + \mathbf{p}V_{\partial\mathcal{R}}^{\text{rel}}$, $\mathbf{S}\mathbf{m}_{\partial\mathcal{R}} + \mathbf{r}V_{\partial\mathcal{R}}^{\text{rel}}$, and $\mathbf{C}\mathbf{m}_{\partial\mathcal{R}} + \mathbf{q}V_{\partial\mathcal{R}}^{\text{rel}}$ as *effective tractions* (CERMELLI & FRIED [3]), a view that we take when we discuss the configurational form of the energy inequality for a migrating control volume.

Using the identity $\dot{\mathbf{G}} = \operatorname{grad}(\dot{\mathbf{n}}) - \mathbf{G}\mathbf{L}$ and the local laws (2.7), we next rewrite (2.10) as a local energy inequality:

$$\dot{\Psi} + \varrho \mathbf{u} \cdot \dot{\mathbf{u}} + \sigma \dot{\mathbf{n}} \cdot \dot{\mathbf{n}} \leq \operatorname{div} \left\{ \mathbf{T}^\top \mathbf{u} + \mathbf{S}^\top \dot{\mathbf{n}} \right\}. \quad (3.4)$$

Integrating (3.4) over $\mathcal{R}(t)$ and using (3.2) and the divergence theorem, we obtain an energy inequality

$$\frac{d}{dt} \int_{\mathcal{R}(t)} \left\{ \Psi + \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{2} \sigma |\dot{\mathbf{n}}|^2 \right\} dv \leq \int_{\partial \mathcal{R}(t)} \left\{ \mathbf{Tm}_{\partial \mathcal{R}} \cdot \mathbf{u} + \mathbf{Sm}_{\partial \mathcal{R}} \cdot \dot{\mathbf{n}} + (\Psi + \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{2} \sigma |\dot{\mathbf{n}}|^2) V_{\partial \mathcal{R}}^{\text{rel}} \right\} da \quad (3.5)$$

for a migrating control volume. This inequality accounts only *implicitly* for the power expended by configurational forces. We now give an equivalent alternative of the energy inequality that accounts *explicitly* for power expended by configurational forces. In this regard it is important to bear in mind that in contrast to a control volume that convects with the liquid, a control volume $\mathcal{P}(t)$ that migrates with respect to the liquid has no intrinsic physical meaning. Consistent with this and the conventional version

$$\frac{d}{dt} \int_{\mathcal{P}(t)} \left\{ \Psi + \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{2} \sigma |\dot{\mathbf{n}}|^2 \right\} dv \leq \int_{\partial \mathcal{P}(t)} \left\{ \mathbf{Tm}_{\partial \mathcal{R}} \cdot \mathbf{u} + \mathbf{Sm}_{\partial \mathcal{R}} \cdot \dot{\mathbf{n}} \right\} da \quad (3.6)$$

of the energy inequality, we rewrite the energy inequality for $\mathcal{R}(t)$ in the form

$$\frac{d}{dt} \int_{\mathcal{R}(t)} \left\{ \Psi + \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{2} \sigma |\dot{\mathbf{n}}|^2 \right\} dv \leq W(\mathcal{R}(t)), \quad (3.7)$$

with

$$W(\mathcal{R}(t)) = \int_{\partial \mathcal{R}(t)} (\mathbf{Cm}_{\partial \mathcal{R}} + \mathbf{q}V_{\partial \mathcal{R}}^{\text{rel}}) \cdot \mathbf{v}_{\partial \mathcal{R}}^{\text{rel}} da + \int_{\partial \mathcal{R}(t)} (\mathbf{Sm}_{\partial \mathcal{R}} + \mathbf{r}V_{\partial \mathcal{R}}^{\text{rel}}) \cdot (\dot{\mathbf{n}} + \mathbf{G}\mathbf{v}_{\partial \mathcal{R}}^{\text{rel}}) da + \int_{\partial \mathcal{R}(t)} (\mathbf{Tm}_{\partial \mathcal{R}} + \mathbf{p}V_{\partial \mathcal{R}}^{\text{rel}}) \cdot \mathbf{v}_{\partial \mathcal{R}} da \quad (3.8)$$

the power expended by the *effective* standard, director, and configurational tractions on $\partial \mathcal{R}(t)$. Each term of $W(\mathcal{R})$ as defined in (3.8) has a definite physical interpretation:

- Configurational forces are introduced to account for power expenditures associated with material transfer due to the migration of $\partial \mathcal{R}$. Specifically, we view the effective configurational traction $\mathbf{Cm}_{\partial \mathcal{R}} + \mathbf{q}V_{\partial \mathcal{R}}^{\text{rel}}$ as a force, per unit area, associated with the transfer of material across $\partial \mathcal{R}$. Since any velocity field $\mathbf{v}_{\partial \mathcal{R}}^{\text{rel}}$ for $\partial \mathcal{R}$ represents the velocity with which material is transferred across $\partial \mathcal{R}$, we take $\mathbf{v}_{\partial \mathcal{R}}^{\text{rel}}$ to be an appropriate power conjugate velocity for $\mathbf{Cm}_{\partial \mathcal{R}}$. Moreover, material is added to \mathcal{R} only along its boundary $\partial \mathcal{R}$; there is no transfer of material to the interior of \mathcal{R} and, for that reason, the configurational body force density \mathbf{f} expends no power.
- Since $\partial \mathcal{R}$ migrates through the liquid, it has no intrinsic material description. Hence, for the velocity conjugate to the effective standard traction $\mathbf{Tm}_{\partial \mathcal{R}} + \mathbf{p}V_{\partial \mathcal{R}}^{\text{rel}}$, we use, in place of \mathbf{u} , the observed velocity $\mathbf{v}_{\partial \mathcal{R}}$ of $\partial \mathcal{R}$.
- Similarly, we assume that the velocity conjugate to the effective director traction $\mathbf{Sm}_{\partial \mathcal{R}} + \mathbf{r}V_{\partial \mathcal{R}}^{\text{rel}}$ is not given by the *material* time derivative $\dot{\mathbf{n}}$ as in the classical theory, but rather by $\dot{\mathbf{n}} + \mathbf{G}\mathbf{v}_{\partial \mathcal{R}}^{\text{rel}}$, the rate of change of the director following the observed motion of $\partial \mathcal{R}$.

By a simple extension of an argument given by CERMELLI, FRIED & GURTIN [5], we may show that the free-energy inequality in the configurational form (3.7) and subject to the requirement that the power $W(\mathcal{R})$ as defined in (3.8) be independent of the velocity field $\mathbf{v}_{\partial \mathcal{R}}$ chosen to characterize the motion of $\partial \mathcal{R}$ is equivalent to the conventional statement (3.6) of the energy inequality if and only if the configurational momentum \mathbf{q} is related to the standard and director momenta by

$$\mathbf{q} = -(\mathbf{p} + \mathbf{G}^T \mathbf{r}) = -\varrho \mathbf{u} - \sigma \mathbf{G}^T \dot{\mathbf{n}} \quad (3.9)$$

and the configurational stress \mathbf{C} is given by the *Eshelby relation*

$$\mathbf{C} = (\Psi - \frac{1}{2} \varrho |\mathbf{u}|^2 - \frac{1}{2} \sigma |\dot{\mathbf{n}}|^2) \mathbf{1} - \mathbf{G}^T \mathbf{S} - \mathbf{T}. \quad (3.10)$$

The argument leading to the relations (3.9) and (3.10) for the configurational momentum and stress make no use of constitutive theory. Hence, in particular, (3.10) applies to broad classes of materials. Moreover, by the momentum balances (2.7), (3.9) and (3.10) yield the result $\dot{\mathbf{q}} - \text{div} \mathbf{C} = \varrho \mathbf{L}^\top \mathbf{u} + \sigma \mathbf{G}^\top \mathbf{L} \dot{\mathbf{n}} - \text{grad} \Psi + \mathbf{S} : \text{grad} \mathbf{G} - \mathbf{G}^\top \mathbf{g}$.⁵ This result, which is independent of constitution, shows that the internal body force \mathbf{f} in the configurational balance (3.1) has the explicit form

$$\mathbf{f} = \varrho \mathbf{L}^\top \mathbf{u} - \text{grad} \Psi + \mathbf{S} : \text{grad} \mathbf{G} + \sigma \mathbf{G}^\top \mathbf{L} \dot{\mathbf{n}} - \mathbf{G}^\top \mathbf{g}. \quad (3.11)$$

Thus, granted (3.9), (3.10), and (3.11), the configurational momentum balance (3.1) is a direct consequence of the momentum balances (2.7).

Using (3.11), the relations (3.9), and (3.10), and the bulk constitutive relations (2.8), we find that the configurational stress \mathbf{C} and internal configurational force \mathbf{f} in the are determined by the fields \mathbf{u} , \mathbf{n} , and p through the relations

$$\left. \begin{aligned} \mathbf{C} &= (\hat{\Psi}(\mathbf{n}, \mathbf{G}) + p - \frac{1}{2} \varrho |\mathbf{u}|^2 - \frac{1}{2} \sigma |\dot{\mathbf{n}}|^2) \mathbf{1} - \mathbf{T}_{\text{dis}}, \\ \mathbf{f} &= \varrho \mathbf{L}^\top \mathbf{u} + \sigma \mathbf{G}^\top \mathbf{L} \dot{\mathbf{n}} - \mathbf{G}^\top \mathbf{g}_{\text{dis}}. \end{aligned} \right\} \quad (3.12)$$

These relations hold regardless of whether \mathbf{T}_{dis} and \mathbf{g}_{dis} take the linear forms (2.5). As we will later find, the interfacial counterparts of \mathbf{C} and \mathbf{f} , which may be viewed as surface excess quantities, do require independent constitutive specification; moreover, whereas the bulk balance merely determines \mathbf{f} , its interfacial counterpart provides an additional evolution equation encompassing the essential physics underlying transitions between the nematic and isotropic phases.

4 Theory for the isotropic phase

We model the isotropic phase as a classical incompressible and viscous liquid. Although the director field and all associated quantities are undefined for such a liquid, we may derive the governing equations for the isotropic phase by formally setting

$$\Psi = 0, \quad \mathbf{r} = \mathbf{0}, \quad \mathbf{S} = \mathbf{0}, \quad \text{and} \quad \mathbf{g} = \mathbf{0}$$

in the equations for the nematic phase. While the momentum balance is then simply (2.7)₁, the angular momentum balance yields in place of (2.9) the usual symmetry relation $\mathbf{T} = \mathbf{T}^\top$.

As is conventional, we assume that the Cauchy stress in the isotropic phase is linearly viscous, whereby

$$\mathbf{T} = -p \mathbf{1} + \mathbf{T}_{\text{dis}}, \quad \mathbf{T}_{\text{dis}} = 2\mu \mathbf{D}, \quad (4.1)$$

with p a constitutively indeterminate pressure and μ the (constant) *shear viscosity*. The governing equations for \mathbf{u} are then the familiar incompressible Navier–Stokes equations

$$\left. \begin{aligned} \text{div} \mathbf{u} &= 0, \\ \varrho \dot{\mathbf{u}} &= -\text{grad} p + \mu \text{div}(\text{grad} \mathbf{u}). \end{aligned} \right\} \quad (4.2)$$

As in the nematic phase, an understanding of the bulk configurational forces in the isotropic phase is superfluous to understanding the flow in that phase but central to the dynamics of the interface. For the isotropic phase, the configurational momentum balance remains (3.1), but now \mathbf{q} and \mathbf{C} are given by

$$\mathbf{q} = -\varrho \mathbf{u} \quad \text{and} \quad \mathbf{C} = -\frac{1}{2} \varrho |\mathbf{u}|^2 \mathbf{1} - \mathbf{T}. \quad (4.3)$$

We therefore have the following isotropic counterpart of (3.12)

$$\mathbf{C} = (p - \frac{1}{2} \varrho |\mathbf{u}|^2) \mathbf{1} - 2\mu \mathbf{D}, \quad \mathbf{f} = \varrho \mathbf{L}^\top \mathbf{u}. \quad (4.4)$$

⁵The expression $\mathbf{S} : \text{grad} \mathbf{G}$ denotes the vector field defined via the identity $(\mathbf{S} : \text{grad} \mathbf{G}) \cdot \mathbf{a} = \mathbf{S} : \text{grad}(\mathbf{G}\mathbf{a})$ for any constant vector \mathbf{a} .

5 Superficial kinematics and basic identities

We suppose that the isotropic and nematic phases are separated by a time-dependent surface \mathcal{S} oriented by a *unit normal field* \mathfrak{m} , defined over \mathcal{S} for all time, directed from the region occupied by the nematic phase into the region occupied by the isotropic phase.

5.1 Superficial fields

A *superficial field* is a smooth field defined on the interface for all time. A superficial tensor field \mathbb{A} is a tensor field on \mathcal{S} such that $\mathbb{A}\mathfrak{m} = \mathbf{0}$.⁶ An example of a superficial tensor field is the *projection* onto the tangent space to \mathcal{S}

$$\mathbb{P} = \mathbf{1} - \mathfrak{m} \otimes \mathfrak{m}. \quad (5.1)$$

Each superficial tensor field \mathbb{A} admits a decomposition of the form

$$\mathbb{A} = \mathbb{A}_{\text{tan}} + \mathfrak{m} \otimes \mathfrak{a}, \quad (5.2)$$

in which $\mathbb{A}_{\text{tan}} = \mathbb{P}\mathbb{A}$ and $\mathfrak{a} = \mathbb{A}^\top \mathfrak{m}$.

5.2 Superficial gradient and divergence

We denote by $\text{grad}_{\mathcal{S}}$ and $\text{div}_{\mathcal{S}}$ the superficial gradient and divergence. Let \mathfrak{a} be a superficial vector field such that $\mathfrak{a} \cdot \mathfrak{m} = 0$, and let \mathbb{A} be a superficial tensor field. Then the surface divergence theorem asserts that, for any subsurface \mathcal{A} of \mathcal{S} ,

$$\int_{\partial\mathcal{A}} \mathfrak{a} \cdot \mathfrak{m}_{\partial\mathcal{A}} \, ds = \int_{\partial\mathcal{A}} \text{div}_{\mathcal{S}} \mathfrak{a} \, da, \quad \int_{\partial\mathcal{A}} \mathbb{A} \mathfrak{m}_{\partial\mathcal{A}} \, ds = \int_{\partial\mathcal{A}} \text{div}_{\mathcal{S}} \mathbb{A} \, da, \quad (5.3)$$

where $\mathfrak{m}_{\partial\mathcal{A}}$ is the outward unit normal to $\partial\mathcal{A}$ in the tangent plane to \mathcal{S} .

The *curvature tensor* \mathbb{K} defined by

$$\mathbb{K} = -\text{grad}_{\mathcal{S}} \mathfrak{m} \quad (5.4)$$

obeys $\mathbb{K}^\top \mathfrak{m} = \mathbf{0}$ and $\mathbb{K} = \mathbb{K}^\top$, and

$$K = \text{tr} \mathbb{K} = \mathbb{P} : \mathbb{K} = -\text{div}_{\mathcal{S}} \mathfrak{m} \quad (5.5)$$

is the *total* (twice the mean) *curvature*. Then, by (5.1), we have the important identity

$$\text{div}_{\mathcal{S}} \mathbb{P} = K \mathfrak{m}. \quad (5.6)$$

5.3 Superficial limits of bulk fields

Consider an arbitrary bulk field Φ continuous up to \mathcal{S} from either phase. For each \mathbf{x} belonging to $\mathcal{S}(t)$, we denote by Φ^+ and Φ^- the limits $\Phi^\pm(\mathbf{x}, t) = \lim_{\epsilon \rightarrow 0^+} \Phi(\mathbf{x} \pm \epsilon \mathfrak{m}(\mathbf{x}, t), t)$. Further, we use $\llbracket \Phi \rrbracket$ and $\langle\langle \Phi \rangle\rangle$ to designate the *jump* and *average* of Φ across \mathcal{S} :

$$\llbracket \Phi \rrbracket = \Phi^+ - \Phi^-, \quad \langle\langle \Phi \rangle\rangle = \frac{1}{2}(\Phi^+ + \Phi^-).$$

Then, given bulk fields Φ and Λ , we have the useful identity:

$$\llbracket \Phi \Lambda \rrbracket = \langle\langle \Phi \rangle\rangle \llbracket \Lambda \rrbracket + \langle\langle \Lambda \rangle\rangle \llbracket \Phi \rrbracket. \quad (5.7)$$

Consider now the fluid velocity field \mathbf{u} . We allow for the possibility that the normal components $\mathbf{u}^\pm \cdot \mathfrak{m}$ of the interfacial limits \mathbf{u}^\pm of \mathbf{u} differ.⁷ However, we assume that the interface is *slip-free*, i.e.,

$$\mathbb{P} \llbracket \mathbf{u} \rrbracket = \mathbf{0} \quad \iff \quad \mathbf{u} := \mathbb{P} \mathbf{u}^+ = \mathbb{P} \mathbf{u}^- = \mathbb{P} \langle\langle \mathbf{u} \rangle\rangle, \quad (5.8)$$

so that the tangential components $\mathbb{P} \mathbf{u}^\pm$ of the interfacial limits \mathbf{u}^+ and \mathbf{u}^- of \mathbf{u} coincide.

⁶A superficial tensor field would generally be defined at each $\mathbf{x} \in \mathcal{S}$ as a linear transformation of the tangent space at \mathbf{x} into \mathbb{R}^3 ; the requirement $\mathbb{A}\mathfrak{m} = \mathbf{0}$ allows us to consider \mathbb{A} at each point as a linear transformation of \mathbb{R}^3 into \mathbb{R}^3 .

⁷By the interfacial mass balance (6.1), the fluid velocity must be discontinuous across \mathcal{S} whenever the densities of the nematic and isotropic phases differ.

5.4 Velocity fields for \mathcal{S}

Let \mathbf{v} denote a *velocity field* for \mathcal{S} . The tangential component of a velocity field for \mathcal{S} can be chosen arbitrarily, since it corresponds to a choice of parametrization of \mathcal{S} , but all velocity fields must have the same normal component

$$V := \mathbf{v} \cdot \mathbf{m},$$

which we refer to as the *normal velocity of the interface*. The fields

$$\mathbf{v} - \mathbf{u}^\pm \quad \text{and} \quad V - \mathbf{u}^\pm \cdot \mathbf{m}$$

represent vector and scalar velocities of \mathcal{S} relative to the isotropic and nematic phases.

Consider an arbitrary subsurface \mathcal{A} of \mathcal{S} . The curve $\partial\mathcal{A}$ evolves in space and its motion is described by a velocity field $\mathbf{v}_{\partial\mathcal{A}}$ on $\partial\mathcal{A}$. Different parametrizations of the curve $\partial\mathcal{A}$ induce different velocity fields, but all velocity fields must have the same components V and $V_{\partial\mathcal{A}}$ with respect to the unit normals \mathbf{m} and $\mathbf{m}_{\partial\mathcal{A}}$, i.e.,

$$\mathbf{v}_{\partial\mathcal{A}} \cdot \mathbf{m} = V \quad \text{and} \quad \mathbf{v}_{\partial\mathcal{A}} \cdot \mathbf{m}_{\partial\mathcal{A}} = V_{\partial\mathcal{A}}. \quad (5.9)$$

The motion of $\partial\mathcal{A}$ relative to the material in the bulk phases is then described by the relative velocities

$$\mathbf{v}_{\partial\mathcal{A}} - \mathbf{u}^\pm \quad \text{and} \quad V_{\partial\mathcal{A}} - \mathbf{u} \cdot \mathbf{m}_{\partial\mathcal{A}}. \quad (5.10)$$

The component of $\mathbf{v}_{\partial\mathcal{A}}$ tangential to $\partial\mathcal{A}$ is not intrinsic and *may be arbitrarily chosen*. We require that the theory not depend on the velocity field $\mathbf{v}_{\partial\mathcal{A}}$ chosen to characterize the migration of $\partial\mathcal{A}$, and therefore that the theory be invariant under transformations of $\mathbf{v}_{\partial\mathcal{A}}$ of the form

$$\mathbf{v}_{\partial\mathcal{A}} \rightarrow \mathbf{v}_{\partial\mathcal{A}} + \mathbf{t}, \quad (5.11)$$

with \mathbf{t} tangent to $\partial\mathcal{A}$.

5.5 Materially intrinsic velocity field for \mathcal{S}

To formulate integral laws for an interface \mathcal{S} migrating through the fluid, what is needed is a velocity field for \mathcal{S} that characterizes its *migration*. Specifically, we seek a single velocity field \mathbf{v} for \mathcal{S} that renders each of the migrational velocities $\mathbf{v} - \mathbf{u}^\pm$ normal. With this in mind, we note that

$$\mathbf{v} - \mathbf{u}^\pm = \mathbf{v} - (\mathbf{u}^\pm \cdot \mathbf{n})\mathbf{m} - \mathbf{u} = (V - \mathbf{u}^\pm \cdot \mathbf{n})\mathbf{n} + (\mathbb{P}\mathbf{v} - \mathbf{u});$$

thus, taking $\mathbb{P}\mathbf{v} = \mathbf{u}$, we arrive at a choice of velocity field \mathbf{v} for \mathcal{S} with each of its migrational velocities $\mathbf{v} - \mathbf{u}^\pm$ normal:

$$\mathbf{v} - \mathbf{u}^\pm = (V - \mathbf{u}^\pm \cdot \mathbf{m})\mathbf{m}. \quad (5.12)$$

Since $\mathbf{u}^\pm - (\mathbf{u}^\pm \cdot \mathbf{n})\mathbf{m} = \mathbf{u}$, the resulting velocity field \mathbf{v} , called the *materially intrinsic velocity-field* for \mathcal{S} , has the specific form

$$\mathbf{v} = V\mathbf{m} + \mathbf{u} \quad (5.13)$$

and is important *because it is normal when computed relative to the material on either side of $\mathcal{S}(t)$* .

For \mathbf{v} the materially intrinsic velocity-field for \mathcal{S} , the migrational velocity $\mathbf{v} - \langle\langle \mathbf{u} \rangle\rangle$ satisfies

$$\mathbf{v} - \langle\langle \mathbf{u} \rangle\rangle = (V - \langle\langle \mathbf{u} \rangle\rangle \cdot \mathbf{n})\mathbf{m}. \quad (5.14)$$

Similarly, by (5.9), (5.10), and (5.12), we also have the identity

$$\begin{aligned} (\mathbf{v}_{\partial\mathcal{A}} - \mathbf{v}) \cdot \mathbf{m}_{\partial\mathcal{A}} &= (\mathbf{v}_{\partial\mathcal{A}} - \mathbf{u}^\pm) \cdot \mathbf{m}_{\partial\mathcal{A}} - (\mathbf{v} - \mathbf{u}^\pm) \cdot \mathbf{m}_{\partial\mathcal{A}} \\ &= V_{\partial\mathcal{A}} - \mathbf{u} \cdot \mathbf{m}_{\partial\mathcal{A}}. \end{aligned} \quad (5.15)$$

Further, in view of (5.4)₁ and (5.13),

$$\text{grad}_\mathcal{S}\mathbf{v} = \mathbf{m} \otimes \text{grad}_\mathcal{S}V - V\mathbb{K} + \text{grad}_\mathcal{S}\mathbf{u}. \quad (5.16)$$

5.6 Materially intrinsic time-derivative. Superficial transport theorem

Let γ be a field on \mathcal{S} and let \mathbf{v} be the materially intrinsic velocity field for \mathcal{S} (cf. (5.13)). Given any time t_0 and any point \mathbf{x}_0 on $\mathcal{S}(t_0)$, let $\mathbf{z}(t)$ denote the curve such that

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{v}(\mathbf{z}(t), t), \quad \mathbf{z}(t_0) = \mathbf{x}_0,$$

and define

$$\overset{\circ}{\gamma}(\mathbf{x}_0, t_0) = \left. \frac{d\gamma(\mathbf{z}(t), t)}{dt} \right|_{t=t_0}. \quad (5.17)$$

The field $\overset{\circ}{\gamma}$ defined in this manner is referred to as the *materially intrinsic normal time-derivative of γ following \mathcal{S}* .

Important to what follows is the *superficial transport theorem* (CERMELLI, FRIED & GURTIN [6]): for $\gamma(\mathbf{x}, t)$ a smooth superficial scalar field,

$$\frac{d}{dt} \int_{\mathcal{A}} \gamma \, da = \int_{\mathcal{A}} \left\{ \overset{\circ}{\gamma} - \gamma(KV - \operatorname{div}_{\mathcal{S}} \mathbf{u}) \right\} da + \int_{\partial \mathcal{A}} \gamma (V_{\partial \mathcal{A}} - \mathbf{u} \cdot \mathbf{m}_{\partial \mathcal{A}}) \, ds. \quad (5.18)$$

5.7 Superficial velocity gradient

We write

$$\mathbb{L} = \operatorname{grad}_{\mathcal{S}} \langle \mathbf{u} \rangle = \langle \mathbb{L} \rangle \mathbb{P} \quad (5.19)$$

for the *superficial velocity gradient* and define the *superficial rate of stretch* \mathbb{D} and the *superficial rate of spin* \mathbb{W} via

$$\mathbb{D} = \frac{1}{2}(\mathbb{P}\mathbb{L} + \mathbb{L}^{\top}\mathbb{P}) = \mathbb{P}\langle \mathbb{D} \rangle \mathbb{P}, \quad \mathbb{W} = \frac{1}{2}(\mathbb{P}\mathbb{L} - \mathbb{L}^{\top}\mathbb{P}) = \mathbb{P}\langle \mathbb{W} \rangle \mathbb{P}. \quad (5.20)$$

Then $\mathbb{L} = \mathbb{D} + \mathbb{W} + \mathbf{m} \otimes \mathbb{L}^{\top} \mathbf{m}$ and, for any superficial tensor field \mathbb{A} ,

$$\mathbb{A} : \mathbb{L} = \mathbb{A}_{\tan} : (\mathbb{D} + \mathbb{W}) + \mathbb{A}^{\top} \mathbf{m} \cdot \mathbb{L}^{\top} \mathbf{m}. \quad (5.21)$$

In the subsequent development we will require a measure of the rate of change of the director field following the interface. A natural choice for such a measure would appear to be $\overset{\circ}{\mathbf{n}}$; however, $\overset{\circ}{\mathbf{n}}$ is not invariant under superimposed rigid-body motions. Among all possible properly invariant choices, we will use the superficial field

$$\mathbf{w} = \overset{\circ}{\mathbf{n}} - \mathbb{W}\mathbf{n} + (\mathbf{n} \cdot \mathbf{m}) \mathbb{L}^{\top} \mathbf{m} - (\mathbf{n} \cdot \mathbb{L}^{\top} \mathbf{m}) \mathbf{m}, \quad (5.22)$$

which obeys $\mathbf{w} \cdot \mathbf{n} = 0$. Thus, introducing

$$\boldsymbol{\xi} = \mathbf{n} \cdot \mathbf{m} \quad (5.23)$$

and noting that $\overset{\circ}{\mathbf{n}} \cdot \mathbf{m} = \mathbf{w} \cdot \mathbf{m} + \mathbf{n} \cdot \mathbb{L}^{\top} \mathbf{m}$, $\overset{\circ}{\mathbf{m}} \cdot \mathbf{n} = -\mathbf{n} \cdot \operatorname{grad}_{\mathcal{S}} (V - \langle \mathbf{u} \rangle \cdot \mathbf{m}) - \mathbf{n} \cdot \mathbb{L}^{\top} \mathbf{m} - \mathbf{n} \cdot \mathbb{K}(\mathbf{v} - \langle \mathbf{u} \rangle)$, $\mathbf{m} \cdot \mathbf{w} = (\mathbf{m} - \boldsymbol{\xi} \mathbf{n}) \cdot \mathbf{w}$, and $\mathbf{n} \cdot \operatorname{grad}_{\mathcal{S}} J = (\mathbf{n} - \boldsymbol{\xi} \mathbf{m}) \cdot \operatorname{grad}_{\mathcal{S}} J$, it follows that

$$\overset{\circ}{\boldsymbol{\xi}} = (\mathbf{m} - \boldsymbol{\xi} \mathbf{n}) \cdot \mathbf{w} - \langle v \rangle (\mathbf{n} - \boldsymbol{\xi}) \cdot \operatorname{grad}_{\mathcal{S}} J - \mathbf{n} \cdot \mathbb{K}(\mathbf{v} - \langle \mathbf{u} \rangle)$$

and thus, by (5.14), that for \mathbf{v} materially intrinsic,

$$\overset{\circ}{\boldsymbol{\xi}} = (\mathbf{m} - \boldsymbol{\xi} \mathbf{n}) \cdot \mathbf{w} - \langle v \rangle (\mathbf{n} - \boldsymbol{\xi} \mathbf{m}) \cdot \operatorname{grad}_{\mathcal{S}} J. \quad (5.24)$$

6 Superficial kinetics

6.1 Pillboxes

Let \mathcal{A} be an arbitrary subsurface of \mathcal{S} . The *superficial pillbox* determined by \mathcal{A} is a control volume of infinitesimal thickness consisting of (Fig. 1):

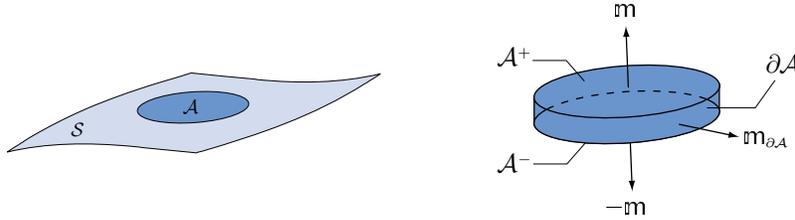


Figure 1: Schematic of a migrating subsurface \mathcal{A} of the interface \mathcal{S} showing an enlarged view of the associated superficial pillbox.

- a surface \mathcal{A}^+ , with unit normal \mathfrak{m} , lying in the isotropic phase;
- a surface \mathcal{A}^- , with unit normal $-\mathfrak{m}$, lying in the nematic phase;
- a lateral bounding surface $\partial\mathcal{A}$ with outward unit normal $\mathfrak{m}_{\partial\mathcal{A}}$.

To write the basic laws for an interfacial pillbox, consider a typical balance in integral form for a time-dependent control volume \mathcal{R} defined so that $\mathcal{A} = \mathcal{R} \cap \mathcal{S}$:

$$\frac{d}{dt} \int_{\mathcal{R}} \mathbf{a} \, dv = \int_{\partial\mathcal{R}} \{ \mathbf{A} \mathbf{m}_{\partial\mathcal{R}} + \mathbf{a} V_{\partial\mathcal{R}}^{\text{rel}} \} \, da + \int_{\mathcal{R}} \mathbf{b} \, dv + \left\{ \begin{array}{c} \text{interfacial} \\ \text{terms} \end{array} \right\},$$

with \mathbf{a} and \mathbf{b} vector fields, \mathbf{A} a tensor field, and $\mathbf{m}_{\partial\mathcal{R}}$ the outward unit normal to $\partial\mathcal{R}$. If these fields admit limit and are smooth up to the interface, standard arguments show that this balance implies a corresponding integral balance

$$\int_{\mathcal{A}} \{ \mathbf{A} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathbf{a} \} \, da + \left\{ \begin{array}{c} \text{interfacial} \\ \text{terms} \end{array} \right\} = \mathbf{0},$$

for \mathcal{A} .

6.2 Mass balance

The isotropic and nematic phases are endowed with constant mass densities ϱ^+ and ϱ^- . We allow for the possibility that these densities differ. The fields $\varrho^\pm (V - \mathbf{u}^\pm \cdot \mathfrak{m})$ then represent flows of mass at the two sides of the interface in the direction \mathfrak{m} , and balance of mass requires that these be equal:

$$\llbracket \varrho (V - \mathbf{u} \cdot \mathfrak{m}) \rrbracket = 0. \quad (6.1)$$

Thus, we may define the *mass flow* J across the interface (in the direction \mathfrak{n}) by the relation

$$J = \varrho^+ (V - \mathbf{u}^+ \cdot \mathfrak{m}) = \varrho^- (V - \mathbf{u}^- \cdot \mathfrak{m}), \quad (6.2)$$

so that

$$J = \langle\langle v \rangle\rangle^{-1} (V - \langle\langle \mathbf{u} \rangle\rangle \cdot \mathfrak{m}) \quad \text{and} \quad \llbracket v \rrbracket J = -\llbracket \mathbf{u} \rrbracket \cdot \mathfrak{m}, \quad (6.3)$$

with

$$v = \frac{1}{\varrho}$$

the *specific volume*.

6.3 Balance of linear momentum

In addition to the bulk Cauchy stress, we account for *interfacial Cauchy stress* through a superficial tensor field \mathbb{T} . The balance of linear momentum for a superficial pillbox \mathcal{A} then requires that

$$\int_{\partial\mathcal{A}} \mathbb{T} \mathfrak{m}_{\partial\mathcal{A}} \, ds + \int_{\mathcal{A}} \left\{ \llbracket \mathbb{T} \rrbracket \mathfrak{m} + \llbracket (V - \mathbf{u} \cdot \mathfrak{m}) \mathbf{p} \rrbracket \right\} \, da = \mathbf{0},$$

or, equivalently, that the superficial field equation

$$\operatorname{div}_s \mathbb{T} + \llbracket \mathbf{T} \rrbracket \mathbf{m} + \llbracket (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{p} \rrbracket = \mathbf{0} \quad (6.4)$$

holds on \mathcal{S} .

6.4 Balance of director momentum

In addition to the bulk director stress \mathbf{S} and the bulk internal director body force \mathbf{g} , we account for a *superficial director force* (density) through a field \mathbf{g} . The balance of director momentum then requires that, for any superficial pillbox \mathcal{A} ,

$$\int_{\mathcal{A}} \left\{ \mathbf{g} - \mathbf{S} \mathbf{m} - (V - \mathbf{u}^- \cdot \mathbf{m}) \mathbf{r} \right\} da = \mathbf{0}$$

or, equivalently, that the superficial field equation

$$\mathbf{g} = \mathbf{S} \mathbf{m} + (V - \mathbf{u}^- \cdot \mathbf{m}) \mathbf{r} \quad (6.5)$$

holds on \mathcal{S} . As with its bulk counterpart (2.7)₂, we may decompose the interfacial director momentum balance into components

$$(\mathbf{1} - \mathbf{n} \otimes \mathbf{n})(\mathbf{g} - \mathbf{S} \mathbf{m}) = (V - \mathbf{u}^- \cdot \mathbf{m}) \mathbf{r} \quad (6.6)$$

and

$$\mathbf{g} \cdot \mathbf{n} = (\mathbf{S}^\top \mathbf{n}) \cdot \mathbf{m} \quad (6.7)$$

perpendicular and parallel to the director.

By (2.8)₂ and the requirement that $(\partial \Psi / \partial \mathbf{G})^\top \mathbf{n} = \mathbf{0}$, (6.7) yields

$$\mathbf{g} \cdot \mathbf{n} = \boldsymbol{\alpha} \cdot \mathbf{m}, \quad (6.8)$$

so that the component of \mathbf{g} parallel to \mathbf{n} is determined by the multiplier field $\boldsymbol{\alpha}$ and the orientation \mathbf{m} of \mathcal{S} . Further, by (2.8)₂, (6.6) can be written as

$$(\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) \mathbf{g} = \frac{\partial \Psi}{\partial \mathbf{G}} \mathbf{m} + (V - \mathbf{u}^- \cdot \mathbf{m}) \mathbf{r}, \quad (6.9)$$

showing that only the component of \mathbf{g} parallel to \mathbf{n} is determined in terms of $\boldsymbol{\alpha}$.

6.5 Balance of angular momentum

The balance of angular momentum requires that, for any superficial pillbox \mathcal{A} and any element \mathbf{o} of \mathbb{R}^3 ,

$$\int_{\partial \mathcal{A}} (\mathbf{x} - \mathbf{o}) \times \mathbb{T} \mathbf{m}_{\partial \mathcal{A}} ds + \int_{\mathcal{A}} (\mathbf{x} - \mathbf{o}) \times \left\{ \llbracket \mathbf{T} \rrbracket \mathbf{m} + \llbracket (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{p} \rrbracket \right\} da - \int_{\mathcal{A}} \mathbf{n} \times \left\{ \mathbf{S} \mathbf{m} + (V - \mathbf{u}^- \cdot \mathbf{m}) \mathbf{r} \right\} da = \mathbf{0}$$

or, equivalently, using (6.4) and (6.9), that the superficial field equation

$$\mathbb{T} + \mathbf{n} \otimes \mathbf{g} = (\mathbb{T} + \mathbf{n} \otimes \mathbf{g})^\top, \quad (6.10)$$

holds on \mathcal{S} . The requirement (6.10) is analogous to the local statement (2.9) of angular momentum balance in the nematic phase. Since \mathbb{T} is a superficial tensor field, $\mathbb{T} \mathbf{m} = \mathbf{0}$ and it follows from (6.10) that

$$\left. \begin{aligned} \mathbb{T}^\top \mathbf{m} &= (\mathbf{g} \cdot \mathbf{m}) \mathbf{n} - \xi \mathbf{g}, \\ \operatorname{Skw}(\mathbb{T}_{\tan}) &= \frac{1}{2} \mathbb{P}(\mathbf{g} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{g}) \mathbb{P}. \end{aligned} \right\} \quad (6.11)$$

6.6 Balance of configurational momentum

In addition to the bulk configurational stress tensor \mathbf{C} and the bulk internal configurational body force \mathbf{f} , we account for *superficial configurational stress* and *superficial internal configurational body force (density)* through fields \mathbb{C} and \mathbf{f} . The balance of configurational momentum then requires that, for any superficial pillbox \mathcal{A} ,

$$\int_{\partial\mathcal{A}} \mathbb{C}\mathbf{m}_{\partial\mathcal{A}} ds + \int_{\mathcal{A}} \left\{ \mathbf{f} + \llbracket \mathbf{C} \rrbracket \mathbf{m} + \llbracket (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rrbracket \right\} da = \mathbf{0}$$

or, equivalently, that the superficial field equation

$$\operatorname{div}_{\mathcal{S}} \mathbb{C} + \mathbf{f} + \llbracket \mathbf{C} \rrbracket \mathbf{m} + \llbracket (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rrbracket = \mathbf{0} \quad (6.12)$$

holds on \mathcal{S} .

The role of the interfacial configurational force \mathbf{f} here is completely different from its bulk counterpart \mathbf{f} . While \mathbf{f} is indeterminate, and must be chosen so that the bulk equation (3.1) is identically satisfied, \mathbf{f} is in general given by constitutive relations describing the ordering kinetics at the phase interface, and this in turn leads from (6.12) an evolution equation for the interface.

6.7 Power

To express the power expended by the tractions, we mimick the reasoning underlying the form (3.8) of the power expenditure on a control volume migrating through the nematic phase. The configurational and standard tractions $\mathbb{C}\mathbf{m}_{\partial\mathcal{A}}$ and $\mathbb{T}\mathbf{m}_{\partial\mathcal{A}}$ are distributed over the boundary $\partial\mathcal{A}$ of the pillbox. As in our discussion of the bulk phases, we take the migrational velocity $\mathbf{v}_{\partial\mathcal{A}} - \langle\langle \mathbf{u} \rangle\rangle$ of $\partial\mathcal{A}$ to be the appropriate power conjugate velocity for $\mathbb{C}\mathbf{m}_{\partial\mathcal{A}}$. For $\mathbb{T}\mathbf{m}_{\partial\mathcal{A}}$, we reason by analogy to our treatment of the power expended by the standard traction on a migrating control volume and take as power conjugate the observed velocity $\mathbf{v}_{\partial\mathcal{A}}$ of $\partial\mathcal{A}$. In addition, we view the configurational tractions $\mathbf{C}^+\mathbf{m}$ and $-\mathbf{C}^-\mathbf{m}$ exerted on \mathcal{A} by the bulk material in the isotropic and nematic phases as forces, per unit area, associated with the transfer of material across \mathcal{S} that occurs with as one phase grows with respect to the other. We therefore take the velocities $\mathbf{v} - \mathbf{u}^{\pm}$ of \mathcal{S} relative to the underlying material to be appropriate power conjugate velocities for $(\mathbf{C}\mathbf{m} + (V - \mathbf{u} \cdot \mathbf{m})\mathbf{q})^{\pm}$. Consistent with our treatment of the power expended by the standard traction on a migrating control volume, we use as a power conjugate velocity for $(\mathbb{T}\mathbf{m} + (V - \mathbf{u} \cdot \mathbf{m})\mathbf{p})^{\pm}$ the velocity \mathbf{v} of \mathcal{S} . Similarly, consistent with our treatment of the power expended by the director traction on a migrating control volume, we use as a power conjugate velocity for $\mathbf{S}\mathbf{m} + (V - \mathbf{u}^- \cdot \mathbf{m})\mathbf{r}$ the velocity $\hat{\mathbf{n}}$. The (net) external power expended on \mathcal{A} then has the form

$$w(\mathcal{A}) = w^*(\mathcal{A}) + \int_{\mathcal{A}} \llbracket \mathbb{T}\mathbf{m} + (V - \mathbf{u} \cdot \mathbf{m})\mathbf{p} \rrbracket \cdot \mathbf{v} da - \int_{\mathcal{A}} \mathbf{g} \cdot \hat{\mathbf{n}} da + \int_{\mathcal{A}} \llbracket (\mathbf{C}\mathbf{m} + (V - \mathbf{u} \cdot \mathbf{m})\mathbf{q}) \cdot (\mathbf{v} - \mathbf{u}) \rrbracket da, \quad (6.13)$$

where we have used the superficial director momentum balance (6.5) and $w^*(\mathcal{A})$ is defined by

$$w^*(\mathcal{A}) = \int_{\partial\mathcal{A}} \left\{ \mathbb{T}\mathbf{m}_{\partial\mathcal{A}} \cdot \mathbf{v}_{\partial\mathcal{A}} + \mathbb{C}\mathbf{m}_{\partial\mathcal{A}} \cdot (\mathbf{v}_{\partial\mathcal{A}} - \langle\langle \mathbf{u} \rangle\rangle) \right\} ds.$$

As in our treatment of the power acting on a migrating control volume, we require that (6.13) be independent of the choice of observed velocity field $\mathbf{v}_{\partial\mathcal{A}}$ chosen to characterize the migration of \mathcal{A} . Necessary and sufficient for this condition to be satisfied is that $w^*(\mathcal{A})$ be invariant under all transformations of the form

$$\mathbf{v}_{\partial\mathcal{A}} \mapsto \mathbf{v}_{\partial\mathcal{A}} + \mathbf{t}, \quad \mathbf{t} \cdot \mathbf{m} = \mathbf{t} \cdot \mathbf{m}_{\partial\mathcal{A}} = 0, \quad (6.14)$$

which, by (5.10), is necessarily accompanied by the transformation

$$\mathbf{v}_{\partial\mathcal{A}} - \langle\langle \mathbf{u} \rangle\rangle \mapsto \mathbf{v}_{\partial\mathcal{A}} - \langle\langle \mathbf{u} \rangle\rangle + \mathbf{t}, \quad (6.15)$$

This invariance holds if and only if

$$\int_{\mathcal{A}} (\mathbb{T} + \mathbb{C})\mathbf{m}_{\partial\mathcal{A}} \cdot \mathbf{t} ds = 0 \quad (6.16)$$

for all subsurfaces \mathcal{A} of \mathcal{S} and all fields \mathfrak{t} tangential to $\partial\mathcal{A}$. It follows that $\mathfrak{A} = \mathbb{T} + \mathbb{C}$ must satisfy

$$\mathfrak{t} \cdot \mathfrak{A} \mathfrak{m}^\perp = 0 \quad (6.17)$$

whenever \mathfrak{t} and \mathfrak{m}^\perp are orthogonal and tangent to \mathcal{S} . Thus $\mathfrak{A} \mathfrak{m}^\perp$ must lie in the direction of \mathfrak{m}^\perp for each \mathfrak{m}^\perp orthogonal to \mathfrak{m} , which is possible if and only if the tangential component $\mathfrak{A}_{\text{tan}}$ of \mathfrak{A} has the form $\mathfrak{A}_{\text{tan}} = \varphi \mathbb{P}$, with φ a superficial scalar field. Invariance therefore implies that the tangential component \mathbb{C}_{tan} of interfacial configurational stress \mathbb{C} must be of the form

$$\mathbb{C}_{\text{tan}} = \varphi \mathbb{P} - \mathbb{T}_{\text{tan}}. \quad (6.18)$$

We view (6.18) as a *superficial pre-Eshelby relation*. Equivalently, bearing in mind (5.2), we have that

$$\mathbb{C} = \varphi \mathbb{P} - \mathbb{T} + \mathfrak{m} \otimes \mathfrak{k}, \quad (6.19)$$

with

$$\mathfrak{k} = (\mathbb{C} + \mathbb{T})^\top \mathfrak{m} \quad (6.20)$$

the (*effective*) *configurational shear*.

By (5.15) and (6.18), it follows that

$$w^*(\mathcal{A}) = \int_{\partial\mathcal{A}} \left\{ \varphi (V_{\partial\mathcal{A}} - \mathbf{u} \cdot \mathfrak{m}_{\partial\mathcal{A}}) + \mathbb{T} \mathfrak{m}_{\partial\mathcal{A}} \cdot \mathbf{v} + \mathbb{C} \mathfrak{m}_{\partial\mathcal{A}} \cdot (\mathbf{v} - \langle \mathbf{u} \rangle) \right\} ds.$$

Further, using the surface divergence theorem (5.3)₃ and the superficial balances (6.4) and (6.12) for standard and configurational momentum, we obtain

$$\begin{aligned} w^*(\mathcal{A}) &= \int_{\partial\mathcal{A}} \varphi (V_{\partial\mathcal{A}} - \mathbf{u} \cdot \mathfrak{m}_{\partial\mathcal{A}}) ds + \int_{\mathcal{A}} \left\{ \mathbb{T} : \text{grad}_s \mathbf{v} + \mathbb{C} : \text{grad}_s (\mathbf{v} - \langle \mathbf{u} \rangle) \right\} da \\ &\quad - \int_{\mathcal{A}} \left[\mathbb{T} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{p} \right] \cdot \mathbf{v} da + \int_{\mathcal{A}} \left\{ \mathfrak{f} + \left[\mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \right] \right\} \cdot (\mathbf{v} - \langle \mathbf{u} \rangle) da \end{aligned}$$

and, thus, using (5.7) and recalling that \mathbf{v} is a superficial field to yield

$$\begin{aligned} \left[\mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \right] \cdot (\mathbf{v} - \langle \mathbf{u} \rangle) &= \left[\mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \cdot (\mathbf{v} - \mathbf{u}) \right] - \langle \mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \rangle \cdot \llbracket \mathbf{v} - \mathbf{u} \rrbracket \\ &= \left[\mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \cdot (\mathbf{v} - \mathbf{u}) \right] + \langle \mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \rangle \cdot \llbracket \mathbf{u} \rrbracket, \end{aligned}$$

find that (6.13) can be expressed as

$$\begin{aligned} w(\mathcal{A}) &= \int_{\partial\mathcal{A}} \varphi (V_{\partial\mathcal{A}} - \mathbf{u} \cdot \mathfrak{m}_{\partial\mathcal{A}}) ds + \int_{\mathcal{A}} \left\{ (\mathbb{T} + \mathbb{C}) : \text{grad}_s \mathbf{v} - \mathbb{C} : \mathbb{L} - \mathfrak{g} \cdot \dot{\mathbf{n}} \right\} da \\ &\quad - \int_{\mathcal{A}} \left\{ \mathfrak{f} \cdot (\mathbf{v} - \langle \mathbf{u} \rangle) + \langle \mathbb{C} \mathfrak{m} + (V - \mathbf{u} \cdot \mathfrak{m}) \mathfrak{q} \rangle \cdot \llbracket \mathbf{u} \rrbracket \right\} da. \quad (6.21) \end{aligned}$$

Toward a further simplification of the expression for the total power expended on \mathcal{A} , consider the second term on the right side of (6.21). By (6.19),

$$\begin{aligned} (\mathbb{T} + \mathbb{C}) : \text{grad}_s \mathbf{v} &= \left\{ (\mathbb{T} + \mathbb{C})_{\text{tan}} + \mathfrak{m} \otimes \mathfrak{k} \right\} : \text{grad}_s \mathbf{v} \\ &= \varphi \text{div}_s \mathbf{v} + (\mathfrak{m} \otimes \mathfrak{k}) : \text{grad}_s \mathbf{v}. \end{aligned}$$

Also, by (5.19), (5.21), (6.11), (6.20), and (6.19),

$$\begin{aligned} \mathbb{C} : \mathbb{L} &= \mathbb{C}_{\text{tan}} : (\mathbb{D} + \mathbb{W}) + \mathbb{C}^\top \mathfrak{m} \cdot \mathbb{L}^\top \mathfrak{m} + \mathfrak{g} \cdot \dot{\mathbf{n}} \\ &= \mathbb{C}_{\text{tan}} : \mathbb{D} - \mathbb{T}_{\text{tan}} : \mathbb{W} + (\mathfrak{k} - \mathbb{T}^\top \mathfrak{m}) \cdot \mathbb{L}^\top \mathfrak{m} + \mathfrak{g} \cdot \dot{\mathbf{n}} \\ &= \mathbb{C}_{\text{tan}} : \mathbb{D} + (\mathfrak{m} \otimes \mathfrak{k}) : \mathbb{L} + \mathfrak{g} \cdot \mathbf{w}. \end{aligned}$$

Thus,

$$(\mathbb{T} + \mathbb{C}) : \text{grad}_S \mathbf{v} - \mathbb{C} : \mathbb{L} - \mathbf{g} \cdot \dot{\mathbf{n}} = \varphi \text{div}_S \mathbf{v} - \mathbb{C}_{\text{tan}} : \mathbb{D} - \mathbf{g} \cdot \mathbf{w} + (\mathbf{m} \otimes \mathbf{k}) : \text{grad}_S (\mathbf{v} - \langle \mathbf{u} \rangle), \quad (6.22)$$

Hereafter, we require that the velocity field \mathbf{v} for \mathcal{S} be migratorially intrinsic. In view of (5.14), (5.16), and (6.3), we may therefore rewrite (6.22) in the form

$$(\mathbb{T} + \mathbb{C}) : \text{grad}_S \mathbf{v} - \mathbb{C} : \mathbb{L} - \mathbf{g} \cdot \dot{\mathbf{n}} = -\varphi(KV - \text{div}_S \mathbf{u}) - \mathbb{C}_{\text{tan}} : \mathbb{D} - \mathbf{g} \cdot \mathbf{w} + \langle v \rangle \mathbf{k} \cdot \text{grad}_S J, \quad (6.23)$$

Next, consider the final term on the right side of (6.21). By (6.3)₁ and (5.14),

$$\mathbf{f} \cdot (\mathbf{v} - \langle \mathbf{u} \rangle) = \langle v \rangle f J,$$

where

$$f = \mathbf{f} \cdot \mathbf{m} \quad (6.24)$$

denotes the *normal configurational force*. Also, by (6.3)₂,

$$\langle \mathbf{Cm} + (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rangle \cdot \llbracket \mathbf{u} \rrbracket = -\mathbf{m} \cdot \langle \mathbf{Cm} + (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rangle \llbracket v \rrbracket J.$$

Thus,

$$\mathbf{f} \cdot (\mathbf{v} - \langle \mathbf{u} \rangle) + \langle \mathbf{Cm} + (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rangle \cdot \llbracket \mathbf{u} \rrbracket = F J, \quad (6.25)$$

with

$$F = \langle v \rangle f - \llbracket v \rrbracket \mathbf{m} \cdot \langle \mathbf{Cm} + (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rangle. \quad (6.26)$$

Finally, using (6.23) and (6.25) in (6.21), we arrive at the intrinsic expression

$$w(\mathcal{A}) = \int_{\partial \mathcal{A}} \varphi (V_{\partial \mathcal{A}} - \mathbf{u} \cdot \mathbf{m}_{\partial \mathcal{A}}) \, ds - \int_{\mathcal{A}} \left\{ \varphi (KV - \text{div}_S \mathbf{u}) + \mathbb{C}_{\text{tan}} : \mathbb{D} \right\} \, da - \int_{\mathcal{A}} \left\{ \mathbf{g} \cdot \mathbf{w} + F J - \langle v \rangle \mathbf{k} \cdot \text{grad}_S J \right\} \, da \quad (6.27)$$

for the total power expended on \mathcal{A} .

6.8 Imbalance of free energy

Since we neglect superficial distributions of momentum, the first and second laws for the interface reduce to an imbalance of free energy. Writing ψ for the *superficial free energy (density)*, measured per unit area of \mathcal{S} , so that

$$\int_{\mathcal{A}} \psi \, da \quad (6.28)$$

represents the net free energy of \mathcal{A} , the imbalance of free energy requires that, for any migrating interfacial pillbox \mathcal{A} ,

$$\frac{d}{dt} \int_{\mathcal{A}} \psi \, da \leq w(\mathcal{A}), \quad (6.29)$$

with $w(\mathcal{A})$ given by (6.13) or its intrinsic equivalent (6.27). Thus, by the superficial transport theorem (5.18),

$$\begin{aligned} \int_{\mathcal{A}} \left\{ \dot{\psi} + (\psi - \varphi)(KV - \text{div}_S \mathbf{u}) \right\} \, da + \int_{\partial \mathcal{A}} (\psi - \varphi)(V_{\partial \mathcal{A}} - \mathbf{u} \cdot \mathbf{m}_{\partial \mathcal{A}}) \, ds \\ \leq - \int_{\mathcal{A}} \left\{ \mathbb{C}_{\text{tan}} : \mathbb{D} + \mathbf{g} \cdot \mathbf{w} + F J - \langle v \rangle \mathbf{k} \cdot \text{grad}_S J \right\} \, da. \end{aligned} \quad (6.30)$$

This inequality can hold for all migrating pillboxes \mathcal{A} only if the coefficient of $V_{\partial \mathcal{A}} - \mathbf{u} \cdot \mathbf{m}_{\partial \mathcal{A}}$ in the integral over $\partial \mathcal{A}$ vanishes. Thus, (6.18) becomes the *superficial Eshelby relation*

$$\mathbb{C}_{\text{tan}} = \psi \mathbb{P} - \mathbb{T}_{\text{tan}} \quad (6.31)$$

and (6.19) becomes

$$\mathbf{C} = \psi \mathbf{P} - \mathbb{T} + \mathbf{m} \otimes \mathbf{k}. \quad (6.32)$$

Moreover, the inequality (6.30) simplifies to

$$\int_{\mathcal{A}} \left\{ \overset{\circ}{\psi} + \mathbf{C}_{\text{tan}} : \mathbb{D} + \mathbf{g} \cdot \mathbf{w} + FJ - \langle\langle v \rangle\rangle \mathbf{k} \cdot \mathbf{grad}_s J \right\} da \leq 0 \quad (6.33)$$

or, equivalently, to the superficial dissipation inequality

$$\overset{\circ}{\psi} + \mathbf{C}_{\text{tan}} : \mathbb{D} + \mathbf{g} \cdot \mathbf{w} + FJ - \langle\langle v \rangle\rangle \mathbf{k} \cdot \mathbf{grad}_s J \leq 0. \quad (6.34)$$

The configurational momentum balance revisited

Adding the configurational momentum balance (6.12) for the interface, decomposes into a normal component

$$\mathbf{m} \cdot \text{div}_s \mathbf{C} + \mathbf{m} \cdot \llbracket \mathbf{Cm} + (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rrbracket + f = 0 \quad (6.35)$$

and a tangential component

$$\mathbb{P} \text{div}_s \mathbf{C} + \mathbb{P} \llbracket \mathbf{Cm} + (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{q} \rrbracket + \mathbb{P} f = \mathbf{0}. \quad (6.36)$$

We refer to (6.35) as the *normal configurational momentum balance*. While this balance is central to the description of the interface, the (nonintrinsic) tangential balance (6.36) is inconsequential to the theory.

An alternative version of the balance (6.35), involving explicitly the field F introduced in (6.26), can be obtained as follows.

Using (6.26) in (6.35) and multiplying through by $\langle\langle v \rangle\rangle$, taking advantage of the identity (5.7), and the relations (3.9), (3.10), and (4.3), we obtain

$$\langle\langle v \rangle\rangle \mathbf{m} \cdot \text{div}_s \mathbf{C} + \llbracket v(\Psi + p - \mathbf{m} \cdot \mathbf{T}_{\text{dis}} \mathbf{m}) \rrbracket - \llbracket \frac{1}{2} |\mathbf{u}|^2 - (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{u} \cdot \mathbf{m} \rrbracket + \frac{1}{2} \sigma v^- |\dot{\mathbf{n}}|^2 + \sigma (v^-)^2 J \dot{\mathbf{n}} \cdot \mathbf{m} + F = 0$$

Thus, since, by (5.8)₁ and (6.3), $\llbracket \frac{1}{2} |\mathbf{u}|^2 - (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{u} \cdot \mathbf{m} \rrbracket = -\frac{1}{2} \llbracket v^2 \rrbracket J^2 = -\langle\langle v \rangle\rangle \llbracket v \rrbracket J^2$, it follows that the normal combined momentum balance is equivalent to

$$\langle\langle v \rangle\rangle \mathbf{m} \cdot \text{div}_s \mathbf{C} + \llbracket v(\Psi + p - \mathbf{m} \cdot \mathbf{T}_{\text{dis}} \mathbf{m}) \rrbracket + \langle\langle v \rangle\rangle \llbracket v \rrbracket J^2 + \frac{1}{2} \sigma v^- |\dot{\mathbf{n}}|^2 + \sigma (v^-)^2 J \dot{\mathbf{n}} \cdot \mathbf{m} + F = 0. \quad (6.37)$$

7 Superficial constitutive equations

Following FRIEDEL (1926) and OSEEN (1933), we assume that the superficial free-energy ψ is given constitutively as a properly invariant function of \mathbf{n} and \mathbf{m} , so that

$$\psi = \psi(\xi), \quad (7.1)$$

where $\xi = \mathbf{n} \cdot \mathbf{m}$ (cf. (5.23)). Then, by (5.24),

$$\overset{\circ}{\psi} = \frac{d\psi}{d\xi} \overset{\circ}{\xi} = \frac{d\psi}{d\xi} \left\{ (\mathbf{m} - \xi \mathbf{n}) \cdot \mathbf{w} - \langle\langle v \rangle\rangle (\mathbf{n} - \xi \mathbf{m}) \cdot \mathbf{grad}_s J \right\}$$

and the superficial dissipation inequality (6.34) reduces to a residual inequality

$$\mathbf{C}_{\text{tan}} : \mathbb{D} + \left\{ \mathbf{g} + \frac{d\psi}{d\xi} (\mathbf{m} - \xi \mathbf{n}) \right\} \cdot \mathbf{w} + FJ - \langle\langle v \rangle\rangle \left\{ \mathbf{k} + \frac{d\psi}{d\xi} (\mathbf{n} - \xi \mathbf{m}) \right\} \cdot \mathbf{grad}_s J \leq 0. \quad (7.2)$$

The inequality (7.2) suggests the consideration of constitutive equations giving the symmetric part $\text{Sym}(\mathbf{C}_{\text{tan}})$ of \mathbf{C}_{tan} , $\mathbf{g} + (d\psi/d\xi)(\mathbf{m} - \xi \mathbf{n})$, F , and $\mathbf{k} + (d\psi/d\xi)(\mathbf{n} - \xi \mathbf{m})$ as properly invariant functions of \mathbf{n} , \mathbf{m} , and the rates \mathbb{D} , \mathbf{w} , J , and $\mathbf{grad}_s J$. If we emulate the treatment of the bulk nematic phase and insist that those functions be linear in the rates \mathbb{D} , \mathbf{w} , J , and $\mathbf{grad}_s J$, we arrive at a description involving

29 independent constitutive moduli. For simplicity, we therefore restrict our attention to constitutive equations that result on neglecting coupling between the operative dissipative mechanisms or, equivalently, on replacing the inequality (7.2) with the more stringent requirements

$$\left. \begin{aligned} & \mathbb{C}_{\text{tan}} : \mathbb{D} \leq 0, \\ & \left\{ \mathbf{g} + \frac{d\psi}{d\xi} (\mathbf{m} - \xi \mathbf{n}) \right\} \cdot \mathbf{w} \leq 0, \\ & FJ \leq 0, \\ & \left\{ \mathbf{k} + \frac{d\psi}{d\xi} (\mathbf{n} - \xi \mathbf{m}) \right\} \cdot \text{grad}_s J \geq 0, \end{aligned} \right\} \quad (7.3)$$

and assuming that $\text{Sym}(\mathbb{C}_{\text{tan}})$ depends linearly on \mathbb{D} , $\mathbf{g} + (d\psi/d\xi)(\mathbf{m} - \xi \mathbf{n})$ depends linearly on \mathbf{w} , F depends linearly on J , and $\mathbf{k} + (d\psi/d\xi)(\mathbf{n} - \xi \mathbf{m})$ depends linearly on $\text{grad}_s J$. Bearing in mind the relation (6.8) determining $\mathbf{g} \cdot \mathbf{n}$, these constitutive equations have the form⁸

$$\left. \begin{aligned} & \text{Sym}(\mathbb{C}_{\text{tan}}) = -(\lambda_1 \text{tr} \mathbb{D} + \lambda_2 \mathbf{n} \cdot \mathbb{D} \mathbf{n}) \mathbb{P} - (\lambda_3 \text{tr} \mathbb{D} + \lambda_4 \mathbf{n} \cdot \mathbb{D} \mathbf{n}) \mathbb{P} \mathbf{n} \otimes \mathbb{P} \mathbf{n} \\ & \quad - 2\lambda_5 \mathbb{D} - \frac{1}{2} \lambda_6 (\mathbb{P} \mathbf{n} \otimes \mathbb{D} \mathbf{n} + \mathbb{D} \mathbf{n} \otimes \mathbb{P} \mathbf{n}), \\ & \mathbf{g} = (\boldsymbol{\alpha} \cdot \mathbf{n}) \mathbf{n} - \frac{d\psi}{d\xi} (\mathbf{m} - \xi \mathbf{n}) - \lambda_7 \mathbf{w} - \lambda_8 (\mathbf{w} \cdot \mathbf{m}) (\mathbf{m} - \xi \mathbf{n}), \\ & F = -\lambda_9 J, \\ & \mathbf{k} = -\frac{d\psi}{d\xi} (\mathbf{n} - \xi \mathbf{m}) + \lambda_{10} \text{grad}_s J + \lambda_{11} (\mathbf{n} \cdot \text{grad}_s J) \mathbb{P} \mathbf{n}, \end{aligned} \right\} \quad (7.4)$$

where the moduli $\lambda_1, \lambda_2, \dots, \lambda_{11}$ may depend on ξ and — to ensure satisfaction of the inequalities (7.3) — must obey

$$\left. \begin{aligned} & 4(\lambda_1 + \lambda_5) + 2(\lambda_2 + \lambda_3 + \frac{1}{2} \lambda_6)(1 - \xi^2) + \lambda_4(1 - \xi^2)^2 \geq 0, \\ & 4(\lambda_1 + \lambda_5 - \frac{1}{2} \lambda_6(1 - \xi^2)) \lambda_4 \geq (\lambda_2 + \lambda_3 + \lambda_6)^2, \quad 4\lambda_5 + \lambda_6(1 - \xi^2) \geq 0, \\ & \lambda_7 \geq 0, \quad \lambda_7 + \lambda_8 \geq 0, \quad \lambda_9 \geq 0, \quad \lambda_{10} \geq 0, \quad \lambda_{10} + \lambda_{11} \geq 0. \end{aligned} \right\} \quad (7.5)$$

Granted (7.4), the theory provides a supplemental constitutive equation for the interfacial Cauchy stress \mathbb{T} . To derive that relation, we first notice that, as a direct consequence of the expressions (6.11) determining the normal component $\mathbb{T}^\top \mathbf{m}$ and skew part $\frac{1}{2}(\mathbb{T} - \mathbb{T}^\top)$ of \mathbb{T} and the superficial Eshelby relation (6.32),

$$\mathbb{T} = \psi \mathbb{P} - \text{Sym}(\mathbb{C}_{\text{tan}}) + \frac{1}{2} \mathbb{P} (\mathbf{g} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{g}) \mathbb{P} + \mathbf{m} \otimes ((\mathbf{g} \cdot \mathbf{m}) \mathbf{n} - \xi \mathbf{g}). \quad (7.6)$$

Thus, by (7.4)_{1,2,4}, we find by a direct calculation that

$$\mathbb{T} = \psi \mathbb{P} - \frac{d\psi}{d\xi} \mathbf{m} \otimes \mathbb{P} \mathbf{n} + \mathbb{T}_{\text{dis}}, \quad (7.7)$$

with

$$\mathbb{T}_{\text{dis}} = -\text{Sym}(\mathbb{C}_{\text{tan}} + \lambda_7 \mathbf{m} \otimes (\xi \mathbf{w} - (\mathbf{w} \cdot \mathbf{m}) \mathbf{n})) - \frac{1}{2} \lambda_7 (\mathbf{w} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{w}) - \lambda_8 (\mathbf{w} \cdot \mathbf{m}) \mathbf{m} \otimes \mathbb{P} \mathbf{n}. \quad (7.8)$$

Thus, even for uncoupled constitutive equations of the form (7.4), the interfacial Cauchy stress includes terms involving not only the interfacial rate of stretch but also terms involving the invariant measure \mathbf{w} of the rate at which \mathbf{n} changes following the interface. Note that, by (7.7) and (7.8), the surface tension $\gamma = \frac{1}{2} \text{tr} \mathbb{T}$ has the form

$$\gamma = \psi + \alpha_1 \text{tr} \mathbb{D} + \alpha_2 \mathbf{n} \cdot \mathbb{D}_0 \mathbf{n}, \quad (7.9)$$

⁸The constitutive equations should be compared with those provided by REY [22, 23] and PONIEWIERSKI [20] or a *material* interface.

with

$$\left. \begin{aligned} \alpha_1 &= \lambda_1 + \lambda_5 + \frac{1}{2}\lambda_3(1 - \xi^2), \\ \alpha_2 &= \lambda_2 + \frac{1}{2}\lambda_6 + \frac{1}{2}\lambda_4(1 - \xi^2). \end{aligned} \right\} \quad (7.10)$$

Further, in view of the representation (6.32) of the interfacial configurational stress and the expression (7.7) for \mathbb{T} , it follows that

$$\mathbb{C} = \lambda_{10}\mathfrak{m} \otimes \text{grad}_s J + \lambda_{11}(\mathbf{n} \cdot \text{grad}_s J)\mathfrak{m} \otimes \mathbb{P}\mathbf{n} - \mathbb{T}_{\text{dis}} \quad (7.11)$$

and, thus, that the interfacial configurational stress is of an entirely dissipative nature.

Direct calculations show that, granted (7.4), (7.7), and (7.11) the angular momentum balance (6.10) and the tangential component (6.36) are satisfied trivally.

8 Specialized interfacial evolution equations

Aside from the no-slip condition (5.8)₁ and the definition (6.2) of the interfacial mass flow, the general interfacial evolution equations of the theory are determined by inserting the constitutive equations (7.4) in the equations (6.4), (6.9), and (6.37) enforcing the balances of linear momentum, director momentum, and normal combined momentum. Instead of those equations, we present an illustrative specialization. In particular, we neglect all sources of dissipation excepting that associated with the exchange of mass between the phases. Thus, we assume that

$$\lambda_7 = \lambda_8 = \lambda_{10} = \lambda_{11} = 0 \quad (8.1)$$

and that

$$\lambda_9 = \lambda > 0. \quad (8.2)$$

The governing interfacial equations of the theory are then:

(i) the no-slip condition

$$\mathbb{P}[\mathbf{u}] = \mathbf{0}; \quad (8.3)$$

(ii) the definition

$$J = \langle\langle \varrho(V - \mathbf{u} \cdot \mathfrak{m}) \rangle\rangle \quad (8.4)$$

of the interfacial mass flow;

(iii) the tangential component

$$\mathbb{P}(\text{div}_s \mathbb{T}_{\text{dis}} + \llbracket \mathbb{T}_{\text{dis}} \rrbracket \mathfrak{m}) = \mathbf{0} \quad (8.5)$$

of the linear momentum balance;

(iv) the normal component

$$\left\{ \psi - \xi \frac{d\psi}{d\xi} \right\} K + \frac{d^2\psi}{d\xi^2} (\mathbb{K}\mathbf{n} - \mathbb{P}\mathbf{G}^\top \mathfrak{m}) \cdot \mathbf{n} - \frac{d\psi}{d\xi} \text{tr} \mathbf{G} + \mathfrak{m} \text{div}_s \mathbb{T}_{\text{dis}} = \llbracket p - \mathfrak{m} \cdot \mathbb{T}_{\text{dis}} \mathfrak{m} \rrbracket + \sigma v^- J \mathbf{G} \mathfrak{m} \dot{\mathbf{n}} + \llbracket v \rrbracket J^2 \quad (8.6)$$

of the linear momentum balance;

(v) the director momentum balance

$$\frac{\partial \Psi}{\partial \mathbf{G}} \mathfrak{m} + \sigma(V - \mathbf{u}^- \cdot \mathfrak{m}) \dot{\mathbf{n}} = -\frac{d\psi}{d\xi} (\mathfrak{m} - \xi \mathbf{n}); \quad (8.7)$$

(vi) and the normal configurational balance

$$\langle\langle v \rangle\rangle \mathfrak{m} \cdot \text{div}_s \mathbb{C} + \llbracket v(\Psi + p - \mathfrak{m} \cdot \mathbb{T}_{\text{dis}} \mathfrak{m}) \rrbracket + \langle\langle v \rangle\rangle \llbracket v \rrbracket J^2 + \frac{1}{2} \sigma v^- |\dot{\mathbf{n}}|^2 + \sigma(v^-)^2 J \dot{\mathbf{n}} \cdot \mathfrak{m} = \lambda J. \quad (8.8)$$

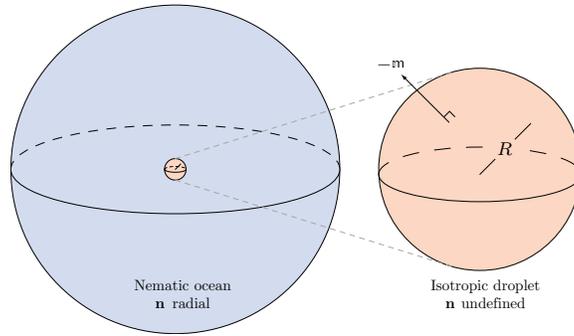


Figure 2: Schematic of an isotropic drop of radius R in a nematic ocean with radial director field. The interfacial unit normal \mathbf{m} is directed outward from the nematic phase toward the origin.

In (8.5), (8.6), and (8.8), the limits $\mathbf{T}_{\text{dis}}^-$ and $\mathbf{T}_{\text{dis}}^+$ are determined from (2.5)₁ and (4.1)₂, respectively. Further, Ψ and $\partial\Psi/\partial\mathbf{G}$ are determined from (2.4). Also, since Ψ measures the free-energy density of the nematic phase relative to that of the isotropic phase, it follows that, in (8.8), $[[v\Psi]] = -v^-\Psi$. Moreover, in (8.5) and (8.6), \mathbb{T}_{dis} is as given in (7.8).

When the densities of the two phases coincide, so that $[[v]] = 0$, (8.3) and (8.4) are replaced by $[[\mathbf{u}]] = \mathbf{0}$ and $J = \varrho(V - \mathbf{u} \cdot \mathbf{m})$, the tangential component of the linear momentum balance is unchanged from (8.5), and the normal component (8.6) of the linear momentum balance, the director momentum balance (8.7), and the combined normal momentum balance (8.8) admit obvious specializations.

9 Radial symmetry: Isotropic inclusion in a nematic ocean

In the absence of flow, CERMELLI, FRIED & GURTIN [5] considered the radially symmetric problem of an isotropic spherical drop growing (or receding) within a nematic ocean in which the director field is purely radial. Here, we reconsider that problem accounting for flow.

In the isotropic phase, the mesogens that make up a nematic liquid-crystalline fluid are randomly aligned. On the other hand, in the nematic phase, the mesogens display orientational order. Assuming that a closer packing of the mesogens accompanies such order, one expects the specific volume of the nematic phase to be slightly less than that of the isotropic phase. This expectation is confirmed by experiment and theory.⁹ We therefore assume that

$$0 < \frac{[[v]]}{v^+} \ll 1. \quad (9.1)$$

For simplicity, we neglect all external forces.

9.1 Kinematics

As in our previous work, we consider an isotropic spherical drop, with time-dependent radius R , surrounded by a nematic ocean with purely radial director field (Figure 2). Then

$$\mathbf{n}(\mathbf{x}, t) = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad |\mathbf{x}| \geq R(t). \quad (9.2)$$

In addition, we assume that the velocity field has the particular form

$$\mathbf{u}(\mathbf{x}, t) = \begin{cases} \mathbf{0}, & |\mathbf{x}| < R(t), \\ \frac{Q(t)\mathbf{x}}{|\mathbf{x}|^3}, & |\mathbf{x}| \geq R(t), \end{cases} \quad (9.3)$$

⁹For para-azoxyanisole (PAA), MAIER & SAUPE (1960) report that $[[v]]/v^+ \approx 0.0036$. For methoxybenzylidene butylaniline (MBBA), PRESS & ARROTT [21] report that $[[v]]/v^+ \approx 0.0011$. For 4'-pentyl-4-biphenylcarbonitrile (5CB), TINTARU, MOLDOVAN, BEICA & FRUNZA [24] report that $\varrho^- \approx 1.0239 \text{ g/cm}^3$ and $\varrho^+ \approx 1.0214 \text{ g/cm}^3$, whereby $[[v]]/v^+ \approx 0.0025$. Moreover, the molecular-statistical theory of MAIER & SAUPE [18] predicts values of $[[v]]/v^+$ consistent with those obtained by experiment.

so that the isotropic drop is quiescent and a purely radial flow prevails in the nematic phase.

By (9.2), it follows that

$$\mathbf{G}(\mathbf{x}, t) = \frac{1}{|\mathbf{x}|} \left\{ \mathbf{1} - \mathbf{n}(\mathbf{x}, t) \otimes \mathbf{n}(\mathbf{x}, t) \right\}, \quad |\mathbf{x}| \geq R(t), \quad (9.4)$$

and

$$\dot{\mathbf{n}}(\mathbf{x}, t) = \ddot{\mathbf{n}}(\mathbf{x}, t) = \mathbf{0}, \quad |\mathbf{x}| \geq R(t). \quad (9.5)$$

Also, by (9.3),

$$\dot{\mathbf{u}}(\mathbf{x}, t) = \begin{cases} \mathbf{0}, & |\mathbf{x}| < R(t), \\ \left\{ \frac{\dot{Q}(t)}{|\mathbf{x}|^2} - \frac{2Q^2(t)}{|\mathbf{x}|^5} \right\} \frac{\mathbf{x}}{|\mathbf{x}|}, & |\mathbf{x}| \geq R(t), \end{cases} \quad (9.6)$$

and

$$\mathbf{L}(\mathbf{x}, t) = \mathbf{D}(\mathbf{x}, t) = \begin{cases} \mathbf{0}, & |\mathbf{x}| < R(t), \\ \frac{Q(t)}{|\mathbf{x}|^3} \left\{ \mathbf{1} - 3 \frac{\mathbf{x}}{|\mathbf{x}|} \otimes \frac{\mathbf{x}}{|\mathbf{x}|} \right\}, & |\mathbf{x}| \geq R(t). \end{cases} \quad (9.7)$$

Moreover, consistent the convention that \mathfrak{m} is directed into the isotropic phase, we have

$$\mathfrak{m}(\mathbf{x}, t) = -\frac{\mathbf{x}}{R(t)}, \quad \mathbb{K}(\mathbf{x}, t) = \frac{1}{R(t)} \mathbb{P}(\mathbf{x}, t), \quad K(\mathbf{x}, t) = \frac{2}{R(t)}, \quad V(\mathbf{x}, t) = -\dot{R}(t), \quad \dot{\mathfrak{m}}(\mathbf{x}, t) = \mathbf{0}, \quad (9.8)$$

on the isotropic-nematic interface $|\mathbf{x}| = R(t)$. Further,

$$\mathbf{n}(\mathbf{x}, t) = -\mathfrak{m}(\mathbf{x}, t) = \frac{\mathbf{x}}{R(t)} \quad (9.9)$$

and, bearing in mind (9.4), (9.5), (9.8)₅, and (9.9),

$$\mathbf{G}(\mathbf{x}, t) = \frac{1}{R(t)} \mathbb{P}(\mathbf{x}, t) \quad \text{and} \quad \dot{\mathbf{n}}(\mathbf{x}, t) = \dot{\mathfrak{m}}(\mathbf{x}, t) = \mathbf{0} \quad (9.10)$$

on $|\mathbf{x}| = R(t)$. Also, by (5.8), (5.13), (9.2), (9.3), and (9.10)₂,

$$\left. \begin{aligned} \mathbf{u}^+(\mathbf{x}, t) \cdot \mathfrak{m}(\mathbf{x}, t) &= 0, & \mathbf{u}^-(\mathbf{x}, t) \cdot \mathfrak{m}(\mathbf{x}, t) &= -\frac{Q(t)}{R^2(t)}, \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{0}, & \mathbf{v}(\mathbf{x}, t) &= \frac{\dot{R}(t)\mathbf{x}}{R(t)}, \end{aligned} \right\} \quad (9.11)$$

where \mathbf{v} is the materially intrinsic velocity-field, while, by (5.19), (5.20)₁, (5.23), (9.7), (9.9), and (9.10),

$$\mathbb{L}(\mathbf{x}, t) = \mathbb{D}(\mathbf{x}, t) = \frac{Q(t)}{|\mathbf{x}|^3} \mathbb{P}(\mathbf{x}, t), \quad \mathbf{w}(\mathbf{x}, t) = \mathbf{0}, \quad \text{and} \quad \xi(\mathbf{x}, t) = -1 \quad (9.12)$$

on $|\mathbf{x}| = R(t)$.

9.2 Bulk results

Granted that \mathbf{u} is as defined in (9.3), the constraint $\text{div} \mathbf{u} = 0$ of incompressibility is satisfied throughout both of the bulk phases.

Next, for the isotropic phase, it follows from (4.1) and (9.7) that $\mathbf{T}_{\text{dis}} = \mathbf{0}$ for $|\mathbf{x}| < R(t)$. Thus, bearing in mind (9.6), the standard momentum balance (4.2)₂ implies that $\text{grad} p = \mathbf{0}$ in the isotropic phase and, hence, that

$$p(\mathbf{x}, t) = p^+(t), \quad |\mathbf{x}| < R(t). \quad (9.13)$$

The pressure in the droplet is therefore uniform.

Next, for on the nematic phase, (9.2) and (9.4) imply that the free-energy density (2.4) specializes to

$$\Psi(\mathbf{x}, t) = \hat{\Psi}(\mathbf{n}(\mathbf{x}, t), \mathbf{G}(\mathbf{x}, t)) = \Psi_0 + \frac{\kappa}{|\mathbf{x}|^2}, \quad |\mathbf{x}| \geq R(t), \quad (9.14)$$

where we have introduced

$$\kappa = 2k_1 - (k_2 + k_4) \geq 0. \quad (9.15)$$

Further, direct calculations show that

$$\frac{\partial \hat{\Psi}(\mathbf{n}(\mathbf{x}, t), \mathbf{G}(\mathbf{x}, t))}{\partial \mathbf{G}} = \frac{\kappa}{|\mathbf{x}|} \left\{ \mathbf{1} - \mathbf{n}(\mathbf{x}, t) \otimes \mathbf{n}(\mathbf{x}, t) \right\}, \quad |\mathbf{x}| \geq R(t), \quad (9.16)$$

and

$$\frac{\partial \hat{\Psi}(\mathbf{n}(\mathbf{x}, t), \mathbf{G}(\mathbf{x}, t))}{\partial \mathbf{n}} = \mathbf{0}, \quad |\mathbf{x}| \geq R(t). \quad (9.17)$$

As a consequence of (9.4), (9.14), and (9.16), we have the important identity

$$\operatorname{div} \left\{ \mathbf{G}^\top(\mathbf{x}, t) \frac{\partial \hat{\Psi}(\mathbf{n}(\mathbf{x}, t), \mathbf{G}(\mathbf{x}, t))}{\partial \mathbf{G}} \right\} = \operatorname{grad} \Psi(\mathbf{x}, t), \quad |\mathbf{x}| \geq R(t). \quad (9.18)$$

Further, using (9.5) and (9.7) in (2.5), we find that

$$\mathbf{T}_{\text{dis}} = \frac{2(\mu_1 + \mu_4)Q(t)}{|\mathbf{x}|^3} \left\{ \mathbf{1} - 3 \frac{\mathbf{x}}{|\mathbf{x}|} \otimes \frac{\mathbf{x}}{|\mathbf{x}|} \right\}, \quad |\mathbf{x}| \geq R(t), \quad (9.19)$$

and

$$\mathbf{g}_{\text{dis}}(\mathbf{x}, t) = \mathbf{0}, \quad |\mathbf{x}| \geq R(t). \quad (9.20)$$

In view of (9.19), it follows that

$$\operatorname{div} \mathbf{T}_{\text{dis}}(\mathbf{x}, t) = \mathbf{0}, \quad |\mathbf{x}| \geq R(t). \quad (9.21)$$

By (9.6), (9.14), (9.18), and (9.21), the standard momentum balance (2.3)₁ implies that $\varrho^- \dot{\mathbf{u}} = -\operatorname{grad}(p + \Psi)$ in the nematic phase and, hence, bearing in mind (9.6), that

$$p(\mathbf{x}, t) = p_\infty - \frac{\kappa}{|\mathbf{x}|^2} + \frac{\varrho^-}{|\mathbf{x}|} \left\{ \dot{Q}(t) - \frac{Q^2(t)}{2|\mathbf{x}|^3} \right\}, \quad |\mathbf{x}| \geq R(t), \quad (9.22)$$

with p_∞ being the (constant) pressure in the far field. In contrast to the situation in the isotropic phase, the pressure in the nematic phase is therefore non-uniform.

Finally, using (9.4), (9.5), (9.16), and (9.17) in (2.3)₂, we find that the director momentum balance is satisfied identically on $|\mathbf{x}| > R(t)$ and, thus, throughout the nematic phase.

9.3 Interfacial mass balance

Using (9.8)₁ and (9.11)_{1,2} in the interfacial mass balance (6.1), we obtain

$$Q = -\frac{[[\varrho]]R^2\dot{R}}{\varrho^-} = \frac{[[v]]R^2\dot{R}}{v^+}, \quad (9.23)$$

which determines Q in terms of the droplet radius R . Using (9.23) in (9.11)₂ and recalling from (9.3) that \mathbf{u} is radial, we obtain

$$\mathbf{u}^- = -\frac{[[v]]\dot{R}}{v^+} \mathbf{m} \quad (9.24)$$

and thus conclude that a liquid particle residing instantaneously on a phase interface that is moving outward must therefore move outward radially with scalar normal velocity $-[[v]]\dot{R}/v^+$ (since \mathbf{m} is directed into the inclusion, the normal velocity is negative for an outward motion). Analogous to what occurs (CERMELLI & FRIED [4]) for a migrating disclination, a backflow therefore accompanies the motion of the phase interface. This suggests that an indirect measurement of the ratio $[[v]]/v^+$ and, thus, of the density difference between the phases might be achieved as a by-product of a backflow measurement. Moreover, by (6.3)₂, the mass flow across the interface is given by

$$J = -\frac{\dot{R}}{v^+}. \quad (9.25)$$

9.4 Interfacial linear momentum balance

Using (9.23) in (9.12)₁ to give

$$\mathbb{D} = \frac{\llbracket v \rrbracket \dot{R}}{v^+ R} \mathbb{P} \quad (9.26)$$

and bearing in mind (9.9) and (9.12)₂, we find from (7.7) and (7.8) that, for the purely radial problem under consideration, the interfacial Cauchy stress reduces to a pure tension of the form

$$\mathbb{T} = \left\{ \sigma + \frac{2\llbracket v \rrbracket (\lambda_1 + \lambda_5) \dot{R}}{v^+ R} \right\} \mathbb{P}, \quad (9.27)$$

with

$$\gamma_0 = \psi(-1). \quad (9.28)$$

Moreover, using (9.23) in the expressions (9.19) and (9.22) for the dissipative stress and pressure in the nematic phase, we find that their interfacial limits are given by

$$\mathbf{T}_{\text{dis}}^- = \frac{2\llbracket v \rrbracket (\mu_1 + \mu_4) \dot{R}}{v^+ R} \left\{ \mathbf{1} - 3\mathbf{m} \otimes \mathbf{m} \right\} \quad (9.29)$$

and

$$p^- = p_\infty - \frac{\kappa}{R^2} + \frac{\llbracket v \rrbracket}{v^+ v^-} \left\{ R \ddot{R} + \left(2 - \frac{\llbracket v \rrbracket}{2v^+} \right) \dot{R}^2 \right\}. \quad (9.30)$$

Further, since $\llbracket (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{p} \rrbracket = J \llbracket \mathbf{u} \rrbracket$, it follows from (9.11)_{1,2}, (9.24), and (9.25) that

$$\llbracket (V - \mathbf{u} \cdot \mathbf{m}) \mathbf{p} \rrbracket = - \frac{\llbracket v \rrbracket \dot{R}^2}{v^+ v^+} \quad (9.31)$$

Using (9.27), (9.29), (9.30), and (9.31) in the tangential and normal components (8.5) and (8.6) of the interfacial linear momentum balance, recalling $\mathbf{T}_{\text{dis}} = \mathbf{0}$ in the isotropic phase, and noting, from (5.6) and (9.8)₃, that $\text{div}_s \mathbb{P} = K \mathbf{m} = 2\mathbf{m}/R$, we find that (8.5) is satisfied trivially and that (8.6) yields an expression

$$p^+ = p_\infty + \frac{2\gamma_0}{R} - \frac{\kappa}{R^2} + \frac{\llbracket v \rrbracket}{v^+ v^-} \left\{ R \ddot{R} + \left(2 - \frac{\llbracket v \rrbracket}{2v^+} \right) \dot{R}^2 \right\} + \frac{4\llbracket v \rrbracket (\mu_1 + \mu_4) \dot{R}}{v^+ R} - \frac{\llbracket v \rrbracket \dot{R}^2}{v^+ v^+} + \frac{4\llbracket v \rrbracket (\lambda_1 + \lambda_5) \dot{R}}{v^+ R^2} \quad (9.32)$$

for the uniform pressure p^+ in the isotropic phase in terms of the droplet radius.

9.5 Interfacial director momentum balance

By (9.9) and (9.12)₃,

$$\frac{d\psi}{d\xi} (\mathbf{m} - \xi \mathbf{n}) = \mathbf{0}. \quad (9.33)$$

Further, evaluating (9.16) on the interface and using (9.10)₁, we find that

$$\frac{\partial \Psi}{\partial \mathbf{G}} \mathbf{m} = \mathbf{0} \quad (9.34)$$

on the interface. Also, by (9.10)₂,

$$\sigma (V - \mathbf{u}^- \cdot \mathbf{m}) \dot{\mathbf{n}} = \mathbf{0} \quad (9.35)$$

on the interface. In view of (9.33), (9.34), and (9.35), the interfacial director momentum balance (8.7) is satisfied identically.

9.6 Normal configurational balance

Using (9.9), (9.10)₂, (9.12)_{2,3}, and (9.26) in (7.4)₂, we find that $\mathbb{P}\mathbf{g} = \mathbf{0}$ and that $\xi\mathbf{g} = (\mathbf{g}\cdot\mathbf{m})\mathbf{n}$ and, thus, from (6.32), (7.7), and (7.11), that

$$\mathbb{C} = \mathbb{C}_{\tan} = -\frac{2\llbracket v \rrbracket(\lambda_1 + \lambda_5)\dot{R}}{v^+R} \mathbb{P}. \quad (9.36)$$

Thus, by (5.6), (9.14), (9.24), (9.25), (9.29), and (9.30), the normal combined momentum balance (6.37) specializes to

$$p^+ = \frac{v^-}{v^+} \left\{ \Psi_0 + p_\infty + \frac{\llbracket v \rrbracket}{v^+v^-} \left\{ R\ddot{R} + \left(2 - \frac{\llbracket v \rrbracket}{2v^+} \right) \dot{R}^2 \right\} + \frac{4\llbracket v \rrbracket(\mu_1 + \mu_4)\dot{R}}{v^+R} \right\} \\ + \frac{\langle\langle v \rangle\rangle \llbracket v \rrbracket \dot{R}}{v^+v^+} \left\{ \frac{4(\lambda_1 + \lambda_5)}{R^2} - \frac{\dot{R}}{v^+} \right\} - \frac{\lambda_9 \dot{R}}{v^+v^+}, \quad (9.37)$$

which, like (9.32), provides an expression for p^+ in terms of R .

9.7 Droplet evolution equation

Eliminating p^+ between (9.32) and (9.37), we arrive at a second-order nonlinear differential equation

$$\frac{\llbracket v \rrbracket^2}{v^+v^-} \left\{ R\ddot{R} + \frac{3\dot{R}^2}{2} + \frac{4v^-(\mu_1 + \mu_4)\dot{R}}{R} + \frac{2v^-(\lambda_1 + \lambda_5)\dot{R}}{R^2} \right\} + \frac{\lambda_9 \dot{R}}{v^+} = v^- \Psi_0 - \llbracket v \rrbracket p_\infty - v^+ \left\{ \frac{2\gamma_0}{R} - \frac{\kappa}{R^2} \right\}. \quad (9.38)$$

When $\llbracket v \rrbracket = 0$, this equation reduces to an equation with structure identical to that of the equation derived and studied by Cermelli, Fried & Gurtin [5] for the droplet evolution problem without flow.

Introducing the dimensionless variables

$$r = R/L \quad \text{and} \quad \tau = t/T,$$

with $L > 0$ a characteristic length and $T > 0$ a characteristic time, and letting

$$\epsilon = \frac{\llbracket v \rrbracket}{v^+}, \quad (9.39)$$

we may rewrite (9.38) in the simpler form

$$\beta_* \dot{r} = \Psi_* - \frac{2\gamma_*}{r} + \frac{\kappa_*}{r^2} - \epsilon(1 + \Psi_*) - \epsilon^2 \left\{ \alpha_* \left(r\ddot{r} + \frac{3\dot{r}^2}{2} \right) + \frac{4\mu_* \dot{r}}{r} + \frac{2\lambda_* \dot{r}}{r^2} \right\}, \quad (9.40)$$

where the superposed dot now stands for differentiation with respect to τ and the parameters α_* , μ_* , λ_* , β_* , Ψ_* , γ_* , and κ_* are defined by

$$\left. \begin{aligned} \alpha_* &= \frac{L^2}{v^- p_\infty T^2}, & \mu_* &= \frac{\mu_1 + \mu_4}{p_\infty T}, & \lambda_* &= \frac{\lambda_1 + \lambda_5}{p_\infty L T}, \\ \beta_* &= \frac{\lambda_9 L}{v^+ v^+ p_\infty T}, & \Psi_* &= \frac{v^- \Psi_0}{p_\infty v^+}, & \gamma_* &= \frac{\gamma_0}{p_\infty L}, & \kappa_* &= \frac{\kappa}{p_\infty L^2}. \end{aligned} \right\} \quad (9.41)$$

We now study equation (9.40) in two steps:

Case (a). Assume that the densities of the nematic and isotropic phases are almost coincident, so that

$$\llbracket v \rrbracket \ll v^+,$$

and neglect the term multiplying ϵ^2 in (9.40): the evolution equation for the interface then reduces to

$$\beta_* \dot{r} = \pi_* - \frac{2\gamma_*}{r} + \frac{\kappa_*}{r^2}, \quad (9.42)$$

with

$$\pi_* = \Psi_* - \epsilon(1 + \Psi_*).$$

We have three regimes:

- When $\pi_* < 0$, i.e. when $p_\infty > v^- \Psi_0 / \llbracket v \rrbracket$, there is a stable radius for the isotropic inclusion, given by the expression

$$r_0 = \frac{\sqrt{\gamma_*^2 + \kappa_* |\pi_*|} - \gamma_*}{|\pi_*|}.$$

Note the competition between the externally applied pressure p_∞ and the free energy difference Ψ_0 . When Ψ_0 is negative, the energy of the nematic phase is lower energy than that of isotropic phase and the drop should grow. On the other hand, the presence of an external pressure stabilizes finite-radius drops, and, since r_0 is a decreasing function of $|\pi_*|$, it follows that an increase of pressure tends to decrease the droplet radius. However, the droplet radius never vanishes because, in the limit as r_0 approaches zero, the droplet becomes a hedgehog defect and the associated defect energy must be taken into account. Moreover, note that for Ψ_0 positive high pressures stabilize a droplet of the isotropic phase surrounded by the nematic phase.

- When $0 < \pi_* < \gamma_*^2 / \kappa_*$, i.e. when $(v^- \Psi_0 - v^+ \gamma_*^2) / \llbracket v \rrbracket \kappa_* < p_\infty < v^- \Psi_* / \llbracket v \rrbracket$, there are two equilibrium radii for the inclusion:

$$r_{0,1} = \frac{\gamma_* \pm \sqrt{\gamma_*^2 - \kappa_* \pi_*}}{\pi_*}.$$

The smallest of these radii is stable (since defect energy prohibits the disappearance of the droplet) but the largest is unstable. Hence, sufficiently large droplets grow without bound and the liquid becomes eventually isotropic.

- Finally, when $\gamma_*^2 / \kappa_* < \pi_*$, i.e. when $p_\infty < (v^- \Psi_0 - v^+ \gamma_*^2) / \llbracket v \rrbracket \kappa_*$, there is no equilibrium radius and the isotropic droplet grows without bound.

Case (b). Consider now the full nonlinear ordinary differential equation (9.40). The equilibria of this equation are the same as those of (9.42).

Let r_{eq} be a generic equilibrium radius for the inclusion, as determined in case (a), and linearize (9.40) in a neighborhood of r_{eq} , letting $Y = r - r_{\text{eq}}$. We obtain a linear second order problem of the form

$$a_1 \ddot{Y} + a_2 \dot{Y} + a_3 Y = 0, \quad (9.43)$$

with

$$a_1 = \epsilon^2 \alpha_* r_{\text{eq}}, \quad a_2 = \epsilon^2 \left(\frac{4\mu_*}{r_{\text{eq}}} + \frac{2\lambda_*}{r_{\text{eq}}^2} + \beta_* \right), \quad a_3 = -\frac{2\gamma_*}{r_{\text{eq}}^2} + \frac{2\kappa_*}{r_{\text{eq}}^3}.$$

The stability of the equilibrium solution may be discussed by studying the sign of the real part of the solution of the characteristic equation associated with (9.43). There are three possibilities:

- When $a_3 < 0$, i.e. when the equilibrium radius r_{eq} is unstable for the reduced equation (9.42), we have

$$\frac{-a_2 + \sqrt{a_2^2 + 4a_1|a_3|}}{2a_1} > 0,$$

so that $Y = 0$, i.e. $r = r_{\text{eq}}$, is unstable also for the fully nonlinear equation (9.40).

- When $0 < a_3 < a_2^2 / 4a_1$, i.e. when the equilibrium radius r_{eq} is stable for the reduced equation (9.42), we have

$$\frac{-a_2 \pm \sqrt{a_2^2 + 4a_1|a_3|}}{2a_1} < 0,$$

so that $Y = 0$, i.e. when $r = r_{\text{eq}}$, is stable also for the fully nonlinear equation (9.40).

- When $a_2^2 / 4a_1 < a_3$, again r_{eq} is stable for the reduced equation (9.42) and we have

$$\frac{-a_2 \pm i\sqrt{4a_1a_3 - a_2^2}}{2a_1},$$

which is complex with negative real part. Thus, again, $r = r_{\text{eq}}$ is stable for the fully nonlinear equation (9.40), but now the solution develops exponentially decaying oscillations near equilibrium.

For the droplet problem in the absence of flow, CERMELLI, FRIED & GURTIN [5] observed the existence of different time scales characterizing the evolution of the droplet. In particular, focusing on the case where the isotropic-to-nematic transition is favored, it was found that immediately subsequent to nucleation the droplet grows with the cube root of time. This is followed by a regime in which the droplet grows like the square root of time. Finally, for sufficiently late stages of growth, the droplet grow linearly with time. Due to the presence of the effects associated with inertia, the viscosities of the nematic phase, and the dilatational viscosity of the interface, we expect the more general evolution equation to display even more time scales. We leave the study of these effects for a future work.

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References

- [1] D. ARMITAGE, F.P. PRICE, Supercooling and nucleation in liquid crystals, *Molecular Crystals and Liquid Crystals* **44** (1978) 33–44.
- [2] D. R. ANDERSON, D. E. CARLSON & E. FRIED, A continuum-mechanical theory for nematic elastomers, *Journal of Elasticity* **56** (1999) 33–58.
- [3] P. CERMELLI & E. FRIED, The influence of inertia on configurational forces in a deformable solid, *Proceedings of the Royal Society of London A* **458** (1997), 1915–1927
- [4] P. CERMELLI & E. FRIED, The evolution equation for a disclination in a nematic liquid crystal, *Proceedings of the Royal Society of London A* **458** (2002) 1–20.
- [5] P. CERMELLI, E. FRIED & M. E. GURTIN, Sharp-interface nematic-isotropic phase transitions without flow, *Archive for Rational Mechanics and Analysis* **174** (2004) 151–178.
- [6] P. CERMELLI, E. FRIED & M. E. GURTIN, Transport relations for surface integrals arising in the formulation of balance laws for evolving fluid interfaces, *J. Fluid Mech.* **544** (2005) 339–351.
- [7] Y.-C. CHEN & E. FRIED, Uniaxial nematic elastomers: constitutive framework and a simple application, *Proceedings of the Royal Society of London A* **462** (2006) 1295–1314.
- [8] A.-G. CHEONG & A. D. REY, Cahn-Hoffman capillarity vector thermodynamics for curved liquid crystal interfaces with application to fiber instabilities, *Journal of Chemical Physics* **117** (2002) 5062–5071.
- [9] A. I. DERZHANSKI & A. G. PETROV, Flexoelectricity in nematic liquid crystals, *Acta Physica Polonica A* **55** (1979) 747–767.
- [10] J. L. ERICKSEN, Anisotropic fluids, *Archive for Rational Mechanics and Analysis* **4** (1960) 231–237.
- [11] J. L. ERICKSEN, Conservation laws for liquid crystals, *Transactions of the Society of Rheology* **5** (1961) 23–34.
- [12] J. L. ERICKSEN, Equilibrium theory of liquid crystals, in *Advances in Liquid Crystals* **2** (1976) 233–289.
- [13] F. C. FRANK, On the theory of liquid crystals, *Discussions of the Faraday Society* **25** (1958) 19–28.
- [14] M. E. GURTIN, The nature of configurational forces, *Archive for Rational Mechanics and Analysis* **131** (1995) 67–100.
- [15] M. E. GURTIN, *Configurational Forces as Basic Concepts of Continuum Physics*, Springer-Verlag, Berlin, 1999.
- [16] F. M. LESLIE, Some constitutive equations for liquid crystals, *Archive for Rational Mechanics and Analysis* **28** (1968) 265–283.
- [17] F. M. LESLIE, Theory of flow phenomena for liquid crystals, *Advances in Liquid Crystals* **4** (1979) 1–81.
- [18] W. MAIER & A. SAUPE, Eine einfache molekular-statistische Theorie der nematischen kristallinflüssigen Phase. Teil II, *Zeitschrift für Naturforschung* **15a** (1960), 287–292.
- [19] W. C. OSEEN, The theory of liquid crystals, *Transactions of the Faraday Society* **29** (1933) 883–899.
- [20] A. PONIEWIERSKI, Shape of the nematic-isotropic interface in conditions of partial wetting, *Liquid Crystals* **27** (2000) 1369–1380.
- [21] M. J. PRESS & A. S. ARROTT, Expansion coefficient of methoxybenzylidene butylaniline through the liquid-crystal phase transition, *Physical Review A* **8** (1973) 1459–1465.
- [22] A. D. REY, Viscoelastic theory for nematic interfaces. *Physical Review E* **61** (2000) 1540–1549.
- [23] A. D. REY, Theory of interfacial dynamics of nematic polymers. *Rheologica Acta* **39** (2000) 13–19.
- [24] M. TINTARU, R. MOLDOVAN, T. BEICA & S. FRUNZA, Surface tension of some liquid crystals in the cynaobiphenyl series, *Liquid Crystals* **28** (2001) 793–797.
- [25] H. ZÖCHER, The effect of a magnetic field on the nematic state, *Transactions of the Faraday Society* **29** (1933) 945–957.

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1014	Brown, E. N., M. R. Kessler, N. R. Sottos, and S. R. White	<i>In situ</i> poly(urea-formaldehyde) microencapsulation of dicyclopentadiene – <i>Journal of Microencapsulation</i> (submitted)	Feb. 2003
1015	Brown, E. N., S. R. White, and N. R. Sottos	Microcapsule induced toughening in a self-healing polymer composite – <i>Journal of Materials Science</i> (submitted)	Feb. 2003
1016	Kuznetsov, I. R., and D. S. Stewart	Burning rate of energetic materials with thermal expansion – <i>Combustion and Flame</i> (submitted)	Mar. 2003
1017	Dolbow, J., E. Fried, and H. Ji	Chemically induced swelling of hydrogels – <i>Journal of the Mechanics and Physics of Solids</i> , in press (2003)	Mar. 2003
1018	Costello, G. A.	Mechanics of wire rope – Mordica Lecture, Interwire 2003, Wire Association International, Atlanta, Georgia, May 12, 2003	Mar. 2003
1019	Wang, J., N. R. Sottos, and R. L. Weaver	Thin film adhesion measurement by laser induced stress waves – <i>Journal of the Mechanics and Physics of Solids</i> (submitted)	Apr. 2003
1020	Bhattacharjee, P., and D. N. Riahi	Effect of rotation on surface tension driven flow during protein crystallization – <i>Microgravity Science and Technology</i> 14 , 36–44 (2003)	Apr. 2003
1021	Fried, E.	The configurational and standard force balances are not always statements of a single law – <i>SIAM Journal on Applied Mathematics</i> 66 , 1130–1149 (2006)	Apr. 2003
1022	Panat, R. P., and K. J. Hsia	Experimental investigation of the bond coat rumpling instability under isothermal and cyclic thermal histories in thermal barrier systems – <i>Proceedings of the Royal Society of London A</i> 460 , 1957–1979 (2003)	May 2003
1023	Fried, E., and M. E. Gurtin	A unified treatment of evolving interfaces accounting for small deformations and atomic transport: grain-boundaries, phase transitions, epitaxy – <i>Advances in Applied Mechanics</i> 40 , 1–177 (2004)	May 2003
1024	Dong, F., D. N. Riahi, and A. T. Hsui	On similarity waves in compacting media – <i>Horizons in World Physics</i> 244 , 45–82 (2004)	May 2003
1025	Liu, M., and K. J. Hsia	Locking of electric field induced non-180° domain switching and phase transition in ferroelectric materials upon cyclic electric fatigue – <i>Applied Physics Letters</i> 83 , 3978–3980 (2003)	May 2003
1026	Liu, M., K. J. Hsia, and M. Sardela Jr.	In situ X-ray diffraction study of electric field induced domain switching and phase transition in PZT-5H – <i>Journal of the American Ceramics Society</i> (submitted)	May 2003
1027	Riahi, D. N.	On flow of binary alloys during crystal growth – <i>Recent Research Development in Crystal Growth</i> 3 , 49–59 (2003)	May 2003
1028	Riahi, D. N.	On fluid dynamics during crystallization – <i>Recent Research Development in Fluid Dynamics</i> 4 , 87–94 (2003)	July 2003
1029	Fried, E., V. Korchagin, and R. E. Todres	Biaxial disclinated states in nematic elastomers – <i>Journal of Chemical Physics</i> 119 , 13170–13179 (2003)	July 2003
1030	Sharp, K. V., and R. J. Adrian	Transition from laminar to turbulent flow in liquid filled microtubes – <i>Physics of Fluids</i> (submitted)	July 2003

List of Recent TAM Reports (cont'd)

No.	Authors	Title	Date
1031	Yoon, H. S., D. F. Hill, S. Balachandar, R. J. Adrian, and M. Y. Ha	Reynolds number scaling of flow in a Rushton turbine stirred tank: Part I—Mean flow, circular jet and tip vortex scaling— <i>Chemical Engineering Science</i> (submitted)	Aug. 2003
1032	Raju, R., S. Balachandar, D. F. Hill, and R. J. Adrian	Reynolds number scaling of flow in a Rushton turbine stirred tank: Part II—Eigen-decomposition of fluctuation— <i>Chemical Engineering Science</i> (submitted)	Aug. 2003
1033	Hill, K. M., G. Gioia, and V. V. Tota	Structure and kinematics in dense free-surface granular flow— <i>Physical Review Letters</i> 91 , 064302 (2003)	Aug. 2003
1034	Fried, E., and S. Sellers	Free-energy density functions for nematic elastomers— <i>Journal of the Mechanics and Physics of Solids</i> 52 , 1671–1689 (2004)	Sept. 2003
1035	Kasimov, A. R., and D. S. Stewart	On the dynamics of self-sustained one-dimensional detonations: A numerical study in the shock-attached frame— <i>Physics of Fluids</i> (submitted)	Nov. 2003
1036	Fried, E., and B. C. Roy	Disclinations in a homogeneously deformed nematic elastomer— <i>Nature Materials</i> (submitted)	Nov. 2003
1037	Fried, E., and M. E. Gurtin	The unifying nature of the configurational force balance— <i>Mechanics of Material Forces</i> (P. Steinmann and G. A. Maugin, eds.), 25–32 (2005)	Dec. 2003
1038	Panat, R., K. J. Hsia, and J. W. Oldham	Rumpling instability in thermal barrier systems under isothermal conditions in vacuum— <i>Philosophical Magazine</i> , in press (2004)	Dec. 2003
1039	Cermelli, P., E. Fried, and M. E. Gurtin	Sharp-interface nematic–isotropic phase transitions without flow— <i>Archive for Rational Mechanics and Analysis</i> 174 , 151–178 (2004)	Dec. 2003
1040	Yoo, S., and D. S. Stewart	A hybrid level-set method in two and three dimensions for modeling detonation and combustion problems in complex geometries— <i>Combustion Theory and Modeling</i> (submitted)	Feb. 2004
1041	Dienberg, C. E., S. E. Ott-Monsivais, J. L. Ranchero, A. A. Rzeszutko, and C. L. Winter	Proceedings of the Fifth Annual Research Conference in Mechanics (April 2003), TAM Department, UIUC (E. N. Brown, ed.)	Feb. 2004
1042	Kasimov, A. R., and D. S. Stewart	Asymptotic theory of ignition and failure of self-sustained detonations— <i>Journal of Fluid Mechanics</i> (submitted)	Feb. 2004
1043	Kasimov, A. R., and D. S. Stewart	Theory of direct initiation of gaseous detonations and comparison with experiment— <i>Proceedings of the Combustion Institute</i> (submitted)	Mar. 2004
1044	Panat, R., K. J. Hsia, and D. G. Cahill	Evolution of surface waviness in thin films via volume and surface diffusion— <i>Journal of Applied Physics</i> (submitted)	Mar. 2004
1045	Riahi, D. N.	Steady and oscillatory flow in a mushy layer— <i>Current Topics in Crystal Growth Research</i> , in press (2004)	Mar. 2004
1046	Riahi, D. N.	Modeling flows in protein crystal growth— <i>Current Topics in Crystal Growth Research</i> , in press (2004)	Mar. 2004
1047	Bagchi, P., and S. Balachandar	Response of the wake of an isolated particle to isotropic turbulent cross-flow— <i>Journal of Fluid Mechanics</i> (submitted)	Mar. 2004
1048	Brown, E. N., S. R. White, and N. R. Sottos	Fatigue crack propagation in microcapsule toughened epoxy— <i>Journal of Materials Science</i> (submitted)	Apr. 2004
1049	Zeng, L., S. Balachandar, and P. Fischer	Wall-induced forces on a rigid sphere at finite Reynolds number— <i>Journal of Fluid Mechanics</i> (submitted)	May 2004
1050	Dolbow, J., E. Fried, and H. Ji	A numerical strategy for investigating the kinetic response of stimulus-responsive hydrogels— <i>Computer Methods in Applied Mechanics and Engineering</i> 194 , 4447–4480 (2005)	June 2004
1051	Riahi, D. N.	Effect of permeability on steady flow in a dendrite layer— <i>Journal of Porous Media</i> , in press (2004)	July 2004

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No.	Authors	Title	Date
1052	Cermelli, P., E. Fried, and M. E. Gurtin	Transport relations for surface integrals arising in the formulation of balance laws for evolving fluid interfaces – <i>Journal of Fluid Mechanics</i> 544 , 339–351 (2005)	Sept. 2004
1053	Stewart, D. S., and A. R. Kasimov	Theory of detonation with an embedded sonic locus – <i>SIAM Journal on Applied Mathematics</i> (submitted)	Oct. 2004
1054	Stewart, D. S., K. C. Tang, S. Yoo, M. Q. Brewster, and I. R. Kuznetsov	Multi-scale modeling of solid rocket motors: Time integration methods from computational aerodynamics applied to stable quasi-steady motor burning – <i>Proceedings of the 43rd AIAA Aerospace Sciences Meeting and Exhibit</i> (January 2005), Paper AIAA-2005-0357 (2005)	Oct. 2004
1055	Ji, H., H. Mourad, E. Fried, and J. Dolbow	Kinetics of thermally induced swelling of hydrogels – <i>International Journal of Solids and Structures</i> 43 , 1878–1907 (2006)	Dec. 2004
1056	Fulton, J. M., S. Hussain, J. H. Lai, M. E. Ly, S. A. McGough, G. M. Miller, R. Oats, L. A. Shipton, P. K. Shreeman, D. S. Widrevitz, and E. A. Zimmermann	Final reports: Mechanics of complex materials, Summer 2004 (K. M. Hill and J. W. Phillips, eds.)	Dec. 2004
1057	Hill, K. M., G. Gioia, and D. R. Amaravadi	Radial segregation patterns in rotating granular mixtures: Waviness selection – <i>Physical Review Letters</i> 93 , 224301 (2004)	Dec. 2004
1058	Riahi, D. N.	Nonlinear oscillatory convection in rotating mushy layers – <i>Journal of Fluid Mechanics</i> , in press (2005)	Dec. 2004
1059	Okhuysen, B. S., and D. N. Riahi	On buoyant convection in binary solidification – <i>Journal of Fluid Mechanics</i> (submitted)	Jan. 2005
1060	Brown, E. N., S. R. White, and N. R. Sottos	Retardation and repair of fatigue cracks in a microcapsule toughened epoxy composite – Part I: Manual infiltration – <i>Composites Science and Technology</i> (submitted)	Jan. 2005
1061	Brown, E. N., S. R. White, and N. R. Sottos	Retardation and repair of fatigue cracks in a microcapsule toughened epoxy composite – Part II: <i>In situ</i> self-healing – <i>Composites Science and Technology</i> (submitted)	Jan. 2005
1062	Berfield, T. A., R. J. Ong, D. A. Payne, and N. R. Sottos	Residual stress effects on piezoelectric response of sol-gel derived PZT thin films – <i>Journal of Applied Physics</i> (submitted)	Apr. 2005
1063	Anderson, D. M., P. Cermelli, E. Fried, M. E. Gurtin, and G. B. McFadden	General dynamical sharp-interface conditions for phase transformations in viscous heat-conducting fluids – <i>Journal of Fluid Mechanics</i> , in press (2006)	Apr. 2005
1064	Fried, E., and M. E. Gurtin	Second-gradient fluids: A theory for incompressible flows at small length scales – <i>Journal of Fluid Mechanics</i> (submitted)	Apr. 2005
1065	Gioia, G., and F. A. Bombardelli	Localized turbulent flows on scouring granular beds – <i>Physical Review Letters</i> , in press (2005)	May 2005
1066	Fried, E., and S. Sellers	Orientational order and finite strain in nematic elastomers – <i>Journal of Chemical Physics</i> 123 , 044901 (2005)	May 2005
1067	Chen, Y.-C., and E. Fried	Uniaxial nematic elastomers: Constitutive framework and a simple application – <i>Proceedings of the Royal Society of London A</i> 462 , 1295–1314 (2006)	June 2005
1068	Fried, E., and S. Sellers	Incompatible strains associated with defects in nematic elastomers – <i>Journal of Chemical Physics</i> 124 , 024908 (2006)	Aug. 2005
1069	Gioia, G., and X. Dai	Surface stress and reversing size effect in the initial yielding of ultrathin films – <i>Journal of Applied Mechanics</i> , in press (2005)	Aug. 2005
1070	Gioia, G., and P. Chakraborty	Turbulent friction in rough pipes and the energy spectrum of the phenomenological theory – <i>Physical Review Letters</i> 96 , 044502 (2006)	Aug. 2005

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1071	Keller, M. W., and N. R. Sottos	Mechanical properties of capsules used in a self-healing polymer – <i>Experimental Mechanics</i> (submitted)	Sept. 2005
1072	Chakraborty, P., G. Gioia, and S. Kieffer	Volcán Reventador's unusual umbrella	Sept. 2005
1073	Fried, E., and S. Sellers	Soft elasticity is not necessary for striping in nematic elastomers – <i>Journal of Applied Physics</i> , in press (2006)	Sept. 2005
1074	Fried, E., M. E. Gurtin, and Amy Q. Shen	Theory for solvent, momentum, and energy transfer between a surfactant solution and a vapor atmosphere – <i>Physical Review E</i> , in press (2006)	Sept. 2005
1075	Chen, X., and E. Fried	Rayleigh–Taylor problem for a liquid–liquid phase interface – <i>Journal of Fluid Mechanics</i> (submitted)	Oct. 2005
1076	Riahi, D. N.	Mathematical modeling of wind forces – In <i>The Euler Volume</i> (Abington, UK: Taylor and Francis), in press (2005)	Oct. 2005
1077	Fried, E., and R. E. Todres	Mind the gap: The shape of the free surface of a rubber-like material in the proximity to a rigid contactor – <i>Journal of Elasticity</i> 80 , 97–151 (2005)	Oct. 2005
1078	Riahi, D. N.	Nonlinear compositional convection in mushy layers – <i>Journal of Fluid Mechanics</i> (submitted)	Dec. 2005
1079	Bhattacharjee, P., and D. N. Riahi	Mathematical modeling of flow control using magnetic fluid and field – In <i>The Euler Volume</i> (Abington, UK: Taylor and Francis), in press (2005)	Dec. 2005
1080	Bhattacharjee, P., and D. N. Riahi	A hybrid level set/VOF method for the simulation of thermal magnetic fluids – <i>International Journal for Numerical Methods in Engineering</i> (submitted)	Dec. 2005
1081	Bhattacharjee, P., and D. N. Riahi	Numerical study of surface tension driven convection in thermal magnetic fluids – <i>Journal of Crystal Growth</i> (submitted)	Dec. 2005
1082	Riahi, D. N.	Inertial and Coriolis effects on oscillatory flow in a horizontal dendrite layer – <i>Transport in Porous Media</i> (submitted)	Jan. 2006
1083	Wu, Y., and K. T. Christensen	Population trends of spanwise vortices in wall turbulence – <i>Journal of Fluid Mechanics</i> (submitted)	Jan. 2006
1084	Natrajan, V. K., and K. T. Christensen	The role of coherent structures in subgrid-scale energy transfer within the log layer of wall turbulence – <i>Physics of Fluids</i> , in press (2006)	Jan. 2006
1085	Wu, Y., and K. T. Christensen	Reynolds-stress enhancement associated with a short fetch of roughness in wall turbulence – <i>AIAA Journal</i> (submitted)	Jan. 2006
1086	Fried, E., and M. E. Gurtin	Cosserat fluids and the continuum mechanics of turbulence: A generalized Navier–Stokes- α equation with complete boundary conditions – <i>Journal of Fluid Mechanics</i> (submitted)	Feb. 2006
1087	Riahi, D. N.	Inertial effects on rotating flow in a porous layer – <i>Journal of Porous Media</i> (submitted)	Feb. 2006
1088	Li, F., and D. E. Leckband	Dynamic strength of adhesion surfaces – <i>Journal of Chemical Physics</i> (submitted)	Mar. 2006
1089	Chen, X., and E. Fried	Squire's theorem for the Rayleigh–Taylor problem with a phase transformation – <i>Proceedings of the Royal Society of London A</i> (submitted)	Mar. 2006
1090	Kim, T.-Y., J. Dolbow, and E. Fried	A numerical method for a second-gradient theory of incompressible fluid flow – <i>Journal of Computational Physics</i> (submitted)	Apr. 2006
1091	Natrajan, V. K., Y. Wu, and K. T. Christensen	Spatial signatures of retrograde spanwise vortices in wall turbulence – <i>Journal of Fluid Mechanics</i> (submitted)	Apr. 2006
1092	Natrajan, V. K., E. Yamaguchi, and K. T. Christensen	Statistical and structural similarities between micro- and macro-scale wall turbulence – <i>Microfluidics and Nanofluidics</i> (submitted)	Apr. 2006
1093	Fried, E.	Sharp-interface nematic–isotropic phase transitions with flow – <i>Archive for Rational Mechanics and Analysis</i> (submitted)	June 2006