CONSTRAINED STOCHASTIC POWER FLOW ANALYSIS

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>LIST OF TABLES</th>
<th>vi</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF FIGURES</td>
<td>vii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>viii</td>
</tr>
<tr>
<td>CHAPTER 1  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Literature Summary</td>
<td>2</td>
</tr>
<tr>
<td>CHAPTER 2  THE CONstrained STOCHASTIC POWER FLOW PROBLEM</td>
<td>6</td>
</tr>
<tr>
<td>2.1 Deterministic Load Flow Solutions By Newton's Method</td>
<td>6</td>
</tr>
<tr>
<td>2.2 Voltage Controlled Bus Constraints</td>
<td>9</td>
</tr>
<tr>
<td>2.3 Linear Load Flow Methods</td>
<td>11</td>
</tr>
<tr>
<td>2.4 Unconstrained Stochastic Load Flow</td>
<td>12</td>
</tr>
<tr>
<td>CHAPTER 3  SOLUTION OF THE CONstrained STOCHASTIC POWER FLOW PROBLEM</td>
<td>15</td>
</tr>
<tr>
<td>3.1 Multiple Linear Transformations</td>
<td>15</td>
</tr>
<tr>
<td>3.2 Single Constraint Solution Algorithm</td>
<td>18</td>
</tr>
<tr>
<td>3.2.1 Fully Correlated Normally Distributed Loads</td>
<td>18</td>
</tr>
<tr>
<td>3.2.2 Fully Correlated Non Normally Distributed Loads</td>
<td>33</td>
</tr>
<tr>
<td>3.2.3 Non Correlated Normally Distributed Loads</td>
<td>37</td>
</tr>
<tr>
<td>3.2.4 Non Correlated Non Normally Distributed Loads</td>
<td>41</td>
</tr>
<tr>
<td>3.3 Multiple Constraints Solution</td>
<td>42</td>
</tr>
<tr>
<td>CHAPTER 4  EXAMPLES AND EVALUATION OF RESULTS</td>
<td>43</td>
</tr>
<tr>
<td>4.1 Eleven Bus System Example</td>
<td>43</td>
</tr>
<tr>
<td>4.2 Six Bus System Example</td>
<td>53</td>
</tr>
<tr>
<td>4.3 Comparison of Results</td>
<td>53</td>
</tr>
<tr>
<td>CHAPTER 5  CONCLUSIONS AND RECOMMENDATIONS</td>
<td>64</td>
</tr>
<tr>
<td>5.1 Conclusions</td>
<td>64</td>
</tr>
<tr>
<td>5.2 Recommendations</td>
<td>64</td>
</tr>
</tbody>
</table>
### TABLE OF CONTENTS (continued)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF REFERENCES</td>
<td>66</td>
</tr>
<tr>
<td>APPENDIX A DERIVATION OF ELEMENTS OF THE JACOBIAN</td>
<td></td>
</tr>
<tr>
<td>MATRIX</td>
<td>68</td>
</tr>
<tr>
<td>A.1 Jacobian Matrix Elements</td>
<td>68</td>
</tr>
<tr>
<td>APPENDIX B DERIVATION OF THE REAL POWER GENERATION</td>
<td></td>
</tr>
<tr>
<td>TRANSFORMATION</td>
<td>70</td>
</tr>
<tr>
<td>B.1 Real Power Generation Transformation</td>
<td>70</td>
</tr>
</tbody>
</table>
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Eleven bus system bus data</td>
<td>45</td>
</tr>
<tr>
<td>4.2</td>
<td>Eleven bus system line data</td>
<td>46</td>
</tr>
<tr>
<td>4.3</td>
<td>Eleven bus system operating constraints</td>
<td>47</td>
</tr>
<tr>
<td>4.4</td>
<td>Eleven bus system base case operating point</td>
<td>48</td>
</tr>
<tr>
<td>4.5</td>
<td>Eleven bus system expected values</td>
<td>49</td>
</tr>
<tr>
<td>4.6</td>
<td>Eleven bus system standard deviations</td>
<td>50</td>
</tr>
<tr>
<td>4.7</td>
<td>Eleven bus system range minimums</td>
<td>51</td>
</tr>
<tr>
<td>4.8</td>
<td>Eleven bus system range maximums</td>
<td>52</td>
</tr>
<tr>
<td>4.9</td>
<td>Six bus system bus data</td>
<td>56</td>
</tr>
<tr>
<td>4.10</td>
<td>Six bus system line data</td>
<td>57</td>
</tr>
<tr>
<td>4.11</td>
<td>Six bus system operating constraints</td>
<td>58</td>
</tr>
<tr>
<td>4.12</td>
<td>Six bus system base case operating point</td>
<td>59</td>
</tr>
<tr>
<td>4.13</td>
<td>Six bus system expected values</td>
<td>60</td>
</tr>
<tr>
<td>4.14</td>
<td>Six bus system standard deviations</td>
<td>61</td>
</tr>
<tr>
<td>4.15</td>
<td>Six bus system range minimums</td>
<td>62</td>
</tr>
<tr>
<td>4.16</td>
<td>Six bus system range maximums</td>
<td>63</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>2.1</td>
<td>Flow chart for conventional constrained power flow problem</td>
<td>10</td>
</tr>
<tr>
<td>3.1</td>
<td>An ideal transmission line</td>
<td>23</td>
</tr>
<tr>
<td>3.2</td>
<td>Probability density function of $P_{TOT}^L$</td>
<td>25</td>
</tr>
<tr>
<td>3.3</td>
<td>Probability density function of $Q_2^G$</td>
<td>28</td>
</tr>
<tr>
<td>3.4</td>
<td>Flow chart for a single constraint multiple linear transformations algorithm</td>
<td>32</td>
</tr>
<tr>
<td>4.1</td>
<td>Eleven bus system one line diagram</td>
<td>44</td>
</tr>
<tr>
<td>4.2</td>
<td>Six bus system one line diagram</td>
<td>55</td>
</tr>
</tbody>
</table>
ABSTRACT

The stochastic load flow methods normally employ a linear load flow. This is necessary since the statistics of voltage and line power flows cannot be determined through the nonlinear load flow equations. However, the linear load flow does not include the finite ranges of generator excitation systems and tap changing under load transformers. These constraints can cause a significant error in the statistical distribution of power system voltages in response to uncertain loading. In this work, the statistics of bus voltage and line power flow are computed for systems with such constraints. Multiple linear transformations are employed, with var or tap limits providing the discrete point at which each transformation is utilized. Statistical distributions are described by the moments of the random variables which facilitates their transformation through multiple linear transformations.

The method is applied to two power systems and the results are compared with Monte Carlo simulations.
1

CHAPTER 1
INTRODUCTION

1.1 Motivation

In planning and operating a power system, it is necessary to perform power flow studies in order to monitor and assess the security of a proposed or operational system. These studies are of the most frequently performed network computations, and there are many well developed techniques that permit them to be made very quickly, efficiently and accurately. However, these methods are inherently deterministic and during each power flow solution the nodal loads, generation and topology of the system are assumed constant. In many cases the value of load at each node in the system is not known completely. Examples of such cases are the short and long range forecast of future loads, and the value of load added to a system after restoring from an outage. In order to assess accurately the state of the system using deterministic techniques, it would be necessary to compute the power flows for all possible combinations of generation and loads. The computation time for this would be astronomical and the analysis and synthesis of the results would be nearly impossible.

In practice, the difficulties are overcome by selection of a limited number of variations of loads which often is being done arbitrarily, depending on the intuition and
experience of engineers. The results are based on partial information, therefore they are inaccurate and they can affect the ability of a power system to recover rapidly from an outage, and influence the future time for investment in capital equipment.

Stochastic power flow analysis has emerged as a hopeful alternative, but being limited by the nonlinear nature of the power flow equations. The objective of this research is to extend the stochastic power flow analysis to the broad area of constrained power systems which must respond to the limitation of mechanical devices such as the reactive power capabilities of generating units and tap ranges of tap-changing-under-load transformers. Results in this area would provide assistance to operators and planners when critical decisions are required.

1.2 Literature Summary

The systematic incorporation of load uncertainty into the load flow problem, has only been given attention since the early 1970's. B. Borkowska [1] is generally credited as the pioneer in this area. In her work, the load forecast is described by a general distribution curve which is subjected to linearized power flow equations. Real and reactive power components were assumed independent, making the computation of distribution functions possible by numerical convolution.

Dopazo et al [2] propose a similar set of computations. The law of large numbers is used to justify the use of a
normally distributed load forecast density function. This assumption, along with the linear transformation produce normal distributions of voltages and flows, which are used to calculate confidence limits on load flow output.

Allan, Grigg and Al-Shakarchi [3] employing a d.c. load flow model, developed a technique to implement convolutions involved in statistical computations, efficiently on a digital computer. In other works they extended the same concept to include both voltage magnitude and angle, and complex line power flows [4,5]. They proposed that the errors introduced in the convolution techniques could be compensated by shifting the computed density curve until its expected (mean) value coincides with the value obtained from a conventional deterministic analysis.

Aboytes and Cory [6] recognized the error introduced by assuming independence between load and generation uncertainty. In fact by using the known correlation between load and generation it is possible and practical to share the load uncertainty among all the generators.

Flam and Sasson [7] used concepts of decoupling, sparsity, ordering, and recursive formulation to arrive at an efficient implementation of stochastic load flow problem. Hayes [8] also uses the concept of decoupled load flow equations with partial recoupling where recoupling allows for increased accuracy with little increase in computation time.

Heydt and Katz [8] incorporated the non-deterministic aspect of available generation into simultaneous interchange
calculation. The random generation margins were assumed multivariate normal based on a generation reserve analysis. Heydt [9] in another work proposes a method which employs a multivariate normal distribution and an infinite series of Hermite polynomials to evaluate the joint probabilities of line power flows.

Heydt and Sauer [10,11] noted two major problem areas. First, the error introduced by the linear approximation which transforms jointly normal variables into other jointly normal variables. Even for exact normal input variables, the nonlinear properties of the load flow equations have a noticeable effect on the mean value. The second problem is a criticism which appears in virtually all past research efforts in stochastic power flow analysis. That is, the assumption of normality of variables.

Sauer [12] proposes an algorithm which employs quadratic transformations to consider large uncertainties in load schedules. A multivariate Gram-Charlier series is developed to represent random loading of virtually any statistical distribution. Statistical moments are used as a means for describing the probability density functions of both input and output variables of the stochastic load flow problem. Prior to this work, virtually no attempts had been made to account for limitations of constrained variables in stochastic load flow solution. Sauer proposes a multiple linear transformation and a multivariate Gram-Charlier series to be used to account
for such constraints. The feasibility of this method will be studied in the following chapters.
CHAPTER 2
THE CONSTRAINED STOCHASTIC POWER FLOW PROBLEM

2.1 Deterministic Load Flow Solutions By Newton's Method

The deterministic load flow problem is a steady state analysis during which the nonlinear power flow equations are solved using iterative techniques, yielding single valued output (voltages) for single valued input (loads). In the general sense a set of nonlinear equations can be expressed as

\[ Y = F(X) \] (2.1)

Equation (2.1) can be solved using a multivariate form of Newton's method where \( F(X) \) is expanded in a Taylor series about the operating point \( X^O \) and neglecting derivatives of order greater than 1.

\[ Y = F(X) = F(X^O) + \frac{\partial F}{\partial X} \bigg|_{X=X^O} (X-X^O) \] (2.2)

The matrix of derivatives is referred to as the jacobian evaluated at the kth solution.

\[ J(i,j) = \frac{\partial F_i}{\partial X_j} \bigg|_{X=X^O(k)} \] (2.3)

Solving Equation (2.2) for \( X \),

\[ X^{(k+1)} = X^{(k)} - J^{-1}F(X^{(k)}) \] (2.4)

assuming \( J \) has an inverse.
The nonlinear network equations expressing the specified real and reactive powers in terms of bus voltages can be written as,

\[
S_i = V_i \sum_{j=1}^{n} (Y_{ij} V_j)^* \quad i = 1, 2, \ldots, n \quad (2.5)
\]

where \( S_i = P_i + j Q_i \) is the net injected complex power at bus \( i \), \( V_j \) is the phasor voltage at bus \( j \), \( Y_{ij} \) is an element of the bus admittance matrix, \( n \) is the number of network busses, and \((.)*\) denotes complex conjugation.

Expressing Equation (2.5) in rectangular coordinates, the real and reactive powers at bus \( i \) are

\[
P_i = \sum_{j=1}^{n} |V_i||V_j||Y_{ij}| \cos (\delta_i - \delta_j - \theta_{ij}) \\
Q_i = \sum_{j=1}^{n} |V_i||V_j||Y_{ij}| \sin (\delta_i - \delta_j - \theta_{ij}) \quad i = 1, 2, \ldots, n \quad (2.6)
\]

For a system of \( n \) busses of which none are voltage controlled (PV or TCUL), the real and reactive powers are specified and the magnitude and angle of bus voltages are unknown for all busses (PQ bus) except the slack (swing) bus where the voltage is specified and remains fixed. Thus there are \( 2(n-1) \) equations and \( 2(n-1) \) unknowns to be solved for a load flow solution if there are \( (n-1) \) PQ busses and a slack bus in the system.

Equation (2.6) can be expressed as,
\[
\begin{bmatrix}
\Delta P \\
\Delta Q
\end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix} - \begin{bmatrix} \delta \\ |V|\end{bmatrix}
\]  
(2.7)

where it should be equal to zero at the solution. Employing Newton's method and Equation (2.4) the load flow equations can be formulated as follows:

\[
\begin{bmatrix}
\delta^{(k+1)}_i \\
|V_i^{(k+1)}|
\end{bmatrix} = \begin{bmatrix} \delta^{(k)}_i \\ |V_i^{(k)}| \end{bmatrix} - J^{-1} \begin{bmatrix} \Delta P^{(k)}_i \\ \Delta Q^{(k)}_i \end{bmatrix}
\]

\[i = 1, 2, \ldots, n \neq \text{swing}\]  
(2.8)

where

\[
J = \begin{bmatrix}
\frac{\partial P_i}{\partial \delta_j} & \frac{\partial P_i}{\partial |V_j|} \\
\frac{\partial Q_i}{\partial \delta_j} & \frac{\partial Q_i}{\partial |V_j|}
\end{bmatrix}
\]  
(2.9)

\[
d = \delta^{(k)}
\]

\[i = 1, 2, \ldots, n \neq \text{swing}\]

\[j = 1, 2, \ldots, n \neq \text{swing}\]

and

\[
\Delta P^{(k)}_i = p_i \text{ scheduled} - \sum_{j=1}^{n} |V_i^{(k)}||V_j^{(k)}||Y_{ij}| \cos (\delta^{(k)}_i - \delta^{(k)}_j - \theta_{ij})
\]

\[i = 1, 2, \ldots, n \neq \text{swing}\]  
(2.10)

\[
\Delta Q^{(k)}_i = q_i \text{ scheduled} - \sum_{j=1}^{n} |V_i^{(k)}||V_j^{(k)}||Y_{ij}| \sin (\delta^{(k)}_i - \delta^{(k)}_j - \theta_{ij})
\]

\[i = 1, 2, \ldots, n \neq \text{swing}\]
The first estimates of bus voltages are taken as the swing bus voltage, then Equation (2.8) is solved in an iterative process where the iteration is continued until the mismatch vector \([\Delta P \ \Delta Q]^t\) becomes zero or small enough for the required accuracy. The derivation of the jacobian elements is given in Appendix A.

2.2 Voltage Controlled Bus Constraints

A modification of, or deviation from the normal computational procedures for the solution of the load flow problem is required to take into account voltage controlled busses. If the voltage magnitude at a generation bus is maintained constant by the generator reactive power available at that bus (PV bus), the fixed voltage is an input to the load flow equations, and the required reactive power generation is an output variable. In the process of the iterative solution, if the required reactive power generation violates the physical limits then the voltage is no longer considered fixed and is allowed to vary as an output variable in response to the newly created input variable which is taken to be the limit value of reactive power. Similarly, if the voltage magnitude at a bus is being maintained fixed by a tap changing under load (TCUL) transformer, the voltage is considered an input, and the required tap is an output variable. However, the input and output variables might interchange roles if the value of tap violates its limits. A flow chart of the conventional constrained power flow solution is shown in Figure 2.1.
Figure 2.1 Flow chart for conventional constrained power flow problem
2.3 Linear Load Flow Methods

The iterative solution method used in the deterministic load flow is not applicable in the stochastic power flow problem. All of the present methods proposed by the work in the literature employ a linearization about an expected operating point.

The random input variables $Y = [y_1, \ldots, y_n]^t$ can be assumed to be the sum of a deterministic vector $Y^0$, and a random component $\Delta Y$.

$$Y = F(X) = Y^0 + \Delta Y$$

Taylor series expansion expressed in Equation (2.2), and the voltages $X^0$ which result from the input loads $Y^0$ comprising the base case operating point can be used to linearize the load flow equations,

$$Y^0 + \Delta Y = F(X^0) + \left. \frac{\partial F}{\partial X} \right|_{X = X^0} (X^0 + \Delta X - X^0)$$

after simplification we have

$$\Delta Y = \left. \frac{\partial F}{\partial X} \right|_{X = X^0} (\Delta X)$$

or

$$\Delta X = J^{-1} \Delta Y \quad (2.11)$$

where
Thus the random component of the output voltages $\Delta X$ is a linear function of the random component of the input loads $\Delta Y$.

### 2.4 Unconstrained Stochastic Load Flow

A stochastic process may be defined as a process involving random variables whose statistics are time dependent. The variables describing an electric power system are influenced by all the random actions of the consumers and the environment in which they live. Furthermore, these variables (bus voltages, line flows, and bus power demands) are time varying quantities thus qualifying the power system as a stochastic process.

Stochastic load flow analysis is the solution of the conventional load flow problem wherein the loads are considered random variables of known statistical distributions. The objective of such analysis is to obtain a range of output quantities, (voltages and line flows), to reflect the system response to a range of input quantities (loads). The ranges are manifested in the statistical distributions and often take the form of probabilities and expected values.

The linearized load flow equation expressed by Equation (2.11) may be employed to solve the unconstrained stochastic load flow problem as follows:
where \([\Delta P \Delta Q]^t\) are random variations about an expected (mean) value, and

\[
E\{\Delta P\} = E\{\Delta Q\} = 0
\]

where \(E\) is the expectation operation defined for any function \(g(x)\) as,

\[
E\{g(x)\} = \int_{-\infty}^{\infty} g(x) f(x) dx
\]

and \(f(x)\) is the probability density function of a random variable \(x\).

Let \(C_p\) be the covariance of \([\Delta P \Delta Q]^t\)

\[
C_p = E\left\{\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^t\right\}
\]

and \(C_v\) be the covariance of \([\Delta \delta \Delta |V|]^t\)

\[
C_v = E\left\{\begin{bmatrix} \Delta \delta \\ \Delta |V| \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |V| \end{bmatrix}^t\right\}
\]

substituting Equation (2.13) for \([\Delta \delta \Delta |V|]^t\) we get

\[
C_v = [J^{-1}] E\left\{\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^t\right\} [J^{-1}]^t
\]

and
Equation (2.15) is a solution to the general stochastic load flow problem; however, in practice determination of $C_p$ is cumbersome due to the complex statistical distributions of the input variables. Thus many assumptions may be needed to be made in order to simplify the problem. Furthermore, the linearization of the power flow equations does not reflect the constraints of reactive power sources, tap changing under load (TCUL) transformers and other mechanical devices with physical limitations.

When Equations (2.11) and (2.12) are used to compute variations in output variables about their operating point values, they may produce solutions which violate the physical limitations. Such problems are inherent in the linear solutions of nonlinear equations, and are generally considered one of the sources of error. Since the presence of voltage controlled busses in a network tends to maintain a flat voltage profile throughout the system, the loss of voltage control at one bus can seriously affect the voltage level of all uncontrolled busses. Since the central objective of stochastic power flow analysis is to provide information about all possible values of output for all possible values of input, the response of the system under all practical conditions must be considered. The following chapter proposes a method which considers the physical constraints of equipment in the linear stochastic power flow analysis.
CHAPTER 3
SOLUTION OF THE CONSTRAINED STOCHASTIC POWER FLOW PROBLEM

3.1 Multiple Linear Transformations

To illustrate the method of multiple linear transformations, consider a three bus power system with swing bus numbered one, and a voltage controlled (PV) bus numbered two. Let the operating point be such that the reactive power source at bus two is within its prescribed limits. Using a linear transformation to compute variations about the operating point,

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix}
= J^{-1}
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta Q_3
\end{bmatrix}
\] (3.1)

and

\[
\begin{bmatrix}
\Delta P_1 \\
\Delta Q_1 \\
\Delta Q_2
\end{bmatrix}
= J' \begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix}
\] (3.2)

or

\[
\begin{bmatrix}
\Delta P_1 \\
\Delta Q_1 \\
\Delta Q_2
\end{bmatrix}
= J' J^{-1}
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta Q_3
\end{bmatrix}
\] (3.3)
where,

\[ \Delta \delta_i = \text{The change in voltage angle at bus } i \]
\[ \Delta |V_i| = \text{The change in voltage magnitude at bus } i \]
\[ \Delta P_i = \text{The change in net injected real power at bus } i \]
\[ \Delta Q_i = \text{The change in net injected reactive power at bus } i \]

and \( J^{-1} \) is the 3 x 3 jacobian inverse matrix used in the final iteration of the operating point load flow. Similarly, \( J' \) is a matrix of sensitivities evaluated at the base case operating point.

\[
J^{-1} = \begin{bmatrix}
\frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial |V_3|} \\
\frac{\partial P_3}{\partial \delta_2} & \frac{\partial P_3}{\partial \delta_3} & \frac{\partial P_3}{\partial |V_3|} \\
\frac{\partial Q_3}{\partial \delta_2} & \frac{\partial Q_3}{\partial \delta_3} & \frac{\partial Q_3}{\partial |V_3|}
\end{bmatrix}
\]

(3.4)

\[
J' = \begin{bmatrix}
\frac{\partial P_1}{\partial \delta_2} & \frac{\partial P_1}{\partial \delta_3} & \frac{\partial P_1}{\partial |V_3|} \\
\frac{\partial Q_1}{\partial \delta_2} & \frac{\partial Q_1}{\partial \delta_3} & \frac{\partial Q_1}{\partial |V_3|} \\
\frac{\partial Q_2}{\partial \delta_2} & \frac{\partial Q_2}{\partial \delta_3} & \frac{\partial Q_2}{\partial |V_3|}
\end{bmatrix}
\]

(3.5)

If the operating point load flow solution had been unable to maintain constant voltage at bus 2 due to excessive demands
of reactive power at bus 2, the linear model for variations about the operating point would be,

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_2| \\
\Delta |V_3|
\end{bmatrix}
= \begin{bmatrix}
J^{-1}
\end{bmatrix}
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta Q_2 \\
\Delta Q_3
\end{bmatrix}
\]

(3.6)

and

\[
\begin{bmatrix}
\Delta P_1 \\
\Delta Q_1
\end{bmatrix}
= \begin{bmatrix} J' J^{-1} \end{bmatrix}
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta Q_2 \\
\Delta Q_3
\end{bmatrix}
\]

(3.7)

where \( J^{-1} \) is the 4 x 4 Jacobian inverse matrix used in the final iteration of the operating point load flow and \( J' \) is a 2 x 4 matrix of sensitivities evaluated at the base case operating point.

When the base case operating point of the 3 bus system above is such that \( Q_2^G \), the reactive power generation at bus 2 is just below it's maximum limit, there may be many cases where changes in the input loads would produce a value of \( Q_2^G \) which would violate the maximum limit and thereby force \( |V_2| \) to vary. Under such circumstances the linear model of Equations (3.1) and (3.3) would not reflect the low voltage at bus 3 (PQ bus) which may occur due to the loss of voltage
control. The use of a transformation of the type given in Equations (3.6) and (3.7) would more readily represent the uncontrolled voltages.

When the statistical distribution of the input variables is known, Equations (3.1-3.3) and (3.6-3.7) can be used to obtain the statistical distributions of the output variables in two modes of operation separately.

3.2 Single Constraint Solution Algorithm

For the simplicity of the problem, consider a system having only one controlled bus. To illustrate the solution technique, the 3 bus system in section 3.1 may be used, where the constrained variable is the reactive power supply at bus 2 ($Q_2^G$).

3.2.1 Fully Correlated Normally Distributed Loads

In order to simplify the transformation of the input load statistics thru the linearized load flow equations we assume loads are fully correlated and are a linear function of only one random variable which is taken to be the change in total real power load demand ($\Delta PL_{TOT}$). Therefore, the input variables may be expressed as follows:

$$\Delta P_{TOT}^{L} = \sum_{k=1}^{NB} \Delta P_{k}^{L}$$

$$\Delta P_{i}^{G} = \alpha_i \Delta P_{TOT}^{L}$$

(3.8)
\[ \Delta P_i^L = \beta_i \Delta P_{TOT} \]
\[ \Delta Q_i^L = \Gamma_i \Delta P_{TOT} \]

where

\[ \beta_i = \frac{P_i^L}{P_{TOT}} \]
\[ \Gamma_i = \frac{Q_i^L}{Q_{TOT}} \]  \hspace{1cm} (3.9)

\[ \alpha_i = C_i \frac{P_i^G P_{TOT}}{P_{TOT} \sum_{k=1}^{NG} C_k P_k^G} \]

where

\[ \Delta P_i^G = \text{the change in real power generation at bus } i \]
\[ \Delta P_i^L = \text{the change in real power demand at bus } i \]
\[ \Delta Q_i^L = \text{the change in reactive power load demand at bus } i \]
\[ P_i^G = \text{the base case real power generation at bus } i \]
\[ P_i^L = \text{the base case real power load demand at bus } i \]
\[ Q_i^L = \text{the base case reactive power load demand at bus } i \]
\[ P_{TOT}^G = \text{the base case total real power generation} \]
\[ P_{TOT}^L = \text{the base case total real power load demand} \]
\[ C_i = 1 \text{ for cyclic generation units} \]
\[ C_i = 0 \text{ for noncyclic generation units} \]

Derivation of \( \alpha_i \) is given in Appendix B.

After having the input variables expressed in terms of \( \Delta P_{TOT}^L \), we can express Equations (3.1) and (3.2) in terms of
\( \Delta P_{TOT} \). Equation (3.1) is repeated below for reference.

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix} =
\begin{bmatrix}
J^{-1}_{\delta P} & J^{-1}_{\delta Q} \\
J^{-1}_{VP} & J^{-1}_{VQ}
\end{bmatrix}
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta Q_3
\end{bmatrix}
\]

(3.10)

where

\[
\Delta P_i = \Delta P_{i}^{G} + \Delta P_{i}^{L}
\]

\[
\Delta Q_i = \Delta Q_{i}^{G} + \Delta Q_{i}^{L}
\]

or

\[
\Delta P_i = \Delta P_{i}^{L} + \alpha_i \sum_{k=1}^{NB} \Delta P_k
\]

Therefore, Equation (3.10) may be written as

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix} =
\begin{bmatrix}
J^{-1}_{\delta P} & J^{-1}_{\delta Q} \\
J^{-1}_{VP} & J^{-1}_{VQ}
\end{bmatrix}
\begin{bmatrix}
\Delta P_2 + \alpha_2 \sum_{k=1}^{NB} \Delta P_k \\
\Delta P_3 + 0 \\
\Delta Q_3 + 0
\end{bmatrix}
\]

(3.11)

or

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix} =
\begin{bmatrix}
T_{\delta P} & T_{\delta Q} \\
T_{VP} & T_{VQ}
\end{bmatrix}
\begin{bmatrix}
\Delta P_1^{L} \\
\Delta P_2^{L} \\
\Delta P_3^{L}
\end{bmatrix}
\]

(3.12)

where
Substituting Equation (3.8) for $\Delta P_1^L$ and $\Delta Q_1^L$, Equation (3.12) becomes,

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix} = \begin{bmatrix} T_{\delta P} & T_{\delta Q} \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \Delta P_{TOT}^L
\] (3.14)
therefore

\[ \Delta \delta_i = T_{\delta_i} \Delta P_i^{\text{TOT}} \]

\[ \Delta |V_1| = T_{V_1} \Delta P_i^{\text{TOT}} \]  \hspace{1cm} (3.15)

where the voltage angle transformation \( T_\delta \) and the voltage magnitude transformation \( T_V \) can be obtained from Equation (3.14).

Using the same approach Equation (3.2) can be expressed as follows:

\[
\begin{bmatrix}
\Delta P_1^G + \Delta P_1^L \\
\Delta Q_1^G + \Delta Q_1^L \\
\Delta Q_2^G + \Delta Q_2^L
\end{bmatrix}
= \begin{bmatrix}
J' & \Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix}

\]

After substituting Equation (3.10) for \([\Delta \delta_2 \ \Delta \delta_3 \ \Delta |V_3|]^T\) and rearranging we get,

\[
\begin{bmatrix}
\Delta P_1^G \\
\Delta Q_1^G \\
\Delta Q_2^G
\end{bmatrix}
= \begin{bmatrix}
J' & \frac{T_{\delta P} T_{\delta Q}}{T_{VP} T_{VQ}} & \Delta P_1^L \\
& \Delta P_2^L & \Delta P_3^L \\
& \Delta Q_1^L & \Delta Q_2^L
\end{bmatrix}
- \begin{bmatrix}
\Delta P_1^L \\
\Delta Q_1^L \\
\Delta Q_2^L
\end{bmatrix}
\]  \hspace{1cm} (3.16)

or

\[
\begin{bmatrix}
\Delta P_1^G \\
\Delta Q_1^G \\
\Delta Q_2^G
\end{bmatrix}
= \begin{bmatrix}
J' & \frac{T_{\delta P} T_{\delta Q}}{T_{VP} T_{VQ}} & \frac{\beta_1}{\gamma_1} \\
& \frac{\beta_2}{\gamma_2} & \frac{\beta_3}{\gamma_3} & \Delta P_i^{\text{TOT}} \\
& \frac{\beta_2}{\gamma_2} & \frac{\beta_3}{\gamma_3} & \Delta P_i^{\text{TOT}}
\end{bmatrix}
\]  \hspace{1cm} (3.17)

therefore
\[ \Delta P_s^G = T_p \Delta P_{TOT} \]
\[ \Delta Q_i^G = T_q \Delta P_{TOT} \]

where the real power generation transformation \( T_p \) and the reactive power generation transformation \( T_q \) can be obtained from Equation (3.17).

In order to find the line power flow transformation we can use a d.c. load flow model as follows:

\[ V_i \]
\[ jX_{ij} \]
\[ V_j \]

Figure 3.1 An ideal transmission line

Assuming negligible resistance, the real power flow in the above transmission line is

\[ P_{ij} = \frac{-|V_i||V_j|}{X_{ij}} \sin(\delta_j - \delta_i) \]

(3.19)

at \( j \)

assuming \((\delta_j - \delta_i)\) is small and the voltage magnitudes are nearly unity Equation (3.19) becomes,

\[ P_{ij} = \frac{\delta_i - \delta_j}{X_{ij}} \]

(3.20)

at \( j \)

therefore,

\[ \Delta P_{ij} = \frac{\Delta \delta_i - \Delta \delta_j}{X_{ij}} \]

(3.21)

at \( j \)
replacing $\Delta \delta$'s by Equation (3.15) we get,

$$\Delta P_{ij} = \frac{[T_{\delta_i} - T_{\delta_j}]}{X_{ij}} \Delta P^L_{\text{TOT}}$$

at $j$

or

$$\Delta P_{ij} = T_{p_{ij}}^{\lambda i} \Delta P^L_{\text{TOT}} \tag{3.22}$$

at $j$

where

$$T_{p_{ij}}^{\lambda i} = \frac{T_{\delta_i} - T_{\delta_j}}{X_{ij}} \tag{3.23}$$

at $j$

So far we have expressed the output variables of the stochastic load flow problem as linear functions of a single input random variable $\Delta P^L_{\text{TOT}}$, assuming loads are fully correlated random variables,

$$\begin{bmatrix} \Delta \delta \\ \Delta |V| \\ \Delta P_S^G \\ \Delta P^G \\ \Delta Q^G \\ \Delta P^{\lambda i} \\ \Delta P^L_{\text{TOT}} \end{bmatrix} = \begin{bmatrix} T_{\delta} \\ T_V \\ T_{p_S} \\ \alpha \\ T_Q \\ T_{p}^{\lambda i} \end{bmatrix} \begin{bmatrix} \Delta P^L_{\text{TOT}} \end{bmatrix} \tag{3.24}$$

where $S$ denotes swing bus and $\lambda i$ denotes line. The next step would be to compute the statistics of the output random
variables using Equation (3.24) and the statistics of $\Delta p_L^{TOT}$. In order to establish the statistics of $\Delta p_L^{TOT}$ we may assume that the total real power load demand ($p_L^{TOT}$) forecast has a normal distribution with a certain specified error of $\pm Er\%$.

![Probability density function of $p_L^{TOT}$](image)

Figure 3.2 Probability density function of $p_L^{TOT}$

Figure 3.2 represents the density function of $p_L^{TOT}$ where

$$p_L^{TOT} = p_L^{OL} + \Delta p_L^{TOT}$$

and

$$E\{\Delta p_L^{TOT}\} = 0$$

$$E\{(\Delta p_L^{TOT})^2\} = \sigma^2$$

where $E$ is the expectation operation defined in Equation (2.14).

thus

$$3\sigma = \left(\frac{Er\%}{100}\right)p_L^{OL}^{TOT}$$
or \[ \sigma = \frac{E_{\tilde{r}_{L}^{\Gamma}}}{300} \frac{P_{oL}^{\Gamma}}{P_{TOT}} \] (3.25)

Therefore \( \Delta P_{TOT}^{L} \) is also a normally distributed random variable with mean zero, and variance \( \sigma^2 \). Now Equation (3.24) can be used to compute mean and variances of the output variables. Equation (3.24) is of the form

\[ X = T_{2}Y \] (3.26)

where \( T_{2} \) is the base case transformation

Therefore

\[ E\{X\} = T_{2}E\{Y\} \] (3.27)

\[ E\{X^2\} = T_{2}^{2}E\{Y^2\} \]

where \[ E\{Y\} = E\{\Delta P_{TOT}^{L}\} = 0 \]

\[ E\{Y^2\} = E\{(\Delta P_{TOT}^{L})^2\} = \sigma^2 \]

Thus output variables are also normally distributed random variables with zero mean, and variances as follows:

\[ E\{(\Delta \delta_{1})^2\} = T_{\delta_{1}}^{2}\sigma^2 \]

\[ E\{(\Delta |V_{1}|)^2\} = T_{V_{1}}^{2}\sigma^2 \] (3.28)

\[ E\{(\Delta P_{S}^{G})^2\} = T_{P_{S}}^{2}\sigma^2 \]

\[ E\{(\Delta P_{L}^{G})^2\} = \alpha_{1}^{2}\sigma^2 \]
However, Equation (3.27) alone does not reflect the constraints of PV bus 2. Depending upon the status of the PV bus at the base case, $\dot{Q}_2^G$ or $|V_2|$ may be employed to obtain a probability estimate of the status of bus 2 due to a future change in the input loads. This probability estimate serves as a criterion to determine whether a second set of linear transformations are required for the computation of the output variables statistics. For example let $T_2$ be the set of transformations where $Q_2^G$ is within constraints and $T_1$ be the second set of transformations where $Q_2^G$ has violated constraints and bus 2 is no longer a PV bus. Furthermore, let Equation (3.27) represent $T_2$. The density function of $Q_2^G$ obtained from Equation (3.28) may be used to compute the probability of $Q_2^G$ violating its constraints as follows:

\[
P(Q_2^G > Q_2^{G\text{ max}}) = \int_{Q_2^{G\text{ max}}}^{\infty} f(Q_2^G)\,dQ_2^G
\]

or

\[
P(Q_2^G > Q_2^{G\text{ max}}) = \text{ERF} \left[ \frac{(Q_2^{G\text{ max}} - Q_2^{G\text{ max}})}{\sigma_{Q_2}} \right]
\]

where $\text{ERF}$ is an error function described as,

\[
\text{ERF} (x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt
\]
\[ \text{ERF}(a) = \int_{a}^{\infty} G(x) \, dx \]

where \( G(x) \) is a zero mean unit variance normal probability density function.

\[ f(Q_G^G) \]

Figure 3.3: Probability density function of \( Q_G^G \)

Figure 3.3 represents the density function of \( Q_G^G \) and the shaded area under the curve represents the probability of \( Q_G^G \) violating \( Q_{G_{\text{max}}} \).

If \( P(Q_G^G > \text{limit}) \) is smaller than a specified tolerance, bus 2 is most likely to maintain its PV status. Therefore transformation \( T_2 \) is sufficient for computation of output statistics, otherwise transformation \( T_1 \) where bus 2 is a PQ bus, must be used along with \( T_2 \) to obtain the statistics of the output variables. In order to do so, we have to establish limits of integration for \( \Delta P_{\text{TOT}}^L \), which must be used in the computation of "expected values".

Let \( T_{2Q_{G^2}} \) be the base case reactive power transformation at bus 2. From Equation (3.24) we have,

\[ \Delta Q_{G^2} = T_{2Q_{G^2}} \Delta P_{\text{TOT}}^L \quad (3.30) \]
where
\[
\Delta Q_2^G \leq Q_2^{G\text{max}} - Q_2^{G}\]

therefore for \( T_{2Q_2} > 0 \)
\[
\Delta P_{\text{TOT}}^L \leq \frac{Q_2^{G\text{max}} - Q_2^{G}}{T_{2Q_2}}
\]  \( (3.31) \)

thus the limit is
\[
\ell = \frac{Q_2^{G\text{max}} - Q_2^{G}}{T_{2Q_2}}
\]  \( (3.32) \)

For example, the statistics of \( \Delta |V_3| \) may be computed as follows:

\[
\Delta |V_3| = T_{2V_3} \Delta P_{\text{TOT}}^L \quad \text{if} \quad \Delta P_{\text{TOT}}^L < \ell
\]

\[
\Delta |V_3| = T_{1V_3} \Delta P_{\text{TOT}}^L \quad \text{if} \quad \Delta P_{\text{TOT}}^L > \ell
\]

and
\[
E\{\Delta |V_3|\} = \int_{-\infty}^{\infty} \Delta |V_3| f(\Delta |V_3|) d\Delta |V_3|
\]  \( (3.33) \)

\[
E\{(\Delta |V_3|)^2\} = \int_{-\infty}^{\infty} (\Delta |V_3|)^2 f(\Delta |V_3|) d\Delta |V_3|
\]

or
\[
E\{\Delta |V_3|\} = \int_{-\infty}^{\ell} (T_{2V_3} \Delta P_{\text{TOT}}^L) f(\Delta P_{\text{TOT}}^L) d\Delta P_{\text{TOT}}^L
\]

\[
+ \int_{\ell}^{\infty} (T_{1V_3} \Delta P_{\text{TOT}}^L) f(\Delta P_{\text{TOT}}^L) d\Delta P_{\text{TOT}}^L
\]  \( (3.34) \)
and

\[
E\{ (\Delta | V_3 |)^2 \} = \int_{-\infty}^{\infty} (T_{2V_3} \Delta P_{TOT}^L)^2 f(\Delta P_{TOT}^L) d\Delta P_{TOT}^L \\
+ \int_{\infty}^{\infty} (T_{1V_3} \Delta P_{TOT}^L)^2 f(\Delta P_{TOT}^L) d\Delta P_{TOT}^L
\]  \hspace{1cm} (3.34)

where

\[
f(\Delta P_{TOT}^L) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ \left(-\frac{(\Delta P_{TOT}^L)^2}{2\sigma^2} \right) \right]
\]  \hspace{1cm} (3.35)

substituting Equation (3.35) in Equations (3.34) and integrating we get,

\[
E\{ (\Delta | V_3 |)^2 \} = \frac{\sigma}{\sqrt{2\pi}} (T_{1V_3} - T_{2V_3}) \exp\left[ -\frac{\lambda^2}{2\sigma^2} \right]
\]

and

\[
E\{ (\Delta | V_3 |)^2 \} = \frac{(T_{1V_3}^2 - T_{2V_3}^2)}{2\pi} \exp\left(-\frac{\lambda^2}{2\sigma^2} \right) + \sigma^2 E\{ L / \sigma \} \\
+ \frac{T_{2V_3}^2 \sigma^2}{2\pi}
\]

Therefore using the same approach the statistics of the other output variables may be obtained.

\[
E\{ \Delta \delta_1 \} = (T_{1\delta} - T_{2\delta})K
\]

\[
E\{ \Delta | V_1 | \} = (T_{1V_1} - T_{2V_1})K \hspace{1cm} (3.36)
\]

\[
E\{ \Delta P_G \} = (T_{1P} - T_{2P})K
\]
\[ E\{\Delta Q_i^G\} = (T_1Q_i^G - T_2Q_i^G)K \tag{3.36} \]

\[ E\{\Delta P_{li}^l\} = (T_1P_{li}^l - T_2P_{li}^l)K \]

where

\[ K = \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\lambda^2}{2\sigma^2}\right) \tag{3.37} \]

and

\[ E\{(\Delta \delta_i^2\} = (T_1\delta_i^2 - T_2\delta_i^2)K' + T_2\delta_i^2 \]

\[ E\{(\Delta |V_i|^2\} = (T_1V_i^2 - T_2V_i^2)K' + T_2V_i^2 \]

\[ E\{(\Delta P_S^G)^2\} = (T_1P_S^G - T_2P_S^G)K' + T_2P_S^2 \tag{3.38} \]

\[ E\{(\Delta Q_1^G)^2\} = (T_1Q_1^G - T_2Q_1^G)K' + T_2Q_1^2 \]

\[ E\{(\Delta P_{li}^l)^2\} = (T_1P_{li}^l - T_2P_{li}^l)K' + T_2P_{li}^2 \]

where

\[ K' = \lambda K + \sigma^2 \text{ERF}(\lambda/\sigma) \tag{3.39} \]

If at the base case \( Q_2^G \) violates its limits, \( |V_2| \) and its desired value \( |V_2\text{des}| \) become the determining factors in determining the status of bus 2. Therefore by replacing \( Q_2^G \) by \( |V_2| \), \( Q_2^\text{max} \) by \( |V_2\text{des}| \), \( T_2Q_2 \) by \( T_1V_2 \), and by interchanging \( T_{1X_i} \) and \( T_{2X_i} \), Equations (3.30) thru (3.39) hold. Figure 3.4 represents
perform the base case operating point load flow employing the Newton's method

compute the variances of output variables using the base case linearized transformation

compute
\[ P(\left| V_2 \right| > \left| V_{2\text{des}} \right|) \]
\[ P(\left| V_2 \right| < \left| V_{2\text{des}} \right|) \]

Is
\[ Q_2^G < Q_2 - Q_{2\text{des}} \]?

The base case transformation is sufficient

Obtain transformation \( T_2 \) where bus 2 is a PV bus

Employ both sets of transformations to obtain the actual statistics of output variables

Obtain transformation \( T_1 \) where bus 2 is a PQ bus

Stop

Figure 3.4 Flow chart for a single constraint multiple linear transformations algorithm
the various steps involved in a multiple linear transformations algorithm.

3.2.2 Fully Correlated Non Normally Distributed Loads

Assuming all loads are fully correlated, the approach taken in section 3.2.1 may be used to represent the output variables in terms of a single random variable $\Delta P^L_{TOT}$. However, $\Delta P^L_{TOT}$ may be a non normally distributed random variable, and depending upon the probability density function of $\Delta P^L_{TOT}$ and its order of complexity the computation of the output statistics might be cumbersome or in some cases impossible.

The difficulties in representation of the probability density function of $\Delta P^L_{TOT}$ may be overcome through the use of the Gram-Charlier series which represents the probability density function of a random variable $x$ as,

$$f(x) = \sum_{s=0}^{\infty} \frac{1}{s!} E[H_s(y)] H_s(y)G(y) \quad (3.40)$$

where $H_s(y)$ is the order $s$ Hermite polynomial of argument $y$ and $G(y)$ is the univariate normal density function in standard measure (zero mean, unit variance), where $y$ is normalized as $y = \frac{x}{\sigma}$. Since the Hermite polynomial is an algebraic sum of ordered powers of $x$, the Gram-Charlier series utilizes the weighted moments of $x$ to describe its density function. The advantage of such a representation is the ease of integration of the resulting density, and the flexibility of the series to handle standard as well as non-standard statistical distributions.
The Hermite polynomials can be computed using the recursive formula,

\[ H_s(y) = yH_{s-1}(y) - (s-1)H_{s-2}(y) \]  

for \( s \geq 2 \) the first six Hermite polynomials are

\[
\begin{align*}
H_{-1}(y) &= \frac{\text{ERF}(y)}{G(y)} \\
H_0(y) &= 1 \\
H_1(y) &= y \\
H_2(y) &= y^2 - 1 \\
H_3(y) &= y^3 - 3y \\
H_4(y) &= y^4 - 6y^2 + 3 \\
H_5(y) &= y^5 - 10y^3 + 15y
\end{align*}
\]

In order to represent the probability density function of \( \Delta P_{\text{TOT}}^L \) we may use Hermite polynomials of order \( 0 \geq s \geq 4 \) as an approximation. Therefore, Equation (3.40) results in

\[
f(x) = G(y)\{E\{1\} + yE\{y\} + \frac{1}{2}(y^2 - 1)E\{y^2 - 1\} + \frac{1}{6}(y^3 - 3y)E\{y^3 - 3y\} + \frac{1}{24}(y^4 - 6y^2 + 3)E\{y^4 - 6y^2 + 3\}\}
\]

where \( x = \Delta P_{\text{TOT}}^L \) and \( y = \frac{x}{\sigma} \)

or
\[ f(x) = \left[ \frac{1}{\sigma \sqrt{2\pi}} \right] \left[ 1 + \frac{m_3}{6\sigma^3} \left( \frac{x^3}{\sigma^3} - \frac{3x}{\sigma} \right) \right] \]

\[ + \frac{1}{24} \left( \frac{m_4}{\sigma^4} - 3 \right) \left( \frac{x^4}{\sigma^4} - \frac{6x^2}{\sigma^2} + 3 \right) \]  

(3.44)

where \( m_i \) is the \( i \)th moment of random variable \( x \) defined as

\[ m_i = E[x^i] \]  

(3.45)

Equation (3.44) may be simplified where the result is

\[ f(x) = G_o(x)\left[ k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0 \right] \]  

(3.46)

where

\[ k_4 = \frac{m_4 - 3\sigma^4}{24\sigma^8} \]

\[ k_3 = \frac{m_3}{6\sigma^6} \]

\[ k_2 = \frac{3\sigma^4 - m_4}{4\sigma^6} \]  

(3.47)

\[ k_1 = \frac{-m_3}{2\sigma^4} \]

\[ k_0 = \frac{5\sigma^4 + m_4}{8\sigma^4} \]

and \( G_o(x) \) is the normal density function with zero mean and variance \( \sigma^2 \).

Equations (3.27) and (3.28) represent the means and variances of the output variables for a single linear
transformation. The statistics of the output variables for the multiple linear transformations may be obtained using the same approach taken in section 3.2.1 and by representing the probability density function of $\Delta P_{TOT}^L$ by $f(x)$ described in Equation (3.46). The final results after integration are as follows:

$$E\{\Delta \delta_i\} = (T_1 \delta_i - T_2 \delta_i)C$$

$$E\{\Delta |V_i|\} = (T_1 V_i - T_2 V_i)C$$

$$E\{\Delta P_S^G\} = (T_1 P_S - T_2 P_S)C$$

$$E\{\Delta Q_1^G\} = (T_1 Q_1 - T_2 Q_1)C$$

$$E\{\Delta P_k^G\} = (T_1 p_k - T_2 p_k)C$$

where

$$C = [\frac{\sigma}{\sqrt{2\pi}} \exp(-x^2/2\sigma^2)] [k_4 x^4 + k_3 x^3 + k']$$

and

$$k' = 8\sigma^4 k_4 + 2\sigma^2 k_2 + k_0$$

and

$$E\{(\Delta \delta_i)^2\} = (T_1 \delta_i^2 - T_2 \delta_i^2)C' + T_2^2 \delta_i^2$$

$$E\{(\Delta |V_i|)^2\} = (T_1 V_i^2 - T_2 V_i^2)C' + T_2^2 V_i^2$$
\[ E\{(\Delta P_S^G)^2\} = (T_{1P_S}^2 - T_{2P_S}^2) C' + T_{2P_S}^2 \sigma^2 \]

\[ E\{(\Delta Q_1^G)^2\} = (T_{1Q_1}^2 - T_{2Q_1}^2) C' + T_{2Q_1}^2 \sigma^2 \]  \hspace{1cm} (3.50)

\[ E\{(\Delta P_k^i)^2\} = (T_{1P_k}^2 - T_{2P_k}^2) C' + T_{2P_k}^2 \frac{\sigma^2}{2} \]  \hspace{1cm} (3.51)

where

\[ C' = \left[ \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\lambda^2/2\sigma^2\right) \right] C'' + \sigma^2 \text{ERF}(\frac{\lambda}{\sigma}) \]

and

\[ C'' = \left[ k_4 \lambda^5 + k_3 \lambda^4 + \frac{k_2}{6} \lambda^3 - \frac{k_1}{3} \lambda^2 + (k_0 + 3\sigma^2 k_2) \lambda \right. 

\[ + \left. (15\sigma^4 k_4^4 - \frac{2}{3} \sigma^2 k_1) \right] \]

3.2.3 Non Correlated Normally Distributed Loads

When loads are considered non correlated random variables, input variables may not be expressed by Equation (3.8) and somewhat a different approach must be taken in order to solve the constrained stochastic load flow problem employing the multiple linear transformations algorithm. Consider again the 3 bus example of previous section. The linear transformations in Equations (3.1) and (3.2) are,

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3| \\
\end{bmatrix} = 
\begin{bmatrix}
J^{-1}
\end{bmatrix}
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta Q_3 \\
\end{bmatrix} \hspace{1cm} (3.52)
\]
expressing above equations in terms of generation and load powers where $\Delta P_i = \Delta P_i^G + \Delta P_i^L$ and $\Delta Q_i = \Delta Q_i^G + \Delta Q_i^L$, 

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix} = 
\begin{bmatrix}
J^{-1}
\end{bmatrix} 
\begin{bmatrix}
\Delta P_2^L \\
\Delta P_3^L \\
\Delta Q_3^L
\end{bmatrix} + 
\begin{bmatrix}
J^{-1}
\end{bmatrix} 
\begin{bmatrix}
\Delta P_2^G \\
\Delta P_3^G \\
\Delta Q_3^G
\end{bmatrix}
\]

(3.54)

where $\Delta P_3^G = \Delta Q_3^G = 0$ since bus 3 is a PQ bus. Therefore Equation (3.54) is of the form

\[
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3|
\end{bmatrix} = 
\begin{bmatrix}
\begin{bmatrix}
J^{-1}
\end{bmatrix} & \begin{bmatrix}
J^{-1} \\
J^{-1}
\end{bmatrix} \\
\begin{bmatrix}
J^{-1} \\
J^{-1}
\end{bmatrix} & \begin{bmatrix}
J^{-1} \\
J^{-1}
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\Delta P_2^L \\
\Delta P_3^L \\
\Delta Q_3^L
\end{bmatrix}
\]

(3.56)

including Equation (3.55) in Equation (3.56) we get
where $a_{ij}$ is an element of the $3 \times 3$ jacobian inverse of the operating point load flow and $b_{ij}$'s are sensitivities evaluated at the operation point. If $Q_{G}^{Q}$ violated constraints the voltage at bus 2 would no longer be considered constant and the linear model using Equations (3.6) and (3.7) would be,

$$
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta |V_3| \\
\Delta |V_2| \\
\Delta Q_{G}^2
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{11} & 0 \\
a_{21} & a_{22} & a_{23} & a_{21} & 0 \\
a_{31} & a_{32} & a_{33} & a_{31} & 0 \\
0 & 0 & 0 & 0 & 0 \\
b_{11} & b_{12} & b_{13} & b_{11} & -1
\end{bmatrix}
\begin{bmatrix}
\Delta P_{2}^L \\
\Delta P_{3}^L \\
\Delta Q_{3}^L \\
\Delta P_{2}^G \\
\Delta Q_{2}^G
\end{bmatrix}
$$

(3.57)

where $c_{ij}$ is an element of the $4 \times 4$ jacobian inverse of the operating point load flow.

Using the linearization of Equation (3.57) the change in reactive power generation at bus 2 is,

$$
\Delta Q_{2}^G = b_{11} \Delta P_{2}^L + b_{12} \Delta P_{3}^L + b_{13} \Delta Q_{3}^L + b_{11} \Delta P_{2}^G - \Delta Q_{2}^L
$$

Let $\Delta Q_{2}^G$ be denoted as $z_5$. Furthermore, introduce four other variables as, $z_1 = \Delta P_{2}^L$, $z_2 = \Delta P_{3}^L$, $z_3 = \Delta Q_{3}^L$, $z_4 = \Delta P_{2}^G$ or in the vector form we have
\[
\begin{bmatrix}
\Delta P_2^L \\
\Delta P_3^L \\
\Delta Q_3^L \\
\Delta P_2^G \\
\Delta Q_2^L
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
b_{11} & b_{12} & b_{13} & b_{11} & -1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5
\end{bmatrix}
\] (3.59)

for \( \Delta Q_2^G = z_5 < Q_2^{G\text{max}} - Q_2^{G} \) Equation (3.57) gives,
\[
\Delta |V_3| = a_{31}z_1 + a_{32}z_2 + a_{33}z_3 + a_{31}z_4
\] (3.60)

and for \( \Delta Q_2^G = z_5 > Q_2^{G\text{max}} - Q_2^{G} \) Equation (3.58) gives,
\[
\Delta |V_3| = c_{31}z_1 + c_{32}z_2 + c_{33}z_3 + c_{31}z_4 + c_{34} \Delta Q_2^L
\]
or
\[
\Delta |V_3| = (c_{31} + c_{34}b_{11})z_1 + (c_{32} + c_{34}b_{12})z_2 +
\]
\[
(c_{33} + c_{34}b_{13})z_3 + (c_{31} + c_{34}b_{11})z_4 - c_{34}z_5
\] (3.61)

Equation (3.60) is of the form
\[
\Delta |V_3| = g_1(Z) \text{ if } z_5 < Q_2^{G\text{max}} - Q_2^{G}
\] (3.62)

and Equation (3.61) is of the form
\[
\Delta |V_3| = g_2(Z) \text{ if } z_5 > Q_2^{G\text{max}} - Q_2^{G}
\] (3.63)

where \( Z = [z_1, z_2, z_3, z_4, z_5]^t \).
In terms of the transformed variables Z, the dual linearization has a boundary which is only a function of $z_5$. The statistical moments and hence the distribution of $\Delta|V_3|$ can readily be determined from the joint density of Z using the two functions $g_1(Z)$ and $g_2(Z)$. Then the $i$th moment of $\Delta|V_3|$ is,

$$E[\Delta|V_3|] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [g_1(Z)]^i f_Z(Z) dz_1 dz_2 dz_3 dz_4 dz_5$$

(3.64)

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [g_2(Z)]^i f_Z(Z) dz_1 dz_2 dz_3 dz_4 dz_5$$

where $l = Q_{2G}^{G_{max}} - Q_{2G}^{G_{G}}$, and $f_Z(Z)$ is the joint probability density function of Z. If loads are considered normally distributed multivariate random variables, $f_Z(Z)$ is

$$f(z_1...z_n) = (2\pi)^{-n/2} |C|^{-1/2} \exp\left[-\frac{1}{2}(Z-M)^t C^{-1} (Z-M)\right]$$

(3.65)

where M is the vector mean of Z, C is the covariance matrix of Z defined by

$$C = E[(Z-M)(Z-M)^t]$$

(3.66)

3.2.4 Non Correlated Non Normally Distributed Loads

If a sequence of random loads $Y = [y_1...y_n]^t$ possess a non normal multivariate density function, the multiple linear transformations method developed in section 3.2.3 and a multivariate Gram-Charlier series may be used to solve the constrained stochastic load flow problem. The multivariate Gram-Charlier
series is used to represent \( f_Z(Z) \), the joint probability density function of \( Z \). The multivariate Gram-Charlier series is a generalization of the single variate series described in section 3.2.2. This series and the conditions for its use are presented in reference [13].

3.3 **Multiple Constraints Solution**

For a system with \( k \) critical constraints, there are \( 2^k \) different operating conditions to be considered. However, simplifications may be made to handle the problem. We may consider only those operating conditions which are most probable to occur. Thus reducing the number of linear transformations required to solve the problem. An orthogonal transformation will present an effective extension of the work on one variable.
CHAPTER 4
EXAMPLES AND EVALUATION OF RESULTS

The multiple linear transformations method developed for fully correlated normally distributed loads, was applied to two different power systems. The results were compared with Monte Carlo simulations. A Monte Carlo simulation of the stochastic power flow problem is the repeated solutions of iterative conventional load flow problems, where the statistical distribution of the variations in input load represent those used in the stochastic power flow problem.

4.1 Eleven Bus System Example

The one line system diagram of the eleven bus system is shown in Figure 4.1. The system bus and line data are given in Tables 4.1 and 4.2 respectively. The system's operating constraints are given in Table 4.3. The upper limit on the reactive power at bus 11 is specified such that \( Q_{11} \) violates its upper limit during the very first iterations of the base case load flow. Thus the upper limit on \( Q_s \) serves as the critical constraint for the analysis.

The means and variances of the output variables are computed corresponding to a percentage change in the total real power load demand. Probable ranges are taken as \((\eta \pm 3\sigma)\) where \( \eta \) is the mean value. The results are tabulated in Tables 4.5 - 4.8. The results correspond to a 10% increase in \( P_{LO} \).
Figure 4.1 Eleven bus system one line diagram
Table 4.1
Eleven bus system bus data

<table>
<thead>
<tr>
<th>Bus No. (1)</th>
<th>L-L RMS Base KV (2)</th>
<th>Scheduled Gen (Inj) Pu</th>
<th>Scheduled Load (Inj) Pu</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>345</td>
<td>6.000 + j1.000</td>
<td>-8.960 + j1.3100</td>
</tr>
<tr>
<td>2</td>
<td>345</td>
<td>0 + j0</td>
<td>-3.720 - j0.8300</td>
</tr>
<tr>
<td>3</td>
<td>345</td>
<td>0 + j0</td>
<td>-1.340 - j0.1570</td>
</tr>
<tr>
<td>4</td>
<td>345</td>
<td>0 + j0</td>
<td>0 + j0</td>
</tr>
<tr>
<td>5</td>
<td>345</td>
<td>11.060 + j1.184</td>
<td>-0.602 - j0.0683</td>
</tr>
<tr>
<td>6</td>
<td>138</td>
<td>0 + j0</td>
<td>-3.700 - j0.1220</td>
</tr>
<tr>
<td>7</td>
<td>138</td>
<td>0 + j0</td>
<td>-1.000 - j0.0670</td>
</tr>
<tr>
<td>8</td>
<td>345</td>
<td>0 + j0</td>
<td>0 + j0</td>
</tr>
<tr>
<td>9</td>
<td>345</td>
<td>0 + j0</td>
<td>0 + j0</td>
</tr>
<tr>
<td>10</td>
<td>138</td>
<td>0 + j0</td>
<td>-1.370 - j0.0410</td>
</tr>
<tr>
<td>11</td>
<td>345</td>
<td>9.250 + j0.102</td>
<td>-5.290 - j0.3600</td>
</tr>
</tbody>
</table>

(1) slack bus = bus no. 1
(2) base MVA = 100.0
<table>
<thead>
<tr>
<th>Line No.</th>
<th>Sending bus no.</th>
<th>Receiving bus no.</th>
<th>Line impedance pu</th>
<th>Line charging B pu</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.0015 + j0.0212</td>
<td>0.38400</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0.0016 + j0.0242</td>
<td>0.42630</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3</td>
<td>0.0034 + j0.0496</td>
<td>0.82600</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>0.0023 + j0.0333</td>
<td>0.60230</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>0.0011 + j0.0152</td>
<td>0.27370</td>
</tr>
<tr>
<td>6 (1)</td>
<td>4</td>
<td>6</td>
<td>0.0008 + j0.0262</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>7</td>
<td>0.0203 + j0.1553</td>
<td>0.07218</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>10</td>
<td>0.0284 + j0.1330</td>
<td>0.03060</td>
</tr>
<tr>
<td>9 (2)</td>
<td>9</td>
<td>10</td>
<td>0 + j0.0277</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>5</td>
<td>0.0001 + j0.0024</td>
<td>0.04330</td>
</tr>
<tr>
<td>11 (3)</td>
<td>7</td>
<td>8</td>
<td>0 + j0.0277</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>11</td>
<td>0.0014 + j0.0142</td>
<td>0.26580</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>5</td>
<td>0.0011 + j0.0178</td>
<td>0.18420</td>
</tr>
</tbody>
</table>

(1) Fixed tap transformer, tap bus 4 off nominal ratio = 1.0
(2) Fixed tap transformer, tap bus 9 off nominal ratio = 1.0
(3) Tap changing under load transformer, tap bus 7
Table 4.3

Eleven bus system operating constraints

<table>
<thead>
<tr>
<th>BUS NO.</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Desired voltage = 1.02 pu</td>
</tr>
<tr>
<td>1</td>
<td>Minimum P = -2.00 pu</td>
</tr>
<tr>
<td>1</td>
<td>Maximum P = +3.00 pu</td>
</tr>
<tr>
<td>1</td>
<td>Minimum Q = -2.00 pu</td>
</tr>
<tr>
<td>1</td>
<td>Maximum Q = +3.00 pu</td>
</tr>
<tr>
<td>5</td>
<td>Desired voltage = 1.02 pu</td>
</tr>
<tr>
<td>5</td>
<td>Minimum Q = -2.00 pu</td>
</tr>
<tr>
<td>5</td>
<td>Maximum Q = +1.00 pu</td>
</tr>
<tr>
<td>7</td>
<td>Desired voltage = 1.020 pu</td>
</tr>
<tr>
<td>7</td>
<td>Minimum off nominal tap = 0.90</td>
</tr>
<tr>
<td>7</td>
<td>Maximum off nominal tap = 1.10</td>
</tr>
<tr>
<td>11</td>
<td>Desired voltage = 1.020 pu</td>
</tr>
<tr>
<td>11</td>
<td>Minimum Q = -2.0 pu</td>
</tr>
<tr>
<td>11</td>
<td>Maximum Q = +0.01 pu</td>
</tr>
</tbody>
</table>
Table 4.4

Eleven bus system base case operating point

<table>
<thead>
<tr>
<th>Bus No.</th>
<th>Voltage magnitude (pu)</th>
<th>Voltage angle (deg)</th>
<th>Tap setting</th>
<th>Required generation (pu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0200</td>
<td>0.000</td>
<td></td>
<td>5.906 - j0.3897</td>
</tr>
<tr>
<td>2</td>
<td>0.9776</td>
<td>-1.486</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.0030</td>
<td>3.721</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.0070</td>
<td>10.780</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.0180</td>
<td>16.310</td>
<td></td>
<td>11.060 + j1.0000</td>
</tr>
<tr>
<td>6</td>
<td>0.9982</td>
<td>6.659</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.0200</td>
<td>14.970</td>
<td>0.9879</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.0120</td>
<td>18.190</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.0180</td>
<td>16.140</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.0170</td>
<td>14.200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.0130</td>
<td>21.350</td>
<td></td>
<td>9.250 + j0.0100</td>
</tr>
<tr>
<td>X</td>
<td>E{X}</td>
<td>T1</td>
<td>T2</td>
<td>MLT</td>
</tr>
<tr>
<td>------------</td>
<td>------------</td>
<td>------</td>
<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>$P_{G1}$</td>
<td></td>
<td>5.9060</td>
<td>5.9060</td>
<td>5.9045</td>
</tr>
<tr>
<td>$Q_{G1}$</td>
<td></td>
<td>-0.3897</td>
<td>-0.3897</td>
<td>-0.4724</td>
</tr>
<tr>
<td>$Q_{G5}$</td>
<td></td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0458</td>
</tr>
<tr>
<td>$Q_{G11}$</td>
<td></td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
</tr>
<tr>
<td>$</td>
<td>V_{2}</td>
<td>$</td>
<td></td>
<td>0.9776</td>
</tr>
<tr>
<td>$</td>
<td>V_{3}</td>
<td>$</td>
<td></td>
<td>1.0030</td>
</tr>
<tr>
<td>$</td>
<td>V_{4}</td>
<td>$</td>
<td></td>
<td>1.0070</td>
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<tr>
<td>$</td>
<td>V_{5}</td>
<td>$</td>
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<td>1.0180</td>
</tr>
<tr>
<td>$</td>
<td>V_{6}</td>
<td>$</td>
<td></td>
<td>0.9982</td>
</tr>
<tr>
<td>$t_7$</td>
<td></td>
<td>0.9879</td>
<td>0.9879</td>
<td>0.9931</td>
</tr>
<tr>
<td>$</td>
<td>V_{8}</td>
<td>$</td>
<td></td>
<td>1.0120</td>
</tr>
<tr>
<td>$</td>
<td>V_{9}</td>
<td>$</td>
<td></td>
<td>1.0180</td>
</tr>
<tr>
<td>$</td>
<td>V_{10}</td>
<td>$</td>
<td></td>
<td>1.0170</td>
</tr>
<tr>
<td>$</td>
<td>V_{11}</td>
<td>$</td>
<td></td>
<td>1.0130</td>
</tr>
</tbody>
</table>
Table 4.6

Eleven bus system standard deviations

<table>
<thead>
<tr>
<th>Variable</th>
<th>P^G</th>
<th>Q^G</th>
<th>Q^G</th>
<th>Q^G</th>
<th>IV^2</th>
<th>IV^3</th>
<th>IV^4</th>
<th>IV^5</th>
<th>IV^6</th>
<th>IV^7</th>
<th>IV^8</th>
<th>IV^9</th>
<th>IV^10</th>
<th>IV^11</th>
</tr>
</thead>
<tbody>
<tr>
<td>T_1</td>
<td>0.3074</td>
<td>0.2369</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0073</td>
<td>0.0058</td>
<td>0.0082</td>
<td>0.0082</td>
<td>0.0076</td>
<td>0.0141</td>
<td>0.0112</td>
<td>0.0087</td>
<td>0.0072</td>
<td>0.0115</td>
</tr>
<tr>
<td>T_2</td>
<td>0.3036</td>
<td>0.0243</td>
<td>0.1178</td>
<td>0.0000</td>
<td>0.0027</td>
<td>0.0045</td>
<td>0.0009</td>
<td>0.0000</td>
<td>0.0013</td>
<td>0.0007</td>
<td>0.0003</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0006</td>
</tr>
<tr>
<td>MLT</td>
<td>0.3055</td>
<td>0.1470</td>
<td>0.0694</td>
<td>0.0000</td>
<td>0.0052</td>
<td>0.0039</td>
<td>0.0052</td>
<td>0.0052</td>
<td>0.0048</td>
<td>0.0085</td>
<td>0.0067</td>
<td>0.0052</td>
<td>0.0043</td>
<td>0.0070</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>0.3060</td>
<td>0.1670</td>
<td>0.0595</td>
<td>0.0000</td>
<td>0.0056</td>
<td>0.0043</td>
<td>0.0058</td>
<td>0.0059</td>
<td>0.0055</td>
<td>0.0098</td>
<td>0.0071</td>
<td>0.0059</td>
<td>0.0049</td>
<td>0.0079</td>
</tr>
</tbody>
</table>
Table 4.7

Eleven bus system range minimums

<table>
<thead>
<tr>
<th>X</th>
<th>Range minimum of X</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T_1 )</td>
<td>( T_2 )</td>
<td>MLT</td>
<td>Monte Carlo</td>
<td></td>
</tr>
<tr>
<td>( P_1^G )</td>
<td>4.9840</td>
<td>4.9960</td>
<td>4.9880</td>
<td>5.3150</td>
<td></td>
</tr>
<tr>
<td>( Q_1^G )</td>
<td>-1.1010</td>
<td>-0.4627</td>
<td>-0.9135</td>
<td>-0.4817</td>
<td></td>
</tr>
<tr>
<td>( Q_5^G )</td>
<td>1.0000</td>
<td>0.6465</td>
<td>0.8376</td>
<td>0.8098</td>
<td></td>
</tr>
<tr>
<td>( Q_{11}^G )</td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
<td></td>
</tr>
<tr>
<td>(</td>
<td>v_2</td>
<td>)</td>
<td>0.9557</td>
<td>0.9695</td>
<td>0.9637</td>
</tr>
<tr>
<td>(</td>
<td>v_3</td>
<td>)</td>
<td>0.9859</td>
<td>0.9991</td>
<td>0.9934</td>
</tr>
<tr>
<td>(</td>
<td>v_4</td>
<td>)</td>
<td>0.9819</td>
<td>1.0040</td>
<td>0.9940</td>
</tr>
<tr>
<td>(</td>
<td>v_5</td>
<td>)</td>
<td>0.9915</td>
<td>1.0200</td>
<td>1.0058</td>
</tr>
<tr>
<td>(</td>
<td>v_6</td>
<td>)</td>
<td>0.9754</td>
<td>0.9944</td>
<td>0.9860</td>
</tr>
<tr>
<td>( t_7 )</td>
<td>0.9455</td>
<td>0.9857</td>
<td>0.9675</td>
<td>0.9632</td>
<td></td>
</tr>
<tr>
<td>(</td>
<td>v_8</td>
<td>)</td>
<td>0.9785</td>
<td>1.0110</td>
<td>0.9962</td>
</tr>
<tr>
<td>(</td>
<td>v_9</td>
<td>)</td>
<td>0.9916</td>
<td>1.0180</td>
<td>1.0060</td>
</tr>
<tr>
<td>(</td>
<td>v_{10}</td>
<td>)</td>
<td>0.9949</td>
<td>1.0170</td>
<td>1.0070</td>
</tr>
<tr>
<td>(</td>
<td>v_{11}</td>
<td>)</td>
<td>0.9784</td>
<td>1.0110</td>
<td>0.9963</td>
</tr>
</tbody>
</table>
Table 4.8
Eleven bus system range maximums

<table>
<thead>
<tr>
<th>X</th>
<th>Range maximum of X</th>
<th>T1</th>
<th>T2</th>
<th>MLT</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1G</td>
<td>6.8290</td>
<td>6.8170</td>
<td>6.8210</td>
<td>6.3870</td>
<td></td>
</tr>
<tr>
<td>Q1G</td>
<td>0.3213</td>
<td>-0.3167</td>
<td>-0.0313</td>
<td>0.02153</td>
<td></td>
</tr>
<tr>
<td>Q5G</td>
<td>1.0000</td>
<td>1.3530</td>
<td>1.2540</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>Q11G</td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
<td></td>
</tr>
<tr>
<td>[v_2]</td>
<td>0.9995</td>
<td>0.9857</td>
<td>0.9950</td>
<td>0.9838</td>
<td></td>
</tr>
<tr>
<td>[v_3]</td>
<td>1.0210</td>
<td>1.0080</td>
<td>1.0170</td>
<td>1.0070</td>
<td></td>
</tr>
<tr>
<td>[v_4]</td>
<td>1.0310</td>
<td>1.0100</td>
<td>1.0250</td>
<td>1.0100</td>
<td></td>
</tr>
<tr>
<td>[v_5]</td>
<td>1.0450</td>
<td>1.0200</td>
<td>1.0390</td>
<td>1.0200</td>
<td></td>
</tr>
<tr>
<td>[v_6]</td>
<td>1.0210</td>
<td>1.0020</td>
<td>1.0150</td>
<td>1.0020</td>
<td></td>
</tr>
<tr>
<td>[v_7]</td>
<td>1.0300</td>
<td>0.9900</td>
<td>1.0190</td>
<td>0.9922</td>
<td></td>
</tr>
<tr>
<td>[v_8]</td>
<td>1.0460</td>
<td>1.0130</td>
<td>1.0360</td>
<td>1.0150</td>
<td></td>
</tr>
<tr>
<td>[v_9]</td>
<td>1.0440</td>
<td>1.0180</td>
<td>1.0370</td>
<td>1.0200</td>
<td></td>
</tr>
<tr>
<td>[v_{10}]</td>
<td>1.0380</td>
<td>1.0170</td>
<td>1.0320</td>
<td>1.0180</td>
<td></td>
</tr>
<tr>
<td>[v_{11}]</td>
<td>1.0480</td>
<td>1.0150</td>
<td>1.0380</td>
<td>1.0170</td>
<td></td>
</tr>
</tbody>
</table>
T_1 is the base case transformation where bus 5 is a PQ bus and T_2 represents the transformation where bus 5 has a PV status. The Monte Carlo results correspond to 10 normally distributed random samples.

4.2 Six Bus System Example

In order to verify the results obtained from the eleven bus example, the multiple linear transformations method was applied to an ideal six bus system. The system's one line diagram is shown in Figure 4.2 and the system's data are given in Tables 4.9 - 4.11. Table 4.12 represents the base case operating point of the system where the upper limit on Q^G serves as the critical constraint in the analysis. The statistical results are tabulated in Tables 4.13 - 4.16, and they correspond to an increase of 10% in the base case total real power load demand.

4.3 Comparison of Results

The results of mean values, standard deviations, and range minimums clearly indicate that the multiple linear transformations method offers a great improvement over transformations T_1 and T_2. The range maximums indicate that T_2 provides results which are closer to the Monte Carlo solution; however, the multiple linear transformations results are closer to the actual solution than those of the base case transformation T_1. Therefore we may conclude that the multiple linear transformations method generates better results.
than the base case transformation of the conventional stochastic load flow analysis.

The results of voltage angles and real power line flows are not presented; however, they indicated that no drastic differences existed between the two methods and either method would provide acceptable results. This is primarily due to the fact that voltage angles are not as sensitive to the loss of voltage controlled busses as voltage magnitudes are.
Figure 4.2 Six bus system one line diagram
Table 4.9

Six bus system bus data

<table>
<thead>
<tr>
<th>Bus No. (1)</th>
<th>L-L RMS Base KV (2)</th>
<th>Scheduled Gen (Inj) Pu</th>
<th>Scheduled Load (Inj) Pu</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>345</td>
<td>4.25 + j0</td>
<td>0 + j0</td>
</tr>
<tr>
<td>2</td>
<td>138</td>
<td>0 + j0</td>
<td>-2.24 + j0</td>
</tr>
<tr>
<td>3</td>
<td>138</td>
<td>1.60 + j0.360</td>
<td>-0.35 + j0</td>
</tr>
<tr>
<td>4</td>
<td>345</td>
<td>0 + j0</td>
<td>-1.47 + j0</td>
</tr>
<tr>
<td>5</td>
<td>138</td>
<td>0 + j0</td>
<td>-2.24 + j0</td>
</tr>
<tr>
<td>6</td>
<td>345</td>
<td>0.80 + j0.367</td>
<td>0.70 + j0</td>
</tr>
</tbody>
</table>

(1) slack bus = bus no. 1
(2) base MVA = 100.0
Table 4.10
Six bus system line data

<table>
<thead>
<tr>
<th>Line No.</th>
<th>Sending bus no.</th>
<th>Receiving bus no.</th>
<th>Line impedance Pu</th>
<th>Line charging B Pu</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>0 + j0.3003</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
<td>0 + j0.3003</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0 + j0.3000</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>0 + j0.3000</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>0 + j0.3000</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>5</td>
<td>0 + j0.6097</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>3</td>
<td>0 + j0.2000</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>3</td>
<td>0 + j0.2000</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>6</td>
<td>0 + j0.2000</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>2</td>
<td>0 + j0.4000</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>4</td>
<td>0 + j0.5988</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>2</td>
<td>0 + j0.4000</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>3</td>
<td>0 + j0.2000</td>
<td>0</td>
</tr>
</tbody>
</table>
### Table 4.11

**Six bus system operating constraints**

<table>
<thead>
<tr>
<th>Bus No.</th>
<th>Constraint Description</th>
<th>Value 1</th>
<th>Value 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bus No. 1</td>
<td>Desired voltage</td>
<td>1.00 pu</td>
<td>1.00 pu</td>
</tr>
<tr>
<td>Bus No. 1</td>
<td>Minimum $P$</td>
<td>-2.00 pu</td>
<td>-2.00 pu</td>
</tr>
<tr>
<td>Bus No. 1</td>
<td>Maximum $P$</td>
<td>+3.00 pu</td>
<td>+3.00 pu</td>
</tr>
<tr>
<td>Bus No. 1</td>
<td>Minimum $Q$</td>
<td>-2.00 pu</td>
<td>-2.00 pu</td>
</tr>
<tr>
<td>Bus No. 1</td>
<td>Minimum $Q$</td>
<td>+3.00 pu</td>
<td>+3.00 pu</td>
</tr>
<tr>
<td>Bus No. 3</td>
<td>Desired voltage</td>
<td>1.00 pu</td>
<td>1.00 pu</td>
</tr>
<tr>
<td>Bus No. 3</td>
<td>Minimum $Q$</td>
<td>-0.200 pu</td>
<td>-0.200 pu</td>
</tr>
<tr>
<td>Bus No. 3</td>
<td>Maximum $Q$</td>
<td>+0.334 pu</td>
<td>+0.334 pu</td>
</tr>
<tr>
<td>Bus No. 6</td>
<td>Desired voltage</td>
<td>1.00 pu</td>
<td>1.00 pu</td>
</tr>
<tr>
<td>Bus No. 6</td>
<td>Minimum $Q$</td>
<td>-0.200 pu</td>
<td>-0.200 pu</td>
</tr>
<tr>
<td>Bus No. 6</td>
<td>Maximum $Q$</td>
<td>+0.020 pu</td>
<td>+0.020 pu</td>
</tr>
</tbody>
</table>
Table 4.12

Six bus system base case operating point

<table>
<thead>
<tr>
<th>Bus No.</th>
<th>Voltage magnitude (pu)</th>
<th>Voltage angle (deg)</th>
<th>Tap setting</th>
<th>Required generation (pu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>0.000</td>
<td></td>
<td>3.200 + j0.615</td>
</tr>
<tr>
<td>2</td>
<td>0.9845</td>
<td>-9.736</td>
<td></td>
<td>1.600 + j0.334</td>
</tr>
<tr>
<td>3</td>
<td>0.9996</td>
<td>-6.672</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.9793</td>
<td>-10.500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.9795</td>
<td>-12.460</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.9682</td>
<td>-1.435</td>
<td></td>
<td>0.800 + j0.020</td>
</tr>
</tbody>
</table>
Table 4.13

Six bus system expected values

<table>
<thead>
<tr>
<th>$X$</th>
<th>$E{X}$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$T_1$</td>
</tr>
<tr>
<td>$q_1^G$</td>
<td>0.6150</td>
</tr>
<tr>
<td>$q_3^G$</td>
<td>0.3340</td>
</tr>
<tr>
<td>$q_6^G$</td>
<td>0.0200</td>
</tr>
<tr>
<td>$</td>
<td>V_2</td>
</tr>
<tr>
<td>$</td>
<td>V_3</td>
</tr>
<tr>
<td>$</td>
<td>V_4</td>
</tr>
<tr>
<td>$</td>
<td>V_5</td>
</tr>
<tr>
<td>$</td>
<td>V_6</td>
</tr>
</tbody>
</table>
Table 4.14
Six bus system standard deviations

<table>
<thead>
<tr>
<th>Variable</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>MLT</th>
<th>Monte Carlo</th>
</tr>
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<tbody>
<tr>
<td>$P^G_1$</td>
<td>0.1244</td>
<td>0.1244</td>
<td>0.1244</td>
<td>0.1244</td>
</tr>
<tr>
<td>$Q^G_1$</td>
<td>0.0700</td>
<td>0.0444</td>
<td>0.0577</td>
<td>0.0580</td>
</tr>
<tr>
<td>$Q^G_3$</td>
<td>0.0000</td>
<td>0.0223</td>
<td>0.0130</td>
<td>0.0124</td>
</tr>
<tr>
<td>$Q^G_6$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$</td>
<td>V_2</td>
<td>$</td>
<td>0.0025</td>
<td>0.0010</td>
</tr>
<tr>
<td>$</td>
<td>V_3</td>
<td>$</td>
<td>0.0041</td>
<td>0.0000</td>
</tr>
<tr>
<td>$</td>
<td>V_4</td>
<td>$</td>
<td>0.0021</td>
<td>0.0014</td>
</tr>
<tr>
<td>$</td>
<td>V_5</td>
<td>$</td>
<td>0.0046</td>
<td>0.0012</td>
</tr>
<tr>
<td>$</td>
<td>V_6</td>
<td>$</td>
<td>0.0042</td>
<td>0.0017</td>
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Table 4.15
Six bus system range minimums

<table>
<thead>
<tr>
<th>X</th>
<th>Range minimum of X</th>
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<tbody>
<tr>
<td></td>
<td>T₁</td>
</tr>
<tr>
<td>P₁</td>
<td>2.8270</td>
</tr>
<tr>
<td>Q₁</td>
<td>0.4050</td>
</tr>
<tr>
<td>Q₃</td>
<td>0.3340</td>
</tr>
<tr>
<td>Q₆</td>
<td>0.0200</td>
</tr>
<tr>
<td></td>
<td>V₂</td>
</tr>
<tr>
<td></td>
<td>V₃</td>
</tr>
<tr>
<td></td>
<td>V₄</td>
</tr>
<tr>
<td></td>
<td>V₅</td>
</tr>
<tr>
<td></td>
<td>V₆</td>
</tr>
</tbody>
</table>
Table 4.16
Six bus system range maximums

<table>
<thead>
<tr>
<th>X</th>
<th>Range maximum of X</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T₁</td>
</tr>
<tr>
<td>P₁</td>
<td>3.5730</td>
</tr>
<tr>
<td>Q₁</td>
<td>0.8251</td>
</tr>
<tr>
<td>Q₃</td>
<td>0.3340</td>
</tr>
<tr>
<td>Q₆</td>
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</tr>
<tr>
<td></td>
<td>V₂</td>
</tr>
<tr>
<td></td>
<td>V₃</td>
</tr>
<tr>
<td></td>
<td>V₄</td>
</tr>
<tr>
<td></td>
<td>V₅</td>
</tr>
<tr>
<td></td>
<td>V₆</td>
</tr>
</tbody>
</table>
5.1 Conclusions

The multiple linear transformations method presented in this thesis can be used to account for the discrete non-linearities introduced by the physical constraints of sources and mechanical equipments in the stochastic solution of the load flow problem. The algorithm has been tested for fully correlated normally distributed random loads considering only a single constrained variable. The results obtained from the computation of the output variables statistics and probable ranges indicate that the algorithm offers significant improvements over the base case single transformation method. While a method employing the Gram-Charlier series has been developed to represent the probability density function of non normally distributed loads, the performance of the algorithm has not been tested for such loads as well as the non correlated loads. The following section offers recommendations for future development of the algorithm.

5.2 Recommendations

The multiple linear transformations method for a single constrained variable may be generalized to provide a theoretical solution for multiple constrained variables. A method for determining the probable ranges of non normal output variables, using a multivariate Gram-Charlier series is needed. Also a
method to compute the statistics of reactive line power flows may be devised.

Since the method for determining the limits of reactive power available at a voltage controlled bus is a function of the real power generation, the limiting values may vary as the generation varies in cyclic units. Therefore, the constrained reactive power supply limits may be considered as random quantities causing a considerable effect on the statistical solution. Thus the effects of different types of generation on the stochastic load flow solution are needed to be studied.
LIST OF REFERENCES


APPENDIX A

DERIVATION OF ELEMENTS OF THE JACOBIAN MATRIX

A.1 Jacobian Matrix Elements

Injected current $\mathbf{I}_i$ is related to all $n$ node voltages through the network bus admittance matrix $\mathbf{Y}$, referenced to ground such that,

$$
\mathbf{S}_i = \mathbf{V}_i \sum_{j=1}^{n} \mathbf{Y}^*_{ij} \mathbf{V}_j^* \quad (A.1)
$$

The real and reactive injected powers are,

$$
P_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{V}_j||Y_{ij}|\cos(\delta_i - \delta_j - \theta_{ij}) \quad (A.2)
$$

$$
Q_i = \sum_{j=1}^{n} |\mathbf{V}_i||\mathbf{V}_j||Y_{ij}|\sin(\delta_i - \delta_j - \theta_{ij}) \quad (A.3)
$$

where $\delta_i$ is the angle of the phasor voltage at bus $i$ referenced to the swing bus, $\theta_{ij}$ is the angle of admittance matrix element $i, j$, and $|\mathbf{V}_i|$ is the magnitude of phasor voltage $\mathbf{V}_i$ referenced to ground. The jacobian derivatives in polar notation are,

$$
J_{P_\delta}(i,i) = \frac{\partial P_i}{\partial \delta_i} = - \sum_{j=1}^{n} \frac{|\mathbf{V}_i||\mathbf{V}_j||Y_{ij}|\sin(\delta_i - \delta_j - \theta_{ij})}{|\mathbf{V}_i||\mathbf{V}_j||Y_{ij}|} \quad (A.4)
$$

$$
J_{P_\delta}(i,k) = \frac{\partial P_i}{\partial \delta_k} = |\mathbf{V}_i||\mathbf{V}_k||Y_{ik}|\sin(\delta_i - \delta_k - \theta_{ik})(k\neq i) \quad (A.5)
$$
\[ J_{PV}(i,i) = \frac{3P_i}{3V_1} = \frac{n}{j=1, \neq i} |V_j||Y_{ij}| \cos(\delta_i - \delta_j - \theta_{ij}) \]
\[ + 2|V_i||Y_{ii}| \cos(-\theta_{ii}) \]  
\[ \text{(A.6)} \]

\[ J_{PV}(i,k) = \frac{3P_i}{3V_k} = |V_i||Y_{ik}| \cos(\delta_i - \delta_k - \theta_{ik}) \]  
\[ (k \neq i) \]  
\[ \text{(A.7)} \]

\[ J_{Q\delta}(i,i) = \frac{3Q_i}{3\delta} = \frac{n}{j=1, \neq i} |V_i||V_j||Y_{ij}| \cos(\delta_i - \delta_j - \theta_{ij}) \]
\[ \text{(A.8)} \]

\[ J_{Q\delta}(i,k) = \frac{3Q_i}{3\delta_k} = -|V_i||V_k||Y_{ik}| \cos(\delta_i - \delta_k - \theta_{ik}) \]  
\[ (k \neq i) \]  
\[ \text{(A.9)} \]

\[ J_{QV}(i,i) = \frac{3Q_i}{3V_1} = \frac{n}{j=1, \neq i} |V_j||Y_{ij}| \sin(\delta_i - \delta_j - \theta_{ij}) \]
\[ + 2|V_i||Y_{ii}| \sin(-\theta_{ii}) \]  
\[ \text{(A.10)} \]

\[ J_{QV}(i,k) = \frac{3Q_i}{3V_k} = |V_i||Y_{ik}| \sin(\delta_i - \delta_k - \theta_{ik}) \]  
\[ (k \neq i) \]  
\[ \text{(A.11)} \]
APPENDIX B

DERIVATION OF THE REAL POWER GENERATION TRANSFORMATION

B.1 Real Power Generation Transformation

Assuming the real power generation at bus i is a function of total real power load demand,

\[ P^G_i = K_i P^L_{TOT} \quad (B.1) \]

where \( K_i \) is a constant.

Let \( X \) be the percent change in \( P^L_{TOT} \) and \( P'_L_{TOT} \) be the new operating point where,

\[ P'_L_{TOT} = (1+X) P^L_{TOT} \quad (B.2) \]

Therefore the corresponding total generation would be,

\[ P'_G_{TOT} = \sum_{i=1}^{NG} P'_G_i = \sum_{i=1}^{NG} K_i P'_L_{TOT} \quad (B.3) \]

where \( NG \) is the number of generation units. Assuming some units are non cyclic where they do not contribute to the required increase in generation, \( P'_G_{TOT} \) can be expressed as,

\[ P'_G_{TOT} = \sum_{i=1}^{NG} \sum_{j=1}^{ng} K_i (1+X) P^L_{TOT} \quad \#n.c \quad (B.4) \]

where \( n.c \) denotes the non cyclic units, or

\[ P'_G_{TOT} = \left[ \sum_{i=1}^{NG} K_i (1+X) - \sum_{j=1}^{\#c} K_j \right] P^L_{TOT} \quad (B.5) \]
where \( c \) denotes the cycling units. Therefore the corresponding real power generation of unit \( k \) is,

\[
P_k' = \frac{K_k}{\sum_{i=1}^{NG} K_i} \left[ \sum_{i=1}^{NG} \frac{K_i(1+X)}{K_i} - \sum_{j=1}^{NG} K_j \right] p_{OL}^{pL} \quad (B.6)
\]

where

\[
P_k' = P_k^{pG} + \Delta P_k^{pG} \quad (B.7)
\]

Thus the change in real power generation at unit \( k \) corresponding to an \( X \) percent change in \( p_{OL}^{pL} \) is,

\[
\Delta P_k^{pG} = \left\{ \frac{K_k}{\sum_{i=1}^{NG} K_i} \left[ \sum_{i=1}^{NG} \frac{K_i(1+X)}{K_i} - \sum_{j=1}^{NG} K_j \right] - K_k \right\} p_{OL}^{pL} \quad (B.8)
\]

or

\[
\Delta P_k^{pG} = \frac{K_k}{\sum_{i=1}^{NG} K_i} \left[ \sum_{i=1}^{NG} \frac{K_i(1+X)}{K_i} - \sum_{j=1}^{NG} K_j - \sum_{l=1}^{NG} K_l \right] p_{OL}^{pL} \quad (B.9)
\]

but

\[
\sum_{j=1}^{NG} K_j + \sum_{l=1}^{NG} K_l = \sum_{i=1}^{NG} K_i
\]

\( \neq n.c \)

thus
\[ \Delta P_k = \frac{K_k}{\sum_{l=1}^{\text{NG}} K_l} \left[ \sum_{i=1}^{\text{NG}} K_i \right] (x \cdot P_{T O T}^{OL}) \]  

(B.10)

or

\[ \Delta P_k = K_k \left[ \sum_{i=1}^{\text{NG}} K_i \right] \Delta P_{T O T}^{L} \]  

(B.11)

Substituting Equation (B.1) for \( K_i \),

\[ \Delta P_k = \frac{P_{G}^{O}}{P_{O L}^{T O T}} \left[ \sum_{i=1}^{\text{NG}} P_i^{G} \right] \left[ \sum_{i=1}^{\text{NG}} P_i^{O} \right] \]  

(B.12)

or

\[ \Delta P_k = C_k \left[ \frac{P_{G}^{O} \cdot P_{O L}^{G}}{P_{O L}^{T O T}} \right] \Delta P_{T O T}^{L} \]  

(B.13)

where \( C_i \) is unity for cyclic units and zero for non cyclic units. Therefore

\[ \Delta P_k = \alpha_k \Delta P_{T O T}^{L} \]  

(B.14)

where

\[ \alpha_k = C_k \left[ \frac{P_{G}^{O}}{\sum_{i=1}^{\text{NG}} C_i P_i^{G}} \right] \left[ \frac{P_{O L}^{G}}{P_{O L}^{T O T}} \right] \]  

(B.15)