BOUNDARY CONTROL, APPLIED TO DC-TO-DC CONVERTER CIRCUITS

RUDIGER MUNZERT

PAP-TR-95-8
July 1995

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FOREWORD

This technical report is a reprint of the thesis written by Rudiger Munzert submitted as ‘Studienarbeit’ at the Technische Hochschule Darmstadt, Germany.

P. T. Krein
Thesis Advisor
July 1995
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ACKNOWLEDGEMENTS

With regard to the inception, development and completion of this thesis, I wish to express my deep gratitude to the following persons:

Prof. P. T Krein, who was my thesis advisor at the University of Illinois. I would like to thank him for bringing me into contact with my interesting topic and for the continued support, advice and understanding I received from him. His pertinent assistance was inestimable to me.

Prof. P. Mutschler at the Technische Hochschule Darmstadt, Germany, who gave me the opportunity to carry out such a research project in the United States in the context of the Darmstadt curriculum.

Prof. G. Poser at the Technische Hochschule Darmstadt and all the people from the International Programs in Engineering at the University of Illinois at Urbana-Champaign, who primarily enabled my stay here in Illinois.

Dipl.-Ing. M. Marcks, Technische Hochschule Darmstadt, who gave fast and valuable advice over the Internet.
To all the treasured persons who I left behind in Europe in order to study in the United States of America.
CHAPTER 1

Introduction

Since mankind has learned to satisfy large parts of its energy needs by the means of electricity, availability and ease of electrical power conversion has become a major factor in technological and thus economical development. In recent years many new appliances have relied heavily on one special subgroup of these conversion processes: dc-to-dc conversion. The variety and number of battery powered applications is virtually exploding. From small handheld electronic devices like portable telephones over laptop computers up to electrical vehicles, it is the capability of supplying a set of different dc voltage needs from one single dc source that enables them to work effectively.

1.1 Dc-to-Dc Converters as Fundamentally Nonlinear Systems

Unlike the old, disadvantageous dissipative regulators, modern conversion circuits are all designed as switching converters. These circuits do not split power flow, but turn it on or off completely, at least ideally. The result is an intermittent energy transfer by switch action. In a mathematical sense such systems are discontinuous and thus automatically nonlinear. Unfortunately it is this nonlinearity that, at the same time, represents both the main problem in circuit analysis as well as the bedrock for successful circuit design.

Conventional converter analysis approaches have therefore often tried to suppress the nonlinear characteristics. Techniques such as averaging and linearization have seen frequent application (see for example [1], [2]). The resulting loss in information is significant. Not only does the issue of ripple get neglected in these approaches, but also most of the ensuing mathematical models solely apply to the converters’ behavior in the small.
This thesis pursues an opposite approach. Rather than eliminating the system's nonlinearity it makes it the focus of the study. The concept of a limit cycle serves as the basis of the examination. Accordingly, a state space representation is chosen as a comprehensive means of analysis.

1.2 Previous Use of State Space Analysis in the Study of Dc-to-Dc Converters

As far back as 1964, Schweitzer and Rosenstein [3] studied state plane trajectories in order to approximate transient behavior of a dc-to-dc converter. Four years later, Babaa, Wilson and Yu enlarged the use of this technique by applying it to a converter's steady state solution. They examined limit cycles in a voltage step-down circuit [4].

In the next decade, the work of Burns can be considered as a major milestone. He achieved significant insight in the way switched dc-to-dc converters operate when he studies a boost converter's behavior near the desired output voltage. He interpreted customary control methods (constant on-time control, constant frequency control) in the state plane and conceived a novel control law for a free running converter [5]. In this context, the concept of a switching boundary was introduced and discussed. In two subsequent publications Burns broadened his results on all three common converter types (boost, buck and buckboost circuit), included parasitics into the analysis [6] and showed the practical feasibility of his boundary control law [7].

Burns' idea of a control boundary is not pursued further. Bass researched the topic at that time, and introduced a concept of Boundary Control. In 1988, Bass reconsidered the tool of state plane analysis in general [8]. One year later, he presented the geometric concepts of a switching boundary and a loadline, with a converter's operating point as the intersection of these two [9]. He also enlarged the concept of switching boundaries by suggesting time varying
boundaries [10]. In 1990, Bass used animation on top of the state plane representation in order to achieve an even more powerful analysis and design tool [11].

The control approach known as *sliding mode control* has also made substantial use of the state plane and the switching boundary concept (see for example [12], [13]).

### 1.3 Stationary Boundary Control

The term *Boundary Control* describes the geometrical interpretation of converter operation and control in the state space, using concepts such as trajectories, switching boundaries and loadlines. A control derived from this theoretical framework will be named Boundary Control. Of course, the output of such a boundary control law is simple. In the buck, boost and buckboost converter case there is only one controlled switch. Thus the only means of converter control is to turn this switch completely on or completely off.

With such an a priori limitation in the ability to continuously control the flow of energy, further restrictions imposed on the control process such as fixing the transistor on-time or, even more restrictive, fixing the converter switching frequency can only cause degradation from the theoretically achievable converter performance. Thus, only by utilizing all of the information available from the system, and by allowing the transistor switch to remain on or off for variable time periods can the theoretical limits in converter performance be approached.

*Burns, 1976, [5]*

This statement suggests not to impose restrictive time-dependent switch action on the circuit, which calls for an autonomous, self-oscillating converter.

Self-oscillating converters are the most favorable for geometric analysis, because the switching surfaces in the state-space are time-invariant. [...] Much work is still needed for this class of converters.

*Bass, 1990 [10]*

In accordance with these deliberations this thesis will, after a part of general boundary control considerations, concentrate on a special autonomous boundary control law.
1.4 The Concerns of This Thesis

This thesis recaps the major concepts of Boundary Control, deepens them and applies a special class of boundaries. Chapter 2 provides a brief overview as well as references on the mathematical foundations the subsequent theory of the thesis is based on. Chapter 3 furnishes a comprehensive analysis of the three standard dc-to-dc converter types, the buck, the boost, and the buckboost converter. In the next Chapter, the loadline is introduced in the conventional way. In addition, two new notions of it are worked out. The type of switching boundaries that the thesis will concentrate on is presented. Chapter 5 investigates the implications of this class of boundaries on the converters' global stability behavior. Stability in the large is studied closely and a set of four stability theorems is formulated. After that, the theorems are tested by computer visualization and animation. The simulation procedure and selected results are presented in Chapter 6. Chapter 7 discusses various performance aspects of the chosen boundary control method. Finally, these results are verified in laboratory experiments. The experiments along with their results are shown in Chapter 8. Finally, the Thesis concludes with a summarizing evaluation of the outcomes that were achieved and suggests directions for future work.
2.1 Power Converters and the Systems Approach

Since the 1950s a general mathematical theory of systems has been developed [14], [15]. It provides a useful scheme for describing and understanding power converters. For a controlled electronic converter it seems natural to distinguish between its power stage and its control circuitry. Both can be viewed as subsystems of the complete system "power converter".

Figure 2.1 visualizes the overall converter model which forms the underlying basis for this thesis. Simultaneously it introduces the fundamental notation: $u$ for the converter's input variables, $y$ for its outputs, $x$ for the power stage's state and $q$ for the controller action.

Figure 2.1 Overall model of a power converter

The three main issues in studying a system are the elements of which the system is formed, the laws by which its behavior can be described and the state it can be in.
2.1.1 A Power Converter’s Elements

The elements of a power conversion system are the circuit components constituting the power and the control circuitry. Based on Bass [16], this thesis will assume that power conversion systems can be meaningfully modeled as networks of the following elements:

- The power stage: linear time invariant (LTI) lumped components, independent and linear controlled sources, and ideal switches
- The control stage: LTI lumped components, independent and linear controlled sources, ideal operational amplifiers, logic gates, and ideal switches.

By an (ideal) switch we mean a device with the following characteristic:

\[
R_s = \begin{cases} 
( v_s, i_s): i_s = 0 \quad \text{for } q = 0 \\
( v_s, i_s): v_s = 0 \quad q = 1
\end{cases}
\] (2.1)

2.1.2 A Power Converter’s State

At any moment in time, a system can be characterized by its state.

**Definition 2.1 (state)** The *state* of a system is the smallest, ordered set of numbers (the so-called *state variables*) which is valid at some given instant and contains sufficient information such that all system processes following this instant can be calculated from these numbers knowing the inputs and using the laws governing the system [17].

The method of boundary control analyzes only the state of the converter’s power stage. It is therefore sufficient to restrict the notion of state variables to this subsystem. Yet, for a given system Definition 2.1 often leaves alternative sets of variables that could be used as state variables. This freedom, however, is not useful for our purposes. We will therefore add a rule that clearly specifies the state variables.

\[\text{Note: } q \text{ is called the state of the switch, and can be either 0 ("open", "off") or 1 ("closed", "on"). Although the same word, this switch state is not linked to the concept of state of the converter, to be introduced in paragraph 2.1.2!}\]
Definition 2.2. (state variables)  State variables shall only be those variables that cannot change their values instantaneously. Such variables are all those who appear in mathematical expressions for energy, for example current in the expression $\frac{1}{2} L I^2$. Thus obvious candidates for state variables in a dc-dc-converter are its inductor currents and its capacitor voltages.

All input, output and state variables are tied together by the laws that govern the system. In physical systems these laws can usually be formulated mathematically. Very often they can be expressed in terms of differential equations (d.e.).

2.1.3 A Power Converter's General Equations

It is possible to express a converter's behavior mathematically. The difficulty involved can be significantly different for the power stage and for the control stage depending on the complexity of the control circuitry. The general equations always have the following form:

- The power stage:
  \[
  \dot{x} = f(x,u,q) \quad \dot{x} = dx/dt, \quad x(t_0) = x_0 \quad (2.2)
  \]
  \[
  y = g(x,u,q) \quad (2.3)
  \]
  Keeping the terminology of Bass [16], equation (2.2) will be called the network equation. It is the primary law describing the power stage's characteristics. The output variables are simply a linear combination of the input and state variables. Equation (2.3) will be referred to as the output equation. (2.2) and (2.3) together form the state equations.

- The control stage:
  \[
  q = c(x,u,t,Q) \quad \text{with} \quad Q \text{ previous value(s) of } q \quad (2.4)
  \]
  \[
  Q(t_0) = Q_0
  \]

\footnotetext[2]{Any system of d.e. of order > 1 can be reduced to a first-order system, so (2.2) is in fact universal.}
As this study will be restricted to switching power converter control it is legitimate to call equation (2.4) the *switching function equation*.

The independent variable t (time) may or may not occur explicitly. In the first case the system is called *nonautonomous*, whereas in the second it is named *autonomous*.

### 2.2 State Plane Analysis

Boundary Control essentially consists of applying state plane analysis to the network equation. State plane analysis is a mathematical tool developed to characterize the behavior of systems described by differential equations. It seems necessary to introduce state plane analysis in a rather formal way before boundary control itself is presented. One restriction, however, will be made right from the beginning: As the converters to be studied all have exactly two state variables, the following mathematical introduction will be tailored to this case.

Consider the autonomous differential equation

\[ \dot{x} = f(x), \quad x = \begin{pmatrix} i \\ v \end{pmatrix}, \quad x(t_0) = x_0 \] (2.5)

The vector notation can be dissolved to yield

\[ \dot{i} = f_1(i, v), \quad \dot{v} = f_2(i, v) \] (2.6)

Starting at some initial point, both i and v change their values as time progresses. Assuming that \( f_1 \) and \( f_2 \) are bounded, the rate of change of \( i(t) \) and \( v(t) \) remains finite. This is exactly the condition introduced for state variables. Let us therefore call \( x \) a state.

Basically there are two possibilities of visualization. The first one, very often the immediate choice of scientists, is to plot two separate graphs with \( v(t) \) and \( i(t) \) (Figure 2.2 a)). The second possibility is to understand i and v as defining a coordinate system. The orthogonal system is shown in Figure 2.2 b). It is called the *state plane*. The resulting curve is called a...
Figure 2.2  Representation over time and phase plane representation\(^3\).

\textit{trajectory}. A familiar example for this form of visualization are the well-known Lissajous-figures.

A clear disadvantage is that time then becomes an implicit parameter and cannot be regained from the graph alone. But, as will be seen later, there can be many advantages which outweigh this drawback.

\subsection{2.2.1 Linear Systems}

If equation (2.5) has a linear right-hand side (rhs), then one has to do with a system of usual linear differential equations for which a large body of theory exists (for example [18], [19], [20]). Some essential points are given in this paragraph.

\textbf{Definition 2.3 (Singular points)}  Singularities are those points for which \( \dot{x} = f(x) = 0 \). These points represent the equilibrium points of the dynamic system.

\textbf{Rule 2.1}  One and only one trajectory passes through every point of the state space, unless the point is a singularity. In this case, either no or an infinite number of trajectories pass through it.

\footnote{It is mathematically unusual to write \( \begin{bmatrix} i \\ v \end{bmatrix} \), but to choose the coordinate system as \( v \cdot \tilde{e}_1 + i \cdot \tilde{e}_2 \), i.e. \( v \) as abscissa and \( i \) as ordinate. Previous work in the field of boundary control, however, has worked with this convention [Burns, Bass]}
Four basic forms of singular points occur. They are shown in Figure 2.3 and will not be defined formally (see for example [21]). We note that no singularity can be reached in finite time.

**Definition 2.4 (asymptotic stability)**  
Singularity $s$ is asymptotically stable if and only if $\lim_{t \to \infty} x = s$.

Closed trajectories exist around a center. There trajectories form a continuum of concentric loops in a way that none of the possible motions approaches the center. It is the initial condition that determines the size of the loop [22]. Closed trajectories are of special interest because they represent periodic behavior.
2.2.2 Nonlinear Systems with Continuous Right-Hand Sides

In nonlinear systems, closed trajectories can exist as isolated curves [22]. By this it is meant that in a finite, nonzero neighborhood of one closed loop there is only a finite number of other loops. This is not the case in linear systems, where initial conditions and therefore closed loops can be made arbitrarily close. In nonlinear systems, frequency and amplitude corresponding to the closed trajectory are determined by the nonlinear properties of the system, not by the initial conditions.

Definition 2.5 (limit cycle) Let $C$ be a closed loop in the state plane. Let $C'$ be a non-closed trajectory in the state plane. If for $\varepsilon > 0$ one can find a value $t_0$, so that any point on $C'$ is at a Euclidean distance $\leq \varepsilon$ for either all $t > t_0$ or all $t < t_0$, then C is a limit cycle.

Definition 2.6 (orbital stability) In the first case of Definition 2.5, $C$ is called orbitally stable; in the second it is called orbitally unstable. If $C$ is orbitally stable in its inside and orbitally unstable on its outside (or vice versa), then $C$ is called orbitally semistable.

Figure 2.4 depicts these definitions graphically.

Linear systems can be stable only at or around singular points. Nonlinear systems allow additional (oscillating) stability at limit cycles.

![Figure 2.4 Limit cycle and orbital stability](image)

a) stable  

b) unstable  

c) semistable
With the rhs being continuous \( \dot{x} \) can only change gradually. Therefore the trajectory will show a continuously turning tangent that is defined in the whole state space. The trajectory is said to be \textit{smooth}.

\subsection*{2.2.3 Nonlinear Systems with Discontinuous Right-Hand Sides}

When \( f: x \rightarrow x \) is continuous but complicated in some neighborhood, it is attractive to replace it by a simpler discontinuous function [23]. This is exactly what is done by introducing ideal switch action to switching converters as modeled in (2.1).

Systems with discontinuous rhs are automatically nonlinear. Until recently, such systems have received only occasional attention in mathematics [24], [25]. Only lately comprehensive mathematical books on the topic have been published [23], [26]. [27] is considered as a standard. Although implications for control theory have been considered early [28], actual application in switching power control practice has not been reported, supposedly due to the high degree of analytical abstraction.

A geometric interpretation is more convenient and will be pursued in this thesis. When the rhs is discontinuous ("jumps"), the derivative \( \dot{x} \) changes its value abruptly. At these locations the trajectory will not be smooth any more. Closed trajectories in form of limit cycles can still occur. Definitions 2.5 and 2.6 also hold for discontinuous systems. Figure 2.5 illustrates cases of this kind.

![Figure 2.5 Limit cycles for piecewise linear differential equations](image)

\[\text{a) no limit cycle} \quad \text{b) limit cycle}\]
This thesis is concerned with the three meanwhile traditional types of dc-dc-converters, the buck, the boost, and the buckboost converter. Figure 3.1 shows simple models of these types. The first task is to furnish a comprehensive set of their network equations, to depict the resulting trajectories graphically and to analyze their shapes.

Figure 3.1 Simple converter models
3.1 Derivation of the Network Equations of Buck, Boost and Buckboost Converter

The task of deriving their respective network equation is basically the same for all of them. An example may give sufficient insight in the necessary procedure. The following paragraph derives the buck converter's network equation, based on the simple model shown above.

In general, four topological structures can be distinguished when the dc-to-dc converters are operating:

a) none of the switches closed (occurring in so-called discontinuous operation)
b) controlled switch closed, diode open (the "on-state" of the converter)
c) controlled switch open, diode conducts (the "off-state" of the converter)
d) both switches closed

In the converter types considered here, however, the fourth topology does not occur. Accordingly Figure 3.2 shows only the three networks that result in the buck case. In conformance with Definitions 2.1 and 2.2, we choose the inductor current and the capacitor voltage as state variables. For each of the topologies, Kirchhoff's Laws and the laws valid for the respective circuit components lead to two d.e., one for each state variable. Their derivation can be seen in Figure 3.3. Next, each switch is assigned a switch state indicator $q_i$. We will use the common convention that $q_1$ shall denominate the state of the controlled switch and $q_2$ the state of the diode. Using this convention, it is possible to describe the converter's behavior in one single equation: its network equation (see Figure 3.4).
Figure 3.2 The three topologies of a simple buck converter

\[
\begin{align*}
L \frac{di}{dt} &= 0 \\
C &= \frac{\int i \, dt}{v} \\
E &= L \frac{di}{dt} + v \\
C &= \frac{\int i_c \, dt}{v} \\
L \frac{di}{dt} &= -v \\
C &= \frac{\int i_c \, dt}{v}
\end{align*}
\]

\[
\begin{align*}
C \frac{dv}{dt} &= -\frac{v}{R} \\
L \frac{di}{dt} &= E - v \\
C \frac{dv}{dt} &= -i_c \\
i_c + i &= \frac{v}{R} \\
C \frac{dv}{dt} &= -\frac{v}{R} + i
\end{align*}
\]

\[
\begin{align*}
\frac{di}{dt} &= 0 \\
\frac{di}{dt} &= \frac{1}{L} \cdot E - \frac{1}{L} \cdot v \\
\frac{di}{dt} &= -\frac{1}{L} \cdot v
\end{align*}
\]

\[
\begin{align*}
\frac{dv}{dt} &= -\frac{1}{C \cdot R} \cdot v \\
\frac{dv}{dt} &= -\frac{1}{C \cdot R} \cdot v + \frac{1}{C} \cdot i \\
\frac{dv}{dt} &= -\frac{1}{C \cdot R} \cdot v + \frac{1}{C} \cdot i
\end{align*}
\]

Figure 3.3 Derivation of the simple buck converter's network equation
As this procedure is very formal, it can be automated. Webster and Ngo presented a computer program to perform this task [29].

The combination \( q_1 = 0, q_2 = 0 \) corresponds to topology \( a_1 \), the combination \( q_1 = 1, q_2 = 0 \) to \( b_1 \), and \( q_1 = 0, q_2 = 1 \) to \( c_1 \). The first combination only occurs in the discontinuous mode of converter operation. Very often, this mode is consciously avoided when a converter is designed as it may give rise to performance problems. Whenever it is ensured that the circuit does not operate in this mode, only topologies \( b_1 \) and \( c_1 \) can occur. We see that in this case \( q_2 \) is fully determined by \( q_1 \): \( q_2 = 1 - q_1 \) This text will then make use of the following simplification in notation:

\[
q_1 := q \quad q_2 := 1 - q
\]

Whenever the letter \( q \) is used for the state of a switch, it will refer to the (only) controlled switch in the circuit. Continuous mode of operation is then always assumed.

The derivation of the network equations of the other two converters is analogous. Their equations are given in Appendix A.

3.2 Enhancement of the Network Equations

The simple converter models used in the previous Section do not reflect reality very well. All parasitics were neglected. However, every component of a real converter shows nonideal behavior, which should be taken care of by the model. Therefore, for purposes other than demonstrating calculation techniques, a second set of converter models will be used. It will be referred to as the enhanced converter models. It includes the series resistance of the inductor.
(\(R_L\)), the transistor’s on state resistance (\(R_{ds}\)), the output capacitor’s equivalent series resistance \(R_c\) and a diode forward voltage drop \(V_{\text{drop}}\). The resulting circuit diagrams are given in Figure 3.5.

![Figure 3.5 Enhanced converter models](image)

Naturally, the network equations change as well and become slightly more complicated. Their overall structure, however, remains the same. It is the matrix elements that become more complex. The enhanced network equations can also be seen in Appendix A.
3.3 The Converters' Trajectories in the State Plane

The network equation determines the evolution of a converter's state over time, from any given initial condition on. As time is incremented by a marginal moment dt, the converter's inductor current changes by \( di \) and its capacitor voltage by \( dv \). Using the state plane representation introduced in Section 2.2, the converters' trajectories can be plotted. Of course, a trajectory depends on the initial condition the system starts from. A meaningful plot for a converter will therefore show a whole "family" of trajectories, that is, show trajectories determined by the same d. e. but different initial conditions.

With respect to the dc-dc-converters studied, three families can be distinguished for each converter, corresponding to the three possible topologies. However, one of them is trivial: when both switches are off, the inductor current is zero, and the output capacitor will be discharged in the usual exponential way. In the state plane, we will see the state move along the voltage axis in direction to the origin. Since moreover this topology occurs only in discontinuous operation, the plots concentrate on the two nontrivial families, the "on-state" trajectories and the "off-state" trajectories. With the aid of Matlab™, these trajectories were plotted\(^4\).

3.3.1 Trajectories for the Simple Converter Models

The plots presented throughout this thesis are all based on the circuit-diagrams presented above. The following parameter values were used for all three converter types:

<table>
<thead>
<tr>
<th>L</th>
<th>C</th>
<th>R</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 mH</td>
<td>12 mF</td>
<td>20 Ω</td>
<td>10 V</td>
</tr>
</tbody>
</table>

These values were chosen such that characteristic state plane features are clearly recognizable.

\(^4\) For details on the algorithm used see Appendix B.
Figure 3.6. shows the trajectories for the simple buck converter.

![Figure 3.6 Simple buck converter -- trajectories](image)

The initial conditions were placed on the x-axis for the transistor being on and on the y-axis for it being off. Both families show a focus, and, since both states approach their focus with time increasing, the foci are stable.

The location of these equilibrium points for the two families can be calculated by equating the network equation with zero [11].

The results are:

- $q = 1$ (on-state): $\mathbf{x}_1 = \begin{pmatrix} E/R \cr E \end{pmatrix} = \begin{pmatrix} 0.5A \cr 10V \end{pmatrix}$
- $q = 0$ (off-state): $\mathbf{x}_0 = \begin{pmatrix} 0 \cr 0 \end{pmatrix}$

In a similar manner, Figure 3.7 shows the boost case.

The on-state has a stable node in the infinite, whereas the off-state is still characterized by a stable focus. The equilibrium points are:

- $q = 1$ : $\mathbf{x}_1 = \begin{pmatrix} \infty \cr 0 \end{pmatrix}$
- $q = 0$ : $\mathbf{x}_0 = \begin{pmatrix} E/R \cr E \end{pmatrix} = \begin{pmatrix} 0.5A \cr 10V \end{pmatrix}$

---

For a detailed derivation of the equilibrium points refer to appendix C.
The buckboost trajectories can be seen in Figure 3.8. As in the boost case, a stable node occurs for $q=1$ and a stable focus for $q=0$. The latter, however, is now located in the origin:

- $q=1: \mathbf{x}_s^1 = \begin{pmatrix} \infty \\ 0 \end{pmatrix}$
- $q=0: \mathbf{x}_s^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Figure 3.8  Simple buckboost converter -- trajectories

The trajectories, especially the locations of the equilibrium points, reveal a serious inexactness of the simple models. They do predict currents going to infinity. In reality, though,
some resistance will always be present and impose a limit. Second, as far as the shape of the trajectories is concerned, parasitics may very well change it to some degree. In order to include these effects, the enhanced converter models were used.

### 3.3.2 Trajectories for the Enhanced Converter Models

The enhanced models were already presented above. The following parasitics were added to the simple circuits:

Table 3.2 Parasitic values used in the simulations

<table>
<thead>
<tr>
<th>$R_L$</th>
<th>$R_C$</th>
<th>$R_{ds}$</th>
<th>$V_{drop}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 mΩ</td>
<td>10 mΩ</td>
<td>30 mΩ</td>
<td>1 V</td>
</tr>
</tbody>
</table>

The new trajectories for the buck converter are depicted in Figure 3.9.

![Figure 3.9 Enhanced buck converter — trajectories](image-url)
The equilibrium points are now:

\[
\begin{align*}
q &= 1 : \mathbf{x}_r^1 = \left( \frac{E}{R + R_L + R_{de}} \right) = \left( \frac{0.499A}{9.980V} \right) \\
q &= 0 : \mathbf{x}_r^0 = \left( \frac{0}{0} \right)
\end{align*}
\]

Two effects of additional resistance can be seen directly. First, additional resistance limits current. Second, it causes voltage drops. A closer comparison of the trajectories in Figure 3.5 and Figure 3.9 shows that the losses introduced by the parasitics reflect in every part of the trajectories. All trajectories are dragged towards the origin a little bit. When the state “moves”, energy is consumed in the parasitics. These losses persistently reduce the capacitor voltage and inductor current values.

Figures 3.10 shows the enhanced trajectories for the other two converters.

![Figure 3.10 Enhanced boost converter and enhanced buckboost converter-- trajectories](image)

Figure 3.10 Enhanced boost converter and enhanced buckboost converter-- trajectories

Still, within the usual operating area of the converters, the trajectories’ overall shape remains nearly unchanged.

**Outcome 3.1** The trajectories of buck, boost, and buckboost converter are relatively robust to parasitics.
Thus a control design in the large based on the geometry of the converters’ trajectories will in
general also be robust with regard to parasitics.

Equilibrium points for the boost and the buckboost circuit are given in Appendix C.
CHAPTER 4
Loadline and Switching Boundary

4.1 The Loadline

In whatever way the power stage is controlled, its state has to evolve according to the
network equation. Graphically, this means that at any given time it has to move along exactly
one trajectory belonging to one of the three families. The family is determined by the switch
states, and the trajectory itself is identified by the initial condition. Whenever a switch turns on
or off, the valid family of trajectories changes and the state of that instant serves as initial
condition for the new trajectory.

It is desired that the converter converge to a periodic behavior: some output voltage
ripple is allowed, but its average output value shall be constant. The same is desired for the
inductor current. As pointed out, this means that in the state plane a limit cycle has to occur.
This Chapter tries to answer where in the state plane such limit cycles are possible. Based on
Bass [9] it will be shown that stable periodic behavior can only occur in a certain region.

4.1.1 Asymptotic Points

In Figure 4.1 a) an idealized limit cycle is shown. Stable operation has already been
reached, and a very definite voltage ripple can be observed ($\Delta v$). Also, the inductor current
shows some specific amount of ripple ($\Delta i$). It is well known that such ripple can be decreased by
increasing the switching frequency ($f_s := 1/T_s$) with which the converter works. In the state
plane this is reflected by the limit cycle getting smaller (Figure 4.1 b)). If in theory $f_s$ is brought
to infinity (high frequency limit) the limit cycle will deform to a point (Figure 4.1 c)).
**Definition 4.1 (asymptotic point)**

If the rhs of (2.2) is periodic with $T_s$ and $L(x)$ is a limit cycle for (2.2),(2.3),(2.4), then the limiting set $x^a \equiv \lim_{T_s \to 0} L(x)$ is called an asymptotic point [16].

Whenever the switching function (1.4) is periodic, a duty ratio can be assigned in the conventional way.

**Definition 4.2 (duty ratio)**

The duty ratio $D$ always refers to $q$. It is defined as the ratio $\frac{T_{on}}{T_{on} + T_{off}} = \frac{T_{on}}{T_s}$ for one switching cycle $T_s$.

The duty ratio is a possibility to control the output voltage of the converter. For a given (dc-) output, it is the load that will sustain a certain (dc-) output current, which in turn will result in a very definite (dc-) inductor current. Once the duty ratio has determined the asymptotic voltage in $x^a$, the load determines the asymptotic current. This additional algebraic restriction imposed by the load can be nicely included in the graphical concept of Boundary Control. The restriction will be visualized as a load line.

**Definition 4.3 (load line)**

The loadline of a converter is the set $LL(x)$ of all its asymptotic points for $D \in [0,1]$ and for a given load value.
4.1.2 Calculation of the Loadline

This Section gives three methods of calculating LL(x). First it illustrates the method used until now by means of an example. Then it will suggest two different methods of obtaining the same result. They are not necessarily more practicable, but they reveal two new thoughts in Boundary Control.

4.1.2.1 Method of the “Low Frequency Describing Equation”

Bass [16] has stated that LL(x) can be obtained by using Ngo’s “low frequency describing equation” [2]. This method replaces \( q_1 = q \) with \( D \) and \( q_2 = (1-q_1) \) with \( (1-D) \) in the state space equation, equals it with zero and solves for \( x \) to yield \( x^a \), the asymptotic point.

The simple buck converter may again serve as an example. Starting point is its network equation, already given in Figure 3.4.

The calculation follows the three steps just mentioned (Figure 4.2):

\[
\begin{align*}
(0) & = \left[ \begin{array}{c}
0 \\
0 \\
-\frac{1}{C \cdot R}
\end{array} \right] \cdot (i_v) + \left[ \begin{array}{c}
0 \\
-\frac{1}{L} \\
\frac{1}{C}
\end{array} \right] \cdot (i_v) + \left[ \begin{array}{c}
\frac{1}{L} \\
0 \\
0
\end{array} \right] \cdot E \cdot D + \left[ \begin{array}{c}
0 \\
-\frac{1}{L} \\
\frac{1}{C}
\end{array} \right] \cdot (1-D) \\
(1) & = -\frac{1}{L} \cdot v^a \cdot D + \frac{1}{L} \cdot E \cdot D - \frac{1}{L} \cdot v^a \cdot (1-D) = 0 \\
(II) & = -\frac{1}{C} \cdot v^a + \frac{1}{C} \cdot i^a \cdot D - \frac{1}{C} \cdot i^a \cdot (1-D) = 0 \\
(I') & = E \cdot D - v^a = 0 \\
\quad & \quad v^a = E \cdot D \\
(II') & = -\frac{1}{R} \cdot v^a + i^a = 0 \\
& \quad i^a = \frac{1}{R} \cdot v^a \\
& \quad i^a = \frac{E}{R} \cdot D, \; D \in [0,1]
\end{align*}
\]

Figure 4.2 Calculation of LL(x) via the “Low Frequency Describing Equation”
Plotted with D as a parameter this set of points is a straight line interconnecting the two equilibrium points of the converter. Figure 4.3 shows the loadline graphically. An overview of the load lines for all the converter types is given in Appendix D.

![Simple Buck Converter - Load Line](image)

Figure 4.3 Loadline of the simple buck converter

### 4.1.2.2 Method of Power Matching

In the high frequency limit, we have nonvarying inductor current and capacitor voltage (zero ripple). Therefore the energy contained in the converter is perfectly constant. Moreover, since all other currents and voltages are just linear combinations of these state variables, all values are constant. Thus any energy dissipated in eventual parasitic resistances is also constant. The equation $P_{\text{in}} = P_{\text{out}} + P_{\text{loss}}$ is not only valid on average, but holds true for every moment in steady state operation. The loadline can therefore also be viewed and calculated as the set of those points in the state space that fulfill the power balance equation without ripple. This interpretation comes closest to the conventional understanding of a loadline in electrical engineering: It is a graphical depiction of the physical constraint that energy flow has to be consistent with a given load. Figure 4.4 follows this idea, again for the simple buck type.
\begin{align*}
\text{P}_{\text{in}} &= \text{P}_{\text{out}} + \text{P}_{\text{loss}} = \text{P}_{\text{out}} + 0 = \text{P}_{\text{out}} \\
\text{Capacitor:} \\
(\text{I}) & \quad E \cdot i^a \cdot D = \frac{(v^a)^2}{R} \\
(\text{II}) & \quad i^a = i_c + i_R = 0 + i_R = i_R = \frac{v^a}{R} \\
(\text{II}) \quad \text{in (I):} & \quad E \cdot \frac{v^a}{R} \cdot D = \frac{(v^a)^2}{R} \quad v^a = E \cdot D \quad i^a = \frac{E}{R} \cdot D, \quad D \in [0,1]
\end{align*}

Figure 4.4 Calculation of LL(x) via the power balance equation

4.1.2.3 Method of Tangential Trajectories

This third method is based on the following observation:

Conjecture 4.1  On the load line the trajectories of the "on-state"-family and of the "off-state"-family are mutually tangential.

Motivation  The trajectory plots reveal that any limit cycle is composed of two pieces of convex trajectories -- two spirals in the buck case, one spiral and one hyperbola in the boost and buckboost case. This general structure is exemplified in Figure 4.5 a).

\begin{figure}
\centering
\subfigure[i]{}
\subfigure[i]{}
\subfigure[i]{}
\caption{Limit cycle in the high frequency limit}
\end{figure}
When $f$, increases, the limit cycle becomes smaller. Graphically the points of intersection move closer together (Figure 4.5 b). When $f \rightarrow \infty$, the limit cycle degenerates to a point. With $f$, huge, but still finite, the limit cycle still has to be composed of two trajectory pieces intersecting twice. For this structure to degenerate to a point, these two points of intersection have to fall together. This is exactly the condition of tangentiality for the two trajectories (Figure 4.5 c).

**Graphical Simulation**

In order to back this test this conjecture and the argumentation, trajectories through load line points have been plotted. Figure 4.6 shows these plots. Trajectories of the one family are drawn to the left of LL(x), trajectories of the other family on the right. These plots are a strong indicator that Conjecture 4.1 is correct.

![Graphical Simulation](image)

**Figure 4.6 Trajectories through load line points**
Analytical Example

Using again the simple buck converter, the following calculation (Figure 4.7) evaluates the trajectories' slopes at the load line points obtained above.

\[
\begin{align*}
\frac{d}{dt}(i) &= \left[\begin{array}{c} 0 \\ -\frac{1}{C \cdot R} \end{array}\right] \cdot (i) + \left[\begin{array}{c} 0 \\ \frac{1}{L} \end{array}\right] \cdot \left(\begin{array}{c} v \\ 0 \end{array}\right) + \left[\begin{array}{c} \frac{1}{C} \\ 0 \end{array}\right] \cdot E \cdot \left[\begin{array}{c} q_1 \\ \frac{1}{L} \end{array}\right] \cdot (i) \cdot q_2 \\
q_1 &= 1 \\
\frac{di}{dt} &= -\frac{1}{L} \cdot v + \frac{1}{L} \cdot E \\
\frac{dv}{dt} &= -\frac{1}{CR} \cdot v + \frac{1}{C} \cdot i \\
q_2 &= 1 \\
\frac{di}{dt} &= -\frac{1}{L} \cdot v \\
\frac{dv}{dt} &= -\frac{1}{CR} \cdot v + \frac{1}{C} \cdot i \\
\end{align*}
\]

at \( LL(x) \) \( i^a = \frac{v_0}{R} \):

\[
\begin{align*}
\frac{di}{dv}_{v^a,i^a,q_1=1} &= \frac{CR \cdot (E - v^a)}{L(R \cdot \frac{v^a}{R} - v^a)} \rightarrow +\infty \\
\frac{di}{dv}_{v^a,i^a,q_2=1} &= -\frac{CR \cdot v^a}{L(R \cdot \frac{v^a}{R} - v^a)} \rightarrow -\infty
\end{align*}
\]

Figure 4.7 Trajectory-slopes of the buck converter on its loadline

Both trajectories are parallel to the i-axis and thus tangential.

Applying Conjecture 4.1, it is possible to understand \( LL(x) \) as the set of points where the on- and the off-trajectories are mutually tangential. It can then be calculated by “inverting” the calculation shown in Figure 4.7. This is done in Figure 4.8, again for the simple buck converter.
\[ \frac{d}{dt}(i) = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \cdot (i) + \left[ \begin{array}{c} 0 \\ \frac{1}{C} \end{array} \right] \cdot (v) + \left[ \begin{array}{c} \frac{1}{L} \\ 0 \end{array} \right] \cdot (i) \cdot q_1 + \left[ \begin{array}{c} 0 \\ \frac{1}{C} \end{array} \right] \cdot (v) \cdot q_2 \]

\[ q_1 = 1 \quad \frac{di}{dt} = \frac{1}{L} \cdot v + \frac{1}{C} \cdot E \]
\[ \frac{dv}{dt} = -\frac{1}{CR} \cdot v + \frac{1}{C} \cdot i \]

\[ q_2 = 1 \quad \frac{di}{dt} = \frac{1}{L} \cdot v \]
\[ \frac{dv}{dt} = -\frac{1}{CR} \cdot v + \frac{1}{C} \cdot i \]

at \( v^a, i^a \):
\[ \left. \frac{di}{dv} \right|_{v^a, i^a, q_1 = 1} = \frac{di}{dv} \bigg|_{v^a, i^a, q_2 = 1} \]
\[ \frac{CR(E-v)}{L(R \cdot i-v)} = \frac{-CR \cdot v}{L(R \cdot i-v)} \]

\[ a) \text{ Denominator } = 0: \]
\[ R \cdot i^a - v^a = 0 \quad \Rightarrow \quad i^a = \frac{v^a}{R} \]

\[ b) \text{ Denominator } \neq 0: \]
\[ CR(E-v^a) = -CR \cdot v^a \quad \Rightarrow \quad E = 0 \quad \downarrow \]

\[ \Rightarrow \quad i^a = \frac{v^a}{R} , \quad D \in [0,1] \]

Figure 4.8 Calculation of \( L(x) \) via the condition of tangential trajectories

Additional calculations along this line have shown that also in the cases of the enhanced boost and buckboost converter Conjecture 4.1 holds true. The correctness of the Conjecture seems to be certain.
4.1.3 Limit Cycle Operation

This section analyzes where in the state plane limit cycles can occur. The question is whether a small limit cycle in steady state always is such that it surrounds or includes its asymptotic point.

Imagine a switching frequency decreasing from a hypothetical starting value of infinity ($f_s \to \infty$). Voltage begins to be applied across the inductor and current begins to flow through the capacitor for a nonzero time. This implicates that both state variables have to vary around the asymptotic point (Figure 4.9 a)): ripple occurs. The output voltage now contains harmonics, and the balance of power does now apply for rms-values, but not for the average dc-values of inductor current and capacitor voltage. Thus the asymptotic point $x^a$ which was calculated precisely for the dc-values does not describe the converter behavior correctly any more. Still it retains its practical meaning. Every reasonable converter is designed in order to keep the harmonics extremely low. Therefore the dc-model is still accurate to a large degree. However, two sorts of limit cycles are imaginable: those that contain the asymptotic point and those that do not (Figure 4.9 b) and c)).

By the means of computer simulation it has been possible to show that in fact both types do occur. Figure 4.10 shows limit cycles for the enhanced boost power stage introduced in Chapter 3. It is controlled by conventional voltage Pulse Width Modulation, with a switching

---

$^6$ Explication on the algorithm used is given in Appendix B.
frequency of 60 kHz and a duty ratio of 0.5. The difference between the two plots lies in the capacitor value. The plot on the left was taken with \( C = 2.4 \, \mu F \), the one on the right with \( C = 1.2 \, \mu F \).

![Graph showing limit cycles for the boost converter under PWM](image)

**Figure 4.10**  Limit cycles for the boost converter under PWM

As can be expected, with the decreased capacitance the current ripple remains nearly unchanged, but the voltage ripple increases. Thus output harmonics do increase and the limit cycle is carried off the asymptotic point.

**Outcome 4.1**  A limit cycle in steady state operation will not always be such that it surrounds or includes its asymptotic point.

However, output harmonics considerations suggest that it will be off only slightly. The important implication is that a reasonably small steady state limit cycle always occurs in the proximity of some part of the loadline.
4.2. The Switching Boundary

The only means of control for the buck, boost and buckboost converter is to command its controlled switch. All control methods therefore construct some kind of law that tells when to switch it on and when to switch it off.

The method of Boundary Control relies on state feedback to accomplish this continuous decision process. Its basic idea is to define a curve in the state space -- the so-called (switching) boundary. Whenever the converter's state is located on one side of the boundary, the switch is closed. Whenever it is on the other side, it is open. Thus switch action occurs at the moment when the state, as it evolves over time, crosses the switching boundary. Figure 4.11 depicts this scheme.

![Figure 4.11 The principle of Boundary Control](image)

Boundaries can be designed stationary (autonomous case) or moving (nonautonomous case). Some implementations may additionally require nonboundary switching. For details see Bass [10], [16].

The use of this graphical concept is twofold. First, traditional control methods such as open loop PWM or voltage feedback PWM can be interpreted and studied with this approach. This has been done in [5] and in [16]. Second, this concept allows the design of new control laws by defining some appropriate boundary. Burns has done that in a very imaginative way [6].
In the following this thesis will pursue the second application further. Burns restricted his examinations to converter operation around the nominal operating point, using a stationary boundary. His studies will be enlarged by studying converter behavior in the large.

**Restrictions**

This thesis will concentrate on stationary boundaries. All boundaries studied will be straight lines.

Departing from a desired operating point, which in the terminology of Boundary control is the desired asymptotic point, such a linear boundary can be mathematically described [16]:

\[
\sigma(x) = k^T \cdot (x^a - x) = 0
\]  

(4.1)

This equation will be named *switching boundary equation*. \( x^a \) is determined by the application. \( k^T = (k_i, k_v), k_i > 0, k_v > 0 \) are design coefficients (feedback gains) that determine the slope of the boundary. Equation (4.1) can be dissolved to yield an explicit representation of the switching boundary:

\[
i = (i^a + \frac{k_v}{k_i} \cdot v^a) - \frac{k_v}{k_i} \cdot v
\]  

(4.2)

This switching boundary equation will become a switching function equation in the sense of (1.4) when \( k_i \) and \( k_v \) are specified (the slope of the boundary) and when it is stated on which side of the boundary the switch is to be kept on and on which it is to be kept off. Unfortunately these specifications might bring along stability problems.
CHAPTER 5

Stability Issues

As stated above, this thesis will be concerned with stability in the large.

**Definition 5.1 (large signal stability)** A converter is regarded large signal stable if, starting from any physically valid initial condition (i.c.), it will get to a $\lambda$-neighborhood of the asymptotic point, with $\lambda > 0$ small, $|\lambda| = f(x_0)$ and $\lim_{\beta \to \infty} |\lambda| = 0$.

Note: This definition is in accordance with the definition of asymptotic stability of a limit cycle (Definition 2.6) as long as $|\lambda| \geq R$, $R$ being the maximum Euclidean distance of the limit cycle from $x^0$.

5.1 Basic Stability Considerations

Bass gives a suggestion how to lay the boundary and define the switch action properly:

**Theorem 5.1** In order to allow large signal stability it is necessary that the boundary divide the state space in such a way that $x^0$ and $x^1$ lie in separate half-planes and that the switch-action is controlled in such a manner that in every half plane the family of trajectories be valid that has its singularity in the opposite half plane. [16]

It has not been evaluated yet whether this restriction is a necessary or a sufficient condition for stability. This thesis will give evidence that in fact it is necessary, but not sufficient. First, it will be shown formally here that Theorem 5.1 is indeed a necessary condition for global stability. The proof starts out with the second statement of the Theorem.

**Motivation** According to Definition 5.1 any physically valid i.c. must lead to the operating point. Let us assume $x_i^0$ as i.c. Although it has been pointed out that the converter will never reach this point, this state value may very well be imposed externally. It is physically
valid, as it does not violate KVL or KCL constraints. If the d.e. with its singularity in this point were valid, the converter state would never change and thus not reach \( x^a \). Assuming now the other, "correct" d.e. to be valid, the state will tend towards \( x^1 \). If \( x^1 \) and \( x^o \) were allowed to lie in a joint region of the state plane, then one of the two singularities would necessarily lie in the region where its d.e is valid. But this is exactly what is not allowed, as has just been shown. Thus the conditions formulated in Theorem 5.1 are in fact necessary ones.

Based on Theorem 5.1, mathematical requirements on \( k^T \) can be established (Figure 5.1).

\[
\begin{align*}
\frac{k_v}{k_i} &< -\frac{(i^1_e - i^0_e)}{(v^1_e - v^0_e)} < 0 \\
0 &> -\frac{(i^a_e - i^0_e)}{(v^a_e - v^0_e)} < \frac{k_v}{k_i} < -\frac{(i^a_e - i^1_e)}{(v^a_e - v^1_e)} > 0
\end{align*}
\]

Figure 5.1 Basic stability requirements for a linear boundary

It remains the question whether any such boundary will force the system to act at the asymptotic point in a \( f_\infty \) limit. In other words, is the condition a sufficient one. Chapter 6 will show that this is not the case.

### 5.2 Advanced Stability Considerations

Theorem 5.1 secures that the state will always come back towards the boundary.

According to Conjecture 4.2 stable operation has to occur at the loadline. The only point where a limit cycle can arise around the loadline and the boundary is the point of intersection of these
two curves. Theorem 5.1 does not yet determine whether the converter state will approach this operating point. It is the purpose of this Section to establish results on this issue.

5.2.1 Overview of Stability Techniques

Many graphical and analytical methods have been suggested and partly applied in order to examine stability behavior of systems of d.e. with noncontinuous right-hand sides.

- Linearization
  As a large body of theory exists on linear d.e. an obvious approach is to linearize the system around its operating point and then to rely on conventional concepts. This procedure has been widely used (see for example [30], [31], [32]). Its disadvantage is that it solely refers to small signal stability and is therefore not useful in our context.

- Discontinuous Theory of Vogel
  A true large signal tool is provided by the Discontinuous Theory of Vogel [33], [34], [35]. Vogel suggests a coordinate mapping from the v,i-space to an r,s-coordinate system, with r and s being the two families of trajectories. His stability concept is easy in theory, but seems to require some amount of computer calculation. Second, his theory does not perfectly match the case given with the dc-dc-converters and would have to be extended or modified. It has not yet been applied to power electronics.

- Successor Functions
  The method of successor functions has been used successfully by Kulawik to analyze stability of a boost converter [36].

  **Theorem 5.2** Let \( \sigma \) be a differentiable arc, given in parametric form \( \sigma(s) \), with \( s \) being strictly monotone on \( \sigma \). If a trajectory through a point \( \sigma_0 = \sigma(s_0) \) of \( \sigma \) has a subsequent intersection with \( \sigma \) at \( \sigma_1 = \sigma(s_1) \), then \( \sigma_1 \) is called the **successor** of \( \sigma_0 \); the function \( s_1 = \phi(s_0) \) is termed **successor function** (Figure 5.2).
Obviously the condition for a limit cycle is that \( s_1 = \phi(s_0) = s_0 = s_0^* \). The condition for asymptotic stability in the large is

\[
\frac{ds_1}{ds_0} \bigg|_{s_0 = s_0^*} = \frac{d\phi}{ds_0} \bigg|_{s_0 = s_0^*} < 1 \tag{5.1}
\]

Geometrically stated, the limit cycle will be stable if for any \( s_0 \) its successor \( s_1 \) will be closer to \( s_0^* \) than \( s_0 \) itself. For details see [35].

- Lyapunov Functions

Lyapunov's direct method has long been applied to power electronics, though usually in a tangled manner (see e.g. [37]).

**Theorem 5.3** If a positive definite function \( V(x) \) can be found such that \( \dot{V}(x) \) is negative definite, then the origin is asymptotically stable.

A function that fulfills these requirements is called a Lyapunov function (see [38] for details). Sanders [39] introduces a Lyapunov function which is useful for analyzing switching power converters. Midya calls it the *excess energy function* [40]:

\[
V(x) = \frac{1}{2} C(v - v^a)^2 + \frac{1}{2} L(i - i^a)^2 \tag{5.2}
\]

It represents the energy difference stored in the power stage between the actual state and the operating point. A similar, but simpler Lyapunov function derived from (5.2) is (5.3):

\[
V(x) = (v - v^a)^2 + (i - i^a)^2 \tag{5.3}
\]
Its meaning is not physical any more, but geometrical. It represents the Euclidean distance of the actual state from the operating point in the state plane.

5.2.2 A Geometrically Based Approach to Stability

The advanced stability argumentation drawn in this paper is a combination of the Method of Successor functions with Lyapunov's Method. Its underlying ideas are the following:

Let firstly Theorem 5.1 be satisfied. Let the switching boundary given by equation (4.1) be the differentiable arc $\sigma$ demanded in the Theory of Successor Functions. Let its parameter $s$ have the value zero at the asymptotic point $x^a = \sigma^a$. When the converter is operating, its trajectory will have intersections with $\sigma$ whenever switching action occurs. According to Theorem 5.2 we will show that, if the converter operation is to be stable, subsequent successors $\sigma_i$ will be located ever closer to $\sigma^a$. The distance $\sigma_i \leftrightarrow \sigma^a$, however, is exactly what equation (5.3) defines as Lyapunov function. Thus values for $V(x)$ will not be taken continuously, but only at points $s$. It would therefore be more correct in this context to speak of a Lyapunov series than of a Lyapunov function. By proving a decreasing distance $\sigma_i \leftrightarrow \sigma^a$ it is simultaneously shown that subsequent successors have decreasing Lyapunov values $V(x)$. This property corresponds to the requirement of negative definiteness of $\dot{V}(x)$. Theorem 5.3 will therefore allow the conclusion of asymptotic stability. It is these lines that the subsequent discussion of stability will follow.

Let us first denote the three types in which points on any switching boundary can be classified.

- One trajectory directed towards $\sigma$, the other one directed away from it:

  The state $x$ reaches $\sigma$, switch action occurs and the now valid d.e. at first leads the state away from the boundary (Figure 5.3 a)). In analogy to the field of optics such a loadline
point will be said to be refractive. The switch action in this point will be termed Refractive Switching.

- Both trajectories are directed towards $\sigma$:
  
The state reaches $\sigma$, switch action occurs and the now valid d.e. redirects the state to the boundary immediately (Figure 5.3 b)). Such a loadline point will be called reflective\(^7\). The switch action it provokes will be referred to as Reflective Switching.

- Both trajectories are directed away from $\sigma$:
  
The state will never be able to reach such a point in undisturbed operation (Figure 5.3 c)). The point will be called rejective.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{refractive_reflective_rejective_points.png}
\caption{Refractive, reflective and rejective switching boundary points}
\end{figure}

Formal definitions can be formulated in a straightforward manner based on the sign of the normal component of $\frac{dx}{dt}$ in the boundary point. Mathematical formulations will be treated in Chapter 6, where these three cases are analyzed by computer simulation.

\(^7\) The terms "refractive" and "reflective" for points of discontinuity in d.e. were used by Vogel [33], the English translation being found in [35].
5.2.3 Operation with One Single Boundary

All of the converters considered have one common boundary already by virtue of their power stage design. The diode in the circuit does not permit the inductor current to become negative at any moment. Instead, the diode will turn off and keep the current at zero. This case is commonly known as the discontinuous mode of operation. In the state plane, the points where the diode automatically turns off is given by the horizontal axis (i = 0). Thus from a Boundary Control point of view this axis represents a stationary switching boundary. As in other approaches of converter analysis it is indicated in Boundary Control as well to study this discontinuous mode of operation separately from the continuous case. The first part of the analysis considers the continuous case, where only the boundary σ, set by the control designer, plays a role.

5.2.3.1 Refractive Operation

Let the converter operate in the refractive mode. As seen from the plots in Chapter 3 and stated above, trajectories are convex to the boundary and do not intersect each other as long as they belong to the same family. Every half-plane is, according to Theorem 5.1, assigned one of the two families of trajectories, which we will call the "correct" family. The other family will be designated to be the "wrong" one for this half-plane. Trajectories will be named \( t_i \) and \( b_i \), with a and b denoting the families and \( i = 1,2,3,\ldots \) the subsequent trajectories followed by the state. The subsequent switching locations will be called \( \sigma_i \).

- Refractive Operation, Correct Families

This mode of operation can be seen in Figure 5.4.

case a) Let the state arrive on \( t_1 \). Assume that \( b_1 \) is such that \( \sigma_2 \) is closer to \( \sigma^a = x^a \) than \( \sigma_0 \) would have been. From the fact that \( t_2 \) cannot intersect \( t_1 \), it follows that \( \sigma_3 \),
Figure 5.4 Refractive operation, correct families

is closer to $\sigma^a$ than $\sigma_1$ was. Thus the series $s_i$ with $i = 1,3,5,...$ is decreasing. In the same way $b_{t_2}$ cannot intersect $b_{t_1}$ and $\sigma_2$ is closer to $\sigma^a$ than $\sigma_0$ would have been. Thus the series $s_i$ with $i = 0,2,4,...$ is decreasing. Therefore the Lyapunov function $V(x)$ also decreases monotonously and $x^a$ is asymptotically stable.

case b) Let the state again arrive on $a_{t_1}$. Assume this time that $b_{t_1}$ is such that $\sigma_2$ is farther away from $\sigma^a = x^a$ than $\sigma_0$ would have been. From the fact that $b_{t_2}$ cannot intersect $a_{t_1}$ it now follows that $\sigma_3$ is farther from $\sigma^a$ than $\sigma_1$ was. Thus the series $s_i$ with $i = 1,3,5,...$ is increasing. Similarly $b_{t_2}$ cannot intersect $b_{t_1}$ either and $\sigma_2$ is closer to $\sigma^a$ than $\sigma_0$ would have been: the series $s_i$ with $i = 0,2,4,...$ is also increasing. The Lyapunov function increases monotonously and $x^a$ is asymptotically unstable.

**Theorem 5.4** A buck-, boost-, or buckboost-converter whose switch is controlled by a stationary, straight line boundary that obeys Theorem 5.1 will be large signal stable if
- it operates in refractive mode and
- for a pair of any two successors $\sigma_n, \sigma_{n+2}$ on one side of the operating point $s_{n+2} < s_n$ holds true.

**Limit Cycles** A transition from case a) to case b) or vice versa is impossible, again due to the fact of nonintersection. It is a notable result that a limit cycle is not possible either. A true limit cycle would imply that, beginning from some moment in time, $\sigma_{n+2} = \sigma_n$. For this to occur some trajectory $b_{t_{n+2}}$ would have to hit the switching boundary at the same
point where \( b_t_n \) hit (or \( a_{t_n + 2} \) where \( a_{t_n} \) hit), which would require a common point of two trajectories of the same family. As introduced in Chapter 2, this is not possible (without violating Theorem 5.1). A stable refractive operation under a stationary boundary as displayed in Figure 5.3 a) will therefore always lead to a chattering problem. Some method of limiting the switching frequency will have to be applied in actual implementation [41].

- Refractive Case, Wrong Families

Although the wrong families of trajectories are consciously avoided through Theorem 5.1, an indispensable switching frequency limitation will lead to this situation (Figure 5.5 a)).

![Figure 5.5 Refractive operation, wrong families](image)

Due to the chattering problem, some method of limiting the switching frequency will have to be applied in actual implementation. Any method of limiting \( f_s \), however, will forbid a switching action unless sufficient time has elapsed since the last switching. Thus the boundary may demand a switching action which the limiting circuit suppresses: the state will continue to evolve along the “old” trajectory \( a_{t_1} \) in the “new” half-plane. Now the stability argumentation of case a) above is not valid any more. The “switching overshoot” brings the state to a trajectory that passes further away from the asymptotic point ( \( b_t_{1'} \) instead of the proper \( b_t_1 \)). Although \( b_t_{1'} \) in Figure 5.4 clearly fulfills case a) above, \( b_t_1 \) will lead to case b).

Regarding stability, Theorem 5.4 holds true unmodified even allowing switching overshoot.

**Limit Cycles**

A stable case a)-converter will lead to a steadily smaller cycle in the state plane and thus to a continuously increasing switching frequency. At some point, \( f_s \)
will reach the preset limit $f_{\text{max}}$ and the frequency-limiting device will become active. As just seen, the cycle will become bigger under frequency-limiting. This increase will occur until $f_s < f_{\text{max}}$ again. Thus under $f_s$-limiting a limit cycle occurs as a decrease-increase equilibrium. The limit cycle's size will be determined by $f_{\text{max}}$. Its switching points, the points of discontinuity, will lie on the switching boundary. Figure 5.5. b) depicts such a characteristic limit cycle.

5.2.3.2 Reflective Operation

Let the converter operate in the reflective mode. As both families of trajectories are directed towards the boundary, the state will not be able to move away from it. The only movement possible is along the switching boundary. This mode of operation is commonly known as *sliding mode* and has received wide attention in the last years. It has always been viewed as a separate field of control. This thesis, in contrary, perceives sliding theory as part of the larger framework of Boundary Control.

- Reflective Operation, Correct Families

The notion of a "correct" family becomes unclear in sliding mode. Ideally, the converter stays exactly on the boundary, which requires an infinite switching frequency. In theory every on- and off-trajectory is followed for a zero length. Thus there should be neither "correct" nor "wrong" trajectories, and only the boundary itself governs the dynamics. As soon as the switching frequency is limited, however, wrong trajectories will be followed. "Correct" in the reflective case will therefore designate the ideal operation with $f_s \to \infty$.

When the converter state moves in a sliding mode, there are basically two possibilities: the state moves towards the operating point $x^a$ (then the Liapunov Function $V(x)$ decreases and Theorem 5.3 indicates asymptotic stability) or it moves away from $x^a$ (then $V(x)$ increases and the movement is unstable). Successors $x_i$ are now not separate discrete points any
more. Rather the value of the parameter $s$ changes continuously. Instead of determining whether a series of $s_i$ is increasing or decreasing, it is now possible to use differential calculus. Evaluating the sign of the time derivative $\frac{ds}{dt}$ will tell whether the sliding motion on the boundary is directed towards $x^a$ or away from it. Figure 5.6 visualizes different situations. Three situations can be distinguished.

Figure 5.6  Direction of movement in sliding mode

First, assume that both trajectories have, at their points of intersection with $\sigma$, components tangential to $\sigma$ that show in the same direction. A sliding mode will then invariably drive the state in this very direction (Figure 5.6 a)). Second, let the respective tangential components be of opposite sign (Figure 5.6 b)). In this case the movement can be either way. It is known that the resulting motion can be calculated as a weighed sum of the tangential components. See [27], [42] for formulas. The third and last case is the one depicted in Figure 5.6 c). Here the trajectories are mutually tangential. An ideal reflective operation will move the state back and forth on a straight line given by the direction of the trajectories in that point. No lateral movement will occur. The state has reached its sliding mode equilibrium point.
As has been conjectured above, it is the points on the load line where the trajectories are mutually tangential. The combination of Conjecture 4.1 with the three sliding cases just presented leads to the stability evaluation in the reflective case (Figure 5.7).

![Diagram showing stable, semi-unstable, and unstable sliding regimes](image)

**Figure 5.7 Possibilities of sliding regimes**

**Theorem 5.5** A buck-, boost-, or buckboost-converter whose switch is controlled by a stationary, straight line boundary that obeys Theorem 5.1 will be large signal stable if
- it operates in reflective mode and
- for a pair of any two successors $\sigma_{t1}, \sigma_{t2}$ on one side of the operating point $s_{t2} < s_{t1}$ holds true, with $t_2 > t_1$.

**Limit cycles** Due to the infinite switching in the “correct” case, no limit cycles with nonzero geometrical dimensions will occur.

- **Reflective Operation, Wrong Families**
  Trajectories of wrong families are followed as soon as the switching frequency is limited. Unlike in the refractive case, a switching frequency limiting device will have to be active all the time. The result is that the state does in fact not slide on the boundary, but oscillates in some close vicinity of it. Figure 5.8 a) depicts this case.
  Now discrete successors $\sigma_i$ occur again. Stability properties will be preserved as long as the switching overshoot is sufficiently small. Utkin elaborates on this topic in theory [43].
Cardoso and others have reported on corresponding experiments [44]. Thus Theorem 5.5 still holds, though the formulation has to be changed to discrete successors again, as it was introduced in the refractive case.

**Limit Cycles**

Under switching frequency limiting a limit cycle will occur. Its size is determined by the switching frequency that is set. Its points of discontinuity, in contrast to the refractive case, are situated off the boundary, one on each side. Figure 5.8 b) shows a typical example.

### 5.2.3.3 Rejective Points

As becomes clear from Figure 5.3 c) points on the loadline that are rejective can never be reached. No trajectory leads towards them. If there is some contiguous region of rejective points, no point within that region can be attained by the system and no successor $\sigma$ will occur on that part of the loadline. If the operating point $x^a$ happens to lie within that region, it clearly can never be approached. This leads to the following instability theorem.

**Theorem 5.6**

A buck-, boost-, or buckboost-converter whose switch is controlled by a stationary, straight line boundary that obeys Theorem 5.1 will be large signal unstable if

- its operating point $x^a$ is situated within a region of the switching boundary where no successors $\sigma_i$ occur ("rejective region").
5.2.3.4 Transitions in the Mode of Operation

Thus far it has been shown that two stable modes of operations can occur, each one yielding its own form of a limit cycle. In addition, there is the rejective case. In both stable cases a limit cycle is made possible by the interaction of switching frequency limitation and switching overshoot. Stability criteria have been established individually for these two modes. It remains to see whether and when a converter changes between the three modes and what impact such changes have on stability.

• Transition refractive $\rightarrow$ rejective
  As has been argued above, an undisturbed system will never reach a rejective region. This first type of change will not happen.

• Transition rejective $\rightarrow$ refractive
  If due to a disturbance a converter is placed on a trajectory that passes through a rejective point, its state will be led to a nonrejective point by definition. As Figure 5.9 a) shows, this can very well be a refractive point.

• Transition reflective $\rightarrow$ rejective
  As a converter system will not reach a rejective region as long as it is not brought there by external forces, this type of change will not occur.

• Transition rejective $\rightarrow$ reflective
  Again, if an external disturbance places the state in a rejective neighborhood, its state will have to approach a nonrejective point. This can also be a reflective point (Figure 5.9 b)).

If the operating point is included in a rejective region, a refractive or reflective operation cannot direct the state towards it. Thus the system necessarily has to be unstable. In such a case two continued two-boundary operation will occurs, which will be discussed in Chapter 6.
It remains now to look at possible transitions between the two modes that can actually be stable under stationary boundary control. Figure 5.10 depicts the two transitions that are possible in theory.

- Transition reflective $\rightarrow$ refractive

Such a transition requires that one family of trajectories change the sign of its component normal to the boundary. After being directed towards $\sigma$, the trajectories have to be directed away from it. As the trajectories are smooth, this change requires a trajectory to be tangential to $\sigma$ in some point. This geometry can be seen in Figure 5.10 a). If the boundary is linear and the basic stability requirement formulated in Theorem 5.1 is satisfied, such a geometry cannot occur as far as buck, boost, and buckboost converter are concerned. Consider first the spirals occurring in all three converter types. The spiral will always be
tangential to a straight line from that side where its center is located. Theorem 5.1 requires
the spiraled trajectory to be valid i.1 that half-plane where its center is not located. Thus the
situation depicted in Figure 5.10 a) cannot occur with the spirals. Consider next the
hyperbolas arising in the boost and buckboost case. A hyperbola touching a straight line will
never reach that line again. It thus lies completely on one side of the line, including the
equilibrium point. Again Theorem 4.1 demands that this trajectory may not be valid in this
very half-plane.

It can therefore be concluded that a sliding mode, once it occurs, will not be abandoned by a
power stage controlled by a stationary straight line boundary.

- Transition refractive $\rightarrow$ reflective

This type of transition is depicted in Figure 5.10 b). It may very well occur. Figure 5.10 b)
in fact is taken from the plot of the buckboost converter's trajectories shown in Chapter 3.

This one example shows that the refractive mode may be abandoned at some point.

The following table may provide a summary of this discussion.

<table>
<thead>
<tr>
<th>Transition</th>
<th>Possible ?</th>
<th>Figure</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>refractive $\rightarrow$ rejective</td>
<td>no</td>
<td>--</td>
<td>transition can be caused by disturbance</td>
</tr>
<tr>
<td>rejective $\rightarrow$ refractive</td>
<td>yes</td>
<td>5.9 a)</td>
<td>transition only possible after disturbance</td>
</tr>
<tr>
<td>reflective $\rightarrow$ rejective</td>
<td>no</td>
<td>--</td>
<td>transition can be caused by disturbance</td>
</tr>
<tr>
<td>rejective $\rightarrow$ reflective</td>
<td>yes</td>
<td>5.9 b)</td>
<td>transition only possible after disturbance</td>
</tr>
<tr>
<td>reflective $\rightarrow$ refractive</td>
<td>no</td>
<td>5.10 a)</td>
<td>--</td>
</tr>
<tr>
<td>refractive $\rightarrow$ reflective</td>
<td>yes</td>
<td>5.10 b)</td>
<td>--</td>
</tr>
</tbody>
</table>
5.2.3.5 Reaching the Operating Point

It can be concluded from the above data that only certain ways are possible to reach the operating point.

Table 5.2 Successive transitions possible to reach $x^a$

<table>
<thead>
<tr>
<th>№</th>
<th>Transitions</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(rejective→) refractive</td>
<td>rejective</td>
</tr>
<tr>
<td>2</td>
<td>(rejective→) reflective</td>
<td>only after</td>
</tr>
<tr>
<td>3</td>
<td>(rejective→) refractive → reflective</td>
<td>disturbance</td>
</tr>
</tbody>
</table>

Overall it is interesting to note that, all parameters of operation remaining equal and no disturbance occurring, only one transition between modes can occur. It is also possible to say which one: It has to be a change from the refractive to the sliding mode.

Considering the fact that a sliding mode cannot be left again, it is natural to ask for the likelihood with which, in transient operation, the operating point will be reached in this mode. By combining the results obtained so far it is easy to show that in almost all cases a sliding mode will bring the converter towards the loadline and thus to the asymptotic point: In refractive operation $x^a$ is approached by a sequence of switch actions on the boundary. With $f_1$ increasing, the distance between two subsequent switchings decreases. In the limit two subsequent switchings fall on the same point, namely the asymptotic point. In order to allow for this phenomenon, the trajectories do not only have to be tangent mutually, but also tangent to the switching boundary. See Figure 5.11 a) for a visualization. In sliding mode, any angle between switching boundary and trajectories is allowed.

Conjecture 4.1 stated that on the loadline the trajectories are mutually tangential. Reaching the asymptotic point via a refractive mode does additionally require the switching boundary to have that exact same slope.
This equality, however, will rarely -- in practice never -- be the case. Thus we conclude that almost always while approaching the asymptotic point the converter will change into a sliding mode at some point. The expected operation thus is a sequence of first a refractive operation (if at all) and then a sliding mode in order to approach the operating point.
5.2.4 Operation with Two Boundaries

As mentioned above the restriction that the diode can only sustain forward current can be interpreted as a boundary as well. It will be named Diode Boundary (δ) in order to contrast it with the switching boundary σ. This section will now extend the stability considerations to cases where δ plays a role in addition to σ. It is those cases in which the converter operates, at least for some time, in the discontinuous mode.

In contrast to a boundary that passes through the state space, the points of this special boundary fall only in two categories: reflective and rejective. This limitation is due to the fact that $i = 0$ can only be approached from one side, namely $i > 0$. Figure 5.12 depicts this situation.

![Figure 5.12 Types of diode boundary points](image)

It is known and can be inferred from the network equation that the state begins to move towards the origin when it reaches the discontinuous mode: The inductor current is zero and the capacitor is discharged by the load.

5.2.4.1 Reflective Operation

In all the dc-dc converters considered here the "on" trajectories are spirals that are, at least in some region, directed towards the diode boundary in the manner that Figure 5.12 a) depicts. Let us look at this family of trajectories. They are valid for the transistor being off, which is the case if the state is above the switching boundary. Considering that σ is regularly implemented with a negative slope, this means that the state has to hit δ to the right of the intersection σ-δ. It will thereupon slide on δ towards the origin until it hits σ. We will call this...
point the successor $\sigma_0$. In $\sigma_0$ the family of on-trajectories will become valid. Two cases have now to be distinguished.

- Refractive Operation with regard to $\sigma$

Here the state moves away from both $\sigma$ and $\delta$. Thus the switching is refractive with respect to $\sigma$. This phenomenon is shown in Figure 5.13.

![Figure 5.13 Refractive operation with regard to $\sigma$](image)

**Figure 5.13** Refractive operation with regard to $\sigma$

**case a)** If the next off-trajectory leads once more to the diode boundary, the sliding on $\delta$ will invariably bring the state to $\sigma_0$ again. Hence Theorem 5.4 is violated: $s_{n+2}$ is not smaller than $s_n$. In fact, $s_0$ given by $\sigma_0(s_0)$ is exactly what was termed $s_0^*$ when the Successor Functions were introduced in Section 5.2.1. It denotes that successor that is reached over and over and thus is part of a limit cycle. Figure 5.13 indicates that Theorem 5.4 still holds true: The stability requirement $s_2 < s_0$ can be taken directly from the one-boundary case.

This requirement can be restated in terms of the diode boundary. This transformation will be done here in order to allow for unified stability theorems for the two-boundary cases, similar to those of the one-boundary situation.

Let the diode boundary $\delta$ also be a differentiable arc demanded by the Theory of Successor Functions. Let its parameter $d$ assume the value zero in the origin of the state plane.
Theorem 5.7  A buck-, boost-, or buckboost-converter whose switch is controlled by a stable stationary, straight line boundary and the diode boundary will be large signal stable if

- for a successor $\delta_0$ there is no successor $\delta_1$ that is not reached via a sliding mode on the diode boundary.

**Limit cycles**  This case furnishes two interesting results.

- This is the first case where a true limit cycle occurs.
- Still, the operating point is not large signal stable under that limit cycle as it can not be reached with any desired closeness.

  case b) If the next off-trajectory redirects the state to the switching boundary instead, Theorem 5.4 is satisfied and stability can be concluded. The same result can be obtained by applying Theorem 5.7. An interesting consequence is the following.

- Continued discontinuous operation and stability of the operating point in the sense of Definition 5.1 are mutually exclusive

- Reflective Operation with regard to $\sigma$

In this situation the state enters into a sliding regime on $\sigma$ upon arriving at $\sigma_0$ (Figure 5.14).

Figure 5.14 Reflective Operation with regard to $\sigma$

  case a) The sliding mode is directed upwards towards $x^a$. Now the discontinuous operation has been quit and Theorem 5.5 is applicable again.

  case b) Let us assume the sliding mode be directed downwards, away from $x^a$. In that case, the state would have to remain stationary in $\sigma_0$. This, however, is physically
impossible. $\sigma_0$ is characterized by a nonzero capacitor voltage, which translates into a nonzero output voltage. The load will take up energy. Still the capacitor is not discharged and the transfer source, the inductor, does not carry energy. These statements are contradictory, which proves that case b) will not occur in the dc-dc converters studied.

5.2.4.2 Transitions

As in the continuous case it has to be determined whether changes between the reflective to the rejective case can occur. One situation has just been studied: changes at the intersection $\sigma$-$\delta$. It remains to investigate whether transitions can occur at other points of the diode boundary.

- **Rejective $\rightarrow$ Reflective Operation**

If the state starts out in a rejective region, it may reach a reflective part of $\delta$ afterwards (see Figure 5.15 a)).

![Figure 5.15 Transitions rejective $\rightarrow$ reflective](image)

This form of transition necessarily requires an intersection with $\sigma$. A return to $\delta$ on one trajectory will definitely not be possible on a hyperbola. As far as the spirals are concerned, they circle around their equilibrium point (Figure 5.15 b)). But Theorem 5.1 specifies that a family may only be valid in that region where its singularity does not lie. Thus such a transition is impossible.

From a stability point of view case a) is fundamentally the same as the one depicted in
Figure 5.13 a). The argumentation drawn there still holds and stability criteria are unchanged.

- Reflective $\rightarrow$ Rejective Operation

Once the state is driven to $\delta$, it begins to slide on it. We have shown that on the switching boundary such a sliding mode cannot be left any more. On the diode boundary, the situation is different. Figure 5.16 a) displays a spiral that crosses $\delta$ twice, the first time in reflective mode, the second time in rejective mode.

![Figure 5.16 Transitions reflective $\rightarrow$ rejective](image)

Of course, the inductor current cannot become negative. Thus the state slides on the boundary while the trajectory is located below it (Figure 5.16 a)). The sliding will last until the state reaches that part of the diode boundary where it is rejective. Let us localize the point on $\delta$ that separates its reflective from its rejective region. The reasoning is simple and has already been used in a similar manner in this thesis. At that very point, the component of $\frac{dx}{dt}$ that is normal to $\delta$ changes its sign. Due to the fact that the trajectories are smooth, such a change has to be continuous. This means that in the point of change, the normal component has to be zero. Thus the point of change is the point where a trajectory of that family is tangential to the $x$-axis. The state will leave the diode boundary in this very point and will continue on a different trajectory of the same family than it arrived on. In this way it is possible that a continued operation reflective-rejective arises.
5.2.4.3 Reflective-Rejective Operation

Such a reflective-rejective operation is visualized in Figure 5.17.

![Figure 5.17 Reflective-rejective operation](image)

**case a)** As Figure 5.17 a) shows, the state can reach the diode boundary more than one time. According to the argumentation above, it does not matter where the state hits the diode boundary: it will always leave it on that trajectory that is tangential to $\delta$.

**Limit Cycle** As soon as the state hits the diode boundary a second time, it will enter a limit cycle.

**Stability** The operating point will therefore never be approached further and $x^\alpha$ is unstable.

**case b)** If the diode boundary is not reached a second time, it does not alter the stability behavior of the switching boundary (Figure 5.17 b)).

These results can be formulated via Theorem 5.7. In fact, the requirements given there already include the reflective-rejective case: stability forbids lasting discontinuous operation.
CHAPTER 6

Computer Visualization and Animation

So far, a substantial number of thinkable operating modes have been studied and classified. Some of them have proven unstable, some of them stable. Some cases were shown to give rise to extended limit cycles. Of others it has even been argued that they do not occur with the converters under consideration. Finally, a stability theory culminating in one plus four stability theorems has been conceived. Throughout this process, computer generated curves have served as a vital basis for establishing these results.

6.1 The Motives for Computer Simulation

In a next step, these results have to be verified, or at least supported by some additional means. Computer simulation was chosen as a first tool for this undertaking. There are several reasons that make computer visualization and animation the obvious choice.

- The converter models have already been brought on the computer. This implementation was necessary in order to solve the converters' differential equations, to which no closed analytical solutions exist.
- The principle of Boundary Control is to visualize a system's state in the state space. In the case of the converters studied here, the state space is of dimension two and reduces to the state plane. The computer cathode ray tube indeed is a wonderfully suited realization of a two dimensional plane.
- Compared to real world experiments, the computer permits the testing of ideal cases. For example, it is not possible to test, in reality, a theory that assumes no parasitics on an electrical circuit. Also there is no uncertainty in parameter values due to measurement inexactness.
• Computer simulations may not be faster to construct than actual experiments, but they are easier to adapt. Changes of parameter values can be performed within seconds.

• A computer animation can be run in speeds different from those that occur in practice. It can be very useful to watch an event in slow motion in order to recognize essential points in its operation.

This list is definitely not complete, but it motivates the usefulness of digital simulation at this point very well. Besides proving the stability theory set up so far, there is an additional purpose that is worth mentioning: Ultimately, real world experiments will be performed in the laboratory. Computer simulation shall help to find operating conditions that are interesting to study in the real world. Suitable parameter values can be predetermined so that the laboratory work can be organized in an efficient manner.

In the following we will describe the two computer simulations that were developed along these aims. The first one is termed Computer Visualization. It serves as a static, graphical depiction of Boundary Control and its stability criteria. The second one is Computer Animation. Animation produces a moving picture of the state when the converter is operating.

Regarding the software, it was decided to use Matlab™. First, Matlab is an application package that is widely used in engineering. Thus it can be assumed that programs written during this thesis can easily be understood, used, transferred and extended later on. Second, the matrix features of Matlab support the network equation nicely. And third, it possesses built-in graphics support that is well suited for visualization and animation.
6.2 Computer Visualization: Boundary Evaluation

In Chapter 5 we distinguished three classes of boundary points: refractive, reflective and rejective points. When the state hits the boundary, the class that hitting point belongs to determines the type of stability and of transient behavior that occurs. In order to obtain a clear picture of what converter behavior a specific control (i.e. specific boundary) will provoke it is helpful to visualize that classification. A first computer simulation tool is therefore a static one. The purpose of this evaluation is to furnish a comprehensive idea of how the converter will behave in operation, independent from the particular initial condition.

6.2.1 Formal Definition of Refractive, Reflective and Rejective Points

In Section 5.2.2 we introduced the three classes of points based on a more or less intuitive understanding. It is now necessary to establish exact mathematical criteria for a clear distinction.

The foundation of the definition is a Cartesian coordinate transformation. The state space in our case is represented as a two-dimensional Cartesian coordinate system whose unit vectors are \( \vec{v} \) (abscissa) and \( \vec{t} \) (ordinate). This original system is visualized in Figure 6.1 a) together with an example of a \( \frac{dx}{dt} \)-vector.

![Figure 6.1 v,i- and t,n-coordinate system](image)

Figure 6.1  v,i- and t,n-coordinate system
Let us now introduce a second, boundary related Cartesian coordinate system. Let its abscissa be tangential to $\sigma$ and consequently $\tau$ be called $t$. Its ordinate then is normal to $\sigma$. We will call it $n$. Figure 5.11 b) shows this new coordinate system.

State velocity vectors can be transformed from the $\tilde{v}, \tilde{t}$- into the $\tilde{t}, \tilde{n}$-system by a conventional coordinate rotation (see for example [46]):

$$
\begin{align*}
t &= v \cdot \cos(\gamma) - i \cdot \sin(\gamma) \\
n &= v \cdot \sin(\gamma) + i \cdot \cos(\gamma)
\end{align*}
$$

(6.1)

The value for $\gamma$, the angle between $\sigma$ and the $v$-axis is easily determined from the switching boundary’s explicit equation (4.2). The boundary slope is seen to be $-k_v/k_i$. Consequently

$$
\gamma = \arctan\left(-\frac{k_v}{k_i}\right)
$$

(6.2)

Every point on the boundary is assigned two state velocity vectors $dx/dt$, one for the on-family and one for the off-family. Let us call them $^{on}\dot{x}$ and $^{off}\dot{x}$.

**Definition 6.1 (refractive point)** A point $\dot{x}$ that fulfills equation (4.1) is called refractive if - both the $\tilde{n}$-component of $^{on}\dot{x}$ and $^{off}\dot{x}$ are positive or - both the $\tilde{n}$-component of $^{on}\dot{x}$ and $^{off}\dot{x}$ are negative.

In the first case, the point will be called "refractive up", in the second "refractive down".

**Definition 6.2 (reflective point)** A point $\dot{x}$ that fulfills equation (4.1) is called reflective if - the $\tilde{n}$-component of $^{on}\dot{x}$ is positive and the $\tilde{n}$-component of $^{off}\dot{x}$ is negative.

**Definition 6.3 (rejective point)** A point $\dot{x}$ that fulfills equation (4.1) is called rejective if - the $\tilde{n}$-component of $^{on}\dot{x}$ is negative and the $\tilde{n}$-component of $^{off}\dot{x}$ is positive.

For the refractive points the Definition itself yields a valuable distinction. It is possible to determine to which side of the boundary the state will be refracted. It would be nice to have a corresponding distinction for the reflective case. It would help to know in which direction the
state is going to slide on $\sigma$. This problem has already been mentioned in Chapter 5. Let us now adapt from Fillippov [27] in order to establish a criterion on this point$^8$.

\[
t_{\text{res}} = \delta \cdot \text{on} t + (1 - \delta) \cdot \text{off} t
\]

\[
\delta = \frac{\text{on} n}{\text{off} n - \text{on} n}
\]

with $\text{on} t$, $\text{off} t$ being the absolute value of the $\vec{T}$ -component of $\text{on} \vec{x}$, $\text{off} \vec{x}$

$\text{on} n$, $\text{off} n$ being the absolute value of the $\vec{n}$ -component of $\text{on} \vec{x}$, $\text{off} \vec{x}$

According to Fillippov $t_{\text{res}}$ is the resulting vector of motion of the boundary. Its sign can be evaluated to determine the direction of the sliding action.

**Extension of Definition 6.2**

A point that fulfills Definition 6.2 can show $t_{\text{res}}$ positive or $t_{\text{res}}$ negative. In the first case, it will be named "reflective right", in the second "reflective left".

We note that $t_{\text{res}} = 0$ indicates a sliding equilibrium point.

### 6.2.2 The Algorithm for the Visualized Boundary Evaluation

We now have established the mathematical foundations for an evaluation of the load line points. What has been said in the previous section already gives some idea of what a corresponding program will have to accomplish. In the following this notion will be deepened and the structure of the algorithm will be described verbally. A Nassi-Shneiderman diagram of the algorithm can be found in Appendix B.4.

Although it is not necessary, it is very helpful to first plot the loadline. It will be seen later that a meaningful interpretation of the boundary characteristics is only possible with respect

---

$^8$ The original formulas can be found in [27], p. 52. They were simplified to the two-dimensional case and adapted to the $t,n$-system. In addition, the notation was adjusted.
to the operating point. As the intersection boundary-loadline marks this point, plotting the latter is a nice way to indicate \( x^a \).

Second, we need to determine all the points that qualify for evaluation according to Definitions 6.1 to 6.3, i.e. the boundary points. Due to the discrete operation of any digital computer it is necessary to specify a certain number of points on the line that shall be evaluated. In order to permit maximum accuracy the point density should be such that the resolution of the graphics is fully utilized.

Third, each of these points has to be tested against the criteria set up in the Definitions 6.1, 6.2 and 6.3. This is where the coordinate transformation comes in. At each boundary point, the rhs of the network equation (2.2) assigns values for the vector \( \frac{dx}{dt} \) in the \( \vec{v}, \vec{t} \) -system. By first setting \( q_1 = 1, q_2 = 0 \) and then setting \( q_1 = 0, q_2 = 2 \), one obtains \( \text{on} \cdot \vec{x} \) and \( \text{off} \cdot \vec{x} \).

Having established \( \gamma \) by equation (6.2), these two vectors can be mapped into the \( \vec{v}, \vec{r} \) -system as given by (6.1). It is now elementary to evaluate the signs of their components in accordance with the Definitions. In the reflective case, one has to go one step further in order to fully classify the boundary point. The sign of \( t_{res} \) is also needed. Hence one calculates first the weighing factor \( \delta \) from (6.4), and then \( t_{res} \) by applying (6.3).

Now the classification can be done exhaustively. Table 6.1 summarizes the cases for a quick overview.

Table 6.1 Classes of boundary points and their criteria

<table>
<thead>
<tr>
<th>( \text{on} \cdot t )</th>
<th>( \text{off} \cdot t )</th>
<th>( t_{res} )</th>
<th>Class</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&lt;&gt; #</td>
<td>refractive up</td>
<td>*</td>
</tr>
<tr>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>reflective right</td>
<td>x</td>
</tr>
<tr>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>reflective left</td>
<td>o</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>&lt;&gt;</td>
<td>rejective</td>
<td>-</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt;&gt;</td>
<td>refractive down</td>
<td>+</td>
</tr>
</tbody>
</table>

\* # <> means: sign does not matter
Once a boundary point has been classified, it has to be plotted. In order to indicate its type, colors (on the screen) or symbols (allowing for black and white printouts) can be used. The symbols used in the following are indicated in the table's rightmost column.

6.2.3. Example of an Evaluated Boundary

The result of this evaluation are plots similar to the following.

![Example of an evaluated boundary](image)

Figure 6.2 Example of an evaluated boundary

The simple line is the loadline \( LL(x) \), whereas the marked one is the switching boundary \( \sigma \). As stated above, the operating point \( x^a \) is given by the point of intersection of these two.

6.3 Computer Animation: Converter Operation

So far, we have constructed a fragmented picture of the buck, boost and buckboost converter controlled by stationary, straight line boundary. The families of trajectories are known. The set of possible operating points has been determined and visualized as the loadline. A boundary can be chosen and evaluated to reveal its properties. Yet all this information still has to be assembled to give an integral picture of a converter's operation. This is done by computer animation (see [11] as basis).
6.3.1 The Algorithm for the Animated Converter Operation

At any moment in time the converter state $x$ can be represented as a point in the state space. The core task of an animation program therefore is to calculate the evolution of the state over time and to record this evolution by plotting subsequent dots at the respective locations in the state space. The complete algorithm, however, integrates many of the concepts studied so far. It will be described below. A diagram of it can be seen in Appendix B.5.

Before starting the actual animation, two central static elements are plotted, namely the means of control and the target of the control. The first one, of course, is the switching boundary. The program therefore calculates a sufficient number of boundary points from equation (4.2) and displays them on the screen in order to give a smooth boundary line. The second one is the operating point $x^o$. It was chosen to locate it as the point of intersection between the switching boundary and the loadline. For this reason, the next step is to calculate an adequate number of loadline points and to plot them. Having established the immobile part of the picture, the actual animation is carried out.

This iterative process is accomplished by the classical programming means of a loop. It starts out with the initial value $x = x_0$ specified by the user. The following steps are then performed within the loop:

- The first task is to determine the switch positions that the chosen boundary control law prescribes for that point of the state space:
  - $v$ is taken from $x$ and the ordinate of the boundary is determined from (4.2) to give $i_{\text{boundary}}$.
  - $i$ is taken from $x$ and compared to $i_{\text{boundary}}$. If it is smaller than $i_{\text{boundary}}$, then the state is below $\sigma$ and the controlled switch is turned on ($q_1 = 1$, $q_2 = 0$). If it is greater, then the state is located above $\sigma$ and the controlled switch is off ($q_1 = 0$, $q_2 = 1$). There is one special case, however: if $i_{\text{boundary}}$ is negative, then the diode boundary $\delta$ may become active. As soon
as i tries to become negative, discontinuous mode occurs and both the controlled switch and the diode turn off \( (q_1 = 0, q_2 = 0) \).

- Next the change in position of \( x \) has to be determined:
  - The network equation (2.2) specifies \( dx \) for an infinitesimally small time \( dt \). By extending the time to a finite \( \Delta t \), the state change \( \Delta x \) is computed\(^9\).
  - The updates state is calculated as \( x_{n+1} = x_n + \Delta x_n \).

- Lastly the new location of the state has to be marked in the state plane:
  - A dot is displayed at that point of the screen that corresponds to the updated state value\(^{10}\).

Despite the low complexity of this algorithm, there are some points that require special attention when implementing it. These issues will be discussed in the following Section.

### 6.3.2 Issues in Implementing the Animation

The problems that are encountered with such an animation algorithm have many facets. First, there are those concerns that arise from the necessity of applying an iterative technique. Second, there are difficulties caused by the constraints of the animation software. And third, there is the issue of convenience for the user to watch the animated sequence. These three aspects are related to a large extent and very often implementation decisions bring along various tradeoffs. Let us illuminate some of them.

- **The Choice of the Integration Formula**

  Since no analytical solution to the network equation is known, one is required to make use of some approximate method. Inaccuracy is therefore unavoidable. The task is to keep it within an acceptable range. Many different iterative approaches for solving systems of first-order differential equations have been elaborated and represent possible choices.

\(^9\) A discussion of this procedure will follow in the next section.

\(^{10}\) Due to limitation in graphics speed it can be indicated to display only one out of \( d \) subsequent points, with \( d \) being an animation parameter. See discussion below.
The best-known are Runge-Kutta-, Adams-, Gear- and Euler-Cauchy algorithms. These algorithms trade simplicity for accuracy. The decision rule for the choice is simple and difficult at the same time: Basically any of these methods will do the job as long as it is sufficiently accurate. Based on previous work of Bass, the uncomplicated Euler-Cauchy-Method, also called Forward Euler Integration, was chosen. A brief description of this technique can be found in Appendix B.1.

- The Choice of the Integration Step Size $\Delta t$

Once Forward Euler Integration is selected, the accuracy of the animated state evolution is determined by the degree to which the infinitely small $dt$ is changed into a $\Delta t$ with a real, positive value. The smaller $\Delta t$ is made the less deviation between animated and veritable trajectory is encountered. On the other hand, a small step size increases computation time significantly. The following was done in order to handle this problem: a very small integration step size was chosen in order to furnish a reference trajectory regarded as the veritable one. Then $\Delta t$ was increased in order to obtain fast computation until some deviation became remarkable. At this point, the step size was decreased again to a value where virtually no error occurs.

- Integrating over the Switching Boundary

Every iterative integration technique that operates with a fixed step size $\Delta t$ will lead to a problem when integrating over a discontinuity such as the switching boundary. Figure 6.3 depicts the dilemma.

The state is calculated at discrete points in time. The points, plotted as "•" in the Figure, are spaced on the trajectory (---) according to the value of $\Delta t$. When the switching boundary is approached (⊙,⊙) it is very likely that no state position will be calculated on the boundary itself. Instead, point ⊙ is evaluated as being situated below the boundary and
the switch states will not be altered with respect to $\Theta$. They should be altered exactly on $\sigma$. Still this change does not happen before $\Theta$. Consequently the new trajectory is $f$, and not $f'$ as it should be. This situation has been termed ‘wrong family’ and has been studied in Section 5.2.3 in theory. From there we know that such an overshoot impedes the state to reach $x^a$. Two solutions are feasible: Either choose $\Delta t$ sufficiently small so that, within the desired level of accuracy, the overshoot does not cause a problem. Or recur to techniques that will notice the discontinuity, recede and decrease the iteration step size around $\sigma$ sufficiently to achieve the desired exactness (such algorithms are discussed for example in [33]). Both solutions have the drawback of increasing the computation time substantially. For the animations that were run in the context of this thesis, the first method was chosen and a slightly slower animation accepted.

- Animation Speed

It turned out that in Matlab™ the slow element is not the computation routine for Forward Euler Integration but the plotting routine. Calculating $x_{n+1}$ is fast, but displaying a dot on the corresponding location of the screen takes considerable time. In order to address this shortcoming the displaystep $d$ was introduced. A large density of subsequent trajectory points is calculated (thus accuracy is preserved), but only every $d$-th point is plotted in order to speed up the animation. As long as $d$ is chosen sufficiently small considering $\Delta t$, the observer can still reconstruct the trajectory in his mind.
Maximization and Ease of Information

On the screen, colors can be used effectively to create a picture that is instructive and easy to grasp. For example, the trajectory points can, by virtue of their color, indicate the switch positions valid at each point. Of course, this information can be recovered by the spectator using exactly the same evaluation performed by the computer during the first step of the loop (see previous Section). Yet this process is tedious and an animation program should pay attention to such human aspects.

Another question is whether to create animation “with memory” or without it. The state of a real converter is unique at any time, and thus a realistic animation would display exactly one point at a time. Yet it has shown that the information lies in the complete trajectory, which a human is not able to retain from seeing a moving point. Thus the animation was implemented with memory, that is all the past state points are indicated together with the most recent point. This memory also enables printouts of the animation results.

Graph Parameters

Whenever the parameters of the converter power stage or the control stage are changed for a new run of the animation program, it is likely that the interesting phenomena occur in a different part of the state plane. It must therefore be possible to adjust the display parameters, especially the portion of the state plane that is displayed, accordingly in an easy manner. Matlab does provide easy setting of these parameters, but unfortunately does not possess a rescaling capability once plots are accomplished.
6.3.3 Example of an Animation

Naturally animation results have to be reproduced in this thesis as static printouts. The trajectory points appear subsequently on the screen, but simultaneously on paper. The information that is lost is the direction in which the state moves. In order to overcome this disadvantage, a small arrow has been added by hand before copying the plots.

Figure 6.4 exhibits an example of such an animation result.

![Figure 6.4](image)

Here both the boundary and the loadline appear as simple lines. Note that the time elapsed between subsequent points is constant. Their distance therefore is a precise indicator of the state speed at any section of the trajectory.
6.4 Visualization and Animation Results

The boundary evaluation and the state animation are graphical tools that support each other mutually. When looking at the first, it is not yet obvious how a state will evolve from a certain point on. The information on the families of trajectories of which a specific trajectory has to be constituted is missing. When only analyzing the second, it is not clear what effects an alteration of the boundary will have on the state evolution. It seems therefore appropriate to present the results of these simulations as pairs of plots. Based on these pairs the stability theory of Chapter 5 will be assessed. In addition, interesting phenomena such as overshoot will be studied.

Due to the infinite number of possible simulations, the plots exhibited in this Chapter are necessarily just examples. The plots reprinted here were chosen because they represent characteristic features nicely. However, corresponding graphs can be and were obtained for the other converters as well.

The converter parameters used in the simulation are the same as those used for the plots of families of trajectories, given in Chapter 3. The values were preserved in order to permit for comparisons.

6.4.1 Operation with One Single Boundary

The results representing the one-boundary cases were taken from the buck converter. Figure 6.5 displays a series of three different boundaries, all designed to achieve a 5V output voltage from a 10V input.
Figure 6.5  Sample trajectories for one-boundary operation
• Refractive Operation

Section 5.2.3.5 claims that, assuming stability, it is both necessary and sufficient for a purely refractive operation that the slope of the boundary equals the slope of the trajectories in the asymptotic point. From the calculation in Figure 4.6 we know that the simple buck converter's trajectories are vertical in $x^a$. In the enhanced model the angle is slightly altered. With the parasitics parameters used it is shifted to 89.97°. In Figure 6.5 a) such a boundary is implemented. And as a matter of fact, the evaluation of that boundary indicates purely refractive properties. To the left (top) of $x^a$ the boundary points are refractive up (right), and to its right (bottom) they are refractive down (left).

This permits us to apply and test Theorem 5.4 in order to evaluate stability. The plot to the right displays a part of the converter's operation under that very boundary (the initial condition was chosen to be $(0.3A, 4.5V)$). Regarding successors, it is evident that the stability condition $s_n + 2 < s_n$ is fulfilled. And really, the operation shows to be stable since the cycle becomes continually smaller. Thus Figure 6.5 a) is the archetype of the refractive stability predicted in Chapter 5.

It is noteworthy to comment on the length of the simulation. It is obvious that the simulation was stopped long before the operating point was approached. 500 display iteration steps, corresponding to a simulated time of 5 milliseconds, were carried out. The reason for this limitation will become clear while studying the cases b) and c).

• Transition refractive $\leftrightarrow$ reflective

In Figure 6.5.b) the weight of current feedback is increased, which in geometrical terms decreases the boundary's slope. In the proximity of the operating point, a sliding mode has developed. This is what is expected: The angle between $\sigma$ and the trajectories has become unequal zero, which according to Section 5.2.3.5 translates into a reflective mode around $x^a$. In contrast to refractive switching, in reflective operation it is possible to determine stability
directly from the evaluated boundary. Theorem 5.5 demands $s_{22} < s_{41}$. Thus as soon as $\sigma$ is reflective towards the operating point at both sides, it is stable. For the refractive part of the operation Theorem 5.4 is applicable again. The complete operation turns out to be of the refractive $\rightarrow$ reflective type. At first the converter adopts refractive operation. Successor values are decreasing and the operating point is approached. As soon as a successor falls into the reflective region, a sliding mode is adopted which then brings the state towards the operating point.

Again, 500 display iterations were completed. We note that this time the operating point is virtually reached.

- **Reflective Operation**

  The last pair of plots reveals an entirely reflective operation. As the boundary has become more and more horizontal, the reflective region has extended. Already the first successor hits $\sigma$ where the boundary is reflective, and the operating point is approached directly in a sliding motion. Also this time, 500 trajectory points were plotted.

- **General Outcomes**

  Besides the confirmation of the stability theory developed in Chapter 5, there are two results that are interesting to note.

  **Outcome 6.1** A stationary, straight-line boundary changes its regions of refractive, reflective and rejective mode continuously and not abruptly as its slope is changed.

  Portions of the line with the same characteristics extend or contract, but they never do that in a discontinuous manner. When a new mode appears, it starts out as a point. This result corresponds well with intuition. As the trajectories themselves are smooth and appear with infinite closeness in families, a gradual change in the boundary slope should provoke only a gradual change in its properties. Outcome 6.1 can be further observed in the trajectories.
presented later in this Chapter.

This outcome has important consequences for the design process of a boundary controlled converter: It is ensured that gradual changes in the feedback gains parameters will only provoke gradual change in the converters general behavior. This property is very helpful when tuning a converter for best operation and performance. Thus converters controlled by a stationary straight-line boundary should be not too difficult to tune.

**** Outcome 6.2 ****

Purely refractive operation reaches the operating point much slower than refractive→reflective or purely reflective operation.

All three plots show the same number of iteration steps. In the first case, the state spirals around the operating point, coming closer very slowly. As can be seen, the speed with which the size of the cycle becomes smaller decreases very fast. The top two successors are a significant distance away from each other, whereas the centermost ones are very close. After 5 ms, the state is still ±0.1V off the target point. In the two sliding mode cases, however, x^a can be considered as attained after that time. We can even infer from the plots that the ±0.1V region is reached in considerably less than 5 ms.

This outcome will be considered further in the next Chapter, when analyzing the transient behavior of the converters.

6.4.2 Operation with Two Boundaries

A second series of plots examines the buckboost converter. Emphasize is put on the two-boundary operation. Figure 6.6 is given as an initial point. It shows that with a relatively horizontal boundary there exists a considerable reflective zone around the operating point. A trajectory started out in (1A, 4V) hits the boundary in this region immediately and a sliding mode drives the state towards the operating point. Operation from this initial condition with a
boundary slope below and slightly above the value shown in this Figure is not yet different from the case discussed above.

Figure 6.6 Reference trajectory for two-boundary operation

Two-boundary operation begins as the boundary slope in increased (Figure 6.7).

- Stable Reflective Operation with regard to $\sigma$

The boundary becoming steeper, the reflective region gets smaller. In Figure 6.7 a) it has nearly disappeared to the left of the operating point. Yet on its right the boundary is still composed entirely of reflective points. However, the change in slope has significant impact on the state evolution. Since the intersection $\sigma$-$\delta$ has moved towards the origin, the state now hits the diode boundary when started out in $(1\text{A}, 4\text{V})$: the converter operates in discontinuous mode. At first, the state slides on $\delta$ until it arrives at $\sigma$. At this point, as it was discussed in the previous Chapter, several cases can occur. Here, the boundary $\sigma$ is reflective. It was argued in Section 5.2.4.1 that in such a case the sliding direction will always be towards the origin and the motion will be stable. This is in fact what happens in the case presented. The animation reveals that $x^\sigma$ is reached in a reflective manner.
Figure 6.7 Sample trajectories for two-boundary operation
• Stable Refractive Operation with regard to $\sigma$

In Figure 6.7 b), the slope is very little below the purely refractive case ($24.3^\circ, 25.5^\circ$). The reflective surrounding of $x^a$ now begins to disappear also to the right of the operating point. Although it is difficult to recognize on this animation plot alone, the two plots together indicate that a reflective mode of operation is entered at the intersection $\sigma-\delta$. The state is driven up very close to the switching boundary (just as can be recognized more clearly in part c) of the Figure), intersects it just above $x^a$, enters into a sliding mode and finally reaches the operating point. Only one time the state has arrived on $\delta$. Thus it fulfills the stability criterion set up by Theorem 5.6. And in fact the simulation proves that the controller-converter system is running stable, hereby sustaining the Theorem.

• Unstable Refractive Operation, Rejective Points

When the boundary slope is still increased, the reflective points on the switching boundary disappear. After the discussion of the buck converter in the last Section this is what is expected. However, after the boundary has become fully reflective, an interesting phenomenon occurs: a region of rejective points arises (Figure 6.7 c)). Before discussing that region let us look at the animation result. As in the preceding situation, the intersection $\sigma-\delta$ is refractive. The state is driven away from $\sigma$ at first to hit it at a current somewhat higher than $1A$. The next section of the trajectory brings the state back to the diode boundary. It repeats its sliding towards $\sigma$ and in fact it repeats the complete last cycle. The state has locked into a limit cycle operation around $x^a$. Obviously this motion is stable. But since it is not possible to increase the switching frequency in order to get the cycle smaller, the operating point itself is unstable according to our definition. The controller-converter system acts more like an oscillator than like a dc-dc converter.
• General Outcome

In Figure 6.7 c) the boundary’s section of rejection is of special interest. What is especially astonishing is the fact that the operating point is included in that region. This would mean that such a boundary would not allow the operating point to be reached from any initial condition. Thus some boundaries that fulfill the basic stability requirement formulated prior to this thesis will not drive the state towards the asymptotic points. In other words, Theorem 5.1 would not be a sufficient condition for large signal stability.

For the reason of the importance of this matter, let us not solely rely on the geometrical representation. An analytic analysis of the case shown in Figure 6.7 c) reveals the following.

The equation of the loadline is

\[ R(R_{ds} + R_L) \cdot i^a \cdot (R + R_c) \cdot V - R \cdot \frac{R \cdot R_c}{R + R_c} \cdot i^a \cdot v^a - (E + V_{drop}) \cdot v^a + \frac{R}{R + R_c} \cdot v^a = 0 \]

With the parameter values given in Chapter 3 and \( V_{out} = 5 \text{V} \) one obtains

\[ x^a = \begin{pmatrix} 0.402 \text{A} \\ 5.000 \text{V} \end{pmatrix} \quad \text{and} \quad \alpha^a = \begin{pmatrix} 0.0506 \text{A/s} \\ -0.1041 \text{V/s} \end{pmatrix} \]

\[ \text{off} = \begin{pmatrix} -0.0300 \text{A} \\ 0.0632 \text{V/s} \end{pmatrix} \]

This translates into an angle \( \gamma \) of \( \approx 25.5^\circ \). The three contingencies are visualized in Figure 6.8.

Figure 6.8 Reflective, refractive and rejective zones around \( x^a \)
As soon as the trajectories angle passes over the bound of $25.5^\circ$, the geometry is such that the on-off control of the converter’s active switch is in fact such that the state is driven way from $x^a$ in both topologies.

With this example at hand, it is possible to evaluate the basic stability criterion introduced by Bass in a final manner.

**Outcome 6.3** The stability condition formulated in Theorem 5.1 is a necessary, but not a sufficient one.

This result is decisive for the design of a stationary boundary control. It makes clear that the question of how to lay the boundary is not only a matter of converter performance, but, more important yet, a question of stability. Laying out a boundary control mechanism requires careful reflection.

- Reflective→Rejective Operation

This is the last mode of operation that was observed. It is exhibited in Figure 6.9.

![Two-boundary operation, reflective→rejective case](image)

**Figure 6.9** Two-boundary operation, reflective→rejective case

As predicted this mode has only been observed with the buck converter. The example given here is a buck converter with pure current feedback, again designed to furnish 5V output from 10V input. The loadline is reflective for $v \leq v^i_1$, but it is refractive downwards...
for higher capacitor voltages. In the animation on the right the on-trajectory actually
decreases the inductor current and brings it to zero. The state then slides on the v-axis for a
short while (from $\approx 10.2\text{V}$ to $\approx 9.8\text{V}$) and then leaves it tangential to it. Upon hitting the
boundary it enters into a sliding regime towards the operating point. This case is different
from the other two-boundary cases as $\delta$ is left not at the intersection $\sigma-\delta$ but at a point
defined by the requirement of trajectory tangentiality as described in Chapter 5. The stability
theorem put forth for the two-boundary case also handles this case: The diode boundary is
hit only once, therefore the state motion is stable. Theory and Simulation are in accordance.

6.4.3 Special Cases of Interest

During the simulation two phenomena were encountered that have an impact on the
converter performance. As they might create implementation problems they are presented
below.

- **Switching Overshoot**

Switching overshoot was introduced theoretically in Sections 5.2.3.1 and 5.3.2.2. Wrong
trajectories are followed when the switch is not toggled exactly at the boundary but a little
bit after it. It was argued that such a delay can be introduced consciously by adding some
frequency limiting device in order to obtain a stable limit cycle. However, switching
overshoot can occur unplanned. The plot in Figure 6.10 a) shows again the stable refractive
buck converter operation.

The integration step was chosen to be 5 $\mu\text{s}$, so when crossing the boundary there is a 5 $\mu\text{s}$
uncertainty in ‘when’ -- in then state plane ‘where’ -- the switch action occurs. This
switching uncertainty produces enough overshoot to hinder the cycle from becoming smaller
than a ripple of $\pm 0.1\text{V}$.
Outcome 6.4  Feedback delays, nonzero switching times of real devices and noise impose a lower limit on the size of the limit cycle that can be reached.

![Enhanced Buck Converter Load Line and Boundary](image)

Figure 6.10  Switching overshoot and output overshoot

In this case, an uncertainty of 5 µs produces approximately 2% of ripple. This value sounds acceptable. However, constraints will have to be expected when working with actual circuits.

- Output Overshoot

Figure 6.10 b) exhibits the startup trajectory of a boost converter, designed to deliver 25V output from 10V input, operating in refractive mode. When the voltage first rises, it reaches a value of ≈ 44V which represents an output overshoot of 76% of the desired voltage. The second and third rise still attain 36V and 33V respectively.

Outcome 6.5  In transient mode stationary straight line boundary control can lead to considerable output overshoot.

This outcome gives a strong indication for both the loads and the devices that must be used. The loads must be insensitive to temporary overvoltage and the electronic converter devices must be able to block substantial reverse voltage. Of course, output overshoot is also a
6.4.4. Assessment of the Simulations

The Visualization and Animation has brought substantial information with regard to both of its purposes.

First, the stability theorems worked out in Chapter 5 have been sustained. Although only a small number of individual simulation results could be included here, a large number of them has been obtained during the process of research. All of the animation results could be assigned to one of the categories conceptualized before. In so far, the example plots given in Sections 6.4.1 and 6.4.2 are comprehensive. Of course, the fact that no cases have occurred that could not be handled with Theorems 5.4 to 5.7 does neither prove their correctness nor their completeness in a strict way. Yet it is a strong indicator for their validity.

Second, limitations of the stationary boundary approach have been detected that were not obvious before. For example, the outcome that there are boundaries that place the operating point in an unreachable region will urge for careful control design. More specifically, results such as the occurrence of large overshoots in \( V_{\text{out}} \) permit to determine necessary device ratings for the actual circuits. In addition, an interesting oscillating mode of operation has been detected. A series of lab experiments will definitely want to provoke one of the large limit cycles encountered in the two-boundary case in order to implement an oscillator.

However, an important field largely uncovered so far are performance considerations. These will follow in the next Chapter.
CHAPTER 7
Performance Issues

In power electronics one usually discerns two basic types of performance: steady state performance and transient performance. The first one studies a system’s behavior once it has already reached a small neighborhood of its operating point. The second is concerned with major, permanent changes in the operating conditions such as an input or a load step. The converter’s startup also falls into this category.

7.1 Steady State Performance

By steady state performance we mean the system’s behavior once it operates in the vicinity of the operating point. No large transients do occur, but the system adjusts itself to small disturbances. The following discussion has to be seen under that restriction. All deviations are small, although in some plots the geometry has been exaggerated in order to demonstrate the problem appropriately.

7.1.1 Small Signal Stability

Small signal stability designates the ability of a system to return to a vicinity of the operating point with a desired closeness, given a “small” disturbance. We have already discussed large signal stability intensively. The geometrical state plane representation of the Boundary Control approach served as a powerful tool. It permitted conclusions that could not be drawn using conventional analytical methods.

Of course, a converter may be small signal stable although it is not stable in the large. Cases of small signal stability have seen extensive discussion and led to techniques such as averaging and linearization in power electronics. They will not be repeated here.
As large signal stability was defined as the system's ability to reach a neighborhood of \( x^* \) from any physically valid initial condition (Definition 5.1), it also includes those points as initial conditions that are considered to represent "small" disturbances. Furthermore it was already shown that in the \( f_\infty \)-limit the limit cycle degenerates to a point which just is the operating point. Thus large signal stability is a sufficient condition for small signal stability.

**Theorem 7.1** All converters that fulfill Theorems 5.4, 5.5 or 5.7 are small-signal stable.

### 7.1.2 Recovery Speed

As a consequence of small signal stability a state that is slightly set off its stable operation cycle will revert to that cycle. Yet, nothing is said about the time it takes the converter-control combination to get back. A first small signal performance indicator will therefore be the *recovery speed*, or *recovery time*. As has been emphasized, any real world boundary will lead to a reflective region around the operating point. Thus a sliding speed or time can be used as the desired performance indicator. Such a recovery time index will be introduced formally in Section 7.2.2. Anticipating the result presented there we can state:

**Design Rule 7.1** In order to realize a fast recovery speed choose the boundary slope so that a small recovery time index will be achieved.

### 7.1.3 Output Ripple

A very important performance criterion for a dc-to-dc converters naturally is the degree to which it is able to furnish true dc output. Due to the switch action, some output voltage ripple will always be present. As any ripple is an unwanted deviation from smooth dc output, control design desires to minimize its value. Conventional methods comprise increasing the output capacitance and increasing the switching frequency. This Section interprets output ripple
in terms of the state plane and evaluates approaches in order to achieve the desired ripple reduction.

7.1.3.1 Altering the Switching Boundary

An apparent approach in Boundary Control is to address a problem in terms of the geometry of the switching boundary. This is finally what Boundary Control is mainly about. In general there are two parameters that can be adjusted using a straight line switching boundary: its slope and its location. Once the desired output voltage is specified, however, the slope is the only parameter that can be altered since the boundary has to pass through the operating point.

At the end of Section 5.3.2 it was shown that in the neighborhood of $x^a$ the slope of $\sigma$ has one major impact: it determines whether the limit cycle will be of the refractive (switch actions occur on the boundary) or the reflective type (switch actions off the boundary). In the first case, the slope of $\sigma$ is fixed because it has to equal the slope of the trajectories in the operating point. In the second case, any slope other than that will provoke reflective operation. As the switch actions are now not initiated by the boundary any more, but rather by some switching frequency limiting device, the boundary does not have an impact on the size of the limit cycle. Changing the boundary does not alter the switch behavior. In summary, the slope of the boundary can not be used as a means to diminish the steady state ripple because it is primatily the switching frequency limiting device that governs the small signal behavior, not the boundary.

7.1.3.2 Increasing the Switching Frequency

A stationary boundary creates an autonomous system. The switching frequency is therefore not set externally, at least in principle. Nevertheless, in an actual implementation there will be a means to influence $f_s$. In Chapter 5 it has been demonstrated that a stable limit cycle
with nonzero geometrical dimension in the vicinity of \( x^a \) is realized by some sort of switching frequency limitation. It is \( f_{ \text{max} } \) that determines the minimum size of the limit cycle, for unless \( f_{ \text{max} } \) is finite, the limit cycle will diminish to a point.

Figure 7.1 shows the effect of increasing \( f_{ \text{max} } \). In the refractive case (a), the points of discontinuity move closer together on the switching boundary. In the reflective one (b), the limit cycle becomes smaller due to less "sliding oscillation" around the boundary.

![Figure 7.1 Effects of increasing the maximum switching frequency](image)

We see that, with a stationary boundary, increasing the switching frequency in fact means increasing the maximum switching frequency. Yet, this increase performs the desired ripple reduction.

Linked to this method are the usual drawbacks of high switching frequencies, the most important of which certainly is the loss in converter efficiency due to increased switching losses.

### 7.1.3.3 Increasing the Output Capacitance

Another customary way to reduce output ripple is to mount a larger output capacitor into the converter. It then stores more energy which can be used to deliver power to the load. Usually the effects of this change are evaluated analytically:

\[
C = \frac{Q}{U} \approx \frac{\Delta Q}{\Delta U} \Rightarrow \Delta U = \frac{1}{C} \cdot \Delta Q
\] (7.1)
Assuming the discharge to be equal, the higher the capacitance the smaller the voltage change
will be. Boundary Control again takes a geometrical approach to this phenomenon. The
capacitance \( C \) appears as a parameter in the converter’s network equation. Therefore a change
in \( C \) will modify the equation and consequently the trajectories in the state plane. Of course,
equation (7.1) can not be violated and an increased capacitance has to lead to a smaller ripple.

Yet this quick statement does not give a deep insight into what actually happens.

Figure 7.2 shows the meanwhile well-known plots of families of trajectories. The buck and the
boost converter have been chosen as examples. On the left side the plots from Chapter 4 are
repeated as a basis for comparison. Limit cycles have been marked by picking two intersecting
trajectories. On the right, similar plots for the same converters are shown. Two changes have
been made: \( C \) has been increased by a factor of five and the value of the iteration step by a factor
of two. Limit cycles are marked again at approximately the same output voltages.

It is easy to analyze the major changes. Firstly, the trajectories are oriented much more
vertically now. Thus intersecting trajectories produce cycles with far less horizontal extension at
the same average output voltage and the same current ripple. In fact, limit cycles with the same
switching frequency (twice the number of points in the left plots) show much smaller voltage
ripple now. Increasing the output capacitance is in fact a feasible way to reduce ripple in a
stationary boundary controlled converter. We note that this result coincides with the analytical
one from equation (7.1). Second, the movement of the state has become slower. Sets of points
belonging to the corresponding trajectory on the two plots have approximately the same
distance. Due to the change in the integration step, however, this equal span indicates a state
speed reduction by factor two. Thus we immediately see that the drawback of this method is
slowed transient behavior of the converter. Of course, an analytical argumentation using the
abstraction of time constants would have led to similar conclusions, but the state plane
representation definitely is easier to grasp and gives direct insight.
7.1.4 Line Regulation

The term line regulation designates the degree to which a system maintains a constant output value as the input (slowly) changes\textsuperscript{11}. In the case of dc-to-dc converters this input consists of the dc input voltage. This section analyzes qualitatively the effects of a small input voltage change on the converters' output. Remaining consistent with the concepts of Boundary Control, this is done in geometrical terms.

\textsuperscript{11} For details on line and load regulation see [47].
Let us first study the buck converter. Here, the loadline is a straight line that connects the two equilibrium points of the two families of trajectories. From Sections 3.3.1 and 3.3.2 we see that in $x^1$, the voltage $v^1$ is proportional $E$, the input voltage. The value for $i^1$, in turns, is proportional $v^1$. Thus we expect the loadline to stretch with $E$, but not to change its slope. This expectation is confirmed by the plots reproduced in Figure 7.3.

Due to the fact that the loadline's slope does not change, its point of intersection

![Enhanced Buck Converter — Load Line](image1)

![Enhanced Buck Converter — Load Line](image2)

Figure 7.3 The buck loadline under input voltage change

with the switching boundary remains the same (Figure 7.4 a)). Consequently the operating point and with it the output voltage is not being dislocated. This holds true for any slope of the switching boundary: A stationary switching boundary leads to perfect line regulation.

![Line regulation, schematic](image3)

Figure 7.4 Line regulation, schematic
Unfortunately the situation is different for boost and the buckboost converter. Figure 7.5 displays these converters’ loadlines for different input voltages. Here an input voltage change will translate to an output voltage change with the same sign. A typical situation is depicted in Figure 7.4 b). It helps evaluating the two extreme cases.

Let first $\sigma$ be vertical. This boundary represents pure voltage feedback. When the loadline changes, its point of intersection with the switching boundary will change. Yet, as the line is only defined for one voltage value, namely $v^a$, the voltage coordinate of this point cannot change. Pure voltage feedback leads to perfect line regulation. On the other hand, the less vertical the boundary, the higher is the change in voltage. The second extreme is reached with a horizontal boundary (pure current feedback). Here, $i^a$ is kept constant, and $v^a$ sees the maximum alteration. Still it can be seen from the plots that even in this worst case the system does not have zero line regulation. The reason lies in the parabola-like shape of the loadline and the fact that its turningpoint moves upward with increasing input voltage. Although the voltage maximum approximately doubles when the input is doubled, any point below that turning point will see less than a voltage doubling. The exact value though is hard to determine. Any
boundary slope between the two extremes leads to an intermediate value of line regulation. The bigger the slope, the better is the line regulation.

As a summary, a conclusion for design can be formulated.

**Design Rule 7.2** For the buck converter, any stationary boundary provides perfect line regulation; for the boost and the buckboost type, good line regulation demands a steep boundary.

### 7.1.5 Load Regulation

Load regulation describes the degree to which a converter’s output voltage remains constant under a (small) load change.

Again we start out with the buck converter. The same considerations on the formula for $x_1$ that were done above now lead to the conclusion that its slope will change with $R$. In fact, the loadline’s slope is inverse proportional to the load $R$. The computer plots confirm this result (Figure 7.6).

![Figure 7.6 The buck loadline under load change](image)

The load regulation of the buck converter is, in contrast to its line regulation, not ideal. Figure 7.7 a) depicts the geometry.
Figure 7.7 Load regulation, schematic

The geometry in fact is easy. The exact value of load regulation could be readily evaluated in terms of the slope of $LL(x)$ and the slope of $\sigma$. Still, the number obtained by this effort would not mean too much. Instead, it is more important to look at the overall behavior again. For a vertical boundary, load regulation is perfect, meaning that $V_{out}$ remains fixed. A horizontal boundary, however, produces the worst regulation. The behavior corresponds to the one seen with the boost and buckboost converters' line regulation behavior.

Let us now study these two latter converters. Figure 7.8 shows their loadlines under a load change.

Figure 7.8 The boost and the buckboost loadline under load change
The situation is redrawn in Figure 7.7 b) and, in terms of output voltage change, looks very similar to the line regulation case studied in the previous section. Thus we confer to the discussion above.

**Design Rule 7.3** Concerning load regulation, all three converters behave in a uniform way. In order to obtain good load regulation, one has to design the boundary very steep.

### 7.2 Transient Performance

Transient behavior, or simply a *transient*, occurs whenever the converter faces major changes in its externally imposed operating conditions. Examples would be important load changes or alterations in the supply voltage. The latter includes the converter startup since it can be viewed as an input voltage step from zero to the rated supply voltage.

#### 7.2.1 Transient Stability

In a transient, the converter state leaves its previous steady operation point or cycle in order to reach a new region of the state plane. At the moment when the change occurs, the state is located in the old operating region at some very definite point. It is perfectly legitimate to consider this point as initial condition and to neglect the state's prior movement since the converter power stage is a system without memory [33]. Now the large signal stability theory developed in Chapter 5 applies directly.

**Theorem 7.2** All converters that fulfill Theorems 5.4, 5.5 or 5.7 are transient stable.
7.2.2 Transient Speed

The transient time is the time a system needs to reach a certain neighborhood of the (new) operating point from a particular initial condition. Of course, the faster the system converges, the better it is. For stationary Boundary Control the transient speed along the boundary is presented as a promising tool.

Definition 7.1 (transient speed) The time derivative $ds/dt$ is defined as

$$v_\sigma = \frac{ds}{dt}$$

(7.2)

It provides a measure for how fast successors approach or leave the operating point. If the parameter $s$ on the boundary $\sigma$ is chosen linear, the value of $v_\sigma$ has direct meaning. Such a choice is assumed in the following. Two statements can be made about the transient speed.

Rule 7.1 If $x^a$ is stable, then $v_\sigma$ is negative definite.

Motivation The parameter $s$ is an indicator of the distance of the successor from the operating point. This Euclidean distance was shown to be a Liapunov-Function $V(x)$ in Chapter 5. Also the equivalence between Liapunov’s stability theory and the Theory of Successor Functions was shown. Liapunov’s theorem on stability requires negative definiteness for $V(x)$ for a system to be stable. This requirement translates into a negative definite transient speed.

Rule 7.2 In $x^a$ the transient speed is zero.

Motivation For a point or a cycle to be stable it is necessary that $s_m = s_n = s^*$, with some $m > n$. This is equivalent to $s = $ constant, which in turn yields $ds/dt = 0$.

Both rules hold true for refractive as well as for reflective stability. Figure 6.5 can be reviewed as a nice visualization. The application of Definition 7.1 is of considerably different difficulty for the two modes of operation. Definition 7.1 seems complicated to evaluate in the refractive case. However, it was shown that in general a sliding mode approaches the operating
point much faster that a refractive convergence (see Outcome 6.2). This motivates the following guideline:

**Design Rule 7.4** In order to achieve fast transient speed aim for a boundary that provides a large reflective zone around the operating point.\(^{12}\)

Having established this heuristic result, it is now relatively easy to apply Definition 7.1 to the reflective case.

**Rule 7.3** In the reflective case \(v_\sigma\) reduces to \(t\), with \(t\) being the \(t\)-component of \(\dot{x}\)
as introduced in Section 6.2.1.

**Motivation** The coordinate \(t\) was defined as the linear state space coordinate for \(\dot{x}\), parallel to \(\sigma\). Thus \(t\) indicates the state speed in direction of \(\sigma\). This is exactly what the time derivation of the parameter \(s\) on \(\sigma\) does. All that has to be considered is an eventual length coefficient in case that one unit of \(s\) does not have the same value as the unit vector of the \(v, i\)- and thus the \(n, t\)-system.

In the process of loadline evaluation (Chapter 6) \(t\) was already used. At that time solely its sign was needed. Now its value can serve as a valuable performance indicator. Since a single speed is not very meaningful, we suggest to calculate an actual reference time based on (7.2) as follows.

\[
\begin{align*}
\dot{v}_\sigma &= v_\sigma(s) = \frac{ds}{dt} \\
\dot{t} &= \frac{1}{v_\sigma} ds \\
t &= \int \frac{1}{v_\sigma} ds \\
T_{\text{trans.ref}} &= \int_{s_0}^{s_{\text{prox}}} \frac{1}{v_\sigma(s)} ds
\end{align*}
\]

In (7.3) \(s_0\) describes some starting distance that is reasonable in the particular case and \(s_{\text{prox}}\) defines the neighborhood of \(x^a\) that shall be reached. Various boundary slopes can then be compared.\(^{13}\)

---

\(^{12}\) On reflective control of dc-dc-converters ("sliding mode control"), a substantial number of publications exists. As a first reference see for example [34], [35].
Design Rule 7.5 For fast transient behavior choose a boundary slope that yields a low transition time reference.

7.2.3 Output Overshoot

As has been seen in Figure 6.10, large voltage overshoots can occur during transient operation. From the plots of the trajectory families we see that it is only the off-trajectories that can cause such overvoltages. In all three converter types they are spirals. In the following, two ways to address the problem will be presented.

7.2.3.1 Altering the Switching Boundary

With one special boundary it is possible to ensure no overshoot at all: If the boundary can be chosen in a way that it is totally reflective, then the state will slide on it from the old operating point to the new one. Thus such a boundary would be the first choice. Yet, it is not guaranteed that such a boundary always exists.

If the boundary shows refractive regions, output overshoot can still be reduced by laying the boundary appropriately. Spirals that bring the output voltage up high are those that have their voltage maxima far right in the state plane. For the reason that trajectories cannot intersect, spirals that have higher voltage maxima necessarily have higher current maxima and vice versa. Thus to avoid high overvoltages, boundaries have to be designed in such a way that they prohibit very high inductor current. This requirement means that, for a given $x^a$, a boundary must not be very steep.

Design Rule 7.6 If voltage overshoots shall be minimized, the boundary has to be designed such that a sliding mode occurs. In refractive operation, a rather flat boundary is promising.

---

13 For a small signal transient, the definition of transient time can be used for the recovery time in an analogous manner.
7.2.3.2 Increasing the Output Capacitance

Another valid option is to modify not the boundary, but the trajectories. The voltage maxima will be lower for the same boundary if the spirals are less extended horizontally. The plots created in the context of the steady state output ripple (Figure 7.2) have shown that this can be achieved by increasing the output capacitor value. Thus, for a given boundary, output overshoot can be reduced by adding additional capacitance.
CHAPTER 8
Laboratory Experiments

At this point, substantial theoretical insight has been gained with regard to autonomous straight line boundary control. Computer simulation has helped to find and sustain the results. Yet, no physical theory is valid without being confirmed by real world experiments. Therefore a set of laboratory experiments has been carried out. Its purposes are to verify the stability theory put forth in Chapter 5 and to test the accuracy of the state animations obtained in Chapter 6.

8.1 Experimental Implementation

This paragraph describes the actual circuitry that has been built in order to implement a boundary controlled converter. Using the conceptual converter model presented in Figure 2.1, it first presents the power stage and after that the boundary control stage.

8.1.1 The Power Stage

The boost converter type was chosen for the realization, sized at approximately 50W power level which is convenient for laboratory testing. Several aspects have to be taken into account when designing a boundary controlled power stage.

- The Switches

In choosing the switches two issues are important. First, the switching devices must act sufficiently fast. Delay in the switch action, this is a time delay between turn-on /-off signal at the control gate of the device and its power path being actually on /off (delay time plus rise time), will lead to switching overshoot as pointed out in Section 5.2.3.1. The corresponding animation result (Section 6.4.3) indicates that delays around 5 \( \mu \)s can already lead to significant performance reductions. Since there is additional time delay in the control
circuitry, the switches' delays should be well below that value. Modern devices do in general fulfill this requirement. Yet, fast devices should be chosen. Second, the animations have shown that in refractive operation large voltage and current overshoots can occur during transients. For this reason the devices must have ratings considerably above the voltage and current value of the operating point.

The power switches used in the experiments are the following:

Table 8.1 Switch device data

<table>
<thead>
<tr>
<th>Switch</th>
<th>Device</th>
<th>t_{on}, rated</th>
<th>t_{on}, meas.</th>
<th>t_{off}, rated</th>
<th>t_{off}, meas.</th>
<th>V_{max}</th>
<th>I_{max}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transistor</td>
<td>MTH 15N35</td>
<td>180 ns</td>
<td>63 ns</td>
<td>180 ns</td>
<td>123 ns</td>
<td>350 V</td>
<td>15 (60) A</td>
</tr>
<tr>
<td>Diode</td>
<td>MUR 460</td>
<td>≤ 75 ns</td>
<td>160 ns</td>
<td>≤ 75 ns</td>
<td>58 ns</td>
<td>400 V</td>
<td>6 A</td>
</tr>
</tbody>
</table>

- The Input Source

Due to the transient overshoot, the input voltage source may see heavy current peaks during converter startup and other transient situations. These peaks should be buffered by the means of a large capacitance at the power stage's input.

Accordingly, a 2 μF input capacitor was inserted into the circuit.

- The Snubber

Switching losses can represent a significant part of the power dissipated in a switching device. This power heats up the transistor and is proportional to f. In the extreme, thermal destruction of the switch will occur. In Boundary Control, stable operation will lead to a chattering problem (f → ∞) unless some form of switching frequency limitation is added. Although this will be done, it is not guaranteed in an experimental laboratory situation that it will perform properly. Therefore a snubber is advisable in order to reduce the risk of immediate destruction of the MOSFET in case of a circuit malfunction.

It was implemented using the following elements: Diode = MUR 160, C = 1.0 μF, R = 48 Ω.
The Sense Resistor

Since the inductor current is one of the states, it has to be measured. A sense resistor transforms current into a proportional voltage signal and is a customary means.

Thus a small resistor of 11.6 mΩ was included in the inductor current path.

A comprehensive circuit diagram is provided in Figure 8.1.

![Figure 8.1 The implemented boost power stage](image)

8.1.2 The Control Stage

The task of the control stage is to determine the switching function \( q \), that is, turn the transistor switch on and off. In Boundary Control this means that it has to measure the state variables, generate a switching boundary, compare the first to the latter, and toggle the MOSFET accordingly. In addition, it has to impose an upper limit on the switching frequency.

The way in which these primary tasks were accomplished will be outlined in the following. Figure 8.1 displays a schematic circuit diagram of the complete control stage.
Figure 8.2  The implemented control stage
8.1.2.1 Sensing the State Values

- Capacitor Voltage

Strictly speaking, the voltage state variable is defined over a pure capacitance. Only in this way it can be guaranteed that its value does not change instantaneously. A real capacitor, however, always shows some equivalent series resistance (ESR). Since the ESR is internal to the physical capacitor, it is only possible to access the series of \(C\) and \(R_C\). The output voltage that is measured in the experiment therefore is not exactly the capacitor voltage used in the simulations. In addition, there is a second source of inaccuracy: due to the current sense resistor in the power stage, the output capacitor is not perfectly grounded any more. Thus the measured value is

\[ V_{\text{out}} = V_C + I_C \cdot R_C + I_L \cdot R_{\text{sense}}. \]

This voltage is reduced by a resistive divider and fed through a voltage follower. In this manner a scaleable Signal \(V_{\text{out,ampl}}\) is achieved.

- Inductor Current

In order to allow for unified signal processing, the current value has to be transformed into a voltage signal. As stated above, a current sense resistor is used for this task. Power considerations require that the sense resistor be very small. The consequence is that the primary current signal (voltage over \(R_{\text{sense}}\)) is very small (mV) compared to the output voltage signal (V). The necessary amplification device is special in two ways. Firstly, it is designed as a true difference amplifier. \(V_{\text{sense}}\) is not measured to the common ground. Such a measurement would include the wire that lead the inductor current, and the wire's stray resistance and inductance would significantly distort the current signal. Secondly, the amplifier was built in two stages. It showed that, with the amplification factor and the frequency needed, a one-stage operational amplifier would run into a gain-bandwidth problem. The signal \(V_{\text{sense,ampl}}\) would be severely distorted with regard to the input, \(V_{\text{sense}}\).
8.1.2.2 Obtaining the Switching Boundary

The switching boundary's vector equation (4.1) can be transformed to yield the following form:

\[ k_i \cdot (i - i^a) + k_v \cdot (v - v^a) = 0 \] \hspace{1cm} (8.1)

Equation (8.1) reveals that such a boundary can be implemented as an analog circuit consisting of two difference amplifiers with an adjustable gain \( k_v \) and \( k_i \) respectively, followed by a summing amplifier with a unity gain. The operating point \( x^a = (i^a, v^a) \) can be given by setting reference voltages. On the switching boundary, the left-hand side of (8.1) is equal to zero.

When the state is above the boundary, the left-hand side of (8.1) becomes negative; when it is below \( \sigma \), it becomes positive. Thus the circuit computes an error voltage as follows:

\[ k_i \cdot (v_{out,amp} - v_{out,ref}) + k_v \cdot (v_{sense,amp} - v_{sense,ref}) = v_{err} \] \hspace{1cm} (8.2)

It is possible to determine on which side of the boundary a state is on at any given moment in time by just looking at the sign of the error voltage. A simple comparator stage was implemented in order to obtain this digital signal. Since the subsequent switching frequency limitation is accomplished using 5V-TTL chips, the comparator signal is adjusted to the TTL-level using an and-gate.

Figure 8.3 shows this digitizing procedure for pure current feedback (horizontal boundary, \( k_v = 0 \)) in the steady state. Part a) of the figure displays a current signal over time, together with the constant reference value. Part b), upper plot, shows that whenever the state is above the boundary, a low signal is produced by the comparator and vice versa (the plot shows inverted signals). This figure also reveals a serious problem in circuit operation. Just after the transistor is toggled, significant switching spikes occur in the inductor current signal. They translate into erroneous short high and low impulses in the switching signal. Unless they are eliminated during the following signal-processing, erratic switch action will occur and the control will not work at
all. Figure 8.3.b), lower plot, shows an approximation of the correct switching signal that has to be obtained by suppressing the disturbing pulses.

![Graph](image.png)

Figure 8.3  Sensed signal and derived comparator output
8.1.2.3 Limiting the Switching Frequency

Three alternatives are obvious when a maximum switching frequency is to be imposed on the converter:

a) minimum on-time, any off-time
b) minimum off-time, any on-time
c) minimum on-time and minimum off-time.

From the previous Figure it can be seen that the problematic switching spikes in the current signal last for approximately 5 μs after the switch action has occurred. If one chooses alternative c) and sets both the minimum off-and on-time to a value larger that 5 μs, the effects of the switching spikes are automatically suppressed. For this reason, possibility c) was implemented. The minimum on- and off-time were set to 18 μs each, resulting in \( f_{\text{max}} = 27.8 \) kHz.

Pulses of 18 μs length are generated by a dual monostable (74LS221). The minimum on-time pulse is triggered by the rising slope of the comparator-output or, if the comparator-indicated off-time is shorter than 18 ms, by the rising slope of the minimum off-time pulse. The minimum off-time pulse is triggered by the falling slope of the comparator output or, if the comparator-indicated on-time is shorter than 18 ms, by the falling slope of the minimum on-time pulse. This sophisticated triggering is made possible by using the built-in input flip-flops of the 74LS221: the output of each of the two monostables is inverted and fed back to the second input of the other monostable’s flip-flop.

In a second step, the comparator’s signal and the two monostables’ signals have to be overlaid in such a manner that the correct switching function is obtained. Figure 8.4 derives the truth table that leads to a suitable logic circuit. This circuit can be also be seen in Figure 8.2.
Figure 8.4 Desired digital signal processing to yield the switching function \( q \)

Figure 8.5 finally displays an example of the operating logic circuitry where the minimum on-time is in effect (the pulse trains are slightly mutually delayed due to signal capturing and storage in the oscilloscope).
8.1.2.4 Driving the MOSFET

Once the correct switching function $q$ is obtained it has to be applied to the transistor's gate. Since a MOSFET needs around 10 V in order to turn fully on, the TTL-signal has to be amplified. The easiest way to do this is to use an integrated MOSFET driver chip. In the realized circuit the standard DS0026 type is used. It is inverting, which means that it has to be provided not with $q$ but with $\overline{q}$. Thus the signal obtained from the switching frequency limiting circuitry is inverted before it reaches the driver. The driver stage is shown together with the complete hybrid boundary control stage in Figure 8.2.
8.2 Experimental Results

If the animation results can be seen as the confirmation of the theoretical framework on stability and performance, then the next step is to test the validity of these simulations. It is the purpose of this Section to compare actual experimental data with the predictions furnished by both the theory and the animations. Various boost converters, differing in their inductance values, were studied in depth. The common converter parameters are displayed in Table 8.2.

<table>
<thead>
<tr>
<th>Converter</th>
<th>$R_{ds} (\Omega)$</th>
<th>$V_{drop} (V)$</th>
<th>$C (\mu F)$</th>
<th>$R_c (m\Omega)$</th>
<th>$R (\Omega)$</th>
<th>$E (V)$</th>
<th>$V_{out} (V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boost</td>
<td>0.194</td>
<td>1*</td>
<td>2.007</td>
<td>37.5</td>
<td>151.3</td>
<td>10</td>
<td>20</td>
</tr>
</tbody>
</table>

* estimated

Inductors had to be changed whenever saturation effects deteriorated the converter operation. Results will be reported here on operation with three of the inductors. The resulting power stages were numbered to allow for easy referencing.

<table>
<thead>
<tr>
<th>Converter 1</th>
<th>Converter 2</th>
<th>Converter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L (m\text{H})$</td>
<td>$R_L (m\Omega)$</td>
<td>$L (m\text{H})$</td>
</tr>
<tr>
<td>1.897</td>
<td>176</td>
<td>1.883</td>
</tr>
</tbody>
</table>

From these values the basic boundary stability requirements, formulated in Theorem 5.1, can be evaluated. The converters' equilibrium points can easily be calculated using Table C.1 and plugged into the formula provided in Figure 5.1. Table 8.4 contains the outcome of this process.

Experimental results on three boundaries within these feedback ranges will be reported in the following. For every boundary control law presented below a transient behavior as well as the converter's steady state will be shown. The transient behavior will always be represented by the converter.
startup. The reason is that a startup situation is easy to reproduce since the initial value (0,0) can be fixed with virtually perfect accuracy. In addition to the state plane representation, the startup situation will also be given in the familiar depiction over time. The steady state will be characterized by the limit cycle and can be seen over time when the startup plot when the steady state operation is reached.

8.2.1. One-Boundary Operation, Stable with Transition

Refractive $\rightarrow$ Reflective

The first example is taken from converter 1, operating under pure current feedback. The voltage gain $k_v$ is zero and thus $\sigma$ is horizontal. The switching boundary shows an extended reflective region around the operating point (Figure 8.6). Consequently it can be expected that $x^a$ will be reached via a sliding mode. The corresponding animation plot in the right part of the Figure confirms this expectation.
Figure 8.6 Evaluated boundary and simulated startup for converter 1, $k_o/k_i = 0$

The experimental results can be seen in Figures 8.7 and 8.8 on the next two pages (x-axis: $1V \leftrightarrow 1V$; y-axis: $1V \leftrightarrow 0.4 A$). The startup in the state plane shows three main differences from the simulation:

- A large current spike is indicated right at the start. Such a spike will be seen in all the plots. It is believed that it can be attributed to the measurement and is in fact not ‘real’. The inductance value forbids such a steep rise in inductor current. In this and in all further plots this spike will be disregarded.

- The animation predicts that the boundary is hit for the second time at $v = 17 V$. In reality this intersection lies close to $14 V$.

- The sliding mode in the lab converter does not occur on the boundary, but around it. This behavior is what is expected taking into account the switching frequency limitation. In fact the oscillation around $\sigma$ towards $x^*$ can be seen nicely.

Part b) of Figure 8.7 exhibits the steady state limit cycle. It reveals an output voltage ripple of approximately $1.5 V$ and the typical spikes in the current signal whenever switch action occurs.
Figure 8.7 Measured startup and steady state operation for converter 1, $k_i/k_o = 0$, state plane
Figure 8.8 gives the somewhat more familiar startup plot of current and voltage over time.

![Figure 8.8 Measured startup for converter 1, k/jk = 0, state variables over time](image)

A couple of features are notable:

- The assumed 'real' current signal has been drawn in by hand.
- It is clearly recognizable how first the current rises and then the voltage. A little below 15 V, the transition to reflective operation begins and the voltage rises more slowly than before.
- The complete startup time lies around 6 ms.
- The steady state switching period is 0.2 ms / 5 = 40 μs. This comes close to the predicted minimum switching period of 2·18 μs = 36 μs.

Overall, the computer animation and the experimental data do match closely. The fact that the boundary is reached at a slightly lower voltage value that may be explained by inaccuracy in the circuit modeling.
8.2.2. Two-Boundary Operation, Stable, Refractive with regard to σ

The second boundary control involves both states in the feedback. Converter 2 is used, and the feedback gains are set to $k_v/k_i = 8/100$. This boundary slope exactly equals the slope of the converter's trajectories in $x_1$ ($7.99/100$). Thus a purely refractive operation is expected.

Figure 8.9 indeed indicates that the reflective region around the operating point has completely disappeared. The animation result for the startup suggests that the converter will enter into discontinuous operation once and will then approach its steady state in a refractive way.

![Evaluated boundary and simulated startup for converter 2, $k_v/k_i = 8.0/100$](image)

The real startup in the state plane looks very similar (Figure 8.10):

- The current reaches 1.8 A before the first switch action occurs.
- Then a heavy voltage overshoot follows. The predicted maximum value of 66 V is not fully reached. The output voltage attains a little over 50 V.
- The limit cycle is reached in a spiraled way. Since the on-trajectories are very close to the switching boundary, noise and other disturbances worsen the steady state converter operation substantially. Multiple limit cycles (see Section 8.2.4 below) occur most frequently in this refractive mode of operation.
Figure 8.10 Measured startup and steady state operation for converter 2, \( k_j/k_i = 8.0/100 \)
8.2.3 Two-Boundary Operation, Unstable

For continued two-boundary operation, the slope of the boundary is increased to 20/100. Converter 3 is used to test for the continued two-boundary case. As can be seen in Figure 8.11, the inductor current is now allowed to reach 4 A initially.

In Figure 8.11 a) it can be seen that a large rejective region has developed. The operating point cannot be stable. The startup animation in part b) of the Figure predicts an ensuing large limit cycle.

![Figure 8.11 Evaluated boundary and simulated startup for converter 3, \( k_r/k_i = 20/100 \)]

The corresponding laboratory result can be compared in Figure 8.12 (x-axis: 1 V ↔ 1 V; y-axis: 1 V ↔ 1 A). The simulation and the experimental curves do match closely.
- The final limit-cycle is reached in one on-off cycle as the stability theory states.
- During startup the output voltage reaches its maximum at 87 V, whereas 97 V are simulated. The difference can be explained by some uncertainty in the exact location and slope of the boundary. Although a slope of 20/100 was set, the first switch action occurs at (3.8 A, 0 V), which is 0.4 A too low.

Thus the boundary was either flatter that 0.2 or it was located off to the left.
- The limit cycle matches well.
Figure 8.12 Measured startup and steady state operation for converter 3, $k/k_i = 20/100$
When the states are plotted over time, one recognizes the true nature of the output. The "dc-to-dc" converter produces a voltage signal that can be described much better as a sawtooth than as a dc-voltage. Since the transistor on-times are very short, the voltage rise is very steep. Then, the RC-output combination determines the voltage decay, which is of the typical exponential form. In some sense, the converter system definitely is stable: the signals are bounded, and they are periodic. Yet no stable behavior in the sense of a dc-to-dc-converter and Definition 5.1.

8.2.4 A Special Case of Interest: Multiple Limit Cycles

A phenomenon that occurred persistently during converter operation is the occurrence of multiple limit cycles. In theory, one unique limit cycle should arise in stable converter operation at a frequency set by the $f_r$-limiting device. In practice, this is not the case. In addition to the desired limit cycle, one or occasionally two other cycles can be distinguished. Figure 8.13 provides an example. The thick, small cycle is the desired one. The bigger, thinner cycle represents the second kind of limit cycles that occurs. In this situation, the unwanted limit cycle has about three times the horizontal and vertical extension as the original one.

This phenomenon has already been noticed by Bass. He investigated its nature in [48]. In accordance with his findings, virtually no other cycles than the two or three that are characteristic for a particular converter under a particular boundary occur during its operation. Although the tested converters do change between the cycles almost at random, a statistical result can be established.

**Outcome 8.1** In boundary-controlled boost converters, multiple limit cycles occur. They are the more frequent the lower the boost ratio is.

From simple countings, given in Table 8.5, it can be inferred that multiple limit cycles can appear frequently.
Figure 8.13 Multiple limit cycles, converter 1, \( k_v/k_i = 5/100 \)

Table 8.5 Repetition of multiple limit cycles

<table>
<thead>
<tr>
<th>( V_{out} ) (V)</th>
<th>( k_v/k_i )</th>
<th>0/100</th>
<th>5/100</th>
<th>6/100</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1 out of 2</td>
<td>1/3</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1/15</td>
<td>1/5</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>1/22</td>
<td>1/25</td>
<td>1/5</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>1/40</td>
<td>1/40</td>
<td>1/9</td>
<td></td>
</tr>
</tbody>
</table>

The plot in Figure 8.13 makes clear that multiple limit cycles can introduce a ripple of easily three times as high as one would encounter without this phenomenon. Thus multiple limit cycles represent a serious degradation in steady state performance of Boundary Control. Unfortunately, no effective remedy has been reported yet.
8.2.5 Assessment of the Experiments

The series of experiments with different boost converters has largely sustained the theory of Boundary Control as described and established in this Thesis. With regard to stability, converters do behave as predicted. But not only the predicted qualitative behavior is achieved. Also, the experimental data and the animation results do match quantitatively to a high degree. Thus it can be concluded that state plane animation, which is relatively easy to carry out, is an appropriate tool for control design.

The experiments have also sustained the conjecture that the power stages change their behavior only gradually when the boundary’s slope or location is modified. In fact, by the means of four potentiometers \( k_i, k_v, V_{\text{sense,ref}}, V_{\text{out,ref}} \) the limit cycle and the transient behavior can be incrementally adapted.

Of course, this series is far from complete. Two types of converters included in the theory and simulations have not been tested. Their implementation might yield fundamentally different results, although this is not expected.

Also, a real application would have to determine the reference value for the inductor current automatically. Thus an additional control loop would be needed to generate this value. Since the primary boundary control phenomena could only have been worsened by introducing such auxiliary multiloop circuitry, this automation was not undertaken.

Overall, these first testings have indeed fulfilled their purpose to sustain the general validity of the previous work.
CHAPTER 9
Conclusions

9.1 Contributions of This Thesis

This Thesis has studied a family of nonlinear converter systems using a geometric approach that fully retains their discontinuous characteristics. State plane analysis techniques were applied in order to achieve novel qualitative insight into converter operation in general and Boundary Control in particular.

The first contribution is a comprehensive overview over the converters' trajectories in state space. They give a clear picture of the possibilities and restrictions any control mechanism for these power stages encounters. Second, the notion of one key element that can be derived from these trajectories, the loadline, was broadened. Two alternative interpretations of it could be found: a very conventional one, based on energy considerations, and a purely geometric one, based on reflections on the limit cycle.

In its main part, the Thesis is concerned entirely with Boundary Control. For one special form of boundaries, stationary straight-line boundaries, substantial contributions were made concerning their stability behavior. A basic stability requirement established previously was proven to be necessary, but not sufficient. A theoretical framework on converter stability has been worked out which is believed to be exhaustive for this kind of control. Four intelligible theorems indicate large signal stability or instability. Effects of nonideal converter behavior such as switching overshoot have been illustrated and a stable limit cycle has been explained as result of external switching frequency limiting. A major contribution in fact is the result that the Boundary Control studied does not itself allow stable limit cycles. If limit cycles occur, the system is unstable in the sense of a dc-to-dc converter. If it is stable, chattering is the definite consequence.
A practical contribution for circuit analysis and design has been made by writing a number of simulation and animation programs using a standard software package. The code can easily be adjusted to other control mechanisms and thus provides a good basis as an explicative as well as predictive tool.

9.2 Assessment of Stationary Straight-line Boundary Control

Although stability and not performance of Boundary Control has been at the center of the work presented here, a first assessment of the considered control method can be undertaken. The big advantage of this control law of course is that it can guarantee large-signal stability. A simple evaluation for one one-boundary and one two-boundary case of operation permits the conclusion of stability for any value of disturbance. The drawback on the other hand is that there are also boundaries that provoke large-signal unstable behavior.

Boundary Control is very robust with regard to parasitics. Parasitics do not change stability behavior significantly.

Furthermore, stationary boundaries can yield very fast transient behavior. Unlike with Pulse With Modulation, for example, voltage and current values can reach their final values in one on-off cycle if the boundary is placed appropriately. The side effect is that at the same time, large output overshoots can occur.

A major limitation, however, is the need for an additional switching frequency limiting device as well as for an outer control loop. The first is made necessary by the inevitability of chattering, and the second by the need to generate a reference value for the inductor current. Such additional circuitry can easily outweigh the original simplicity of the boundary control law.

Even with these devices, an important operational problem remains: the phenomenon of multiple limit cycles significantly increases the converter output ripple.

Mainly due to the last two facts, Boundary Control is still far from commercial use.
9.3 Directions for Future Work

The field of Boundary Control is very open. The scope of this Thesis has still been very restricted: to stationary boundaries, and within that domain to straight-line boundaries.

Already the basic underlying geometry, the converter trajectories in the state space, are in need of detailed investigation. It remains to study systematically how the families of trajectories change their shape under a change of the main parameters L, C and R. Especially nonresonant combinations, where in the state plane foci become nodes, seem interesting.

As far as the boundary law control studied here is concerned, several questions remain. For example, it is unclear whether a rejective region does always include the operating point and, if not, what will happen in that case. Also, very generally, performance and robustness considerations have to be deepened. Furthermore, boundaries with a positive slope should be examined closer.

Remaining within the field of self-oscillating dc-to-dc converters, stationary boundaries other than a straight line may be promising. In fact, looking at the norm of the state velocity on the state plane suggests that boundary shapes can be designed in order to optimize transient or steady state performance. Thus stationary, curved boundaries represent a related area of research.

And finally there is the vast possibilities that arise when moving boundaries are introduced. A third dimension, time, enters the model. Studying such systems will definitely be a very complex, but at the same time a very promising undertaking.
APPENDIX A

Simple and Enhanced Network Equations for all three Converter Types

The procedure outlined in Section 3.1 leads to the network equation of basically any switching power converter. Following the steps shown in Figure 3.2 and Figure 3.3 the equations for all the converter types and models used in this thesis can be obtained. The results are given below.

A.1 Buck Converter

\[
\frac{d}{dt}(i) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{C \cdot R} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} \begin{bmatrix} v \end{bmatrix} \begin{bmatrix} \frac{1}{L} \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \cdot q_1 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{C} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} \begin{bmatrix} q_2 \end{bmatrix}
\]

Figure A.1 Network equation of the simple buck converter

\[
\frac{d}{dt}(i) = \begin{bmatrix} 0 & -\frac{1}{C \cdot R} \\ 0 & \frac{1}{C \cdot (R + R_c)} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \end{bmatrix} \begin{bmatrix} \frac{R}{L(R + R_c)} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \cdot q_1 + \begin{bmatrix} \frac{1}{L} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \end{bmatrix} \begin{bmatrix} \frac{1}{L} \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} V_{\text{drop}} \end{bmatrix} \cdot q_2
\]

Figure A.2 Network equation of the enhanced buck converter
A.2 Boost Converter

\[ \frac{d}{dt}(i(v)) = \left[ \begin{array}{c} 0 \\ 0 \\ -\frac{1}{C \cdot R} \end{array} \right] \cdot (i(v)) + \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \cdot E \cdot q_1 + \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \cdot \left[ \begin{array}{c} \frac{1}{L} \\ 0 \\ 0 \end{array} \right] \cdot (i(v)) + \left[ \begin{array}{c} 0 \\ 0 \\ -\frac{1}{L} \end{array} \right] \cdot E \cdot q_2 \]

Figure A.3 Network equation of the simple boost converter

\[ \frac{d}{dt}(i(v)) = \left[ \begin{array}{c} 0 \\ 0 \\ -\frac{1}{C \cdot (R + R_c)} \end{array} \right] \cdot (i(v)) + \left[ \begin{array}{c} -\frac{R_c + R_d}{L} \\ 0 \\ 0 \end{array} \right] \cdot (i(v)) + \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \cdot \left( \begin{array}{c} E \\ V_{drop} \end{array} \right) \cdot q_1 \\
+ \left[ \begin{array}{c} -\frac{1}{L} \cdot \left( R \cdot R_c \right) \\ \frac{R}{L(R + R_c)} \\ \frac{R}{C(R + R_c)} \end{array} \right] \cdot (i(v)) + \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \cdot \left( \begin{array}{c} E \\ V_{drop} \end{array} \right) \cdot q_2 \]

Figure A.4 Network equation of the enhanced boost converter
A.3 Buckboost Converter

\[
\frac{d}{dt} (i) = \left[ \begin{array}{cc} 0 & 0 \\ 0 & \frac{1}{C \cdot R} \end{array} \right] \cdot \begin{bmatrix} i \\ v \end{bmatrix} + \left[ \begin{array}{c} \frac{1}{L} \\ 0 \end{array} \right] \cdot E \cdot q_1 + \left[ \begin{array}{c} 0 \\ \frac{1}{C} \end{array} \right] \cdot \begin{bmatrix} i \\ v \end{bmatrix} \cdot q_2
\]

Figure A.5 Network equation of the simple buckboost converter

\[
\frac{d}{dt} (i) = \left[ \begin{array}{cc} 0 & 0 \\ 0 & \frac{1}{C(R + R_c)} \end{array} \right] \cdot \begin{bmatrix} i \\ v \end{bmatrix} + \left[ \begin{array}{c} \frac{-R_i + R_{ds}}{L} \\ 0 \end{array} \right] \cdot \begin{bmatrix} i \\ v \end{bmatrix} + \left[ \begin{array}{c} \frac{1}{L} \\ 0 \end{array} \right] \cdot \begin{bmatrix} E \\ V_{drop} \end{bmatrix} \cdot q_1 \\
+ \left[ \begin{array}{c} \frac{-R}{L(R + R_c)} \\ \frac{R}{C(R + R_c)} \end{array} \right] \cdot \begin{bmatrix} i \\ v \end{bmatrix} + \left[ \begin{array}{c} 0 \\ \frac{-1}{L} \end{array} \right] \cdot \begin{bmatrix} E \\ V_{drop} \end{bmatrix} \cdot q_2
\]

Figure A.6 Network equation of the enhanced buckboost converter
APPENDIX B

The Computer Implementations

B.1 Forward Euler Integration

Both topologies that occur when a converter is operating in continuous mode individually yield a system of two differential equations of first order. One equation describes voltage, the other one current (see equations (2.6) in Chapter 2). The two equations are coupled since the current occurs in the voltage equation and vice versa. Thus they have to be solved simultaneously.

Usually a closed analytic solution for such a system does not exist. One is therefore restricted to iterative techniques. Various methods have been developed, all differing in complexity and accuracy. See [21] for a good overview. The technique used here is known as Forward Euler Integration or Euler-Cauchy Integration. The idea is simple:

\[
\frac{dx}{dt} = f(x) \quad \Rightarrow \quad \frac{\Delta x}{\Delta t} \approx f(x) \quad \Delta x = f(x) \cdot \Delta t
\]

\[
x_{n+1} = x_n + \Delta x_n
\]

\[
x_{n+1} = x_n + f(x_n) \cdot \Delta t , \ n \text{ being the step of iteration}
\]

\[
\Delta t \text{ the step size}
\]

\[
x_0 \text{ a specified initial value}
\]

The interval \( \Delta t \) is kept fix. This iteration process furnishes the solution as a polygonal approximation. A more detailed discussion of this method can be found in [48].
### B.2 The Plots of Families of Trajectories

As long as $\Delta t$ is chosen sufficiently small, successive points of the trajectory can be calculated with any desired closeness. Different initial values will lead to different trajectories although the same d.e. apply. All trajectories specified by the same d.e., differing only by virtue of their initial value, are said to belong to one *family*.

The task in establishing a comprehensive picture of the trajectories of any of the converters consists therefore of plotting several trajectories of each family. The following algorithm was used (simplified):

1. **Specify network equation**
2. **Specify converter parameter values**
3. **Select first topology: $q_1 = 1$, $q_2 = 0$**
4. **Select different initial values: $x = x_0$**
   - **m iterations**
   - $x_{n+1} = x_n + f(x_n) \cdot \Delta t$
   - **Plot $x_{n+1}$**
5. **Select second topology: $q_1 = 0$, $q_2 = 1$**
6. **Select different initial values: $x = x_0$**
   - **m iterations**
   - $x_{n+1} = x_n + f(x_n) \cdot \Delta t$
   - **Plot $x_{n+1}$**

---

**Figure B.1** Algorithm for obtaining the plots of a converter’s families of trajectories (simplified)
B.3 The Plots of Loadlines

In the computer calculation the “Method of the Low Frequency Describing Equation” is used. The switch states \( q_i \) are replaced by the duty ratios \( D_i \). Setting \( dx/dt \) equal zero yields a system of two linear equations. Thanks to its matrix orientation it is very easy to solve for \( x \) in Matlab\textsuperscript{TM}. The loadline is obtained in parametric form as \( x(D) \) just as was seen in Chapter 4. Spanning \( D \) over its range from zero to one gives the loadline.

In order to obtain both a continuous load line as well as marks for special points (\( D = 0.05, 0.1, 0.15, \ldots, 0.95, 1.00 \)) a repetition of the program body was added. Its only purpose is to mark these points when plotting the loadline.

The following Nassi-Shneiderman diagram depicts the algorithm (simplified).

<table>
<thead>
<tr>
<th>Specify network equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specify converter parameter values</td>
</tr>
<tr>
<td>Assign ( q_1 = D, q_2 = (1-D) )</td>
</tr>
<tr>
<td>Equate network equation with 0</td>
</tr>
<tr>
<td>Counter = 1 to 1000</td>
</tr>
<tr>
<td>Solve for ( x(D = \text{counter}/1000) )</td>
</tr>
<tr>
<td>Store ( x ) subsequently as ( x_{\text{counter}} )</td>
</tr>
<tr>
<td>Plot all ( x_{\text{counter}} ) as smooth loadline *</td>
</tr>
<tr>
<td>Counter = 1 to 20</td>
</tr>
<tr>
<td>Solve for ( x(D = \text{counter}/20) )</td>
</tr>
<tr>
<td>Store ( x ) subsequently as ( x_{\text{counter}} )</td>
</tr>
<tr>
<td>Plot all 20 ( x_{\text{counter}} ) as marks on loadline *</td>
</tr>
</tbody>
</table>

* This routine is specified for Matlab and need modification for conventional programming languages.

Figure B.2 Algorithm for obtaining the plot of a converter’s loadline (simplified)
B.4 The Plots of Limit Cycles under PWM

Pulse Width Modulation yields true limit cycles. A constant duty ratio D and a constant switching frequency \( f \) translate into a constant sequenced number of iteration on-steps and off-steps. A substantial total of switching periods (m) has to be computed before displaying the cycle in order to allow the system to reach its steady state. When the limit cycle is plotted, several (n) of them are actually drawn. This permits some control: Only if they coincide one can be assured that the converter operates very near its steady state.

This is the algorithm that was programmed (simplified):

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>specify network equation</td>
</tr>
<tr>
<td>2</td>
<td>specify converter parameter values</td>
</tr>
<tr>
<td>3</td>
<td>assign ( q_1 = D ), ( q_2 = (1-D) )</td>
</tr>
<tr>
<td>4</td>
<td>equate network equation with 0</td>
</tr>
<tr>
<td>5</td>
<td>solve for ( x = x^\alpha )</td>
</tr>
<tr>
<td>6</td>
<td>plot ( x^\alpha )</td>
</tr>
<tr>
<td>7</td>
<td>determine # of on-time steps and # of off-time steps</td>
</tr>
<tr>
<td>8</td>
<td>select initial value ( x = x_0 )</td>
</tr>
</tbody>
</table>

\[
\text{m iterations (no plotting)}
\]

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>on-steps iterations</td>
</tr>
<tr>
<td>10</td>
<td>( q_1=1, q_2=0 )</td>
</tr>
<tr>
<td>11</td>
<td>( x_{n+1} = x_n + f(x_n) \cdot \Delta t )</td>
</tr>
<tr>
<td>12</td>
<td>off-steps iterations</td>
</tr>
<tr>
<td>13</td>
<td>( q_1=0, q_2=1 )</td>
</tr>
<tr>
<td>14</td>
<td>( x_{n+1} = x_n + f(x_n) \cdot \Delta t )</td>
</tr>
</tbody>
</table>

To be continued
### Algorithm for obtaining the plots of a converter's families of trajectories (simplified)

<table>
<thead>
<tr>
<th>n iterations (with plotting)</th>
</tr>
</thead>
<tbody>
<tr>
<td>on-steps iterations</td>
</tr>
<tr>
<td>$q_1 = 1, q_2 = 0$</td>
</tr>
<tr>
<td>$\mathbf{x}_{n+1} = \mathbf{x}_n + f(\mathbf{x}_n) \cdot \Delta t$</td>
</tr>
<tr>
<td>plot $\mathbf{x}_{n+1}$</td>
</tr>
<tr>
<td>off-steps iterations</td>
</tr>
<tr>
<td>$q_1 = 0, q_2 = 1$</td>
</tr>
<tr>
<td>$\mathbf{x}_{n+1} = \mathbf{x}_n + f(\mathbf{x}_n) \cdot \Delta t$</td>
</tr>
<tr>
<td>plot $\mathbf{x}_{n+1}$</td>
</tr>
</tbody>
</table>

Figure B.3 Algorithm for obtaining the plots of a converter's families of trajectories (simplified)
B.5 The Evaluation of the Switching Boundary

This is the Nassi-Shneiderman diagram (simplified) for the programs that classify the points on $\sigma$ and visualize the result. A verbal description of the algorithm is given in Section 6.2.2.

| Specify network equation                      |                                  |
| Specify converter parameter values           |                                  |
| Equate network equation with 0               |                                  |
| Counter $= 1$ to $1000$                      |                                  |
| Solve for $^{m}x(D = \text{counter}/1000)$  |                                  |
| Store $^{m}x$ subsequently as $^{m}x_{\text{counter}}$ |                          |
| Plot all $^{m}x_{\text{counter}}$ as smooth loadline * |                      |
| Specify switching boundary equation          |                                  |
| Specify switching boundary parameters        |                                  |
| Calculate $\gamma$                          |                                  |
| Counter $= 1$ to $1000$                      |                                  |
| Solve for $^{bd}x(v=(\text{counter}/1000) \cdot v_{\text{max}})$ |                  |
| Store $^{bd}x$ subsequently as $^{bd}x_{\text{counter}}$ |                |
| Counter $= 1$ to $1000$                      |                                  |
| Calculate $^{a}x$ and $^{b}x$ at $^{bd}x_{\text{counter}}$ |                        |
| Map $^{a}x$ and $^{b}x$ into the $\vec{I}, \vec{I}$-system |                    |
| EVALUATION-AND- PLOT*                        |                                  |
| Add explications to graph                    |                                  |
| (Calculate trajectory slope at $x^{\sigma}$)** |                        |

* This part of the diagram had to be portrayed separately for layout reasons: see figure B.5.

** This routine does not serve immediate evaluation purposes.

Figure B.4  Algorithm for evaluating a converter's boundary (simplified)
<table>
<thead>
<tr>
<th>Condition</th>
<th>Calculate $\delta$</th>
<th>Calculate $t_{res}$</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_t &gt; 0 \land b_t &gt; 0$</td>
<td></td>
<td></td>
<td>Plot $b_d x_{counter}$ with symbol for 'refractive up'</td>
</tr>
<tr>
<td>$a_t &lt; 0 \land b_t &lt; 0$</td>
<td></td>
<td></td>
<td>Plot $b_d x_{counter}$ with symbol for 'reflective right'</td>
</tr>
<tr>
<td>$a_t &lt; 0 \land b_t &lt; 0$</td>
<td></td>
<td></td>
<td>Plot $b_d x_{counter}$ with symbol for 'reflective left'</td>
</tr>
<tr>
<td>$a_t &gt; 0 \land b_t &gt; 0$</td>
<td></td>
<td></td>
<td>Plot $b_d x_{counter}$ with symbol for 'refractive down'</td>
</tr>
</tbody>
</table>

Figure B.5 Algorithm for discerning the point classes and plotting the points
B.6 Animation of the State

The evolution of the converter state over time is animated on the computer screen. The heart of the algorithm is an iterative integration and display loop. The most important parts of the algorithm are given in the diagram below.

<table>
<thead>
<tr>
<th>Specify network equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specify converter parameter values</td>
</tr>
<tr>
<td>Specify switching boundary equation</td>
</tr>
<tr>
<td>Specify switching boundary parameters</td>
</tr>
<tr>
<td>Calculate $\gamma$</td>
</tr>
<tr>
<td>Counter = 1 to 1000</td>
</tr>
<tr>
<td>Solve for $\frac{bdx}{v=(counter/1000) \cdot v_{max}}$</td>
</tr>
<tr>
<td>Store $\frac{bdx}{x}$ subsequently as $\frac{bdx}{x_{counter}}$</td>
</tr>
<tr>
<td>Plot all $\frac{bdx}{x_{counter}}$ as smooth switching boundary</td>
</tr>
<tr>
<td>Equate network equation with 0</td>
</tr>
<tr>
<td>Counter = 1 to 1000</td>
</tr>
<tr>
<td>Solve for $\frac{llx}{(D = counter/1000)}$</td>
</tr>
<tr>
<td>Store $\frac{llx}{x}$ subsequently as $\frac{llx}{x_{counter}}$</td>
</tr>
<tr>
<td>Plot all $\frac{llx}{x_{counter}}$ as smooth loadline</td>
</tr>
</tbody>
</table>

Select initial value $x = x_0$, step size $\Delta t$ and display step $d$

$m$ iterations

<table>
<thead>
<tr>
<th>Calculate $i_{boundary}(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_{boundary} &gt; 0$ ?</td>
</tr>
<tr>
<td>Yes</td>
</tr>
<tr>
<td>$i &gt; i_{boundary}$ ?</td>
</tr>
<tr>
<td>No</td>
</tr>
<tr>
<td>$q_1 = 1$, $q_2 = 0$</td>
</tr>
<tr>
<td>$q_1 = 0$, $q_2 = 1$</td>
</tr>
</tbody>
</table>

$x_{n+1} = x_n + f(x_n) \cdot \Delta t$

Plot every $d$-th $x$

Figure B.6 Algorithm for evaluating a converter's boundary (simplified)
APPENDIX C

Equilibrium Points and Their Derivation

Every topology of a converter yields one set of two differential equations. Each of these sets has exactly one equilibrium point. Interpreted physically, this is the steady state the converter tends to in this topology. The time changes $dv/dt$ and $di/dt$ then are zero. This consideration motivates that the coordinates of an equilibrium point can be obtained by equating the d.e. of that topology with zero. The equilibrium for 'both switches off' is trivial (zero current, zero voltage), so that one can concentrate on the two nontrivial cases.

The simple buck converter is again chosen to exemplify the process of derivation. The derivation can be seen in Figure C.1.

The results for all the converters studied are given in Table C.1.
\[
\frac{di}{dt}(v) = \begin{bmatrix} 0 & -\frac{1}{C \cdot R} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} 0 & \frac{L}{1} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{C} & 0 \\ 0 & \frac{L}{1} \end{bmatrix} \begin{bmatrix} v \\ E \end{bmatrix} + \begin{bmatrix} \frac{1}{C} & 0 \\ 0 & \frac{L}{1} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} q_2
\]

Transistor on, Diode off:

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C \cdot R} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i^1_e \\ v^1_e \end{bmatrix} + \begin{bmatrix} 0 & \frac{L}{1} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} i^1_e \\ v^1_e \end{bmatrix} + \begin{bmatrix} \frac{1}{C} & 0 \\ 0 & \frac{L}{1} \end{bmatrix} \begin{bmatrix} v^1_e \\ E \end{bmatrix}
\]

\[
0 = -\frac{1}{L} v^1_e + \frac{1}{L} E \\
0 = -\frac{1}{CR} v^1_e + \frac{1}{C} i^1_e
\]

\[
x^1_e = \begin{bmatrix} \frac{E}{R} \\ E \end{bmatrix}
\]

Transistor off, Diode on:

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C \cdot R} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i^0_e \\ v^0_e \end{bmatrix} + \begin{bmatrix} 0 & \frac{L}{1} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} i^0_e \\ v^0_e \end{bmatrix} + \begin{bmatrix} \frac{1}{C} & 0 \\ 0 & \frac{L}{1} \end{bmatrix} \begin{bmatrix} v^0_e \\ 0 \end{bmatrix} q_2
\]

\[
0 = -\frac{1}{L} v^0_e \\
0 = -\frac{1}{CR} v^0_e + \frac{1}{C} i^0_e
\]

\[
x^0_e = \begin{bmatrix} \frac{E}{R} \\ E \end{bmatrix}
\]

Figure C.1 Derivation of the equilibrium points exemplified for the simple buck converter
<table>
<thead>
<tr>
<th>Converter</th>
<th>Model</th>
<th>$x_0^1$</th>
<th>$x_0^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buck</td>
<td>Simple</td>
<td>$x_0^1 = \begin{pmatrix} E / R \ E \end{pmatrix}$</td>
<td>$x_0^0 = \begin{pmatrix} 0 \ 0 \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>Enhanced</td>
<td>$x_0^1 = \begin{pmatrix} E \ R + R_L + R_d \end{pmatrix}$</td>
<td>$x_0^0 = \begin{pmatrix} 0 \ 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>Boost</td>
<td>Simple</td>
<td>$x_0^1 = \begin{pmatrix} \infty \ 0 \end{pmatrix}$</td>
<td>$x_0^0 = \begin{pmatrix} E / R \ E \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>Enhanced</td>
<td>$x_0^1 = \begin{pmatrix} E \ R_L + R_d \end{pmatrix}$</td>
<td>$x_0^0 = \begin{pmatrix} \frac{E - V_{drop}}{R + R_L} \ \frac{R \cdot (E - V_{drop})}{R + R_L} \end{pmatrix}$</td>
</tr>
<tr>
<td>Buckboost</td>
<td>Simple</td>
<td>$x_0^1 = \begin{pmatrix} \infty \ 0 \end{pmatrix}$</td>
<td>$x_0^0 = \begin{pmatrix} 0 \ 0 \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>Enhanced</td>
<td>$x_0^1 = \begin{pmatrix} E \ R_L + R_d \end{pmatrix}$</td>
<td>$x_0^0 = \begin{pmatrix} 0 \ 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
APPENDIX D

Plots of Loadlines for all Three Converter Types, Simple and Enhanced Models

Either of the three methods presented in Chapter 4 can be used for calculating the loadlines for all of the converter models. The results for the converters under consideration are given below.

D.1 Buck Converter

For the buck converter, there are no decisive differences between the simple and the enhanced model. In the enhanced model, the maximum output voltage ($D=1$) reduces slightly due to the converter losses. Yet the shape, especially the slope, remains unchanged.
D.2 Boost Converter

In the boost converter case the situation is different. The simple model does predict an infinitely high output voltage for $D=1$. This converts into a loadline rising above all limits. In reality stray resistances keep voltage and current values limited. The boost converter consequently does not furnish an arbitrarily high output voltage. Rather, at some point, the output voltage “breaks down”. This phenomenon is reflected in the enhanced model’s loadline. The geometric difference is immense. The lines significantly diverge when the boost factor exceeds a value around three. It should be obvious that for boost converters above that range a geometrically based control design has to operate with the enhanced converter model. Parasitic values have to be known rather well in order to design a precise boundary control. This requirement definitely represents a disadvantage of this control method.
D.3 Buckboost Converter

As is true with regard to the families of trajectories, the buckboost converter is very similar to the boost type also in respect to the loadline. What has been said for the boost converter in Section D.2 holds equally true for the buckboost circuit.
REFERENCES


