A MULTITUBE DIFFERENTIAL PRESSURE MANOMETER
FOR MEASURING THE AVERAGE FLOW OF FLUIDS
IN CLOSED DUCTS

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UNIVERSITY OF ILLINOIS BULLETIN
Price: Thirty Cents

UNIVERSITY OF ILLINOIS BULLETIN

Volume 50, Number 51; March, 1952. Published seven times each month by the University of Illinois. Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Office of Publication, 338 Administration Building, Urbana, Illinois.
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Published by the University of Illinois, Urbana
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I. INTRODUCTION

Various methods have been devised to measure the flow of fluids in closed ducts — especially the flow of water, air, and other gases. Measuring the quantity of air flowing through mine openings or the flow of air in standardized fan tests involves the type of fluid flow with which this Bulletin is primarily concerned. Many standard methods of measuring air flow are inaccurate, time-consuming, or both. Particularly time-consuming are the methods used in fan testing; for example, such testing is often difficult to incorporate in a three-hour laboratory period such as is customary in undergraduate teaching.

There are few satisfactory means of measuring the average flow or the quantity of flow past a given cross-section in a duct; most measuring methods are not compatible with all test conditions. Thus the use of calibrated orifices cannot be readily adapted to fan testing and to some other operations that require accurate flow metering, because the orifice constricts the flow of fluid. Again, calibration of the duct and the use of a center factor may cause a large error if the pattern of flow changes.

A manometer developed in the University of Illinois mine ventilation laboratory uses twenty fixed pitot tubes, measures the velocity pressures simultaneously at twenty points at a given cross-section of the duct, extracts the square root of each velocity pressure, averages the twenty square roots, and multiplies the result by a constant which is adjusted for the specific weight of air in the duct during the test. This procedure gives the average velocity or quantity in one reading. In addition the manometer does not seriously obstruct the duct, and automatically adjusts for any changes in flow pattern at the position of the pitot tubes.

The present Bulletin describes the theory and construction of this average-velocity gage and analyzes possible sources of error in readings which are made with it.
II. METHODS OF MEASURING AIR FLOW

1. Orifice Metering

In this type of flow measurement, the pressure drop caused by the constricting effect of an orifice in the duct is known to be a function of the quantity of fluid flowing through the orifice. The measurement of the pressure drop across a calibrated orifice will give the quantity flowing or the average velocity of the fluid.

2. Vane-Type Anemometers

These instruments consist of either propeller or radial-blade designs, the blades being attached to a rotating axis which in turn is geared to a mechanism that will count the revolutions of the axis. When this anemometer is placed in an air stream the rotation of the blade assembly will measure the amount of air passing the blades, which in turn gives the velocity. The instrument measures the approximate amount of air passing through its whole cross-sectional area, and consequently does not measure the velocity at a point. It is not adaptable to the measurement of average flow in relatively small ducts. However, it is accurate enough for flow measurements in mine drifts, where great precision is not required.

3. Hot-Wire Anemometer

In this instrument the cooling power of moving air is used to measure the velocity. A small length of platinum wire is included in a bridge circuit in order to measure its resistance accurately. The heated platinum wire is placed in the air stream and the amount of electric current necessary to maintain its temperature at a fixed value is measured. The current necessary to maintain this temperature is a function of the velocity of the air passing the wire. An ammeter properly connected and calibrated may be used to measure the velocity of the air which is passing the heated wire.

The hot-wire anemometer measures the velocity of air flowing past a very small area; this velocity so determined can be considered to be the velocity at a given point. If the velocity of air is measured at enough properly spaced points across the cross-section of a closed duct, the average of these readings will be close to the average velocity of the air. (See NAFM code on page 7.)
4. Pitot Tube

The pitot tube and the pressure-indicating instruments employed with it are widely used to measure air flow. Since the pitot tube is a basic part of the particular anemometer with whose development this paper is concerned, it is described in some detail in this section.

The two openings of a pitot tube connected to the legs of a U-tube manometer give the difference between the total pressure and the static pressure which is the velocity pressure. The relation between the velocity pressure so measured and the velocity of the air flowing past the end of the pitot tube is

\[ V = 1097.4 \sqrt{\frac{h_i}{w}} \]  

(1)

where

- \( V \) is velocity in ft/min
- \( h_i \) is velocity pressure in inches water gage
- \( w \) is the specific weight of air in lb/cu ft

The lengths of tubing and the U-tube constitute a closed system. That is, the static plus the velocity pressure on one leg of the tube are opposed by the static pressure only on the other leg; a reading on the scale of the U-tube thus measures velocity pressure. The same principle is employed in the average velocity manometer. It should be noted, however, that the static pressure is assumed to be the same at all points across the cross-section of the duct as it is at the end of the pitot tubes. Consequently, the static pressure is measured at the skin of the duct instead of at the position of the pitot tube within the duct.

The operation of making the pitot tube traverse of a closed duct in order to determine the average velocity as well as the quantity of the air flow is tedious under the most ideal working conditions. Enough points must be taken as velocity measuring stations so that the average of these readings will approximate closely the average velocity of the air flowing through the duct, provided that each velocity thus determined can be taken as representative of equal portions of the cross-sectional area of the duct. The number of points prescribed by the National Association of Fan Manufacturers is twenty.

The individual velocities are ordinarily calculated in feet per minute by the use of Eq. 1.

The average velocity is then determined as follows:

\[ V_{av} = \frac{V_1 + V_2 + V_3 \ldots + V_n}{n} \]  

(2)
or

\[ V_{av} = \frac{1097.4}{n} \sqrt{W} \sum_{i=1}^{n} \frac{1}{\sqrt{h_i}} \]  \hspace{1cm} (3)

The problem suggested by this mathematical statement is to design an instrument which will do mechanically the operation indicated by \( \sum_{i=1}^{n} \frac{1}{\sqrt{h_i}} \), the quantity in front of the summation sign being a constant for a given psychrometric condition of the air at the time of the test. Adjustment for the variation in the weight of air is only a matter of proper calibration, as is shown in Chap. IV.
III. THEORY OF AVERAGE-VELOCITY MANOMETERS

In an ordinary manometer the volume of liquid displaced by the velocity pressure of the air is directly proportional to the velocity pressure. However, if by appropriate design of the manometer tube the quantity of liquid is made proportional to the square root of a given velocity pressure and if all the required velocity pressures in a duct are measured simultaneously with the same number of manometers, the total quantity of liquid displaced represents the quantity on the right of the summation sign. If an appropriate variable scale is provided so that the quantity of liquid thus displaced is measured for the appropriate weight of air on this scale, which also takes the values 1097.4 and \( n \) into account, the average velocity may be read directly.

These operations are easily performed. First the static pressure is taken from points on the surface of the duct; this means that the static pressure is assumed to be constant over the cross-section. The total pressure (static plus velocity) is measured by placing at the proper points in the duct, fixed facing total-pressure tubes of small diameter. Each is connected to the top openings of a system of an equal number of manometer tubes of appropriate design. The lower ends of these are connected to a common reservoir and are opposed in a closed system by the static pressure through a tube which is connected to the top of the volume-measuring tube. For reading the average velocity the liquid in the measuring tube must always be brought back to its zero reference level, which is also the zero coordinate of the curved tubes, so that the static pressure will cancel properly (reach null balance), and so that the oil displaced in each tube will be proportional to the square root of its applied velocity pressure.

Each manometric tube is so designed that the volume of the oil displaced from it is proportional to the square root of the height of the oil displaced (velocity pressure).

That is,

\[ Q = K \sqrt{h} \]  (4)

where

- \( Q \) is the quantity of oil displaced
- \( K \) is a constant of proportionality
- \( h \) is the height of oil displaced (velocity pressure)

Three types of manometric tubes fulfill this requirement.
5. Volume of Revolution

Differentiating Eq. 4,

\[ dQ = \frac{1}{2} K h^{-\frac{3}{2}} dh \]  

Also, for any infinitesimal height the volume is

\[ dQ = \pi r^2 dh \]  

Equating Eqs. 5 and 6 and solving for \( r \), the variable radius of the required volume of revolution is

\[ r = \sqrt{\frac{K}{2\pi}} \cdot h^{-\frac{3}{4}} \]  

This is the equation of the required curve.

The fact that this hyperbola is asymptotic to both the \( h \) and \( r \) axes introduces a difficulty in designing a manometer which will cover the range of velocities desired — from zero to some appropriate fixed value. It is seen that the integral

\[ Q = \int_0^{h_2} \pi r^2 \, dh \]  

does not exist. However, an integral from a very small lower limit does exist:

\[ Q = \int_{h_1}^{h_2} \pi r^2 \, dh \]  

For actual mechanical design of the manometer, the lower limit was made equal to 0.01 in. That is, a small upper portion of the tube was made as a right circular cylinder of equivalent volume required by Eq. 4, and the rest of the tube was designed according to Eq. 7. Depressions of oil within the circular cylinder do not give correct readings in the range up to about 400 ft per min for air at 0.075 lb/cu ft. However, readings taken above this velocity are correct. (See Fig. 1.)

6. Right Circular Cylinder with Core

A second type of manometer tube which will comply with Eq. 4 may be designed by placing a shaped core within a right circular cylinder. In this case a differential volume is

\[ dQ = \pi (r_1^2 - r^2) \, dh \]
Equating with Eq. 5 and solving for \( r \),

\[
r = \sqrt{r_1^2 - \frac{K}{2\pi h^{1/2}}} \tag{11}
\]

As in the first type, the integral Eq. 8 does not exist, and similar provisions—choice of a very small lower limit for \( h \)—must be made in the design of the core.

7. Curved Small Bore Cylindrical Tubing

A third design of tube which satisfies Eq. 4 is a curved small bore tube in which the surface of the liquid is assumed to remain perpendicular to the axis of the tube. The volume of liquid displaced is then proportional to the length of the tube.

Here

\[
dQ = K_1 \sqrt{(dx)^2 + (dh)^2} \tag{12}
\]

Equating with Eq. 5

\[
\frac{K^2 (dh)^2}{4h} = (dx)^2 + (dh)^2 \tag{13}
\]

\[
dx = \sqrt{\frac{K^2}{4h} - 1} \, dh \tag{14}
\]
or, by integration

\[ x = \sqrt{h} \sqrt{\frac{K^2}{4} - h} + \frac{K^2}{4} \sin^{-1} \frac{2\sqrt{h}}{K} + C \]  

(15)

If it is assumed that the curve goes through the origin, the constant \( C \) is zero.

Equation 15 yields a whole family of curves. A given curve is obtained by assigning a value to \( K \). It is noted that the curve does not exist for negative values of \( h \) or values of \( h \) larger than \( K^2/4 \). The upper limiting value of the family of curves is found by setting \( K^2/4 = h \). Equation 15 becomes

\[ x = h \sin^{-1} (1) = h \frac{\pi}{2} \]  

(16)

Equation 16 is a straight line whose slope is \( 2/\pi \). (See Fig. 2.)
IV. CONSTRUCTION AND TESTING OF THE MANOMETER

In the early development of the gage only the first type (volume of revolution) of manometer tube had been conceived. A test model using six containers of this type, machined from plastic, proved successful. The scale factor was $K = 1$, so that the tubes were too large for a working model with twenty tubes. Reducing the scale factor to $K = \frac{1}{2}$ introduced problems in machining which made it very difficult to produce containers with the exact hyperbolic shape to a required accuracy of $\pm 0.001$ in. on the radius.

The second type of container with core does not appear practical for gage construction, although its possibilities have not been fully investigated.

The curved small bore tube was subsequently conceived, and with an appropriate design of frame for mounting, the whole was designed and assembled. (Figs. 3 and 4) The glass tubing had to be small enough so that the meniscus of the fluid would remain normal to the axis of the tube.
even at very low inclinations, and large enough so that capillarity effects would be neglected. It was found that the range of diameter from about 0.15 to 0.20 in. would satisfy this condition for Ellison gage oil.

Glass tubing was selected from standard stock and bent to a template. Uniform heating of the tubing during bending was necessary in order to obtain a smooth, accurate curve and to prevent excessive distortion of the bore. The tubes were mounted individually, held by wire clamps, and supported on small tabs of rubber for adjustments.
The movable measuring tube was mounted on a vertical screw in turn mounted parallel to a fixed plastic rod 1 in. in diameter, on which was secured a reading scale. The scale was drawn on vellum. Its basis of construction is as follows:

When

\[ d \text{ is the diameter of manometer tubes} \]
\[ D \text{ is the diameter of measuring tube} \]
\[ H \text{ is the height of oil displaced into measuring tube} \]
\[ L \text{ is the length of oil column displaced in small tubes} \]

Then

\[ L_i = K \sqrt{h_i} \quad (17) \]
\[ Q_i = \frac{\pi d^2}{4} L_i \quad (18) \]
\[ V_{av} = \frac{1097.4}{20} \sqrt{\frac{w}{w}} \sum_{i=1}^{n} \sqrt{h_i} \quad (19) \]

or

\[ V_{av} = \frac{1097.4}{20K} \sqrt{\frac{w}{w}} \cdot \frac{D^2}{d^2} \cdot H \quad (20) \]

In Eq. 20 appropriate values may be chosen for \( D, d \) and \( K \) and a scale constructed for values of \( w \), the weight of air, for the range of weights that may be found in a given laboratory. That is, the weight of air must be determined, and the scale on the vertical plastic rod must be rotated to the proper position for a given weight of air. If the weight of air changes during a fan test, for example, the scale may be immediately adjusted to the proper weight of air. The present test model employs small tubes with an inside diameter of about 0.175 in. and a measuring tube of approximately 1 in. inside diameter.

The best method available for testing the average velocity gage is by comparison with a standard pitot tube traverse. The gage was first leveled in the plane normal to the plane of the tubes and then as closely as possible in the plane of the tubes. When the gage was leveled properly, the readings compared very favorably with the results of the pitot tube traverse. The results of three final tests are given in the following table. The last test reported shows the difference in velocities to be consistently less than 1 percent, a remarkable agreement in view of the fact that the gage was constructed with standard rather than precision tubing.
After this test was completed, the instrument was also tested to determine whether the relationship between the volume of oil displaced and the pressure applied was properly maintained over a range of velocities from 0 to 6000 ft/min. The same pressure was applied to all the tubes in increasing increments. The quantity of oil displaced was proportional to the square root of the applied pressure, approaching an accuracy of plus or minus 1 percent.

<table>
<thead>
<tr>
<th>Test No. 1</th>
<th>Test No. 2</th>
<th>Test No. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pitot</td>
<td>Gage</td>
<td>Pitot</td>
</tr>
<tr>
<td>Tube</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>4480</td>
<td>4390</td>
</tr>
<tr>
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<td>1190</td>
<td>1075</td>
<td>1385</td>
</tr>
<tr>
<td></td>
<td>870</td>
<td>880</td>
</tr>
</tbody>
</table>

* Velocities in ft/min
V. ERROR OF SIMPLE DIFFERENTIAL PRESSURE MANOMETERS

Certain investigators have found that under given conditions of flow the errors in the measurement of air and other fluids by differential pressure manometers may be of the order of 100 percent or even greater. Judd and Pheley\(^{(1)}\) in their study of the flow of air with Venturi meters, orifices, nozzles and pitot tubes concluded that:

1. Pulsations in fluid flow are made up of sudden changes in velocity and pressure.
2. Largest pulsations are caused by pressure changes.
3. Pressure changes are in the form of wave fronts which travel with the velocity of sound.
4. Velocity of pulsations is independent of the velocity of the fluid.
5. The effect of pulsations is to increase the manometer reading. The magnitude of the error so caused is a function of frequency, static pressure, type of meter and component devices.
6. Pulsation must be reduced or eliminated to insure correct readings.

Lindahl\(^{(2)}\) showed that if the pressure varies as a sine wave, the instantaneous pressure is given by

\[
P = h + p \sin \theta
\]  

(21)

where

- \(h\) is the average velocity pressure reading
- \(p\) is the maximum variation in pressure
- \(P\) is the instantaneous velocity pressure
- \(K\) is a constant

The value of the instantaneous velocity is given by

\[
V = K (h + p \sin \theta)^{1/2}
\]

(21a)

From these expressions the relation between the mean and average velocities is found to be

\[
\frac{V_m}{V_{av}} = \frac{K h^{1/2}}{K h^{1/2} \left[ 1 - \frac{1}{16} \left( \frac{P}{h} \right)^2 - \frac{15}{1024} \left( \frac{P}{h} \right)^4 \right]}
\]

(22)

*Parenthesized superscripts refer to correspondingly numbered entries in the References.*
Therefore, the average of the velocity pressures is higher than that of the velocities. The amount of this error is a function of the ratio of the amplitude of oscillations to the average pressure. For a ratio of $p/h = \frac{1}{4}$ the error introduced is only of the order of 0.4 percent — very small in comparison with errors actually present. Thus, uniform pulsation does not appear to account for all the errors encountered in metering the flow of fluids. Pulsation accounts for large errors only when the amplitude of the pulsating meter is great; this may happen when the frequency of the pulsation is very close to the characteristic frequency of pulsation of the meter itself.

This development assumes that the meter is oscillating with the same amplitude as the velocity pressure. For a non-damped meter this condition will obtain only if the restoring force is equal to maximum amplitude of the applied pressure (see below) and the frequency of the pressure oscillations is much greater than the natural frequency of the meter. When the meter is of the damped type (as most are), the maximum amplitude depends on the relationship of a number of factors in the equation of motion. The assumption in both cases is also that the type of oscillation of the meter is simple harmonic in character, that is, the differential equation of motion is linear.

These points on oscillating characteristics as the causes of basic error can be best demonstrated by analyzing the behavior of a simple type of manometer. The simple U-tube may be treated mathematically with essentially the same equations as an elastically-restrained body. Its characteristics are also important for comparison with the more difficult case of the manometer tubes employed in the average-velocity gage. The three important types of behavior to be analyzed are (1) free vibration, (2) forced vibration, and (3) forced-damped vibration.

8. Free Vibration

The differential equation of motion of the liquid in a U-tube is derived as follows.

The mass multiplied by the acceleration is given by

$$m \frac{d^2 l}{dt^2} = \frac{\rho A L}{g} \frac{d^2 l}{dt^2}$$

(23)

where

- $l$ is the displacement
- $\rho$ is the density of fluid
- $A$ is the cross-sectional area of tube
- $L$ is the length of column of fluid
- $g$ is the acceleration of gravity
The restoring force is

\[ -2\rho Al \] \hspace{1cm} (24)

Hence, the equation of motion is

\[ \frac{\rho AL}{g} \cdot \frac{d^2l}{dt^2} + 2\rho Al = 0 \] \hspace{1cm} (25)

or

\[ \frac{d^2l}{dt^2} + \frac{2g}{L} l = 0 \] \hspace{1cm} (26)

The solution is of the form

\[ = B \cos \omega_0 t + C \sin \omega_0 t \] \hspace{1cm} (27)

where the frequency is

\[ \omega_0 = \sqrt{\frac{2g}{L}} \] \hspace{1cm} (28)

and the period

\[ T = 2\pi \sqrt{\frac{L}{2g}} \] \hspace{1cm} (29)

If the following boundary conditions are assumed, the behavior may be more completely demonstrated:

\[ t = 0 \]
\[ \frac{dl}{dt} = v_0 = 0 \]
\[ l = l_0 \]

and, from these

\[ l = l_0 \cos \omega_0 \] \hspace{1cm} (30)

Hence, the oscillation is simple harmonic in character. If the tube is inclined at an angle \( \alpha \) the frequency becomes

\[ \omega_0 = \sqrt{\frac{2g \sin \alpha}{L}} \] \hspace{1cm} (31)

9. Forced Vibration

If a periodic force or pressure, \( P \sin \omega t \), acts on the liquid on one side of the tube, the appropriate equation of motion is

\[ m \frac{d^2l}{dt^2} + kl = P \sin \omega t \] \hspace{1cm} (32)
where \[ m = \frac{\rho AL}{g} \]
\[ k = 2\rho A \]

One complete solution of Eq. 32 is

\[ l = B \cos \omega_0 t + C \sin \omega_0 t + \frac{P \sin \omega t}{m (\omega_0^2 - \omega^2)} \]  \hspace{1cm} (33)

The last term may be rewritten

\[ \frac{P \sin \omega t}{k} \left( \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} \right) \]  \hspace{1cm} (34)

If \( \omega^2/\omega_0^2 \) is small compared to unity then the displacement is that due to the instantaneous value of the force. If the ratio is greater than unity the force and deflection are opposite in sign. Finally, if the ratio is equal to unity, the displacement becomes infinite — that is, a condition of resonance exists. Then a new solution is required:

\[ l = B \cos \omega_0 t + C \sin \omega_0 t - \frac{P}{2m\omega_0} \cos \omega_0 t \]  \hspace{1cm} (35)

When the frequencies are only slightly different, beats will occur. (See Fig. 5.)

The last term in this solution shows that when the frequency of the applied pressure is the same as that of the manometer, the amplitude increases indefinitely. (See Fig. 6.)

If either of these conditions exists, the amplitude of oscillation of the manometer will be greater than that of the applied pressure and the error of the velocity reading will increase accordingly.

10. Forced-Damped Vibration

Because most differential pressure meters have some damping effect present, this case is of greater interest than those previously considered.
In a U-tube the damping effect of the viscosity of the manometric fluid should be taken into account. It is assumed that the motion is opposed by an additional force proportional to the velocity. The equation of motion for this case is

\[ m \frac{d^2 l}{dt^2} + \beta \frac{dl}{dt} + kl = P \sin \omega t \]  

(36)

where

\[ \beta = \frac{\mu \rho \ln c}{d^2} \]

\( c \) is the constant

\( \mu \) is the absolute viscosity

\( d \) is the diameter of manometer tube

The complete solution of Eq. 36 is

\[ l = Be^{\lambda_1 t} + Ce^{\lambda_2 t} + \frac{P (k - m \omega^2) \sin \omega t - \beta \omega \cos \omega t}{(k - m \omega^2)^2 + (\beta \omega)^2} \]  

(37)

This may be rewritten in the form

\[ l = Be^{\lambda_1 t} + Ce^{\lambda_2 t} + \frac{P \sin (\omega t - \psi)}{\sqrt{(k - m \omega^2)^2 + (\beta \omega)^2}} \]  

(38)

where

\[ \tan \psi = \frac{\beta \omega}{k - m \omega^2} \]

and the angle \( \psi \) is the phase lag.

The expressions for \( \lambda_1 \) and \( \lambda_2 \) are

\[ -\frac{\beta}{2m} + \sqrt{\left(\frac{\beta}{2m}\right)^2 - \frac{k}{m}} \]  

(39)
and

\[-\frac{\beta}{2m} - \sqrt{\left(\frac{\beta}{2m}\right)^2 - \frac{k}{m}} \tag{40}\]

For values of \(\beta^2 > 4km\) the motion is not oscillatory but is a subsidence. For \(\beta^2 < 4km\) the quantities become complex and the motion becomes oscillatory. The point at which \(\beta^2 = 4km\) is that at which the change from subsident to oscillatory motion occurs, and the quantity \(\beta = 2\sqrt{km}\) is known as the critical damping factor. For this condition (of critical damping) it is found that

\[\lambda_1 = \lambda_2 = -\sqrt{\frac{k}{m}} \tag{41}\]

Fig. 7. Amplitudes of Forced Oscillations as the Function of the Frequency for Various Values of the Damping Factor \(\beta\)
Further, if $\beta < 2\sqrt{k/m}$ the maximum displacement $c$ is given by

$$c_{\text{max}} = \frac{P}{k} \frac{2km/\beta^2}{\sqrt{4km/\beta^2 - 1}}$$

If values of the amplitude ratio $c/F_0/k$ are plotted as a function of the ratio $\omega\sqrt{k/m}$ for different values of the ratio $\beta/\beta_c$ where $\beta_c = 2\sqrt{k/m}$ is the critical damping (Fig. 7), it may be shown that the maximum amplitude occurs for small values of $\beta/\beta_c$ and, in the case of the manometer, the magnitude of error increases.

It is also pertinent to compare these characteristics of force-damped vibration with those of high damping for unforced oscillation. (See Figs. 8 and 9.) In the latter case the motion is a subsidence.
VI. ERROR OF THE CURVED-TUBE MANOMETER

To evaluate the accuracy of the average-velocity gage its oscillating characteristics must be considered. The simplest case is that in which damping is neglected and there is no imposed force acting on the manometric fluid. That is, the surface of the liquid is considered to be making small oscillations about the origin as shown in Fig. 10.

![Fig. 10. Scheme for Oscillation of the Liquid Level about the Origin in a Manometric Tube](image)

The differential equation of motion for this system is

\[
\frac{(L - l)}{g} \rho A \frac{d^2 l}{dt^2} + \rho A h = 0
\]

(43)

The restoring force is identical with that for a vertical tube of the same diameter and height. As a first approximation the displacement is considered to be small as compared to \( L \); that is, the mass of moving fluid is assumed to be constant. This supposition permits the equation of motion to be placed in an analytically solvable form. The tube is assumed to be connected, as shown, to a very large reservoir of fluid, which will provide the physical condition necessary to make the equation applicable. This condition effectively exists in the average-velocity gage. The loss in velocity head at the entrance to the reservoir is neglected, as is the effect of centrifugal force along the curved tube.

If the value of \( h \) is substituted with Eq. 43 and the above approximation is made, the equation of motion becomes

\[
\frac{d^2 l}{dt^2} + \frac{g}{L K^2} l \frac{\partial}{\partial t} = 0
\]

(44)
The order of the equation is reduced by letting \( v = \frac{dl}{dt} \) and solving the resulting equation:

\[
v \cdot \frac{dv}{dt} = -\frac{g}{LK^2} l^2
\]  

(45)

\[
\int_v^v \frac{dv}{v} = -\frac{g}{LK^2} \int_l^l l^2 dl
\]  

(46)

For the conditions \( t = 0 \) \( l = l_1 \)

\[
\left( \frac{dl}{dt} \right)^2 = \frac{2g}{3LK^2} (l_1^3 - l^3)
\]  

(47)

or

\[
\int_0^l dt = \sqrt{\frac{3LK^2}{2g}} \int_{l_1}^l \frac{dl}{\sqrt{l_1^3 - l^3}}
\]  

(48)

This may be placed in a recognizable form\(^{(a)}\) by substituting

\[
x = \frac{l}{l_1}
\]  

(49)

Equation 48 becomes

\[
 t = \sqrt{\frac{3LK^2}{2gl_1}} \int_x^1 \frac{dx}{\sqrt{1 - x^2}}
\]  

(50)

This is an elliptic integral, the member under the radical having one real and two complex roots, which may be placed in the standard Legendre form by making the substitution

\[
\cos \phi = \frac{\sqrt{3} - 1 + x}{\sqrt{3} + 1 - x}
\]  

(51)

This converts Eq. 50 to

\[
 t = \sqrt{\frac{\sqrt{3} LK^2}{2gl_1}} \int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 75^\circ \cdot \sin^2 \phi}}
\]  

(52)

If the value of the integral is denoted by \( F \), the following table indicates the time displacement values for one-quarter cycle. This is plotted in Fig. 11 for comparison with a circular cosine curve of the same frequency.
The time for a complete cycle is four times that for a quarter-cycle, or

\[ T = 4K \sqrt{\frac{\sqrt{3} L}{2gl_1}} \times 1.843 \]  

The solution of the oscillation of the meniscus of the manometer fluid about some point other than the origin may be accomplished by the same method, that is, by means of elliptic integrals.
Considering the oscillation about the point $l_0$ in Fig. 12, which condition corresponds to a free vibration with the liquid in the reservoir at the same level as the point, the equation of motion is

\[
\frac{d^2l}{dt^2} + \frac{g}{LK^2} (l^2 - l_0^2) = 0
\]  

(54)

If $l < l_0$ this leads to the following upon integration:

\[
t = \sqrt{\frac{3LK^2}{2g}} \int_{l_1}^{l} \frac{dl}{\sqrt{-(l - l_1) (l^2 + ll_1 + l_1^2 - 3l_0^2)}}
\]  

(55)

where the roots of the quantity under the radical sign are real, and in order of magnitude are:

\[
\alpha = \frac{-l_1 + \sqrt{12l_0^2 - 3l_1^2}}{2}, \quad \beta = l_1, \quad \gamma = \frac{-l_1 - \sqrt{12l_0^2 - 3l_1^2}}{2}
\]  

(56)

Equation 55 leads, in turn, to a solution in the LeGendre form of

\[
t = \sqrt{\frac{3LK^2}{2g}} \frac{2\sqrt{2}}{\sqrt{12l_0^2 - 3l_1^2}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - K_i^2 \sin^2 \phi}}
\]  

(57)

where

\[
K_i^2 = \frac{\alpha - \beta}{\alpha - \gamma} = \frac{-3l_1 + \sqrt{12l_0^2 - 3l_1^2}}{2 \sqrt{12l_0^2 - 3l_1^2}}
\]  

(58)

and

\[
\phi = \cos^{-1} \frac{(\beta - \gamma) (\alpha - l)}{(\alpha - \beta) (l - \gamma)}
\]  

(59)

\[
= \cos^{-1} \frac{(3l_1 + \sqrt{12l_0^2 - 3l_1^2}) (-l_1 - 2l + \sqrt{12l_0^2 - 3l_1^2})}{(-3l_1 + \sqrt{12l_0^2 - 3l_1^2}) (2l + l_1 + \sqrt{12l_0^2 - 3l_1^2})}
\]  

Fig. 12. Condition for Oscillation about a Point Other than the Origin
When \( l > l_0 \), the solution of Eq. 54 leads to

\[
t = \sqrt{\frac{3LK^2}{2g}} \int_{l}^{l_2} \frac{dl}{\sqrt{-(l - l_2) (l^2 + l_2^2 + l_2^2 - 3l_0^2)}}
\]  

The quantity under the radical sign has three real roots for values of \( l_0 > \frac{1}{2} l_2 \). They are, in order of magnitude:

\[
\alpha = l_2 \quad \gamma = \frac{-l_2 - \sqrt{12l_0^2 - 3l_2^2}}{2} \\
\beta = \frac{-l_2 + \sqrt{12l_0^2 - 3l_2^2}}{2} \quad \alpha > l > \beta > \gamma
\]

This development leads to the Legendre form of

\[
t = \sqrt{\frac{3LK^2}{2g}} \frac{2 \sqrt{2}}{3l_2 + \sqrt{12l_0^2 - 3l_2^2}} \int_{\phi}^{\phi_0} \frac{d\phi}{\sqrt{1 - K_2^2 \sin^2 \phi}}
\]  

where

\[
K_2^2 = \frac{\alpha - \beta}{\alpha - \gamma} = \frac{3l_2 - \sqrt{12l_0^2 - 3l_2^2}}{3l_2 + \sqrt{12l_0^2 - 3l_2^2}}
\]

and

\[
\phi = \cos^{-1} \sqrt{\frac{l - \beta}{\alpha - \beta}} = \cos^{-1} \sqrt{\frac{2l + l_2 - \sqrt{12l_0^2 - 3l_2^2}}{3l_2 - \sqrt{12l_0^2 - 3l_2^2}}}
\]

These waves are also elliptic cosines, and their periods are four times the expression in front of the integral sign. The frequencies of the upward displacement and the downward are different, the latter being greater. A comparison shows that the time of oscillation increases as the central point of oscillation is moved downward along the curve because \( l_0 \) increases and \( L \) decreases.

A condition of resonance therefore could not occur for the oscillation of the manometer fluid unless a very complex pressure wave with similar frequency characteristics were imposed upon it. This is very unlikely to occur in practice. Even in the case of free forced vibration (without viscous damping), then, the curved manometer tubes are self-damping.

A simple test was performed with the test model of the gage to observe its oscillating characteristics. The liquid in one of the tubes was displaced and its movement observed when it was released. The movement was clearly a subsidence for the gage as constructed, probably because of viscous damping of the manometric fluid.

From the observed fluctuations of the velocity pressures in the curved tubes it appears, superficially at least, that the largest variations in
velocity are not necessarily periodic but consist of intermittent surges. While such occurrences could cause a considerable error in a standard pitot tube traverse, they seem to cause no appreciable error in the average-velocity gage. That is, the total flow of air through a duct may be assumed to be reasonably constant for given fan speed and delivery conditions. Therefore, an increase in velocity in one portion of the duct cross-section will be accompanied by a corresponding decrease in velocity elsewhere. Such a supposition is borne out by the fact that for a given fan delivery the liquid in the volume-measuring tube remains constant even though the velocity in individual tubes is fluctuating considerably. This fluctuation is logically due to constant total flow conditions which in turn result in a displacement of fluid in one tube being compensated by a rise in one or more of the others. Actually, the behavior of the fluid is very complex and could be evaluated rigorously only by the solution of twenty simultaneous equations. The information obtained for one tube, however, makes a more detailed examination appear unnecessary.

Working models of the average-velocity gage have been constructed and these are now in use in the mine ventilation laboratory at the University of Illinois.
REFERENCES

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