FLEXURAL VIBRATIONS OF PIEZOELECTRIC QUARTZ BARS AND PLATES

BY

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AND
MARION W. WOODRUFF

BULLETIN No. 291
ENGINEERING EXPERIMENT STATION
PUBLISHED BY THE UNIVERSITY OF ILLINOIS, URBANA
PRICE: FORTY CENTS
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THE ENGINEERING EXPERIMENT STATION,
UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS
FLEXURAL VIBRATIONS OF PIEZOELECTRIC QUARTZ BARS AND PLATES

BY

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ENGINEERING EXPERIMENT STATION
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## CONTENTS

### I. INTRODUCTION

1. Applications of Piezoelectrically-Controlled Oscillators .................................................. 5
2. Difficulties Encountered in Stabilizing Low-Frequency Oscillators .................................... 5
3. Object of Investigation ............................................................................................................. 6
4. Acknowledgments ................................................................................................................... 6

### II. FLEXURAL VIBRATIONS OF Y-CUT QUARTZ CRYSTALS.

5. Method of Cutting Quartz Bars ............................................................................................... 6
6. Material Investigated .............................................................................................................. 10
7. Crystal Mountings .................................................................................................................. 11
8. Wavemeters .......................................................................................................................... 11
9. Methods of Frequency Determination .................................................................................... 12
10. Results of Measurements ....................................................................................................... 15
11. Comparison with Theory ........................................................................................................ 18
12. Derivation of Design Formula ............................................................................................... 22
13. Comparison of Calculated and Measured Values of Frequency ............................................. 25
14. Frequency Factors .................................................................................................................. 26
15. Application of Formula to Typical Cases ............................................................................... 27
16. Secondary Frequencies .......................................................................................................... 30

### III. FLEXURAL AND LONGITUDINAL VIBRATIONS OF X-CUT QUARTZ CRYSTALS

17. Material and Results of Measurements .................................................................................. 32
18. Frequency Factors .................................................................................................................. 32

### IV. CONCLUSION

19. Summary and Conclusions ...................................................................................................... 32
LIST OF FIGURES

NO. PAGE
1. Apparatus for Cutting Quartz Bars .................................................. 7
2. Orientation of Main Dimensions of Quartz Bars Cut from a Crystal ........ 8
3. Tool for Grinding Series of Bars to Precise Height .......................... 9
4. Collection of Quartz Bars and Plates ................................................ 10
5. Methods for Determination of Natural Frequencies of Flexural Vibrations of Quartz Bars ................................................................. 12
6. Fundamental Frequencies of Piezoelectric Quartz Bars in Flexural and Length Vibrations ................................................................. 14
7. Curves for Frequency Factors for Flexural Vibrations of Piezoelectric Quartz Bars ................................................................. 23
8. Curves for Frequency Factors for Flexural Vibrations of Piezoelectric Quartz Bars ................................................................. 24

LIST OF TABLES

NO. PAGE
1. Data of Measurements and Calculations for Twenty-Five Y-Cut Quartz Bars and Plates ................................................................. 16
2. Data of Measurements and Calculations for Five X-Cut Quartz Bars ....... 31
FLEXURAL VIBRATIONS OF PIEZOELECTRIC QUARTZ BARS AND PLATES*

I. INTRODUCTION

1. Applications of Piezoelectrically-Controlled Oscillators.—One of the important applications of piezoelectricity consists in the use of vibrating quartz crystals for the stabilization of high frequency oscillators. Nearly all radio broadcasting stations use piezoelectrically-controlled oscillators in order to enable simultaneous radio transmission of a great number of stations and to keep each of them within ± 50 cycles of the allotted frequency channel. The frequency of oscillators thus controlled is so constant that the variations amount to less than one part in a hundred thousand. The range of stabilized frequencies used in broadcasting and relay communication covers 550 to 21,520 kc. There are, however, many applications of low-frequency resonators, filters and oscillators in the range from 1 to 100 kc., whose stabilization would be useful. Very little has been done to extend piezoelectric control for the lower frequencies.

2. Difficulties Encountered in Stabilizing Low-Frequency Oscillators.—The main difficulty consisted in the necessity of using very large crystals. In the usual method of controlling the frequency the interaction of an oscillator with the longitudinal vibration of a piezoelectrically-excited quartz bar is utilized. The stability of frequency is obtained by choosing the length dimension \( b \) of the vibrating crystal as shown in Fig. 2 so that its natural frequency \( f_b \) will be precisely equal to the required constant frequency. The natural frequency \( f_b \) of a vibrating bar is known to depend on the velocity of sound propagation \( v \), so that \( f_b = \frac{v}{2b} \). Therefore, the lower the frequency required, the greater the length of the bar has to be made. The limit is set by the size of crystals obtainable at reasonable cost. If, for instance, the required frequency \( f_b \) should be 10,000 cycles per second, the length would have to be 27 cm., and for \( f_b = 1000 \) cycles, the length would have to be 270 cm. Quartz crystals sufficiently large to make bars of such dimensions are not available.

In order to reduce the dimensions of the bars, Harrison† suggested the use of flexural vibrations. Preliminary experiments disclosed, however, that below 50 kc. the usual circuits did not yield complete piezo-

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*The results of this investigation were presented in a paper before the annual meeting of the Illinois State Academy of Science, May 1, 1936 at Quincy. An abstract was published in the Trans. Illinois State Academy of Science. Vol. 29, 1936, No. 2, pp. 225-227.
electric control, and that the frequency of the oscillator was not sufficiently independent of its circuit constants.

For the development of efficient, low frequency, piezoelectrically-stabilized oscillators, it was first of all necessary to obtain knowledge of the natural frequencies of flexurally-vibrating quartz bars, and how these frequencies depend on the dimensions of the crystal. Besides the investigation of Harrison the only references found in literature* relate to crystals of small height compared to their length. They were used in vacuum tubes for the excitation of glow discharges and served as visible indicators of resonance. An experimental study showed that such crystals are hardly adaptable for sustained powerful oscillations of constant frequency and amplitude as required for power oscillators. The formulae given by the authors proved to be approximations applicable only for a narrow range of crystal dimensions, and could not serve as a basis for the calculation required for the design of oscillators.

3. Object of Investigation.—To find, experimentally, the precise relation between natural frequency and the geometrical dimensions of quartz bars and plates sustained in flexural vibrations by piezoelectric excitation was the object of this investigation. Its further aim was to derive an expression which could be used for calculating the dimensions of quartz resonators in connection with low-frequency oscillators and filters.

4. Acknowledgments.—This investigation has been carried on as part of the work of the Engineering Experiment Station under the general administrative direction of DEAN M. L. ENGER, Director of the Engineering Experiment Station, and PROF. ELLERY B. PAINE, Head of the Electrical Engineering Department of the College of Engineering. Acknowledgment is made of the able service rendered by W. W. BROOKS, Research Graduate Assistant, and to J. A. STEWART, Research Graduate Assistant, for his assistance in the final stage of the investigation.

II. FLEXURAL VIBRATIONS OF Y-CUT QUARTZ CRYSTALS

5. Method of Cutting Quartz Bars.—A Brazilian quartz crystal of unusual size was first sliced into thin parallel slabs. The method of cutting into slices is shown in Fig. 1. The apparatus consisted of a sheet iron cutting wheel $D$, 0.16 cm. ($\frac{1}{16}$ in.) thick and 19.6 cm. (8 in.)

in diameter, which was driven by an electric motor $B$, and rotated at 164 r.p.m. The circumference of the wheel moved with a speed of about 0.5 m. per sec. through a trough $C$ filled with a mixture of finely granulated No. 50 carborundum and water. A small amount of powdered soap was added to increase the cutting efficiency.

The original crystal stock $A$, which measured 18 cm. in the direction of the optical axis and 14 cm. in the direction of one of the electrical axes, was cemented to a wooden base $E$. The latter was then clamped near one end of a flat iron lever $F$, which was attached by means of a hinge $G$ to the frame $H$ of the apparatus. The other end of the lever carried a weight $I$, movable along a rod $J$. By shifting this weight the pressure of the crystal against the cutting wheel could be adjusted so that the force exerted was about 6 kg.

Slabs of quartz $S$ (see Fig. 2) were thus obtained which required more or less time to cut, depending upon their area. Their dimensions $a$, $b$, and $t$ were respectively 5 to 15 cm. in the direction of the optical O-Z axis, about 14 cm. in the direction of the electrical O-X axis, and 0.2 to 0.3 cm. in the direction of the mechanical O-Y axis.

The rough slabs $S$ were then trued up on an iron lapping wheel $K$,
which could be driven by the same motor $B$ (see Fig. 1). When all the rough cutting marks made by the iron wheel were thus ground out, the slab was placed on glass plates and ground flat on one face by using finer and finer grades (from No. 200 to No. 500) of carborundum and a succession of grinding glass plates. In order to obtain glass plates of
satisfactory flatness, they were ground one against the other. A succession of plates was used also in this process. Whenever one side of the crystal slab showed a uniformly cut surface after the application of a very fine grade of abrasive (No. 500), it was judged to be flat. The other side was then repeatedly ground and checked by a micrometer caliper until it proved to be parallel to the first side.

From such slabs of uniform thickness bars $B$ (see Fig. 2) were cut by means of a hacksaw frame into which had been placed a smooth copper strip. The quartz slab was cemented to a metal backing. Guide pins fixed in a supporting base served to keep the copper strip aligned during the cutting operation. After this operation the edges of the bars were rough and in most cases they were not exactly parallel, nor were the angles exactly square. In order to avoid the lengthy work of finishing each of a series of bars individually to a precise height and symmetry, a specially-machined iron tool was used, as shown in Fig. 3. It consisted of an iron block, into which a groove was carefully machined to be parallel and square with the ends. The width $W$ of the groove was sufficient to accommodate six quartz bars $q$. For each required height of the quartz bars, a pair of such tools was made, of which one was slightly different in the dimension $H$ from the other by ten thousandths of an inch. This allowed the use of the tool with the deeper groove for the finishing of one set of edges and then, by removing the bars and placing them into the shallower trough of the other tool, with the finished side in the bottom of the groove, the other edges could be finished. A number of such pairs of tools, of various dimensions, were prepared to serve for the finishing of bars of different lengths. In Fig. 1 three of such tools $L$ are shown placed on the shelf below the lapping wheel $K$. The left tool could be used for
both of the operations. It was provided with two grooves of different depth, one cut into the upper, the other into the lower face of the block. By allowing the ends of the quartz bars to protrude beyond the squared end of the tool, the ends of the bars could be ground to become parallel and squared. The grinding action of the carborundum and water on a glass plate had little effect on the tool, as the iron member, loaded with carborundum, ground the glass, but did not change its dimension or shape. In this way all of the crystals were finished more quickly and more exactly than could have been done singly, without the use of the tool. Even in connection with single crystals this method of finishing was found advantageous.

6. Material Investigated.—The experimental study started with five bars. In the course of the investigation the number of quartz bars and plates increased gradually. In order to verify certain conclusions for as large a range of dimensions as possible more crystals of intermediate sizes were found necessary. Finally, a collection of twenty-five Y-cut and five X-cut bars and plates was available. Twenty-one of them were photographed (see Fig. 4) together with a large quartz crystal similar to one from which most of the specimens originated. The Y-cut crystals were classified into four groups, each including crystals of various lengths, but of constant height. The height (a in
Fig. 2), coincident with the optical axis, was, for the four groups, about 0.75, 1.00, 1.5, and 1.8 cm., respectively. The thickness (t in Fig. 2), coincident with the mechanical axis, varied from 0.14 to 0.27 cm. It was chosen approximately the same for each of the groups, with the exception of the second group, which contained crystals of two thicknesses. The length (b in Fig. 2), coincident with the electrical axis, varied from 0.7 to 13 cm. Among the specimens there were five in the collection (Nos. 1, 2, 3, 4, and 5) which were cut from another quartz rock, and differed from the rest in appearance because they were highly polished. All other bars were cut from one single quartz rock. Their surface had a dull and smooth appearance produced by grinding with the finest abrasive. Some of them were etched with hydrofluoric acid.

7. Crystal Mountings.—A number of four-electrode crystal holders mounted on supports made of bakelite were used, each adaptable for the particular range of bar dimensions. The space between the electrodes was adjusted to be about 0.02 cm. larger than the thickness of the quartz bar investigated, so that the latter could freely slide on the bakelite base within the electrodes. For the purpose of stabilizing the vibrations each bar had a loop of silk thread tightened around its nodal plane. In Fig. 4 these supporting silk loops appear as two white lines on each of the quartz bars. The distance of the two nodal planes from the nearest end faces of the bars was, for the fundamental mode of flexural vibrations, \( d = 0.224b \), \( b \) denoting the total length of the bar. This distance \( d \) for each of the bars investigated was included in column 6 of Table 1.

8. Wavemeters.—For radio frequency ranges, precision wavemeters of the General Radio Company’s Type 224 were used. For lower frequencies from 2 to 50 kc. similar tuning circuits were applied, which consisted of variable air condensers and inductance coils coupled to aperiodic circuits. The latter contained copper oxide rectifying units connected with d-c. indicating instruments.

Low-frequency wavemeters applied for flexural vibrations were calibrated by absolute measurements. For this purpose an electromagnetically driven tuning fork, whose natural frequency was 992 cycles per second, was used in connection with a two-stage amplifier to drive a synchronous clock. The latter indicated its number of revolutions within a given interval of time, checked by the standard time as broadcasted by the Arlington Radio Station. The output of the first amplifier was also fed into another one-stage amplifier whose grid excitation was made abnormally large for the purpose of producing
harmonics of the fundamental frequency of the tuning fork. These harmonics supported by the tuned output circuit of the amplifier were used for the calibration of the wavemeter. For frequencies above 10 kc., more stages of amplification were used, each stage being in connection with a tuned circuit of a higher range of harmonic frequencies.

9. Methods of Frequency Determination.—Two methods were used for the determination of natural frequencies of bars and plates vibrating flexurally.

For the frequencies above 15 kc., the first method represented diagrammatically in Fig. 5a was used. An oscillator circuit a whose frequency was variable by means of a condenser $C_1$ was driven by a thermionic tube. The inductance $L_1$ served to couple $a$ with circuit $b$. The latter consisted of a coupling inductance $L_2$, the quartz bar $q$
mounted in a four-electrode holder, and a copper oxide rectifier output meter \( d \). A wavemeter \( w \) was loosely coupled with circuit \( b \). The quartz bar acted as an electromechanical resonator, which absorbed energy from the circuit \( a \) at a rate which depended on how close its frequency (varied by varying the capacitance \( C_1 \)) approached the fixed natural frequency of the vibrating bar \( q \). The polarity of the four electrodes placed parallel to the electrical axis \( X \) of the bar was such that the direction of the electric field across the upper pair of electrodes was opposite to the field direction in the lower pair of electrodes. Due to the converse effect the piezoelectrically-stressed quartz bar was thus subjected to compression along the upper part whenever tension was produced along the lower part, and vice versa, in accordance with the phase of alternating potentials induced by the oscillator at the two pairs of electrodes. Under these conditions the bar \( q \) vibrated flexurally in the plane determined by the electrical axis \( X \), and the optical axis \( Z \). At resonance the bar with its circuit \( b \) responded vigorously, and the output meter indicated accordingly a sharply-defined pulse \( p \). Thus, by tuning with the condenser \( C_1 \), its resonance setting-point could be found, and the frequency determined by adjusting the variable condenser \( C_2 \) of the wavemeter circuit \( w \). The reaction of the latter upon the crystal circuit \( b \) influenced the current through the output meter, which served also as resonance indicator for the wavemeter. Whenever greater sensitiveness was required, an aperiodic circuit with a current indicator in series was coupled to the wavemeter.

For the determination of frequencies of flexurally-vibrating bars, whose natural frequency was below 15 kc., the second method (represented in Fig. 5b) proved more sensitive. A similar oscillating circuit \( a \) was used for the excitation of the circuit \( b \). In the latter, however, a balancing bridge \( e \) was inserted. The resistance \( R' \) and the quartz bar \( q \) with its four-electrode holder formed the piezoelectric branch of the bridge, and the resistance \( R'' \) with a variable capacitance \( C'' \) formed the compensating branch of the bridge. The purpose of the bridge was to balance at the points \( g \) and \( h \) that part of the potential across the quartz crystal \( q \) which was independent of the piezoelectric reaction, and which was caused by the excitation of the circuit \( b \) at the points \( i \) and \( j \) by the oscillator \( a \). The bridge could be balanced roughly by adjusting the condenser \( C_1 \) of the circuit \( a \) so that it was off by about 1 per cent from the resonance frequency \( f_0 \) of the bar \( q \). Then a finer adjustment was made by varying the condenser \( C'' \) until the input of the three-stage amplifier \( f \) was not energized at that frequency. The balance was regarded as established with sufficient accuracy when the
output meter indicated less than 0.5 volt. Under these conditions precise frequency determinations could be made by adjusting the condenser $C_1$ for a setting at which a sudden increase to several volts in the deflection of the output meter $d$ was observed.

Besides the fundamental flexural vibrations, the two modes of
longitudinal length and thickness vibrations of the bars were also investigated.

For the determination of the natural frequencies of longitudinal vibrations the quartz bars were placed in two-electrode holders; otherwise the method resembled that shown in Fig. 5a. Whenever the inductive coupling of the circuits \( a \) and \( b \) was not sufficient for the excitation of the bars, the coil \( L_2 \) was removed and the circuit \( b \) connected directly to the coil \( L_1 \). For the detection of frequencies other than fundamental it was found necessary to insert (see Fig. 5a) in the circuit \( b \) an amplifier in place of the meter \( d \), and to connect the latter to the output terminals of the amplifier.

10. Results of Measurements.—When the measured fundamental frequencies were plotted against the length \( b \) of the quartz bars, the curves, I, II, IV, and L, shown in Fig. 6, were obtained. Curves I, II, and IV represent the natural frequencies \( f_p \) of the flexural mode of vibrations of three groups of bars which differed from each other by the magnitude of the \( a \) dimension. All bars of group I had a height of 0.75 cm., the bars of group II had a height of 1.06 cm., and the bars of group III a height of 1.8 cm. The respective number marked on each bar is indicated on the curves.

In contrast to the flexural vibration, the longitudinal vibrations in the direction of the length of these three groups of bars gave frequencies \( f_b \) which could be represented by a single curve L. The results of measurements confirmed the known relation that the natural frequency of bars vibrating longitudinally in the length direction is independent of the height \( a \) of the bar, and is a linear function of the length \( b \) of the bar, namely,

\[
f_b = \frac{A_b}{b} = \frac{1}{2b} \sqrt{\frac{E_1}{\rho}}
\]

where the average frequency constant, \( A_b \), was found to be \( 2.7 \times 10^5 \) per cm. \(^{-1} \). Also, for the frequency \( f_t \) of the longitudinal thickness vibration, the measurements were in accord with the relation

\[
f_t = \frac{A_t}{l}
\]

and gave an average value for \( A_t = 1.96 \times 10^5 \) per cm. \(^{-1} \).

As to the flexural vibrations of the bars, it was found that their measured natural frequencies \( f_f \) did not correspond to the known relation

\[
f_f = A_f \frac{a}{b^2}
\]
Frequencies calculated according to this relation gave values from 2 to 75 per cent larger than the measured ones. The frequency factor $A_f$ was found to be a function of the ratio $a/b$ of the bar. For crystals whose lengths were over ten times larger than their heights the measured fundamental frequency was in close agreement with this relation. For crystals with a ratio $a/b = 0.5$ the discrepancy amounted to 40 per cent, and for a square plate, $a/b = 1$, the discrepancy reached 60 per cent.
TABLE 1.—Concluded
DATA OF MEASUREMENTS AND CALCULATIONS FOR TWENTY-FIVE Y-CUT QUARTZ BARS AND PLATES

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<th>$f_p$</th>
<th>$B/A$</th>
<th>$A/B$</th>
<th>$\sqrt{1 + \frac{222}{a^2} \times 10^9}$</th>
<th>$\sqrt{1 + \frac{222}{a^2} \times 10^9}$</th>
<th>Calculated Frequency $f_p$</th>
<th>Calculated Frequency from Equation (25)</th>
<th>Frequency Factor</th>
<th>Deviation of $f_p$ from $f_p$</th>
<th>Deviation in Per Cent</th>
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The results of measurements and calculations, systematically arranged, are included in Table 1. The table consists of three main groups of columns: dimensions (columns 2-6), longitudinal vibrations (columns 7-11), and flexural vibrations (columns 12-23). The specimens investigated are divided into four groups, I, II, III, IV. Each group contains quartz bars and plates of a definite height $a$.

Column 1 indicates the numbers by which the specimens were marked in the consecutive order in which they were cut.
Columns 2, 3, and 4 give in centimeters the bar dimensions of thickness \(t\), length \(b\), and height \(a\), respectively.

Column 5 gives the ratio of height to length, \(a/b\).

Column 6 indicates the distance, \(d = 0.224b\), from both ends of the bars, at which nodal zones were formed when the bars vibrated flexurally.

Column 7 enumerates the measured natural frequencies \(f_t\) of thickness vibrations.

Column 8 shows the frequency constant for thickness vibrations, \(A_t = f_t t\), calculated from the data of columns 2 and 7.

Column 9 enumerates the measured natural frequencies \(f_b\) of the length vibrations.

Column 10 gives the period, \(T_b = 1/f_b\), calculated from the data in column 9.

Column 11 gives the frequency factor, \(A_b = f_b t\), for length vibrations calculated from the data in columns 3 and 9.

Column 12 enumerates the measured natural frequency \(f'_p\) of the flexural vibrations.

The remaining columns will be discussed in Sections 10 and 13.

11. Comparison with Theory.—The theory of lateral vibrations of bars, as given by Rayleigh for isotropic material, in its simplest form leads to a differential equation of the fourth order

\[
\frac{\partial^2 y}{\partial t^2} + \frac{\kappa^2 E}{\rho} \frac{\partial^4 y}{\partial x^4} = 0
\]

where \(y^*\) is the transversal displacement of the bar at any point \(x\) along the length axis, \(\kappa\) the radius of gyration of the bar’s cross section, \(E\) Young’s modulus of elasticity, and \(\rho\) the density of the bar’s material. The ratio \(E/\rho\) is the square of the velocity of propagation of longitudinal vibrations. For a bar free at both ends with the boundary conditions \(\frac{\partial^2 y}{\partial x^2} = 0\) \(\frac{\partial^3 y}{\partial x^3} = 0\) Equation (4) yields the very well known relation for the natural frequencies of bars

\[
f_p = \frac{m^2}{2\pi} \frac{\kappa}{b^2} \sqrt{\frac{E}{\rho}}
\]

in which \(b\) is the length of the bar in centimeters, and \(m\) depends on the mode of vibrations, \(K = 1, 2, 3, \ldots\) so that

\[
m = (K + 1/2)\pi
\]

*In the particular case of the Y-cut quartz bars investigated the displacement \(y\) is directed along the optical axis marked O-Z in Fig. 2. The dimension of the bar in this direction is denoted \(a\).*
For the fundamental vibration of a bar with an area $ab$ of rectangular cross section, $K = 1$, $m^2 = 22.37$ and $\kappa = a/\sqrt{12}$, the theoretical frequency of flexural vibrations according to Equation (5) assumes a form similar to that of Equation (3), namely,

$$f = 1.028 \sqrt{\frac{E_1}{\rho}} \frac{a}{b^2}$$  \hspace{1cm} (7)

The values calculated from this formula do not agree at all with the measured values obtained from the experimental study of crystals, as enumerated in column 12 of Table 1. As mentioned in Section 10, the deviations become considerable for $a/b > 0.1$, and reach 75 percent for a crystal plate with $a/b = 1.5$.

The differential equation which served as a basis for the derivation of Equation (7) evidently does not take into account all the physical phenomena which enter into the process of flexural vibrations of quartz bars. It was assumed that the displacements of each cross section along a vibrating bar are strictly normal to the X-Y plane (see Fig. 2). Actually, however, rotational to and fro motions of the cross sections about axes perpendicular to the plane of vibration, O-X, had to be taken into consideration, in addition to the motion of translation. A corresponding term

$$-\kappa^2 \frac{\partial^4 y}{\partial x^2 \partial t^2}$$

added by Rayleigh on the left side of Equation (4) produced a more complete differential equation.

$$\frac{\partial^2 y}{\partial t^2} + \frac{\kappa^2 E}{\rho} \frac{\partial^4 y}{\partial x^4} - \frac{\kappa^2 \partial^4 y}{\partial x^2 \partial t^2} = 0$$  \hspace{1cm} (8)

Rayleigh* showed, for the case of a bar free at one end, how the introduction of rotational energy influences the expression (7) for the frequency. The required correction for a bar free at both ends, as given by Goens,† consists of a factor by which Young's modulus of elasticity $E_1$ must be decreased, namely,

$$E_2 = \frac{E_1}{1 + \sigma(m) \frac{\kappa^2}{b^2}}$$  \hspace{1cm} (9)

The physical meaning of this correction has to do with the realization that the assumed velocity of propagation, $v_1 = \sqrt{\frac{E_1}{\rho}}$, which is

true for pure longitudinal waves, applies only for infinitely thin bars. With increasing cross sectional dimensions, a different velocity of propagation, namely, that of transversal waves comes more and more into play. The resultant velocity is then

$$v_2 = \sqrt{\frac{E_2}{\rho}} = \sqrt{\frac{E_1}{\frac{1 + \sigma(m)\frac{\kappa^2}{b^2}}{1 + \frac{a^2}{12}}}}$$

(10)

and depends on the frequency of vibrations, because the function $\sigma(m)$ depends upon the mode of vibration. For the fundamental flexural vibration, $\sigma(m) = 49.48$. By inserting this value for $\sigma(m)$ and $a^2/12$ for $\kappa^2$ in Equation (9), the modulus of elasticity becomes

$$E_2 = \frac{E_1}{1 + 4.12(a/b)^2}$$

(11)

When this correction was used in connection with Equation (7) for the calculation of the frequency of the quartz bars the disagreement with measured values decreased, but did not disappear altogether. Especially for the larger values of $a/b$, discrepancies of about $+7$ to $+11$ per cent resulted in spite of Rayleigh's correction.

A notable contribution to the theory of flexural vibrations was made by Timoshenko,* who extended still further the differential equation, by considering the effect of shearing forces in addition to the translatory and rotary motions of the cross-sectional elements. Correspondingly, he inserted two more terms with the modulus of shearing $G$ and a coefficient $\xi$ to account for the non-uniform distribution of shearing forces throughout each cross section of the vibrating bar. He derived the following differential equation:

$$\frac{\partial^2 y}{\partial t^2} + \frac{\kappa^2 E}{\rho} \frac{\partial^4 y}{\partial x^4} - \kappa^2 \left(1 + \frac{E}{G} \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho \xi \kappa^2}{G} \frac{\partial^4 y}{\partial t^4} \right) = 0$$

(12)

A complete solution of this equation was given by Goens.† However, its application for the calculation of natural frequencies was found to be very involved. Therefore Goens' simplified solution, in form of a correction factor for the modulus of elasticity,

$$E_3 = \frac{E_1}{1 + 2m \varphi(m) \frac{\kappa^2}{b^2} \left(3 - \frac{E_1}{G} \right) + m^2 \varphi^2(m) \frac{\kappa^2}{b^2} \left(1 + \frac{\xi E_1}{G} \right)}$$

(13)

†See footnote p. 10.
was applied for checking the results of frequency measurements made on 25 bars and plates enumerated in column 12 of Table 1.

For the fundamental mode of flexural vibrations of a bar of rectangular cross section, the specific values of $m$, $\varphi(m)$, $\kappa$, and $\xi$ are as follows:

\begin{align*}
  m &= 4.73 & \varphi(m) &= 0.9825 & \kappa^2 &= a^2/12 \\
  m^2 &= 22.373 & \varphi^2(m) &= 0.9653 & \xi &= 1.2
\end{align*}

As to the ratio $E_1/G$, reliable data for crystalline quartz are not available, because it depends on the Poisson's ratio $\lambda$, which is too small for dependable measurements. Considering that in the known relation

$$E/G = 2(1 + \lambda)$$

$\lambda$ is practically equal to zero for quartz, the value 2 for the ratio $E_1/G$ was used. When these specific values were inserted in Equation (13), the modified modulus

$$E_3 = \frac{E_1}{1 + 6.55(a/b)^2}$$

was obtained. The frequency of the quartz bars was then checked by Equation (7) in which $E_3$ was substituted for $E_1$.

The deviations of values thus calculated from measured values of fundamental frequencies were in most cases negative, and indicated that the values calculated according to the Timoshenko-Goens correction were too small, while the positive deviations of the corresponding Rayleigh correction gave values which were too large compared with measured frequencies.

The question suggested itself whether the deviations might have been caused by assuming, incorrectly, the Poisson ratio for quartz to be $\lambda = 0$. Inspection of Equation (13) showed, however, that for $\lambda > 0$ the deviations become still larger. By assuming $\lambda < 0$ the deviations become smaller, and approach zero for $\lambda = -0.54$. Such a large negative value for $\lambda$ is contrary to what is known of the mechanical properties of quartz.

It may be inferred that the Timoshenko differential equation applies only to pure mechanical vibrations in isotropic materials, while in piezoelectric vibrations of quartz bars additional electro-mechanical phenomena have to be considered. Dielectric stresses, caused by the electric potential gradient set up in quartz by the voltage applied to the electrodes, and also the distribution of space charges within the materials, may require additional terms.
The derivation of a differential equation which could take into account the electrostrictive phenomena within a lattice of a flexurally vibrating crystalline quartz bar is a problem in itself.*

12. Derivation of Design Formula.—For purposes of designing low-frequency resonators for self-sustained stabilized oscillators, it was found sufficient to derive design formulae on the basis of the experimental data obtained from measurements on the 25 quartz bars enumerated in Table 1. Search in literature has revealed two empirical formulae; one was given by Harrison† and the other by Giebe and Scheibe.‡ But natural frequencies calculated on the basis of these formulae gave discrepancies even larger than those obtained by applying Rayleigh’s (Equation 11) and Timoshenko-Goens’ (Equation 15) corrections. For example, a bar with a comparatively small ratio \( a/b = 0.209 \) (bar No. 5 in Table 1), when calculated according to the Harrison or Giebe-Scheibe formula, gave discrepancies from the measured values amounting to, respectively, +28.8 per cent and −3.18 per cent. The deviations assumed rapidly increasing values with increasing ratios, \( a/b \). For \( a/b = 0.376 \) (bar No. 1 in Table 1), the corresponding values reached +75.2 per cent and −10.4 per cent.

It was therefore necessary to obtain new formulae.

The procedure in deriving the frequency formula was as follows:

The distribution of the deviations indicated that the general character of Equations (11) and (15) might not be at fault. It was therefore assumed that these equations hold for piezoelectric quartz, and that only the elastic modulus and the numerical value in the term connected with \( a/b \) (marked in what follows by \( D \)) is influenced by a factor as yet unknown. Denoting in Equation (7)

\[
1.028 \sqrt{E_1/\rho} = A
\]  

and in Equations (11) and (15)

\[
1 + D(a/b)^2 = B^2
\]  

the fundamental natural frequency of a flexurally-vibrating quartz bar may be expressed in the form

\[
f'_F = \frac{A}{B} \frac{a/b}{b}
\]  

*Further contribution to the theory of flexural vibration of bars was made by W. P. Mason in the Journal of the Acoustical Society of America, Vol. VI, p. 246, 1933. This article came to our attention when the present publication was in press.

†See footnote p. 5.

‡See footnote p. 6.
From the frequency $f'_p$ and the geometrical dimensions $b$ and $a$ of 25 bars which were determined by precise measurements (columns 3, 4, and 12 in Table 1) the ratio

$$\frac{A}{B} = \frac{f'_p b}{a/b}$$

was calculated, and the results plotted against $a/b$. In Fig. 7, the curve V is shown for values of $a/b$ limited to the useful range 0 to 0.4 so as to avoid crowding of the points marked for each bar. Curve VI in Fig. 8 gives the $A/B$ ratios for the complete range of $a/b$ from 0 to 1.4 investigated.

Considering that for $a/b = 0$ the frequency factors will be $B = 1$ and $A/B = A$, it follows that the numerical value of the factor $A$ may be obtained by extrapolating the full-line curve V until it intersects the $A/B$ axis. It was thus determined that $A = 5.52 \times 10^5$. Due to the uncertainty connected with extrapolation, the numerical value of $A$ was checked experimentally from the data of measurements of natural frequency $f_b$ and the frequency factor $A_b$ of the longitudinally-vibrating bars (columns 9 and 11 of Table 1). From Equations (1) and (16)
FIG. 8. CURVES FOR FREQUENCY FACTORS FOR FLEXURAL VIBRATIONS OF PIEZOELECTRIC QUARTZ BARS

Curve No. III,

\[ B = \sqrt{1 + 5.22 \left( \frac{b}{d} \right)^2} \]

Curve No. IV,

\[ \frac{A}{B} = \frac{5.52 \times 10^{-5}}{\sqrt{1 + 5.22 \left( \frac{b}{d} \right)^2}} \]
the ratio between the frequency factor \( A \) for flexural vibration and the factor \( A_b \) for longitudinal length vibration was

\[
\frac{A}{A_b} = \frac{1.028\sqrt{\frac{E_i}{\rho}}}{1/2\sqrt{\frac{E_i}{\rho}}}, \text{ or } A = 2.056A_b
\]  \hspace{1cm} (20)

With the average values of \( A_b = 2.7 \times 10^5 \), the factor \( A = 5.55 \times 10^5 \) was obtained. From the two values for \( A \), the former was chosen because it proved to give closer agreement when used for the calculation of flexurally-vibrating bars. This constant factor \( A \) is indicated in Fig. 7 by a line drawn parallel to the \( a/b \) axis for the ordinate \( A/B = 5.52 \times 10^5 \).

In order to obtain the frequency factor \( B \) as a function of \( a/b \), it was only necessary to divide for each bar the value of \( A \) by the corresponding value of \( A/B \) (column 15, Table 1). The result was curve VII in Fig. 8. That the mathematical expression of the curve was in accordance with Equation (17), where

\[
B = \sqrt{1 + \frac{D(a/b)^2}{(a/b)^2}}
\]

was verified by checking the numerical value of the coefficient

\[
D = \frac{B^2 - 1}{(a/b)^2}
\]  \hspace{1cm} (21)

for each of the specimens from the experimental values of \( B \) and \( a/b \). This coefficient was found to be a constant for all values of \( a/b \), and the corresponding \( B \) values represented by the curve VII in Fig. 8. Its average value was \( D = 5.22 \). Accordingly, the expression for the frequency factor \( B \) was obtained

\[
B = \sqrt{1 + 5.22(a/b)^2}
\]  \hspace{1cm} (22)

In column 17 of Table 1 are given the values of \( B \), calculated from Equation (22) for each of the bars investigated. With two numerical factors \( A \) and \( D \) determined, the complete expression for the fundamental natural frequency of flexurally-vibrating piezoelectric quartz bars and plates follows from Equations (18), (20), and (22)

\[
f'_p = \frac{5.52 \times 10^5}{\sqrt{1 + 5.22(a/b)^2}} \frac{a/b}{b}
\]  \hspace{1cm} (23)

13. Comparison of Calculated and Measured Values of Frequency.

—By calculating the ratio

\[
\frac{A}{B} = \frac{5.52 \times 10^5}{\sqrt{1 + 5.22(a/b)^2}}
\]  \hspace{1cm} (24)
and plotting against values of $a/b$ for each bar, the broken-line curve, VIII in Fig. 7, and the curve VI (column 16, Table 1) in Fig. 8 were obtained. Both show the relatively good agreement with experimental values indicated by circles. The values of frequencies $f'$ calculated from Equation (23) and tabulated in column 18, along with the measured values of $f'$ shown in column 12, may serve for a quantitative comparison of discrepancies. The deviations $\delta$ (in per cent) are indicated in column 20, Table 1. Of the 25 crystals investigated,

- 3 bars showed deviations from calculated values within 0 to 0.25 per cent
- 5 bars showed deviations from calculated values within 0.25 to 0.5 per cent
- 10 bars showed deviations from calculated values within 0.5 to 1.0 per cent
- 3 bars showed deviations from calculated values within 1.0 to 1.5 per cent
- 2 plates showed deviations from calculated values within 1.5 to 2.5 per cent
- 2 plates showed deviations from calculated values above 2.0 per cent

The classification of the crystals into bars and plates was made on the basis of the ratio $a/b$ of height to length in the plane of vibration. Crystals with a ratio $a/b < 0.45$ to 0.5 were designated as bars; those with a ratio $a/b > 0.45$ to 0.5 as plates. The deviations were partly positive, partly negative.

For 9 bars, calculated values were larger than measured by $+0.36$ to 1.33 per cent
For 13 bars, calculated values were smaller than measured by $-0.22$ to 1.92 per cent
For 1 bar, the calculated value was equal to the measured value.

Generally the deviations were slight for bars with dimensions of practical significance. They may be ascribed to threefold sources: First, errors in observation connected with methods of measurements of frequency and length; second, the lack of homogeneity of quartz rock, which is known to show variation not only from one crystal specimen to another, but within different parts of the same specimen; and third, the limitations of Equation (23), which, although derived from theoretical considerations in conjunction with experimental data, cannot be regarded as representing rigorously all phenomena involved in sustained piezoelectric oscillations of quartz bars and, especially, of plates.

14. Frequency Factors.—For the purpose of designing piezoelectric quartz resonators it may be stated that, as the result of measurements (see Table 1), the following average frequency factors were established for Y-cut quartz bars:

For flexural vibrations, \[ \frac{A}{B} = \frac{5.52 \times 10^6}{\sqrt{1 + 5.22(a/b)^2}} \]
For longitudinal length vibrations, \[ A_l = 2.70 \times 10^5 \]
For longitudinal thickness vibrations, \[ A_t = 1.96 \times 10^5 \]
15. Application of Formula to Typical Cases.—

(a) It may be inferred from the smallness of the deviations $\delta$, tabulated in column 20, Table 1, that, for all practical purposes, Equation (23) may safely be used for the calculation of frequency $f_F$ from given dimensions of bars as well as of plates, whose ratio $a/b$ is smaller than 0.75.

(b) For the case when the length $b$ and frequency $f_F$ are given, and it is required to calculate the height $a$ of the crystal, it follows directly from Equation (23) that

$$a = \sqrt{\frac{bf_F^2}{A^2 - Dbf_F^2}} = \sqrt{\frac{bf_F^2}{30.47 \times 10^{10} - 5.22b^2f_F^2}}$$

(c) For the case when the height $a$ and frequency $f_F$ are given, and it is required to calculate the length $b$ of the crystal, it follows similarly

$$b^2 = \sqrt{\frac{D^2a^4 + 4A^2a^2}{f_F^2} - \frac{Da^2}{2}} = \sqrt{6.81a^4 + 30.47 \times 10^{10}(a/f_F)^2 - 2.61a^2}$$

(d) In the course of this investigation, it was often necessary to calculate the dimensions $a$ and $b$ for a given ratio $a/b$ and frequency $f_F$. The following relations were applied for this purpose:

$$a = \frac{A}{B} \frac{(a/b)^2}{f_F} = \frac{5.52 \times 10^5}{\sqrt{1 + 5.22(a/b)^2}} \cdot \frac{(a/b)^2}{f_F}$$  

or

$$b = \frac{A}{B} \frac{a/b}{f_F} = \frac{5.52 \times 10^5}{\sqrt{1 + 5.22(a/b)^2}} \cdot \frac{a/b}{f_F}$$

Numerical calculations were considerably simplified by the use of curves plotted in Figs. 7 and 8. The curves, VII, VIII, and IX, give the values of $B$, $A/B$, and $A/B \cdot a/b = bf_F$, respectively, for any value of $a/b$. For instance, instead of calculating $b$ from Equation (28), the value of $bf_F$ is looked up on curve IX for a given $a/b$ value and divided by the given frequency $f_F$ to obtain $b$.

(e) Another problem presented itself when it was necessary to cut bars of different dimensions, but of equal prescribed natural frequency.
Two possible cases were investigated. The first may be formulated as follows:

If the length \( b \) of a bar be changed by multiplying by \( n \), what must be the factor of multiplication \( m \) for the height \( a \), in order that the frequency \( f_F \) may remain unchanged?

The principle of similitude requires that the dimensions \( a \) and \( b \) of one bar and the dimensions \( ma \) and \( nb \), of the other bar of equal frequency \( f_F \), be determined by an equation similar to Equation (23), namely,

\[
f_F = \frac{A}{\sqrt{1 + D(a/b)^2}} \frac{a/b}{b} = \frac{A}{\sqrt{1 + D(m/n)^2(a/b)^2}} \frac{m/n \cdot a/b}{nb}
\]

Solving for \( m \), the factor for the height \( a \) is obtained

\[
m = \frac{n^2}{\sqrt{1 - D(a/b)^2(n^2 - 1)}}
\]

(f) Similarly, the other case may be thus formulated: If the height \( a \) of a bar be changed by multiplying by a factor \( m \), what must be the factor \( n \) for the length \( b \) in order that the frequency \( f_F \) may remain unchanged? The answer is obtained by solving for \( n \), Equation (29)

\[
n^2 = -\frac{D}{2} \left( \frac{ma}{b} \right)^2 + m \sqrt{D \left( \frac{a}{b} \right)^2 + \left[ \frac{D}{4} \left( \frac{ma}{b} \right)^2 + 1 \right] + 1}
\]

(g) The process of cutting and grinding quartz bars of precise dimensions to produce the prescribed frequency involves accurate and time-consuming work. As a result of too much grinding, the needed dimensions may be slightly overstepped, and thus a costly specimen may be made useless for the designed purpose. It was, therefore, desirable to obtain formulae for the relative variation of frequency with varying dimensions, and vice versa. The following problem presented itself:

A bar has a natural fundamental frequency \( f_F \), its dimensions \( a \) and \( b \) may be varied by amounts expressed by the factors \( q \) and \( r \), respectively. What will be the relative factor \( p \) by which the natural frequency will change?

The changed frequency is

\[
pf_F = \frac{A}{\sqrt{1 + D(q/r)^2(a/b)^2}} \frac{q/r \cdot a/b}{rb}
\]
By substituting for \( f_p \) the expression from Equation (18) and solving for \( p \), the following result is obtained:

\[
p = \sqrt{\frac{1 + D(a/b)^2}{1 + D(q/r)^2(a/b)^2}} \cdot \frac{q/r}{r} \quad (33)
\]

In case only \( a \) is to be changed, \( r = 1 \) and

\[
p = q \sqrt{\frac{1 + D(a/b)^2}{1 + D(qa/b)^2}} \quad (33a)
\]

Similarly, if only \( b \) is to be changed, \( q = 1 \) and

\[
p = \frac{1}{r^2} \sqrt{\frac{1 + D(a/b)}{1 + D(a/rb)^2}} \quad (33b)
\]

(h) During the final lapping of bars for a prescribed frequency, repeated measurements of frequency must be made. In order to reduce the number of such tests to a minimum, it is useful to calculate by what amount both or either of the two dimensions \( a \) and \( b \) must be changed in order to reduce the frequency just measured to the required one. The problem may be thus formulated: Given a bar whose natural frequency is \( f_p \) and whose height and length dimensions are \( a \) and \( b \), respectively; it is required that its frequency be changed by a given factor \( p \) and its length changed also by a given factor \( r \). What must be the factor \( q \) by which the height \( a \) must be changed?

The answer is obtained by solving Equation (32) for \( q \); that is

\[
q = \frac{pr^2}{\sqrt{1 + D(a/b)^2(1 - p^2r^2)}} \quad (34)
\]

In case no change is required for \( b \), \( r = 1 \) is to be substituted in Equation (34),

\[
q = \frac{p}{\sqrt{1 + D(a/b)^2(1 - p^2)}} \quad (34a)
\]

(i) Finally, a similar case was considered, when the natural frequency of a bar had to be changed by a given factor \( p \), and also the height \( a \) by a given factor \( q \). It is required to calculate the factor \( r \) by which the length \( b \) must be changed.

By solving Equation (32) for \( r \), the following expression is obtained:

\[
r^2 = -\frac{D}{2} q^2 \left(\frac{a}{b}\right)^2 + q \sqrt{\frac{D^2}{4} \left(\frac{a}{b}\right)^4 p^2 q^2 + \left[1 + D \left(\frac{a}{b}\right)^2 \right]} \quad (35)
\]
For the special case where the height is to remain unchanged, \( q = 1 \) is to be substituted in Equation (35) so that

\[
  r^2 = -\frac{D}{2} \left( \frac{a}{b} \right)^2 + \frac{1}{p} \sqrt{\frac{D^2}{4} \left( \frac{a}{b} \right)^4 b^2 + \left[ 1 + D \left( \frac{a}{b} \right)^2 \right]} \quad (35a)
\]

Numerical calculations may be simplified by remembering that

\[
  1 + D \left( \frac{a}{b} \right)^2 = B^2 \quad (\text{see Equations (17) and (22)})
\]

where \( B \) is a factor given by curve VII in Fig. 8.

16. Secondary Frequencies.—Of the 25 samples investigated, 10 bars showed additional natural frequencies which were observed by merely varying the frequency of the excitation circuit without changing the four-electrode arrangement of the crystal holder, or modifying the diagram of connection of the resonator circuit. These frequencies, denoted by \( f''_p \), are indicated in column 21 of Table 1, and their percentage deviation from the normal flexural vibration is listed in column 22. The relative intensity of these vibrations was much smaller than that of the normal flexural vibration. No definite indication was found of the origin of the secondary frequencies. They were probably due to a coupling effect. As a clue, which may lead to their source, the following observations may be of interest.

The frequency data in column 21 may be divided into three groups. In the first group, the secondary frequencies \( f''_F \) are related to the length vibrations \( f_b \) by multiple numbers as follows:

<table>
<thead>
<tr>
<th>Crystal No.</th>
<th>16</th>
<th>15</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio ( f_b/f''_F )</td>
<td>3.02</td>
<td>2.98</td>
<td>6.90</td>
</tr>
</tbody>
</table>

These secondary frequencies may represent subharmonics of the longitudinal vibrations. In the second group, the secondary frequencies \( f''_F \) are related to the longitudinal ones by repeating nonharmonic ratios:

<table>
<thead>
<tr>
<th>Crystal No.</th>
<th>8</th>
<th>10</th>
<th>11</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio ( f_b/f''_F )</td>
<td>5.73</td>
<td>5.72</td>
<td>2.44</td>
<td>2.45</td>
</tr>
</tbody>
</table>

The third group of bars is characterized by a ratio between the longitudinal \( f_b \) and the normal flexural vibration \( f'_F \) which is close to a multiple number:

<table>
<thead>
<tr>
<th>Crystal No.</th>
<th>14</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio ( f_b/f'_F )</td>
<td>6.1</td>
<td>4.07</td>
<td>5.15</td>
<td>5.98</td>
</tr>
</tbody>
</table>
### Table 2

**Data of Measurements and Calculations for Five X-Cut Quartz Bars**

<table>
<thead>
<tr>
<th>Crystal Number</th>
<th>Thickness</th>
<th>Length</th>
<th>Height</th>
<th>Ratio of Dimensions</th>
<th>Natural Frequency of Thickness Vibration, Measured</th>
<th>Frequency Factor $f_t \times t$</th>
<th>Natural Frequency of Length Vibration, Measured</th>
<th>Frequency Factor $f_b \times b$</th>
<th>Natural Frequency of Flexural Vibration, Measured</th>
<th>Frequency Factor $f_r$</th>
<th>Frequency from Equation (23)</th>
<th>Calculated Frequency from $f_r$</th>
<th>Deviation of Natural Frequency $%$ from Calculated Frequency $f_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.213</td>
<td>2.475</td>
<td>0.744</td>
<td>0.3005</td>
<td>1338.0</td>
<td>2.860</td>
<td>110.1</td>
<td>2.732</td>
<td>55.72</td>
<td>1.21</td>
<td>18.60</td>
<td>-0.21</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>0.2135</td>
<td>4.320</td>
<td>0.8825</td>
<td>0.2100</td>
<td>1347.0</td>
<td>2.976</td>
<td>61.8</td>
<td>2.751</td>
<td>55.60</td>
<td>1.09</td>
<td>22.70</td>
<td>-0.84</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>0.2135</td>
<td>4.0587</td>
<td>0.7455</td>
<td>0.1825</td>
<td>1335.0</td>
<td>2.580</td>
<td>50.5</td>
<td>2.728</td>
<td>56.40</td>
<td>1.40</td>
<td>18.60</td>
<td>-0.32</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>0.214</td>
<td>3.155</td>
<td>1.475</td>
<td>0.4575</td>
<td>1344.0</td>
<td>2.576</td>
<td>84.6</td>
<td>2.665</td>
<td>56.40</td>
<td>1.40</td>
<td>18.60</td>
<td>-0.32</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.2135</td>
<td>4.185</td>
<td>1.474</td>
<td>0.352</td>
<td>1346.0</td>
<td>2.572</td>
<td>65.1</td>
<td>2.723</td>
<td>56.40</td>
<td>1.40</td>
<td>18.60</td>
<td>-0.83</td>
<td></td>
</tr>
</tbody>
</table>
III. Flexural and Longitudinal Vibrations of X-Cut Quartz Crystals

17. Material and Results of Measurements.—All quartz bars investigated thus far and enumerated in Table 1 were Y-cut. Their length dimension $b$ was made parallel to the direction of the electrical axis $OX$, (see Fig. 2). In order to extend the investigation to bars in which the length direction formed an angle of 30 deg. with the electrical axis, so as to coincide with a mechanical axis, five X-cut crystals (Nos. 26 to 30) were prepared from the quartz stock used for the Y-cut samples. Because the moduli of elasticity in the direction of the two respective axes are known to differ only slightly from each other, the flexural frequencies of the two types of crystals having equal dimensions were not expected to differ from each other by more than 0.5 per cent. Measurements were made, however, for the purpose of ascertaining that the design formulae (23 to 35) may be applied for X-cut crystals. The results of measurements of the frequency of flexural and longitudinal vibrations are given in Table 2. The notations and numbering of the columns are similar to those in Table 1. Column 18 shows that the deviation $\delta$ of measured values of frequency in flexural vibrations from calculated values was less than 1 per cent. Hence it may be concluded that the expression (23), with the auxiliary design formulae, may be applied to X-cut quartz bars also.

18. Frequency Factors.—From the data in Table 2 the average frequency factors for X-cut crystals were found to be as follows:

For flexural vibrations

$$\frac{A}{B} = \frac{5.52 \times 10^5}{\sqrt{1 + 5.22(a/b)^2}}$$

For longitudinal length vibrations $A_\ell = 2.7 \times 10^6$
For longitudinal thickness vibrations $A_t = 2.86 \times 10^5$

Only the last factor differs from the corresponding factor for the Y-cut bars, the first two are identical for both X-cut and Y-cut quartz bars.

IV. Conclusion

19. Summary and Conclusions.—

(1) A series of 30 piezoelectric quartz bars and plates were specially cut and investigated with a view of obtaining data for the design of low-frequency stabilized oscillators and filters.
(2) The fundamental natural frequency of flexurally-vibrating Y-cut quartz bars and plates within a range of from 3.5 to 306 kc. was determined, along with frequencies of longitudinal vibrations.

(3) For the measurement of frequency of flexurally-vibrating quartz resonators below 15 kc. per second a sensitive bridge method was developed.

(4) On the basis of experimental data obtained on Y-cut quartz bars and plates, and in conjunction with theoretical considerations, a formula (Equation 23) was derived for the calculation of the fundamental natural frequency of flexurally-vibrating bars and plates.

(5) The frequency formula was found to hold within 1 per cent for quartz bars whose ratio $a/b$ did not exceed 0.4, and within 1.5 per cent for bars and plates whose ratio $a/b$ did not exceed 0.75.

(6) Measurements on X-cut quartz bars were made which show that the frequency formula may be used also for the calculation of dimensions of this type of resonator.

(7) Further formulae (Equations 25 to 35) were derived for calculating the dimensions and frequency of flexurally-vibrating quartz bars for particular cases which occur in the course of designing and grinding of crystals.

(8) Average frequency factors applicable for flexural and longitudinal vibrations were determined for the quartz rock used in this investigation (Sections 14 and 18).
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