STABILITY OF PERIODIC SOLUTIONS TO NONLINEAR KLEIN-GORDON EQUATIONS

BY

NANJUNDAMURTHY VENKATASUBBU

THESIS

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Adviser:

Professor Jared C. Bronski

Abstract

We study the stability of periodic travelling wave solutions to nonlinear Klein-Gordon equations, subject to small and localized perturbations. Using the periodic Evans function technique, we analyze the spectrum of a linearized quadratic eigenvalue pencil associated with the linearized equation, in a neighbourhood of the origin on the spectral plane. The stability criterion is expressed in terms of signature of an index involving physical parameters of the wave, thereby holding the result valid for a general nonlinearity $\mathrm{F(u)}$. The result is then verified for the following cases:

- cubic nonlinearity; $\mathrm{F(u) = u^3 - u}$
- sine Gordon equation; $\mathrm{F(u) = -\sin(u)}$

To my father Mr. Venkatesan Venkatasubbu, mother Mrs. Padma Venkatasubbu, and my dearest little sister Sushma Venkatasubbu.

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Chapter 1

Introduction

In this paper, we study the stability of periodic traveling wave solutions to nonlinear Klein-Gordon (NLKG) equation (1.0.1) with a general nonlinearity $F(u)$.

$$u_{tt} = u_{\xi\xi} + F(u) \tag{1.0.1}$$

$$u(\xi, t) : \mathbb{U} \times [0, \infty) \to \mathbb{R}$$

$$\text{where} \quad \mathbb{U} \subset \mathbb{R} \quad \text{is open}$$

The nonlinearity $F(u)$ is assumed to be $C^{\infty}(\mathbb{R})$. We consider periodic solutions of a finite period and the perturbations are assumed to be small and localized i.e. integrable.

1.1 Applications

The nonlinear Klein-Gordon equation can be perceived as the most natural generalization of the wave equation [6] and also as a relativistic generalization of the Schroedinger equation [9]. It has applications in a wide range of physical phenomena with specific applications in quantum physics and field theory. In a mathematical context, the nonlinear Klein-Gordon equation first arose in Liouville's work on theory of constant surfaces with an exponential nonlinearity $\mathrm{F(u) = e^u}$ [6]. For cubic nonlinearity $\mathrm{F(u) = u^3 - u}$, it has been used as a model equation in field theory by R. F. Dashen et al [4]. They have used a kink-solution ϕ of the cubic-NLKG to model an extended hadron with the boundary condition $|\phi| = 1$ at infinity. Classical applications of the NLKG are detailed in the work of P. Gravel and C. Gauthier [5]. They deal with the application of Klein-Gordon equation to problems in classical mechanics and the resemblance of the equations, parameters and solutions to these problems to analogous phenomena in quantum mechanics. The Klein-Gordon equation, in this case, models the elastic coupling of two linear waves in an ideal isotropic medium. Mathematically, it arises as an inhomogeneous problem associated with the homogeneous wave equation. The coupling of waves reduces symmetry of the problem and is shown to give rise to an inertial mass, analogous to the mass of zero-spin particle in quantum mechanics. A historical account of the Klein-Gordon equation can be found in the paper by H. Kragh [9] in which he refers to it as the *relativistic wave equation*.

For $F(u) = -\sin(u)$, (1.0.1) becomes the more popular cousin of the NLKG, the sine-Gordon equation (SGE) which is a well-studied member of the NLKG family. Among its many applications, a few are mentioned here in brief. The phase difference between the wave functions of two superconductors in a large Josephson junction is governed by the SGE. A moving kink-solution in this case represents the propagation of a single flux quantum through the junction. The SGE is also used to model Bloch-wall motion in magnetic crystals wherein it describes the behaviour of rotation of the orientation of magnetization within a Bloch-wall; steady SGE models a stationary wall while the unsteady counterpart models a moving wall. In this example, the SGE arises out of a variational problem associated with the minimization of free energy of a static Bloch-wall. Yet another interesting application of SGE can be found in the propagation of dislocations in crystals wherein a kink-solution of the SGE represents a *slip* propagating through the crystal. Two other applications of SGE are propagation of ultra-short optical pulses and a unitary theory of elementary particles. All the applications listed above are described in greater detail in [1], accompanied by an exhaustive theory of SGE.

1.2 Motivation

The stability of periodic wave solutions to the SGE has been analyzed in [12] and the result has been categorized into four types based on the wave speed and an arbitrary constant of integration, E. In that paper, the author uses "eigenfunction expansion technique" wherein the perturbations are expanded in an eigenfunction-series. Following a change of dependent variable, these eigenfunctions are shown to solve a "Schroedinger" equation and the stability of periodic waves to the SGE is linked to the spectrum of this Schroedinger equation. However, the change of dependent variable involves a spectral parameter which is also used in the eigenfunction-series expansion of perturbations. This leads to an inconsistency while concluding on the stability of the periodic waves of SGE based on the *Schroedinger spectrum*. This method may also cause alterations to the boundary conditions of the original problem.

More recent studies on stability of periodic waves of Klein-Gordon equations include those of Natali [10] and Natali & Pastor [11]. In [10] the author studies orbital stability of periodic waves associated with sine-Gordon and sinh-Gordon equations, while [11] deals with orbital stability of *dnoidal*, *cnoidal* and *snoidal* solutions to two types of Klein-Gordon equations. In this paper, we propose to approach the stability analysis in an alternate way by studying the spectrum of a linearized quadratic eigenvalue pencil obtained by linearizing the NLKG about a periodic travelling wave solution, while maintaining consistency in boundary conditions. By stability, we mean spectral stability throughout the paper and a formal definition to this effect will be provided in the next section. The motivation for our work arises from this possibility of an alternate approach and the scope of generalizing it to nonlinearities of generic type. Also, the stability result derived in this work is based on physical parameters of periodic waves and therefore, it is easy to compute this index for a general nonlinearity. We investigate the stability of periodic traveling wave solutions to (1.0.1) using the periodic Evans function technique. This technique was employed in [2] to study the modulational stability of periodic waves to the gKdV equation and we adapt the same framework for nonlinear Klein-Gordon equation. An overview of our analysis is presented in the next section.

The organization of the paper is as follows. In Section 2, we present a short note on the existence of periodic solutions to (2.1.2) followed by a few basic definitions and a short overview of the stability analysis. Section 3 consists of the eigenvalue problem associated with a quadratic pencil and a perturbation-analysis of its spectrum leading to the main result of the paper. In section 4, we apply our stability result to two specific forms (examples) of the nonlinearity $F(u)$.

Chapter 2

Preliminaries and Definitions

2.1 Existence and Bounds of Periodic Solutions

The existence of periodic solutions to (1.0.1) can be established as follows.

Proposition 1. *The NLKG equation* (1.0.1) *has periodic solutions of the form*

$$u(\xi, t) = \phi\left(\xi - ct\right) \tag{2.1.1}$$

$$\phi\colon \mathbb{R} \to [0, T]$$

with a finite period $T \in \mathbb{R}$*. Moreover,* $\phi \in C^2([0, T])$ *when* $u(\xi, t)$ *is a smooth solution of* (1.0.1).

Remark 1. *Here, c is the wave speed and we only consider waves with speeds less than unity i.e.,* $|c| < 1$.

Proof. Substituting (2.1.1) in (1.0.1), we obtain the following travelling wave ordinary differential equation (ODE)

$$\left(1 - c^2\right)\phi_{xx} + F\left(\phi\right) = 0 \tag{2.1.2}$$

where $x = \xi - ct$. Integrating the ODE with respect to x we see that $\phi(x)$ satisfies

$$\frac{\left(1 - c^2\right)}{2}\phi_x^2 + P\left(\phi\right) = E \tag{2.1.3}$$

where E is a constant of integration and $\mathrm{P}(\phi)$ is the antiderivative of the nonlinearity $\mathrm{F}(\phi)$. Without any loss of generality, we can assume $E > 0$ and rewrite (2.1.3) as

$$\phi_x^2 = G\left(\phi\right) = \frac{2}{\left(1 - c^2\right)}\left(E - P\left(\phi\right)\right) \tag{2.1.4}$$

Observe that if $u(\xi, t)$ is a smooth solution of (1.0.1), then $\phi(x)$ is continuous. Let $\phi(x)$ be continuous and periodic over an arbitrary finite closed interval $[a, b] \in \mathbb{R}$. Applying mean value theorem on this interval, we conclude that there exist extremal states of $\phi(x)$ corresponding to $\phi_x = 0$, for some $x \in [a,b]$. Due to translational invariance of the problem, we can fix this interval to $[0,T]$. Therefore, the set of zeros of $G(\phi)$ is non-empty and there is at least one T-periodic $\phi(x)$ which satisfies (2.1.3). In other words, periodic solutions to (1.0.1) exist such that $\phi\colon \mathbb{R} \to [0,T]$. Further, if $\phi(x)$ is continuous and solves (2.1.2), clearly it is $C^2([0,T])$. $\square$

In the current periodic setup, the constant of integration E corresponds to total energy of the system, $P(\phi)$ corresponds to the (periodic) potential and the first term on the left hand side of (2.1.3) corresponds to the kinetic energy. We now proceed to establish bounds on $\phi(x)$.

Proposition 2. *The solutions to* (2.1.2) *are bounded by two consecutive simple zeros of* $G(\phi)$ *and wave motion occurs only if* $G(\phi) > 0$ *between these zeros. Further, the wave will have a finite amplitude.*

Proof. Let ϕ_0 and ϕ_1 be two zeros of $G(\phi)$ such that $\phi_0 < \phi_1$ (refer fig.1). It is essential that $G'(\phi_i) \neq 0$ $(i = 0, 1)$ since we are not considering homoclinic or heteroclinic cycles here; in other words, ϕ_0 and ϕ_1 are required to be simple zeros. From (2.1.4), we see that $G(\phi) \geq 0\ \forall \phi(x) \in \mathbb{R}$ and since $G(\phi_0) = 0 = G(\phi_1)$, it is required that $G(\phi) > 0$ for all $\phi_0 < |\phi(x)| < \phi_1$. If this condition is satisfied, then wave motion occurs in the interval $[\phi_0, \phi_1]$. Outside this interval, $G(\phi)$ changes sign as it varies across its simple zeros and becomes negative. Therefore, it follows that $\phi_0 \leq |\phi(x)| \leq \phi_1\ \forall x \in [0,T]$ and wave motion is confined to this region if $G(\phi) > 0$.

Further, note that $\left|\int_{\mathbb{R}} F(u)\right| < \infty$. We therefore have $|P(\phi)| = \left|\int_{\mathbb{R}} F(u)\right| < \infty \implies |G(\phi)| < \infty$. Moreover, observe that the continuity of $\phi(x)$ imparts continuity to $G(\phi)$ and by applying Weierstrass' definition of continuity to $G(\phi)$, we have $|\phi(x)| < \infty\ \forall x \in [0,T]$. It follows that $|\phi_1 - \phi_0| < \infty$ i.e., the periodic solution has a finite amplitude (refer fig.1). $\square$

Remark 2. *The simple zeros of* $G(\phi)$ *have an interesting physical interpretation: they correspond to the energy states* $P(\phi) = E$ *wherein the kinetic energy is zero (refer fig.1). For example, if the periodic solution describes the motion of an oscillating particle, these are the states where it comes to a rest and reverses its direction of motion. Also, in this perspective, it is physically reasonable to expect that* $G(\phi) \geq 0$ *since this implies the potential energy of the particle should never exceed the total energy of the system.*

2.2 Definitions

At this juncture, we are in a position to formulate a precise notion of periodic solutions to the travelling wave ODE as follows.

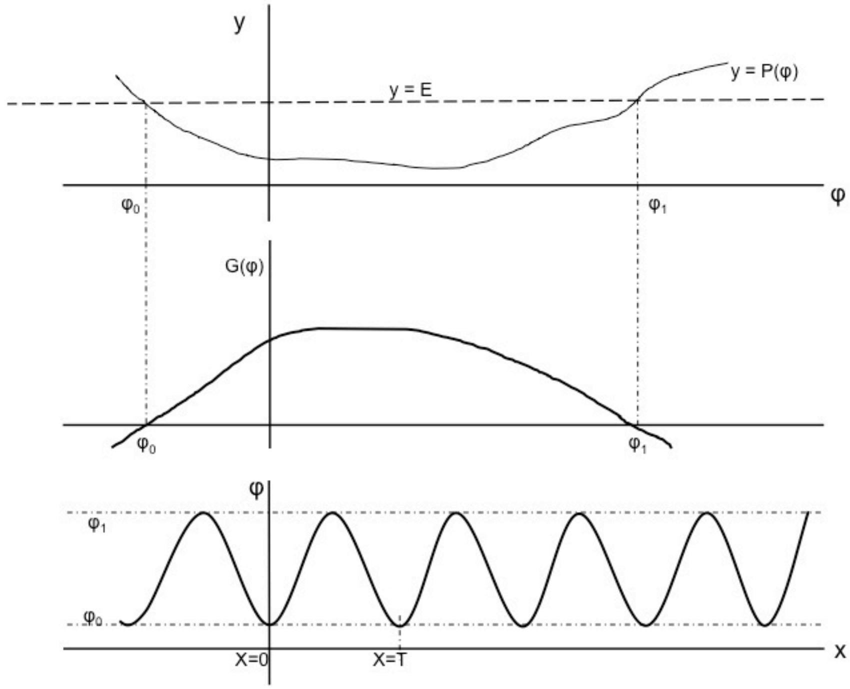

Figure 2.1: A Schematic diagram showing the region of wave motion as determined by the zeros of $G\phi$). Observe that the zeros correspond to *zero kinetic energy* states. The wave motion is bounded and oscillatory between ϕ_0 and ϕ_1.

Definition 1. (2.1.1) *is said to be a periodic solution to the travelling wave ODE* (2.1.2) *if it solves* (2.1.2), *subject to* $\phi(0) = \phi_0 = \phi(T)$ *and* $\phi_x(0) = 0 = \phi_x(T)$.

Note that the boundary conditions specified in the definition above are not unique. Owing to the translational invariance and periodicity of the problem, it is always possible to reduce any other values for boundary conditions to the ones specified above; in other words, all the curves in fig.1 can be translated along vertical and horizontal axes without altering the nature and properties of the problem. Therefore, the validity of stability analysis presented in this paper is not restricted only to the boundary conditions specified above. We have just picked one among the many possibilities, fixed our reference frame to it and continued with the analysis. The solution to (2.1.2) can generally be expressed implicitly as a periodic integral in the form

$$x = \sqrt{\frac{(1-c^2)}{2}} \int_{\phi_0}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{(E - P(\phi))}}$$

and by choosing the period points as limits of integration, we obtain an expression for the period as

$$T = \sqrt{2\left(1-c^2\right)} \int_{\phi_0}^{\phi_1} \frac{\mathrm{d}\phi}{\sqrt{(E-P(\phi))}} \tag{2.2.1}$$

The above integral can be regularized at the square root branch points (ϕ_0 and ϕ_1) which enables direct differentiation of the period with respect to the parameters E and c ([2], [7]). Another quantity which will play an important role in our analysis is:

$$K := \int_0^T \phi_x^2 \mathrm{dx} \tag{2.2.2}$$

We next define the monodromy matrix as follows.

Definition 2. *Let $\Psi(x;\mu)$ be a T-periodic solution of a linear system $\Psi_x = \mathbb{H}(x;\mu)\Psi$ subject to $\Psi(0;\mu) = \mathbb{I}$, where $\mathbb{H}(x;\mu)$ is a periodic coefficient, $\mathbb{I}$ is the identity matrix and $\mu \in \mathbb{C}$ is a complex parameter. Then, the monodromy matrix $\mathbb{M}(\mu)$ is defined as the matrix obtained by evaluating $\Psi(x;\mu)$ at the period point T; i.e., $\mathbb{M}(\mu) = \Psi(T;\mu)$*

The monodromy matrix may be seen as a linear operator that maps the solution at every period point to its value at the next period point. In the next section, it will be shown that a monodromy matrix can be defined for a linear system associated with the travelling wave ODE and it will play a pivotal role in determining the stability of periodic solutions to the NLKG equation.

We next define the periodic Evans function for our problem. The Evans function serves as an analytic tool that counts eigenvalues of a linear operator with which it is associated [8]. It is constructed in such a way that the zeros of this function coincide with the eigenvalues of the operator. In the ongoing analysis, we will be interested in the eigenvalues of the monodromy operator. Hence we look for a definition which incorporates a function involving the monodromy and vanishes at its eigenvalue points on the complex plane. We therefore define the Evans function in terms of the characteristic polynomial of the monodromy operator. Also, since it involves periodic terms in the definition, the Evans function is periodic in this case.

Definition 3. *For the present problem, we define the periodic Evans function as:*

$$D(\mu,\rho) = det\left[\mathbb{M}(\mu) - e^{i\rho}\mathbb{I}\right]$$

where, $\rho \in \mathbb{R}$ and $\mathrm{e}^{\mathrm{i}\rho}$ is an eigenvalue of the monodromy matrix.

Finally, we define stability in the context of current problem as follows.

Definition 4. *Periodic solutions to* (1.0.1) *are said to be spectrally stable under small localized perturbations if the spectrum of a quadratic eigenvalue pencil associated with the linearized periodic problem is purely imaginary.*

Recall that by stability, we mean spectral stability. We now present an overview of the stability analysis carried out in the next section. The nonlinear PDE (1.0.1) is linearized about the traveling wave solution (2.1.1) and a quadratic eigenvalue pencil is obtained from the linearized equation. We then study the spectrum of this pencil with the intention of determining its geometry on the complex plane. This will give an insight into the stability of periodic solutions in the sense of definition 4. We use the periodic Evans function to determine the geometry of the spectrum mentioned above and based on some calculations involving the monodromy $\mathbb{M}(\mu)$, obtain a criterion for stability.

Chapter 3

Stability Analysis

Let $\psi(\xi,t) = u(\xi,t) + \epsilon v(\xi,t)$ be a composite solution of (1.0.1) consisting of an unperturbed part $u(\xi,t)$ and a perturbation $v(\xi,t)$ ($\epsilon \ll 1$ is a small parameter). Substituting this in (1.0.1) and retaining terms of $\mathbb{O}(\epsilon)$, we obtain the linearized equation as

$$v_{tt} = v_{\xi\xi} - f(u)v \tag{3.0.1}$$

where $f(u) = -F'(u)$. Note that since we are considering small localized perturbations, we require $|v(\xi,t)| \ll |u(\xi,t)|$ and that $v(\xi,t)$ be locally integrable. The analysis is then moved into the travelling reference frame by effecting a Galilean transformation in (3.0.1) as

$$x = \xi - ct$$
$$\tau = t$$

which transforms the linearized equation into the travelling reference frame as

$$v_{\tau\tau} - 2cv_{\tau x} = (1-c^2)v_{xx} - f(\phi)v \tag{3.0.2}$$

Defining a linear operator $L[\phi] := (1-c^2)\partial_x^2 - f(\phi)$, we can write (3.0.2) as

$$v_{\tau\tau} - 2cv_{\tau x} = L[\phi]\, v$$

Performing a Fourier transform in time on (3.0.2) yields a linearized quadratic eigenvalue pencil

$$L[\phi]\, v + 2c\mu\partial_x v = \mu^2 v \tag{3.0.3}$$

where μ is a spectral parameter introduced by the Fourier transform. Denoting the spectrum of (3.0.3) by $\mathbb{S}$, we recall that the geometry of $\mathbb{S}$ will decide the stability or instability of periodic waves of (1.0.1) and that the periodic Evans function will help us in determining the geometry of $\mathbb{S}$.

The periodic Evans function involves working with the monodromy matrix $\mathbb{M}(\mu)$ which we need to compute in its normal form. Observe that $\mu = 0$ reduces (3.0.3) to a homogeneous equation

$$L[\phi]\, v = 0 \tag{3.0.4}$$

from which it is easy to compute $\mathbb{M}(0)$. Using this as the "starting point", we obtain the monodromy matrix $\mathbb{M}(\mu)$ through a Taylor series expansion of the monodromy about $\mu = 0$, as

$$\mathbb{M}(\mu) = \mathbb{M}(0) + \mu \mathbb{M}_{\mu}(0) + \frac{\mu^2 \mathbb{M}_{\mu\mu}(0)}{2} + ... \tag{3.0.5}$$

The series expansion shown above is indeed valid because $\mathbb{M}(\mu)$ is analytic in μ ([2]) and $\mu = 0$ is an element of $\mathbb{S}$ (proposition 3 below). Note that the Taylor series expansion is local in μ and it is valid in a small neighbourhood of $\mu = 0$. The second and third terms are obtained from solutions of inhomogeneous problems associated to (3.0.4). Once the normal form of the monodromy $\mathbb{M}(\mu)$ is obtained, it is substituted in the periodic Evans function which yields an inequality. Finally, the stability index is derived based on this inequality.

The quadratic eigenvalue pencil (3.0.3) is a second order homogeneous ordinary differential equation which can be written as a first order system as

$$\vec{v}_x(x) = \mathbb{H}(x;\mu)\ \vec{v}(x)$$

where

$$\vec{v}(x) = \begin{pmatrix} v \\ v_x \end{pmatrix} \quad , \quad \mathbb{H}(x;\mu) = \begin{pmatrix} 0 & 1 \\ \frac{\mu^2 + f(\phi)}{1-c^2} & \frac{2\mu c}{1-c^2} \end{pmatrix}$$

The first order system corresponding to (3.0.4) can therefore be written as

$$\vec{v}_x(x) = \mathbb{H}(x;0)\ \vec{v}(x) \tag{3.0.6}$$

and we will show that it indeed has a fundamental solution matrix of the form

$$\mathbb{V}(x;0) = \begin{pmatrix} v_1 & v_2 \\ v_{1x} & v_{2x} \end{pmatrix}$$

Evaluating this fundamental matrix solution at the period point T will then give us the first term on the right hand side of (3.0.5).

Proposition 3. *Let* $\phi(\mathrm{x}) \in$ *be a periodic solution of* (2.1.2) *in the sense of definition 1. Then the following statements are true*

1. *There exist two linearly independent solutions of* (3.0.6) *and they are given by*

$$\vec{v_1} = \begin{pmatrix} \phi_E \\ \phi_{Ex} \end{pmatrix} \quad , \quad \vec{v_2} = \begin{pmatrix} \phi_x \\ \phi_{xx} \end{pmatrix} \tag{3.0.7}$$

2. *The fundamental solution matrix of* (3.0.6) *indeed exists and its columns consist of the vectors in* (3.0.7).

3. $\mu = 0$ *is in the spectrum of the quadratic eigenvalue pencil* (3.0.3) *and the monodromy matrix* $\mathbb{M}(\mu)$ *evaluated at* $\mu = 0$ *has the form*

$$\mathbb{M}(0) = \begin{pmatrix} 1 & 0 \\ \frac{-P'(\phi_0)^2 T_E}{(1-c^2)} & 1 \end{pmatrix} \tag{3.0.8}$$

Proof. A straightforward calculation shows that $v = \phi_E$ and $v = \phi_x$ satisfy (3.0.4). Next, substituting $v = \phi_{Ex}$ on the left hand side of (3.0.4), we get $L\,[\phi]\,\phi_{Ex} = \partial_E\left((1-c^2)\phi_{xx} + F(\phi)\right) = 0$ from (2.1.2). Similarly for $v = \phi_{xx}$, we get $L\,[\phi]\,\phi_{xx} = \partial_x\left((1-c^2)\phi_{xx} + F(\phi)\right) = 0$ from (2.1.2). Thus, $v = \phi_E$, $v_x = \phi_{Ex}$, $v = \phi_x$ and $v_x = \phi_{xx}$ all solve (3.0.4) and by writing them in vector form, it can be seen that they solve (3.0.6). Moreover, ϕ_E and ϕ_x are linearly independent. Therefore, we can conclude that (3.0.6) has two linearly independent solutions given by (3.0.7) and this proves part 1 of the proposition.

For part 2, we proceed as follows. Observe that the linear operator L$[\phi]$ has a structure resembling a differential equation with a periodic coefficient which is known to have a fundamental solution matrix ([14]). In order for

$$\mathbb{V}(x;0) = \begin{pmatrix} \vec{v_1} & \vec{v_2} \end{pmatrix}$$

with $\vec{v_1}$ and $\vec{v_2}$ as given in (3.0.7) to be a fundamental solution matrix for (3.0.6), it must satisfy the initial condition

$$\mathbb{V}(0;0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.9}$$

It can be verified that although the vectors in (3.0.7) do not satisfy (3.0.9) in their current form, they can be scaled appropriately to do so. For $v_1 = \phi_E$, the scaling is straightforward and the scaled solution is $v_1 = \frac{\phi_E(x)}{\phi_E(0)}$. The scaling for v_2 is obtained by evaluating ϕ_{xx} at $x = 0$ in (2.1.2) and by substituting $P'(\phi)$ for $F(\phi)$ since the potential is antiderivative of the nonlinearity. We thus have the *normalized solutions* as

$$v_1 = \frac{\phi_E\,(x)}{\phi_E\,(0)} \qquad v_2 = \frac{-\left(1-c^2\right)\phi_x(x)}{P'(\phi_0)} \tag{3.0.10}$$

It can be verified that these solutions solve the homogeneous problem (3.0.6) and also satisfy the initial condition (3.0.9). The *fundamental solution matrix* to (3.0.6) is therefore obtained as:

$$\mathbb{V}(x;0) = \begin{pmatrix} \frac{\phi_E(x)}{\phi_E(0)} & \frac{-\left(1-c^2\right)\phi_x(x)}{P'(\phi_0)} \\ \frac{\phi_{Ex}(x)}{\phi_E(0)} & \frac{-\left(1-c^2\right)\phi_{xx}(x)}{P'(\phi_0)} \end{pmatrix} \tag{3.0.11}$$

and this proves part 2 of the proposition.

Next, observe that for $\mu = 0$, (3.0.3) reduces to (3.0.4) which has been shown to have two linearly independent solutions (refer part 1). Therefore, there exist $v \neq 0$ such that $L\,[\phi]\,v = \mu^2 v - 2c\mu\partial_x v = 0 \implies \mu = 0$. Thus $\mu = 0$ is in the spectrum of (3.0.3).

The *monodromy matrix* $\mathbb{M}(T;\mu)$ can now be obtained by evaluating $\mathbb{V}(x;0)$ at $x = T$. Notice that at $x = 0$, we have $P(\phi(0)) = E$, from (2.1.3). Differentiation of this relation with respect to the energy E yields $\phi_E(0)P'(\phi(0)) = 1$. Making use of this fact and calculating $\phi_E(T)$ by the chain rule, we have the monodromy as given in (3.0.8) and thus the proof the proposition is complete. □

Remark 3. *The monodromy matrix* (3.0.8) *has a Jordan block structure and is typically known to "break" under perturbation. Observe that* $\mathbb{M}(\mu)$ *can be obtained analytically as the perturbed form of* $\mathbb{M}(0)$ *via series expansion. The monodromy thus breaks analytically in this case.*

Now that we have the first term on the left hand side of (3.0.5), we move on to obtain the next two terms. The next step is to obtain the derivatives of the monodromy at $\mu = 0$ and to do so, we use the method of *variation of parameters.* Consider

$$\mathbb{V}_\mu\,(x;0) = \mathbb{V}\,(x;0)\,\mathbb{Q}(x) \tag{3.0.12}$$

where $\mathbb{Q}(x)$ is a real-valued variable coefficient matrix. Differentiation of (3.0.12) with respect to x and (3.0.6) with respect to μ results in a system of differential equations from which we can obtain an expression for $\mathbb{Q}_x(x)$ as

$$\mathbb{Q}_x(x) = \mathbb{V}^{-1}\mathbb{H}_\mu\mathbb{V}$$

$\mathbb{Q}(x)$ can now be retrieved from the above expression by integrating it with respect to x as

$$\mathbb{Q}(x) = \frac{2}{(1-c^2)}\int_0^x \begin{pmatrix} -v_{1x}v_2 & -v_2v_{2x} \\ v_1v_{1x} & v_1v_{2x} \end{pmatrix} dx$$

The fact that $\mathbb{V}(x)$ has a symplectic structure makes the calculation of its inverse easy. Substituting for v_1 and v_2 from (3.0.10), we obtain the variable coefficient matrix as

$$\mathbb{Q}(x) = -T\mathbb{I}$$

where $\mathbb{I}$ is a 2×2 identity matrix. Substituting this in (3.0.12) and evaluating it at $x = T$ gives us the second term on the left hand side of (3.0.5):

$$\mathbb{M}_\mu(0) = -T\mathbb{M}(0) \tag{3.0.13}$$

The calculation of the second derivative of the monodromy is more tedious when compared to the first derivative and we resort to variation of parameters once again. Consider

$$\mathbb{V}_{\mu\mu}\,(x;0) = \mathbb{V}\,(x;0)\,\mathbb{G}(x) \tag{3.0.14}$$

where $\mathbb{G}(x)$ is, again, another real-valued variable coefficient matrix. Differentiation of (3.0.14) with respect to x and of (3.0.6) *twice* with respect to μ results in a system of differential equations again, from which we obtain the expression for $\mathbb{G}_x(x)$ as:

$$\mathbb{G}_x(x) = \mathbb{V}^{-1}\mathbb{H}_{\mu\mu}\mathbb{V} + 2\mathbb{V}^{-1}\mathbb{H}_{\mu}\mathbb{V}_{\mu} \tag{3.0.15}$$

Among all the terms on the right hand side of (3.0.15), only $\mathbb{V}_{\mu}$ needs to be computed afresh and we now proceed to do so. Differentiating (2.1.2) with respect to the wave speed c, we obtain a second order differential equation which when written as a system of two first order equations, turns out to be

$$\vec{a}_x(x) = \mathbb{H}(x;0)\vec{a}(x) + \left(\frac{2c}{1-c^2}\right)\vec{b}(x) \tag{3.0.16}$$

where, $\mathbb{H}(x;0)$ is as given in (3.0.6) and

$$\vec{a}(x) = \begin{pmatrix} \phi_c \\ \phi_{cx} \end{pmatrix} \quad , \quad \vec{b}(x) = \begin{pmatrix} 0 \\ \phi_{xx} \end{pmatrix}$$

A close observation indicates that (3.0.16) is an inhomogeneous variant of the homogeneous problem (3.0.6), with $\left(\frac{2c}{1-c^2}\right)\vec{b}(x)$ being the inhomogeneity. Next, differentiation of (3.0.6) with respect to μ and substituting for $\mathbb{H}_{\mu}$ and $\mathbb{V}$ yields

$$\mathbb{V}_{\mu x} = \mathbb{H}\mathbb{V}_{\mu} + \frac{2c}{(1-c^2)}\begin{pmatrix} 0 & 0 \\ v_{1x} & v_{2x} \end{pmatrix}$$

Remark 4. *At this juncture, we remark that eventually when we substitute for the second derivative of monodromy $\mathbb{M}_{\mu\mu}$ in the Taylor series* (3.0.5) *and retain only up to $\mathbb{O}(\mu^2)$ terms, we expect only $M_{\mu\mu}^{(1,2)}$, the* $(1,2)$ *element of $\mathbb{M}_{\mu\mu}$, to contribute towards the leading order terms in the series while other all other elements get modded out as higher order terms. Hence we break the matrix-equation obtained above, into two* vector-equations *corresponding to each column of $\mathbb{V}_{\mu}$ and work with only the second column. We now proceed with the vector-equation corresponding to the second column of $\mathbb{V}_{\mu}$ which we denote by $\vec{v}_{\mu}^{s}$.*

We are now interested in the vector equation

$$\vec{v}_{\mu x}^{s} = \mathbb{H}\vec{v}_{\mu}^{s} + \left(\frac{2c}{1-c^2}\right)\begin{pmatrix} 0 \\ v_{2x} \end{pmatrix} \tag{3.0.17}$$

A comparison of (3.0.17) and (3.0.16) shows that the vector $\vec{v}_{\mu}^{s}$ can be considered as another solution (with appropriate scaling) of the inhomogeneous variant of the homogeneous problem (3.0.6). We can therefore write the vector as a linear combination of $\vec{a}(x)$ and the solutions to (3.0.6). Let

$$\vec{v}^s_\mu = \frac{-(1-c^2)}{P'(\phi(0))}\vec{a}(x) + \alpha\vec{v}_1 + \beta\vec{v}_2$$

The coefficients α and β can be computed by evaluating the above expression at $x = 0$ where the boundary conditions are known. Substituting $\vec{v}^2_\mu(0) = (0,0)^\top$, we get $\alpha = \beta = 0$. We therefore have

$$\vec{v}^s_\mu = \frac{-(1-c^2)}{P'(\phi(0))}\vec{a}(x)$$

Using this vector in (3.0.15) we obtain the variable coefficient matrix $\mathbb{G}(x)$ upon integration. Substituting this in (3.0.14) and evaluating the expression at $x = T$, we obtain the second derivative of monodromy as

$$\mathbb{M}_{\mu\mu}(0) = \begin{pmatrix} \mathrm{M}^{(1,1)}_{\mu\mu} & \frac{-2\left(1-c^2\right)(K+cK_c)}{P'(\phi_0)^2} \\ \mathrm{M}^{(2,1)}_{\mu\mu} & \mathrm{M}^{(2,2)}_{\mu\mu} \end{pmatrix} \tag{3.0.18}$$

As mentioned earlier, only the $M^{(1,2)}_{\mu\mu}$ term has been computed explicitly since only that term will be seen to contribute to the leading order terms in the series expansion of Evans function. We are now ready to state the stability theorem.

Theorem 1. *If the period $T(E)$ of the traveling wave is such that $\left(\frac{dT(E)}{dE}\right) \geq 0$, then the spectrum $\mathbb{S}$ of the quadratic eigenvalue pencil* (3.0.3) *will consist of $\mu = 0$ and $\mu \in i\mathbb{R}$ (purely imaginary eigenvalues) in a small neighbourhood of $\mu = 0$ on the complex plane (see fig.3). In the light of definition 1, it follows that the periodic solutions* (2.1.1) *to the NLKG equation* (1.0.1) *will be stable under small localized perturbations.*

Proof. Using (3.0.8), (3.0.13) and (3.0.18) in (3.0.5), we obtain the monodromy $\mathbb{M}(\mu)$ in its normal form and substituting it in the periodic Evans function, we get

$$\begin{aligned} D(\mu,\rho) = det\left[\mathbb{M}(\mu) - e^{i\rho}\mathbb{I}\right] &= 1 - 2e^{\mathrm{i}\rho} + e^{2\mathrm{i}\rho} - 2T\mu + 2e^{\mathrm{i}\rho}T\mu \\ &\quad - (K + cK_c)\,T_E\mu^2 + T\mu^2 + \mathbb{O}\left(\mu^3\right) \\ &= 0 \end{aligned} \tag{3.0.19}$$

where K is as given in (2.2.2) and $\mathrm{K_c} = \frac{\partial \mathrm{K}}{\partial \mathrm{c}}$. Expanding the exponential terms in a series and retaining only $\mathbb{O}\left(\rho^2\right)$ terms results in a quadratic equation in μ as:

$$-\rho^2 + 2Ti\rho\mu - \left((K + cK_c)\,T_E - T^2\right)\mu^2 + \mathbb{O}\left(\mu^3, \rho^2\mu, \rho^2\mu^2, \rho\mu^2\right) = 0 \tag{3.0.20}$$

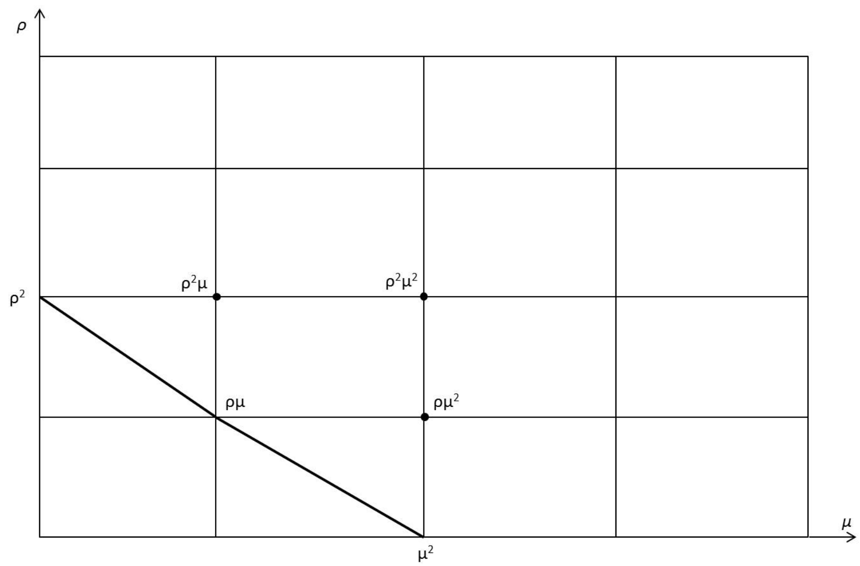

Figure 3.1: Newton's diagram. Terms of $\mathbb{O}(2)$ are connected by a dark solid line in the diagram. All terms on the line and to the left of it are retained in the periodic Evans function expansion while those to the right of it are dropped as they are of higher order.

The choice of leading order terms to be retained in the expansion and subsequently in the quadratic equation has been made with the aid of *Newton's diagram* for ρ and μ (refer fig.2)

The spectral stability of the traveling wave solution to (2.1.2) will be dictated by the nature of roots μ of (3.0.20). The following remark may be appropriate here.

Remark 5. *The relationship between the geometry of* $\mathbb{S}$ *on the complex plane and stability of the travelling wave may be interpreted thus: in order that the perturbations remain bounded or decay as they evolve, the eigenvalues of the quadratic pencil should lie either on the unit circle or inside it on the complex plane; i.e.* $Re(\mu) \leq 0$. *However, the symmetry of the problem ensures that existence of spectrum off the unit circle implies its existence both inside and outside the circle, and the solutions to the traveling wave equations will belong to two sets: one that is asymptotically stable and the other that is exponentially unstable. Hence it is required that the eigenvalues lie 'on' the unit circle i.e.,* $Re(\mu) = 0$. *Observe that in this context, definition 4 indeed makes sense. We now look for roots of* (3.0.20) *such that* $\mu \in i\mathbb{R}$.

For convenience, we change the variable μ in (3.0.20) to $\eta = i\mu$ and rewrite it as

$$-\rho^2 + 2T\rho\eta + \left((K + cK_c)\, T_E - T^2\right)\eta^2 = 0 \tag{3.0.21}$$

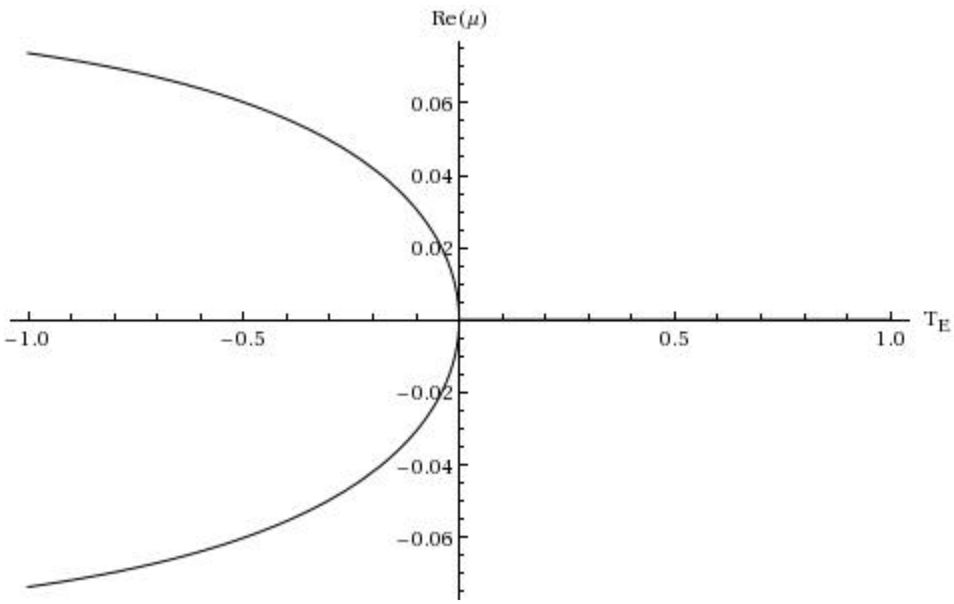

Figure 3.2: Plot of the real component of eigenvalue μ of the linearized quadratic eigenvalue pencil. As $\mathrm{T_E}$ crosses over from negative through zero to positive values, the eigenvalue becomes purely imaginary; i.e. $\mathrm{Re}(\mu) = 0$.

where $T_E = \frac{dT(E)}{dE}$. We now focus our attention to the roots of (3.0.21) and note that $\forall \mu \in i\mathbb{R}$, $\eta \in \mathbb{R}$. Therefore, our stability criterion stated above in terms of μ translates into the requirement $\eta \in \mathbb{R}$. The discriminant for (3.0.21) is

$$\Delta = 4\rho^2 T_E (K + cK_c)$$

The roots η of (3.0.21) are real if $\Delta \geq 0$ and complex if $\Delta < 0$. Recall that $\rho \in \mathbb{R}$ and by differentiating K as given in (2.2.2) with respect to c, we obtain $\mathrm{K_c} = \frac{2\mathrm{cK}}{1-\mathrm{c}^2}$. Thus when $\Delta \geq 0$, (3.0.21) has real roots and the following inequality arises:

$$T_E K \left(\frac{1+c^2}{1-c^2} \right) \geq 0 \tag{3.0.22}$$

Clearly, from (2.2.2) we see that we can define K to be non-negative and since we are considering waves with speeds $|c| < 1$, we have $\frac{1+c^2}{1-c^2} > 0$. Therefore, $T_E \geq 0$ ensures that (3.0.22) is satisfied and $\mu \in i\mathbb{R}$. The proof of the theorem is now complete. □

Remark 6. *1. The monodromy $\mathbb{M}(\mu)$ is obtained by a local expansion in μ in which terms of degree 3 and higher, are dropped. The stability result is therefore valid in a small neighbourhood of $\mu = 0$ on the complex plane.*

2. The terms $\mathrm{M}_{\mu\mu}^{(1,1)}$, $\mathrm{M}_{\mu\mu}^{(2,1)}$ and $\mathrm{M}_{\mu\mu}^{(2,2)}$ contribute to the $\mathbb{O}\left(\mu^3\right)$ terms of the Evans function, which have been dropped. This fact may be considered as the justification for not computing them explicitly.

Chapter 4

Examples

In this section, we consider two different types of F(u) in (1.0.1). Our goal would be to apply Theorem 1 to determine the stability of periodic travelling wave solutions to (1.0.1) corresponding to these two forms of $F(u)$.

4.1 Cubic Nonlinearity

We first consider the case $\mathrm{F}(\phi) = \phi^3 - \phi$ in (2.1.2). This is an example from the category of power-law nonlinearity which has been widely studied by many authors, especially in the context of KdV family. The potential in (2.1.3), corresponding to this cubic nonlinearity, would be $\mathrm{P}\left(\phi\right) = \frac{\phi^4}{4} - \frac{\phi^2}{2}$ and (2.2.1) would give the period as

$$T\left(E\right) = \sqrt{2\left(1-c^2\right)} \int_{\phi_0}^{\phi_1} \frac{d\phi}{\sqrt{\left(E - \frac{\phi^4}{4} + \frac{\phi^2}{2}\right)}} \tag{4.1.1}$$

The period cannot be differentiated with respect to total energy E right away, since the integral is not in a regularized form. The approach, therefore, would be to first express $T(E)$ in terms of an elliptic integral. This will allow us to differentiate the period directly with respect to E.

Consider the polynomial $\mathrm{R}\left(\phi\right) = \mathrm{E} - \frac{\phi^4}{4} + \frac{\phi^2}{2}$. The bi-quadratic form of $\mathrm{R}\left(\phi\right)$ indicates that we can factorize it as a product of two quadratic polynomials in ϕ which in turn will enable the period to be expressed as a complete elliptic integral of first kind.

Consider the following factorization:

$$R\left(\phi\right) = E - \frac{\phi^4}{4} + \frac{\phi^2}{2} = (a - b\phi^2)(g + d\phi^2) \tag{4.1.2}$$

where a, b, g and d are all real, positive and non-zero coefficients. We have three known coefficients in $\mathrm{R}\left(\phi\right)$ but four unknown coefficients in the quadratic factors. Without any loss of generality, we set $\mathrm{a} = 1$ and proceed to obtain the other three unknown coefficients. A straightforward comparison of the two sides of (4.1.2) gives

$$g = E, \qquad d = \frac{1 \pm \sqrt{1+4E}}{4} \qquad b = \frac{1}{1 \pm \sqrt{1+4E}}$$

Thus we have $\mathrm{E} - \frac{\phi^4}{4} + \frac{\phi^2}{2} = (1 - \mathrm{b}\phi^2)(\mathrm{E} + \mathrm{d}\phi^2)$ where b and d are as obtained above. Next, setting $1 - \mathrm{b}\phi^2 = \eta$, we get

$$R(\phi) = \eta^2 \left(\frac{bE + d}{b}\right)\left(1 - k^2\eta^2\right)$$
$$\text{where} \qquad k^2 = \frac{d}{bE + d} \tag{4.1.3}$$

and the period can subsequently be expressed as

$$T(E) = -2\Theta(E)\,K(k) \tag{4.1.4}$$

where

$$\Theta(E) = \sqrt{\frac{1 - c^2}{\Omega}}$$
$$\Omega = \sqrt{1 + 4E}$$
$$K(k) = \int_0^1 \frac{d\eta}{\sqrt{(1 - \eta^2)(1 - k^2\eta^2)}} \tag{4.1.5}$$

$K(k)$ is the complete elliptic integral of first kind with modulus $\mathrm{k} \in (0, 1)$.

Remark 7. 1. *Imposing* $\mathrm{k} \in (0,1)$ *in* (4.1.3)*, we obtain a constraint on the energy as* $E > 0$*. Using this constraint in the expressions for* b *and* d *and recalling that the coefficients are all real positive, we can drop the negative sign and retain only the '+' sign in the expressions for* b *and* d*. Thus we have*

$$k(E) = \sqrt{\frac{1 + \frac{1}{\Omega}}{2}} \qquad b = \frac{1}{1 + \Omega} \qquad d = \frac{1 + \Omega}{4}$$

2. *The constraint* $E > 0$ *obtained for the energy is reasonable for a physical system.*

3. *Change of limits of integration to* 0 *and* 1 *in the expression for period is indeed possible due to translational invariance of the problem.*

The period has thus been expressed in terms of a complete elliptic integral of first kind and in this form, it can be differentiated directly with respect to E. Using the product and chain rules of differentiation, we get

$$T_E(E) = -\left[\Theta_E(E)\,K(k) + \Theta(E)\,K'(k)\,k_E(E)\right] \tag{4.1.6}$$

Differentiating Θ(E) with respect to E we get,

$$\Theta_E(E) = -\frac{\sqrt{\frac{1-c^2}{\Omega}}}{\Omega^2} \tag{4.1.7}$$

Similarly, a direct differentiation of k(E) with respect to the energy E yields,

$$k_E(E) = -\frac{1}{\Omega^3\sqrt{2+\frac{2}{\Omega}}} \tag{4.1.8}$$

Next, the derivative of K(k) with respect to its modulus k can be expressed as [3]:

$$\frac{dK(k)}{dk} = \frac{M(k) - (1-k^2)K(k)}{k(1-k^2)} \tag{4.1.9}$$

where M(k) is the complete elliptic integral of second kind. The commonly used notation for complete elliptic integral of second kind is E(k) [3]. However, in order to avoid confusing it with the total energy E in this paper, we use the notation M(k). Using the series representation for K(k) and M(k) given in [3], we have

$$M(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3k^4}{16} - \frac{45k^6}{64} - \frac{1575k^8}{256} - \frac{99225k^{10}}{1024}...\right)$$

and

$$K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{16} + \frac{225k^6}{64} + \frac{11025k^8}{256} + \frac{893025k^{10}}{1024} + ...\right)$$

Substituting the above series expansions in (4.1.9), we have

$$\frac{dK(k)}{dk} = \frac{\pi}{2k(1-k^2)}\left(\frac{1}{4} + \frac{k^2}{16} + \frac{177k^4}{64} + \frac{9945k^6}{256} + \frac{842625k^8}{1024} + ...\right) \tag{4.1.10}$$

Clearly,

$$K(k) > 0 \quad \text{and} \quad \frac{dK(k)}{dk} > 0 \qquad \forall k \in (0,1) \tag{4.1.11}$$

Thus we now have the following summary: $\forall\, |\mathrm{c}| < 1$ and $\forall \mathrm{E} > 0$ we have $\Theta(\mathrm{E}) > 0$ (by definition), $\Theta_{\mathrm{E}}(\mathrm{E}) < 0$ from (4.1.7) and $\mathrm{k_E}(\mathrm{E}) < 0$ from (4.1.8). These facts, along with (4.1.11), when substituted into (4.1.6), shows that $\mathrm{T_E}(\mathrm{E}) > 0$ and we arrive at the following corollary to Theorem 1:

Corollary 1. *Periodic travelling wave solutions to nonlinear Klein-Gordon equations with cubic nonlinearity are stable to small localized perturbations when the total energy of the base wave is real positive and the speed of the base wave is less than unity.*

4.2 Sine Gordon Equation

We next verify our result for the case of Sine Gordon equation. The nonlinearity here is given by $\mathrm{F}(\phi) = -\sin(\phi)$ and the corresponding potential is given by $\mathrm{P}(\phi) = \cos(\phi)$. The period then turns out to be

$$T(E) = \sqrt{2\left(1-c^2\right)} \int_{\phi_0}^{\phi_1} \frac{d\phi}{\sqrt{\left(E - \cos(\phi)\right)}} \tag{4.2.1}$$

The calculations involved in this case are relatively more straightforward compared to the case of cubic nonlinearity. Writing $\phi = 2\theta$ and working through some algebra, we get the period as

$$T(E) = 2\sqrt{2}\sqrt{\frac{(1-c^2)}{(E+1)}} \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{1 - k^2 \cos^2(\theta)}}$$

where $\mathrm{k}^2 = \frac{2}{\mathrm{E}+1}$. Finally, effecting a change of variables as $\cos(\theta) = \beta$, we can express the period as an elliptic integral of first kind as

$$T(E) = -2\sqrt{2}\ \Psi(E)\, K(k) \tag{4.2.2}$$

where

$$\Psi(E) = \sqrt{\frac{1-c^2}{E+1}}$$
$$K(k) = \int_0^1 \frac{d\beta}{\sqrt{(1-\beta^2)(1-k^2\beta^2)}}$$
$$k(E) = \sqrt{\frac{2}{E+1}}$$

Again, $K(k)$ is the complete elliptic integral of first kind with modulus $k \in (0,1)$. Note that the Ψ mentioned above is not related to that mentioned in definition 2. Also for $k(E) \in \mathbb{R}$, it is required that $E > -1$. Differentiating (4.2.2) with respect to E by employing product and chain rules, we obtain

$$T_E(E) = -2\sqrt{2}\left[\Psi_E(E)\,K(k) + \Psi(E)\,K'(k)\,k_E(E)\right] \tag{4.2.3}$$

Differentiation of $\Psi(\mathrm{E})$ and $\mathrm{k}(\mathrm{E})$ yields respectively

$$\Psi_E(E) = -\frac{\sqrt{1-c^2}}{2(E+1)^{\frac{3}{2}}}$$

and

$$k_E(E) = -\frac{1}{\sqrt{2}(E+1)^{\frac{3}{2}}}$$

We now proceed to analyze the stability of periodic traveling wave solutions to the sine-Gordon equation.

Case 1: When $\mathrm{E} > 1$ and $|\mathrm{c}| < 1$, we have $\mathrm{k} \in (0,1)$ from the definition of $k(E)$. Hence (4.1.11) holds true in this case. Also, $\Psi(\mathrm{E}) > 0$, $\mathrm{K(k)} > 0$, $\Psi_{\mathrm{E}}(\mathrm{E}) < 0$ and $\mathrm{k_E}(\mathrm{E}) < 0$ $\forall \mathrm{E} > 1$. Using all these facts in (4.2.3), we can conclude that $\mathrm{T_E}(\mathrm{E}) > 0$ and by Theorem 1, the periodic travelling wave solutions to the Sine Gordon equation are stable in this case.

Case 2: When $|\mathrm{E}| < 1$ and $|\mathrm{c}| < 1$, we have $\mathrm{k} > 1$. The signs of all terms in (4.2.3) remain same as in the previous case except for the derivative $\mathrm{K}'(\mathrm{k})$. Recalling (4.1.10), we observe that for $k > 1$ the derivative becomes negative and $\frac{\mathrm{dK(k)}}{\mathrm{dk}} \to -\infty$ when $\mathrm{k} \to \infty$. Using these facts in (4.2.3) once again, we come across an interesting phenomenon. The term $\mathrm{T_E}(\mathrm{E})$ consists of a positive and a negative term. We can therefore expect it to change its sign as the energy varies in the interval $(-1,1)$. Physically, this means that the period $\mathrm{T}(\mathrm{E})$ passes through an extremum as the total energy varies between -1 and 1. Applying Theorem 1, we can conclude that the periodic travelling waves switch their stability as the period varies across its extremum.

On the spectral plane, this phenomenon corresponds to the eigenvalue starting in the left and right half planes and collapsing onto the imaginary axis or vice versa.

Two more categories of traveling wave solutions and their stability are considered in [1] and [12]. We will not discuss those cases here since we are interested only in waves with speed $|c| < 1$; although, appropriate scaling can ensure that this constraint on the wave speed can almost always be fulfilled. We now have another corollary to Theorem 1:

Corollary 2. *Periodic traveling wave solutions to the Sine Gordon equation*

1. *are stable to small localized perturbations when* $E > 1$ *and* $|c| < 1$*.*
2. *switch stability when* $|E| < 1$ *and* $|c| < 1$*.*

Chapter 5

Discussion and Conclusions

We have presented an analysis of the stability of traveling wave solutions to the nonlinear Klein-Gordon equation (1.0.1) and derived a stability index whose signature decides the stability of the solution to small localized perturbations. The stability index has been expressed in terms of physical quantities which can be computed directly for a given wave. The main result of the paper is a theorem which states that the base wave does not experience any instability to small perturbations when $\frac{\mathrm{dT(E)}}{\mathrm{dE}} \geq 0$, where T(E) is the period of the base wave. The theorem was then applied to two specific types of nonlinearity and the conclusions obtained thereby, were compared to those obtained in [1] and were observed to be in agreement with the latter. Also, the boundary conditions associated with the original problem have been preserved throughout our analysis, thereby maintaining consistency between the original problem and the linearized eigenvalue problem. The periodic Evans function technique is found to be very useful in the sense that it allows us to work with any general form of the nonlinearity and acts as a tool to study the spectrum of the quadratic eigenvalue pencil conveniently. The simplicity of the stability index is rather interesting and is based on physical quantities which exist and can be computed for any periodic traveling wave; the result is not confined to any specific type of nonlinearity. Interestingly, the stability criterion derived in this paper for periodic travelling waves is similar in nature to that derived for the orbital stability of solitary waves of KdV in [13]. According to the analysis presented in that work, the solitary waves of KdV are stable if the *momentum* (an invariant functional) is a non-decreasing function of wave speed.

The stability theory presented here, however, does not predict a global picture for the stability of the traveling periodic wave. At a distance considerably far from the origin on the complex spectral plane, the spectrum might branch off the imaginary axis, thereby giving rise to exponential growth and decay of perturbations. Also, the stability index derived here does not give a count of number of unstable and stable eigenvalues. It is useful in describing the qualitative behaviour of the spectrum of the quadratic eigenvalue pencil under perturbation.

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