SIMULINK TOOLBOX FOR $\mathcal{L}_1$ ADAPTIVE CONTROL

BY

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THESIS

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ABSTRACT

This document presents our work on Simulink Toolbox for $\mathcal{L}_1$ control. It starts with a brief idea on what parameters define an $\mathcal{L}_1$ controller. The document then explains how our toolbox provides an interface for selecting those parameters. Following the interface is a short review of inner working of the toolbox. Finally, setup and results of validation tests performed on the toolbox are given. For each test, the document explains control problem formulation along with theorems on bounds of controller performance. Setup information for each tests includes controller parameter setting through the toolbox user interface, whereas, validation results include system output as compared to already published $\mathcal{L}_1$ control results.

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CHAPTER 1

INTRODUCTION

When designing controller for any plant, an exact model is often not available. Hard to model and sense uncertainties, such as dynamic friction, external forces or added dynamics due to damaged parts can play a significant part in plant dynamics. A good controller needs to be able to counter those uncertainties.

Adaptive control is used for controlling such plants with uncertainties. The key idea here is to estimate these uncertainties by observing state or output of the system and then adapting to them accordingly [1, 2]. Although, the objective is often not to have perfect estimation. Rather it is to get; what may be a non-accurate estimate of uncertainties; albeit one that makes the overall system behave well.

Model Reference Adaptive Control (MRAC) has been shown to have good features in terms of performance [3]. For example, MRAC that ensure arbitrarily close performance to the desired dynamics have been developed [4]. For MRAC, fast adaptation is often required to ensure close to desired performance. Unfortunately, this can make the system less robust [5]. Furthermore, robustness of conventional MRAC to unmodeled dynamics have been shown to be weak [6, 7].

$\mathcal{L}_1$ control addresses this robustness problem by decoupling it with adaptation rate [8]. Therefore, for an $\mathcal{L}_1$ controller, adaptation rate can be set arbitrarily high, whereas, robustness is ensured by adding a low pass filter to the controller. At the end, $\mathcal{L}_1$ control guarantees bounds on transient performance while providing guaranteed robustness. This makes $\mathcal{L}_1$ control useful in several applications.

Since the introduction of $\mathcal{L}_1$ control in [9], $\mathcal{L}_1$ has found application in safety critical applications such as aviation [10–13], high uncertainty problems such as pressure drilling [14, 15] and high precision problems such as hard disk control. Fortunately, the whole gamut

of $\mathcal{L}_1$ controllers; designed for different problems; has a consistent design structure. This motivates a toolbox for a single $\mathcal{L}_1$ control toolbox that can be used implement $\mathcal{L}_1$ control for a wide variety of problems.

A Matlab based toolbox was developed for $\mathcal{L}_1$ control [16]. Unfortunately, this toolbox simulated $\mathcal{L}_1$ for only a limited number of plants with a few allowed parameter adjustments. Now we have developed a full fledge Simulink [17] toolbox that implements $\mathcal{L}_1$ controller for matched uncertainty problems. This allows using complete functionality of Simulink with the toolbox. Furthermore, the toolbox design allows easy inclusion of $\mathcal{L}_1$ control for a wider array of problems such as those with unmatched uncertainty, but this is left as future work.

The rest of this document is divided as follows:

- Chapter 2 gives a brief introduction to structure of an $\mathcal{L}_1$ control. The chapter explains what parameters are required to completely specify an $\mathcal{L}_1$ controller.
- Chapter 3 introduces Graphical User Interface (GUI) for the toolbox. In this chapter, a thorough explanation of all parameter setting options presented in the GUI are explained.
- Chapter 4 presents a brief text on the inner working of the toolbox.
- Chapter 5 includes several problem formulations presented in [8, 9, 18] along with theorems providing bounds on $\mathcal{L}_1$ performance. For each of these problem formulations, we present details on validation tests that we performed on our toolbox. We present process of setting up the toolbox along with results that match published $\mathcal{L}_1$ control results. Setup details for the toolbox serve as examples of use of the toolbox.
- Chapter 6 is a short conclusion overviewing the document.

CHAPTER 2

$\mathcal{L}_1$ CONTROLLER AND ITS PARAMETERS

An $\mathcal{L}_1$ controller has three main components/blocks as shown in Figure 2.1 and described below:

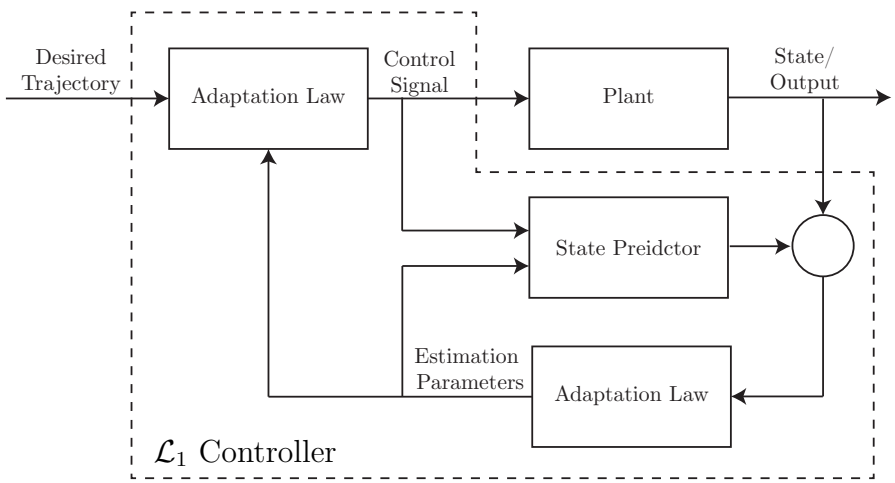


Figure 2.1: An $\mathcal{L}_1$, as shown in the dashed box, has three components: Adaptation Law, State Predictor and Control Law.

2.1 Adaptation Law

Uncertainties in the plant are estimated using the adaptation law. Controller states such as $\hat{\omega}$ and $\hat{\theta}$ etc., also known as estimation parameters, are updated in the adaptation block of the controller. We will see later in the document the meaning of these controller states. Here we just give an overview of what parameters are needed for these controller states.

The adaptation law is implemented for each estimation parameter independently with equations that have the following form:

$$\dot{\hat{\theta}}(t) = \Gamma \text{Proj}(\hat{\theta}, -\tilde{x}^T(t)Pbx(t)), \qquad \hat{\theta}(0) \in \Theta$$

where

$$A_m P^T + PA_m = -Q \qquad \textit{where } Q = Q^T > 0$$

Here Proj is an operator that is used to keep estimation parameter (in this case $\hat{\theta}$) within some specified bounds. See [8] for details on this operator.

Depending on the what type of uncertainties are present in the plant, adaptation of these parameters needs to be turned on or off in the controller.

2.1.1 Parameters for State Predictor

We need to set the following controller parameters for the adaptation law block

- Γ is the adaptation gain.
- Q matrix in the Lyapunov equation for P.
- A boolean coefficient for all estimation parameters (such as $\hat{\omega}$ and $\hat{\theta}$ etc.) for turning them on or off.
- Initial value for estimation parameters.
- Projection bounds for estimation parameters either in the form of minimum and maximum value or a center of projection along with radius from the center.

2.2 State Predictor

Structure of state predictor is dependent on what kind of uncertainties are present in the plant. Fortunately, the difference is only missing or additional terms in the structure. We

write a following example structure for state predictor:

$$\begin{aligned}\dot{\hat{x}} &= A_m\hat{x}(t) + b\,(\hat{\omega}(t)u(t) + \\ &\qquad \hat{\theta}^T(t)x(t) + \hat{\theta}^T_{nonlin}(t)||x(t)||_\infty + \hat{\sigma}(t)\Big) \\ \hat{y} &= c^T\hat{x}(t)\end{aligned} \tag{2.2.1}$$

2.2.1 Parameters for State Predictor

The parameters that need to be set for state predictor are as below:

- $\hat{x}$ is the state of predictor, as described in Equation (2.2.1). We need to set its initial value $\hat{x}_0$.
- A_m, b and c coefficients are derived from the plant but need to be set for the controller. The GUI enforces them to have compatible sizes with each other and $\hat{x}_0$.

Depending on the plant structure, some of the terms that involve adaptation parameters may not be a part of the state predictor. If they are not part of the state predictor, the only modification required is removal of the relevant term(s). For example if $\hat{\theta}$ needs to be removed, we will have a simpler state predictor as follows:

$$\begin{aligned}\dot{\hat{x}} &= A_m\hat{x}(t) + b\,(\hat{\omega}(t)u(t) + \\ &\qquad \hat{\theta}^T_{nonlin}(t)||x(t)||_\infty + \hat{\sigma}(t)\Big) \\ \hat{y} &= c^T\hat{x}(t)\end{aligned}$$

The only exception to this rule is $\hat{\omega}$ which is set equal to 1 instead of removing of $\hat{\omega}(t)u(t)$ term.

We will see exact form of state predictor for different problem formulations later in this document.

2.3 Control Law

This is the part of controller where the controller calculates its output - the control signal. This block also depends on what parameters' adaptation is turned on and off. We here give a rather simple example of what dynamics of controller block may look like

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + b(u_{ad}(t) + \hat{\theta}(t)\hat{x}(t)), \qquad \hat{x}(0) = x_0,$$
$$\hat{y}(t) = c^T\hat{x}(t)$$

Depending on what estimation parameters are turned on or off, the exact dynamics of controller vary. But this does not affect the parameters required by the control law which are given as follows

2.3.1 Parameters for State Predictor

The only parameter required for state predictor is as follows

- A transfer function for the dynamics of controller.

CHAPTER 3

GUI FOR CONTROLLER PARAMETER SETTING

The GUI parameter selection is designed for easy input of controller parameters for matched and unmatched uncertainty problems. Parameters specific to unmatched uncertainty problems are disabled in the current version. As an added feature GUI allows to add extra input and output ports to the controller e.g. ($\hat{x}$) can be input from a separate input port.

The controller block has some values of all parameters stored. As soon as GUI is called, all the fields in the GUI get filled according to the stored parameter values. After modifying them, the user can press the 'Apply' button on the GUI (see, for example, in Figure 3.1). Pressing of apply button does two things

- Check if the values set by user are valid. If some invalid entry (e.g. a string value in place of number or wrong sizing of matrices) is observed, a dialog box appears describing why the entries are invalid. Details on what checks are performed on parameter values follow in this section.

- If all entries are valid, the old parameter values are overwritten by new ones.

There are in total 5 tabs in the GUI interface. We explain them one by one as follows:

3.1 State Predictor Tab

This tab is used to set parameters for the state predictor. There are four fields and one radio box in this tab as shown in Figure 3.1.

- Am: This is the input field for Hurwitz matrix A_m for the state predictor. GUI does not allow to input anything other than a Hurwitz matrix in this field.

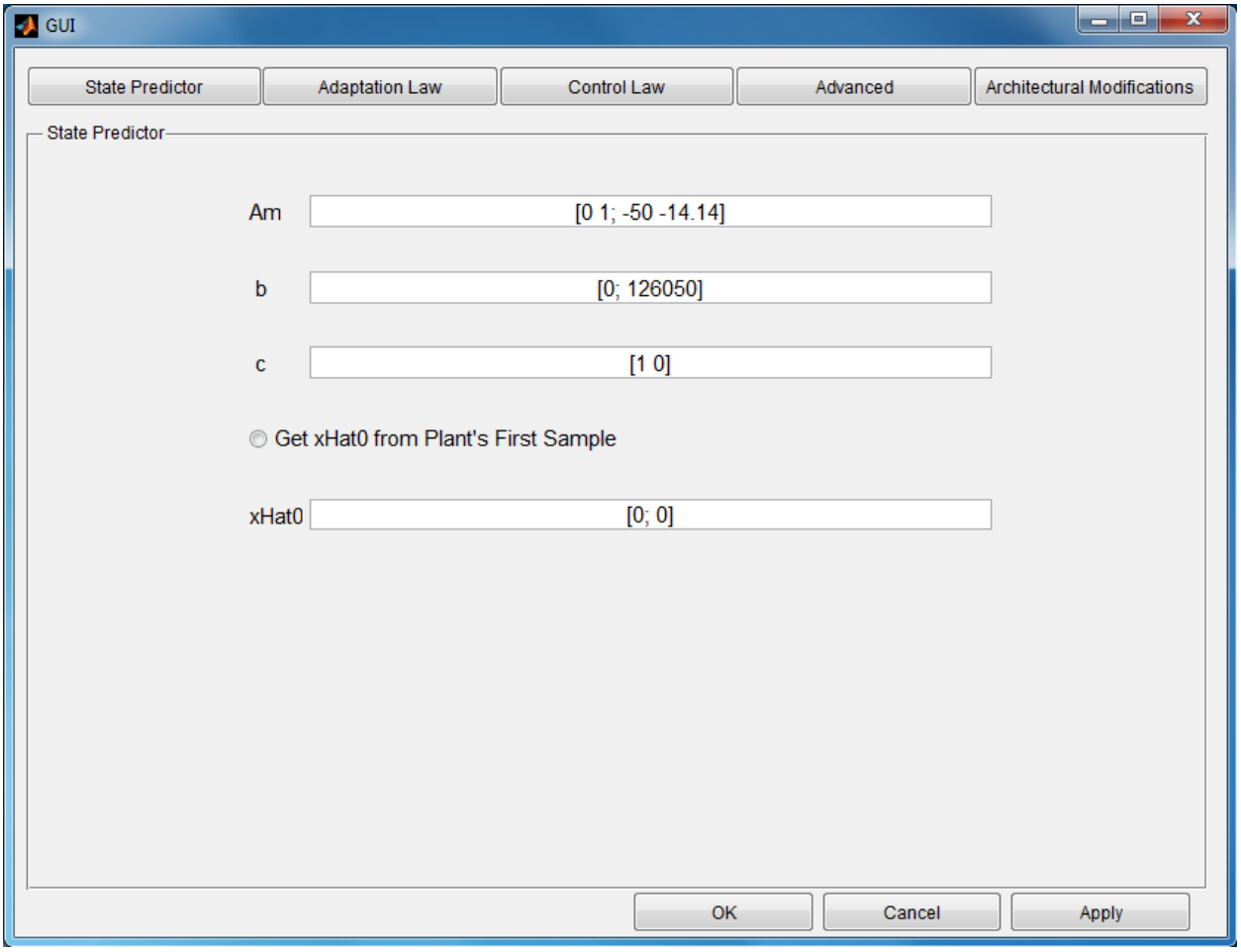


Figure 3.1: A view of State Predictor Tab in the GUI

- b: This is the b matrix for the state predictor. GUI allows only vectors that have dimensionality compatible to A_m.

- c: This is the c matrix for the state predictor. GUI allows only row vectors that have dimensionality compatible to A_m.

- Get xhat0 from Plant's First Sample: This radio box is for future implementation. If it is selected, the next field 'xhat0' gets disabled and initial value of plant state x is used as $\hat{x}_0$.

- xHat0: This is used to set the initial value $\hat{x}_0$ for the state predictor if the previous radio button is disabled. GUI allows only vectors that have dimensionality compatible to that of A_m.

3.2 Adaptation Law Tab

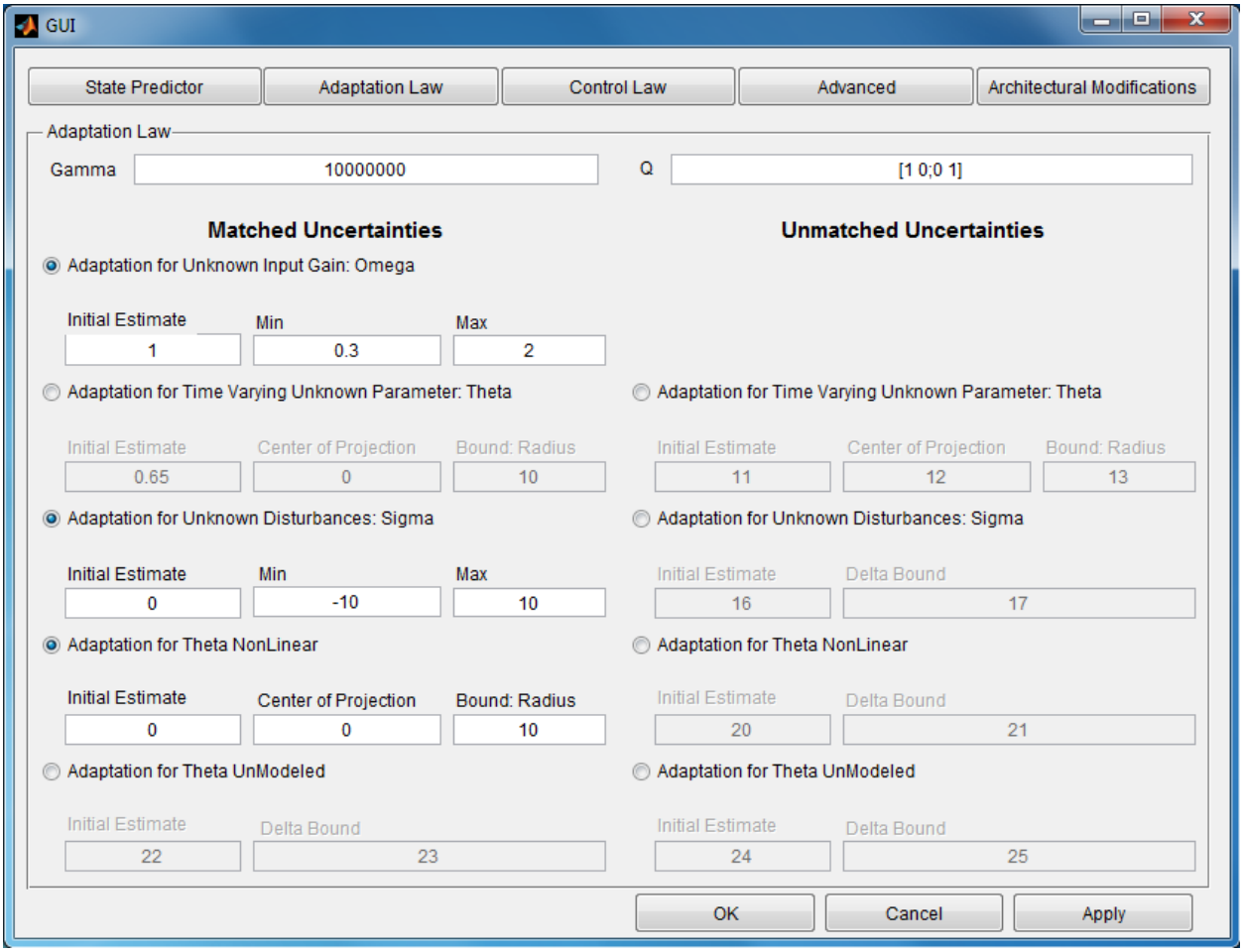


Figure 3.2: A view of Adaptation Law Tab in the GUI

This tab is used to set parameters for the adaptation block of the controller and is shown in Figure 3.2. A description of fields and radio buttons for this tab are as follows:

- Gamma: This is the adaptation gain Γ for the adaptation block. If advanced settings are not used (explained in Section 3.4), this is used as adaptation gain for all estimation parameters. GUI only allows positive values for it.

- Q: This is the Q matrix used to find P matrix in the adaptation law through the Lyapunov equation $A_m P^T + P A_m = -Q$. Square matrix with same dimensionality as A_m are allowed by GUI.

- For all estimation parameters, we have
  - Radio button to turn adaptation on or off. If adaptation is turned off, the rest of the fields for that parameter get disabled.
  - Initial value for the parameter. GUI enforces it to have correct dimension. GUI enforces it to have value within projection limits set by the next two fields.
  - The last two fields for all estimation parameters require either the center and radius for projection, or minimum and maximum value if it is a scalar parameter. GUI enforces these fields to have correct dimensionality. If max and min are used, the value in max field has to be greater than that in min.

  Only the first four estimation parameters are active in the current version of the toolbox.

3.3 Control Law Tab

The control law tab is shown in Figure 3.3 and allows to set parameters for the control block of the controller. The exact meaning of fields is explained below:

- Low Pass Filter: This drop down menu allows to set the way transfer function is entered. Options given are transfer function and state space form. But currently, only transfer function form is accepted.
- Numerator: This field is used to enter the numerator polynomial of the transfer function.
- Denominator: This field is used to enter the denominator polynomial of the transfer function.
- K: This field is redundant but allows a separate constant multiplier for the transfer function. GUI allows only stable transfer functions. Furthermore, whenever adaptation for Omega is turned on, the GUI only allows transfer functions that have a pole at the origin.

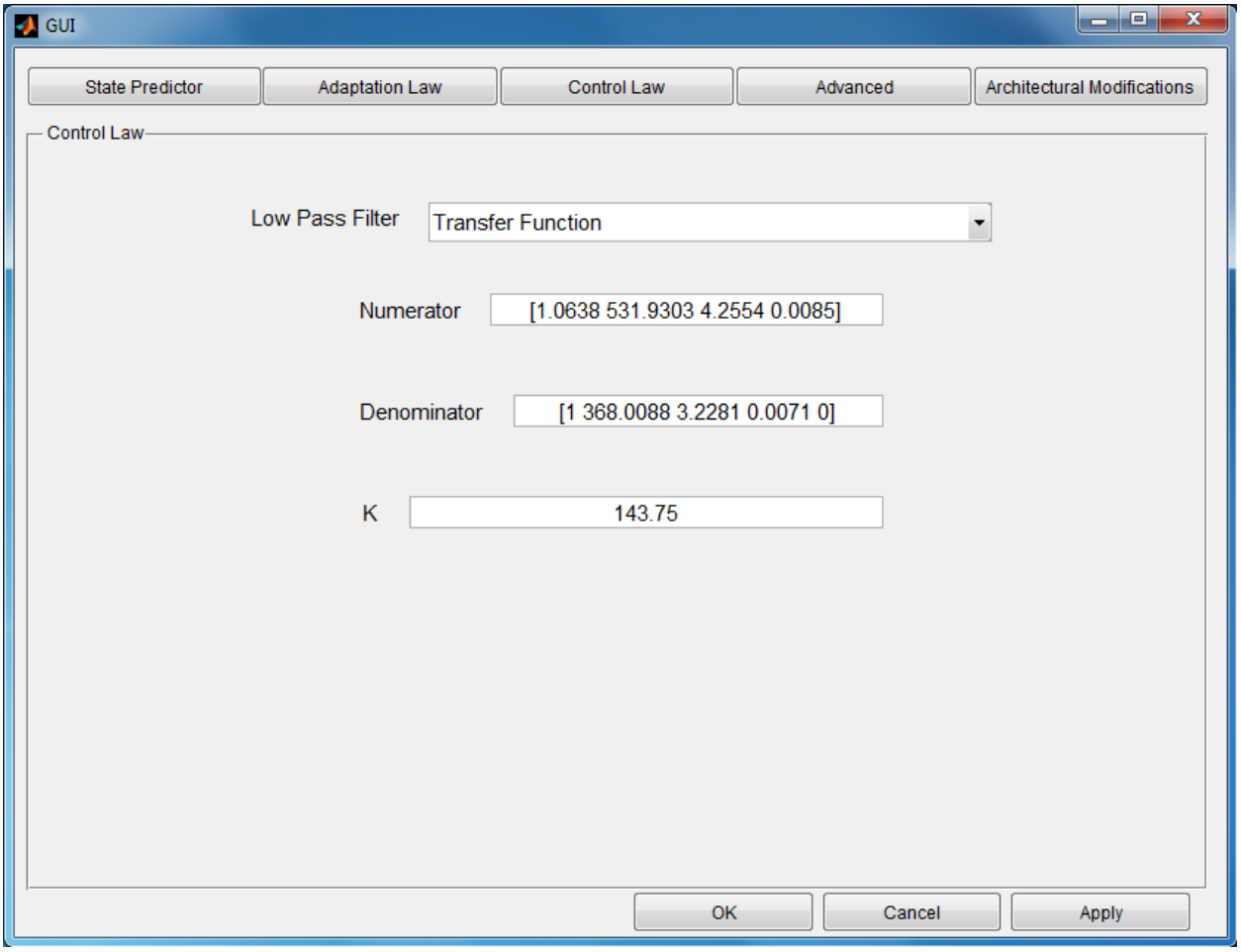


Figure 3.3: A view of Control Law Tab in the GUI

3.4 Advanced Tab

The advanced tab allows a few advanced settings for adaptation block. It is shown in Figure 3.4.

Fields of advanced tab are as below:

- Epsilon for projection: This is a smoothing parameter, always greater than zero, is used for smoothing when the estimation parameter is blocked to go outside projection bounds. See [8] for details.
- Ellipse for transformation: This matrix allows an elliptic boundary for bounds for $\hat{\theta}$ estimation parameter. GUI enforces it to have same dimensions as A_m.
- Use Advanced Options: This radio button allows to use separate values of adaptation

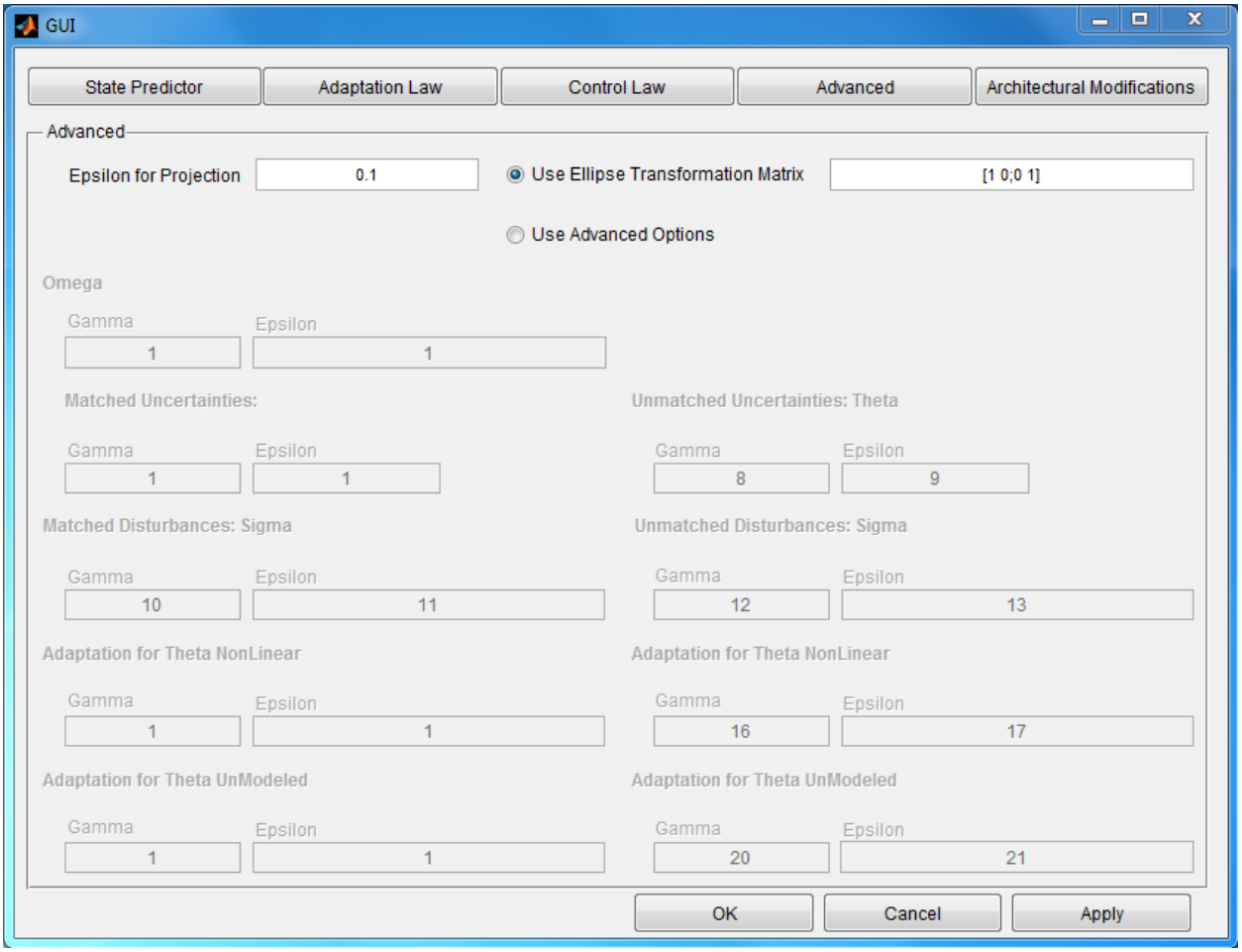


Figure 3.4: A view of Advanced Tab in the GUI

gain for every estimation parameter. Setting it on, enables all the fields below it on the GUI.

- For every estimation parameter, we have
  - Gamma: This allows to set adaptation gamma for that particular estimation parameter. GUI enforces this parameter to be a positive scalar.
  - Epsilon: This allows to set smoothing parameter epsilon to be set for that particular parameter. GUI allows only positive constants in this field.

    Both the above fields are enabled if and only if 'Use Advanced Options' radio button is set to on and estimation for that parameter is turned on.

3.5 Architectural Modifications Tab

This tab allows to add or remove input an output ports from the controller block. It is shown in Figure 3.5.

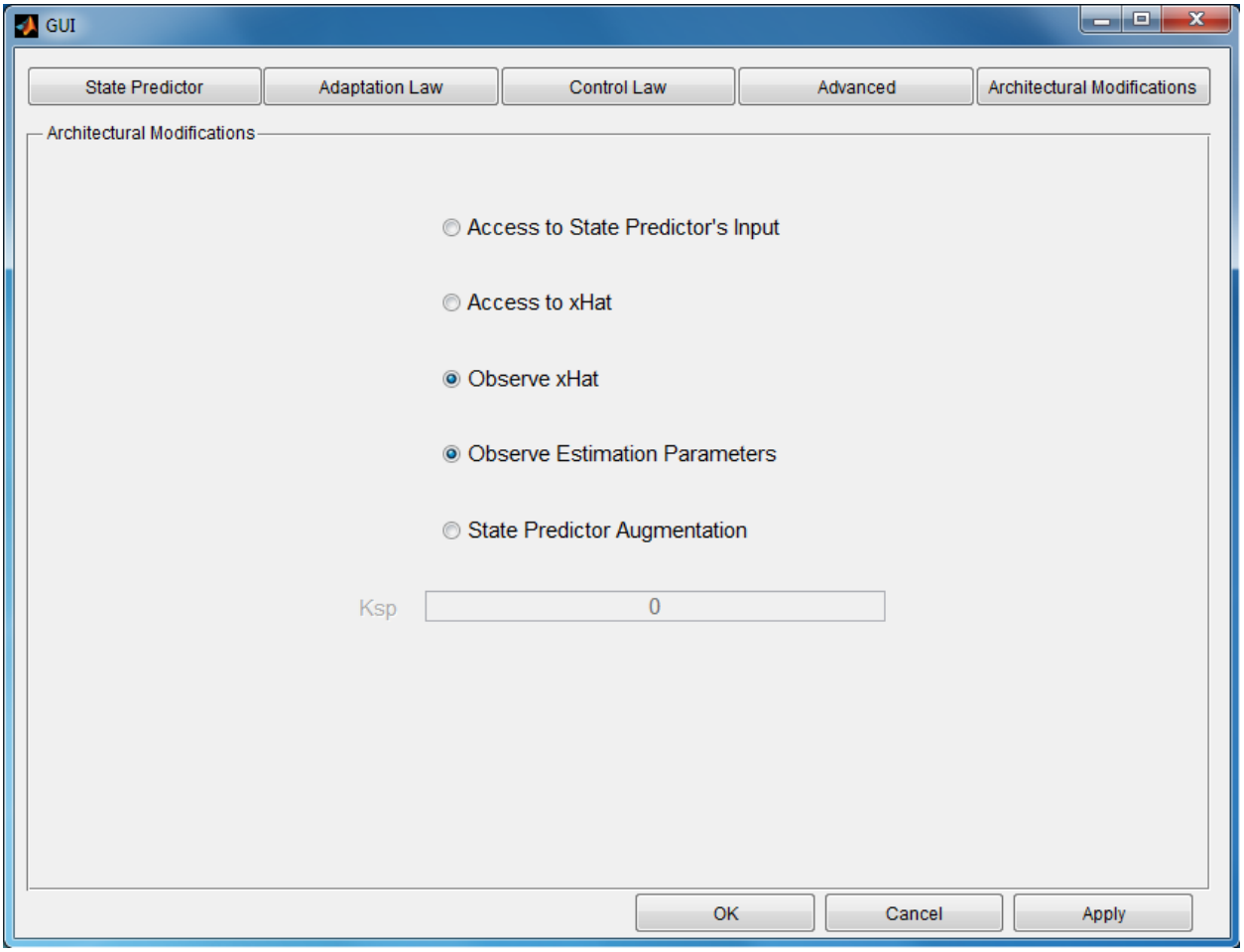


Figure 3.5: A view of Architectural Modifications Tab in the GUI

The exact meaning of all the options in this tab is shown below:

- Access to State Predictor's Input: Turning this option on makes an input port in the controller where the user can feed in the control signal. Then this control signal is used in the state predictor block instead of the actual control signal, the output of the controller.

- Access to xHat: This option turns on an input port that allows user to feed in controller state $\hat{x}$ to the controller. The controller then uses the state fed in by the user instead

of calculating its own.

- Observe xHat: This opens up an output port for state predictor $\hat{x}$
- Observe Estimation Parameters: This opens up an output port for estimation parameters.
- State Predictor Augmentation: This feature is currently unavailable.

CHAPTER 4

INNER WORKING OF THE TOOLBOX

Simulink allows complicated systems to be covered under 'masks' as single box. This way, complicated systems appear as one box that can be used as part of a larger system.

This controller is also a mask for a system which is actually rather simple. A snapshot of the system is shown in Figure 4.1.

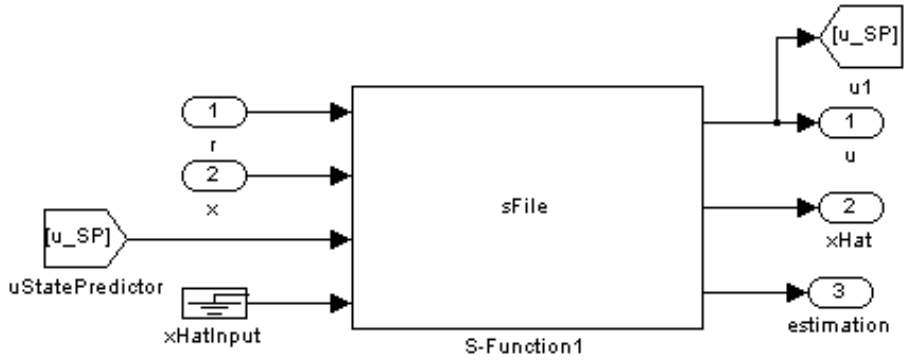


Figure 4.1: A snapshot of the system serving as the controller under the mask

Covering this system (shown in Figure 4.1) under a mask allows changing the input and output ports of the controller. In the figure, we can see a block by the name sFile. This block implements all the dynamics of the controller. This inner block has ports to input everything that can be an input for the controller and outputs everything that can be an output for the controller. But depending on the parameter settings from the architectural modifications tab, these inputs and outputs are either routed to ports of the controller or blocked by dummy variables.

The block by the name sFile runs a mex file. Mex files Matlab compatible file that are compiled from C/C++ or Fortran code. We wrote our code in C++. Furthermore, for

efficient implementation of linear algebra files, we use Eigen 3.1.2 library [19]. This library is available for free under MPL2 license.

For implementation as a Simulink block, the code has to be an S-function. We have developed the code as a level-2 S function. The important functions of such files are as follows:

- void mdlInitializeSizes(SimStruct *S): This function is called to set correct dimensions of the block e.g. number of input and output ports and their type. Furthermore, states of the block are also initialized in this function.
- void mdlInitializeConditions(SimStruct *S): This function is called at the start of simulation to set initial values of controller states.
- void mdlOutputs(SimStruct *S, int_T tid): This function is called during execution and is used to calculate the output of the block.
- void mdlDerivatives(SimStruct *S): This function is used derivatives of block states. All dynamics of controller are implemented here.

CHAPTER 5

TOOLBOX VALIDATION WITH PUBLISHED RESULTS

In this chapter, we will show results of our toolbox and compare them to published results. The purpose is to validate working of the toolbox. Furthermore, we will go over parameter parameter settings through controller GUI for these tests. This will serve as examples for use of GUI.

5.1 System with Unknown Constant Parameters

5.1.1 Problem Formulation

Consider the following LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b(u(t) + \theta x(t)), \qquad x(0) = x_0, \\ y(t) &= c^T x(t) \end{aligned} \tag{5.1.1}$$

Here $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are the state and output of the system, whereas $u \in \mathbb{R}$ represents the control signal. A, b and c are known system parameters such that (A, b) is controllable. θ is the unknown parameter that belongs to convex set Θ.

The objective is to make $y(t)$ follow desired trajectory $r(t)$ with some bounds on transient and steady state performance.

5.1.2 $\mathcal{L}_1$ controller

The $\mathcal{L}_1$ control output is described as follows:

$$u(t) = u_m(t) + u_{ad}(t), \qquad u_m(t) = -k_m^T x(t) \tag{5.1.2}$$

With this input, we can write system dynamics (Equation (5.1.1)) as

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(u_{ad}(t) + \theta(t)x(t)), \qquad \hat{x}(0) = x_0, \\ y(t) &= c^T x(t) \end{aligned}$$

where $A_m \triangleq A - bk_m$ is a Hurwitz matrix of the desired system.

u_{ad} represents the dynamic part of $\mathcal{L}_1$ controller. We have the following dynamics for the controller

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(u_{ad}(t) + \hat{\theta}(t)\hat{x}(t)), \qquad \hat{x}(0) = x_0, \\ \hat{y}(t) &= c^T \hat{x}(t) \end{aligned} \tag{5.1.3}$$

The update law for the unknown parameter $\hat{\theta}$ is as follows:

$$\dot{\hat{\theta}}(t) = \Gamma \text{Proj}(\hat{\theta}, -\tilde{x}^T(t)Pbx(t)), \qquad \hat{\theta}(0) \in \Theta \tag{5.1.4}$$

where $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ and P satisfies the Lyapunov equation $A_m P^T + PA_m = -Q$ with $Q = Q^T > 0$. u_{ad} can now be calculated as:

$$u_{ad}(s) = -C(s)(\hat{\eta}(s) - k_g r(s)) \tag{5.1.5}$$

Here $k_g \triangleq \frac{-1}{c^T A_m^{-1} b}$ defines the feed-forward law and $\eta(s)$ is the Laplace Transform of $\hat{\eta}(t) \triangleq \hat{\theta}^T x(t)$. Finally, $C(s)$ is a strictly proper low-pass filter which is initialized with zero initial conditions. Furthermore, it has unity DC gain $C(0) = 1$ and satisfies the following relation

$$\lambda \triangleq ||G(s)||_{\mathcal{L}_1} L < 1 \tag{5.1.6}$$

where $L \triangleq \max_{\theta \in \Theta} ||\theta||_1$ and $G(s) \triangleq H(s)(1 - C(s))$ with $H(s) = (s\mathbb{I} - A_m)^{-1}b$.

5.1.3 Performance and Design of Controller

Now we present a few theorems that define bounds on performance of the controller. These bounds help in design of the controller as well. Before we go towards theorems, we introduce the following two systems

Reference System:

We consider the reference system defined as follows

$$\begin{aligned} \dot{x}_{ref}(t) &= Ax_{ref}(t) + b(u_{ref}(t) + \theta^T x_{ref}(t)), \qquad x_{ref}(0) = x_0, \\ y_{ref}(t) &= c^T x_{ref}(t) \end{aligned} \tag{5.1.7}$$

where

$$u_{ref}(s) = -k_m^T x_{ref} - C(s)(\theta^T x_{ref}(s) - k_g r(s)) \tag{5.1.8}$$

In this system, we assume that we have knowledge of θ, and therefore, we have it in the expression for $u(t)$. If $C(s) = 1$, we would cancel out all the uncertainties in Equation (5.1.7). But with $C(s)$ filtering out certain frequencies, we cancel out the uncertainties only in the bandwidth of the filter.

Desired System:

Now we consider the desired system defined as follows:

$$\begin{aligned} x_{des}(s) &= k_g H(s)C(s)r(s) + x_{in}(s), \\ u_{des}(s) &= -k_m^T x_{des}(s) + C(s)[k_g r(s) - \theta^T x_{des}(s)], \\ y_{des}(s) &= c^T x_{des}(s) \end{aligned} \tag{5.1.9}$$

where $x_{in}(s) = (s\mathbb{I} - A_m)^{-1}x_0$.

Notice that $k_g H(s)C(s)r(s) = k_g(s\mathbb{I} - A_m)^{-1}C(s)r(s)$ and that $k_g(s\mathbb{I} - A_m)^{-1}$ makes sure that the desired system tracks $C(s)r(s)$ with ideal dynamics. Furthermore, the inclusion of k_g makes the DC gain of this system equal to unity.

The following theorems will use the two system introduced above to provide bounds on performance of $\mathcal{L}_1$ controller.

We first present a theorem about stability of this reference system.

Theorem 1 *The reference system of Equation* (5.1.7) *and* (5.1.8) *is bounded-input bounded-state (BIBS) stable with respect to* x_0 *and* $r(t)$ *provided that Equation* (5.1.6) *with our standard definition of* $G(s)$ *and* L *is satisfied.*

This theorem along with with the proof appears in [8]. Now we present a theorem that bounds performance of the actual system with respect to the reference system.

Theorem 2 *The system of Equation* (5.2.1) *with controller of Equation* (5.1.2), (5.1.3), (5.1.4) *and* (5.1.5) *satisfies the following conditions:*

$$||x_{ref} - x||_{\mathcal{L}_\infty} \leq \frac{\gamma_1}{\sqrt{\Gamma}}$$
$$||u_{ref} - u||_{\mathcal{L}_\infty} \leq \frac{\gamma_2}{\sqrt{\Gamma}}$$
$$\lim_{t\to\infty} ||x_{ref}(t) - x(t)|| = 0$$
$$\lim_{t\to\infty} ||u_{ref}(t) - u(t)|| = 0$$

Here

$$\gamma_1 \triangleq \frac{||C(s)||_{\mathcal{L}_1}\sqrt{\theta_{max}}}{(1 - ||G(s)||_{\mathcal{L}_1}L)\sqrt{\lambda_{min}(P)}}$$
$$\gamma_2 \triangleq ||H_1(s)||_{\mathcal{L}_1}\sqrt{\frac{\theta_{max}}{\lambda_{min(P)}}} + \gamma_1||C(s)\theta^T + k_m^T||_{\mathcal{L}_1}$$

This theorem along with with the proof appears in [8, 9].

Here notice that $||x_{ref} - x||_{\mathcal{L}_\infty}$ can be bounded by arbitrarily small bound by choosing a higher value of Γ.

We now know performance bounds on our system with respect to x_{ref}. We now move to the desired system of Equation (5.1.9) with the following theorem.

Theorem 3 *The system of Equation* (5.1.1) *with controller of Equation* (5.1.2), (5.1.3), (5.1.4) *and* (5.1.5) *satisfies the following conditions*

$$
\begin{aligned}
||y_{ref} - y_{des}||_{\mathcal{L}_\infty} &\leq \frac{\lambda}{1-\lambda}||c^T||_1 \left(||k_g H(s)C(s)||_{\mathcal{L}_1}||r(s)||_{\mathcal{L}_\infty} + ||x_{in}||_{\mathcal{L}_\infty}\right), \\
||x_{ref} - x_{des}||_{\mathcal{L}_\infty} &\leq \frac{\lambda}{1-\lambda}\left(||k_g H(s)C(s)||_{\mathcal{L}_1}||r(s)||_{\mathcal{L}_\infty} + ||x_{in}||_{\mathcal{L}_\infty}\right), \\
||u_{ref} - u_{des}||_{\mathcal{L}_\infty} &\leq \frac{\lambda}{1-\lambda}||C(s)\theta^T + k_m^T||_{\mathcal{L}_1}\left(||k_g H(s)C(s)||_{\mathcal{L}_1}||r(s)||_{\mathcal{L}_\infty} + ||x_{in}||_{\mathcal{L}_\infty}\right)
\end{aligned}
$$

This theorem along with with the proof appears in [8].

It is clear that by reducing λ, x_{ref} comes arbitrarily close to x_{des}. Whereas, we saw earlier that the actual system state x comes arbitrarily close to x_{ref} with high enough Γ.

There is a catch here though. λ can be made smaller by increasing the bandwidth of $C(s)$. This results in reduction of robustness of the system. But with above theorems, we can design filter and adaptation such that bounds on x are acceptable.

5.1.4 Example Simulation on Toolbox

We consider this example published in [8, 9] to test and verify working of our toolbox. We repeat the example problem here as follows.

Plant:

The system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + b\left(u(t) + \theta^T x(t)\right) \qquad x(0) = x_0 \\
y(t) &= c^T x(t)
\end{aligned}
$$

has

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where θ, unknown to the controller is set as

$$\theta = \begin{bmatrix} -4 \\ 4.5 \end{bmatrix}$$

Controller Setting:

Most of the entries in the GUI interface of the toolbox are obvious for this problem. The only non-trivial design parameters are the conservative bound for θ and the filter. We stick to Example in [8, 9] and set bounds for θ to be $\Theta = \begin{bmatrix} \theta_{lim} \\ \theta_{lim} \end{bmatrix}$ where $\theta_{lim} = [-10\ 10]$.

We set filter to be

$$C(s) = \frac{160}{s + 160}$$

For this controller, $||G(s)||_{\mathcal{L}_1} L$ turn out to be $\sim 0.3 < 1$ and thus it satisfies the $\mathcal{L}_1$ controller requirements.

Controller Setting through Toolbox Interface:

With this finalized we can set the parameters of the controller as follows:

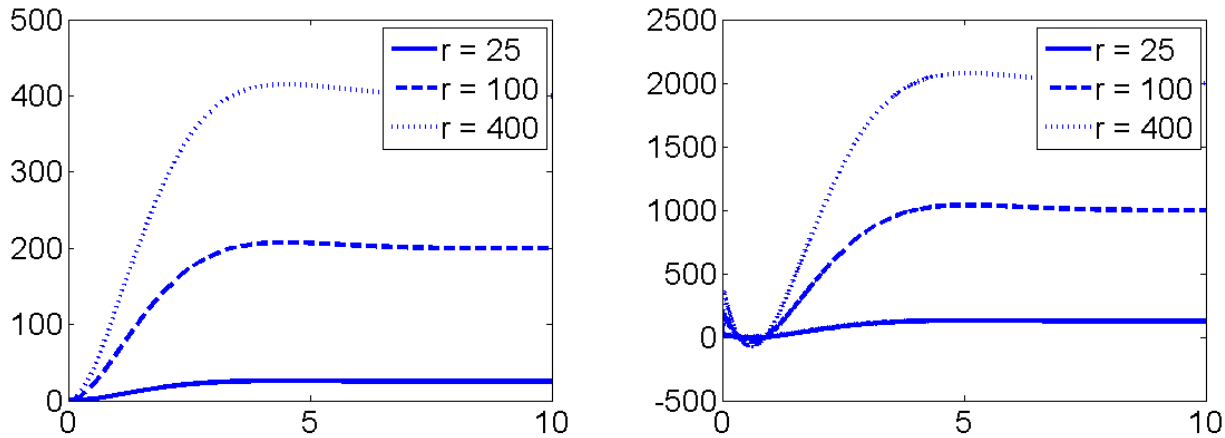
- State Predictor Tab:
  - $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
  - $\hat{x}$ from plant: Disabled
- Adaptation Tab:
  - Γ : 10000

- $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Adaptation for θ: On
- Adaptation for all the rest: Off
- Center of projection for $\hat{\theta}$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Bound for θ : 10

- Control Law Tab:
  - Low Pass Filter: Transfer Function
  - Numerator: [160]
  - Denominator: $\begin{bmatrix} 1 & 160 \end{bmatrix}$
  - K: 1

- Advanced Tab:
  - Epsilon for Projection: 1
  - Use Epsilon Transformation: Off
  - Use Advanced Settings: Off

- Architectural Modifications
  - Access to State Predictor's Input: Off
  - Access to $\hat{x}$: Off
  - Observe $\hat{x}$: On
  - Observe Estimation Parameters: On
  - State Predictor Augmentation: Off



(a) $y(t)$ (b) $u(t)$

Figure 5.1: $\mathcal{L}_1$ controller with constant trajectory tracking and controller setting as described in Section 5.1.4

Simulation Results:

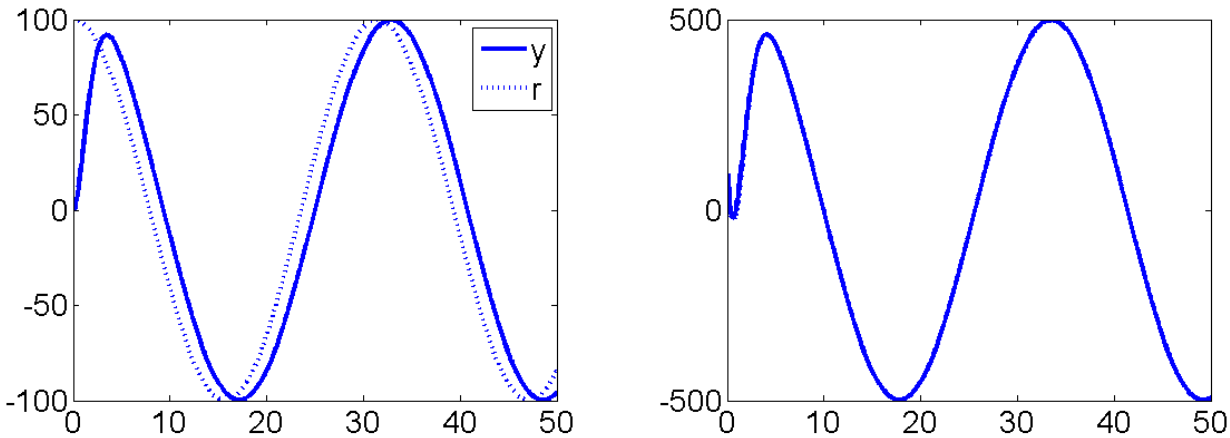
To have an exact comparison with tried and tested examples, we chose the same reference signals as presented in [8,9]. We set $r = 25, 100$ and 400 for testing the system with constant reference tracking. The results are plotted in 5.1a and 5.1b. It is clear that they match the expected result.

Testing with time varying reference was done by setting $r = 100\cos(0.2t)$. The results are plotted in Figure 5.2a and 5.2b. They also match the results shown in [8, 9].

Finally we change Γ and filter setting for a new test. This time we use controller setting as described in Section 5.1.4 with the following exceptions:

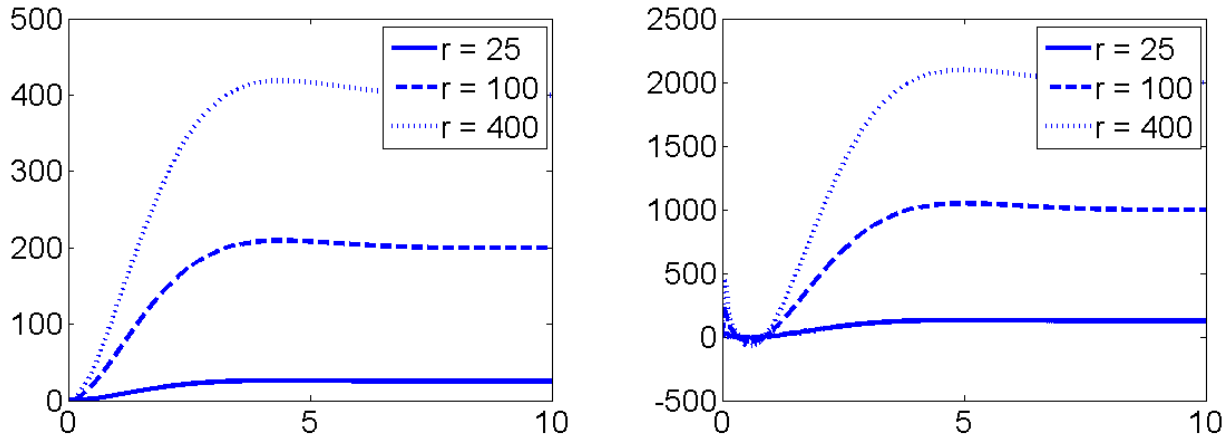
- Adaptation Tab: Γ: 400
- Control Law Tab: Numerator: [7500 125000]
- Control Law Tab: Denominator: [1 150 7500 125000]

The results of with this new parameter settings are shown in Figure 5.3a and 5.3a. They turn out to be as expected.



(a) $y(t)$ (b) $u(t)$

Figure 5.2: $\mathcal{L}_1$ controller with time-varying trajectory tracking and controller setting as described in Section 5.1.4



(a) $y(t)$ (b) $u(t)$

Figure 5.3: $\mathcal{L}_1$ controller with time-varying trajectory tracking and controller setting modified from Section 5.1.4

5.2 System with Uncertain Input Gain

5.2.1 Problem Formulation

Consider the LTI system

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(\omega u(t) + \theta(t)x(t) + \sigma(t)), \qquad x(0) = x_0, \\ y(t) &= c^T x(t) \end{aligned} \tag{5.2.1}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ represent the state and regulated output of the systems. A_m is a Hurwitz matrix specifying the desired dynamics for the system, whereas b and c are known system parameters. $\theta(t) \in \mathbb{R}^n$ and $\sigma(t) \in \mathbb{R}$ are time-dependent unknown parameters such that

$$\omega \in \Omega \qquad \theta(t) \in \Theta \qquad |\sigma(t)| < \Delta \qquad \forall t \geq 0$$

where $\Omega \triangleq (\omega_{lo}, \omega_{uo})$, Θ is a convex compact set and $\Delta \in \mathbb{R}^+$. All ω_{lo}, ω_{uo}, Θ and Δ are known.

Assumption 1 *The time-varying uncertainties $\theta(t)$ and $\sigma(t)$ are continuously differentiable with respect to time with the following bounds*

$$||\dot{\theta}(t)|| \leq d_\theta < \infty \qquad |\dot{\sigma}(t)| \leq d_\sigma < \infty \qquad \forall t \geq 0$$

The objective is again to have $y(t)$ follow a certain desired trajectory $r(t)$ with some bounds on the system's transient and steady state performance.

5.2.2 $\mathcal{L}_1$ controller

Notice that unlike system of Equation (5.1.1), our system under consideration (Equation (5.2.1)) already has A_m driving the system dynamics rather than A. This means that we can ignore

$u_m = -k_m^T x(t)$, this time through the controller design. Therefore, the control input u will now defined in a way that is similar to how u_{ad} was defined for system of Equation (5.1.1).

We have the following dynamics for the controller

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(\hat{\omega}(t)u(t) + \hat{\theta}(t)\hat{x}(t) + \hat{\sigma}(t)), \qquad \hat{x}(0) = x_0, \\ \hat{y}(t) &= c^T \hat{x}(t) \end{aligned} \tag{5.2.4}$$

The update law for the unknown parameters are as follows:

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \Gamma \mathrm{Proj}(\hat{\theta}, -\tilde{x}^T(t)Pbx(t)), \qquad \hat{\theta}(0) = \hat{\theta}_0 \in \Theta, \\ \dot{\hat{\sigma}}(t) &= \Gamma \mathrm{Proj}(\hat{\sigma}, -\tilde{x}^T(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0 \quad |\hat{\sigma}_0| < \Delta, \\ \dot{\hat{\omega}}(t) &= \Gamma \mathrm{Proj}(\hat{\omega}, -\tilde{x}^T(t)Pbu(t)), \qquad \hat{\omega}(0) = \hat{\omega}_0 \in \Omega \end{aligned} \tag{5.2.5}$$

$\tilde{x}$ is defined as before to be $\tilde{x}(t) = \hat{x}(t) - x(t)$ and P satisfies the equation $A_m P^T + PA_m = -Q$ where $Q = Q^T > 0$. The controller output can now be given as:

$$u(s) = -kD(s)(\hat{\eta}(s) - k_g r(s)) \tag{5.2.6}$$

Here $\hat{\eta}(t) = \hat{\omega}(t)u(t) + \hat{\theta}^T(t)x(t) + \hat{\sigma}(t)$ while $k_g = \frac{1}{c^T A_m^{-1} b}$. And $D(s)$ is a strictly proper low-pass filter such that

$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)}$$

is stable and strictly-proper.

Like before, we require

$$\lambda \triangleq ||G(s)||_{\mathcal{L}_1} L < 1 \tag{5.2.7}$$

with our usual definitions of $L \triangleq \max_{\theta \in \Theta} ||\theta||_1$ and $G(s) \triangleq H(s)(1 - C(s))$ where $H(s) = (s\mathbb{I} - A_m)^{-1} b$.

5.2.3 Performance and Design of Controller

Now we will present a few theorems that provide bounds on performance on the system. More theorems can be found in [8]. For, this we first need to define a reference system.

Reference System:

We consider the reference system defined as follows

$$\begin{aligned} \dot{x}_{ref}(t) &= A_m x_{ref}(t) + b(\omega u_{ref}(t) + \theta(t) x_{ref}(t) + \sigma(t)), \quad x_{ref}(0) = x_0, \\ y_{ref}(t) &= c^T x_{ref}(t) \end{aligned} \tag{5.2.8}$$

where

$$u_{ref}(s) = -C(s)(\eta_{ref}(s) - k_g r(s)) \tag{5.2.9}$$

This reference system is again a non-adaptive version of our $\mathcal{L}_1$ controller where we assume knowledge of unknown parameters. Since all uncertainties would cancel out if we were to put $C(s) = 1$, this system approaches the desired system with increasing bandwidth of $C(s)$.

The following theorems will prove stability of the reference system and show how the actual system response is related to the reference system response.

Theorem 4 *If Equation* (5.2.7) *is satisfied, the reference system of Equation* (5.2.8) *and* (5.2.9) *is bounded-input bounded-state (BIBS) stable with respect to* x_0 *and* $r(t)$.

This theorem and its proof is provided in [8].

Theorem 5 *The actual system given in Equation* (5.2.1) *with controller described in Equations* (5.2.4), (5.2.5) *and* (5.2.6) *satisfies the following relations*

$$||x_{ref} - x||_{\mathcal{L}_\infty} \leq \frac{\gamma_1}{\sqrt{\Gamma}}$$
$$||u_{ref} - u||_{\mathcal{L}_\infty} \leq \frac{\gamma_2}{\sqrt{\Gamma}}$$

Here

$$\gamma_1 \triangleq \frac{||C(s)||_{\mathcal{L}_1}}{1 - ||G(s)||_{\mathcal{L}_1} L} \sqrt{\frac{\theta_m}{\lambda_{min}(P)}}$$

$$\gamma_2 \triangleq \left|\left|\frac{C(s)}{\omega}\right|\right|_{\mathcal{L}_1} L\gamma_1 + \left|\left|\frac{H_1(s)}{\omega}\right|\right|_{\mathcal{L}_1} \sqrt{\frac{\theta_m}{\lambda_{min}(P)}}$$

where $H_1(s) = \frac{C(s)}{c_0^T H(s)} c_0$ *is proper and BIBO stable for some* c_0*. The existence of such a* c_0 *is proved as Lemma A.12.1 in [8].*

The proof of this theorem is also provided in [8].

Note that just as before, $||x_{ref} - x||_{\mathcal{L}_\infty}$ can be bounded by arbitrarily small bound by choosing a higher value of Γ.

We already know that the reference system approaches the desired system as bandwidth of C(s) increases. This, along with the theorem above implies that the response of the actual system can be arbitrarily close to that of the desired system with high enough adaptation gain Γ and large enough bandwidth of $C(s)$.

5.2.4 Example Simulation on Toolbox

We use the problem described in [8] to test out our toolbox.

Plant:

The plant that we use for testing is a simple single link robotic arm. The dynamics of this plant can be described as follows

$$I\ddot{q}(t) = u(t)\frac{Mgl\cos(q(t))}{2} + \bar{\sigma}(t) + F_1(t)q(t) + F(t)\dot{q}(t)$$

Here $q(t)$ represent the angular position of the arm. $\dot{q}(t)$ and $\ddot{q}(t)$ are therefore the angular velocity and acceleration of the arm. $u(t)$ represents the input to the system in the form of torque. The rest of the parameters are unknown: namely M representing the mass, l repre-

senting the length and I representing the moment of inertia of the arm. Furthermore, $F(t)$ is time-varying friction coefficient, $F_1(t)$ is external torque coefficient and $\bar{\sigma}(t)$ is disturbance. All of these parameters are also unknown.

By setting

$$\omega = \frac{1}{I} \qquad \theta(t) = \begin{bmatrix} 1 + \frac{F_1(t)}{I} \\ 1.4 + \frac{F(t)}{I} \end{bmatrix} \qquad \sigma(t) = \frac{Mgl\cos(x_1(t))}{2I} + \frac{\bar{\sigma}(t)}{I},$$

we can describe our system with the following equations:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\omega u(t) + \theta^T(t)x(t) + \sigma(t)), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned} \tag{5.2.12}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ where $x_1(t)$ represents $q(t)$, the angular position of the robotic arm.

It is clear that we have this system belonging to the class defined in Equation (5.2.1) with

$$A = A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

For the unknown parameters, we will pick one of the following settings for θ in our experiments:

$$\theta_1(t) = \begin{bmatrix} 2 + \cos(\pi t) \\ 2 + 0.3\sin(\pi t) + 0.2\cos(2t) \end{bmatrix}$$

$$\theta_2(t) = \begin{bmatrix} \sin(0.5\pi t) + \cos(\pi t) \\ -1 + 0.1\sin(3\pi t) \end{bmatrix}$$

$$\theta_3(t) = \begin{bmatrix} 4.5 \\ 3 - \sin(t) \end{bmatrix}$$

Similarly, we will pick σ and ω from the following options:

$$\begin{aligned} \sigma_1(t) &= \sin\left(\tfrac{\pi t}{2}\right), & \omega_1 &= 1, \\ \sigma_2(t) &= \cos(x_1(t)) + 2\sin(\pi t) + \cos\left(\tfrac{7\pi t}{5}\right), & \omega_2 &= 1.5, \\ \sigma_3(t) &= \cos(x_1(t)) + 2\sin(2\pi t) + \cos\left(\tfrac{16\pi t}{5}\right), & \omega_3 &= 0.8 \end{aligned}$$

With the plant described in Equation (5.2.12), we can now finalize our controller settings

Controller Setting:

We define the bounds for unknown parameters of the system as $\Omega = [0.1, 2]$ $\Delta = 50$ and $\Theta = \begin{bmatrix} \theta_{lim} \\ \theta_{lim} \end{bmatrix}$ where $\theta_{lim} = [-5, 5]$. Furthermore, we set $\Gamma = 100000$ and choose filter to be

$$D(s) = \frac{1}{s}$$

and $k = 60$.

With these settings, $||G(s)||_{\mathcal{L}_1} L < 1$ and therefore, we should expect our controller to yield good results.

Controller Setting through Toolbox Interface:

We can simulate these controller settings on our toolbox using using the following settings on its GUI interface:

- State Predictor Tab:
  - Am: $[01; -1 - 1.4]$
  - b: [0; 1]
  - c: [1 0]
  - Get xHat0 from Plant's First Sample: Off

  - xHat0: $[0; 0]$

- Adaptation Tab:
  - Γ : 100000
  - Q: $[1\ 0; 0\ 1]$
  - Adaptation for Omega: On
  - Initial Estimate for Omega: 0.1
  - Min for Omega: 0.1
  - Max for Omega: 2
  - Adaptation for Theta: On
  - Initial Estimate for Theta: $[0; 0]$
  - Center of Projection for Theta: $[0; 0]$
  - Bound for Theta: 5
  - Adaptation for Sigma: On
  - Initial Estimate for Sigma: 0
  - Min for Sigma: -50
  - Max for Sigma: 50
  - Adaptation for all the rest: Off

- Control Law Tab:
  - Low Pass Filter: Transfer Function
  - Numerator: $[1]$
  - Denominator: $[1\ 0]$
  - K: 60

- Advanced Tab:
  - Epsilon for Projection: 0.1

  - Use Epsilon Transformation: Off
  - Use Advanced Settings: Off

- Architectural Modifications
  - Access to State Predictor's Input: Off
  - Access to xHat: Off
  - Observe xHat: On
  - Observe Estimation Parameters: On
  - State Predictor Augmentation: Off

Simulation Results:

The simulation results with constant and time-varying trajectory tracking are shown in this section. To compare confirm validity of out toolbox results, we use the same settings as example in [8].

System response and controller output for different settings of σ are shown in Figure 5.4, 5.5 and 5.6. The reference trajectory is a sinusoid and is also plotted in the figures. The results are the same as published in [8] which validates our toolbox.

The controller performance with different σ and ω settings can be observed in Figure 5.4, 5.7 and 5.8 where σ is kept constant. These results also match the ones published in [8].

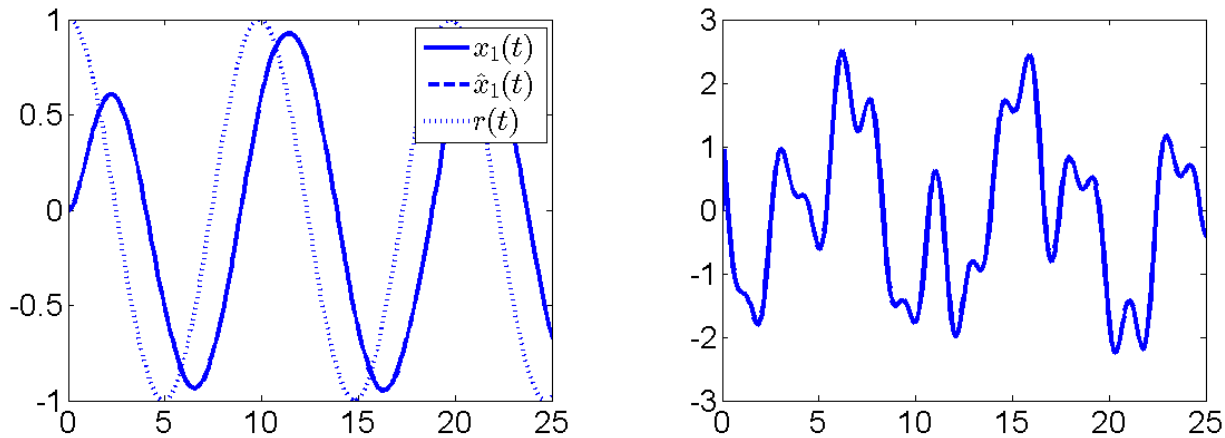
Controller performance for constant trajectory tracking is plotted in Figure 5.9.

To test the system with non-zero initialization error $\tilde{x}(t) = \hat{x} - x$, the parameter settings needed to be changed for the toolbox. Specifically, same settings as described in Section 5.2.4 are used with the following exception:

- State Predictor Tab: xHat0: $[1; 1]$

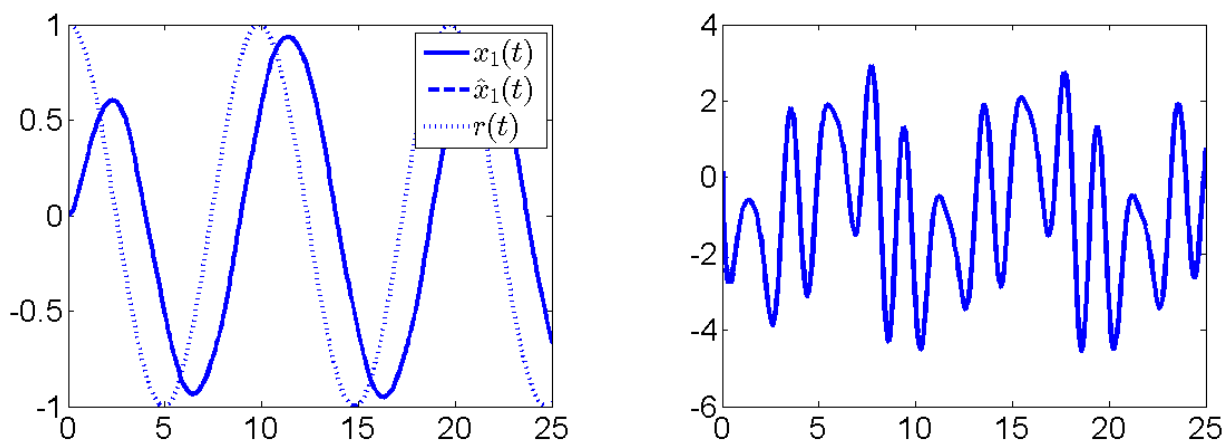
The results for constant and time-varying uncertainties with this new controller setting are shown in Figure 5.10 and 5.11 respectively.

To make the system robust against time-delays, changing the filter or projection bounds can help (See [8] for details). Both of these changes require controller parameters to be



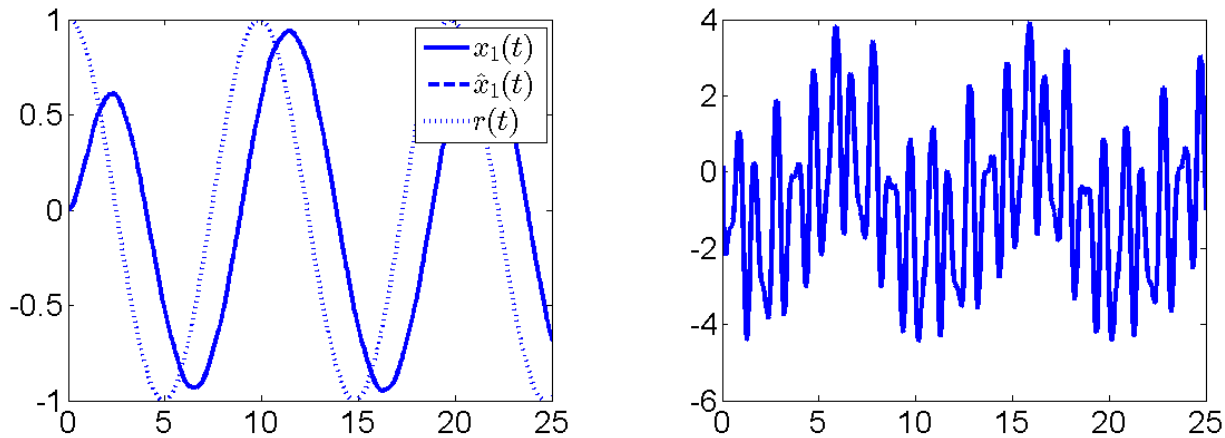
(a) Reference Trajectory and System Response (b) Controller Output $u(t)$

Figure 5.4: $\mathcal{L}_1$ controller with parameter settings as given in Section 5.2.4. Whereas, $\theta = \theta_1(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$



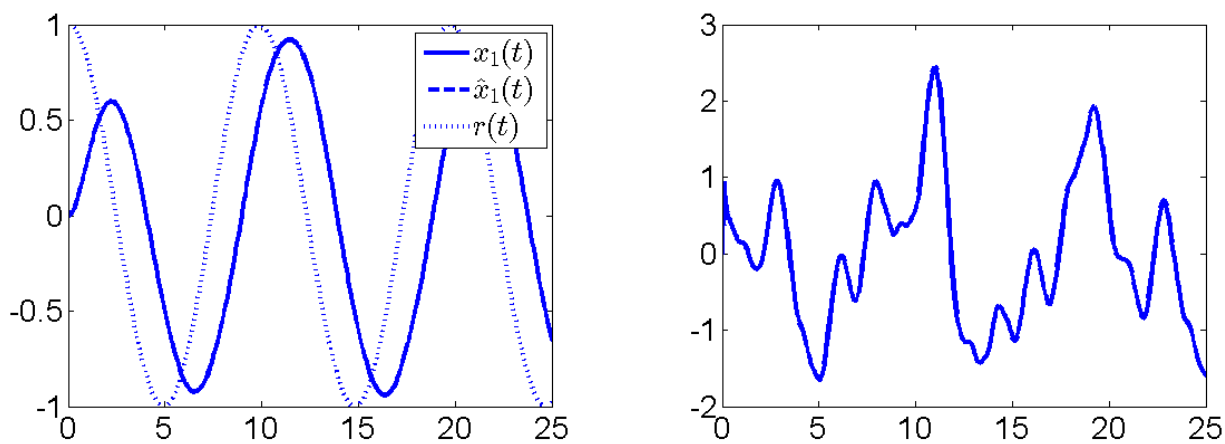
(a) Reference Trajectory and System Response (b) Controller Output $u(t)$

Figure 5.5: $\mathcal{L}_1$ controller with parameter settings as given in Section 5.2.4. Whereas, $\theta = \theta_1(t)$, $\sigma = \sigma_2(t)$ and $\omega = \omega_1$



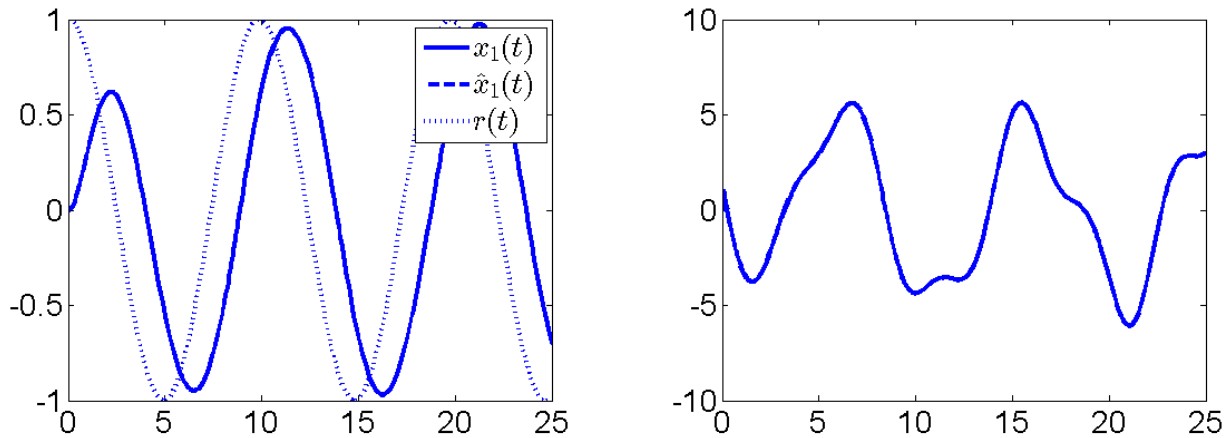
(a) Reference Trajectory and System Response (b) Controller Output $u(t)$

Figure 5.6: $\mathcal{L}_1$ controller with parameter settings as given in Section 5.2.4. Whereas, $\theta = \theta_1(t)$, $\sigma = \sigma_3(t)$ and $\omega = \omega_1$



(a) Reference Trajectory and System Response (b) Controller Output $u(t)$

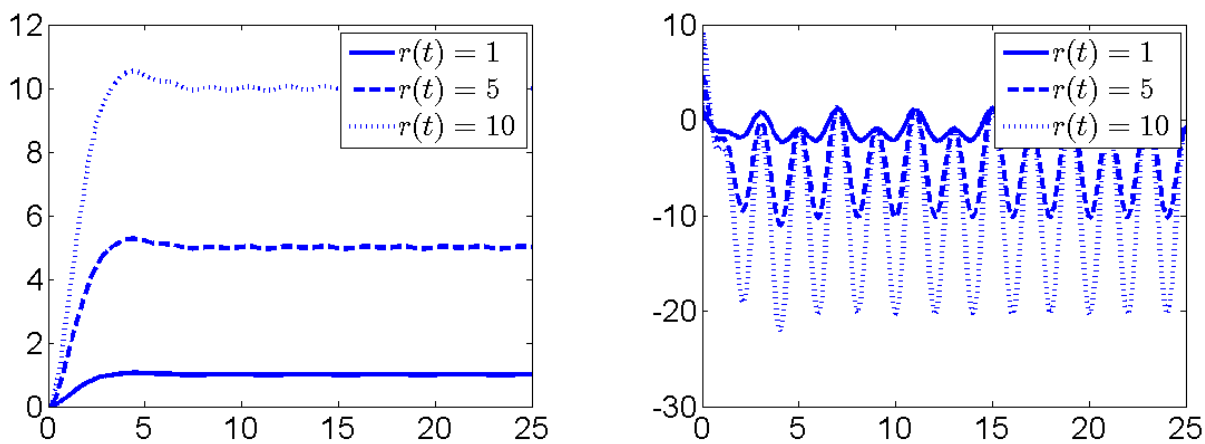
Figure 5.7: $\mathcal{L}_1$ controller with parameter settings as given in Section 5.2.4. Whereas, $\theta = \theta_2(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_2$



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

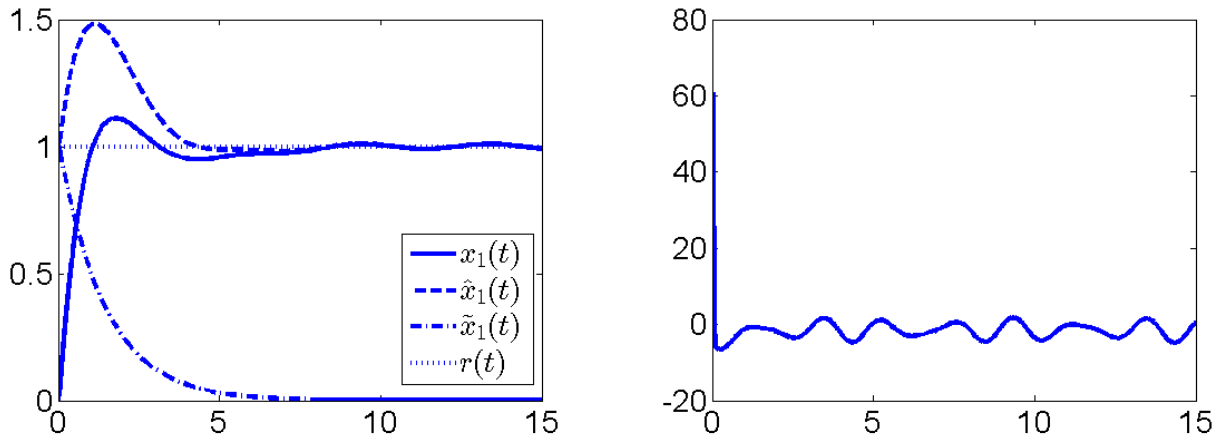
Figure 5.8: $\mathcal{L}_1$ controller with parameter settings as given in Section 5.2.4. Whereas, $\theta = \theta_3(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_3$



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

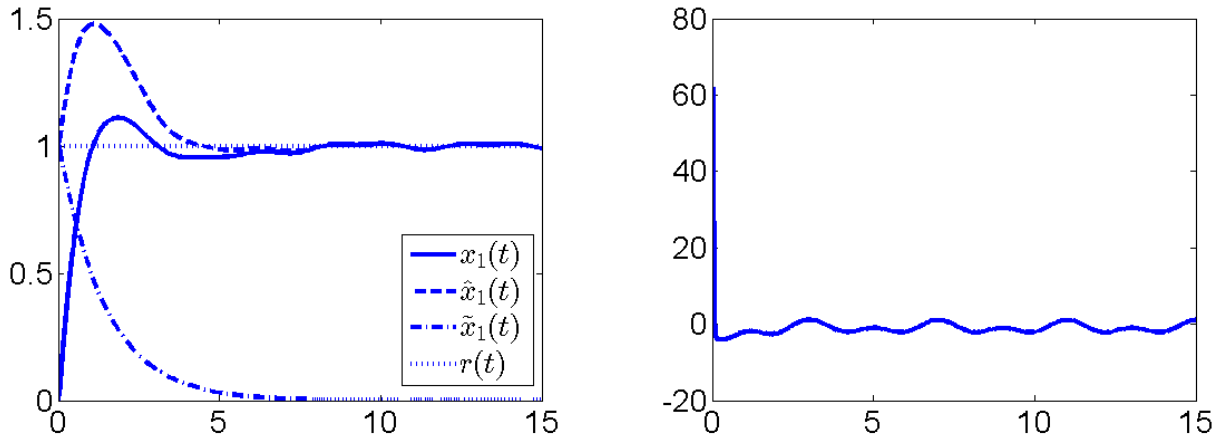
Figure 5.9: $\mathcal{L}_1$ controller with parameter settings as given in Section 5.2.4. Whereas, $\theta = \theta_1(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

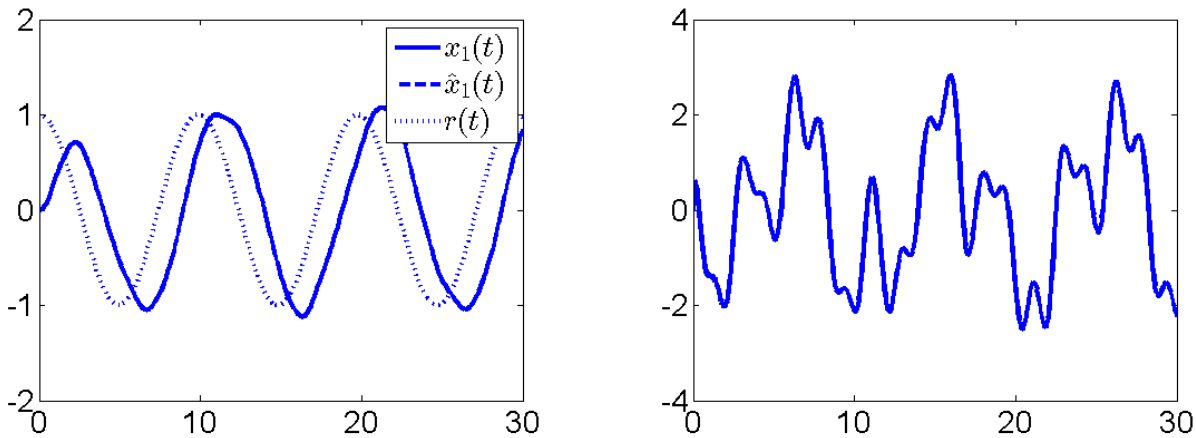
Figure 5.10: $\mathcal{L}_1$ controller with parameter settings modified from Section 5.2.4. Whereas, $\theta = [2\ 2]^T$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

Figure 5.11: $\mathcal{L}_1$ controller with parameter settings modified from Section 5.2.4. Whereas, $\theta = \theta_1(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

Figure 5.12: $\mathcal{L}_1$ controller with parameter settings modified from Section 5.2.4. Whereas, $\theta = \theta_1(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$

changed using its GUI. First, we change the filter and therefore set the parameters for the controller same as in Section 5.2.4 with the following exceptions:

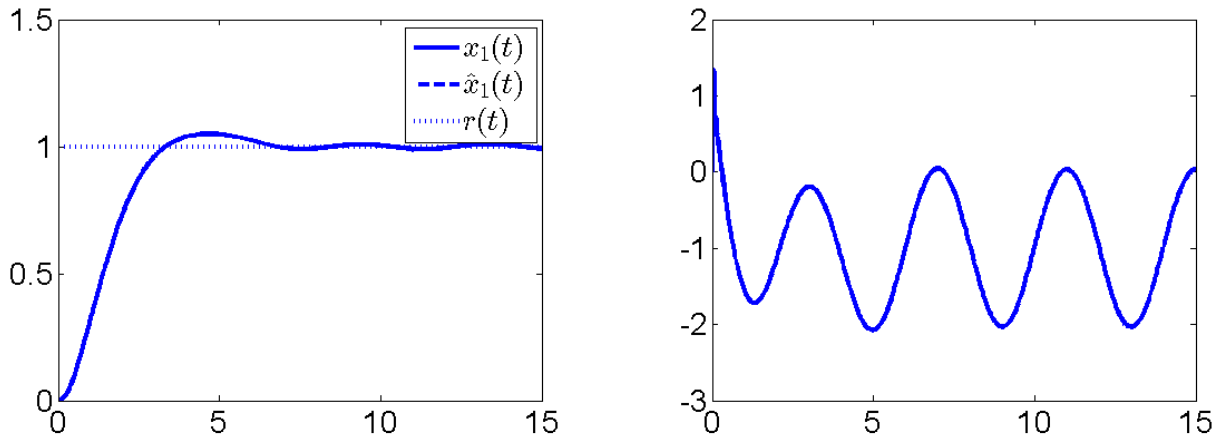
- Control Law Tab: Numerator: [1 0.1]
- Control Law Tab: Denominator: [1 0.9 0]
- Control Law Tab: K: 13

The results with this new setting and time-delayed system are shown in Figure 5.12.

Now we change the projection bounds for the controller. Namely, we set the controller parameters same as described in Section 5.2.4 except for the following:

- Adaptation Law Tab: Gamma: 100000000
- Adaptation Law Tab: Max for Omega: 1000000000
- Adaptation Law Tab: Min for Sigma: -1000000000
- Adaptation Law Tab: Max for Sigma: 1000000000

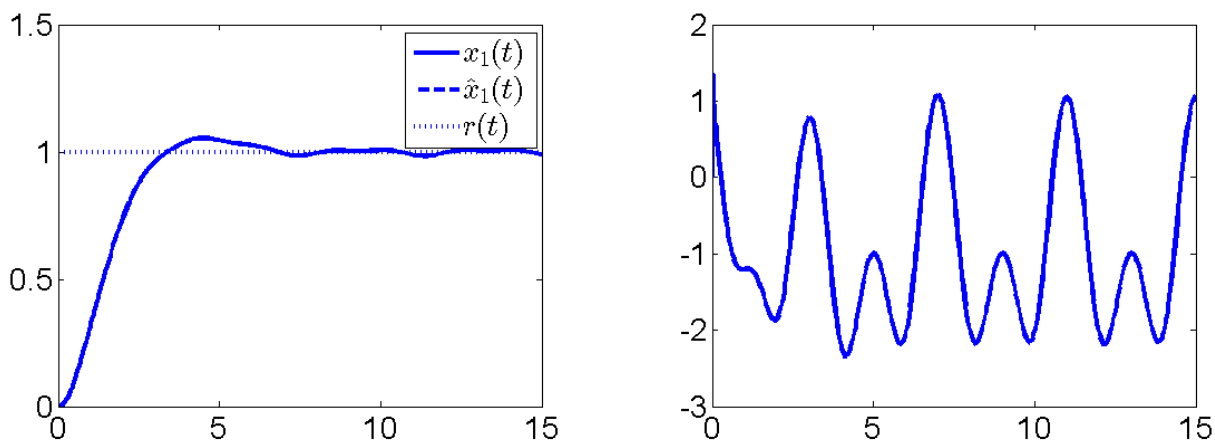
The results with these new controller settings for constant and time-varying uncertainties are plotted in Figure 5.13 and 5.14 respectively.



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

Figure 5.13: $\mathcal{L}_1$ controller with parameter settings modified from Section 5.2.4. Whereas, system has delay of 15ms and $\theta = [2\ 2]^T$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$



(a) Reference Trajectory and System Response

(b) Controller Output $u(t)$

Figure 5.14: $\mathcal{L}_1$ controller with parameter settings modified from Section 5.2.4. Whereas, system has delay of 15ms and $\theta = \theta_1(t)$, $\sigma = \sigma_1(t)$ and $\omega = \omega_1$

5.3 System with Unmodeled Actuator Dynamics

Now we will extend the $\mathcal{L}_1$ controller introduced in the last Section to systems with unmodeled actuator dynamics. The input term will, therefore, now be replaced by response of a system (actuator dynamics) acting on the input. The controller will remain the same but we would require different assumptions to be satisfied by the system and the controller and end up with similar but different bounds.

5.3.1 Problem Formulation

We consider the system

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(\mu(t) + \theta(t)x(t) + \sigma(t)), \qquad x(0) = x_0, \\ y(t) &= c^T x(t) \end{aligned} \tag{5.3.1}$$

such that

$$\mu(s) = F(s)u(s)$$

Once again $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are the state and output of the system. Hurwitz matrix A_m specifies the desired dynamics, whereas b and c are other system parameters that are known. Just as before, $\theta(t) \in \mathbb{R}^n$ and $\sigma(t) \in \mathbb{R}$ are not known but satisfy the following known bounds

$$\begin{aligned} &\theta(t) \in \Theta \\ &||F(s)||_{\mathcal{L}_1} \leq F_L \qquad F(0) \in \Omega \\ &|\sigma(t)| \leq \Delta_0 \qquad \forall t \geq 0 \end{aligned}$$

Here Θ is a known convex compact set, $F_L > 0, \Omega = [\omega_l, \omega_u]$ with $\omega_l, \omega_u \in \mathbb{R}$ and $\Delta_0 \in \mathbb{R}^+$.

Assumption 2 *We assume that both $\theta(t)$ and $\sigma(t)$ are continuously differentiable with re-*

spect to time and satisfy the following bounds on time-derivative

$$||\dot{\theta}(t)|| \leq d_\theta < \infty \qquad |\dot{\sigma}(t)| \leq d_\sigma < \infty \qquad \forall t \geq 0$$

We aim to make $y(t)$ follow $r(t)$ with a certain transient and steady state performance.

5.3.2 $\mathcal{L}_1$ controller

We have the same controller as before which we repeat here for sake of completion as follows.

We have the controller dynamics defined as follows

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b(\hat{\omega}(t)u(t) + \hat{\theta}(t)\hat{x}(t) + \hat{\sigma}(t)), \qquad \hat{x}(0) = x_0, \\ \hat{y}(t) &= c^T\hat{x}(t) \end{aligned} \tag{5.3.4}$$

The update laws also remain the same and are given as

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \Gamma\text{Proj}(\hat{\theta}, -\tilde{x}^T(t)Pbx(t)), \qquad \hat{\theta}(0) = \hat{\theta}_0 \in \Theta, \\ \dot{\hat{\sigma}}(t) &= \Gamma\text{Proj}(\hat{\sigma}, -\tilde{x}^T(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0 \quad |\hat{\sigma}_0| < \Delta, \\ \dot{\hat{\omega}}(t) &= \Gamma\text{Proj}(\hat{\omega}, -\tilde{x}^T(t)Pbu(t)), \qquad \hat{\omega}(0) = \hat{\omega}_0 \in \Omega \end{aligned} \tag{5.3.5}$$

P is the solution to Lyapunov equation $A_m^T P + PA_m = Q$ for an arbitrary $Q > 0$ and $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$. The projection operator keep $\hat{\theta} \in \Theta$, $\hat{\omega} \in \Omega$ and $|\hat{\sigma}_0| < \Delta$ where Δ is a computable bound for which details can be found in [8]. The control signal can now be calculated as

$$u(s) = -kD(s)(\hat{\eta}(s) - k_g r(s)) \tag{5.3.6}$$

We have the same $\hat{\eta}(t) = \hat{\omega}(t)u(t) + \hat{\theta}^T(t)x(t) + \hat{\sigma}(t)$ and $k_g = \frac{1}{c^T A_m^{-1} b}$ but $D(s)$, a strictly proper low-pass filter is now selected such that

$$C(s) = \frac{kF(s)D(s)}{1 + kF(s)D(s)}$$

is also stable and strictly-proper with unity DC gain.

Furthermore, we still require

$$\lambda \triangleq ||G(s)||_{\mathcal{L}_1} L < 1 \tag{5.3.7}$$

where $L \triangleq \max_{\theta \in \Theta} ||\theta||_1$ and $G(s) \triangleq H(s)(1 - C(s))$ where $H(s) = (s\mathbb{I} - A_m)^{-1} b$.

5.3.3 Performance and Design of Controller

Before we provide theorems that define bounds on system performance, we define the following reference system.

Reference System:

We consider the reference system defined as follows

$$\begin{aligned} \dot{x}_{ref}(t) &= A_m x_{ref}(t) + b(\mu_{ref}(t) + \theta(t) x_{ref}(t) + \sigma(t)), \quad x_{ref}(0) = x_0, \\ \mu_{ref}(s) &= F(s) r(s), \\ y_{ref}(t) &= c^T x_{ref}(t) \end{aligned} \tag{5.3.8}$$

where

$$u_{ref}(s) = -C(s)(\eta_{ref}(s) - k_g r(s)) \tag{5.3.9}$$

Continuing on the same strategy as before, we have introduced this reference system as the non-adaptive version of $\mathcal{L}_1$ controller. We assume knowledge of uncertainties here and try to cancel them out through input u. Notice that if $C(s) = 1 \quad \forall s$, all uncertainties would cancel out and this system would behave as the desired system.

Now we prove the stability of this reference system

Theorem 6 *The reference system given in Equation* (5.3.8) *and* (5.3.9) *is bounded-input*

bounded-state stable with respect to x_{in} and r with conservative bounds given as follows

$$||x_{ref}||_{\mathcal{L}_\infty} \leq \frac{||G(s)||_{\mathcal{L}_1}\Delta_0 + ||H(s)C(s)k_g||_{\mathcal{L}_1}||r||_{\mathcal{L}_\infty} + ||x_{in}||_{\mathcal{L}_\infty}}{1 - ||G(s)||_{\mathcal{L}_1}L} = \rho_r$$

Furthermore, bounds on u are given as

$$||u_{ref}||_{\mathcal{L}_\infty} \leq \left|\left|\frac{C(s)}{F(s)}\right|\right|_{\mathcal{L}_1} (|k_g|||r||_{\mathcal{L}_\infty} + L\rho_r + \Delta_0)$$

See [8] for proof of this theorem.

Now we provide a theorem that links actual system performance with that of the reference system.

Theorem 7 *If the adaptive gain is high enough to satisfy the following criterion*

$$\Gamma \geq \frac{(\omega_u - \omega_l)^2 + 4\max_{\theta\in\Theta}||\theta||^2 + 4\Delta^2 + 4\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\max_{\theta\in\Theta}||\theta||d_\theta + \Delta d_\sigma}{\lambda_{min}(P)\gamma_0^2}$$

then

$$\begin{aligned}
||\hat{x} - x||_{\mathcal{L}_\infty} &\leq \gamma_0 \\
||x_{ref} - x||_{\mathcal{L}_\infty} &\leq \frac{||C(s)||_{\mathcal{L}_1}}{1 - ||G(s)||_{\mathcal{L}_1}L}\gamma_0 + \beta = \gamma_1 \\
||u_{ref} - u||_{\mathcal{L}_\infty} &\leq \left|\left|\frac{C(s)}{F(s)}\right|\right|_{\mathcal{L}_1} L\gamma_1 + ||H_1(s)||_{\mathcal{L}_1}\gamma_0 = \gamma_2
\end{aligned}$$

Here γ_0 and β are arbitrarily small constants and H_1 is a BIBO stable system such that

$$H_1(s) = \frac{C(s)}{F(s)}\frac{1}{c_0^T H(s)}c_0^T \tag{5.3.13}$$

for some c_0.

The proof of this theorem can be found in [8]. Furthermore, as mentioned before, the proof of existence of a c_0 that satisfies Equation (5.3.13) is given as Lemma A.12.1 in [8].

We have shown that by selecting arbitrarily small γ_0 and β, we can make x approach x_{ref}

with arbitrarily small bounds. We also know that increasing bandwidth of $C(s)$ makes the reference system approach the desired system. Therefore, it follows that we can approximate the desired system with arbitrarily small errors by choosing correct value of control parameters.

5.3.4 Example Simulation on Toolbox

Plant:

We test our toolbox on Rohrs' example [6, 7]. This example was specifically designed for testing MRAC for robustness. Now we implement $\mathcal{L}_1$ controller using our toolbox on the same example and compare our results with already reported results in [8, 18].

The plant is given as

$$\begin{aligned} y(s) &= \frac{2}{s+1}\mu(s) \\ \mu(s) &= \frac{229}{s^2+30s+229}u(s) \end{aligned}$$

Whereas, the desired model can be given as

$$y_m(s) = \frac{3}{s+3}r(s)$$

To use $\mathcal{L}_1$ controller for this system, we first write it as follows

$$\begin{aligned} \dot{x}(t) &= -3x(t) = 2(\mu(t) + x(t)) \\ y(t) &= x(t) \end{aligned} \tag{5.3.15}$$

Controller Setting:

For controller parameters, we set $\hat{x}_0 = 0$, $\hat{\omega}_0 = 1.14$, $\hat{\theta}_0 = 0.65$ and $\hat{\sigma}_0 = 0$. For the bounds, we set $\Theta = [-10, 10]$, $\Delta = 10$ and $\Omega = [0.5, 5.5]$.

These settings match the settings in [8,18] and therefore, we expect to get the same results.

Controller Setting through Toolbox Interface:

We use the GUI interface to specify the controller settings to the toolbox as follows:

- State Predictor Tab:
  - Am: $[-3]$
  - b: $[2]$
  - c: $[1]$
  - Get xHat0 from Plant's First Sample: Off
  - xHat0: $[0]$
- Adaptation Tab:
  - Γ : 1000
  - Q: $[1]$
  - Adaptation for Omega: On
  - Initial Estimate for Omega: 1.14
  - Min for Omega: 0.5
  - Max for Omega: 5.5
  - Adaptation for Theta: On
  - Initial Estimate for Theta: 0.65
  - Center of Projection for Theta: 0
  - Bound for Theta: 10
  - Adaptation for Sigma: On
  - Initial Estimate for Sigma: 0
  - Min for Sigma: -10
  - Max for Sigma: 10
  - Adaptation for all the rest: Off

- Control Law Tab:
  - Low Pass Filter: Transfer Function
  - Numerator: [1]
  - Denominator: [1 0]
  - K: 5
- Advanced Tab:
  - Epsilon for Projection: 0.1
  - Use Epsilon Transformation: Off
  - Use Advanced Settings: Off
- Architectural Modifications
  - Access to State Predictor's Input: Off
  - Access to xHat: Off
  - Observe xHat: On
  - Observe Estimation Parameters: On
  - State Predictor Augmentation: Off

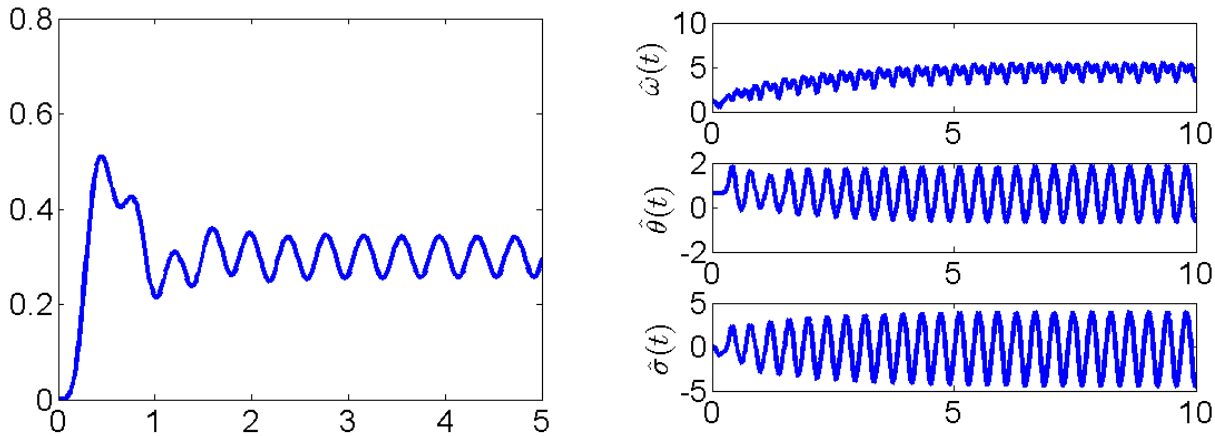
Simulation Results:

Now we provide the results to simulation to this problem.

System response and controller estimates for reference trajectory of

$$r_1(t) = 0.3 + 1.85\sin(16.1t)$$

are shown in Figure 5.15 with controller output shown in Figure 5.17a.



(a) $y(t)$

(b) Estimation Parameters

Figure 5.15: $\mathcal{L}_1$ controller with reference trajectory $r_1(t)$

Whereas, system response and controller estimates when reference trajectory is given as

$$r_2(t) = 0.3 + 2\sin(8t)$$

are plotted in Figure 5.16 and the controller output is given in Figure 5.17b.

All these plots match the $\mathcal{L}_1$ controller simulations in [8, 18].

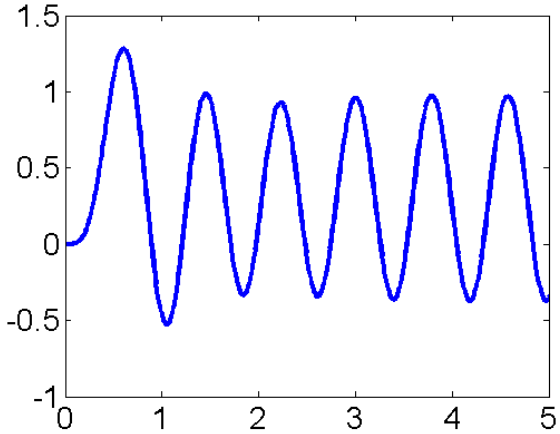
5.4 Nonlinear Systems

We will now consider a certain class of nonlinear system and define $\mathcal{L}_1$ controller for it.

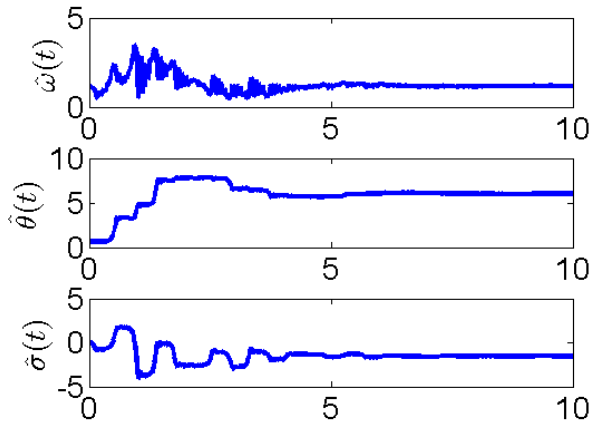
5.4.1 Problem Formulation

Consider the following class of systems

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(\omega u(t) + f(t, x(t))), \qquad x(0) = x_0, \\ y(t) &= c^T x(t) \end{aligned} \tag{5.4.1}$$

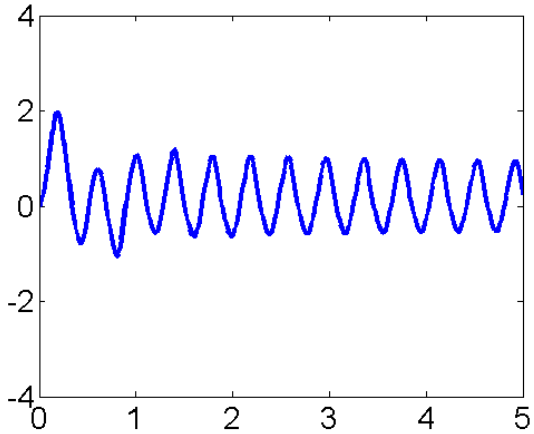


(a) $y(t)$

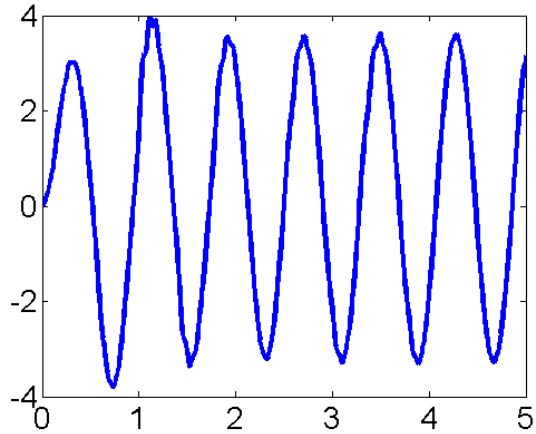


(b) Estimation Parameters

Figure 5.16: $\mathcal{L}_1$ controller with reference trajectory $r_2(t)$



(a) $u(t)$ when $r = r_1(t)$



(b) $u(t)$ when $r = r_2(t)$

Figure 5.17: $\mathcal{L}_1$ Controller Output

Here $y \in \mathbb{R}^n$ is the output of the system. Whereas, $x \in \mathbb{R}^n$ represents the state of the system. Hurwitz matrix A_m described the dynamics desired from the system and b and c are other known parameters.

ω is an unknown constant for which we know the sign and a minimum to maximum range $\omega \in \Omega \triangleq [\omega_l, \omega_u]$. Also, $f : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is an unknown map which is non-linear but is continuous in its arguments.

Assumption 3 *We assume that with $x = 0$, f remains bounded for all time $t \geq 0$. That is, there exists a constant $B > 0$ for which*

$$f(t, 0) \leq B \qquad \forall t$$

Assumption 4 *For every positive $\delta \in \mathbb{R}$, there exist time-independent bounds $d_{f_x}(\delta) > 0$ and $d_{f_t}(\delta) > 0$ such that $||x||_\infty \leq \delta$ ensures that partial derivatives of f are piece-wise continuous and are also bounded in the following sense*

$$\left|\left|\frac{\partial f(t,x)}{\partial x}\right|\right|_1 \leq d_{f_x}(\delta) \qquad \left|\frac{\partial f(t,x)}{\partial t}\right| \leq d_{f_t}(\delta)$$

5.4.2 $\mathcal{L}_1$ controller

We have the controller dynamics defined as follows

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(\hat{\omega}(t)u(t) + \hat{\theta}(t)||\hat{x}(t)||_\infty + \hat{\sigma}(t)), \qquad \hat{x}(0) = x_0, \\ \hat{y}(t) &= c^T \hat{x}(t) \end{aligned} \tag{5.4.4}$$

$\hat{\theta}(t) \in \mathbb{R}$, $\hat{\omega}(t) \in \mathbb{R}$ and $\hat{\sigma}(t) \in \mathbb{R}$ are estimates for unknown system parameters and are updated as follows:

$$\begin{aligned}
\dot{\hat{\theta}}(t) &= \Gamma \mathrm{Proj}(\hat{\theta}, -\tilde{x}^T(t)Pb||x(t)||_\infty), \qquad \hat{\theta}(0) = \hat{\theta}_0 \in \Theta, \\
\dot{\hat{\sigma}}(t) &= \Gamma \mathrm{Proj}(\hat{\sigma}, -\tilde{x}^T(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0 \quad |\hat{\sigma}_0| < \Delta, \\
\dot{\hat{\omega}}(t) &= \Gamma \mathrm{Proj}(\hat{\omega}, -\tilde{x}^T(t)Pbu(t)), \qquad \hat{\omega}(0) = \hat{\omega}_0 \in \Omega
\end{aligned} \tag{5.4.5}$$

where $\tilde{x}(t)$ is the estimation error $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ and P solves the Lyapunov equation $A_m^T P + PA_m = Q$ where Q is an arbitrary positive definite matrix $Q > 0$.

Also

$$\begin{aligned}
&\Theta \triangleq [-\theta_b, \theta_b] \qquad \theta_b \triangleq d_{f_x}(\rho) \\
&\Delta \triangleq B + \epsilon
\end{aligned}$$

Where $\epsilon > 0$ is an arbitrary constant and

$$\rho \triangleq \rho_r + \bar{\gamma_1}$$

Where $\bar{\gamma_1}$ is an arbitrary constant and ρ_r is some constant such that $\rho_r > \rho_{in}$ and $\rho_{in} \triangleq ||s(s\mathbb{I} - A_m)^{-1}||_{\mathcal{L}_1}\rho_0$ with ρ_0 being an arbitrary bound on x_0 i.e. $(||x_0||_\infty < \rho_0 < \infty)$.

The control signal is now given as

$$u(s) = -kD(s)(\hat{\eta}(s) - k_g r(s)) \tag{5.4.7}$$

Here $\hat{\eta}(t) = \hat{\omega}(t)u(t) + \hat{\theta}^T(t)||x(t)||_\infty + \hat{\sigma}(t)$ and k_g is given by $\frac{1}{c^T A_m^{-1} b}$. $D(s)$ is a strictly proper transfer function such that for all $\omega \in \Omega$, the following transfer function

$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)}$$

with some chosen feedback gain $k > 0$ is strictly proper and has zero DC-gain $C(0) = 1$.

Furthermore, we need k and $D(s)$ to be chosen such we satisfy the following

$$||G(s)||_{\mathcal{L}_1} < \frac{\rho_r - ||H(s)C(s)k_g||_{\mathcal{L}_1}||r||_{\mathcal{L}_\infty} - \rho_{in}}{L_{\rho_r}\rho_r + B} \tag{5.4.8}$$

where $G(s) \triangleq H(s)(1 - C(s))$ with $H(s) = (s\mathbb{I} - A_m)^{-1}b$ and

$$L_{\rho_r} \triangleq \frac{\rho_r + \bar{\gamma_1}}{\rho_r} d_{f_x}(\rho_r + \bar{\gamma_1}) \tag{5.4.9}$$

5.4.3 Performance and Design of Controller

We define our reference system as follows

Reference System:

$$\begin{aligned} \dot{x}_{ref}(t) &= A_m x_{ref}(t) + b(\omega u_{ref}(t) + f(t, x_{ref}(t))), \quad x_{ref}(0) = x_0, \\ u_{ref}(s) &= \frac{C(s)}{\omega}(k_g r(s) - \eta_{ref}(s)), \\ y_{ref}(t) &= c^T x_{ref}(t) \end{aligned} \tag{5.4.10}$$

where $\eta_{ref}(t) = f(t, x_{ref}(t))$

As the usual strategy, we have defined the reference system such that if $C(s) = 1 \quad \forall s$ ($C(s)$ bandwidth covers all), the uncertainties would cancel out in the above expression and the reference system will turn into the desired system. Theorem showing stability of this reference system follows:

Theorem 8 *If the system described in Equation* (5.4.10) *satisfies the norm condition of Equation* (5.4.8)*, then its state, input and output satisfy the following*

$$\begin{aligned} ||x_{ref}||_{\mathcal{L}_\infty} &\leq \rho_r, \\ ||u_{ref}||_{\mathcal{L}_\infty} &\leq \rho_{ur} \end{aligned} \tag{5.4.11}$$

where

$$\rho_{ur} \triangleq \left|\left|\frac{C(s)}{\omega}\right|\right|_{\mathcal{L}_1} (k_g||r||_{\mathcal{L}_\infty} + L_{\rho_r}\rho_r + B)$$

with ρ_r defined in Equation (5.4.9).

The proof can be found in [8].

Theorem 9 *Let*

$$\gamma_1 \triangleq \frac{||C(s)||_{\mathcal{L}_1}}{1 - ||G(s)||_{\mathcal{L}_1} L_{\rho_r}} \gamma_0 + \beta$$

with some small constants γ_0 and β such that $\gamma_1 \leq \bar{\gamma_1}$. Furthermore, let us define

$$\begin{aligned} \theta_m(\rho, \rho_u) &\triangleq 4\theta_b^2 + 4\Delta^2 + (\omega_u - \omega_l)^2 + 4\frac{\lambda_{max}(P)}{\lambda_{min}(Q)}(\theta_b d_\theta(\rho, \rho_u) + \Delta d_\sigma(\rho, \rho_u)) \\ \gamma_2 &\triangleq \left|\left|\frac{C(s)}{\omega}\right|\right|_{\mathcal{L}_1} L_{\rho_r}\gamma_1 + \left|\left|\frac{H_1(s)}{\omega}\right|\right|_{\mathcal{L}_1} \gamma_0 \end{aligned}$$

where $d_\theta(\rho, \rho_u)$ and $d_\sigma(\rho, \rho_u)$ are constants that define bounds on the time derivative of the non-linear map f (details can be found in [8]) and H_1 is BIBO stable system that satisfies $H_1(s) = C(s)\frac{1}{c_0^T H(s)} c_0^T$ for some vector c_0.

Then, if Γ is large enough to satisfy

$$\Gamma \geq \frac{\theta_m(\rho, \rho_u)}{\lambda_{min}(P)\gamma_0^2}$$

Then

$$\begin{aligned} ||\hat{x} - x||_{\mathcal{L}_\infty} &\leq \gamma_0 \\ ||x_{ref} - x||_{\mathcal{L}_\infty} &\leq \gamma_1 \\ ||u_{ref} - u||_{\mathcal{L}_\infty} &\leq \gamma_2 \end{aligned}$$

We refer to [8] for the proof of this theorem.

The theorem above shows that the actual system response can be brought arbitrarily close to the reference system. Since the reference system mimics the desired system with large bandwidth of $C(s)$, therefore, correct selection of controller parameters can bring the actual system response arbitrarily close to the desired system as well.

5.4.4 Example Simulation on Toolbox

Now we present our simulations using the toolbox for this class of problems.

Plant:

We use wing rock example from [8] to test our toolbox. Dynamics model for this system was given in [20, 21] as

$$\ddot{\phi}(t) + \frac{a_0}{t_r^2}\phi(t) + \frac{a_1}{t_r}\dot{\phi}(t) + a_2|\dot{\phi}(t)|\dot{\phi}(t) + \frac{a_3}{t_r^3}\phi^3(t)$$
$$+\frac{a_4}{t_r}\phi^2(t)\dot{\phi}(t) + \frac{\omega}{t_r^2}u(t) + d(t, \phi(t), \dot{\phi}(t)) = 0$$

In state space form, we can set $x = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix}$ and write the system above as follows

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0/t_r^2 & -a_1/t_r \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/t_r^2 \end{bmatrix} (\omega u(t) + g_0(t, x(t))), \qquad x(0) = x_0$$
$$y(t) = c^T x(t)$$

This system is similar to the class of system defined in Equation (5.4.1) where

$$A = \begin{bmatrix} 0 & 1 \\ -a_0/t_r^2 & -a_1/t_r \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$g_0(t, x) = -t_r^2 a_2|\dot{\phi}(t)|\dot{\phi}(t) - a_3\phi^3(t) - t_r a_4\phi^2(t)\dot{\phi}(t) - t_r^2 d(t, \phi(t), \dot{\phi}(t))$$

To completely bring it in the same form as Equation (5.4.1), we define

$$Am \triangleq \begin{bmatrix} 0 & 1 \\ -50 & -14.14 \end{bmatrix} = A - bk_m$$

This gives us $k_m = [-0.0046\ 0.0001]$ and we can apply u_m in an inner loop that will make

the overall system to be equivalent to

$$\dot{x}(t) = A_m x(t) + b(\omega u(t) + g_0(t, x(t))), \qquad x(0) = x_0$$
$$y(t) = c^T x(t)$$

Controller Setting:

Now we describe parameters for our controller.

For bounds of parameter estimates, we set $\Theta = [-10, 10]$, $\Delta = 10$ and $\Omega = [0.3, 2]$. We set $\Gamma = 10000000$ and the filter as follows:

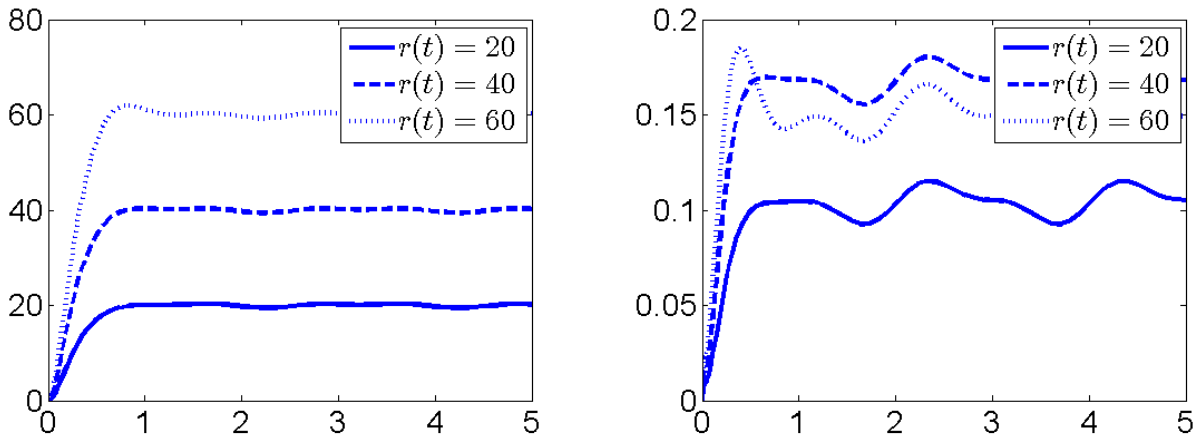
$$D(s) = \frac{(s + 500)(s + 0.004)^2}{s(s + 368)(s + 0.00439)^2}$$

Controller Setting through Toolbox Interface:

We specify these controller settings using the GUI of the toolbox as follows:

- State Predictor Tab:
  - Am: $[0\ 1; -50\ \ -14.14]$
  - b: $[0; 126050]$
  - c: $[1\ 0]$
  - Get xHat0 from Plant's First Sample: Off
  - xHat0: $[0; 0]$
- Adaptation Tab:
  - Γ : 10000000
  - Q: $[1\ 0; 0\ 1]$
  - Adaptation for Omega: On
  - Initial Estimate for Omega: 1

  - Min for Omega: 0.3
  - Max for Omega: 2
  - Adaptation for Theta: Off
  - Adaptation for Sigma: On
  - Initial Estimate for Sigma: 0
  - Min for Sigma: -10
  - Max for Sigma: 10
  - Adaptation for Theta NonLinear: On
  - Initial Estimate for Theta NonLinear: 0
  - Center of Projection for Theta NonLinear: 0
  - Bound for Theta NonLinear: 10
  - Adaptation for all the rest: Off
- Control Law Tab:
  - Low Pass Filter: Transfer Function
  - Numerator: [1.0638 531.9303 4.2554 0.0085]
  - Denominator: [1 368.0088 3.2281 0.0071 0]
  - K: 143.75
- Advanced Tab:
  - Epsilon for Projection: 0.1
  - Use Epsilon Transformation: Off
  - Use Advanced Settings: Off
- Architectural Modifications
  - Access to State Predictor's Input: Off



(a) $y(t)$ (b) $u(t)$

Figure 5.18: $\mathcal{L}_1$ controller for α_1 with uncertainties and zero delay

- Access to xHat: Off
- Observe xHat: On
- Observe Estimation Parameters: On
- State Predictor Augmentation: Off

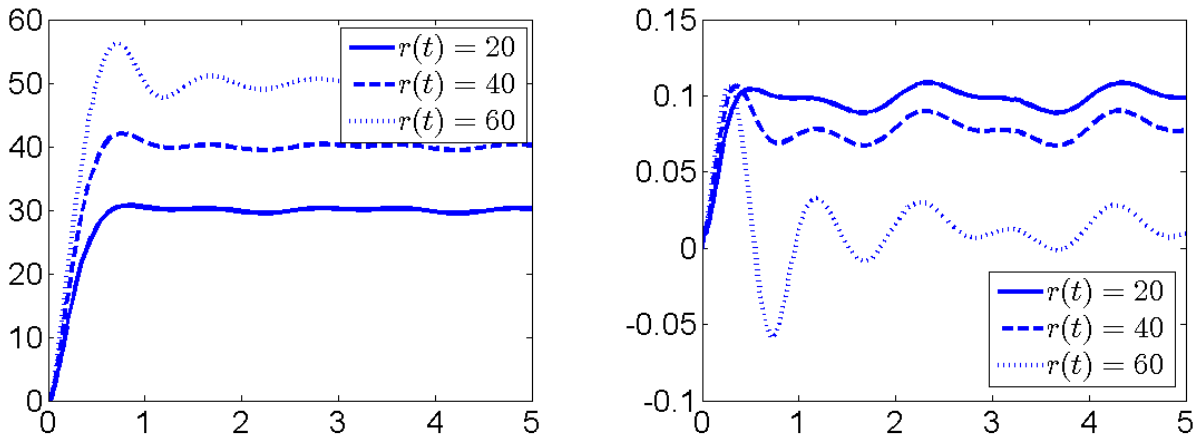
Simulation Results:

We first simulate the system without any time delay. Two different values of angle of attach in the system lead to two different parameter settings in the plant. Namely, we set the angle of attack $\alpha_1 = 27.0$ deg and $\alpha_2 = 35.0$ deg in our experiments.

Simulation results with uncertainties in the systems are shown in Figure 5.18 and Figure 5.19 for α_1 and α_2 respectively.

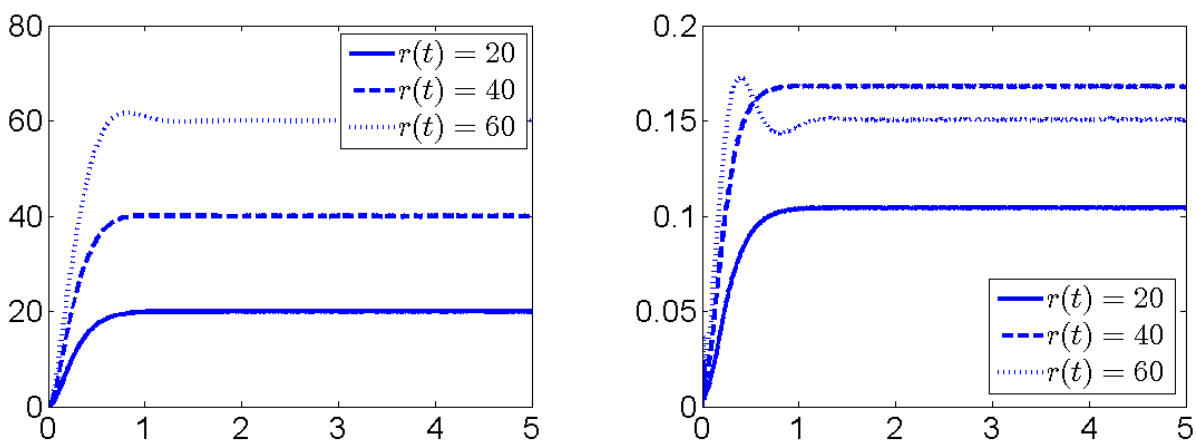
We then introduce a time-delay of 5 ms in the system and simulate the system without uncertainties. The results for α_1 and α_2 are plotted in Figure 5.20 and 5.21 respectively.

All these results match those published in [8] which validates working of our toolbox.



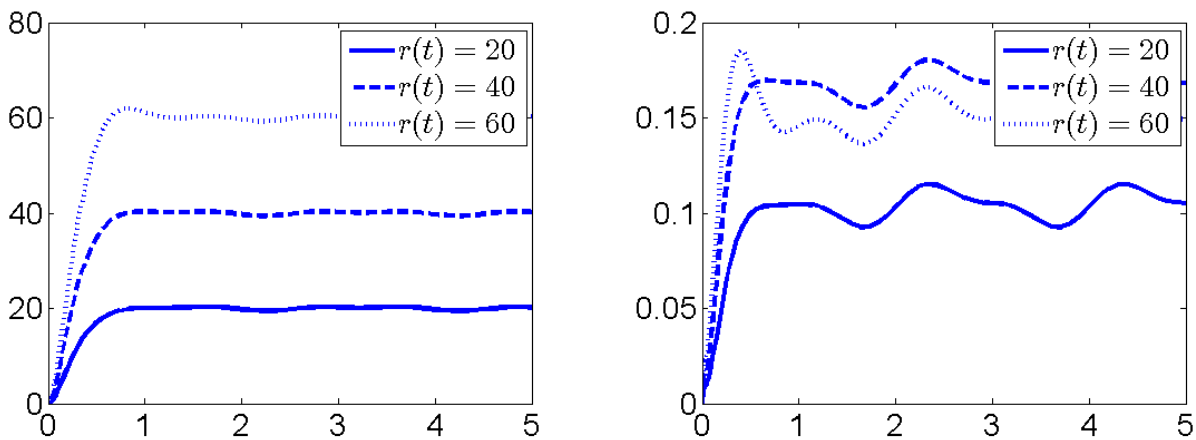
(a) $y(t)$ (b) $u(t)$

Figure 5.19: $\mathcal{L}_1$ controller for α_2 with uncertainties and zero delay



(a) $y(t)$ (b) $u(t)$

Figure 5.20: $\mathcal{L}_1$ controller for α_1 with 5ms delay



(a) $y(t)$ (b) $u(t)$

Figure 5.21: $\mathcal{L}_1$ controller for α_1 with 5ms delay

CHAPTER 6

CONCLUSION

In this document we have shown use, working and validation results for our Simulink Toolbox for $\mathcal{L}_1$ control. We started out with a general idea of what parameters identify the exact structure of an $\mathcal{L}_1$ controller. We showed GUI of our toolbox that lets user an easy way of setting up those parameters. Then after briefing over inner working of the toolbox, we delved into exact problem formulation for which our toolbox can be used in its current form. We went over theorems that claimed performance bounds on $\mathcal{L}_1$ controller on those problems formulations and then validated our toolbox for instances of those problem formulations. Our results showed exact match of controller results with already published results of $\mathcal{L}_1$ control.

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