COMPUTING THE GOODWILLIE-TAYLOR TOWER OF DISCRETE MODULES

BY

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DISSERTATION

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Abstract

Let P_m and D_m be the m^{th} level and layer of the Goodwillie-Taylor tower for discrete modules. In [6] the rank filtration of D_1F is described and the associated spectral sequence is shown to be equivalent to the filtration by rows of the reduced Robinson bicomplex of F. We consider the rank filtrations of P_mF and D_mF and give explicit descriptions of the entries of the E^1 page of the associated spectral sequences. In the case $m = 1$ these entries agree with those of the spectral sequence associated with the reduced Robinson bicomplex.

To my parents, Ray and Nili, for their constant support and encouragement over all the years of my formal mathematical education.

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Chapter 1

Background and Introduction

A framework for studying homotopy functors[1] $F : \mathcal{T}op_+ \to \mathcal{T}op_+$ was introduced by Thomas Goodwillie in a seminal sequence of papers [3, 4, 5]. For each such functor F, Goodwillie associates a tower

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P_2F & D_2F \arrow[l] \\
 & P_1F \arrow[from=2-2, d] & D_1F \arrow[l] \\
F \arrow[r] \arrow[ur] \arrow[uur] & P_0F \arrow[from=3-2, d] &
\end{tikzcd}$$

of functors and natural transformations, natural in F, and whose limit is equivalent, in an appropriate sense, to F. The functor P_nF is called the n^{th} *level* of the tower and the homotopy fiber of the map $P_nF \to P_{n-1}F$, denoted D_nF, is called the n^{th} *layer* of the tower. Goodwillie's tower has been adapted to other classes of functors (by Thomas Goodwillie himself and by others) but all the towers exhibit properties and employ terminology analogous to the classical Taylor series of an analytic function $f : \mathbb{R} \to \mathbb{R}$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & p_2f & d_2f \arrow[l] \\
 & p_1f \arrow[from=2-2, d] & d_1f \arrow[l] \\
f \arrow[r] \arrow[ur] \arrow[uur] & p_0f \arrow[from=3-2, d] &
\end{tikzcd}$$

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[1] A *homotopy functor* is a functor which preserves weak equivalences.

Here, $p_n f$ denotes the n^{th} approximating Taylor polynomial $\sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i$ of f. The arrows in the second diagram are purely cosmetic but if we think of the "fiber" of $p_n f \to p_{n-1} f$ as the difference polynomial $d_n f = p_n f - p_{n-1} f$ then the n^{th} "layer" $d_n f$ of the Taylor series of f turns out to have a strong resemblance to the n^{th} *layer* $D_n F$ of Goodwillie's tower of F. At least partially on account of this resemblance, Goodwillie refered to his tower as a *Taylor* tower. In this document, we refer to Goodwillie's tower, and others towers of its kind, as *Goodwillie-Taylor towers* or *GT towers* for short.

Restricting attention to functors F which satisfy two additional conditions,

continuity The natural map $\hom(X, Y) \to \hom(FX, FY)$ is continuous[2].

colimit axiom If X is a filtered colimit of finite CW spaces $\{X_\alpha\}$, then the natural map $colimit(FX_\alpha) \to FX$ is an equivalence.

Mauer-Oats [8] constructed the *additive* GT tower whose levels P_n^{add} are characterized by a fibration sequence:

$$|\mathrm{cr}_{n+1}^{\bullet+1} F| \to F \to P_n^{add} F \tag{1.1}$$

where $\mathrm{cr}_{n+1}^{\bullet+1} F$ refers to a certain simplicial object constructed from the cross effects of F. The advantage of considering $P_n^{add} F$ over $P_n F$ is that the cross effects of F are relatively easy objects to compute, at least for F a *discrete module* (see below). In particular, $P_n^{add} F(X)$ is determined only by the values of F at the coproducts $\vee^i X$ for $i \leq n$ while $P_n F(X)$ is defined in terms of a larger class of objects including large suspensions of X.

For functors F which commute with geometric realizations[3], Mauer-Oats showed that the additive GT tower of F agrees with Goodwillie's tower of F.

One strategy for studying the additive GT tower is to study the spectral sequence associated to the sequence:

$$\pi_k |\mathrm{cr}_{n+1}^{\bullet+1} F| \to \pi_k F \to \pi_k P_n^{alg} F$$

It turns out that the E^1 page of this spectral sequence can be understood by understanding a related

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[2] Here, we are thinking of $\mathcal{T}op_+$ as a category enriched over itself.

[3] A functor F *commutes with geometric realizations* if for each simplicial space Y, $|F(Y)| \cong F(|Y|)$

sequence:

$$\perp_{n+1}^{\bullet+1}\pi_k F \to \pi_k F \to P_n^{alg}\pi_k F$$

Here P_n^{alg} refers to yet another variant of Goodwillie's tower, called the *algebraic* GT tower, which applies to functors defined on the category of finite pointed sets and taking values in the category of connective chain complexes of R-modules. We refer to such objects as *discrete modules.*[4]

A driving problem in homotopy theory is to characterize ring spectra which admit a multiplication satisfying the E_∞-associativity and commutativity operations. In an extensive (initially unpublished) work, Paul Goerss and Mike Hopkins showed the existence of such a multiplication on Lubin-Tate spectra. Later, in a seminal *Inventiones* paper, Alan Robinson provided a construction which greatly improved their argument and was adopted in a subsequent published version of their result [11, 2]. Robinson's construction was, for any reduced discrete module[5] F concentrated homologically in degree zero, a first quadrant bicomplex ΞF such that the total chain complex of ΞF is quasi-isomorphic to $D_1^{alg}F(S^0)$. The p^{th} column of ΞF is the normalized chain complex associated with the Bar construction:

$$\mathcal{B}(\epsilon Lie_{q+1}^* : \Sigma_{q+1} : F((q+1)_+))$$

where ϵLie_n denotes the Lie representation twisted by the sign representation. Besides aiding in Goerss and Hopkins' work, a bicomplex for $D_1^{alg}F(S^0)$ also gives stable homotopy groups of F and the "Q-construction" for F. The importance of Robinson's result should not be understated.

Intermont, Johnson and McCarthy [6] constructed a *reduced*[6] version $\widetilde{\Xi}F$ of Robinson's complex ΞF in terms of the *cross effects* of F (denoted here by $\perp_\bullet F$) evaluated at S^0. The q^{th} column of $\widetilde{\Xi}F$ is the normalized chain complex associated with the Bar construction:

$$\mathcal{B}(\epsilon Lie_{q+1}^* : \Sigma_{q+1} : \perp_{q+1}F(S^0))$$

They introduce the *rank* filtration of D_1F and make the crucial observation that the spectral sequence determined by the reduced Robinson bicomplex is equivalent to the spectral sequence determined by the rank filtration of D_1F.

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[4] Such functors have been referred to in the literature as Γ-modules.

[5] A discrete module F is said to be *reduced* if $F(+) = 0$.

[6] There is a natural inclusion $\perp_{q+1}F(S^0)) \to F((q+1)_+)$ and hence $\widetilde{\Xi}F \to \Xi F$.

In this thesis, we consider the rank filtrations of both P_mF and D_mF and give explicit descriptions of the entries of the E^1 page of the associated spectral sequences with coefficients in an arbitrary commutative ring R. In the case $m = 1$ these entries agree with those of the spectral sequence associated with the reduced Robinson bicomplex.

In the case of D_mF, with coefficients a field, Richter (see [10]) described the entries of an equivalent spectral sequence. The results of this thesis can be though of as a generalization of her work although the techniques employed here are very different.

If a bicomplex computes[7] F, the E^1 page of its associated spectral sequence does not generally recover F. Generally, the spectral sequence only encodes information about the layers of the filtration of F. It is more desireable to have an actual bicomplex. In future work, we aim to interpret the reduced Robinson complex using our model for the E^1 page of D_1F. It should then be easier to find a bicomplex that computes D_mF and P_mF by extending this interpretation to our models for the E^1 page of the rank filtrations of D_mF and P_mF.

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[7] That is, its total complex is quasi-isomorphic to F

Chapter 2

Prelimenaries

2.1 Basic categories and notational conventions

For n a nonnegative integer, we use the same symbol n for the corresponding ordinal and discrete category. In particular, n is a totally ordered set with cardinality n.

For a set X, $|X|$ denotes the cardinality of X. In particular, if X is finite then $|X|$ is an integer. X_+ denotes the disjoint union of X with $\{+\}$. We often write $+$ as an abbreviation for 0_+, for $\{+\}$ and for arbitrary basepoints when the context is clear. We also write S^0 for 1_+. If X is an ordered set then X_+ inherits the order of X and the element $+$ is assumed to be minimal in X_+. For a pointed set X, X_- denotes the unpointed ordered set resulting from the deletion of the basepoint of X.

Σ_X denotes the symmetric group on the set X.

$\mathbb{F}_+$ denotes the category of finite basepointed sets and basepoint-preserving functions.[1] We view the category $\mathbb{F}_+$ as being enriched over itself by pointing each hom-set $\hom(X, Y)$ with the unique map factoring through $+$.

$\mathbb{E}$ denotes the category of finite nonempty sets and epimorphisms.[2]

Simplicial Sets:

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[1] We sometimes use the same symbol for the skeletal subcategory generated by $0_+, 1_+, 2_+, \cdots$. this later category is sometimes referred to in the literature as Γ.

[2] Elsewhere in the literature, the symbol Ω is used to denote this category.

The nondegenerate simplices of a simplicial object X are called *faces* of X. The faces of X which are not faces of any other face of X are called *facets of* X. The set of facets of X is denoted $\mathcal{M}X$. A simplicial set is *pure of dimension* d if all of its facets are in dimension d. For a pure simplicial set of dimension d, the faces of dimension $d-1$ are called *facelets of* X.

If P is a poset or small category then its nerve is denoted $\mathcal{N}P$. A poset is *pure of dimension* d if its nerve is. In particular, the facets of $\mathcal{N}P$ are the (strictly increasing) chains in P with length d, and the facelets of $\mathcal{N}P$ are the strictly increasing chains of P with length $d-1$.

R-Modules:

Througout this document, R is a fixed commutative ring with unit. $\mathcal{M}od_R$ denotes the category of left R-modules, $c\mathcal{M}od_R$ denotes the category of connective[3] chain complexes of R-modules and $s\mathcal{M}od_R$ denotes the category of simplicial R-modules. We tacitly use the chain of functors

$$\mathcal{M}od_R \hookrightarrow c\mathcal{M}od_R \overset{\cong}{\to} s\mathcal{M}od_R$$

to identify objects in the categories on the left of an arrow with their images on the right. Here, the second map above is the Dold-Kan equivalence.

The left adjoint to the forgetful functor $\mathcal{M}od_R \to \mathcal{S}et$ is denoted $R[-]$. Explicitly, $R[A] := \bigoplus_{a\in A} R$. We denote $R[A] \otimes_R M$ by $A \otimes M$ and $\hom_R(R[A], M)$ by $\hom(A, M)$. Similarly, if $U : \mathcal{M}od_R \to \mathbb{F}_+$ is the functor that forgets all the algebraic structure of an R-module except the zero element (which is taken as the basepoint) then $\widetilde{R}[-]$ denotes the left adjoint of U. Explicitly, $\widetilde{R}[X] := cokernel\left(R[+] \overset{\xi}{\to} R[X]\right)$, where ξ is induced by the unique map $+ \to X$. We abbreviate $\widetilde{R}[X] \otimes_R M$ by $X \otimes M$ and $\hom_R(\widetilde{R}[X], M)$ by $\hom(X, M)$.

2.2 Discrete functors and discrete modules

Definition 1. *A (possibly contravariant) functor* $\mathbb{F}_+ \to \mathbb{F}_+$ *is called a* discrete functor.

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[3] A chain complex is called *connective* if it has nontrivial homology only in nonnegative degrees.

Definition 2. *For* $X, Y \in \mathbb{F}_+$*, we abbreviate* $\hom_{\mathbb{F}_+}(X,Y)$ *by* $\hom(X,Y)$ *and we define:*

$$\hom_\sharp(X,Y) := \frac{\hom(X,Y)}{\{f \in \hom(X,Y) \mid f^{-1}(\{+\}) \neq \{+\}\}}$$

For a finite pointed set X, define:

$$\begin{aligned}
\Pi^n X &:= \hom(n_+, X) \\
\wedge^n X &:= \hom_\sharp(n_+, X) \\
\Delta^n X &:= \{f \in \hom_\sharp(n_+, X) \mid f \text{ is not injective}\} \\
\mathcal{I}^n X &:= \{f \in \hom_\sharp(n_+, X) \mid f \text{ is injective}\}_+ \\
&= \{f \in \hom(n_+, X) \mid f \text{ is injective}\}_+
\end{aligned}$$

In the case that $\alpha : X \to Y$ a noninjective map of pointed sets, we define $\mathcal{I}^n f$ to be the basepoint map.

Note that $\Pi^n X$, $\wedge^n X$ and $\Delta^n X$ are the n-fold product of X, the n-fold smash product of X and the n^{th} fat diagonal of X, respectively.[4] Moreover, $\mathcal{I}^n$ fits in the split exact sequence [5]

$$+ \to \Delta^n \to \wedge^n \to \mathcal{I}^n \to +$$

These are standard constructions, natural in X. In particular, Π^n, $\wedge^n$, Δ^n and $\mathcal{I}^n$ are all discrete functors.

Definition 3. *A (possibly contravariant) functor* $\mathbb{F}_+ \to c\mathcal{M}od_R$ *is called a* discrete module*.*

One of the simplest discrete modules, $\widetilde{R}[-]$, admits the following useful properties:

Proposition 1.

1. $\widetilde{R}[+] = 0$
2. $\widetilde{R}[\bigvee^n] = \bigoplus^n \widetilde{R}[-]$
3. $\widetilde{R}[\bigwedge^n] = \bigotimes^n \widetilde{R}[-]$

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[4] The *fat diagonal* $\Delta^n X$ denotes the subset of $\wedge^n X$ consisting of elements $x_1 \wedge \cdots \wedge x_n$ with $x_i = x_j$ for some $i \neq j$.

[5] An *exact* sequence of discrete functors is a chain of discrete functors such that the image of each map is equal to the inverse image of the basepoint under the next map. A short exact sequence of discrete functors is *split* if the third map admits a section and the second map admits a retraction.

Composing $\widetilde{R}[-]$ with a discrete functor is always a discrete module. Most of the discrete modules mentioned in this paper of this type, but this is not generally the case. For example the discrete module $(\widetilde{R/J})[-]$, where J is any proper ideal of R, does not factor through $\widetilde{R}[-]$. The theory developed in this paper however, applies to *all* discrete modules.

Applying $\widetilde{R}[-]$ to the exact sequence

$$+ \to \Delta^n \to \wedge^n \to \mathcal{I}^n \to +$$

of discrete functors yields the exact sequence of discrete modules:

$$0 \to \widetilde{R}\,[\Delta^n] \to \widetilde{R}\,[\wedge^n] \to \widetilde{R}\,[\mathcal{I}^n] \to 0$$

We make crucial use of this sequence in this paper.

2.3 Cross effects and degree

Roughly speaking, a discrete module F is *of degree* n if the value of F on the coproduct of any $n+1$ objects is determined by its values at coproducts of proper subsets of those $n+1$ objects. In this section we develop the cross effect construction and make this notion of degree precise.

The *zero*th *cross effect* of a discrete functor F is defined as $F(+)$ and denoted cr_0F or cr^0F according to whether F is covariant or contravariant. For $n > 0$ and F covariant the n^{th} *cross effect of* F *at* $X_1, \ldots, X_n$, denoted $\mathrm{cr}_nF(X_1, \ldots, X_n)$, is the homotopy fiber of the map:

$$F\left(X_1 \vee \ldots \vee X_n\right) \to \bigoplus_i F(X_1 \vee \ldots \vee \widehat{X_i} \vee \ldots \vee X_n)$$

induced by the identity maps on X_j, $j \neq i$ and the basepoint map $X_i \to \widehat{X_i} = +$. When F is contravariant the n^{th} *cross effect of* F *at* $X_1, \ldots, X_n$, denoted $\mathrm{cr}^nF(X_1, \ldots, X_n)$, is the homotopy cofiber of the map:

$$\bigoplus_i F(X_1 \vee \ldots \vee \widehat{X_i} \vee \ldots \vee X_n) \to F\left(X_1 \vee \ldots \vee X_n\right)$$

induced by the identity maps on X_j, $j \neq i$ and the basepoint map $+ = \widehat{X_i} \to X_i$.[6]

Equivalently, the cross effects of F (we'll assume F is covariant) can be characterized recursively (up to quasi-isomorphism) by:

$$\begin{aligned} \mathrm{cr}_n F(X_1, \ldots, X_n) \oplus \mathrm{cr}_{n-1} F(X_2, X_3, \ldots, X_n) \oplus \mathrm{cr}_{n-1} F(X_1, X_3, \ldots, X_n) \\ \cong \mathrm{cr}_{n-1} F(X_1 \vee X_2, X_3, \ldots, X_n) \end{aligned}$$

Remark 1. *Note that* $\mathrm{cr}_n F(X_1, \ldots, X_n)$ *admits the obvious right action of* Σ_n*. Also if any* $X_i = +$ *then* $\mathrm{cr}_n F(X_1, \ldots, X_n) \cong 0$*.*

We write $\perp_n F(X)$ for the n^{th} *diagonalized* cross effect of F at X. That is

$$\perp_n F(X) := \mathrm{cr}_n F(X, X, \cdots, X)$$

Computing cross effects of a discrete module F usually involves understanding the effect of F on finite coproducts. For two of the key examples above, the behaviour of F with respect to coproducts is quite easy to describe:

Proposition 2. *For* $F_n = \widetilde{R}[\wedge^n]$, $\widetilde{R}[\mathcal{I}^n]$*:*

$$F_n \left(X_1 \vee \cdots \vee X_m \right) = \bigoplus_{\sigma \in \hom(n,m)} \bigotimes_{i \in m} F_{|\sigma^{-1}(i)|}(X_i)$$

Consequently, it can be shown that the cross effects of F *are given by*

$$\mathrm{cr}_m F_n(X_1, \ldots, X_m) = \bigoplus_{\sigma \in \mathrm{sur}(n,m)} \bigotimes_{i \in m} F_{|\sigma^{-1}(i)|}(X_i)$$

Example 1. *In the particular case* $F = \widetilde{R}[\wedge^n]$ *we have an especially simple expression for the diagonalized*

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[6] By the homotopy cofiber of a map $M \xrightarrow{f} N$ of connective chain complexes, we mean the total complex of the bicomplex

$$\begin{pmatrix} M \\ \downarrow \\ N \end{pmatrix} := \begin{pmatrix} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ M_0 & \leftarrow & M_1 & \leftarrow & M_2 & \leftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ N_0 & \leftarrow & N_1 & \leftarrow & N_2 & \leftarrow & \cdots \end{pmatrix}$$

The *homotopy fiber* of f is a shift by -1 of the homotopy cofiber of f.

cross effects of F:

$$\perp_m F(X) = \operatorname{sur}(n,m) \otimes F(X)$$

Example 2. *The diagonalized cross effects of $\widetilde{R}[\mathcal{I}^3]$ are given by :*

$$\begin{aligned}
\perp_0 \widetilde{R}[\mathcal{I}^3] &= 0 \\
\perp_1 \widetilde{R}[\mathcal{I}^3] &= \operatorname{sur}(3,1) \otimes \widetilde{R}[\mathcal{I}^3] \\
\perp_2 \widetilde{R}[\mathcal{I}^3] &= \operatorname{sur}(3,2) \otimes (\widetilde{R}[\mathcal{I}^1] \otimes \widetilde{R}[\mathcal{I}^2]) \\
\perp_3 \widetilde{R}[\mathcal{I}^3] &= \operatorname{sur}(3,3) \otimes (\widetilde{R}[\mathcal{I}^1] \otimes \widetilde{R}[\mathcal{I}^1] \otimes \widetilde{R}[\mathcal{I}^1]) \\
\perp_m \widetilde{R}[\mathcal{I}^3] &= 0, \text{ for } m > 3
\end{aligned}$$

Definition 4. *We say that a discrete module F is of* degree n *if $\perp_{n+1} F \cong 0$. We say that F is* reduced *if $\perp_0 F \cong 0$.*

The following two results are proven in [7].

Proposition 3. *The diagonalized cross effect $\perp_n$ is an endofunctor in the category of discrete modules.*

Proposition 4. *$\perp_n$ preserves fibration/cofibration sequences in the category of discrete modules.*

Example 3. *The discrete modules $\widetilde{R}[\mathcal{I}^n]$ and $\widetilde{R}[\wedge^n]$ are both reduced. Since $\operatorname{sur}(n,m) = \emptyset$ for $m > n$, they both have degree n. Using the result above, we get a quasi-exact sequence $\perp_m \widetilde{R}[\Delta^n] \to \perp_m \widetilde{R}[\wedge^n] \to \perp_m \widetilde{R}[\mathcal{I}^n]$. Since the map $\perp_m \widetilde{R}[\wedge^n] \to \perp_m \widetilde{R}[\mathcal{I}^n]$ is an isomorphism when $n = m$, $\widetilde{R}[\Delta^n]$ has degree $n-1$.*

The following result is proven in [9].

Theorem 1.

$$Func(\mathbb{F}_+, c\mathcal{M}od_R) \longrightarrow Func(\mathbb{E}, c\mathcal{M}od_R)$$

$$F \mapsto \perp_\bullet F(S^0)$$

is an equivalence of categories. Moreover, this equivalence preserves tensor products in the sense that there is an isomorphism in $c\mathcal{M}od_R$:

$$F \otimes_{\mathbb{F}_+} G \cong \perp_\bullet F(S^0) \otimes_{\mathbb{E}} \perp^\bullet G(S^0)$$

Remark 2. *The derived version of the above equivalence reads:*

$$F \widehat{\otimes}_{\mathbb{F}_+} G \cong \perp_\bullet F(S^0)\ \widehat{\otimes}_{\mathbb{E}} \perp^\bullet G(S^0)$$

2.4 The algebraic Goodwillie-Taylor tower for discrete modules

Note that for any fixed integer n, $\perp_n$ is an endofunctor on the category of discrete modules. The composition

$$\perp_n F(X) \to F(\vee^n X) \to F(X)$$

where the first map is the natural inclusion and the second map is induced by the fold map, determines a natural transformation $\epsilon : \perp_n \to Id$. In [6], McCarthy and Johnson recognize $\perp_n$ and ϵ as part of a cotriple which they use to construct an augmented simplicial object

$$\perp_{n+1}^{\bullet+1} \overset{\epsilon}{\to} Id$$

and a cofibration sequence

$$N(\perp_{n+1}^{\bullet+1}) \xrightarrow{N\epsilon} N(Id) \xrightarrow{p_n} P_n^{alg}$$

of endofunctors of discrete modules[7] characterizing P_n^{alg}. Here N denotes the normalized chain complex functor. McCarthy and Johnson proceed to show that $P_n^{alg}F$ has degree n and that the natural map $F \to P_n^{alg}F$ is initial among maps from F to a degree n discrete module. These properties enable the construction of a tower

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P_2^{alg} \arrow[d] & D_2^{alg} \arrow[l] \\
 & P_1^{alg} \arrow[d] & D_1^{alg} \arrow[l] \\
Id \arrow[r] \arrow[ur] \arrow[uur] & P_0^{alg} &
\end{tikzcd}$$

of endofunctors on the category of discrete modules. Evaluating at a particular discrete module F, the resulting tower of discrete modules is called the *algebraic GT-tower of F*. $P_n^{alg}F$ is called the n^{th} level of the tower and $D_n^{alg}F := hofiber(P_n^{alg}F \to P_{n-1}^{alg}F)$ is called the n^{th} layer of tower. In the remainder of this paper we refer to this algebraic version of the GT-tower exclusively and we drop the qualifier *algebraic*. The following results are proven in [6].

Proposition 5. *D_k and P_k are quasi-exact functors on the category of discrete modules.*

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[7] Actually, their construction of P_n^{alg} and the resulting tower applies to a wider class of functors than just the discrete modules. See [8] for details.

Proposition 6. *Let $F = \widetilde{R}[\wedge^n]$. Then F is* homogeneous *of degree n. In other words:*

$$P_m F = \begin{cases} F, & m \geq n \\ 0, & \textit{otherwise} \end{cases} \qquad D_m F = \begin{cases} F, & m = n \\ 0, & \textit{otherwise} \end{cases}$$

In the remainder of this section, we restrict our attention to the explicit computation of the levels and layers of the GT tower of some especially simple discrete modules.

Example 4. *Let $F = \widetilde{R}[\mathcal{I}^2]$. Since $\widetilde{R}[\mathcal{I}^2]$ is reduced, $P_0\widetilde{R}[\mathcal{I}^2] = \widetilde{R}[\mathcal{I}^2](+) = +$. Since $\widetilde{R}[\mathcal{I}^2]$ has degree 2, $P_2\widetilde{R}[\mathcal{I}^2] = \widetilde{R}[\mathcal{I}^2]$. In order to compute $P_1\widetilde{R}[\mathcal{I}^2]$, apply the quasi-exact functor P_1 to the short exact sequence of discrete modules*

$$0 \to \widetilde{R}[\Delta^2] \to \widetilde{R}[\wedge^2] \to \widetilde{R}[\mathcal{I}^2] \to 0 \tag{2.1}$$

to get a short exact sequence:

$$0 \to P_1\widetilde{R}[\Delta^2] \to P_1\widetilde{R}[\wedge^2] \to P_1\widetilde{R}[\mathcal{I}^2] \to 0$$

From earlier computations we have $P_1\widetilde{R}[\wedge^2] \cong 0$. Also, $P_1\widetilde{R}[\Delta^2] = P_1\widetilde{R}[\wedge^1] \cong \widetilde{R}[\wedge^1] = \widetilde{R}[-]$. Thus, the homology of $P_1\widetilde{R}[\mathcal{I}^2]$ is given by:

$$H_i P_1\widetilde{R}[\mathcal{I}^2] = \begin{cases} \widetilde{R}[-], & i = 1 \\ 0, & i \neq 1 \end{cases}$$

It remains to compute $D_2\widetilde{R}[\mathcal{I}^2]$.[8] *Applying D_2 to the exact sequence (2.1) and observing that $D_2\widetilde{R}[\Delta^2] \cong D_2\widetilde{R}[\wedge^1] \cong 0$ and $D_2\widetilde{R}[\wedge^2] \cong \widetilde{R}[\wedge^2]$ we get $D_2\widetilde{R}[\mathcal{I}^2] \cong \widetilde{R}[\wedge^2] = \bigotimes^2 \widetilde{R}[-]$. This completes the description of the GT-tower of $\widetilde{R}[\mathcal{I}^2]$.*

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[8] The n^{th} layer of any discrete module F can be expressed in terms of D_1F and the $(n+1)^{st}$ cross effect of F:

$$D_n F \cong \left(D_1^{(n)} \mathrm{cr}_n F\right)_{h\Sigma_n}$$

where $D_1^{(n)}$ denotes the n^{th} *multilinearization* of D_1F and $(-)_{h\Sigma_n}$ indicates *homotopy orbits.* See [7] for a proof. We'll take a less sophisticated approach for the simple examples in this section.

Chapter 3

The Left Kan and Rank Filtrations and their Associated Spectral Sequences

We start this chapter with a short review of the construction of a spectral sequence from a filtered chain complex. Later in the chapter, we describe some particular filtrations of discrete modules and more explicit descriptions of the associated spectral sequences.

3.1 The spectral sequence associated with a filtration of chain complexes.

Suppose $L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow \operatorname{colim}_n L_n := L_\infty$ is a filtration of chain complexe with each map $L_n \longrightarrow L_{n+1}$, a cofibration. Then for each n, we get a Puppe sequence:

$$L_n \longrightarrow L_{n+1} \longrightarrow \frac{L_{n+1}}{L_n} \longrightarrow \Sigma L_n \longrightarrow \Sigma L_{n+1} \longrightarrow \Sigma \frac{L_{n+1}}{L_n} \longrightarrow \cdots \tag{3.1}$$

Also, the composition of every consecutive pair of maps in each such Puppe sequence is nullhomotopic.

It is important to note that some of the maps in this Puppe sequence, namely the maps to the suspensions, may only exist in the homotopy category. Nevertheless, these *maps* do induce genuine maps in homology. To say that a composition of such maps is null-homotopic, essentially means that the induced map in homology is the zero map.

Remark 3. *Recall what's really going on with the Puppe Sequence of a cofibration $A \to X$. Let CA denote* *the cone on A and $A \to CA$ the obvious inclusion map. There is a genuine diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X/A \arrow[d, equal] \\
CA \arrow[r] & CA \cup_A X \arrow[r] \arrow[d] & \frac{CA \cup_A X}{CA} \\
& \Sigma A &
\end{tikzcd}$$

Here the map denoted by $=$ is a homeomorphism.

Descending to the homotopy category gives a slightly richer diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X/A \arrow[d, equal] \\
CA \arrow[r] & CA \cup_A X \arrow[r, leftrightarrow, "\cong"] \arrow[d] & \frac{CA \cup_A X}{CA} \\
& \Sigma A &
\end{tikzcd}$$

The extra map in the second diagram may or may not lift to the first diagram, but it does induce a map in homology. In particular, the sequence

$$X \longrightarrow X/A \longrightarrow \Sigma A$$

induces the familiar chain complex:

$$\cdots \longrightarrow H_p A \longrightarrow H_p X \longrightarrow H_p(X/A) \longrightarrow H_{p-1} A \longrightarrow \cdots$$

Selected pieces of the Puppe sequences (3.1) fit into a sequence:

$$\cdots \longrightarrow \Sigma^{-2}\frac{L_3}{L_2} \longrightarrow \Sigma^{-1} L_2 \longrightarrow \Sigma^{-1}\frac{L_2}{L_1} \longrightarrow L_1 \longrightarrow \frac{L_1}{L_0}$$

Note that every four consecutive maps in this sequence contains a pair of consecutive maps from a Puppe sequence. After composing at every other object we get a sequence

$$\cdots \longrightarrow \Sigma^{-2}\frac{L_3}{L_2} \longrightarrow \Sigma^{-1}\frac{L_2}{L_1} \longrightarrow \frac{L_1}{L_0} \tag{3.2}$$

with the property that the composition of any two consecutive maps factors through a pair of Puppe maps and hence is nullhomotopic. It follows that the induced chain in homology;

$$\cdots \longrightarrow H_p\Sigma^{-2}\frac{L_3}{L_2} \longrightarrow H_p\Sigma^{-1}\frac{L_2}{L_1} \longrightarrow H_p\frac{L_1}{L_0} \tag{3.3}$$

is a chain complex. This chain complex is the p^{th} row of the E^1 page of the spectral sequence of L_∞ associated with the given filtration of F.

3.2 The left Kan filtration and the associated spectral sequence

In this section, we give an explicit description of the E^0 page of spectral sequence associated with the (homotopy) left Kan filtration of an arbitrary discrete module F.

Definition 5. *For a map $X \xrightarrow{f} Y$ of finite pointed sets, let $|f|$ denote the unique integer through which f admits an epic-monic factorization in $\mathbb{F}_+$.*

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[rr, bend left, "f"] & {|f|_+} \arrow[r, hook] & Y \end{tikzcd}$$

For r a nonegative integer, we define:

$$\hom^{\leq r}(X,Y) := \{f \in \hom(X,Y) \mid |f| \leq r\}$$
$$\hom^{=r+1}(X,Y) := \frac{\hom^{\leq r+1}(X,Y)}{\hom^{\leq r}(X,Y)}$$
$$\hom^{=0}(X,Y) := \hom^{\leq 0}(X,Y) \cong +$$

Definition 6. *For a discrete module F, and a nonnegative integer n, we define a discrete module L_nF by:*

$$L_nF(X) := \hom^{\leq n}(-,X)\widehat{\otimes}_{\mathbb{F}_+}F(-)$$

Here, as is the convention throughout this document, $\widehat{\otimes}$ denotes the bar construction. Specifically, a k-simplex in $L_nF(X)$ consists of a finite pointed set A_0, a map $f : A_{k+1} \to X$ of finite pointed sets with $|f| \leq n$, and a chain:

$$A_0 \xrightarrow{\beta_0} A_1 \xrightarrow{\beta_1} A_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_k} A_{k+1}$$

of finite pointed sets.

We define a map $L_nF \to F$ of discrete modules by evaluating $F(f \circ \beta_k \circ \beta_{k-1} \circ \cdots \circ \beta_0)$ at $F(A_0)$.

Note that

1. The evaluation of the map $L_nF \to F$ at any set of cardinality at most $n+1$, is a quasi-isomorphism.
2. If $F \to G$ is a quasi-isomorphism of discrete functors, then so is the induced map $L_nF \to L_nG$.

In fact, L_nF, together with the map $L_nF \to F$, is the universal object with these properties.

The map $L_nF \to F$ factors uniquely through the inclusion $L_nF \to L_{n+1}F$ and $\operatorname{colim}(L_1F \to L_2F \to \cdots) \cong F$. We call $L_1F \to L_2F \to \cdots$ the *homotopy left Kan* filtration of F. We refer to the spectral sequence associated with the homotopy left Kan filtration of F as the left Kan spectral sequence of F.

Proposition 7.

$$\frac{L_nF}{L_{n-1}F} \cong \widetilde{R}[\mathcal{I}^n]\widehat{\otimes}_{\Sigma_n}\mathrm{cr}_nF(S^0)$$

Proof.

$$\begin{aligned}\frac{L_nF}{L_{n-1}F}(X) &:= \frac{\widetilde{R}[\hom^{\leq n}(-,X)]\ \widehat{\otimes}_{\mathbb{F}_+}F(-)}{\widetilde{R}[\hom^{\leq n-1}(-,X)]\ \widehat{\otimes}_{\mathbb{F}_+}F(-)}\\ &\cong \frac{\widetilde{R}[\hom^{\leq n}(-,X)]}{\widetilde{R}[\hom^{\leq n-1}(-,X)]}\ \widehat{\otimes}_{\mathbb{F}_+}F(-)\\ &\cong \widetilde{R}[\hom^{=n}(-,X)]\ \widehat{\otimes}_{\mathbb{F}_+}F(-)\\ &\cong \mathrm{cr}^{\bullet}\widetilde{R}[\hom^{=n}(-,X)](S^0)\ \widehat{\otimes}_{\mathbb{E}}\ \mathrm{cr}_{\bullet}F(S^0)\\ &\cong \widetilde{R}[\mathcal{I}^n](X)\ \widehat{\otimes}_{\Sigma_n}\ \mathrm{cr}_{\bullet}F(S^0)\end{aligned}$$

The equivalence in the second line holds since the operation $\widehat{\otimes}_{\mathbb{F}_+}F$ is (right) exact. The equivalence in the fourth line comes from remark 2. The last equivalence uses a straightforward computation:

$$\mathrm{cr}^{\bullet}\,\hom^{=n}(-,X)(S^0) \cong \begin{cases} \widetilde{R}[\mathcal{I}^n](X), & \bullet = n \\ 0, & \mathit{otherwise} \end{cases}$$

□

Since that the action of Σ_n on $\mathcal{I}^n$ is free, we have $\frac{L_nF}{L_{n-1}F} \cong \widetilde{R}[\mathcal{I}^n]\otimes_{\Sigma_n}\mathrm{cr}_nF(S^0)$. It follows that $\frac{L_nF}{L_{n-1}F}$ has homology concentrated in dimension 0. Hence, the E^1 page of the Kan spectral sequence of F is:

$$\begin{bmatrix} \widetilde{R}[\mathcal{I}^1]\otimes\mathrm{cr}_1F(S^0) & 0 & 0 & \\ 0 & \widetilde{R}[\mathcal{I}^2]\otimes_{\Sigma_2}\mathrm{cr}_2F(S^0) & 0 & \\ 0 & 0 & \widetilde{R}[\mathcal{I}^3]\otimes_{\Sigma_3}\mathrm{cr}_3F(S^0) & \\ & & & \ddots \end{bmatrix}$$

with all the differential maps zero.

Remark 4. *It is apparent that the resulting spectral sequence is, in general, too coarse to recover F* as a discrete module.

Consider, for example, the discrete module $F = \widetilde{R}[\wedge^2]$. This discrete module has degree 2 so $\mathrm{cr}_kF(S^0)$ vanishes for $k > 2$. Also, $\mathrm{cr}_1\,F(S^0) = \widetilde{R}[\wedge^2](S^0) = R$ and $\mathrm{cr}_2\,F(S^0) = \widetilde{R}[\wedge^2](S^0) = R[\Sigma_2]$. Thus, the (terminal) E^1 page of the Kan spectral sequence of $F = \widetilde{R}[\wedge^2]$ at X is :

$$\begin{bmatrix} \widetilde{R}[\mathcal{I}^1](X)\otimes\mathrm{cr}_1F(S^0) & 0 \\ 0 & \widetilde{R}[\mathcal{I}^2](X)\otimes_{\Sigma_2}\mathrm{cr}_2F(S^0) \end{bmatrix} = \begin{bmatrix} \widetilde{R}[\Delta^2](X) & 0 \\ 0 & \widetilde{R}[\mathcal{I}^2](X) \end{bmatrix}$$

To say that the E^1 page of the Kan spectral sequence of F recovers F is to say that the following short exact sequence of discrete modules is split.

$$\widetilde{R}[\Delta^2] \to \widetilde{R}[\wedge^2] \to \widetilde{R}[\mathcal{I}^2]$$

Evaluated at a fixed X, this sequence is a split exact sequence of R-modules, but the splitting is not natural in X so the sequence is not split as a sequence of discrete modules.

3.3 The rank filtrations of D_mF and P_mF and their associated spectral sequences.

The homotopy left Kan filtration can be used in the obvious way to get spectral sequences for P_mF and D_mF, but these spectral sequences suffer the same deficiency as the Kan spectral sequence of F - they're too coarse to recover P_mF and D_mF. In this section, we describe the *rank* filtrations for P_mF and D_mF. In the case of D_1F, and conjecturally for all P_mF and D_mF, the associated spectral sequences come from bicomplexes that recover P_mF and D_mF.

In the remainder of this chapter, we take some first steps towards explicit descriptions of the E^1 page of the rank spectral sequences of D_mF and P_mF.

Proposition 8. *$P_mL_1F \to P_mL_2F \to P_mL_2F \to \cdots$ and $D_mL_1F \to D_mL_2F \to D_mL_2F \to \cdots$ are* nice *filtrations of P_mF and D_mF, respectively.*

Proof. In [7], it is shown that P_m and D_m are quasi-exact operators on the category of discrete modules. It follows that P_m and D_m take cofibrations to cofibrations. In particular, the maps

$$P_mL_{n-1}F \to P_mL_nF \qquad D_mL_{n-1}F \to D_mL_nF$$

are cofibrations.

Recall that P_mF is a colimit operation on F. In [7], D_mF is characterized as the *homotopy orbits of the multilinearization of the m^{th} cross effect of F.*. In particular, D_mF is also a colimit operation on F. It follows that P_m and D_m commute with other colimit constructions on F. In particular,

$$\operatorname*{colim}_n P_mL_nF = P_m \operatorname*{colim}_n L_nF = P_mF \qquad \operatorname*{colim}_n D_mL_nF = D_m \operatorname*{colim}_n L_nF = D_mF$$

□

By a slight abuse of terminology, we refer to these filtrations as the *rank* filtrations of P_mF and D_mF.

Proposition 9.

$$\frac{P_m L_n F}{P_m L_{n-1} F} \cong P_m \widetilde{R}[\mathcal{I}^n] \ \widehat{\otimes}_{\Sigma_n} \ \mathrm{cr}_n F(S^0) \qquad \frac{D_m L_n F}{D_m L_{n-1} F} \cong D_m \widetilde{R}[\mathcal{I}^n] \ \widehat{\otimes}_{\Sigma_n} \ \mathrm{cr}_n F(S^0)$$

Proof.

$$\begin{aligned} \frac{P_m L_n F}{P_m L_{n-1} F}(X) &\cong P_m \frac{L_n F}{L_{n-1} F}(X) \\ &\cong P_m \left(\widetilde{R}[\mathcal{I}^n](X) \widehat{\otimes}_{\Sigma_n} \mathrm{cr}_{\bullet} F(S^0) \right) \\ &\cong P_m \widetilde{R}[\mathcal{I}^n](X) \widehat{\otimes}_{\Sigma_n} \mathrm{cr}_{\bullet} F(S^0) \end{aligned}$$

The first equivalence follows from proposition the quasi-exactness of P_m. The second equivalence follows from proposition 7. The equivalence on the last line comes from the quasi-exactness of the construction $\widehat{\otimes}_{\mathbb{F}_+} \mathrm{cr}_{\bullet} F(S^0)$.

A similar calculation establishes the formula for the layers of the rank filtration of $D_m F$. ☐

Note that the action of Σ_n on $P_m \widetilde{R}[\mathcal{I}^n]$ is not free. It follows that the homology of $\frac{P_m L_n F}{P_m L_{n-1} F}$ is not generally concentrated in dimension 0. As a result, the associated spectral sequence has possibly nonzero entries on or above the corresponding nontrivial entries of the E^1 page of the Kan sequence for F:

$$\left[\begin{array}{ccccccc} & & \vdots & & & & \\ \square & \square & \square & \square & \square & \square & \\ \square & \square & \square & \square & \square & \square & \\ 0 & \square & \square & \square & \square & \square & \\ 0 & 0 & \square & \square & \square & \square & \cdots \\ 0 & 0 & 0 & \square & \square & \square & \\ 0 & 0 & 0 & 0 & \square & \square & \\ 0 & 0 & 0 & 0 & 0 & \square & \\ & & \vdots & & & & \end{array} \right]$$

Since $\widetilde{R}[\mathcal{I}^n]$ has degree n, $P_m \widetilde{R}[\mathcal{I}^n] \cong \mathcal{I}^n$ for $n \leq m$. The corresponding columns of the E^1 page are identical to the corresponding columns of the E^1 page of the Kan spectral sequence of F. For instance, when $m = 3$, the E^1 page of the rank spectral sequence for $P_m \widetilde{R}[\mathcal{I}^n]$ has the form:

$$\begin{bmatrix} & & \vdots & & & & \\ 0 & 0 & 0 & \square & \square & \square & \\ \widetilde{R}[\mathcal{I}^1]\otimes \mathrm{cr}_1 F(S^0) & 0 & 0 & \square & \square & \square & \\ 0 & \widetilde{R}[\mathcal{I}^2]\otimes_{\Sigma_2} \mathrm{cr}_2 F(S^0) & 0 & \square & \square & \square & \\ 0 & 0 & \widetilde{R}[\mathcal{I}^3]\otimes_{\Sigma_3} \mathrm{cr}_3 F(S^0) & \square & \square & \square & \cdots \\ 0 & 0 & 0 & \square & \square & \square & \\ 0 & 0 & 0 & 0 & \square & \square & \\ 0 & 0 & 0 & 0 & 0 & \square & \\ & & \vdots & & & & \end{bmatrix}$$

In a later chapter, we'll prove a crucial theorem which states that $P_m\widetilde{R}[\mathcal{I}^n]$ has homology concentrated in dimension $n-m$. This allows us to further refine our picture of the E^1 page of $P_m\widetilde{R}[\mathcal{I}^n]$. For instance, in the case $m=3$, it looks like:

$$\begin{bmatrix} & & \vdots & & & & \\ 0 & 0 & 0 & \square & \square & \square & \\ \widetilde{R}[\mathcal{I}^1]\otimes \mathrm{cr}_1 F(S^0) & 0 & 0 & \square & \square & \square & \\ 0 & \widetilde{R}[\mathcal{I}^2]\otimes_{\Sigma_2} \mathrm{cr}_2 F(S^0) & 0 & \square & \square & \square & \\ 0 & 0 & \widetilde{R}[\mathcal{I}^3]\otimes_{\Sigma_3} \mathrm{cr}_3 F(S^0) & \square & \square & \square & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & \vdots & & & & \end{bmatrix}$$

Consider now the E^1 page of the rank filtration of D_mF. Since the construction of the E^1 page is natural, the E^1 pages of D_m, P_mF and $P_{m-1}F$ are related in the same way as D_m, P_mF and $P_{m-1}F$ are related. Now, since the E^1 pages of P_mF and $P_{m-1}F$ agree on the first $m-1$ columns, D_mF has zeros in those columns. For example, in the case $m=3$, the E^1 page of the rank filtration of D_3F has the form:

$$\begin{bmatrix} & & \vdots & & & & \\ 0 & 0 & 0 & \square & \square & \square & \\ 0 & 0 & 0 & \square & \square & \square & \\ 0 & 0 & 0 & \square & \square & \square & \\ 0 & 0 & 0 & \square & \square & \square & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & \vdots & & & & \end{bmatrix}$$

In later chapters, we'll give explicit descriptions of the entries marked $\square$ by examining $D_3\widetilde{R}[\mathcal{I}^n]$ and hence $\frac{D_3 L_n F}{D_3 L_{n-1} F}$. More generally, we'll examine the layers and levels of Goodwillie-Taylor tower of $\widetilde{R}[\mathcal{I}^n]$.

Providing an explicit description of the differential maps between these columns, and more generally between the columns in the E^1 pages of P_mF and D_mF, requires a careful analysis of the maps in the Puppe sequences associated with $L_{n-1}P_mF \to L_nP_mF$ and $L_{n-1}D_mF \to L_nD_mF$. We leave this task for future work in which we also hope to prove that these E^1-pages are generated by a bicomplex - an "E^0 page" - in much the same way as the E^1 page of the Kan spectral sequence of D_1F is generated by the reduced Robinson bicomplex, $\widetilde{\Xi}F$.

Chapter 4

Trees and their Rank Intervals

We assume throughout this chapter that A is an finite nonempty set whose elements are enumerated by $a_1, \ldots, a_n$. We introduce a poset T_A of labelled trees with a simplicial structure equivalent to the nerve of the category Π_n of partitions of n. We consider certain simplicial subsets of T_n called *rank intervals* and give explicit descriptions for their cohomology groups.

The results of this chapter rely on a combinatorial theorem of Anders Björner and Michelle Wachs [1], [12]. In the first few sections of the chapter we develop the terminology to first state (a special case of) their theorem and then to establish its premise.

4.1 Partitions

For any positive integer r, the set $\mathrm{sur}(A, r)$ of surjections from A to r admits a left action of the symmetric group Σ_r. The elements of the orbit set $\mathrm{sur}(A, r)_{\Sigma_r}$ are called *partitions* of A of *rank* r. For an arbitrary partition ρ, we write $|\rho|$ for the rank of ρ. For a partition ρ with rank r and a representative $f \in \mathrm{sur}(A, r)$, a *block* of ρ is a subset of of A of the form $f^{-1}\{i\}$ for some $i \in r$. We often identify a partition with its set of blocks and use the notation $B \in \rho$ to indicate that B is a block of the partition ρ. Note that the set of blocks of a partition does not depend on the choice of representative f.

We write $\rho \leq \rho'$ if each block of ρ is contained in some block of ρ'. For example, if $A = \{1, 2, 3, 4, 5\}$ then

$$13|25|4 \leq 13|245$$

More precisely, for partitions ρ and ρ' of ranks r and r', we write $\rho \leq \rho'$ if there exist surjections f,f' and ξ

such that f represents ρ, f' represents ρ' and the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
r \arrow[rr, "\xi"'] & & r'
\end{tikzcd}$$

One can verify that this determines a partial order on the set of partitions of A. In particular, the value of $\rho \leq \rho'$ does not depend on the choice of representatives f,f' for ρ and ρ'. The resulting poset is denoted Π_A.

Π_A has a unique element with rank $n = |A|$. This element is initial in Π_A and is denoted by $\hat{0}$. Similarly, Π_A has a unique element with rank 1 which is terminal in Π_A and is denoted $\hat{1}$. For example, if $A = \{1, 2, 3, 4, 5\}$ then $\hat{0} = 1|2|3|4|5$ and $\hat{1} = 12345$.

For $\rho, \rho' \in \Pi_A$, the *interval* $\Pi_A[\rho, \rho']$ is defined by:

$$\Pi_A[\rho, \rho'] := \{\tau \in \Pi_A \mid \rho \leq \tau \leq \rho'\}$$

For integers m, m', the *rank interval* $\Pi_A(m, m')$ is defined by:

$$\Pi_A(m, m') := \{\tau \in \Pi_A \mid m < |\tau| < m'\}$$

Any subposet of Π_A which contains one or both of the universal elements $\hat{0}, \hat{1}$ is topologically[1] a cone and hence has trivial homotopy type. On the other hand, we'll see that the proper rank intervals of Π_A (i.e. those rank intervals $\Pi_A(m, m')$ with $m \geq 1$ and $m' \leq n$) have the homotopy type of a bouquet of spheres.

One particular rank interval $\Pi_A(1, n) = \Pi_A - \{\hat{0}, \hat{1}\}$ has been extensively studied. For $n < 3$, $\Pi_A(1, n)$ is obviously empty but for $n \geq 3$, it is well-known (see [12]) that $\Pi_A(1, n)$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n - 3$ and that the Σ_A-action on its unique reduced cohomology group is given by ϵLie_A. [2]

Example 5. *Consider the rank interval* $\Pi_A(1, n)$ *in the case* $A = \{x, y, z\}$*. It's not hard to see that the*

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[1] The homotopy and homology of a small category, and in particular a poset, is defined to be that of its nerve.

[2] Lie_A denotes the submodule of the free Lie algebra on A, generated (as a module) by the Lie monomials in which each element of A appears exactly once. The right Σ_A-action on this module is given by permutation of the symbols in A. ϵLie_A denotes the tensor product $\mathrm{sgn} \otimes_{\Sigma_n} Lie_A$ of the sign representation with Lie_n.

cohomology of $\Pi_A(1,n)$ *is concentrated in dimension* 0 *and is freely generated by the partitions*

$$x|yx \quad y|xz \quad z|xy$$

The cohomology group $H^0\Pi_A(1,n)$ *admits the obvious right action of* Σ_A*. Moreover, since the coaugmentation map is trivially* Σ_A*-invariant, the reduced cohomology group*

$$\widetilde{H}^0\Pi_A(1,n) = \frac{R[\ x|yx,\ y|xz,\ z|xy\ ]}{x|yz + y|xz + z|xy = 0}$$

inherits the Σ_A*-action.*

Remark 5.

1. $\widetilde{H}^0\Pi_A(1,n)$ *is a free* R*-module and an* $R[\Sigma_A]$*-module but not a free* $R[\Sigma_A]$*-module.*
2. *The map determined by*

$$x|yz \mapsto [x,[y,z]] \qquad y|xz \mapsto [y,[x,z]] \qquad z|xy \mapsto [z,[x,y]]$$

is a Σ_A*-isomorphism between the cohomology of* $\Pi_A(1,n)$ *and* $\epsilon Lie_A := Lie_A \otimes_{\Sigma_n} (-1)$*.*

This Σ_A-action and its generalization to arbitrary finite sets A, are central ingredients in the description of the first layer of the GT-tower for an arbitrary discrete module F. This connection is apparent in Robinson's bicomplex for D_1F.

In the next few sections of this chapter, we give an alternate description of this Σ_A-representation in terms of a certain category of labelled trees. In these terms, we will more readily generalize this description to Σ_A-representations given by cohomology groups of arbitrary rank intervals $\Pi_A(m,m')$. In the next chapter, we'll see that these generalized representations are related to the higher layers and levels of the GT-tower in the same way that the representation on $\Pi_A(1,n)$ is related to the first layer.

4.2 The Category and Simplicial Set of A-trees.

A *tree* is a finite connected poset such that every connected subposet has a unique greatest lower bound.[3] The elements of a tree t are called *vertices*. The maximal vertices of a tree are called *leaves* and the unique minimal vertex is called the *root*. The vertices which are neither roots nor leaves are called *internal vertices*. For t a tree, $|t|$ denotes the set of internal vertices of t. The *edges* of a tree are the closed intervals that contain exactly two vertices.

A basic property of trees is that there is a unique maximal chain between the root and any given vertex. The *height* of a vertex v in a tree t is length of the unique maximal chain between v and the root of t. In particular, the height of the root is zero. The i^{th} *vertex layer* of a t is the set of vertices of t of height i. The *height* of a tree t is the maximum height of its vertices. The *height* of an edge is the height of its minimal element. The i^{th} *edge layer* of t is the set of edges of t of height i. A tree is called *balanced* if all of its leaves have the same height. A map of balanced trees is a map of trees which preserves height.

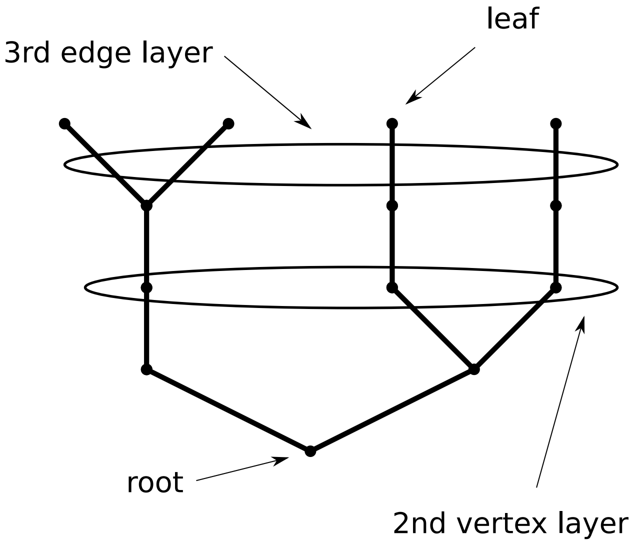

Figure 4.1: A balanced tree of height 4

For vertices v, w of a tree t, we say that v is an *immediate successor* of w if $w < v$ and the interval (w, v) is empty. v is called a *node* if it has exactly one immediate successor and a *branching vertex* if it has at least two immediate successors. The *parent* of v is the maximal branching vertex in the interval $(-\infty, v)$ of t unless v is the root, in which case v is its own parent.

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[3] The objects defined here as *trees* are more accurately described as *finite rooted trees*. Since all the trees discussed in this paper are of this type, this abbreviation should cause no confusion.

For a finite nonempty set A, an *A-labelled tree* is a tree t equipped with a bijective map, called a *leaf labelling*, from A to the set of leaves of t. A map of A-labelled trees is a map of trees which preserves the labelling. The category of isomorphism classes of balanced and A-labelled trees is denoted $\widetilde{\mathcal{T}}_A$ and the objects of $\widetilde{\mathcal{T}}_A$ are called *A-trees.* For simplicity, we don't usually distinguish between A-trees and the A-labelled trees which represent them.

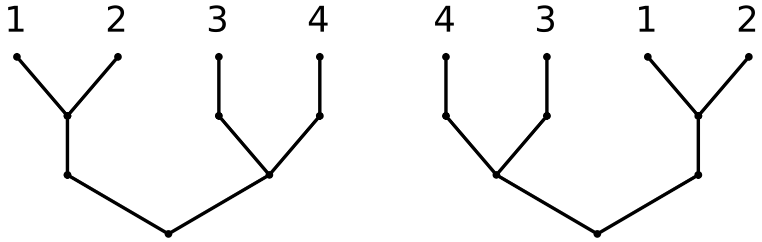


Figure 4.2: two identical 4-trees

Note that $\widetilde{\mathcal{T}}_A$ always has a unique tree of height 1. When $|A| = 1$, $\widetilde{\mathcal{T}}_A$ also has a unique tree of height 0.

$\mathcal{T}_A$, the main category of interest in this chapter, denotes the subcategory of $\widetilde{\mathcal{T}}_A$ generated by the A-labelled trees of height at least 2.

Remark 6. *The category $\mathcal{T}_A$ is actually a poset. That is, there is at most one morphism between any two A-trees. However, we do not study the homotopy/homology of this poset by considering its nerve. Instead, we will (in the next section) identify a simplicial structure on $\mathcal{T}_A$ itself.*

Next, we describe a bijection between the set of A-trees and the nerve of the category of partitions of A. This bijection induces a simplicial structure on $\mathcal{T}_A$.

A balanced tree determines a surjection from any vertex layer to any lower vertex layer. The surjections from the top layer of a A-tree determine partitions of A. Considering surjections from the top layer to successively lower layers determines a chain of partitions of A. This actually defines a set isomorphism between the objects of $\mathcal{T}_A$ and the simplices of $\mathcal{N}\Pi_A$.

This bijection induces a simplicial structure on $\mathcal{T}_A$ and the restriction of this bijection to the 0-simplices of $\mathcal{N}\Pi_A$ gives an equivalence of categories between Π_A and the A-trees of height 2. Moreover, if we define the indiscrete partition of A and the unique A-tree t with $|t| = 1$ to be the basepoints of their respective categories, then this equivalence preserves basepoints.

Henceforth, we refer to the isomorphic categories Π_A and the category of A-trees of height 2 interchangably. We also refer to the isomorphic simplicial sets $\mathcal{N}\Pi_A$ and $\mathcal{T}_A$ interchangably.

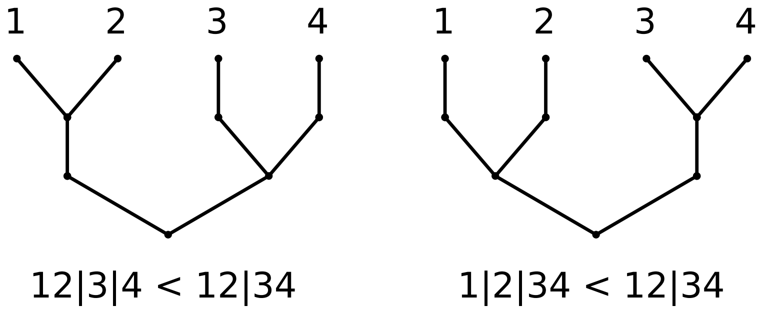


Figure 4.3: corresponding 1-simplices in $\mathcal{T}_4$ and $\mathcal{N}\Pi_4$

Remark 7.

1. *The two definitions for the symbol* $|t|$ *are compatible in the sense that the rank of any partition is the number of internal vertices of the corresponding tree of height* 2*.*

2. *The map* $t \mapsto |t|$ *determines an order-reversing map from each poset of partitions to* $\mathbb{N}$ *and from each poset of trees to* $\mathbb{F}$*.*

We introduce two basic operations on trees and then use them to give an alternate characterization of the simplicial structure on $\mathcal{T}_A$.

Definition 7. *To* subdivide *an edge of a tree* t *is to split the edge into two edges by inserting a new vertex in the middle of the original edge. To* collapse *an edge is to identify the two vertices incident to that edge and then to delete the edge.*

The tree that results from collapsing or subdividing a single edge of a balanced tree is not, in general, balanced. However, simultaneously collapsing all the edges in a single layer of a balanced tree t does produce a balanced tree. Similarly, subdividing all the edges in a single layer of a balanced tree produces a balanced tree. In particular, as long as the top edge layer is not collapsed, subdividing or collapsing edges one layer at a time does not affect the set of leaves of a tree or the leaf labelling.

Proposition 10. *The simplicial structure on* $\mathcal{T}_A$ *induced by the bijection with* $\mathcal{N}\Pi_n$ *consists of* k*-simplices that are the trees of height* $k+2$*, face maps that are the edge level collapses (for all but the top layer) and degeneracy maps that are the edge level subdivisions (for all but the top layer).*

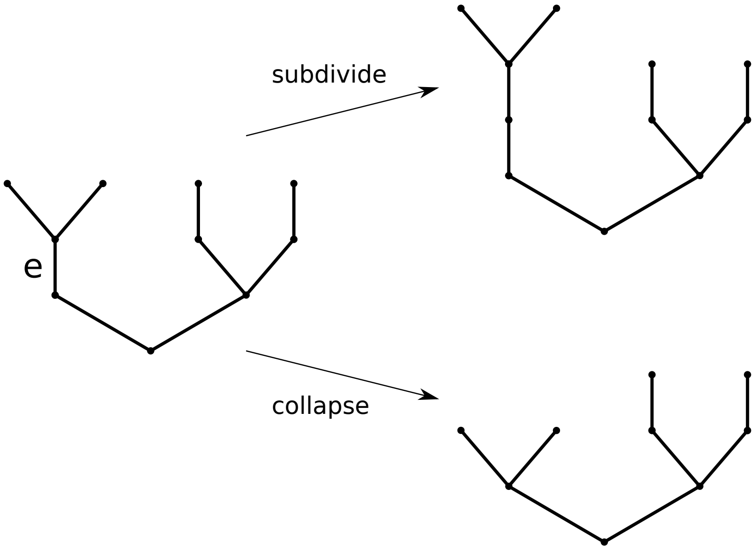


Figure 4.4: collapsing and subdividing the single edge ‘e’

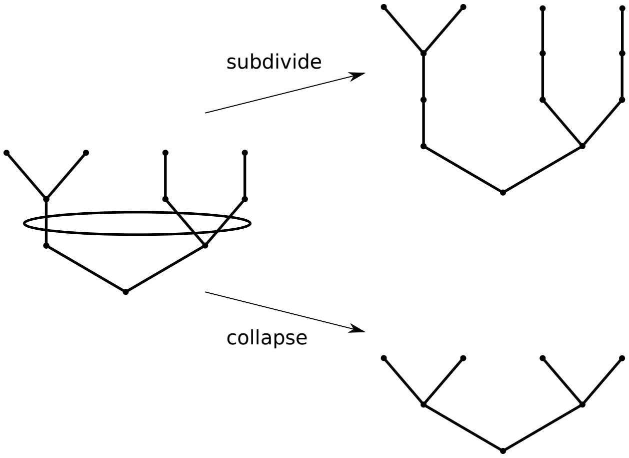


Figure 4.5: collapsing and subdividing an entire edge level

For a k-simplex $t \in \mathcal{T}_A$, define 0-simplices $\lfloor t \rfloor$ (the *floor* of t) and $\lceil t \rceil$ (the *ceiling* of t) by

$$\lfloor t \rfloor := \begin{cases} d_1 d_2 \dots d_k t, & k > 0 \\ t, & k = 0 \end{cases} \qquad \lceil t \rceil := \begin{cases} d_{k-1} d_{k-2} \dots d_0 t, & k > 0 \\ t, & k = 0 \end{cases}$$

Here, the d_i's refer to the face maps of $\mathcal{T}_A$.

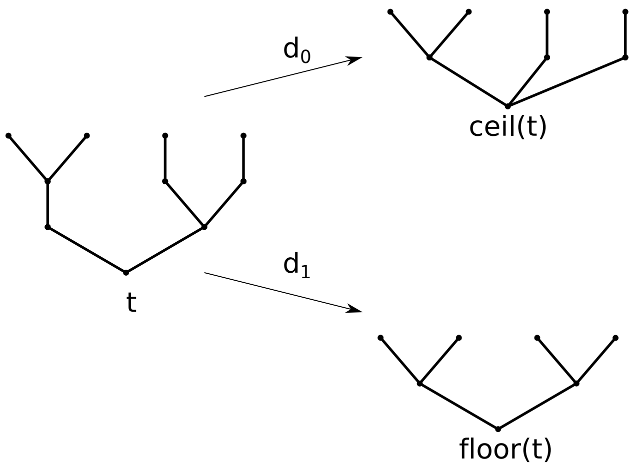


Figure 4.6: $\lfloor t \rfloor = d_1 t$ and $\lceil t \rceil = d_0 t$

We define $(\widetilde{\mathcal{T}}_A)_{-1}$ to be the singleton set consisting of the unique A-tree of height 1. Then there is a trivial augmentation map $(\widetilde{\mathcal{T}}_A)_0 \to (\widetilde{\mathcal{T}}_A)_{-1}$.

Trusting that resulting ambiguities can be resolved from context, we abbreviate the symbol $|\lfloor t \rfloor|$ for the set of internal vertices of $\lfloor t \rfloor$, by $\lfloor t \rfloor$ itself. Similarly, we write $\lceil t \rceil$ for $|\lceil t \rceil|$.

Definition 8. *A* rank interval *of $\mathcal{T}_A$ is given by:*

$$\mathcal{T}_A(m, m') := \{t \in \mathcal{T}_A \mid m < \lfloor t \rfloor \text{ and } \lceil t \rceil < m'\}$$

for each $m < m'$. A rank interval is called proper *if $1 \leq m$ and $m' \leq n$.*

An interval *of $\mathcal{T}_A$ is given by:*

$$\mathcal{T}_A[\rho, \rho'] := \{t \in \mathcal{T}_A \mid \rho \leq \lceil t \rceil \text{ and } \lfloor t \rfloor \leq \rho'\}$$

for each pair $\rho < \rho'$ of 0-simplices of $\mathcal{T}_A$.

It is straightforward to verify that the intervals and rank intervals inherit simplicial structures from $\mathcal{T}_A$. Specifically, $\mathcal{T}_A(m, m')$ and $\mathcal{T}_A[\rho, \rho']$ are the simplicial subsets of $\mathcal{T}_A$ corresponding to $\mathcal{N}\Pi_A(m, m')$ and $\mathcal{N}\Pi_A[\rho, \rho']$, respectively. Henceforth, we refer to nerves of intervals in Π_A and the corresponding "intervals" of $\mathcal{T}_A$ interchangably.

We also define:

$$\mathcal{T}_A(\rho, \rho'] := \{t \in \mathcal{T}_A \mid \rho < \lfloor t \rfloor \text{ and } \lceil t \rceil \leq \rho'\}$$

$$\mathcal{T}_A[\rho, \rho') := \{t \in \mathcal{T}_A \mid \rho \leq \lfloor t \rfloor \text{ and } \lceil t \rceil < \rho'\}$$

$$\mathcal{T}_A(\rho, \rho') := \{t \in \mathcal{T}_A \mid \rho < \lfloor t \rfloor \text{ and } \lceil t \rceil < \rho'\}$$

In the remainder of this section, we examine a few simple examples of rank intervals of $\mathcal{T}_A$.

Example 6. *Consider the simplicial set $\mathcal{T}_A(1, n)$ in the case $A = \{x, y, z\}$. Observe that the set of nondegenerate simplices of $\mathcal{T}_A(1, n)$ are exactly its 0-simplices.*

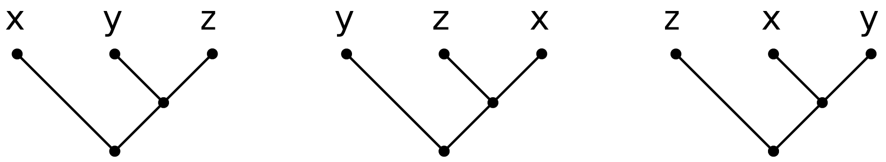
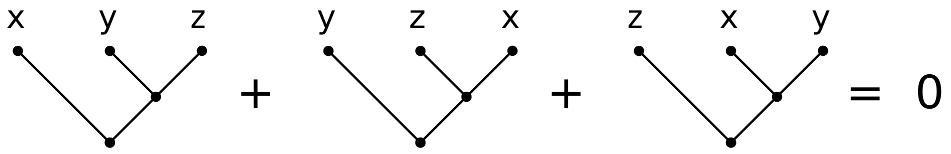


Figure 4.7: The nondegenerate simplices of $\mathcal{T}_A(1, n)$ for $A = \{x, y, z\}$ and $n := |A| = 3$.

The cohomology of $\mathcal{T}_A(1, n)$ is concentrated in dimension 0 and is freely generated by its three 0-simplices. The reduced cohomology is the quotient of the unreduced cohomology by the relation:



The set of 0-simplices admits the action of Σ_A given by leaf permutation and the augmentation map is (trivially) Σ_A-invariant. It follows that the cohomology and the reduced cohomology of $\mathcal{T}_A(1, n)$ both inherit this Σ_A-action.

Henceforth, unless explicitly stated otherwise, we assume that A is endowed with a total ordering: $a_1 < \cdots < a_n$. This induces a total ordering on the set of leaves of each A-tree. We denote by $A^{\#}$, the set of words of A, ordered reverse lexicographically.[4] We call a word *increasing* if the sequence of letters in the word is empty or strictly increasing. We call a word *decreasing* if the sequence of letters in the word is empty or weakly decreasing.

For t an A-tree and v a vertex of t we denote by A_v, the set of labels of leaves in the interval $[v, \infty)$ of t. We denote by $\lambda(v)$, the minimal element of A_v.

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[4] Using the reverse lexicographic ordering we compare words by looking at the first letter from the right at which they differ and then use the ordering on A to determine the order relation.

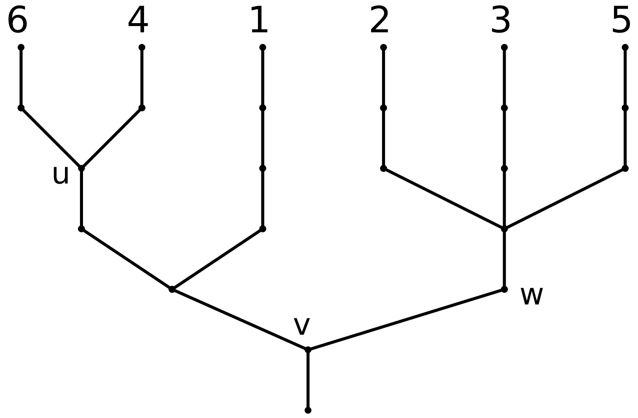


Figure 4.8: $\lambda(u) = 4$, $\lambda(v) = 1$, $\lambda(w) = 2$

For l a vertex layer of t that contains at least one branching vertex[5] we define:

$$\lambda(l) := \min\{\lambda(v) \mid v \text{ is a branching vertex in } l\}$$

Let $l_1, l_2, \ldots, l_k$ be the vertex layers of t that contain branching vertices. Assuming that the layers $l_1, l_2, \ldots, l_k$ are listed in increasing order of height, we write $\phi(t)$ for the word $\lambda(l_2)\lambda(l_3)\cdots\lambda(l_k)$. We call t *increasing* (*decreasing*) if $\phi(t)$ is increasing (decreasing).

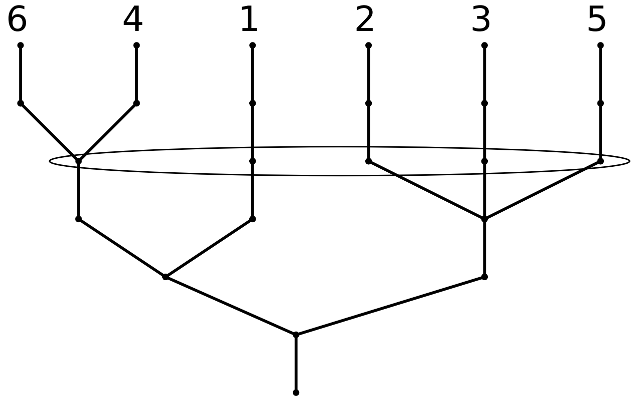


Figure 4.9: An increasing tree. For the indicated layer l, $\lambda(l) = 4$ and $\phi(t) = 124$.

The proof of the following theorem is postponed to a later section in this chapter.

Theorem 2. *Let $\mathcal{T}_A(m, m')$ be a proper rank interval. Then:*

1. *$\mathcal{T}_A(m, m')$ has the homotopy type of a wedge of spheres of dimension $m' - m - 2$, indexed by its decreasing facets.*

---

[5] If l is a vertex layer which does not contain any branching vertices then $\lambda(l)$ is undefined.

2. *The reduced cohomology group of $\mathcal{T}_A(m, m')$ is free and has a basis represented by the decreasing facets of $\mathcal{T}_A(m, m')$.*

Remark 8. *An immediate corollary of this theorem is that the (unique) cohomology group is free of finite rank and so the corresponding homology group is simply the dual R-module.*

4.3 Increasing facets and decreasing facets of rank intervals

Definition 9. *A tree $t \in \mathcal{T}_A$ is said to be* binary *if each of its branching vertices has exactly two immediate successors. t is said to be* regulated *if each of its vertex levels contain at most one branching vertex.*

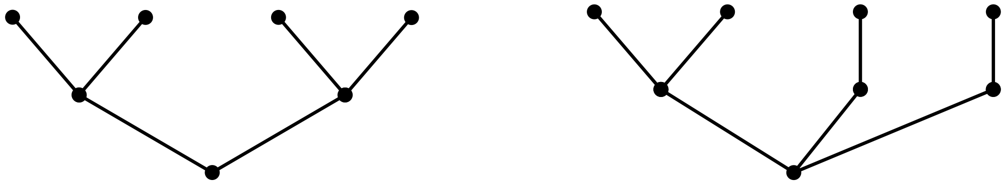


Figure 4.10: The tree on the left is binary but not regulated. The tree on the right is regulated but not binary.

Notice that neither of the trees in figure 4.10 are facets. Indeed, they are faces of the following nondegenerate trees:

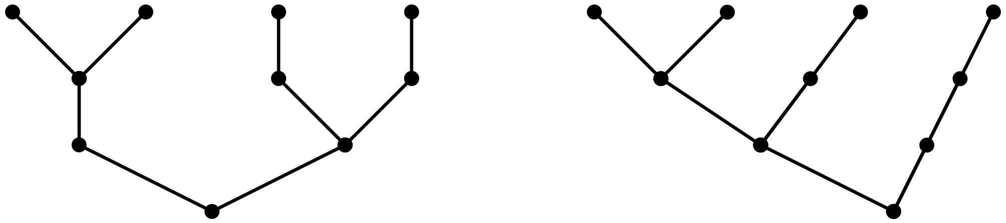


Figure 4.11: These trees witness that the trees in figure 4.10 are not facets.

Proposition 11. *The facets of $\mathcal{T}_A(1,n)$ are exactly the nondegenerate trees which are both binary and regulated.*

Proof. When $n < 3$ this set is empty so we assume $n \geq 3$. Let $t \in \mathcal{T}_A(1,n)$ be nondegenerate. That is, each vertex level of t, other than the leaf level, contains at least one branching vertex. If t has a vertex level with more than one branching vertex, t is not maximal. Similarly, if t contains a branching vertex with more than two immediate successors, then t is not maximal. In either case, t is not a facet. On the other hand, if t is both binary and regulated then t has exactly one branching vertex in each level. Such a tree cannot be a face of a higher tree. ☐

Proposition 12. *$\mathcal{T}_A(1,n)$ is pure of dimension $n-3$.*

Proof. Observe that the cardinality of the vertex levels of a binary regulated tree increases by one with each higher level. Since such trees have exactly n vertices in their top level and a unique vertex at the bottom level, the sequence of cardinalities of the vertex levels of such facets is $1, 2, 3, \cdots, n$. In particular, they have height $n-1$. From proposition 11, it follows that $\mathcal{T}_A(1,n)$ is pure of dimension $n-3$. ☐

Consider the categories $\mathcal{T}_A(0,n+1)$, $\mathcal{T}_A(0,n)$ and $\mathcal{T}_A(1,n+1)$. Each of these are nerves of a poset (specifically, $\Pi_A(0,n+1)$, which contains an initial or a terminal object so their reduced cohomology is trivial. Nevertheless we describe their facets.

Any maximal chain of partitions of $\Pi_A(0,n)$ is simply a maximal chain of partitions of $\Pi_A(1,n)$ with the terminal partition $\hat{1}$ appended to the end. Equivalently, the facets of $\mathcal{T}_A(0,n)$ are simply the result of appending to each facet of $\mathcal{T}_A(1,n)$, a single edge at the root. Similarly the facets of $\mathcal{T}_A(1,n+1)$ are the result of appending to each facet of $\mathcal{T}_A(1,n)$, a single edge at each leaf. Finally, the facets of $\mathcal{T}_A(0,n+1)$ are the result of appending to each facet of $\mathcal{T}_A(1,n)$, a single edge at the root and at each leaf. In particular, $\mathcal{T}_A(1,n+1) = \mathcal{N}\Pi_A(1,n+1)$ is pure of dimension $n-2$ and $\mathcal{T}_A(0,n+1) = \mathcal{N}\Pi_A(0,n+1)$ is pure of dimension $n-1$.

Proposition 13. *$\mathcal{T}_A(m,m')$ is pure of dimension $m'-m-2$.*

Proof. Use induction on $r = (m-1)+(n-m')$. We've established the case $r=0$ in proposition . Incrementing r by one is equivalent to incrementing m by one or decrementing m' by one. Each facet t in $\mathcal{T}_A(m+1,m')$ has one more branching vertex in $\lfloor t \rfloor$ than do facets in $\mathcal{T}_A(m,m')$. As a consequence, the height of facets in $\mathcal{T}_A(m+1,m')$ is one less than the height of facets in $\mathcal{T}_A(m,m')$. Similarly, each facet t in $\mathcal{T}_A(m,m'-1)$ has one more branching vertex in $\lceil t \rceil$ than do facets in $\mathcal{T}_A(m,m')$ and consequently height one less. This proves the result for $r+1$. The result follows by induction. ☐

Next, we work towards a description of the *decreasing* and *increading* facets of $\mathcal{T}_A(1,n)$.

Definition 10. *A tree t is called* left-regulated *if its branching vertices form a linearly ordered subposet of t.*

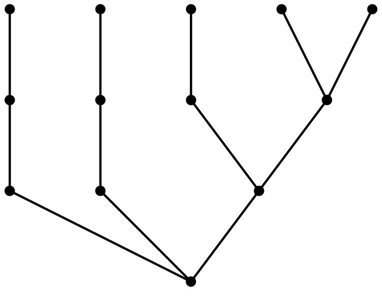


Figure 4.12: A left-regulated tree (leaf labels suppressed)

Lemma 1. *If a tree $t \in \mathcal{T}_A(1,n)$ is increasing or decreasing then t is left-regulated.*

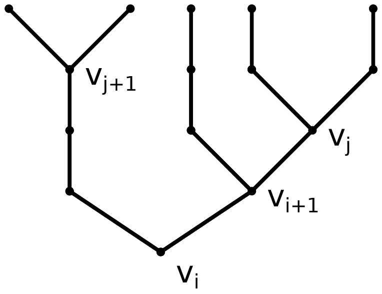
Proof. Let $t \in \mathcal{T}_A(1, n)$ with t *not* left-regulated. Let v_j, v_{j+1} be the first pair of incomparable branching vertices of t, let v_i be the greatest lowest bound for v_j, v_{j+1} in t and let v_{i+1} be the minimum branching vertex in (the linearly ordered set) $(v_i, v_j]$.

Observe that $A_{v_i} = A_{v_{i+1}} \sqcup A_{v_{j+1}}$ (disjoint union) and $A_{v_j} \subseteq A_{v_{i+1}} \subsetneq A_{v_i}$. It follows that

$$\lambda(v_i) = \min\{\lambda(v_{i+1}), \lambda(v_{j+1})\} \quad \text{and} \quad \lambda(v_i) \leq \lambda(v_{i+1}) \leq \lambda(v_j).$$

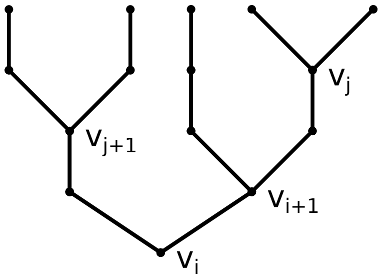
Since v_j, v_{j+1} are incomparable, $A_{v_j} \cap A_{v_{j+1}} = \emptyset$ hence $\lambda(v_j) \neq \lambda(v_{j+1})$. There are two cases to consider.

Case 1 $\lambda(v_j) < \lambda(v_{j+1})$



It is clear that t is not decreasing. Moreover, since $\lambda(v_i) \leq \lambda(v_j) < \lambda(v_{j+1})$ we have that $\lambda(v_i) = \min\{\lambda(v_{i+1}), \lambda(v_{j+1})\} = \lambda(v_{i+1})$ and so t is not increasing.

Case 2 $\lambda(v_{j+1}) < \lambda(v_j)$



It is clear that t is not increasing. Moreover, since $\lambda(v_i) = \min\{\lambda(v_{i+1}), \lambda(v_{j+1})\} \leq \lambda(v_{j+1}) < \lambda(v_j)$, t is not decreasing.

□

It follows from lemma 1 that the decreasing facets of $\mathcal{T}_A(1,n)$ are the trees of the form:

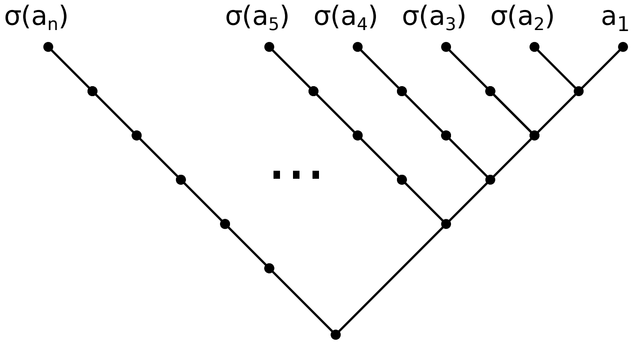


Figure 4.13: a decreasing facet of $\mathcal{T}_A(1,n)$

where $\sigma \in \Sigma_{A-\{a_1\}}$. Equivalently, the decreasing facets in $\mathcal{T}_A(1,n)$ are the facets t with:

$$\phi(t) = \underbrace{a_1 a_1 \dots a_1}_{n-3}$$

A second consequence of lemma 1 is the uniqueness of the following increasing facet in $\mathcal{T}_A(1,n)$:

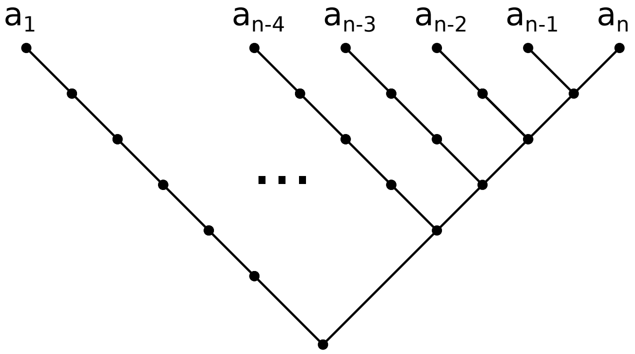


Figure 4.14: The unique increasing facet in $\mathcal{T}_A(1,n)$

4.4 The Σ_A-action on $\widetilde{H}^{n-3}\mathcal{T}_A(1,n)$

There is a natural right Σ_A-action on $\mathcal{T}_A(1,n)$ that permutes leaf labels. In this section, we describe the induced Σ_A-action on $\widetilde{H}^{n-3}\Pi_A(1,n)$. In the case $n=3$ we already gave a description of $\widetilde{H}^{n-3}\mathcal{T}_A(1,n)$ and its associated Σ_3-action (see example 6). Assume then that $n \geq 4$. From theorem 2, we know that the decreasing facets form a basis for $\widetilde{H}^{n-3}\mathcal{T}_A(1,n)$. However, when $n \geq 4$, the decreasing facets are a proper subset of the facets and do not inherit the obvious action of Σ_A on the set of facets. The Σ_A action on the facets does, however, induce an the free R-module generated by the decreasing facets, namely, $\widetilde{H}^{n-3}\Pi_A(1,n)$.

We specify that action by showing how to express the cohomology class represented by an arbitrary facet of $\mathcal{T}_A(1,n)$ as a $\mathbb{Z}$-linear combination of cohomology classes represented by *decreasing* facets.

In the previous section, we characterized the facets of $\mathcal{T}_A(1,n)$. In order to understand the cohomology relations, we also provide an explicit characterization of the facelets of $\mathcal{T}_A(1,n)$. Using proposition 13, these facelets can be characterized as the nondegenerate $n-4$-simplices. However, the following description is a little more explicit. Each of these facelets has a unique vertex layer which fails to be both binary and regulated. Facelets of *type I* are binary and nondegenerate trees which have a vertex layer with exactly two branching vertices. Facelets of *type II* are regular and nondegenerate trees which have a unique branching vertex with exactly three immediate succesors.

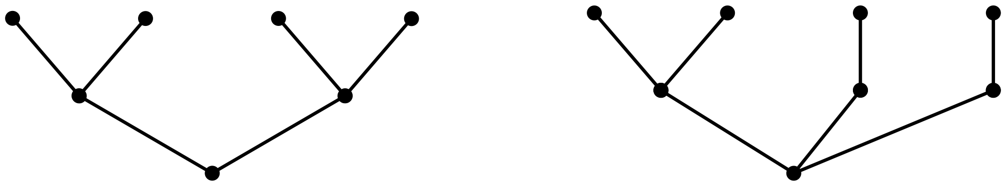


Figure 4.15: For $n = 4$, facelets in $\mathcal{T}_A(1,n)$ of type I and type II

Recall that $\mathcal{MT}_A(1,n)$ denotes the set of facets of $\mathcal{T}_A(1,n)$. Let α be a facelet of $\mathcal{T}_A(1,n)$ and let i be the height of the unique vertex level which isn't both binary and regulated. Then any facet t which has α as a face, must have α as the i^{th} face. In other words, for each facelet α of $\mathcal{T}_A(1,n)$, there is a unique integer i such that:

$$\{t \in \mathcal{MT}_A(1,n) \mid d_i t = \alpha\} \neq \emptyset$$

For that integer i, let:

$$\mathcal{M}_\alpha := \{t \in \mathcal{MT}_A(1,n) \mid d_i t = \alpha\}$$

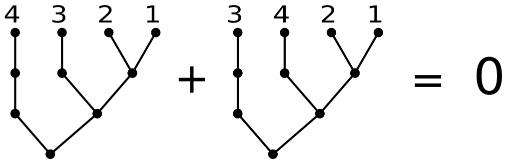
The coboundary relations on $\widetilde{H}^{n-3}\mathcal{T}_A(1,n)$ are exactly those relations of the form

$$\sum_{t \in \mathcal{M}_\alpha} [t] = 0$$

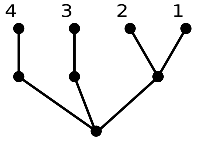
for some facelet α. Moreover, the number of summands in this relation is determined by the *type* of α. More precisely,

$$|\mathcal{M}_\alpha| = \begin{cases} 2, & \text{if } \alpha \text{ is of type I} \\ 3, & \text{if } \alpha \text{ is of type II} \end{cases}$$

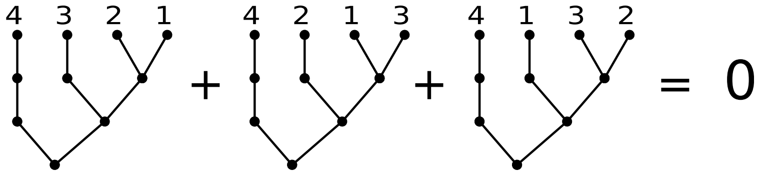
For example,



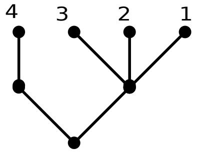
is the coboundary relation (with square brackets suppressed) corresponding to the type I facelet:



Similarly,



is the coboundary relation corresponding to the type II facelet:



There is an obvious action of Σ_n on $\mathcal{T}_A(1,n)$ given by leaf permutation. Since permuting the leaves of a tree does not change its shape (only its labelling), this action induces an action on the facets of $\mathcal{MT}_A(1,n)$. Recall that a decreasing facet of $\mathcal{T}_A(1,n)$ is a left regulated facet whose highest branching vertex is the parent of the maximal leaf a_n of A. Hence, permuting the leaves of this facet may not preserve its decreasing nature. We will give an algorithm with which (the cohomology class represented by) any facet can be expressed as a $\mathbb{Z}$-linear combination of (cohomology classes represented by) decreasing facets of $\mathcal{T}_A(1,n)$. From theorem 2 the decreasing facets form a basis for $\widetilde{H}^{n-3}\mathcal{T}_A(1,n)$. Therefore, this algorithm gives a Σ_n-action on $\widetilde{H}^{n-3}\mathcal{T}_A(1,n)$.

We now give a recursive algorithm for expressing each facet t of $\mathcal{T}_A(1,n)$ as a ($\mathbb{Z}$-linear) combination of decreasing facets using only coboundary relations.

Consider first, the case where t is left regulated and let i be the height of the parent of the leaf labelled n. If $i = n - 2$, the highest possible height of a branching vertex in a facet of $\mathcal{T}_A(1, n)$, then t is already a decreasing facet. Suppose then that $i < n - 2$ and let $\alpha = d_i t$. Then α is a facelet of type II and the coboundary relation associated with α is of the form

$$t + t' + t'' = 0$$

where t' and t'' are trees in which the parent of the leaf labelled n has height $i + 1$. We then recursively express t' and t'' as $\mathbb{Z}$-linear combinations of decreasing facets of $\mathcal{T}_A(1, n)$.

Next, we show how an arbitrary facet t of $\mathcal{T}_A(1, n)$ can be expressed as a $\mathbb{Z}$-linear combination of left regulated facets by recursion on n. In the case $n = 3$, this is apparent from the calculations in example 6. Suppose then that $n > 3$, t is a facet of $\mathcal{T}_A(1, n)$, r is the root of t and $v(t), v'(t)$ are the immediate successors of r. We can decompose the set $V(t)$ of vertices of t as a disjoint union:

$$V(t) = \{r\} \sqcup [v(t), \infty) \sqcup [v'(t), \infty)$$

To be definite, we can assume without loss of generality that $[v(t), \infty)$ contains the minimal leaf of t.

We can decompose the set of branching vertices of t as a disjoint union[6]

$$BV(t) = \{r\} \sqcup B[v(t), \infty) \sqcup B[v'(t), \infty)$$

We proceed by recursion on the cardinality $|B[v(t), \infty)|$ of $B[v, \infty)$. Before we proceed with the case $|B[v(t), \infty)| = 0$, we define an operation on trees.

Let t be a facet in $\mathcal{T}_A(1, n)$ and let a be a leaf not in A. We denote by t_a, the facet in $\mathcal{T}_{A\ cup\{a\}}(1, n + 1)$ obtained by adding a new root vertex r, an edge from r to the root of t and a chain of edges from r to a so that the resulting tree is binary and regulated. Note that the length of this chain is completely determined by the height of t. We extend this operation *by linearity* so that

$$t_a + t'_a = (t + t')_a$$

---

[6] Here, the operator 'B' means 'branching vertices of'.

Now, returning back to the case where t satisfies $|B[v(t), \infty)| = 0$, we observe that t is necessarily of the form $t = t'_a$ where t' is a facet in $\mathcal{T}_{A-\{a\}}(1, n-1)$. We can recursively express t' as a $\mathbb{Z}$-linear combination

$$t' = z_1 t'_1 + \cdots + z_k t'_k$$

where $t'_1, \cdots, t'_k$ are left regulated. Then, we have that

$$t = z_1 (t'_1)_a + \cdots + z_k (t'_k)_a$$

where $z_1(t'_1)_a, \cdots, z_k(t'_k)_a$ are also left regulated.

It remains to consider the case $|B[v(t), \infty)| > 0$. Let $i = i(t)$ be the height of the lowest branching vertex in $[v(t), \infty)$ and denote that branching vertex by v_i. Note that $0 < i < n-1$. We proceed by recursion on i.

In the case $i = 1$, let $\alpha = d_0 t$. Then α is a facelet of type II and the coboundary relation associated to α is $t + t' + t'' = 0$, where $|B[v(t'), \infty)| < |B[v(t), \infty)|$ and $|B[v(t''), \infty)| < |B[v(t), \infty)|$. We can then recursively express t' and t'' as $\mathbb{Z}$-linear combinations of left regulated trees. Using $t + t' + t'' = 0$, we get the required expression for t as a $\mathbb{Z}$-linear combination of left regulated trees.

Finally, we consider the case $i > 1$. Note that the branching vertex of t at height $i-1$, call it v_{i-1}, lies in $[v'(t), \infty)$. In particular, v_{i-1} is not the parent of v_i. Let $\alpha = d_{i-1} t$. Then α is a facelet of type I and the coboundary relation associated with α is $t + t' = 0$ where $|B[v(t'), \infty)| = |B[v(t), \infty)|$ and $i(t') < i(t)$. Thus, we recursively express t' and $\mathbb{Z}$-linear combinations of left regulated trees. Using $t + t' = 0$, we get the required expression for t as a $\mathbb{Z}$-linear combination of left regulated trees.

This completes the algorithm description. Figure 4.16 illustrates this algorithm in a particular case.

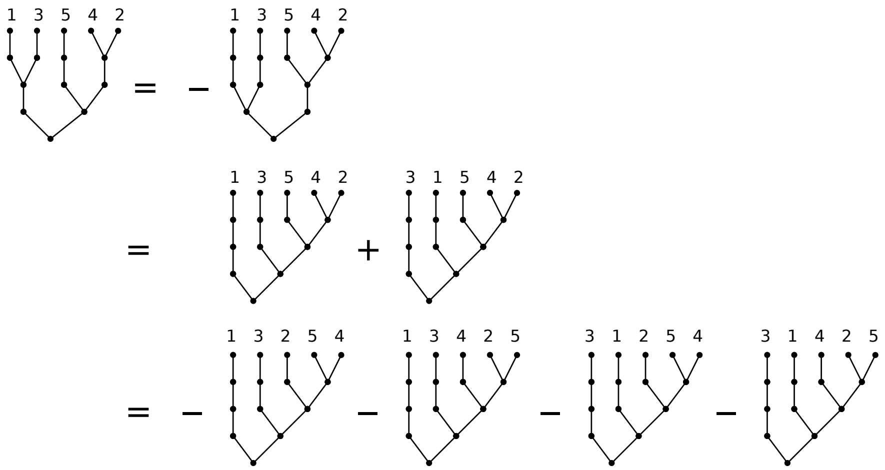


Figure 4.16: Expressing a facet of $\mathcal{T}_A(1,5)$ as a $\mathbb{Z}$-linear combination of decreasing facets.

4.5 Prunning and Splitting

In this section, we introduce two constructions on trees called prunning and splitting and we use these contructions, together with an external theorem of Bjorner and Wachs to prove theorem 2. We also describe an explicit basis for the cohomology of a rank interval $T_A(m, m')$.

Proposition 14. *The facets of $\mathcal{T}_A(m, m')$ are the trees t whose vertex layers are both binary and regulated except possibly the vertex layer $\lceil t \rceil$ and the root layer of t. Specifically, the root layer of t has $m+1$ immediate successors and $\lceil t \rceil$ has $n - m' + 1$ branching vertices.*

Proof. As in the proof of 11, the vertex layers of a facet $t \in \mathcal{T}_A(m, m')$ that lie strictly in between $\lceil t \rceil$ and $\lfloor t \rfloor$ must be both binary and regulated. The condition of maximal nondegeneracy on $\lceil t \rceil$ and $\lfloor t \rfloor$ are determined by m and m'. Specifically, $\lceil t \rceil$ should have exactly $m' - 1$ vertices and $\lfloor t \rfloor$ should have exactly $m + 1$ vertices. ☐

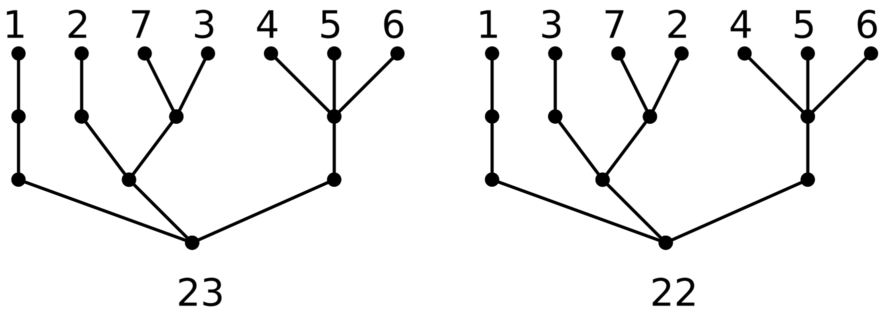


Figure 4.17: An increasing facet and a decreasing facet of $\mathcal{T}_7(2,5)$ with their associated words ϕ

Definition 11. *Let $t \in \mathcal{T}_A$.*

We prune *t as follows. For each $v \in \lceil t \rceil$, we label v with the set of leaf labels of leaves in the interval (v, ∞) of t. We then delete the interval (v, ∞). The result of prunning t is a tree, denoted $p(t)$, of height one less than the height of t.*

We split *t by deleting the root of t and the lowest edge layer of t. The result of splitting t is an unordered tuble of trees, denoted $s(t)$, each component of which is of height one less than the height of t.*

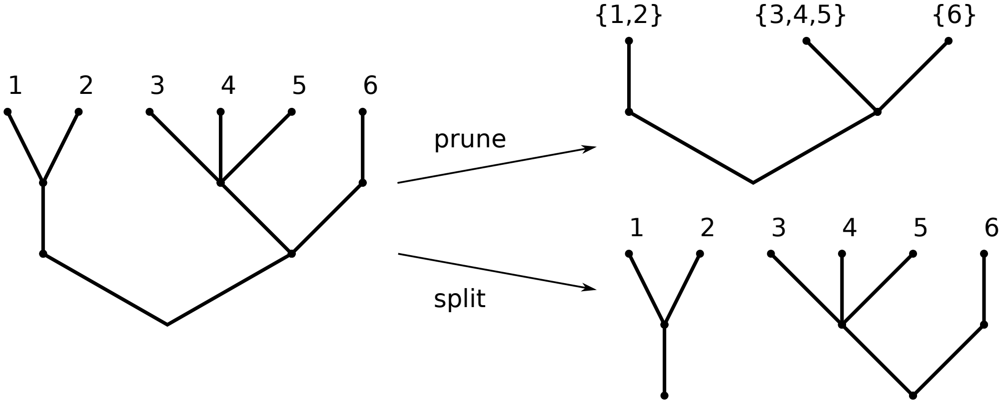


Figure 4.18: Prunning and splitting a tree

For $|A| > 1$, the prunning and splitting operations determine maps:

$$\widetilde{\mathcal{T}}_A \to \coprod_{\tau \in \Pi_A} \widetilde{\mathcal{T}}_\tau \qquad t \mapsto (\lceil t \rceil; p(t))$$

$$\widetilde{\mathcal{T}}_A \to \coprod_{\tau \in \Pi_A} \prod_{B \in \tau} \widetilde{\mathcal{T}}_B \qquad t \mapsto (\lfloor t \rfloor; s(t))$$

Here, $\prod$ denotes the unordered product.

Proposition 15. *Restricting the prunning map the the set of facets of an interval $\mathcal{T}_A[\rho, \rho']$ with $\rho \neq \rho'$ gives a bijection: $\mathcal{MT}_A[\rho, \rho'] \to \mathcal{MT}_\rho(\hat{0}, \rho']$.*

Proof. Let $t \in \mathcal{MT}_A[\rho, \rho']$. Then $\lfloor t \rfloor = \rho$, $\lceil t \rceil = \rho'$, and all other vertex layers of t are binary and regulated. Prunning has the effect of making $\lceil p(t) \rceil$ a binary and regulated vertex layer. Prunning doesn't affect the lower levels of $p(t)$ so by proposition 14, $p(t)$ is still a facet. To see that this map is a bijection, simply note that appending an edge to each leaf for each element in the leaf label is an inverse to prunning. (See figure 4.19.) $\square$

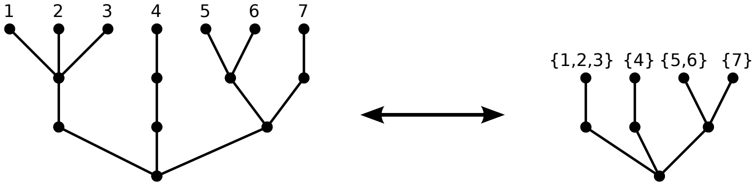


Figure 4.19: pruning and unpruning a facet

We'll be interested in following the pruning map with the splitting map. Note that restricting the splitting map to the set of facets of an interval $\mathcal{T}_A(\hat{0}, \tau]$ gives a map $\mathcal{MT}_A(\hat{0}, \tau] \to \sqcap_{B \in \tau,\ |B|>1} \widetilde{\mathcal{T}}_B(1, |B|)$ but the components of the image of this (restricted) map are not neccesarily facets. The problem is that that splitting a facet generally produces tuples with degenerate components. For example, see figure 4.20.

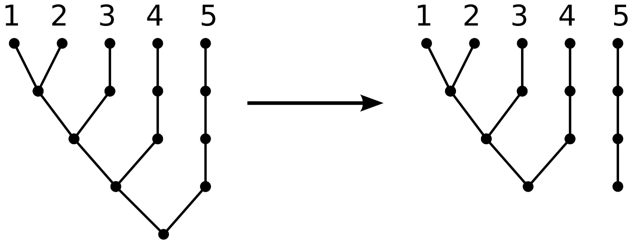


Figure 4.20: splitting produces a degenerate component

This undesirable phenomenon is remedied by following the splitting operation with a "collapsing" operation, κ, that collapses all the degeneracies from a given tree. Also, if the root of t has a single adjacent edge, then κ collapses that edge. Then for any partition $\tau \in \Pi_A$, composing κ with the splitting operation gives a map:

$$\mathcal{MT}_A(\hat{0}, \tau] \to \prod_{B \in \rho'} \mathcal{M}\widetilde{\mathcal{T}}_B(1, |B|)$$

Where, for $|B| < 3$, $\mathcal{M}\widetilde{\mathcal{T}}_B(1, |B|)$ is taken to be the singleton set consisting of the unique B-tree of height $|B| - 1$. See figure 4.21.

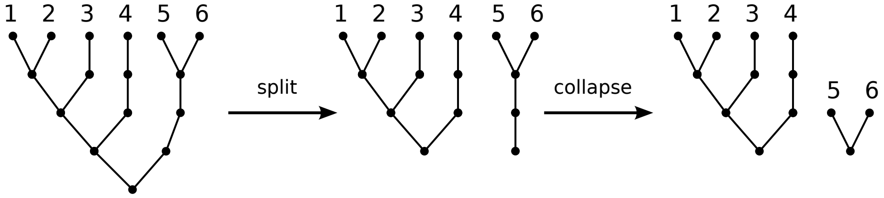


Figure 4.21: splitting then collapsing

Note that because of the collapsing operation, this map is not generally injective. For example, see figure 4.22. We'll see later in this section that this phenomenon does not exist when we restrict to the set of decreasing facets or the set of increasing facets.

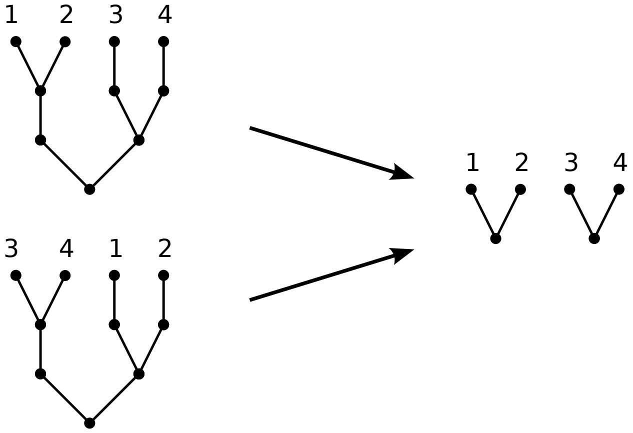


Figure 4.22: $\kappa \circ s$ is not generally injective.

Next, we investigate how prunning and splitting interact with the function ϕ. Recall that ϕ assigns a word to each tree and determines if the tree is decreasing or increasing (or neither).

The total ordering on A induces one on any partition $\tau \in \Pi_A$ by:

$$B \leq B' \ \Leftrightarrow \ min(B) \leq min(B')$$

With this ordering, it makes sense to talk about λ for vertices of trees in $\widetilde{\mathcal{T}}_\tau$. More explicitly, for $t \in \widetilde{\mathcal{T}}_\tau$ and v a vertex of t, define

$$\lambda(v) := min\{min(B) \mid B \in \tau_v\}$$

Remark 9. *For t a facet of $\mathcal{T}_A[\rho, \rho']$, the last letter of the tree word $\phi(t)$ is determined by ρ. Specifically, that letter is the minimal element of a nonsingleton block of ρ. Moreover, $\phi(p(t))$ is exactly the truncation by the last letter of $\phi(t)$.*

Proposition 16. *Prunning takes increasing trees to increasing trees and decreasing trees to decreasing trees.*

Proof. From remark 9, $\phi(p(t))$ is a subword of $\phi(t)$. In particular, if $\phi(t)$ is increasing (decreasing) then so is $\phi(p(t))$. $\square$

Next, we investigate the effect of the map $\kappa \circ s$ on tree words. More specifically, let t be a facet in $\mathcal{T}_A(\hat{0}, \tau]$ and let t'_B denote the projection of $\kappa \circ s(t)$ onto the factor corresponding to the block $B \in \tau$.

Each vertex v of t'_B which contributes a letter to the word $\phi(t'_B)$, also contributes the same letter to $\phi(t)$. In other words, $\phi(t'_B)$ is a subword (although not necessarily a contiguous subword, or even a nonempty word) of $\phi(t)$. As a consequence of this, we have:

Proposition 17. *The map* $\kappa \circ s : \mathcal{MT}_A(\hat{0}, \tau] \to \prod_{B \in \rho'} \mathcal{M}\widetilde{\mathcal{T}}_B(1, |B|)$ *takes increasing trees to increasing trees and decreasing trees to decreasing trees.*

Proposition 18. *The map* $\kappa \circ s : \mathcal{MT}_A(\hat{0}, \tau] \to \prod_{B \in \rho'} \mathcal{M}\widetilde{\mathcal{T}}_B(1, |B|)$ *restricts to a bijection on the set of increasing trees and on the set of decreasing trees.*

Proof. We describe inverses for both these restrictions starting with the decreasing case.

Assume without loss of generality that $\tau = B_1|B_2|\cdots|B_k$ where each $minB_1 > minB_2 > \cdots > \min B_k$. Let $t_i \in \mathcal{MT}_{B_i}(1, |B_i|)$ be decreasing and let h_i be the height of t_i. We construct a facet $t \in \mathcal{T}_A(\hat{0}, \tau]$ which is sent by $\kappa \circ s$ to the unordered tuple $(t_1, \ldots, t_k)$.

For each tree t_i attach a chain of edges of length $1 + \sum_{j<i} h_j$ to its root. Attach the other ends of these chains together. This attachment point will be the root of the constructed tree t. Finally, add edges to the leaves of the resulting (unbalanced) tree so that all the leaves have the same height. The resulting tree t is a decreasing facet of $\mathcal{MT}_A(\hat{0}, \tau]$ and $\kappa \circ s(t) = (t_1, t_2, \ldots, t_k)$. See figure 4.23 for an illustration of this process in a particular case.

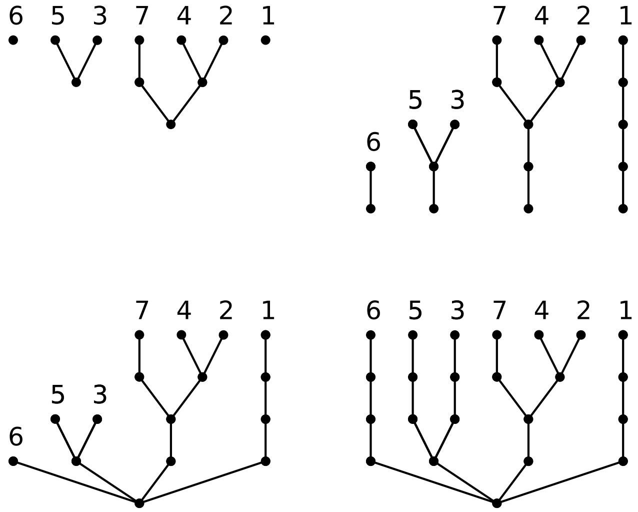


Figure 4.23: Reconstructing a decreasing facet in $\mathcal{T}_A(\hat{0}, 1|742|6|53])$ from its image under $\kappa \circ s$.

Next, we describe the inverse construction to $\kappa \circ s$ on increasing trees. Let $t_i \in \mathcal{MT}_{B_i}(1, |B_i|)$ be increasing with height h_i. Let $v_1, \ldots, v_l$ be all the branching vertices of the t_i's indexed so that $\lambda(v_i) < \lambda(v_j)$ whenever

$i < j$.

To each tree, apply appropriate degeneracies so that $h_j - h_i = j - i$ when ever v_i is the parent of v_j. Each tree of positive height has a branching vertex, say v_i at its root. Attach to that vertex a chain of edges of length i. Attach to the trees of height 0 a chain of edges of length 1. Connect all the trees together by identifying the ends of newly appended edge chains. Finally, append sufficiently many edges to the leaves to make a balanced tree. The resulting tree t is an increasing facet of $\mathcal{MT}_A(\hat{0}, \tau]$ (specifically $\phi(t) = \lambda(v_1) \dots \lambda(v_1)$) and $\kappa \circ s(t) = (t_1, t_2, \dots, t_k)$. See figure 4.24 for an illustration of this process in a particular case.

□

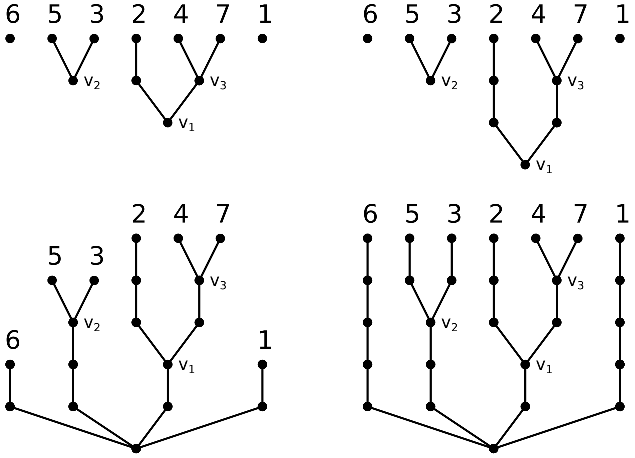


Figure 4.24: Reconstructing an increasing facet in $\mathcal{T}_A(\hat{0}, 1|742|6|53])$ from its image under $\kappa \circ s$.

4.6 A result of Björner and Wachs

Michelle Wachs and Anders Björner proved several combinatorial results having to do with *shellability* of posets. See [12] and [1] for an extensive survey of such results. We make use of this one:

Theorem 3. *Suppose P is a poset for which $\widehat{P} := P \cup \{\hat{0}, \hat{1}\}$ admits an EL-labeling. Then P has the homotopy type of a wedge of spheres, where the number of i-spheres is the number of decreasing maximal $(i+2)$-chains of $\widehat{P}$. The decreasing maximal $(i+2)$-chains, with $\hat{0}$ and $\hat{1}$ removed, form a basis for the cohomology $\widetilde{H}^i(P; \mathbb{Z})$.*

We won't provide a definition of *EL-labelling*. Instead, we use the local terminology to state the theorem in the special case where P is a proper rank interval $\Pi_A(m, m')$ or, equivalently, $\mathcal{T}_A(m, m')$.

Theorem 4. *("The Björner-Wachs theorem") Let* $1 \le m < m' \le n$ *and suppose that every interval* $\mathcal{T}_A[\rho, \rho']$ *of* $\mathcal{T}_A(m, m')$ *contains a unique increasing facet* ι*. Further, suppose that* $\phi(\iota)$ *is minimal (reverse lexicographically in* $A^{\#}$*) among all facets in* $\mathcal{T}_A[\rho, \rho']$*. Then*

1. $\mathcal{T}_A(m, m') = \mathcal{N}\Pi_A(m, m')$ *has the homotopy type of a wedge of spheres of dimension* $m' - m - 2$*.*
2. *A basis for the (unique) reduced cohomology group* $\widetilde{H}^{m'-m-2}\Pi_A(m, m')$ *is represented by the decreasing facets in* $\mathcal{T}_A(m, m')$*.*

In the remainder of this section, we establish the premise of the Björner-Wachs theorem. Theorem 2 then follows immediately.

First note that in the case of $m = 1, m' = n$ the premise obviously holds from figure 4.14.

Lemma 2. *For any partition* $\tau \in \Pi_A$*, the interval* $\mathcal{T}_A(\hat{0}, \tau]$ *has a unique increasing facet.*

Proof. Each element of $t \in \mathcal{T}_A(\hat{0}, \tau)]$ can be specified by a chain of partitions:

$$t : \rho_0 > \rho_1 > \cdots > \rho_k$$

If t is a facet of $\mathcal{T}_A(\hat{0}, \tau]$ then $\rho_0 = \tau$,$|\rho_k| = n - 1$ and ρ_{i+1} is obtained from ρ_i by splitting a single block B of ρ_i into exactly two nonempty pieces.

Suppose t is a facet. Then, without loss of generality, we can write:

$$\rho_i = B_1 \mid B_2 \mid \cdots \mid B_s$$
$$\rho_{i+1} = B_1' \mid B_1'' \mid B_2 \mid \cdots \mid B_s$$

where $B_1 = B_1' \sqcup B_1''$. Let a be the minimal element of A among those blocks B_j of ρ_i with $|B_j| > 1$. Then t is increasing if for every i, $a \in B_1$ and $B_1' = \{a\}$ or $B_1'' = \{a\}$.

This requirement of increasing facets comletely determines as algorithm for constructing an increasing facet of $\mathcal{T}_A(\hat{0}, \tau]$. Namely, start with τ and progressively split of the smallest element of a nonsingleton block. Stop when there are $n - 1$ blocks. The result follows. ☐

Example 7. *Suppose* $\tau = 145|293|67|8$*. The unique increasing facet of* $\mathcal{T}_A(\hat{0}, \tau]$ *is:*

$$145|293|67|8 > 1|45|293|67|8 > 1|45|2|93|67|8 > 1|45|2|9|3|67|8 > 1|4|5|2|9|3|67|8$$

Corollary 1. *Every interval $\mathcal{T}_A[\rho, \rho']$ has a unique increasing facet.*

Proof. By propositions 16, and 15, the prunning map restricts to an injection from the increasing facets of $\mathcal{T}_A[\rho, \rho']$ to the increasing facets of $\mathcal{T}_A(\hat{0}, \rho']$. By lemma 2, there is a unique increasing facet in $\mathcal{T}_A(\hat{0}, \rho']$. The result follows. ☐

Recall from remark 9 that $\phi(p(t))$ is the truncation by the last letter of $\phi(t)$. Moreover, the last letter of $\phi(t)$ is constant for facets t in the same interval. It follows that if t, t' are facets in $\mathcal{T}_A[\rho, \rho']$, then

$$\phi(p(t)) < \phi(p(t')) \implies \phi(t) < \phi(t')$$

Let t be a facet of $\mathcal{T}_A(\hat{0}, \rho']$ and let t_i be a component of $\kappa \circ s(t)$. It follows directly from the splitting construction that $\phi(t_i)$ is a subword of $\phi(t)$. That is, the sequence of letters of $\phi(t_i)$ is a subsequence of $\phi(t)$. Moreover, if we prepend $\phi(t_i)$ by it's minimal leaf label, then that extended word is also a subword of $\phi(t)$. In fact, these extended tree words of the $\phi(t_i)$'s recover the entire word $\phi(t)$. It follows that

$$\forall i \quad \phi(t_i') < \phi(t_i') \implies \phi(t) < \phi(t')$$

Let ι be the unique increasing facet of $\mathcal{T}_A[\rho, \rho']$. Since prunning is a bijection between the increasing facets of $\mathcal{T}_{\rho'}(\hat{(0)}, \rho']$ and $\mathcal{T}_{\rho'}(\hat{(0)}, \rho']$, $p(\iota)$ is the unique increasing facet of $\mathcal{T}_{\rho'}(\hat{(0)}, \rho']$. Since $\kappa \circ s$ bijection from the increasing facets of $\mathcal{T}_{\rho'}(\hat{(0)}, \rho']$ to the unordered product of increasing facets in rank intervals of the form $\widetilde{\mathcal{T}}_B(1, |B|)$. We know in such intervals that the unique increasing facet (pictured in figure 4.14) is reverse lexicographically minimal among the facets of $\widetilde{\mathcal{T}}_B(1, |B|)$. It then follows from the discussion above that ι must have been reverse lexicographically minimal among the facets of $\mathcal{T}_A[\rho, \rho']$.

This establishes the premise of the Björner-Wachs theorem and hence of theorem 2.

4.7 An Explicit description of the reduced cohomology of $\frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)}$

From theorem 2 we have an explicit description for the reduced cohomology of $\mathcal{T}_A(m,n)$ given by the following diagram:

$$\begin{tikzcd}
\widetilde{R}[\mathcal{D}\mathcal{M}\mathcal{T}_A(m,n)] \arrow[r] \arrow[dr, "\cong"'] & \widetilde{R}[\mathcal{M}\mathcal{T}_A(m,n)] \arrow[d] & \\
 & \widetilde{R}\left[\dfrac{\mathcal{M}\mathcal{T}_A(m,n)}{\sim}\right] \arrow[r, "\cong"] & \widetilde{H}^{n-m-2}\mathcal{T}_A(m,n)
\end{tikzcd}$$

Here, the symbol $\mathcal{D}$ means *decreasing elements of* and the symbol $\sim$ denotes the coboundary relations or, in the case $n-m-2=0$, the coaugmentation relation.

Decomposing rank intervals in terms of intervals, we have:

$$\mathcal{T}_A(m,n) = \coprod_{|\rho'|<n} \coprod_{|\rho|>m} \mathcal{T}_A[\rho,\rho']$$

where ρ and ρ' range over the 0-simplices of $\mathcal{T}_A$. A similar decomposition applies to the facets of these simplicial objects.

$$\mathcal{M}\mathcal{T}_A(m,n) = \coprod_{|\rho'|=n-1} \coprod_{|\rho|=m+1} \mathcal{M}\mathcal{T}_A[\rho,\rho']$$

Noticing that the facets of $\mathcal{T}_A(m,n)$ are the same as the facets of $\frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)}$ we get:

$$\begin{aligned}
\mathcal{M}\frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)} &= \mathcal{M}\mathcal{T}_A(m,n) \\
&= \coprod_{|\rho'|=n-1} \coprod_{|\rho|=m+1} \mathcal{M}\mathcal{T}_A[\rho,\rho']
\end{aligned}$$

Since $|\rho'| = n-1$, the condition $t \leq \rho'$ is determined by looking only at the top edge layer of t. This means that the coboundary relations respect the left coproduct.

In fact, the coboundary relations also respect the right coproduct. To see this, it is crucial to realize that the set (denoted by $\sim'$) of coboundary relations of $\frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)}$ is strictly smaller than the set (denoted $\sim$) of coboundary relations of $\mathcal{T}_A(m,n)$. Specifically, the set difference $\sim - \sim'$ consists of the relations of the form $d_0\alpha = 0$ for facelets α. These relations are exactly the ones that involve facets from multiple (actually three) summands of the rightmost coproduct. With these troublesome relations omitted, the reduced cohomology of $\frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)}$ respects the right coproduct and we have a decomposition :

$$\begin{aligned}
\widetilde{H}^* \frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)} &= \widetilde{H}^{n-m-1} \frac{\mathcal{T}_A(m,n)}{\mathcal{T}_A(m+1,n)} \\
&= \widetilde{R}\left[\frac{\mathcal{M}\mathcal{T}_A(m,n)}{\sim'}\right] \\
&= \widetilde{R}\left[\frac{\coprod_{|\rho'|=n-1} \coprod_{|\rho|=m+1} \mathcal{M}\mathcal{T}_A[\rho,\rho']}{\sim'}\right] \\
&= \bigoplus_{|\rho'|=n-1} \bigoplus_{|\rho|=m+1} \widetilde{R}\left[\frac{\mathcal{M}\mathcal{T}_A[\rho,\rho']}{\sim'}\right] \\
&= \bigoplus_{|\rho'|=n-1} \bigoplus_{|\rho|=m+1} \widetilde{R}\,[\mathcal{D}\mathcal{M}\mathcal{T}_A[\rho,\rho']] \\
&= \bigoplus_{|\rho'|=n-1} \bigoplus_{|\rho|=m+1} \widetilde{R}\,[\mathcal{D}\mathcal{M}\mathcal{T}_A(m,n) \cap \mathcal{T}_A[\rho,\rho']]
\end{aligned}$$

In this way we can describe the decreasing facets of a rank interval in terms of the decreasing facets of its intervals. Using the techniques of the last section, we can express these decreasing facets in terms of decreasing facets of rank intervals of the form $\mathcal{T}_B(1,|B|)$, which are exceedingly simple.

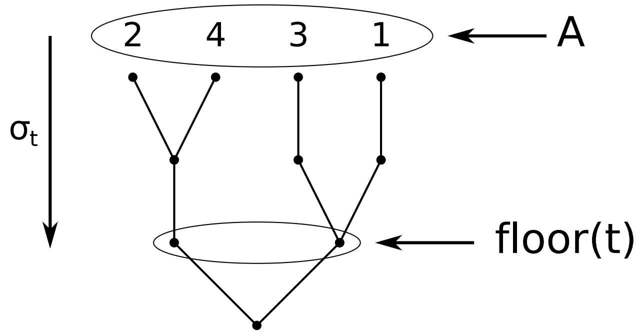
Chapter 5

The E^1 Page of Spectral Sequences for D_mF and P_mF

In this chapter, we use the tree structures of the previous chapter to give more explicit descriptions of the spectral sequences associated with the rank filtrations of D_mF and P_mF.

5.1 The discrete functor $\wedge^t$

For fixed $t \in \mathcal{T}_A$, let σ_t be the surjection $A \to \lfloor t \rfloor$ determined by t.



Recalling that $\wedge^n X := \hom(n, X)$, we define the discrete functor $\wedge^t$ by:

$$\wedge^t X := \{f \in \hom_\sharp(n, X) \mid f \text{ factors through } \sigma_t\}$$

For example, for the tree t illustrated above,

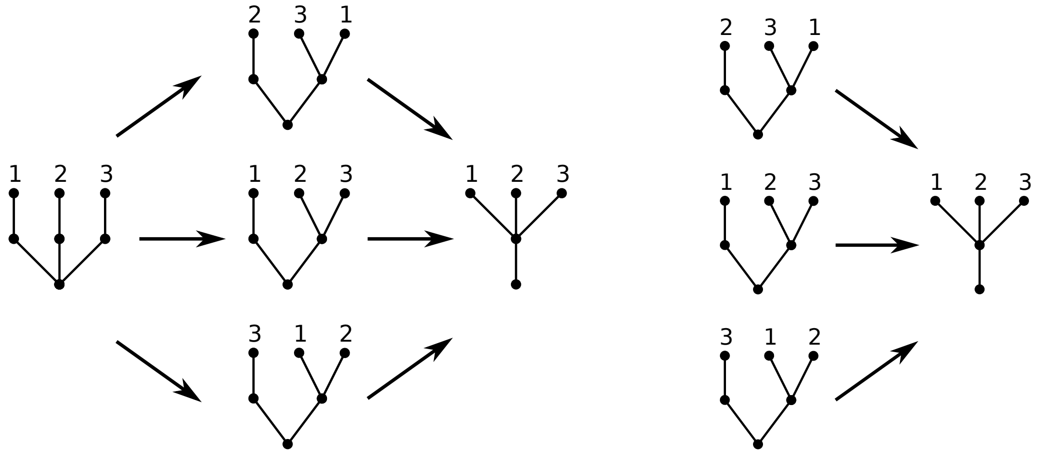
$$\wedge^t X := \{f \in \hom_\sharp(n, X) \mid f(2) = f(4) \text{ and } f(3) = f(1)\}$$

Notice that, for fixed t, the discrete module $\widetilde{R}[\wedge^t]$ is homogeneous of degree $\lfloor t \rfloor$. That is,

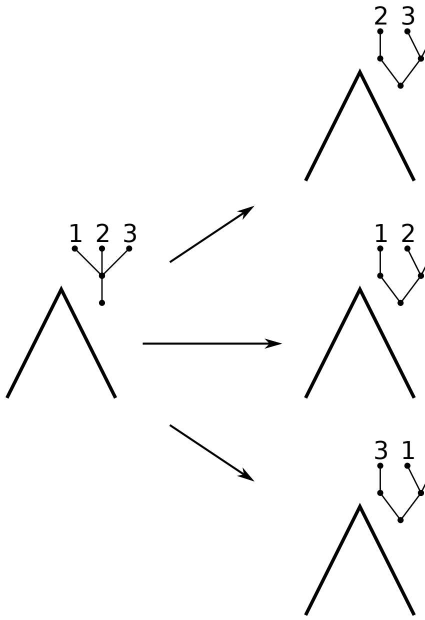
$$D_m\widetilde{R}[\wedge^t] \cong \begin{cases} \widetilde{R}[\wedge^{\lfloor t \rfloor}], & \lfloor t \rfloor = m \\ 0, & \text{otherwise} \end{cases} \qquad P_m\widetilde{R}[\wedge^t] \cong \begin{cases} \widetilde{R}[\wedge^{\lfloor t \rfloor}], & \lfloor t \rfloor \le m \\ 0, & \text{otherwise} \end{cases}$$

Our main strategy for studying the GT towers of $\widetilde{R}[\Delta^n]$ and $\widetilde{R}[\mathcal{I}^n]$ is to express $\widetilde{R}[\Delta^n]$ and $\widetilde{R}[\mathcal{I}^n]$ in terms of homogeneous discrete modules of the type $\widetilde{R}[\wedge^t]$, thus facilitating the easy evaluation of D_m and P_m.

5.2 Simplicial resolutions for Δ^n, $\wedge^n$ and $\mathcal{I}^n$



$(\mathcal{T}_3)_0$ and $(\mathcal{T}_3(0,3))_0$



$\{\wedge^t \mid t \in (\mathcal{T}_3(0,3))_0\}$

The main observation to make about these diagrams is:

$$\operatorname{colim}\{\wedge^t \mid t \in (\mathcal{T}_3(0,3))_0\} = \Delta^3$$

In fact, hocolim$\{\wedge^t \mid t \in (\mathcal{T}_3(0,3))_0\} \cong \{\wedge^t \mid t \in \mathcal{T}_3(0,3)\}$ is a simplicial resolution of Δ^3. More generally, we have a simplicial resolution:

$$\Delta^n \cong \{\wedge^t \mid t \in \mathcal{T}_n(0,n)\}$$

Using the exact sequence $\Delta^n \to \wedge^n \to \mathcal{I}^n$ and the trivial simplicial resolution;

$$\wedge^n \cong \{\wedge^t \mid t \in \mathcal{T}_n\}$$

we get a simplicial resolution for $\mathcal{I}^n$:

$$\mathcal{I}^n \cong \left\{\wedge^t \mid t \in \frac{\mathcal{T}_n}{\mathcal{T}_n(0,n)}\right\}$$

Using the fact that $\widetilde{R}[\wedge^t]$ is homogeneous of degree $\lfloor t \rfloor$, and the above resolution for Δ^n we compute:

$$\begin{aligned}
D_m\widetilde{R}[\Delta^n] &\cong \{D_m\widetilde{R}[\wedge^t] \mid t \in \mathcal{T}_n(0,n)\} \\
&\cong \{\widetilde{R}[\wedge^t] \mid t \in \mathcal{T}_n(0,n),\ \lfloor t \rfloor = m\} \\
&\cong \widetilde{R}[\{\wedge^m \mid t \in \mathcal{T}_n(0,n),\ \lfloor t \rfloor = m\}] \\
&\cong \widetilde{R}\left[\left\{\wedge^m \mid t \in \frac{\mathcal{T}_n(m-1,n)}{\mathcal{T}_n(m,n)}\right\}\right] \\
&= \widetilde{R}[\wedge^m] \otimes \frac{\mathcal{T}_n(m-1,n)}{\mathcal{T}_n(m,n)}
\end{aligned}$$

In particular, it follows that $D_m\widetilde{R}[\Delta^n]$ has reduced cohomology concentrated in the same dimension as the reduced cohomology of $\frac{\mathcal{T}_n(m-1,n)}{\mathcal{T}_n(m,n)}$, dimension $n-m-1$.

Remark 10. *For $m \geq n$ we have $\mathcal{T}_n(m-1,n) = \emptyset$ and $D_m\widetilde{R}[\Delta^n] = 0$.*

Recall from an earlier calculation:

$$D_m\widetilde{R}[\wedge^n] \cong \begin{cases} \widetilde{R}[\wedge^n], & n = m \\ 0, & n \neq m \end{cases}$$

and the quasiexactness of the sequence

$$D_m\widetilde{R}\,[\Delta^n] \to D_m\widetilde{R}\,[\wedge^n] \to D_m\widetilde{R}\,[\mathcal{I}^n]$$

It follows from remark 10 that

$$D_n\widetilde{R}[\mathcal{I}^n] \cong D_n\widetilde{R}[\wedge^n] \cong \widetilde{R}[\wedge^n]$$

When $m < n$, $D_m\widetilde{R}[\wedge^n] \cong 0$ so

$$D_m\widetilde{R}[\mathcal{I}^n] \cong \Sigma D_m\widetilde{R}[\Delta^n] \cong \Sigma\widetilde{R}[\wedge^m] \otimes \frac{\mathcal{T}_n(m-1,n)}{\mathcal{T}_n(m,n)}$$

We compute $P_m\widetilde{R}[\Delta^n]$ and $P_m\widetilde{R}[\mathcal{I}^n]$ similarly, only we don't automatically get as explicit expressions.

$$\begin{aligned}
P_m\widetilde{R}[\Delta^n] &\cong \{P_m\widetilde{R}[\wedge^t] \mid t \in \mathcal{T}_n(0,n)\} \\
&\cong \{\widetilde{R}[\wedge^{\lfloor t \rfloor}] \mid t \in \mathcal{T}_n(0,n),\ \lfloor t \rfloor \leq m\} \\
&\cong \widetilde{R}\left[\left\{\wedge^{\lfloor t \rfloor} \mid t \in \frac{\mathcal{T}_n(0,n)}{\mathcal{T}_n(m,n)}\right\}\right] \\
P_m\widetilde{R}[\wedge^n] &\cong \widetilde{R}\left[\left\{\wedge^{\lfloor t \rfloor} \mid t \in \frac{\mathcal{T}_n}{\mathcal{T}_n(m,n+1)}\right\}\right] \\
&\cong \begin{cases} \widetilde{R}[\wedge^n], & m \geq n \\ 0, & m < n \end{cases} \\
P_m\widetilde{R}[\mathcal{I}^n] &\cong \begin{cases} \widetilde{R}[\mathcal{I}^n], & m \geq n \\ \Sigma P_m\widetilde{R}[\Delta^n], & m < n \end{cases}
\end{aligned}$$

Since $\widetilde{R}[\wedge^i](S^0) \equiv R$, it is apparent from these expressions that $P_m\widetilde{R}[\Delta^n](S^0)$ and $P_m\widetilde{R}[\mathcal{I}^n](S^0)$ have reduced cohomology concentrated in dimensions $n-m-1$ and $n-m$ respectively. In fact, this is true for the discrete modules $P_m\widetilde{R}[\Delta^n]$ and $P_m\widetilde{R}[\mathcal{I}^n]$ (not only their values at S^0), as the next two results show.

Theorem 5. *$P_m\widetilde{R}[\Delta^n]$ has reduced cohomology concentrated in dimension $n-m-1$.*

Proof. We use a double induction argument on m. Since $\Delta^n(+) = 0$, $P_1\widetilde{R}[\Delta^n] \cong D_1\widetilde{R}[\Delta^n]$ which, by an earlier calculuation, has reduced cohomology concentrated in dimension $n-2 = n-m-1$. This is the base case for the forward induction. Also, since $\widetilde{R}[\Delta^n]$ has degree $n-1$, $P_{n-1}\widetilde{R}[\Delta^n] \cong \widetilde{R}[\Delta^n]$ has reduced homology concentrated in dimension $0 = n-(n-1)-1$. This is the base case for the backward induction.

Next, use the quasi-exact sequence:

$$D_m\widetilde{R}[\Delta^n] \to P_m\widetilde{R}[\Delta^n] \to P_{m-1}\widetilde{R}[\Delta^n]$$

and the fact from earlier fact that the reduced cohomology of $D_m\widetilde{R}[\Delta^n]$ is concentrated in dimension $n-m-1$

to generate the long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}^{n-m-2}P_{m-1}\widetilde{R}[\Delta^n] \arrow[r] & \widetilde{H}^{n-m-2}P_m\widetilde{R}[\Delta^n] \arrow[r] & 0 \arrow[dll] \\
& \widetilde{H}^{n-m-1}P_{m-1}\widetilde{R}[\Delta^n] \arrow[r] & \widetilde{H}^{n-m-1}P_m\widetilde{R}[\Delta^n] \arrow[r] & \square \arrow[dll] \\
& \widetilde{H}^{n-m}P_{m-1}\widetilde{R}[\Delta^n] \arrow[r] & \widetilde{H}^{n-m}P_m\widetilde{R}[\Delta^n] \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where $\square$ denotes a potentially nonzero value.

Now assume the forward inductive hypothesis: $P_k\Delta^n$ is concentrated in dimensions $n-k-1$ for $k < m$. Then the section of our long exact sequence where $\widetilde{H}^*P_m\widetilde{R}[\Delta^n]$ does not vanish is:

$$0 \to \widetilde{H}^{n-m-1}P_m\widetilde{R}[\Delta^n] \to \square \to \square \to \widetilde{H}^{n-m}P_m\widetilde{R}[\Delta^n] \to 0$$

This shows that the reduced cohomology of $P_m\widetilde{R}[\Delta^n]$ is concentrated in dimensions $n-m$ and $n-m-1$.

Now forget the forward induction hypothesis and assume the backward inductive hypothesis: $P_k\widetilde{R}[\Delta^n]$ is concentrated in dimensions $n-k-1$ for $k > m$. Then the section of our long exact sequence where $\widetilde{H}^*P_m\widetilde{R}[\Delta^n]$ does not vanish is:

$$0 \to \widetilde{H}^{n-m-2}P_m\widetilde{R}[\Delta^n] \to \square \to \square \to \widetilde{H}^{n-m-1}P_m\widetilde{R}[\Delta^n] \to 0$$

This shows that the reduced cohomology of the discrete module $P_m\widetilde{R}[\Delta^n]$ is concentrated in dimensions $n-m-2$ and $n-m-1$.

$\square$

Using the usual quasi-exact sequences, we get that the discrete modules $P_m\widetilde{R}[\wedge^n]$ and $P_m\widetilde{R}[\mathcal{I}^n]$ have reduced cohomology concentrated in dimension 0 and $n-m$, respectively.

Chapter 6

Conclusions and Future Work

$P_m\widetilde{R}[\mathcal{I}^n]$ and $D_m\widetilde{R}[\mathcal{I}^n]$ have cohomology concentrated in dimension $n-m$. Moreover, we have explicit descriptions of these homology groups and their associated Σ_n-actions. Combining this with the expression in proposition 9 for the layers of the rank filtrations of P_mF and D_mF, we get simpler, more explicit expressions for these layers:

$$\frac{P_mL_nF}{P_mL_{n-1}F} \cong H^{n-m}P_m\widetilde{R}[\mathcal{I}^n] \mathbin{\widehat{\otimes}_{\Sigma_n}} \mathrm{cr}_nF(S^0) \qquad \frac{D_mL_nF}{D_mL_{n-1}F} \cong H^{n-m}D_m\widetilde{R}[\mathcal{I}^n] \mathbin{\widehat{\otimes}_{\Sigma_n}} \mathrm{cr}_nF(S^0)$$

We assume for simplicity (as Robinson does) that the discrete module F is concentrated in dimension 0. For such F, $\mathrm{cr}_nF(S^0)$ can be taken to be an R-module (with a Σ_n action) instead of a chain complex of R-modules. With these assumptions, the expressions above for the layers of the rank filtrations are simplicial R-modules and the homologies of these simplicial R-modules are the entries of the E^1 page of the associated spectral sequences of P_mF and D_mF.

More explicitly, the entry of the E^1 page of the "rank" spectral sequences of P_mF and D_mF in the p^{th} row and the q^{th} column are:

$$H_p\left(\Sigma^{-q}\frac{P_mL_{q+1}F}{P_mL_qF}\right) = H_{p+q}\left(H^{q+1-m}P_m\widetilde{R}[\mathcal{I}^{q+1}] \mathbin{\widehat{\otimes}_{\Sigma_{q+1}}} \mathrm{cr}_{q+1}F(S^0)\right)$$

$$H_p\left(\Sigma^{-q}\frac{D_mL_{q+1}F}{D_mL_qF}\right) = H_{p+q}\left(H^{q+1-m}D_m\widetilde{R}[\mathcal{I}^{q+1}] \mathbin{\widehat{\otimes}_{\Sigma_{q+1}}} \mathrm{cr}_{q+1}F(S^0)\right)$$

Example 8. *The entries of the E^1 page of the rank spectral sequence of D_1F are*

$$H_{p+q}\left(H^qD_1\widetilde{R}[\mathcal{I}^{q+1}] \mathbin{\widehat{\otimes}_{\Sigma_{q+1}}} \mathrm{cr}_{q+1}F(S^0)\right)$$

Recall that the homology of $D_1\widetilde{R}[\mathcal{I}^{q+1}](S^0)$, which is the same as the homology of $\widetilde{R}[\mathcal{T}_n(1,n)]$ and of $\widetilde{R}[\Pi_n(1,n)]$, is well-known to be isomorphic (as a Σ_{q+1}-representation) to ϵLie^*_{q+1}. Thus, we can refor-

mulate this last expression as:

$$H_{p+q}\left(\epsilon Lie^*_{q+1} \mathbin{\widehat{\otimes}_{\Sigma_{q+1}}} \mathrm{cr}_{q+1}F(S^0)\right) \otimes \widetilde{R}[-]$$

This is exactly the same as the corresponding entry of the E^1 page of the spectral sequence obtained from the filtration by columns of the reduced Robinson complex!

Example 8 suggests a possible construction of bicomplexes for D_mF and P_mF. Namely, take as the q^{th} column of these bicomplexes to be the normalized chain complexes of:

$$H^{q+1-m}P_m\widetilde{R}[\mathcal{I}^{q+1}] \mathbin{\widehat{\otimes}_{\Sigma_{q+1}}} \mathrm{cr}_{q+1}F(S^0)$$

$$H^{q+1-m}D_m\widetilde{R}[\mathcal{I}^{q+1}] \mathbin{\widehat{\otimes}_{\Sigma_{q+1}}} \mathrm{cr}_{q+1}F(S^0)$$

Of course, there is one crucial ingredient missing in this construction, the horizontal maps between these columns. These maps are the main source of complexity in Robinson's original description of his bicomplex.

In future work, the author hopes to interpret these horizontal maps in a natural way using the tree representations developed in the previous chapter. Then, a generalization to a bicomplex for the higher levels and layers of the Goodwillie-Taylor tower of F should be more apparent.

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