ELASTODYNAMICS AND WAVE PROPAGATION IN FRACTAL MEDIA

BY

HADY JOUMAA

DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering in the Graduate College of the University of Illinois at Urbana-Champaign, 2012

Urbana, Illinois

Doctoral Committee:

Professor Martin Ostoja-Starzewski, Chair
Professor Alexander Vakakis
Professor Arif Masud
Professor Robert Edward Lee DeVille
ABSTRACT

The elastodynamics and wave propagation in three-dimensional fractal media is explored through the application of analytic and computational methods. In particular, two different mechanical models are introduced; with each one applied to characterize an elastodynamic problem pertaining to some fractal media of distinctive properties.

The first model considers media whose fractality is uniform in all the directions, thus denoted “isotropic”. The formulation which governs the propagation of waves in this model is first developed from fractional hydrodynamic laws, and then, boundary value problems are solved analytically and numerically on spherical domains. In the second model, the fractality is direction dependent, thus the designation “anisotropic”. This model, which implements the concept of product measures to regularize fractional integrals in deriving the balance laws, is assigned to treat the elastodynamics of fractal solid materials. Here, the application of Hookes relation (classical elasticity) in the constitutive law is limited to dilatational wave motion. In order to treat general problems, a non-classical (Cosserat-type) constitutive model is incorporated, featuring the introduction of microrotation and couple-stress variables into the micropolar element and, subsequently, the balance laws. Various eigenvalue-type problems of different kinematic configurations are solved analytically, while a transient analysis based on modal excitation is simulated numerically, resulting in validated computational tools capable of
solving complex elastodynamic problems of arbitrary settings.

The development and verification of these two fractal models promotes the consideration of the more challenging acoustic-solid interaction problems in the fractal paradigm. Indeed an idealized problem is first handled in the continuum framework, where the mathematical steps of the solution are analysed, and then, its demonstration in the fractal domain is performed, illustrating the effectiveness of the fractal models discussed before. In conclusion, our analytical and computational investigation advances the mechanics of fractal media for applications which cannot be studied with classical continuum mechanics.
To all past and current mathematicians who diligently explored the algebra and geometry of fractal sets.
And to all future motivated engineers who will continue in my footsteps in researching the mechanics of fractal media.
ACKNOWLEDGMENTS

That a Doctorate student acknowledges his advisor is quite reasonable but Prof. Martin Ostoja-Starzewski is an extraordinary faculty from many aspects. His character, faith, wisdom, encouragement, patience, and devotion have always made feel in need of working harder to compensate for his support. He was always above my expectations and granted me all what a graduate student can seek from his academic advisor. I truly enjoyed his supervision regarding broadening my research perspective, publishing my papers, and performing the job search.

I am indeed grateful to Prof. Alexander Vakakis of the Mechanical Science and Engineering (MechSE) Department who served as the contingent chair for both my preliminary and final examinations. His constructive remarks helped me perfecting this work. I also thank Prof. Arif Masud of the Department of Civil Engineering for serving in both preliminary and final examination committees, his input was influential in clarifying many points in my work. I also appreciate his guidance in my teaching assistantship for the calculus class. Prof. Robert DeVille form the Mathematics Department service in my committee is of great importance to affirm the mathematical analysis of my research. Finally I would like to acknowledge Prof. Robert Haber of the MechSE Department for his service in my preliminary examination and his assistance in the finite element methods class, along with his support in my teaching assistantship for the dynamics class.
I would like to sincerely acknowledge Dr. Paul Demmie of Sandia National Laboratories. Dr. Demmie was for me more than a research collaborator, he introduced me to new topics in engineering mechanics such as peridynamics and gave me the opportunity to outreach my research in conferences and technical meetings. I am grateful to his insightful comments in our joined publications, and to the recommendation letters he wrote in support of my job applications.

No Doctorate level research can be fulfilled without the crucial financial support of research agencies. I hereby must acknowledge the precious support I received from the National Science Foundation (NSF) and Sandia National Laboratories. Their sponsorship in my college and in the various conferences I attended significantly improved my research outcome.

For the past four years, I had a brilliant and extraordinary friend and labmate, Dr. Jun Li. Our collaboration on and off-campus was indeed memorable. We exchanged many favours and with his help, I always managed to overcome the trouble I sometimes faced. I wish you Jun the best in all your future endeavours and I pray we can once reunite in the future and pursue our exploration of mechanics of fractal media.

I am very grateful to the MechSE Department Head, Prof. Placid Ferreira for his strong support, particularly during moments of hardship. Prof. Ferreira made me feel proud to belong to this Department and planted in me the faith to always work hard to lift its name.

Special thank you to Ms. Kathy Smith of the MechSE graduate office for her wonderful administrative assistance, particularly my thesis review. Ms. Smith helped troubleshoot every problem I faced in my college years.

I would like to thank my companion in the town of Urbana and in UIUC, Lead Scientist Dr. Nahil Sobh of Beckman Institute and the Micro and
Nanotechnology Laboratory for the good time and friendly discussions we used to have. I also appreciate his care for the job opportunities he guided me to.

Finally, it is hard for me to express my gratitude to my parents for their ever long spiritual and financial support to pursue my studies and earn my Doctorate degree. Their devotion for my betterment is always sensed in every forward step I achieve in my life.
# TABLE OF CONTENTS

| LIST OF TABLES | .......................................................... | x |
| LIST OF FIGURES | .......................................................... | xi |
| CHAPTER 1 INTRODUCTION | ....................................................... | 1 |
| 1.1 What are Fractals? | .......................................................... | 1 |
| 1.2 Why Fractal Mechanics? | ....................................................... | 5 |
| 1.3 Objectives and Outline | ....................................................... | 8 |
| CHAPTER 2 ON THE WAVE PROPAGATION IN ISOTROPIC FRACTAL MEDIA | ....................................................... | 10 |
| 2.1 Model Description | .......................................................... | 11 |
| 2.2 Modal Decomposition | .......................................................... | 16 |
| 2.3 Fractal BVP – Case Study | ....................................................... | 21 |
| 2.4 FEM for Reduced Problem | ....................................................... | 29 |
| 2.5 FEM for 3d Problems | .......................................................... | 33 |
| CHAPTER 3 DILATATIONAL WAVE MOTION IN ANISOTROPIC FRACTAL SOLIDS | ....................................................... | 36 |
| 3.1 Mathematical Background | .......................................................... | 37 |
| 3.2 Elastodynamic Model | .......................................................... | 40 |
| 3.3 Carpinteri Column | .......................................................... | 43 |
| 3.4 Modal Analysis | .......................................................... | 45 |
| 3.5 Computational Solutions | ....................................................... | 51 |
| CHAPTER 4 ELASTODYNAMICS IN MICROPOLAR FRAC- TAL SOLIDS | ....................................................... | 60 |
| 4.1 Motivation | .......................................................... | 60 |
| 4.2 Elastodynamic Modelling Procedures | ....................................................... | 63 |
| 4.3 Analytical Approach | .......................................................... | 67 |
| 4.4 Numerical Solutions | .......................................................... | 78 |
| CHAPTER 5 RADIATION DAMPING IN VIBRATION OF SPHER- ICAL STRUCTURES | ....................................................... | 87 |
| 5.1 Background | .......................................................... | 87 |
5.2 Damping Extraction Methods ................. 92
5.3 Pulsating Thin Spherical Shell .............. 95
5.4 Vibrating Solid Sphere .......................... 101
5.5 Radiation Resistance Investigation .......... 113

CHAPTER 6  FRACTAL ACOUSTIC-ELASTODYNAMIC INTERACTION .................. 117
   6.1 Sommerfeld Condition .......................... 117
   6.2 Thin Shell Vibration ............................ 120

CHAPTER 7  CONCLUSIONS .......................... 124
   7.1 Contributions .................................. 124
   7.2 Future Trends .................................. 126

APPENDIX A  SPHERICAL FRACTAL HELMHOLTZ EQUATION 129
APPENDIX B  FRACTAL HELMHOLTZ EQUATION ........... 131
REFERENCES ........................................... 138
LIST OF TABLES

2.1 Exact and asymptotic (WKB) solution for the first kind problem for the first five modes, $D = 2.5$ .......................... 25
2.2 Exact and asymptotic (WKB) solution for the second kind problem for the first five modes, $D = 2.5$ .......................... 28

5.1 Results of the damping factors (in %) for all three cases. Note the strong similarity at low damping ...................... 101
5.2 Summarized damping evaluation results. Clear matching is noted at low $q$ .................................................. 115
LIST OF FIGURES

1.1 Illustration of mathematically constructed fractal objects. The Koch snowflake in (a) is an example of a regular fractal while the 2d Cantor set in (b) is randomly generated. 2
1.2 Illustration of natural fractals. The porous rock in (a) and the human lung in (b) are meaningful demonstrations of fractals in the geophysical and biological systems. 3
1.3 Illustration of the dimensional regularization from geometry configuration. 7

2.1 The fractal radial harmonic function of first kind, shown for different $D, k = 1$. 20
2.2 The fractal radial harmonic function of second kind, shown for different $D, k = 1$. 20
2.3 The spherical shell domain in two views along with the spatial meshing along the radial direction for the one-dimensional reduced problem. 21
2.4 First three orthonormal modal functions. Note the remarkable match between the exact and asymptotic solutions. 26
2.5 First three orthonormal modal functions for the second kind problem. Note the remarkable match between the exact and asymptotic solutions. 28
2.6 The system numerical (1d FEM) and exact solutions for the first three modal excitations. 32
2.7 Convergence plots in time and space. The order of accuracy is shown for each mode. 32
2.8 The system numerical (3d FEM) and exact solutions for the first two modal excitations. 35

3.1 Layout of the Carpinteri column, the physical domain on which the dilatational wave propagation problem is solved. The Hausdorff dimensions are indicated for every direction. 45
3.2 The first three modal functions for the homogeneous Dirichlet BVP shown for Hausdorff dimensions corresponding to those of the Carpinteri column. 48
3.3 The first three modal functions for the first mixed BVP shown for Hausdorff dimensions corresponding to those of the Carpinteri column. 50

3.4 The first three modal functions for the second mixed BVP shown for Hausdorff dimensions corresponding to those of the Carpinteri column. 51

3.5 Transient solution for the Dirichlet BVP. The first two modes are excited in which \( \omega_{1,2} = 5.44, \omega_3 = 3.51 \) for the first mode, and \( \omega_{1,2} = 10.88, \omega_3 = 7.03 \) for the second mode. Frequencies are in rad/sec. Exact solutions are shown in solid line and numerical ones in coloured dots. 54

3.6 The transient solution for the first mixed BVP. The first two modes are excited where \( \omega_{1,2} = 2.72, \omega_3 = 1.76 \) for the first mode, and \( \omega_{1,2} = 8.16, \omega_3 = 5.27 \) for the second mode. Frequencies are in rad/sec. Exact solutions are shown in solid line and numerical ones in coloured dots. 54

3.7 The transient solution for the second mixed BVP. The first two modes are excited where \( \omega_{1,2} = 2.72, \omega_3 = 1.76 \) for the first mode, and \( \omega_{1,2} = 8.16, \omega_3 = 5.27 \) for the second mode. Frequencies are in rad/sec. Exact solutions are shown in solid line and numerical ones in coloured dots. 55

3.8 Spatial convergence plots. The convergence analysis is based on the Dirichlet BVP. The effects of the Hausdorff dimension on convergence are clearly noticed. 56

3.9 Temporal convergence plots. The convergence analysis is based on the Dirichlet BVP. The Hausdorff dimension has very little effect on the convergence rate. 56

3.10 Results for 3d analysis. For the first mode case we set \( \Delta t = 2.0 \times 10^{-3} \) and utilized a uniform mesh discretized by \( N_x = N_y = 121, N_z = 241 \). For the second mode problem, the grid size is kept the same but we set \( \Delta t = 1.4 \times 10^{-3} \). Remarkable matching is observed between the exact (solid black) and the numerical (coloured dots) solution. 57

3.11 Contour plots for the solution of the Dirichlet BVP in first mode excitation at \( t = 0 \) sec. 58

3.12 Contour plots for the solution of the Dirichlet BVP in first mode excitation at \( t = 4 \) sec. 58

3.13 Contour plots for the solution of the Dirichlet BVP in second mode excitation at \( t = 0 \) sec. 59

3.14 Contour plots for the solution of the Dirichlet BVP in second mode excitation at \( t = 2 \) sec. 59

3.15 Legend scale for all contour plots shown in Figures 3.11, 3.12, 3.13, and 3.14. 59
4.1 Layout of Carpinteri column, the physical domain on which
the analytical and numerical problems of elastodynamics
are solved. ............................................................... 64
4.2 Transient solution for the dilatational wave problem. The
exact solution is shown in solid black and the numerical is
shown in coloured dots. ................................................. 79
4.3 First problem contour plots for $u_i$ at different instants. ...... 80
4.4 Transient solution for the torsional wave problem. The
exact solution is shown in solid black and the numerical is
shown in coloured dots. ................................................. 81
4.5 Second problem contour plots for $\phi_i$ at different times. ...... 82
4.6 Transient solution for the in-plane problem. The exact
solution is shown in solid black and the numerical is shown
in coloured dots. .......................................................... 83
4.7 In-plane problem contour plots for $u_1$, $u_2$, and $\phi_3$ at differ-
ent times. ................................................................. 84
4.8 Transient solution for the out-of-plane. The exact solu-
tion is shown in solid black and the numerical is shown in
coloured dots. ............................................................. 85
4.9 Out-of-plane problem contour plots for $\phi_1$, $\phi_2$, and $u_3$ at
different times. .......................................................... 86

5.1 Schematic layout for the general acoustic-structure prob-
lem with shown outward normal vector on the common
interface ................................................................. 90
5.2 Transient response for the under-damped harmonic oscilla-
tor; labelled are the extrema points in (a) the displacement
response and the inflection points in (b) the energy response . 93
5.3 Frequency response plot for the natural response of the
under-damped harmonic oscillator. .................................. 96
5.4 Transient solution for the shell displacement for different $\zeta_0$ . 99
5.5 Transient solution for the acoustic pressure at different rad-
ial locations for $\zeta_0 = 0.025$ and $\frac{\theta}{\psi} = 0.1$ ............... 100
5.6 Modal shapes for the solid sphere radial vibration for $\beta =
-1$ with modal wave numbers indicated ......................... 103
5.7 Bode plot of the sphere’s outer surface displacement for
the first two modes ...................................................... 109
5.8 Transient solutions of the sphere’s outer surface normalized
displacement ............................................................ 112
5.9 Transient solution of the normalized acoustic pressure at
three different radial locations for the first modal excita-
tion, $q = 10^{-2}$, $\frac{\theta}{\psi} = 0.1$ ................................. 113
5.10 Transient solution of the sphere’s volume averaged normal-
ized total energy ....................................................... 114
6.1 Schematic layout for the problem of the thin shell vibration in fractal media ................................................. 121

B.1 Mode shapes corresponding to solutions of the 2d fractal Helmholtz equation where $L_1 = L_2 = 1$, $D_1 = D_2 = \frac{1}{3} \ln \frac{18}{3}$. ................................................. 136

B.2 Mode shapes corresponding to solutions of the 2d fractal Helmholtz equation where $L_1 = 1$, $L_3 = 2$ and $D_1 = \frac{1}{3} \ln \frac{18}{3}$, $D_3 = \frac{\ln 2}{\ln 3}$. ................................................................. 137
CHAPTER 1

INTRODUCTION

1.1 What are Fractals?

The term “fractal” refers to a physical or theoretical entity that is broken or fractured in space or time (1). In general, a fractal object can be continuous or subdivided in parts, each of which is a reduced-size copy of the whole in a deterministic or stochastic sense. Fractal bodies can thus be classified as either geometric (regular) or natural (random). Geometric fractals are idealized sets described as self-invariant, i.e. they are mathematically constructed in a such a way that every small part in the set is a rescaled copy of the larger “mother” part. The Koch snowflake shown in Fig. 1.1(a) and the ternary Cantor set are well known demonstrations of geometric fractals. On the other hand, natural fractals as the ones shown in Fig. 1.2, do not possess this powerful property of self-similarity, they nevertheless display a weaker or statistical version of self-similarity (sometimes called self-affinity), where randomness becomes influential in generating the body’s geometry (2). This type of fractals is abundant in nature (e.g. rocks, tree leaves, island distributions, clouds) and in living bodies (e.g. brains, cardiovascular systems, neural structures) (1; 3).

In principle, fractal bodies are characterized by their complex and highly irregular topology; as a result, the Euclidean geometry, whose basic constituents are essentially smooth, fails to provide a descriptive model to rep-
Figure 1.1: Illustration of mathematically constructed fractal objects. The Koch snowflake in (a) is an example of a regular fractal while the 2d Cantor set in (b) is randomly generated.
Many brittle and/or ductile materials display fractal features: fracture surfaces, dislocation patterns in metals, plastic ridges in ice fields, shear bands in rocks, …

Figure 1.2: Illustration of natural fractals. The porous rock in (a) and the human lung in (b) are meaningful demonstrations of fractals in the geophysical and biological systems.
resent fractal bodies. This reality impelled the development of "fractal geometry" which specializes in explaining the fundamental concepts regarding the geometric characteristics of fractal entities (4). On account of this issue, scientists have sometimes adopted fractal models to study mechanics problems involving bodies which exhibit non-smooth geometric features. The mechanics of random composite structures (5; 6) and the flow in porous materials (7; 8) are two of the many demonstrations of the incorporation of fractal models in mechanics.

Fractal sets are usually attributed by the Hausdorff dimension $D$ which is the scaling exponent characterizing the fractal pattern’s power law. For regular fractals, $D$ is a constant and it is mathematically determined through the self-similarity property. But for the case of random fractals, it becomes a random variable and its evaluation is restricted to statistical methods (4). Fractal media which exhibit uniform fractality in all directions are categorized as isotropic; they reveal fluid-like behaviour with regard to wave propagation phenomena. However, fractal media are in general anisotropic i.e. they display direction-dependent fractality, the issue that compels the assignment of a particular Hausdorff dimension in every direction, $D_i$. It is erroneous to claim that the equivalent Hausdorff dimension of the entire body is $\sum D_i$ as is applicable in $\mathbb{E}^3$. This algebraic sum is physically meaningless and this lemma constitutes a major discrepancy between fractal and Euclidean geometries.

It is important to note that for irregular physical bodies, the fractal feature is observed only within a finite range of length scales. This range is bounded from above by the overall macroscopic length of the body and from below by the microscopic scale of the smallest fractal feature (4). Mechanical investigation within this range must incorporate the fractal structure of the
body; in such a case, the power law scaling would be valid with $D$ being the characteristic dimensionality.

1.2 Why Fractal Mechanics?

The development of theoretical and applied mechanics which was initiated in the late eighteenth century constituted the first establishment on which scientists relied to understand the macroscopic behaviour of matter. This broad and profound topic was nourished as a direct consequence of the mastering of two important mathematical topics: “Euclidean geometry” and “Eulerian calculus”. The nineteenth century witnessed a giant leap in science and engineering regarding the discovery and application of fundamental continuum mechanics principles. In fact, till our days, these principles are still taught and applied in solving engineering problems. Besides, the immense advancement in the computer technology (both hardware and software), promoted the incorporation of “numerical or computational methods” into the field of continuum mechanics. This resulted in the formation of the topic of computational mechanics whereby challenging problems of continuum mechanics which mathematicians cannot find analytical solutions to, admit data-like solutions generated by computer programs with accuracy and cost of production being user-selective. All in all, after almost two centuries of research and applications, the field of continuum mechanics has become well organized and vastly disseminated through the numerous publications made in almost all of its branches.

The ubiquity of continuum mechanics has made of it the most reliable procedure among scientists and engineers to model the mechanical behaviour or explain some phenomena in relevance to solid and fluid bodies. Neverthe-
less, the applicability of this type of mechanics stops at the boundaries of “geometrically irregular” domains. These domains are characterized by a geometry that cannot be contained in a Euclidean-type one, or more precisely, it cannot be comprised of the fundamental elements of Euclidean geometry. We here refer to a quote from Mandelbrot: “clouds are not spheres, mountains are not cones, and coastlines are not circles...”. Ideally, these domains are everywhere irregular, i.e. it is impossible to find a closed set pertaining to this domain where a geometric entity (e.g. curve, surface) is smooth. Thus, they can be described as fractals. This being the case, the differentiability property, vital for the application of continuum mechanics, vanishes. In consequence, the fundamental laws of continuum mechanics cease to apply, opening the gate for new approaches to explore the mechanics of fractal media whereby non-conventional models, taking into consideration this fractal feature, are constructed (7; 8; 9; 10).

The study of fractal mechanics is not as straightforward as that of the continuum mechanics; the mathematical reliance of fractal mechanics laws on challenging subjects in “fractional calculus” and fractal algebra stands behind this difficulty. It is proved that integrals on fractal sets can be approximated by fractional integrals whose order is the dimension of the set (11). As a direct consequence, the fundamental balance laws applied on fractal domains are initially expressed in terms of fractional integrals which is of non-integer order. If these equations are left in this form, the applicability of an analytical procedure or a numerical method to construct a meaningful solution to a certain problem is hopeless. In such a case, fractal mechanics would remain a pure mathematical topic which has no physical substance, and most importantly no direct usefulness in solving real world applications. Therefore, the reproduction of the fractional mathematical description into a
more appreciable form is mandatory to acquire the feasibility of the analytical and numerical solutions and consequently physical significance of fractal mechanics. The procedure applied to transform the mathematical formulations from the fractal topology on which they are originally applied into an equivalent Euclidean domain whereby all equations of the balance laws become continuous, is referred to as “dimensional regularization”. The dimensional regularization is a form of homogenization which regularizes the fractional form of a measurable quantity into an equivalent continuous form through the introduction of a pre-multiplied factor, sometimes denoted as “product measure” (12). This process is schematically illustrated in Fig. 1.3. The application of the dimensional regularization is not unique to all fractal media. It depends on the fractal nature of the media under scrutiny. In our work, we applied a regularization which is based on the Riesz potential for isotropic media, while for anisotropic media the product measures were introduced in the dimensional regularization. As a result of applying the dimensional regularization onto the fractal medium, a “homogenized continuous model” is obtained, describing the mechanics of fractal body by continuous equations. This mechanical model, though linear, involves non-homogeneous differen-
tial equations which contain complex terms portraying the fractal features of the medium. Being the case, the procedures to obtain analytical solutions to general problems are not direct, and only feasible in some coordinate systems.

1.3 Objectives and Outline

The main objective of the research work in this thesis is to advance the knowledge and application of the topic of fractal mechanics which is currently at a formative stage, with ultimate goal to bring this topic to the level where it becomes as disseminated and implemented as is continuum mechanics. Undoubtedly, this goal can be realized once a real application problem, outside the scope of continuum mechanics, is successfully treated by fractal mechanics. But to reach this aim, a necessary step involving the construction of a field theory capable of handling analytical and numerical treatments for ideal models, must be first achieved. This thesis discusses the fulfilment of this step and provides a detailed explanation to two elastodynamic homogenized continuous models, each of which employs a particular mathematical approach when performing the dimensional regularization.

In brief, we will first consider isotropic fractal media discussed in (7) and derive the corresponding wave equation pertaining to their homogenized continuous model. A full modal decomposition in the spherical system is discussed and corresponding boundary value problems on spherical domains are solved analytically. A numerical simulation based on the finite element method is then achieved where the transient response due to modal excitation is obtained. Next we consider the anisotropic fractal solid model, in which the dimensional regularization is achieved based on the concept of product measures (13; 14). For this model, the satisfaction of the angular momen-
tum balance cannot be achieved in the constitutive framework of Hooke’s law. Yet, the simulation of dilatational wave motion, in which the angular momentum balance is trivially satisfied, can be achieved with a classical theory of elasticity. For the general elastodynamic problem, the Cosserat (micropolar) elasticity theory is incorporated where the kinematics is augmented by the microrotation and curvature fields, and couple-stress tensor resulting from moment loading affecting the angular momentum balance (15). Again, the modal decomposition is discussed and various eigenvalue problems are solved analytically and numerically. Finally, we consider the topic of acoustic-solid interaction in fractal media, in which the isotropic fractal model is utilized in the fluid domain. This problem is first presented in the continuous setting to ease the understanding of the mathematical approaches applied in deriving the solution.
In this chapter, we explore the wave propagation phenomenon in three-dimensional (3d) isotropic fractal media. We first overview the basic fractional calculus and the applied dimensional regularization leading to the definition of the corresponding fractal derivative applied in the formulation of the balance laws. The wave equation is then derived, and its modal decomposition in the spherical coordinate system is performed. The analytical procedure is demonstrated through a boundary value problem (BVP) considered on a spherical shell domain. Next, we consider the finite element method (FEM) where the weak formulation is generated and implemented in a numerical scheme. The resulting elastodynamic system is excited and the transient response due to modal excitations is captured. Two solvers capable of handling problems of arbitrary initial and boundary conditions (BC) are developed. The first solver is elementary; it handles problems of purely radial dependence, effectively 1d. This solver is validated in space and time with the exact solution. The second solver deals with general advanced 3d problems of arbitrary spatial dependence.

The topics discussed in this chapter are adopted from the journal paper listed in ref. (16)
2.1 Model Description

2.1.1 Balance Laws

Many complex structures, characterized by an irregular geometry, can be described with the theory of fractal sets of fractional dimensionality (7). A practical and well-known demonstration of this geometric modelling is the problem of random porous media. Indeed a porous material, cannot be truthfully regarded as a continuum because of the successive void regions spread throughout its volume. The crucial parameter which sets whether the fractal nature of this body is to be considered or disregarded, is the “length scale”. More precisely, fractal features must be incorporated in the mechanical analysis only when the latter is conducted within a scale range limited from above by the overall macroscopic length of the body, and from below by the microscopic scale of the smallest fractal feature (4). In addition, the total mass $M$ of a given porous body of characteristic length $R$ does not scale in power law with the dimension of the Euclidean space containing this body. It rather scales with the Hausdorff dimension $D$, which is non-integer and smaller than the Euclidean space’s dimension. For example, $M \sim k R^D$ where $2 < D < 3$ for a fractal body contained in a 3d Euclidean space (2).

The fractal body under examination has a uniform fractality in all directions, in other words, $D$ is unique and direction-independent. It is thus meaningful to designate this type of fractal body as “isotropic”. The approach to perceive isotropic fractal bodies and construct the set of laws that govern their mechanics, requires to first establish the fractional integrals (of order $D$) over fractal sets, then transforming these integrals onto equivalent continuous sets through the application of the appropriate dimensional regularization. As such, the mass of an isotropic fractal volume $V_D$, with density
\( \rho \) is defined as

\[
m(V_D) = \int_{V_D} \rho(\vec{R}) \, dV_D
\]  

(2.1)

By regularization, we have

\[
dV_D = c_V(r, D) \, dV_3
\]  

(2.2a)

\[
c_V(r, D) = \frac{2^{3-D} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \, r^{D-3}
\]  

(2.2b)

where \( r = |\vec{R}| \) and \( \Gamma \) is the Euler Gamma function. This form of \( c_V \) is mathematically derived from Riesz potential theory which is meant to generalize the Riemann-Liouville fractional integral into multi-dimensional form. For the general case, Riesz regularized the \( \alpha \) order fractional integral of a certain function \( f \) to an equivalent integral in the \( m \)th dimensional Euclidean space \( (V_m) \) by the following formula

\[
I^\alpha f = \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{2^{\alpha-\frac{m}{2}} \pi^\frac{m}{2} \Gamma\left(\frac{\alpha}{2}\right)} \int_{V_m} f(\vec{R}) \, r^{\alpha-m} \, dV_m
\]  

(2.3)

For our case, the isotropic fractal domain is embedded in a 3d Euclidean space, thus setting \( m = 3 \) and \( \alpha = D \), and normalizing the above form as to always reproduce the Euclidean solution whenever \( D = 3 \), the form of \( c_V \) is obtained. For a homogeneous \( (\rho \equiv \rho_0 = \text{const.}) \) spherical fractal medium of outer radius \( R \), the mass becomes

\[
m = \frac{2^{5-D} \pi \Gamma\left(\frac{3}{2}\right)}{D \Gamma\left(\frac{D}{2}\right)} R^D
\]

and that is consistent with the power law scaling introduced before.

Before presenting the discussion on the balance laws, we first introduce the fractal representation of Gauss's theorem which we utilize to define the arising fractal derivatives used in formulating these balance laws. In the Euclidean space and over a domain \( W \), we have for a given vector function
$u_i$ and scalar field $f$ the following integral relation

$$
\int_{\partial W} f u_i \, dS = \int_{W} \text{Div} (f u_i) \, dV \quad (2.4)
$$

For a fractal domain $W_D$ that is bounded by a continuous surface, the surface integral is independent of the fractality of $W_D$ while the volume integral can be expressed in the fractal setting with the consideration of Eq. (2.2a). We thus have

$$
\int_{\partial W_D} f u_i \, dS = \int_{W_D} \frac{1}{c_V} \text{Div} (f u_i) \, dV_D \quad (2.5)
$$

thence, a definition for the fractal divergence operator $\text{Div}_D$ and that for a fractal gradient $\nabla^D_i$ operating on a scalar field $f$ can be inferred as follows

$$
\text{Div}_D (u_i) = \frac{1}{c_V} \text{div} (u_i) = \frac{1}{c_V} u_{i,i} \quad (2.6a)
$$

$$
\nabla^D_i (f) = \frac{1}{c_V} \nabla_i (f) = \frac{1}{c_V} f_{,i} \quad (2.6b)
$$

These operators play a major role in the mathematical characterization of the homogenized continuous model and they will emerge in the expressions of the balance laws.

The mass balance law is expressed in the following equation ($u_i$ is the velocity field)

$$
0 = \frac{d}{dt} \int_{W_D} \rho dV_D.
$$

$$
= \int_{W_D} \frac{\partial \rho}{\partial t} \, dV_D + \int_{\partial W_D} \rho u_i n_i \, dS
$$

$$
= \int_{W_D} \left[ \frac{\partial \rho}{\partial t} + \frac{1}{c_V} (\rho u_i)_{,i} \right] \, dV_D \quad (2.7)
$$
But since $W_D$ is arbitrary, the integrand must vanish, thus \( \frac{\partial \rho}{\partial t} + \frac{1}{c_V} (\rho u_i)_i = 0 \). In this regard, the fractal material derivative can be defined as

\[
\left( \frac{d}{dt} \right)_D = \frac{\partial}{\partial t} + \frac{1}{c_V} u_i \frac{\partial}{\partial x_i} \tag{2.8}
\]

Finally, the mass balance law reduces to

\[
\left( \frac{d}{dt} \right)_D \rho = -\rho \text{Div}_D (u_i) \tag{2.9}
\]

For the momentum balance derivation, we consider the body forces $g_i$ and surface forces $\tau_{ij}$ acting on the fractal body. By Newton’s law, the rate of change of the momentum equals the total forces acting on the body thus

\[
\frac{d}{dt} \int_{W_D} \rho u_i dV_D = \int_{W_D} \rho g_i dV_D + \int_{\partial W_D} \tau_{ij} n_j dS \tag{2.10}
\]

Applying Gauss’s theorem to substitute for the surface integral, and incorporating the fractal’s material derivative defined in Eq. (2.8), the momentum balance reduces to

\[
\left( \frac{d}{dt} \right)_D u_i = g_i + \frac{1}{\rho} \nabla^D \tau_{ji} \tag{2.11}
\]

In our consideration of wave propagation, the stress tensor is reduced to its isotropic form $\tau_{ji} = -p \delta_{ji}$. Here, we are ignoring bulk flows and and its resulting viscous effects.

### 2.1.2 Formulation of wave equation

The wave equation in isotropic fractal media is derived by considering small perturbations to the governing balance laws describing the hydrodynamics of the fractal medium. Assuming isentropic (reversible and adiabatic) processes...
for small amplitude oscillation (small disturbance variables: $u_k$, $p^*$, and $\rho^*$) about an equilibrium point $(\bar{p}, \bar{\rho})$, we have, for a fluid of compressibility $\kappa$

$$p - \bar{p} = \kappa \frac{\rho - \bar{\rho}}{\bar{\rho}} \Rightarrow p^* = \kappa \frac{\rho^*}{\bar{\rho}}$$  \hspace{1cm} (2.12)

Introducing the fractal gradient operator into Eq. (2.11) and disregarding body forces, we obtain

$$\nabla^D_i \left[ \rho \left( \frac{d}{dt} \right)^D u_i \right] = -\nabla^D_i \nabla^D_i p^*$$ \hspace{1cm} (2.13)

Neglecting the higher order terms (convective ones and $\rho^* u_k$ product), we simplify the above equation to

$$\bar{\rho} \nabla^D_i \left[ \frac{\partial u_i}{\partial t} \right] = -\nabla^D_i \nabla^D_i p^*$$ \hspace{1cm} (2.14)

Similarly, the mass balance in (2.9) can be simplified to the following form,

$$\frac{\partial \rho^*}{\partial t} = -\bar{\rho} \nabla^D_i u_i$$ \hspace{1cm} (2.15)

Differentiating the above equation with respect to time, we obtain

$$\frac{\partial^2 \rho^*}{\partial t^2} = -\bar{\rho} \frac{\partial}{\partial t} \left[ \nabla^D_i u_i \right] = -\bar{\rho} \nabla^D_i \left[ \frac{\partial u_i}{\partial t} \right]$$ \hspace{1cm} (2.16)

Combining (2.14) and (2.16), we finally have

$$\frac{\partial^2 p^*}{\partial t^2} = \nabla^D_i \nabla^D_i p^* \Rightarrow \frac{\partial^2 p^*}{\partial t^2} = \frac{\kappa}{\bar{\rho}} \nabla^D_i \nabla^D_i p^*$$ \hspace{1cm} (2.17)
If we assign the wave celerity \( \vartheta^2 = \frac{\kappa}{\rho} \), and expand the \( \nabla^D_k \) operator twice on \( p^* \), we obtain the fractal wave equation in its expanded form

\[
\frac{\partial^2 p}{\partial t^2} = \vartheta^2 \left[ \frac{2^{D-3} \Gamma \left( \frac{D}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} |\vec{R}|^{-D} \right]^2 \left[ (3 - D) \vec{R} \cdot \nabla p + |\vec{R}|^2 \nabla^2 p \right] \quad (2.18)
\]

While linear in principle, equation (2.18) is inhomogeneous by the presence of complex terms reflecting the fractal nature of the medium, hindering the chances of obtaining analytical solutions to general problems. In addition to the wave celerity \( \vartheta \), \( D \) plays a significant role in wave propagation phenomena. We can readily verify that this equation reduces to the classical 3d wave equation in the classical continuum case. Due to the isotropic nature of the homogenization, the position vector \( \vec{R} \), appears in the equation. For this reason, obtaining analytical solutions to this equation in the Cartesian or cylindrical coordinate system is impossible. However, the spherical coordinate system allows one to decompose the equation into three independent modal equations, each of which can be solved analytically.

2.2 Modal Decomposition

In eigenvalue problems, modal decomposition serves to formulate analytical solutions using the technique of modal superposition. In the case of continuous wave equation, modal decomposition in Cartesian \((x, y, z)\), cylindrical \((r, \theta, z)\), and spherical \((r, \theta, \phi)\) coordinates generates three decoupled ordinary differential equations (17, ch. 9). In the fractal domain, the first two frames of references are futile; they fail to produce independent equations. Fortunately, the spherical system produces the desirable decoupled equations. We start our modal analysis by separating time and space variables
by introducing the frequency $\omega$, thus

$$p\left(\vec{R},t\right) = P\left(\vec{R}\right) e^{i\omega t}$$ (2.19)

Substituting (2.19) into (2.18), we obtain what we will refer to as the *fractional Helmholtz equation* expressed in (a) compact and (b) expanded forms, respectively,

$$\nabla_D^2 P + k^2 P = 0$$ (2.20a)

$$\vartheta^2 \left[ 2^{D-3} \Gamma\left(\frac{D}{2}\right) |\vec{R}|^{2-D} \right] \left[ (3-D) \vec{R} \cdot \nabla P + |\vec{R}|^2 \nabla^2 P \right] + k^2 P = 0$$ (2.20b)

where the wavenumber $k = \omega/\vartheta$. Expressing the dependent variable $P$ in (2.19) as a product of three independent modal functions, $P(r,\theta,\phi) \equiv F(r) \cdot G(\theta) \cdot H(\phi)$, and substituting back into (2.20b), we obtain three independent Sturm-Liouville equations for a 3d eigenvalue problem expressed as

$$\phi : \ H'' + m^2 H = 0$$ (2.21a)

$$\theta : \ G'' + \frac{G'}{\tan \theta} + \left[ l(l+1) - \frac{m^2}{\sin^2 \theta} \right] G = 0$$ (2.21b)

$$r : \ r^2 F'' + (5-D) r F' + \left[ (\lambda k)^2 r^{2D-4} - l(l+1) \right] F = 0$$ (2.21c)

Recall that, in spherical coordinates, we have

$$r = |\vec{R}|$$ (2.22a)

$$\vec{R} \cdot \nabla P = r \frac{\partial P}{\partial r}$$ (2.22b)

$$\nabla^2 P = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial P}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial P}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 P}{\partial \phi^2}$$ (2.22c)
The constant $\lambda$ showing in (A.1) will always be pre-multiplied by the wave number $k$, it is defined as follows

$$\lambda = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} 2^{3-D}$$

(2.23)

and $l$ and $m$ are two independent integers characterizing the modes, so-called mode numbers.

Let us physically interpret these equations. First, we realize that the Hausdorff dimension $D$ appears in the radial equation only. This is because the homogenization, as conducted, relies solely on the radial distance from the origin to the point of interest, rendering the azimuthal and transcendental modes free of fractal effects (7). Being the case, the azimuthal and transcendental equations are exactly identical to those of the continuum problem, see (17, pp. 251). In addition, upon setting $D = 3$, the radial mode equation reduces to the spherical Bessel equation, which offers an additional verification of this analysis. Having realized that the radial mode equation is similar to the spherical Bessel equation triggers an important hint in forming its solution as we will be later explained.

The analytical solutions to the modal equations are now discussed. Concerning (2.21a), it has the well-known harmonic solution

$$H(\phi) = h_1 \sin(m\phi) + h_2 \cos(m\phi)$$

(2.24)

The transcendental equation (2.21b) can be transformed into a Legendre differential equation if we apply the transformation $\eta = \cos \theta$. The transformed
equation is

\[(1 - \eta^2) \frac{d^2G}{d\eta^2} - 2\eta \frac{dG}{d\eta} + \left[l(l+1) - \frac{m^2}{1 - \eta^2}\right] G = 0 \quad (2.25)\]

One solution to this equation is the Legendre function \(P^{|m|}_{l}(\eta)\). Note that, for the case \(m = 0\), we obtain the Legendre polynomial, denoted as \(P_{l}(\eta)\). The other homogeneous solution to (2.25) is not analytic at \(\eta = \pm 1\). This renders it physically meaningless for this problem, therefore the Legendre function is solely considered for the solution of the spherical wave problem.

The radial mode equation (A.1) is the most challenging to solve – no closed form solution is obvious at a glance. A detailed procedure to formulate the analytic solution of this equation is provided in the Appendix A. These two linearly independent homogeneous solutions are given as

\[F^{(1)}_{\nu}(r; k, D) = r^{\frac{D-4}{2}} J_{\nu} \left( \frac{\lambda kr^{D-2}}{D-2} \right)\]

\[F^{(2)}_{\nu}(r; k, D) = r^{\frac{D-4}{2}} Y_{\nu} \left( \frac{\lambda kr^{D-2}}{D-2} \right)\]

\[F(r) = f_{1}F^{(1)}_{\nu}(r) + f_{2}F^{(2)}_{\nu}(r) \quad (2.26c)\]

where the constant \(\nu = \sqrt{4l(l+1)+(D-4)^2} > 0\), while \(J_{\nu}\) and \(Y_{\nu}\) are, respectively, Bessel functions of the first and second kind of order \(\nu\). Effectively, in (A.7a) and (A.7b) we have introduced the fractal radial harmonic functions of the first and second kind. Figures 2.1 and 2.2 show some plots of these two functions for several values of \(D\) and \(l\). The unnormalized sinc function is reproduced for the case \(D = 3, l = 0\).

Having solved all three equations (2.21a – A.1), we can now express the solution to wave propagation problems in terms of these functions while
Figure 2.1: The fractal radial harmonic function of first kind, shown for different \( D, k = 1 \).

Figure 2.2: The fractal radial harmonic function of second kind, shown for different \( D, k = 1 \).
accounting for the initial and boundary conditions. This concept is demonstrated in the next section where two BVP are solved.

2.3 Fractal BVP – Case Study

We demonstrated in the modal decoupling analysis that fractal effects are observable in the radial harmonics only. Therefore, 3d problems can be reduced to 1d (in the radial coordinate) while preserving the fractal effects that we intend to explore. As an application, a Sturm-Liouville (eigenvalue) problem of two different kinds is considered. The domain of interest is a spherical shell centered at the origin with inner radius $R_{in} = 1$ and outer radius $R_{out} = 2$, as shown in Fig. 2.3. In the first kind problem, homogeneous Dirichlet BCs are enforced on both shell’s surfaces, while in the second kind problem, the BC on the outer surface is replaced with a homogeneous Neumann type. The analytic solution is presented in terms of the fractal radial harmonic functions introduced before. In addition, the WKB asymptotic method is applied, providing a meaningful approximate solution to the problem.
2.3.1 First kind problem

When suppressing the dependence of $\theta$ and $\phi$ on $p$, the resulting wave equation for this reduced problem simplifies to

$$\frac{\partial^2 p}{\partial t^2} = \left( \frac{\vartheta}{\lambda} \right)^2 r^{4-2D} \left[ (5 - D) r \frac{\partial p}{\partial r} + r^2 \frac{\partial^2 p}{\partial r^2} \right]$$ (2.27)

The corresponding reduced fractal Helmholtz equation is

$$r^2 P'' + (5 - D) r P' + \lambda^2 k^2 r^{2D-4} P = 0$$ (2.28)

The BC are of homogeneous Dirichlet type at both ends of the domain, i.e. $P(R_{in}) = P(R_{out}) = 0$. The general homogeneous solution for $P(r)$, expressed in terms of the harmonic functions, is

$$P_n(r) = a_n F^{(1)}_{\nu}(r, k_n, D) + b_n F^{(2)}_{\nu}(r, k_n, D)$$ (2.29)

where $\nu = \frac{4-D}{2(D-2)}$. The nontrivial values (eigenvalues) of $k_n$, along with the corresponding relation between constants $a_n$ and $b_n$, are determined through the application of the BC which result in the following relations

$$F^{(1)}_{\nu}(1, \lambda k_n, D) F^{(2)}_{\nu}(2, \lambda k_n, D) - F^{(1)}_{\nu}(2, \lambda k_n, D) F^{(2)}_{\nu}(1, \lambda k_n, D) = 0$$ (2.30a)

$$\frac{a_n}{b_n} = -\frac{F^{(2)}_{\nu}(1, \lambda k_n, D)}{F^{(1)}_{\nu}(1, \lambda k_n, D)}$$ (2.30b)

Equation (2.30a) is of transcendental nature; it admits infinite solutions in $k$. A general root-finding algorithm for nonlinear algebraic equations (bisection) is applied to solve for the eigenvalues, which are then substituted into Eq. (2.30b) to solve for $\frac{a_n}{b_n}$. This exact solution is provided in Tab. 2.1 for the
particular case \( D = 2.5 \). Rewriting Eq. (2.28) in the Sturm-Liouville form (18, pp. 29) as

\[
\frac{d}{dr} \left[ r^{5-D} \frac{dP}{dr} \right] + (\lambda k)^2 r^{D-1} P = 0 \quad (2.31)
\]

leads to the orthonormality condition expressed as

\[
\int_1^2 P_n P_m r^{D-1} dr = \delta_{mn} \quad (2.32)
\]

The first three orthonormal modal functions are shown in Fig. 2.4.

### 2.3.2 Asymptotic modal analysis

The determination of the higher order modes can be quite challenging if performed analytically, and here comes the effectiveness of the asymptotic approach in fulfilling this procedure. The WKB asymptotic method is well-known in predicting the higher modes of eigenvalue problems in a simplified mathematical operation. This method is thoroughly explained in reference (18, ch. 10). We briefly present the major steps followed to reach the asymptotic solution. The WKB method expresses the solution \( P(r) \) as an infinite series of the form

\[
P(r) = e^{i\lambda k \sigma(r)} \sum_{j=0}^{\infty} \frac{\Psi_j(r)}{(\lambda k)^j} \quad (2.33)
\]

Substituting this form into (2.28), we obtain a series of decreasing power of \( k \). Satisfying each and every order of \( k \), the following relations are obtained

\[
O \left( k^2 \right) : -r^{6-2D} \sigma'^2 + 1 = 0 \Rightarrow \sigma(r) = \pm \frac{r^{D-2}}{D-2} \quad (2.34a)
\]

\[
O \left( k \right) : 2r\Psi'_0 + 2\Psi_0 = 0 \Rightarrow \Psi_0(r) = \frac{1}{r} \quad (2.34b)
\]
\[
\frac{d(r \Psi_{j+1})}{dr} = \frac{i}{2r} \frac{d(r^{5-D} \Psi_j')}{dr}
\]

(2.34c)

Equation (2.34c) provides the recursive relation to solve for \( \Psi_{j+1} \) knowing \( \Psi_j \). By truncating the series to one term only (\( \Psi_0 \)), the resulting \( n \)th mode corresponding to the \( n \)th wavenumber \( k_n \) is approximated by:

\[
\tilde{P}_n(r) = \frac{1}{r} \left[ A_n \cos \left( \frac{\lambda k_n r^{D-2}}{D - 2} \right) + B_n \sin \left( \frac{\lambda k_n r^{D-2}}{D - 2} \right) \right]
\]

(2.35)

Applying the BC on the asymptotic solution \( \tilde{P}_n \), a transcendental equation for the wavenumber \( k_n \), along with the \((A_n, B_n)\) relation are obtained as follows

\[
\tilde{k}_n = \frac{n\pi (D-2)}{\lambda (2^{D-2} - 1)}
\]

(2.36a)

\[
\frac{A_n}{B_n} = -\tan \left( \frac{n\pi}{2^{D-2} - 1} \right)
\]

(2.36b)

If the asymptotic modes were to be orthonormalized by the same condition of (2.32), the resulting form is obtained

\[
\tilde{P}_n(r) = \sqrt{\frac{2(D-2)}{2^{D-2} - 1}} \frac{1}{r} \sin \left( n\pi \frac{r^{D-2} - 1}{2^{D-2} - 1} \right)
\]

(2.37)

As realized, solving for \( \tilde{k}_n \) and \( \frac{A_n}{B_n} \) is straightforward and involves no additional tool as was the case with the exact form. The asymptotic solution of the first five modes for the case \( D = 2.5 \) is listed in Tab. 2.1. Even though the WKB method is meant to target higher modes, the lower modes are still well approximated. The remarkable matching between the exact and the asymptotic solution is strong, particularly for higher modes. The plot of the modal functions in Fig. 2.4 illustrates the significance of the asymptotic approximation.

Interestingly, the components of the WKB homogeneous solution shown
Table 2.1: Exact and asymptotic (WKB) solution for the first kind problem for the first five modes, $D = 2.5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda k_n$</th>
<th>$\lambda \hat{k}_n$</th>
<th>rel. error</th>
<th>$a_n/b_n$</th>
<th>$\Delta a_n/B_n$</th>
<th>rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.8377</td>
<td>3.7922</td>
<td>0.0118</td>
<td>-20.4082</td>
<td>-3.6202</td>
<td>0.8226</td>
</tr>
<tr>
<td>2</td>
<td>7.6076</td>
<td>7.5844</td>
<td>0.0030</td>
<td>0.4551</td>
<td>0.5981</td>
<td>0.3142</td>
</tr>
<tr>
<td>3</td>
<td>11.392</td>
<td>11.376</td>
<td>0.0013</td>
<td>-1.1084</td>
<td>-0.9548</td>
<td>0.1386</td>
</tr>
<tr>
<td>4</td>
<td>15.181</td>
<td>15.169</td>
<td>0.0007</td>
<td>1.6348</td>
<td>1.8624</td>
<td>0.1222</td>
</tr>
<tr>
<td>5</td>
<td>18.971</td>
<td>18.961</td>
<td>0.0004</td>
<td>-0.2749</td>
<td>-0.2271</td>
<td>0.1739</td>
</tr>
</tbody>
</table>

in (2.35) are nothing but the first terms of the asymptotic expansion of the harmonic functions $F^{(1)}$ and $F^{(2)}$ for large $k$. References, (18, pp. 572) and (17, App. A4) provide a detailed asymptotic expansion for the Bessel functions leading to a first order approximation for $F^{(1)}$ and $F^{(2)}$ given as

$$F^{(1)}_{1.5} \approx \tilde{F}^{(1)}_{1.5} (r, k, D = 2.5) = -\frac{1}{r} \sqrt{\frac{1}{\pi \lambda k}} \cos \left(2\lambda k \sqrt{r}\right) \text{ as } k \to +\infty \quad (2.38a)$$

$$F^{(2)}_{1.5} \approx \tilde{F}^{(2)}_{1.5} (r, k, D = 2.5) = -\frac{1}{r} \sqrt{\frac{1}{\pi \lambda k}} \sin \left(2\lambda k \sqrt{r}\right) \text{ as } k \to +\infty \quad (2.38b)$$

As such, we can perform the asymptotic analysis by applying the BC on $\tilde{P}_n (r) = A_n \tilde{F}^{(1)} + B_n \tilde{F}^{(2)}$ and reproduce the same results of the WKB work. The inconvenience of this approach lies in its reliance on the exact solution, nevertheless, it remains significant through its strong approximation to the higher modes while avoiding the differential equation solving dictated by the WKB and the implementation of the root finding algorithm required by the exact analysis. In conclusion, the two asymptotic approaches are consistent, both providing a meaningful prediction to the higher modes.
2.3.3 Second kind problem

In this problem, we enforce homogeneous Dirichlet condition on the shell’s inner surface and homogeneous Neumann condition on the outer surface, i.e. \( P(R_{in}) = 0 \) and \( \frac{dP}{dr}|_{R_{out}} = 0 \). The procedures to solve for the modes, exactly and asymptotically, are no different from those conducted for the first kind problem. The general solution for \( P(r) \) is given in equation (2.29). Applying the relevant BC to solve for \( k_n \) and the \((a_n, b_n)\) relation, we obtain the following equations

\[
F_{\nu}^{(1)}(1, \lambda k_n, D) \frac{dF_{\nu}^{(2)}}{dr}(2, \lambda k_n, D) - F_{\nu}^{(2)}(1, \lambda k_n, D) \frac{dF_{\nu}^{(1)}}{dr}(2, \lambda k_n, D) = 0
\]

(2.39a)

\[
\frac{a_n}{b_n} = -\frac{F_{\nu}^{(2)}(1, \lambda k_n, D)}{F_{\nu}^{(1)}(1, \lambda k_n, D)}
\]

(2.39b)

Again, a numerical method is required to solve the transcendental equation of (2.39a) to obtain the exact eigenvalues \( k_n \), and consequently the \((a_n, b_n)\) relation. The modal orthonormality condition for this problem is no different from that of the previous problem (Eq. (2.32)). The first three modes are shown in Fig. 2.5.

The asymptotic modal analysis, whether realized by the WKB method or
by considering the asymptotic expansion of the fractal harmonic functions, leads to the asymptotic homogeneous solution provided in Eq. (2.35). To solve for the asymptotic wavenumber $\tilde{k}_n$, and the $A_n, B_n$ relation, we apply the corresponding BC and obtain the following equations,

$$
\lambda \tilde{k}_n 2^{D-2} \cos \left[ \frac{\lambda \tilde{k}_n (1 - 2^{D-2})}{D - 2} \right] + \sin \left[ \frac{\lambda \tilde{k}_n (1 - 2^{D-2})}{D - 2} \right] = 0 \quad (2.40a)
$$

$$
\frac{A_n}{B_n} = -\tan \left( \frac{n\pi}{2^{D-2} - 1} \right) \quad (2.40b)
$$

In contrary to the asymptotic case of the first kind problem, $\tilde{k}_n$ cannot be evaluated directly. The application of a root finding algorithm is necessary to fulfil this purpose. This issue slightly weakens the significance of the asymptotic approach for this problem. Applying the bisection method, we solve for the wavenumbers; they are provided for the first five modes in Tab. 2.2. We would like here to make an important note about the solutions of Eq. (2.40a). Since this equation is only meaningful to characterize higher modes, the simplification is to approximate its solution, taking into consideration that only the large values of $\tilde{k}_n$ are relevant. In this regard, for $D = 2.5$, Eq. (2.40a) becomes equivalent to $\sqrt{2} \lambda \tilde{k}_n = \tan \left[ 2 \left( \sqrt{2} - 1 \right) \lambda \tilde{k}_n \right]$. An approximation for the large roots of this equation is given as $\lambda \tilde{k}_n = \frac{1}{2(\sqrt{2}-1)} \left[ n\pi - \frac{\pi}{2} - \frac{\sqrt{3}(\sqrt{2}-1)}{n} \right]$, whereby we have $\lambda \tilde{k}_4 \approx 13.096$ and $\lambda \tilde{k}_5 \approx 16.924$. In comparison with the exact roots of Tab. 2.2, we conclude that this technique provides an acceptable approximation to the wavenumbers, and its accuracy improves as we go higher in modes. The asymptotic modal functions are plotted in Fig. 2.5.
Table 2.2: Exact and asymptotic (WKB) solution for the second kind problem for the first five modes, $D = 2.5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda k_n$</th>
<th>$\lambda \tilde{k}_n$</th>
<th>rel. error</th>
<th>$a_n/b_n$</th>
<th>$A_n/B_n$</th>
<th>rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4023</td>
<td>1.2914</td>
<td>0.0791</td>
<td>-0.0055</td>
<td>0.6252</td>
<td>114.9882</td>
</tr>
<tr>
<td>2</td>
<td>5.5665</td>
<td>5.5350</td>
<td>0.0057</td>
<td>4.3295</td>
<td>13.4198</td>
<td>2.0996</td>
</tr>
<tr>
<td>3</td>
<td>9.4086</td>
<td>9.3899</td>
<td>0.0020</td>
<td>-0.0207</td>
<td>0.0699</td>
<td>4.3713</td>
</tr>
<tr>
<td>4</td>
<td>13.222</td>
<td>13.208</td>
<td>0.0010</td>
<td>-4.4201</td>
<td>-3.3882</td>
<td>0.2335</td>
</tr>
<tr>
<td>5</td>
<td>17.025</td>
<td>17.015</td>
<td>0.0006</td>
<td>0.5175</td>
<td>0.5828</td>
<td>0.1261</td>
</tr>
</tbody>
</table>

Figure 2.5: First three orthonormal modal functions for the second kind problem. Note the remarkable match between the exact and asymptotic solutions.
2.4 FEM for Reduced Problem

In this section, we introduce the FEM as a numerical alternative to solve the reduced problem of wave propagation. We discuss the FEM scheme construction through the weak formulation. We next solve transient problems based on modal excitation. The numerical results are verified and convergence is confirmed through error analysis.

2.4.1 FEM construction

The reduced wave equation is presented in Eq. (6.12b). We seek an FEM transient solution to the reduced problem whose domain and BC are already presented in the first kind problem of the modal analysis. On the temporal side, the system is initially excited with a particular mode, i.e. \( p|_{t=0} = P_n(r) \), a modal function. As such, the expected response is a single frequency harmonic, corresponding to the excited mode.

The weak formulation is obtained following standard FEM procedures explained in (19; 20). We introduce an admissible function \( \hat{p} \), continuous throughout the domain of interest and satisfying the BC. Consider the strong (differential) form of (6.12b), then multiplying both sides by \( \hat{p} \) and integrating over the domain \( \Omega \), the weak (integral) form is generated.

\[
\left( \frac{\lambda}{\partial} \right)^2 \int_{\Omega} \frac{\partial^2 p}{\partial t^2} \hat{p} \, dr = \int_{\Omega} (5 - D) r^{5-2D} \frac{\partial p}{\partial r} \hat{p} \, dr + \int_{\Omega} r^{6-2D} \frac{\partial^2 p}{\partial r^2} \hat{p} \, dr \quad (2.41)
\]

Integrating by parts and performing additional calculus (integration by part),
we obtain
\[
\left(\frac{\lambda}{\vartheta}\right)^2 \int_{\Omega} \frac{\partial^2 p}{\partial t^2} \hat{p} \, dr = \int_{\Omega} (D - 1) r^{5-2D} \frac{\partial p}{\partial r} \hat{p} \, dr + \frac{\partial p}{\partial r} r^{6-2D} \hat{p} \left.\right|_{R_{in}}^{R_{out}} - \int_{\Omega} \frac{\partial p}{\partial r} \frac{\partial \hat{p}}{\partial r} r^{6-2D} \, dr
\] (2.42)

Upon applying the BC, the middle term on the right-hand side vanishes. The integration is then performed on an elemental basis, \( \Omega_e \) where \( \bigcup_{e} \Omega_e = \Omega \). The weak form is thus rewritten as
\[
\left(\frac{\lambda}{\vartheta}\right)^2 \sum_{e=1}^{N_e} \int \frac{\partial^2 p}{\partial t^2} \hat{p} \, dr + (1 - D) \sum_{e=1}^{N_e} \int r^{5-2D} \frac{\partial p}{\partial r} \hat{p} \, dr + \sum_{e=1}^{N_e} \int \frac{\partial p}{\partial r} \frac{\partial \hat{p}}{\partial r} r^{6-2D} \, dr = 0
\] (2.43)

For simplicity, the domain is meshed with 2-node linear elements. The integration can be performed either analytically or by implementing the Gaussian quadrature rule (19, ch. 3). The former method is chosen because the latter one fails to produce the exact value due to the presence of the non-integer powers in the kernels. The elemental integration of the first term of (2.43) generates the elemental mass matrix \( M_e \), the second term generates the fractal elastic matrix \( L_e \), while the third term generates the regular elastic matrix \( K_e \). The nomenclature of these matrices will be better appreciated when discussing the 3d formulation in the upcoming section. Summing all the resulting elemental matrices, the discrete formulation is produced
\[
\sum_{e=1}^{N_e} M_e \ddot{p} + \left( \sum_{e=1}^{N_e} L_e + \sum_{e=1}^{N_e} K_e \right) p = 0 \Rightarrow M\ddot{p} + (L + K)p = 0
\] (2.44)

The obtained model is constitutively similar to its continuum elastodynamic
counterpart which balances inertial forces with elastic ones. As such, the numerical treatments of the continuous system remain applicable in the fractal field. In this problem, the mass matrix $M$ and the regular elastic matrix $K$ are symmetric positive definite (SPD), while the fractal elastic matrix $L$ is not. Thus, the total stiffness matrix $L+K$ is in general not SPD. These remarks can be proved by observing the elemental forms of these matrices. Having formulated the discrete elastodynamic model, eigenvalue (frequency) analysis in addition to transient analysis can be conducted. The latter topic is discussed in the next section.

2.4.2 Time-marching solution

We apply a time-marching scheme to solve (2.44) and obtain its corresponding transient response. The numerical analysis literature in (19, ch. 9) and (20, ch. 9) describes a variety of methods to solve this linear hyperbolic system, whereby we note the Newmark method. It can be implemented either implicitly or explicitly depending on its parametric adjustments. For our analysis, we have chosen the trapezoidal (implicit) case of Newmark’s method; it is unconditionally stable, does not generate any dissipative errors, and has a good accuracy. In conclusion, the numerical algorithm of the trapezoidal method can be implemented for fractal problems; the matching between numerical and analytical solutions is a clear proof of the method’s validity. The first three modal excitations are reported in Fig. 2.6. The error analysis in time and space is conducted; convergence plots are shown in Fig. 2.7 where the $L_2$ norm of the error is plotted with respect to the element size and time step.
Figure 2.6: The system numerical (1d FEM) and exact solutions for the first three modal excitations.

Figure 2.7: Convergence plots in time and space. The order of accuracy is shown for each mode.
2.5 FEM for 3d Problems

In this section, we extend the application of the FEM to general 3d problems. The formulation is conceptually similar to that of the reduced problem but mathematically more complex. We adhere to the same problem solved on the spherical shell to ease the perception of the obtained solutions. Needless to say, the designed solver can handle general problems of arbitrary domains and BC.

The weak form for this problem is obtained by considering an admissible function \( \hat{\rho} \), multiplying both sides of the strong form shown in (2.18) by this function, and then integrating both sides of the equation over the domain of interest, \( \Omega \). The corresponding weak form is given as

\[
\left( \frac{\lambda}{\vartheta} \right)^2 \int_{\Omega} \frac{\partial^2 p}{\partial t^2} \hat{\rho} d\Omega = \int_{\Omega} (3 - D) \hat{\rho} |\vec{R}|^{4-2D} \vec{R} \cdot \vec{\nabla} p d\Omega + \int_{\Omega} \hat{\rho} |\vec{R}|^{6-2D} \nabla^2 p d\Omega
\]

(2.45)

Calculus rearrangements are made for the kernel of the third integral to engage the admissible function into the gradient operator. Applying the Green-Gauss theorem to this integral, we obtain

\[
\hat{\rho} r^{6-2D} \nabla^2 p = \vec{\nabla} \cdot \left( \hat{\rho} r^{6-2D} \vec{\nabla} p \right) - r^{6-2D} \vec{\nabla} \hat{\rho} \cdot \vec{\nabla} p - (6 - 2D) \hat{\rho} r^{4-2D} \vec{R} \cdot \vec{\nabla} p \quad (2.46a)
\]

\[
\int_{\Omega} \hat{\rho} r^{6-2D} \nabla^2 p d\Omega = \int_{S} \hat{\rho} r^{6-2D} \vec{\nabla} p dS - \int_{\Omega} r^{6-2D} \vec{\nabla} \hat{\rho} \cdot \vec{\nabla} p d\Omega + (2D - 6) \int_{\Omega} \hat{\rho} r^{4-2D} \vec{R} \cdot \vec{\nabla} p d\Omega \quad (2.46b)
\]

Applying the Dirichlet BC on the inner and outer spherical boundaries, the surface integral vanishes. Incorporating (2.46b) into (2.45), the weak form
becomes

\[
\left( \frac{\lambda}{\partial} \right)^2 \int_{\Omega} \frac{\partial^2 p}{\partial t^2} \hat{p} \, d\Omega = \int_{\Omega} (D - 3) \hat{p} r^{4-2D} \vec{R} \cdot \vec{\nabla} p \, d\Omega - \int_{\Omega} r^{6-2D} \vec{\nabla} \hat{p} \cdot \vec{\nabla} p \, d\Omega
\]

(2.47)

The integration is conducted on the elemental level. We thus have

\[
\left( \frac{\lambda}{\partial} \right)^2 \sum_{e=1}^{N_e} \int_{\Omega_e} \frac{\partial^2 p}{\partial t^2} \hat{p} \, d\Omega_e = \sum_{e=1}^{N_e} \int_{\Omega_e} (D - 3) \hat{p} r^{4-2D} \vec{R} \cdot \vec{\nabla} p \, d\Omega_e - \sum_{e=1}^{N_e} \int_{\Omega_e} r^{6-2D} \vec{\nabla} \hat{p} \cdot \vec{\nabla} p \, d\Omega_e
\]

(2.48)

The above equation is similar to that of (2.43). The three governing matrices \((M, L, \text{ and } K)\) introduced in the reduced problem are easily identified. For the continuum case \((D = 3)\), the middle term, which produces the fractal elastic matrix \(L\) vanishes. It is clear by now why \(L\) is named as such: it disappears from the formulation when fractal effects are absent. On the other hand, the symmetric regular elastic matrix \(K\) regains its conventional form and reflects regular elastic effects in the case of a continuous medium. The mass matrix \(M\) is indifferent to fractal effects; this is a direct consequence of the homogeneity (constant density) property of the fractal medium. The procedure to derive the elemental matrices requires integrating over 3d elements, 4-node tetrahedral being the simplest choice. The integrals cannot be evaluated exactly; the use of a quadrature method becomes a must. A useful mathematical algorithm for the derivation of quadrature points and corresponding weights over tetrahedral elements is provided in (21). Assembling the governing matrices, the discrete elastodynamic form presented in Eq. (2.44) is obtained but with a much larger number of degrees of freedom. The Newmark method is applied to solve for the transient solution. The first
Figure 2.8: The system numerical (3d FEM) and exact solutions for the first two modal excitations.

two modal excitations are simulated and the response is shown in Fig. 2.8.
In this chapter, we report a study on wave motion in a generally anisotropic fractal medium whose constitutive response is represented by classical Hooke’s law. First, the governing elastodynamic laws are formulated on the basis of dimensional regularization. It is discovered that the satisfaction of the angular momentum equation precludes the implementation of the classical elasticity theory which manifests in symmetry of the Cauchy stress tensor. Nevertheless, the classical elastic constitutive model can still be applied to explore dilatational wave propagation; in such a case, the angular momentum balance is “trivially” satisfied. The resulting problem, of eigenvalue type, is solved analytically. A computational finite element method solver is developed to simulate the problem in its 1d form, it is validated by the reference solutions generated through the modal analysis. A 3d finite-difference-based solver is also developed and the obtained results match those of the 1d simulations.

The topics discussed in this chapter are adopted from the journal paper listed in ref. (22)
3.1 Mathematical Background

The development of a field theory in which the fundamental laws of mechanics (conservation of mass, linear and angular momentum) are reproduced in the fractal setting, resulting in continuum-type equations which can be treated analytically and numerically, constitutes the core of the study achieved in this chapter. While this analysis was performed on isotropic fractal media in the Chapter 2, our focus will now switch to a different category of fractals classified as “anisotropic”, i.e. their Hausdorff dimension is direction-dependent.

The formulation of a three-dimensional (3d) elastodynamic model for anisotropic fractal solids was developed in (13) based on product measures. The latter work utilized two different approaches to derive the governing elastodynamic equations of this material model: the fractal mechanics approach and the variational energy approach with both approaches producing consistent results. This attribute is not realized in previous fractal analysis where a single approach is followed to describe the mechanics of the model. In brief, Li and Ostoja-Starzewski (13) adopted the modified Riemann-Liouville fractional integral formulated by Jumarie (23) to derive the mass power law, Gauss’s law and the transport theorems in fractal domains. Accordingly, the fractional integral of a certain function $f(x)$ over a fractal set of Hausdorff dimension $D \in [0, 1]$, with support on $[0, L]$, is regularized as follows

$$\int_0^L f(x) (dx)^D = \int_0^L D(L - x)^{D-1} f(x) dx \quad (3.1)$$

In this regard, the mass power law of a 3d fractal body $(V_D)$ embedded in
the Cartesian system $Ox_1x_2x_3$ is derived as follows

\[
m = \int_{V_D} \rho(x_i) \, dV_D
= \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} \rho(x_1, x_2, x_3) (dx_3)^{D_3} (dx_2)^{D_2} (dx_1)^{D_1}
= \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} \rho(x_1, x_2, x_3) c_1 c_2 c_3 \, dx_3 \, dx_2 \, dx_1
= \int_V \rho(x_i) \, c_V \, dV
\]  

where $\rho$ is the local mass density of the body (measured in $\text{Kg/m}^D$) and $c_i$ is the product measure along the $i$th direction defined as

\[
c_i = D_i (L_i - x_i)^{D_i - 1}, \quad i = 1, 2, 3
\]  

Here, $c_V$ is a volumetric product measure defined as the product of all three product measures, thus $c_V = c_1 c_2 c_3$. Note that for a uniform density $\rho \equiv \rho_0$, the total mass becomes $m = \rho_0 L_1^{D_1} L_2^{D_2} L_3^{D_3}$. This finding is mathematically consistent with the universal mass power law scaling of a fractal body with characteristic length $L$ and Hausdorff dimension $D$, $m \sim L^D \ (7)$. In conclusion, $c_i$ is applied to regularize 1d or length integrals, while its equivalent, $c_V$, regularizes 3d or volume integrals. Consistent with this, a surface product measure is introduced to regularize surface integrals as will be later shown in deriving Gauss’s theorem on fractal media.

At this point, we discuss the fractional reproduction of Gauss’s theorem, a widely applied integral identity in formulating many continuum mechanics principles. This analysis leads to defining the mathematical form of the fractal derivative, which will be utilized in generalizing the fundamental balance laws of mechanics in anisotropic fractal media. In continuous domains, Gauss’s theorem correlates the integral of a vector function $f_i$ over a closed
surface $\partial W$, to the integral of its divergence over the volume $W$ enclosed by this surface (24), thus

$$\int_{\partial W} f_i n_i dS = \int_W f_{i,i} dV \quad (3.4)$$

For fractal domains, the surface integral is regularized by the surface product measure in similarity with the regularization performed along the length as shown in Eq. (3.1). Thus, after considering the surface projection onto each coordinate plane as illustrated in (13), the surface integral becomes

$$\int_{\partial W_D} f_i n_i dS_D = \int_{\partial W} f_i c_i^S n_i dS \quad (3.5)$$

where $c_i^S = \frac{c_i}{c_i}$. Note that $c_i^S$ is independent of the $i$th Cartesian coordinate $x_i$. Applying the conventional theorem of Gauss to the right-hand side of Eq. (3.5), we obtain

$$\int_{\partial W} f_i c_i^S n_i dS = \int_W \left( f_i c_i^S \right)_i dV = \int_W f_{i,i} c_i^S dV \quad (3.6)$$

and using the volume integral regularization as was done in Eq. (3.2), the volume integral of Eq. (3.6) is expressed in the fractal configuration as

$$\int_W f_{i,i} c_i^S dV = \int_{W_D} f_{i,i} c_i^S dV_D = \int_{W_D} \frac{1}{c_i} f_{i,i} dV_D \quad (3.7)$$

Combining the integral identities of equations (3.5), (3.6), and (3.7), the generalization of Gauss’s theorem on fractal domains is reproduced. Thus

$$\int_{\partial W_D} f_i n_i dS_D = \int_{W_D} \frac{1}{c_i} f_{i,i} dV_D \quad (3.8)$$

By comparing equations (3.4) and (3.8), the mathematical form of the reg-
ularized fractal derivative can be clearly inferred. This fractal derivative, denoted by $\nabla^D_i$, is defined as

$$\nabla^D_i = \frac{1}{c_i} \frac{\partial}{\partial x_i} \quad \text{(no sum on } i)$$ \hspace{1cm} (3.9)

Having constructed the fractal derivative, the fundamental balance laws of mechanics can now be established, eventually leading to the development of a continuum-like model for a generalized anisotropic 3d fractal body defined in the Cartesian coordinate system. The governing equations of elastodynamics for this model are discussed in detail in (13). We will briefly present the basic equations from which we deduce the one relevant to the propagation of dilatational waves, the problem we intend to explore. The resulting eigenvalue type problem, is treated analytically and numerically in three different settings of boundary conditions (BC).

3.2 Elastodynamic Model

In this section, we apply the fractal derivative we just identified to introduce the fractal version of the balance laws. Then, we construct the appropriate elastodynamic model which suits the simulation of dilatational waves. The outcome is a Navier like equation which is non-homogeneous and grasps the fractal features of the medium.

With regard to kinematics, the strain tensor field for this fractal solid model resulting from an infinitesimal deformation $u_i$ is defined as

$$\epsilon_{ij} = \frac{1}{2} \left( \nabla^D_j u_i + \nabla^D_i u_j \right)$$ \hspace{1cm} (3.10)

Similarly to the classical continuum case, the symmetry property of the strain
The linear momentum balance incorporating inertia, elastic, and body forces is expressed as

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \nabla_j D \sigma_{ij} + f_i \]  \hspace{1cm} (3.11)

while the angular momentum balance is given as

\[ e_{ijk} \frac{\sigma_{jk}}{c_j} = 0 \]  \hspace{1cm} (3.12)

where \( e_{ijk} \) is the Levi-Civita permutation tensor. In the continuum case \((D_i = 1\) and thus \(c_i \equiv 1\) for all \(i\)), the angular momentum balance is automatically satisfied whenever the adopted constitutive law generates a symmetric Cauchy stress tensor. However, in the fractal case, the satisfaction of the angular momentum balance becomes hard to achieve because \(c_i \neq c_j\) (the product measures depend on different coordinates and they can never be equivalent). Therefore, in mechanics of fractal solid bodies, the stress tensor cannot be symmetric, and as such, the well known constitutive law of Hooke’s model defined as \((\delta_{ij}\) being the Kronecker tensor),

\[ \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \]  \hspace{1cm} (3.13)

can no more be applied since it generates a symmetric stress tensor out of a symmetric strain tensor. Consequently, a fundamental attribute of the classical theory of elasticity is lost due to fractal effects, and this is remedied by the application of a non-classical elastic constitutive model (e.g. Cosserat model) to achieve a meaningful satisfaction of the balance laws for general problems (14). Nevertheless, a careful observation of the above model reveals a possibility to implement Hooke’s law while still satisfying the angular mo-
momentum balance. This satisfaction is achieved *trivially* by setting all shear
components of the stress tensor to be identically zero i.e. $\sigma_{ij} \equiv 0$ for $i \neq j$. A
feasible kinematic configuration that forces all the shear stress components
and, thus, all the shear strain components to vanish exists and is unique; it
is prescribed as

$$u_i \equiv u_i(x_i) \quad (3.14)$$

We thus have the displacement in a given direction dependent only on the
coordinate of that direction. As a result, the propagated waves in this partic-
ular problem are of *dilatational* type. This type of waves is also designated
as irrotational or primary (25). It is worth noting that propagation of distor-
tional or secondary waves cannot be studied within the framework of Hooke’s
model since this type of waves is based on shear effects which cannot be con-
sidered because of the symmetry problem discussed before.

The incorporation of Hooke’s law into the linear momentum balance pro-
duces the fractal Navier equation. In the general anisotropic case, three
different Hausdorff dimensions are present in this equation. The continuum
model is reproduced whenever all three Hausdorff dimensions are set to one
(and consequently all product measures become identically one). This is
consistent with the initial assumptions about this fractal solid model. The
indicial form of the fractal Navier equation is stated as

$$\rho \ddot{u}_i = \frac{\lambda + \mu}{c_V} \left( \frac{c_V u_{j,i}}{c_i c_j} \right)_{,j} + \frac{\mu}{c_V} \left( \frac{c_V u_{i,j}}{c_j c_j} \right)_{,j} \quad (3.15)$$

In contrary to the classical continuum case, Eq. (3.15) is only meaningful
to study dilatational wave propagation problems. Enforcing the prescribed
kinematics that simulates the propagation of this type of waves (Eq. (3.14)),
we obtain a set of three 1d decoupled equations that are similar in form, their
expression in terms of (a) fractal and (b) conventional derivatives is

\[ \frac{\rho \ddot{u}_i}{\lambda + 2\mu} = \nabla_i^D (\nabla_i^D u_i) \]

(3.16a)

\[ \frac{\rho \ddot{u}_i}{\lambda + 2\mu} = \frac{u_{i,ii}}{c_i^2} - \frac{c_{i,i}u_{i,i}}{c_i^3} \]

(3.16b)

The above equation contains one index \( i \) and no summation should be considered if index is repeated. The inherent relation between the dilatational wave propagation and model decoupling makes the solution of a 3d problem feasible by solving three similar but independent problems of 1d type. This significantly facilitates the computational analysis with regard to problem cost and complexity.

3.3 Carpinteri Column

All the analytical and numerical problems we consider in this work are solved on a box-shaped domain referred to as the Carpinteri column; it is shown in Fig. 3.1. The Carpinteri column is a regular fractal body (having mathematically well defined Hausdorff dimensions in all three directions) that is embedded in a 3d Euclidean space. It was considered by Carpinteri et al. (5) to model concrete columns composed of fractal-like microstructures. Concerning our application, the Carpinteri column is a meaningful illustration of a fractal solid body that exhibits the property of fractal anisotropy. The cross-section of the column, which is square in shape, constitutes of a Sierpiński carpet. It is “fractally” swept along the longitudinal direction in conjunction with a ternary Cantor set. Clearly, the Hausdorff dimension of this body along the longitudinal direction, \( x_3 \), is equivalent to that of the ternary Cantor set. The determination of the Hausdorff dimensions for

43
the in-plane directions $x_1$ and $x_2$ is not as evident as is the case with the longitudinal direction. Nevertheless, substantial knowledge and significant assumptions help conjecturing viable values for these dimensions. First, the axial and planar symmetries revealed in the Sierpiński carpet’s geometry indisputedly induce that $D_1 = D_2$. This fractal set is built in a 2d Euclidean space and its Hausdorff dimension ($D_{\text{carpet}} = \frac{\ln 8}{\ln 3}$) is derived within the 2d framework. Superficial knowledge in fractal geometry might lead to the erroneous assumption obtained by analogy with the continuous case where $D_1 = \frac{1}{2}D_{\text{carpet}}$. The inapplicability of this simple supposition is briefly explained. The theory of products of fractal sets infers that if a fractal set (denoted as $G$) is generated from the Cartesian product of two primary fractal sets (denoted as $E$ and $F$), i.e. $G = E \times F$, their dimensions satisfy the following inequality (it becomes equality for continuous and Cantor sets),

$$D_G \geq D_E + D_F$$  \hspace{1cm} (3.17)

See Chapter 7 of (4) for more about this topic. Unluckily, the Sierpiński carpet is not constructed as the product of two independent fractal sets. Thus Eq. (3.17) cannot even predict an upper limit for $D_1$. If we consider the horizontal slice (set formed by intersecting a horizontal line with the carpet) and attempt to obtain the dimension of that slice, we realize that it depends on the position where the line intercepts the carpet. For example, on side length corresponding to $x = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $D_{\text{slice}} = 1$ since no void is to occur at this position. On the other hand, if $x = \frac{1}{2}$, we obtain $D_{\text{slice}} = D_{\text{ternary Cantor set}}$. Clearly, the dimension is not uniform along the edge, it is evaluated statistically in (26) and this is how $D_1$ is estimated for our problem. In conclusion, the utilized Hausdorff dimension along each direction
is provided below.

\[ D_1 = D_2 = \frac{1 \ln 18}{3 \ln 3} \]
\[ D_3 = \frac{\ln 2}{\ln 3} \]  

(3.18)

The Carpinteri column exhibits a planar fractal isotropy that renders the analysis along the \( x_1 \) and \( x_2 \) directions redundant especially since \( L_1 = L_2 \).

### 3.4 Modal Analysis

The propagation of dilatational waves is described in Eq. (3.16). We now present the analytical solution to this equation and conduct modal analysis. The fulfilment of the modal analysis allows for the treatment of variety of boundary value problems (BVP), whereby we discuss three problems characterized by their distinct BC. The solutions to these problems will be later used in validating the computational tools constructed to simulate some complex
problems which do not admit closed-form solutions.

Defining the dilatational wave speed to be

\[ \vartheta_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]  

(3.19)

and expressing the displacement in its modal form \((j = \sqrt{-1})\)

\[ u(x, t) = U(x) e^{j\omega t} \]  

(3.20)

The resulting modal equation, general for all directions, becomes

\[ -\left( \frac{U_x}{c} \right)_x = ck^2 U \]  

(3.21)

or more explicitly,

\[ (L - x)^2 U''(x) + (D - 1)(L - x) U'(x) + k^2 D^2 (L - x)^2 U(x) = 0 \]  

(3.22)

where \(k\) is the wavenumber

\[ k = \frac{\omega}{\vartheta_d} \]  

(3.23)

This modal equation admits the following general solutions (the derivation of the solution is explained in detail in the Appendix B)

\[ f_1(x, k) = \cos \left[ k (L - x)^D \right] \]  

(3.24a)

\[ f_2(x, k) = \sin \left[ k (L - x)^D \right] \]  

(3.24b)

hence the modal function \(U_m(x)\) for a given BVP can be expressed as the
weighted sum of $f_1$ and $f_2$,

$$U_m(x) = C_1 f_1(x, k_m) + C_2 f_2(x, k_m)$$  \hfill (3.25)$$

where constants $C_1$ and $C_2$ are determined via application of the appropriate BC. In the case of homogeneous BC, the Sturm-Liouville problem presented in Eq. (3.21) admits the following modal orthogonality property

$$\int_0^L U_m(x) U_n(x) c(x) \, dx = \delta_{mn}$$  \hfill (3.26)$$

This orthogonality is indeed verified in our discussed eigenvalue problems.

Having obtained the homogeneous solution to the general BVP, we now proceed to discuss three different eigenvalue-type problems, each of which has its own setting of BC. For all these problems, the solutions in terms of modal functions and corresponding wavenumbers are evaluated.

### 3.4.1 Dirichlet BVP

In this section, we intend to solve the problem where homogeneous BC on $u$ are enforced on both sides of the domain. Thus, the modal function $U(x)$ must satisfy

$$U|_{x=0} = 0 \quad \& \quad U|_{x=L} = 0$$  \hfill (3.27)$$

Applying the BC of Eq. (3.27) into the general solution presented in Eq. (3.25), we obtain the non-trivial solution described as follows

$$U_m(x) = \sin \left[ k_m (L - x)^D \right]$$  \hfill (3.28a)$$

$$k_m = \frac{m\pi}{L^D} \quad m \in \mathbb{N}$$  \hfill (3.28b)$$
Figure 3.2: The first three modal functions for the homogeneous Dirichlet BVP shown for Hausdorff dimensions corresponding to those of the Carpinteri column.

This general modal solution is valid for all three directions. It depends on $D$ and $L$ which are direction characteristics and by applying their appropriate values, the modal solution for a particular direction is obtained. The first three modal functions corresponding to the Carpinteri column are shown in Fig. 3.2. A thorough observation of the modal functions reveals that they are all infinitely steep at the right edge ($x = L$). This is corroborated mathematically where we have

$$\left. \frac{dU_m}{dx} \right|_{x=L} \approx (L - x)^{D-1}$$

Per Eq. (3.29), the singularity of the first derivative of $U_m$ at the right edge intensifies for smaller $D$. This is clearly manifested in Fig. 3.2 where the plots become more “asymmetric” with respect to $x = 0.5$ and squeezed near $x = L$ for higher modes and lower $D$. 
3.4.2 First mixed BVP

In this section, we consider the problem where a homogeneous Dirichlet BC is applied on the right side and a homogeneous Neumann BC on the left end. As a result, the following conditions

\[ \frac{dU}{dx} \bigg|_{x=0} = 0 \quad \& \quad U \bigg|_{x=L} = 0 \quad (3.30) \]

must be satisfied in the general modal solution of Eq. (3.25). This eigenvalue problem admits the following solution

\[ U_m(x) = \sin \left[ k_m (L - x)^D \right] \quad (3.31a) \]

\[ k_m = \frac{(2m + 1) \pi}{2L^D} \quad m \in \mathbb{N} \quad (3.31b) \]

The first three modal functions corresponding to the Carpinteri column are shown in Fig. 3.3. Similarly to the Dirichlet BVP, the modal functions are all infinitely steep at the right edge \((x = L)\). This is also corroborated mathematically where we have

\[ \frac{dU_m}{dx} \bigg|_{x=L} \simeq (L - x)^{D-1} \quad (3.32) \]

Note that the steepness of the modal functions intensifies for higher modes and lower \(D\). In such a case, a coarse uniform mesh along \([0, L]\) may not sufficiently resolve the mode near the right edge, leading to poor numerical accuracy. This problem is remedied by applying a biased refinement in which the nodal concentration is higher near the right edge where steepness occurs.
Figure 3.3: The first three modal functions for the first mixed BVP shown for Hausdorff dimensions corresponding to those of the Carpinteri column.

3.4.3 Second mixed BVP

In this problem, we swap the locations of the BC applied in the first mixed BVP. In other words, we have

$$\frac{dU}{dx} \bigg|_{x=L} = 0 \quad \& \quad U \bigg|_{x=0} = 0$$

At first glance, nothing about this problem should be different from its predecessor in terms of analysis and solution form. Nevertheless, the particularity of this problem lies in the fact that modal solutions exist only if $D > 0.5$ as will be shown. For $D \leq 0.5$ the BC cannot be simultaneously applied and thus the eigenvalue problem has no solution. Applying the Neumann BC to the general solution of Eq. (3.25), we obtain

$$\lim_{x \to L^-} \left[ C_1 (L - x)^{(D-1)} \sin \left( k (L - x)^D \right) - C_2 (L - x)^{(D-1)} \cos \left( k (L - x)^D \right) \right] = 0$$

the problem is now to solve for $C_1$, $C_2$, and $D$ for the limit to be satisfied. Clearly the second term of the equation diverges for any $D < 1$, it thus
Figure 3.4: The first three modal functions for the second mixed BVP shown for Hausdorff dimensions corresponding to those of the Carpinteri column.

must perish from the solution and this is achieved by setting \( C_2 \equiv 0 \). The first term asymptotically behaves like \((L - x)^{(2D-1)}\); it vanishes only if the power is positive, necessitating \( D > 0.5 \). Concerning the wavenumber \( k \), it is assigned via the application of the Dirichlet BC. In conclusion the modal solution is described as

\[
U_m(x) = \cos \left[ k_m (L - x)^D \right] \quad D > 0.5 \quad (3.35a)
\]

\[
k_m = \frac{(2m + 1)\pi}{2L^D} \quad m \in \mathbb{N} \quad (3.35b)
\]

The first three modal functions are shown in Fig. 3.4. Fortunately, the Hausdorff dimensions of our domain are all above 0.5 making the second mixed BVP simulation feasible.

3.5 Computational Solutions

The finite element method (FEM) is ubiquitously applied to simulate elasto-
dynamic problems arising in mechanics of continuous materials (20; 19). In
this work, we prove that FEM is robust in handling mechanics problems in fractal solids too. In the first subsection, we discuss the application of the FEM in solving the 1d problem resulting from decoupling of the governing equation. A transient analysis in addition to testing convergence in time and space are performed. Next, we simulate the 3d problem in its full version, confirming the agreement between the 1d and 3d simulations.

3.5.1 1d FEM problem

As previously shown, the propagation of pure dilatational waves simplifies the general elastodynamic equation to three decoupled universal equations for all directions. Effectively, a large 3d problem can be reduced to three small 1d problems, each to be solved independently. This is a significant simplification from a computational perspective and it is worth being exploiting as achieved below.

The reduced 1d equation (Eq. (3.16)) represents the strong form of the problem whose domain of interest is $\Omega = ]0, L[$. This equation is first rewritten as

$$\rho c \ddot{u} = \left( \lambda + 2 \mu \right) \left( \frac{u_{,x}}{c} \right)_{,x} \tag{3.36}$$

To obtain the weak form, we multiply the above equation by an admissible function $\hat{u}$ where $\hat{u} \in H^1(\Omega)$, the space of all test functions. Integrating over $\Omega$, we obtain

$$\int_{\Omega} \rho c \ddot{u} \hat{u} d\Omega + \int_{\Omega} \left( \lambda + 2 \mu \right) \frac{u_{,x} \hat{u}_{,x}}{c} d\Omega = 0 \tag{3.37}$$

Partitioning the domain $\Omega$ into a countable union of subdomains (elements) $\Omega_e$, where $\bigcup_e \Omega_e = \Omega$, we obtain a discrete formulation based on two gov-
erning matrices, the inertia matrix $M$, and the stiffness matrix $K$. On the elemental level, these matrices are evaluated as

$$M_{ij}^e = \int_{\Omega_e} \rho \chi_i h_j \, dx \quad (3.38a)$$

$$K_{ij}^e = \int_{\Omega_e} \left( \lambda + 2\mu \right) \frac{h_{i,x} h_{j,x}}{c} \, dx \quad (3.38b)$$

where $h(x)$ is the elemental hat (polynomial) function utilized in constructing the numerical solution, i.e. $u_{num}(x) = \sum_{i=1}^{N_e} h_i(x) U_i$. As noticed, the governing matrices are symmetric; they can be evaluated exactly without performing any numerical integration (e.g. Gaussian quadrature rule). Once the assembly process is fulfilled, the discrete elastodynamic model is obtained, it is expressed as

$$M \ddot{U} + K \cdot U = 0 \quad (3.39)$$

The trapezoidal time stepping method is implemented to acquire the transient solution (19). This time march scheme is implicit, unconditionally stable and second-order accurate. The numerical results for all three discussed problems are shown in Figures 3.5, 3.6, and 3.7. The Lamé constants, in addition to the density, are all set to one; the exciting frequencies are found accordingly. The solutions of both mixed BVP oscillate at the same frequency, and, indeed, their wavenumber forms are identical. All problems exhibit a remarkable agreement between the exact and numerical solutions.

The numerical solver is further validated in space and time to ensure convergence. We have analysed the decay power of the $L_2$-norm of the global error with respect to element size and time step. For the spatial error, the convergence rate decreases as $D$ decreases most likely due to the skewness of
Figure 3.5: Transient solution for the Dirichlet BVP. The first two modes are excited in which $\omega_1,2 = 5.44$, $\omega_3 = 3.51$ for the first mode, and $\omega_1,2 = 10.88$, $\omega_3 = 7.03$ for the second mode. Frequencies are in rad/sec. Exact solutions are shown in solid line and numerical ones in coloured dots.

Figure 3.6: The transient solution for the first mixed BVP. The first two modes are excited where $\omega_1,2 = 2.72$, $\omega_3 = 1.76$ for the first mode, and $\omega_1,2 = 8.16$, $\omega_3 = 5.27$ for the second mode. Frequencies are in rad/sec. Exact solutions are shown in solid line and numerical ones in coloured dots.
Figure 3.7: The transient solution for the second mixed BVP. The first two modes are excited where $\omega_{1,2} = 2.72$, $\omega_3 = 1.76$ for the first mode, and $\omega_{1,2} = 8.16$, $\omega_3 = 5.27$ for the second mode. Frequencies are in rad/sec. Exact solutions are shown in solid line and numerical ones in coloured dots.

The mode shape for smaller $D$. The temporal error is nevertheless independent of $D$ and uniform for all the modes. The convergence plots are shown in Figures 3.8 and 3.9.

3.5.2 3d FDM analysis

In this section, we ignore the computational advantage granted by the handling of dilatational waves which simplifies 3d problems to only 1d. Effectively, we seek a numerical solution to Eq. (3.15) for a general problem setting. Even though waves of non-dilatational type can be analysed in this solver, numerical solutions for only dilatational wave problems are physically meaningful. Hence, the 3d analysis adds no originality to our computational work which, as shown in the previous section, can be adequately accomplished with the 1d solver. Nevertheless, problems involving non-classical material models where general waves are permissible cannot be subject to decoupling (as is the case here) and thus 3d analysis becomes mandatory. Since we plan to consider these problems in Chapter 4, we decided to briefly
Figure 3.8: Spatial convergence plots. The convergence analysis is based on the Dirichlet BVP. The effects of the Hausdorff dimension on convergence are clearly noticed.

(a) $D = \frac{\Delta x}{\ln 18}$

(b) $D = \frac{\Delta x}{\frac{\ln 2}{\ln 3}}$

Figure 3.9: Temporal convergence plots. The convergence analysis is based on the Dirichlet BVP. The Hausdorff dimension has very little effect on the convergence rate.

(a) $D = \frac{\Delta t}{\frac{1}{3} \ln 18}$

(b) $D = \frac{\Delta t}{\frac{\ln 2}{\ln 3}}$
Figure 3.10: Results for 3d analysis. For the first mode case we set $\Delta t = 2.0 \times 10^{-3}$ and utilized a uniform mesh discretized by $N_x = N_y = 121, N_z = 241$. For the second mode problem, the grid size is kept the same but we set $\Delta t = 1.4 \times 10^{-3}$. Remarkable matching is observed between the exact (solid black) and the numerical (coloured dots) solution.

In the numerical framework, Eq. (3.15) can be most effectively handled in a finite difference discretization with explicit time march setting (27). In brief, we adopted the 2nd order accurate spatial discretization and the 1st order accurate Euler’s forward method to respectively approximate derivatives arising in elastic and inertia terms. Results for the first two modal excitations of the Dirichlet BVP are shown in Fig. 3.10. In addition, contour plots for the displacement field at different instants are captured in Figures 3.11 and 3.12 for the first modal excitation and in Figures 3.13 and 3.14 for the second modal excitation.
Figure 3.11: Contour plots for the solution of the Dirichlet BVP in first mode excitation at $t = 0$ sec.

Figure 3.12: Contour plots for the solution of the Dirichlet BVP in first mode excitation at $t = 4$ sec.
Figure 3.13: Contour plots for the solution of the Dirichlet BVP in second mode excitation at $t = 0$ sec.

Figure 3.14: Contour plots for the solution of the Dirichlet BVP in second mode excitation at $t = 2$ sec.

Figure 3.15: Legend scale for all contour plots shown in Figures 3.11, 3.12, 3.13, and 3.14.
In this chapter, we explore the elastodynamics and wave propagation in fractal micropolar solid media. Such media incorporate a fractal geometry while being constitutively modelled by the Cosserat elasticity. The formulation of the balance laws which govern the mechanics of fractal micropolar solid media is first presented. Four eigenvalue type elastodynamic problems admitting closed form analytical solutions are introduced and discussed. A numerical procedure to solve general initial boundary value wave propagation problems in three-dimensional micropolar bodies exhibiting geometric fractality is then applied. Verification of the numerical procedure is discussed using the analytical solutions.

4.1 Motivation

We continue the exploration of elastodynamics in anisotropic fractal solids, the topic we introduced in Chapter 3. Our focus now extends to investigate on general problems involving the propagation of arbitrary waves. We learned

The topics discussed in this chapter are adopted from the journal paper listed in ref. (28)
in Chapter 3 about the caveat regarding the impossibility of satisfying the angular momentum balance within the framework of classical elasticity. As a result, the consideration of a non-classical constitutive model becomes evident to fulfil the physical and mathematical meaningfulness of the problem. Being the case, we adopt the elastodynamic model developed in (13; 14; 29) and aimed to describe the wave motion in fractal (non-continuous) micropolar (non-classical) solid materials. This model is useful for solving complex mechanics problems involving fractal materials composed of microstructures of inherent length scale, generalizing the universally applied classical theory of elastodynamics for continuous bodies.

Undoubtedly, the incorporation of the micropolar elasticity theory into this anisotropic fractal model complicates the elastodynamic problem by introducing additional degrees of freedom, microrotations and curvatures, and at the same time, augmenting the kinetic analysis via the mandatory consideration of the angular momentum balance equation. It is well known that for the case of classical elasticity, the satisfaction of the angular momentum balance is intuitively achieved through the symmetry of the stress tensor, a property that is ruled out in micropolar elasticity. We should also remark that the study of dilatational waves (topic discussed in Chapter 3) can be achieved within the framework of micropolar elasticity by applying the necessary kinematic constraints to nullify all the shear components of the stress tensor. In such a case, the micropolar constitutive law is suppressed to mimic Hooke’s law and this problem is our first demonstration when considering generalized waves in micropolar fractal solids.

The wave propagation problem is basically governed by the linear and angular momentum equations which, in essence, constitute an extended version of the three-dimensional (3d) Navier field equations; the variables being the
linear displacements and microrotations. Due to the complexity of these field equations, general modal analysis based upon the Helmholtz decomposition cannot be performed analytically. As a result, exact solutions, crucial for the numerical solver’s verification, seem unfeasible at first sight. However, particular problems, in which the dependence of the displacement and microrotation variables on the Cartesian coordinates is suppressed, generate a system of decoupled equations that is solvable analytically and can be used for verification.

The development of the fundamental balance laws in the fractal framework requires the application of the dimensional regularization discussed in Chapter 3. It transforms fractional integrals over fractal sets to equivalent continuous integrals over Euclidean sets (23). This regularization results in having the balance laws expressed in continuous form, thereby simplifying their mathematical manipulation in both analytical and numerical procedures. The product measure $c_i$ is utilized to achieve this transformation. The mathematical analysis demonstrating the application of $c_i$ to regularize the balance laws is thoroughly explained in (13; 14). The differentiation process in this homogenized model is obtained while preserving the mathematical consistency of Gauss’s law in the fractal setting. We recall that the regularized fractal derivative operator and the product measures are defined as

$$\nabla_i^D = \frac{1}{c_i} \frac{\partial}{\partial x_i} \quad \text{(no sum on } i) \quad , \quad i = 1, 2, 3 \quad (4.1a)$$

$$c_i(x_i) = D_i(L_i - x_i)^{D_i-1} \quad , \quad 0 < D_i \leq 1 \quad (4.1b)$$

$$c_V = c_1c_2c_3 \quad (4.1c)$$

where $L_i$ and $D_i$ are the length and the Hausdorff dimension in the $i$th
direction, and \( c_V \) is the product of all three product measures.

For our work, we consider idealized problems characterized by an anisotropic fractal structure embedded in a 3d topological domain. This domain, introduced in Chapter 3 as the Carpinteri column, is shown in Fig. 4.1. It is composed of a Sierpiński carpet in the \((x_1 x_2)\) cross-section, and the latter is swept longitudinally (along \( x_3 \)) in conjunction with a ternary Cantor set (5). For simplicity, we set \( L_1 = L_2 = 1 \) and \( L_3 = 2 \). The corresponding Hausdorff dimensions for this column are determined to be (26)

\[
D_1 = D_2 = \frac{1}{3} \ln 18 \quad \frac{1}{3}
D_3 = \frac{\ln 2}{\ln 3}
\] (4.2)

The consideration of a parallelepipedic (box shaped) domain is imperative for problem solving as will be asserted later from the field equations. Indeed, the domain characteristics \( L_i \) and \( D_i \) are inherent in the product measure’s definition and consequently in the entire formulation. As a result, the field equations are only valid in a Cartesian coordinate system.

4.2 Elastodynamic Modelling Procedures

In this section, we apply the constitutive, kinematic and kinetic laws to characterize the general elastodynamic model under investigation. The outcome of this work is two field equations that constitute the cornerstone on which the entire analytic and computational work is based. The detailed formulation of this model is discussed in (14).

The development of the micropolar theory of elasticity is mainly attributed to the work of the Cosserat brothers in 1909 (15). With regard to kinematics, this theory grants every “quasi-infinitiesimal” point in the so-called “micro-
continuum” two degrees of freedom: a displacement $u_i$, and a microrotation $\phi_i$. Accordingly, two independent measures for the continuum’s deformation are generated: the strain tensor $\gamma_{ij}$, and the curvature tensor $\kappa_{ij}$. These tensors are defined as ($e_{ijk}$ being the Levi-Civita permutation tensor)

$$\gamma_{ij} = \nabla_i^D u_j - \epsilon_{kij} \frac{\phi_k}{c_i} \quad \text{(no sum on } i) \quad \text{ (4.3a)}$$

$$\kappa_{ij} = \nabla_i^D \phi_j \quad \text{ (4.3b)}$$

The kinetic interaction between quasi-infinesimal elements is not limited to forces as is the case with Hookean (classical) elasticity; it includes moment loading too. As a result, a couple-stress tensor $\mu_{ij}$ associated with this loading, must be incorporated into the fundamental balance laws. The Cosserat force-stress tensor, or simply stress tensor $\tau_{ij}$, along with the newly introduced couple-stress tensor $\mu_{ij}$ are, for the case of small deformation, linearly

Figure 4.1: Layout of Carpinteri column, the physical domain on which the analytical and numerical problems of elastodynamics are solved.
related to the strain and curvature tensors (15),

\[ \tau_{ij} = C_{ijkl}^{(1)} \gamma_{kl} \]  \hspace{1cm} (4.4a)

\[ \mu_{ij} = C_{ijkl}^{(2)} \kappa_{kl} \]  \hspace{1cm} (4.4b)

where \( C_{ijkl}^{(1)} \) and \( C_{ijkl}^{(2)} \) are two elastic moduli tensors that vary in complexity. For the case of isotropic media, they are defined as (\( \delta_{ij} \) being the Kronecker delta)

\[ C_{ijkl}^{(1)} = (\beta - \alpha) \delta_{jk} \delta_{il} + (\beta + \alpha) \delta_{jl} \delta_{ik} + \lambda \delta_{ij} \delta_{kl} \]  \hspace{1cm} (4.5a)

\[ C_{ijkl}^{(2)} = (\psi - \epsilon) \delta_{jk} \delta_{il} + (\psi + \epsilon) \delta_{jl} \delta_{ik} + \eta \delta_{ij} \delta_{kl} \]  \hspace{1cm} (4.5b)

Here \( \beta \) and \( \lambda \) are the usual Lamé’s constants, measured in Pa, while \( \psi \), \( \epsilon \), and \( \eta \) are micropolar constants, measured in Pa.m\(^2\), and \( \alpha \) is also a micropolar constant but measured in Pa. Consequently, the direct stress vs. strain and couple-stress vs. curvature relations become

\[ \tau_{ij} = (\beta + \alpha) \gamma_{ij} + (\beta - \alpha) \gamma_{ji} + \lambda \gamma_{kk} \delta_{ij} \]  \hspace{1cm} (4.6a)

\[ \mu_{ij} = (\psi + \epsilon) \kappa_{ij} + (\psi - \epsilon) \kappa_{ji} + \eta \kappa_{kk} \delta_{ij} \]  \hspace{1cm} (4.6b)

This theory, in contrast to classical continuum theories, possesses an intrinsic length scale that can be easily detected if we scrutinize the units of the components of the stiffness tensors. This remark explains the significance of the micropolar theory in modelling problems involved with finite microstructures (e.g. granular media, lattice structures, liquid crystals).

In classical elasticity problems, the angular momentum balance is ignored in the mathematical analysis since it is satisfied through the symmetry property of the Cauchy stress tensor. However, in micropolar elasticity, \( \tau_{ij} \) and \( \mu_{ij} \)
are in general non-symmetric. As a result, the angular momentum equation becomes an indispensable part of the problem and has to be considered alongside with the linear momentum balance to fully achieve the elastodynamic setting of the problem. According to (14), the balance equations are

\[ \rho \ddot{u}_i = \nabla_j D_j \tau_{ji} \quad (4.7a) \]

\[ I \ddot{\phi}_i = \nabla_j D_j \mu_{ji} + e_{ijk} \frac{\tau_{jk}}{c_j} \quad (4.7b) \]

and if expanded in terms of the displacement and rotation fields, they become

\[ \rho \ddot{u}_i = (\beta + \alpha) \left[ \nabla_j D_j \nabla_j u_i - \nabla_j D_j \left( e_{kji} \frac{\phi_k}{c_j} \right) \right] + (\beta - \alpha + \lambda) \nabla_i D_j \nabla_j u_j \]

\[ + (\beta - \alpha) \nabla_j D_j \left( e_{kji} \frac{\phi_k}{c_i} \right) \quad \text{(no sum on } i) \quad (4.8a) \]

\[ I \ddot{\phi}_i = (\psi + \epsilon) \nabla_j D_j \nabla_j \phi_i + (\psi - \epsilon + \eta) \nabla_i D_j \nabla_j \phi_j - (\beta + \alpha) \phi_i \sum_{j \neq i} \frac{1}{c_j^2} \]

\[ + (\beta + \alpha) e_{ijk} \frac{\nabla_k u_j}{c_j} + (\beta - \alpha) e_{ijk} \frac{\nabla_k u_j}{c_j} \]

\[ + 2(\beta - \alpha) \frac{\phi_i c_i}{c_v} \quad \text{(no sum on } i) \quad (4.8b) \]

where \( \rho \) is the mass density and \( I \) is the mass moment of inertia assuming, and for simplicity, an isotropic inertia tensor \( I_{ij} = I \delta_{ij} \). Clearly, no Helmholtz decomposition seems viable, the issue that renders the mathematical analysis cumbersome. We verify the correctness of our work by examining the case where \( D_i = 1 \) for all \( i \) in which fractal effects vanish and the field equations proposed by Eringen for elastodynamics in micropolar continua are reproduced (15). For more details, review Chapter 5 of (15) where a variety of fundamental problems in wave motion in micropolar media are discussed. From this perspective, this research builds upon the findings of Eringen by
extending the exploration of micropolar elastodynamics into fractal structures.

4.3 Analytical Approach

In this section, we consider some special problems, the kinematics of which is imposed in such a way to obtain reduced field equations admitting analytical solutions. These problems may not always be physically meaningful. Nevertheless, their solutions are mathematically valuable, especially when it comes to verifying the computational solver. For some problems, the kinematic assignment is not sufficient to obtain closed-form solutions and additional constraints on the stiffness constants have to be enforced to fulfill this purpose. All the discussed problems are of eigenvalue type, and the approach followed in analysing them is through modal decomposition. The first two problems are similar because they both reduce to solving a 1d modal equation, while the last two are more complex by requiring the solution of a higher dimensional modal equation.

4.3.1 Dilatational wave problem

Consider the elastodynamic problem where the following \( u_i \) and \( \phi_i \) fields are imposed

\[
    u_i \equiv u(x_i) \quad (4.9a)
\]

\[
    \phi_i \equiv 0 \quad (4.9b)
\]

This kinematic setting actually models the propagation of waves that are of dilatational type. This problem, though discussed in Chapter 3, will be briefly introduced here as to ease the understanding of the subsequent prob-
lems, particularly with regard to modal analysis. Applying the prescribed
kinematics into Eq. (4.3) and (4.6), the only non-zero components are the
axial stress and strain tensors. Thus, the induced deformation is volumetric
(dilatation or contraction) and non-distortional; the problem designation is
consistent with the one introduced in Chapter 5 of (25) where wave propaga-
tion in continuous domains is investigated. Having the curvature, the shear
strain, and consequently, the shear stress and the couple-stress components
identically zero, micropolar effects become abolished, the issue that makes
the consideration of this particular problem in the paradigm of Hookean elas-
ticity fully consistent with the micropolar study. The angular momentum
balance stated in Eq. (4.8b) is trivially satisfied, while the linear momentum
equation (4.8a) branches into three identical decoupled equations, one for
each displacement variable. Consider the governing equation for $u_1$

$$\frac{\rho}{\lambda + 2\beta} \ddot{u}_1 = \nabla^D_1 \nabla^P_1 u_1 = \frac{1}{c_1} \left( \frac{u_{1,1}}{c_1} \right)_{,1} \quad (4.10)$$

Performing the Fourier analysis by setting $u_1(x_1, t) = U_1(x_1) e^{j\omega t}$, ($j = \sqrt{-1}$), we obtain an eigenvalue problem governed by the 1d \textit{fractal Helmholtz}
equation. The modal function $U_1(x_1)$ and its corresponding wavenumber $k$
obey the following equation ($\vartheta_d$ being the dilatational wave speed already
defined in Chapter 3)

$$k^2 U_1 = -\nabla^D_1 \nabla^P_1 U_1 = -\frac{1}{c_1} \left( \frac{U_{1,1}}{c_1} \right)_{,1} \quad (4.11a)$$

$$k = \frac{\omega}{\vartheta_d}, \quad \vartheta_d = \sqrt{\frac{\lambda + 2\beta}{\rho}} \quad (4.11b)$$
The general modal solution for the eigenvalue problem, derived in detail in the Appendix B, is given by

\[ U_1(x_1; k) = C_1 f_1(x_1, k) + C_2 f_2(x_1, k) \] (4.12)

where \( f_1 \) and \( f_2 \) are the homogeneous solutions of the fractal Helmholtz equation, also denoted as fractal harmonic functions,

\[ f_1(x, k) = \cos \left[ k (L - x)^D \right] \] (4.13a)

\[ f_2(x, k) = \sin \left[ k (L - x)^D \right] \] (4.13b)

The solutions for \( u_2(x_2, t) \) and \( u_3(x_3, t) \) have the same functional form of that of \( u_1 \), but with a different frequency which is \( L \) and \( D \) dependent. The application of the problem boundary conditions (BC) determines the eigenvalue \( k \), the constants \( C_1 \) and \( C_2 \). Clearly for \( D = 1 \), the fractal harmonics become the standard harmonic functions which are indeed the solutions for the continuum case. In the case of homogeneous BC, the eigenvalue problem presented in Eq. (4.11a) admits the following modal orthonormality property in every direction,

\[ \int_0^{L_i} U_m U_n c(x_i) \, dx_i = \delta_{mn} \] (4.14)

where \( U_m = U_m(x_i, k_m) \) is the \( m \)th mode in the \( i \)th direction, and \( U_n \) is the \( n \)th mode for the same direction.

4.3.2 Torsional wave problem

Consider the problem where the displacement field is suppressed to zero and the microrotation field is permitted in such a way that every rotation solely
depends on the direction about which it occurs. Mathematically, we have

\[ u_i \equiv 0 \quad (4.15a) \]

\[ \phi_i \equiv \phi_i (x_i) \quad (4.15b) \]

This problem resembles the one discussed in the previous section with slight differences. The shear components of the curvature and couple-stress tensors are all identically zero. The axial components of the strain and stress tensors are also zero (rendering the deformation equivoluminal), but not the shear components. As a result, the elastodynamic equations for the rotation field cannot be solved exactly unless certain restrictions on the elastic moduli are enforced. For example, if we consider the governing equation for \( \phi_1 \), we have

\[ I \ddot{\phi}_1 = (2\psi + \eta) \nabla^D_1 \left( \nabla^D_1 \phi_1 \right) + \phi_1 \left[ 2 \frac{\beta - \alpha}{c_2 c_3} - (\beta + \alpha) \left( \frac{1}{c_2^2} + \frac{1}{c_3^2} \right) \right] \quad (4.16) \]

The presence of \( c_2 \) and \( c_3 \) which, respectively, depends on \( x_2 \) and \( x_3 \), is not allowed in the \( \phi_1(x_1) \) equation. As a result, the second term on the right-hand side must be completely eliminated and this can be realized only by setting \( \alpha = \beta = 0 \). This restriction applies to the equations of \( \phi_2 \) and \( \phi_3 \) as well. In such a case, we reproduce the same modal equation of the dilatational wave problem. Consequently, by setting \( \phi_i (x_i, t) \equiv \Phi (x_i) e^{i\omega t} \), \( \Phi \) satisfies the 1d fractal Helmholtz equation but with a wave speed and wavenumber given as

\[ \vartheta_t = \sqrt{\frac{\eta + 2\psi}{I}} \quad , \quad k = \frac{\omega}{\vartheta_t} \quad (4.17) \]
As a result, the general solution for \( \Phi \), derived in the Appendix B, is given in terms of the fractal harmonic functions, by

\[
\Phi (x_i) = C_1 f_1 (x_i, k) + C_2 f_2 (x_i, k) \tag{4.18}
\]

### 4.3.3 In-plane problem

The so-called first planar problem is a generalization of the in-plane elasticity where the displacement and the rotation fields are prescribed to be

\[
\begin{align*}
&u_1 \equiv u_1 (x_1) \quad u_2 \equiv u_2 (x_2) \quad u_3 \equiv 0 \tag{4.19a} \\
&\phi_1 \equiv 0 \quad \phi_2 \equiv 0 \quad \phi_3 \equiv \phi_3 (x_1, x_2) \tag{4.19b}
\end{align*}
\]

The resulting strain and curvature tensors become

\[
\begin{align*}
\gamma_{ij} &= \begin{bmatrix}
\nabla_D u_1 & -\frac{\phi_3}{c_1} & 0 \\
\phi_3 & \nabla_D^2 u_2 & 0 \\
0 & 0 & 0
\end{bmatrix} \\
\kappa_{ij} &= \begin{bmatrix}
0 & 0 & \nabla_D \phi_3 \\
0 & 0 & \nabla_D \phi_3 \\
0 & 0 & 0
\end{bmatrix} \tag{4.20}
\end{align*}
\]

The corresponding non-zero components of the stress tensor expressed in terms of the unknown fields are

\[
\begin{align*}
\tau_{11} &= (2\beta + \lambda) \nabla_D u_1 + \lambda \nabla_D u_2 \tag{4.21a} \\
\tau_{22} &= \lambda \nabla_D u_1 + (2\beta + \lambda) \nabla_D u_2 \tag{4.21b} \\
\tau_{33} &= \lambda (\nabla_D u_1 + \nabla_D u_2) \tag{4.21c}
\end{align*}
\]
\[ \tau_{12} = (\beta + \alpha)\gamma_{12} + (\beta - \alpha)\gamma_{21} \quad (4.21d) \]

\[ \tau_{21} = (\beta - \alpha)\gamma_{12} + (\beta + \alpha)\gamma_{21} \quad (4.21e) \]

Similarly, for the non-zero couple-stress tensor components, we obtain

\[ \mu_{13} = (\psi + \epsilon)\nabla^D_1 \phi_3 \quad (4.22a) \]

\[ \mu_{31} = (\psi - \epsilon)\nabla^D_1 \phi_3 \quad (4.22b) \]

\[ \mu_{23} = (\psi + \epsilon)\nabla^D_2 \phi_3 \quad (4.22c) \]

\[ \mu_{32} = (\psi - \epsilon)\nabla^D_2 \phi_3 \quad (4.22d) \]

The resulting linear momentum equations for \( u_1 \) and \( u_2 \) along with the angular momentum equation for \( \phi_3 \) become

\[ \rho \ddot{u}_1 = (2\beta + \lambda) \nabla^D_1 \nabla^D_1 u_1 + (\beta - \alpha) \nabla^D_2 \left( -\frac{\phi_3}{c_1} \right) + (\beta + \alpha) \nabla^D_2 \left( \frac{\phi_3}{c_2} \right) \quad (4.23a) \]

\[ \rho \ddot{u}_2 = (2\beta + \lambda) \nabla^D_2 \nabla^D_2 u_2 + (\beta - \alpha) \nabla^D_1 \left( \frac{\phi_3}{c_2} \right) + (\beta + \alpha) \nabla^D_1 \left( \frac{\phi_3}{c_1} \right) \quad (4.23b) \]

\[ I \ddot{\phi}_3 = (\psi + \epsilon) \left[ \nabla^D_1 \nabla^D_1 \phi_3 + \nabla^D_2 \nabla^D_2 \phi_3 \right] - (\beta + \alpha) \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} \right) \phi_3 \]

\[ + (\beta - \alpha) \frac{\phi_3}{c_1 c_2} \quad (4.23c) \]

Since the in-plane displacement \( u_1 \) is set to only depend on \( x_1 \) and similarly for \( u_2 \) and \( x_2 \), the above equations must be configured to preserve this mathematical consistency. In other words, the \( x_2 \) terms of the \( u_1 \) equation must vanish and similarly the \( x_1 \) terms of the \( u_2 \) equation. This consistency is achieved once we set \( \alpha = \beta = 0 \). Thus, the field equations become the 1d
fractal wave equation for $u_1$ and $u_2$, and the 2d fractal wave equation for $\phi_3$.

\[
\frac{\rho}{\lambda} \ddot{u}_1 = \nabla^D_1 \nabla^D_1 u_1 = \frac{1}{c_1} \left( \frac{u_{1,1}}{c_1} \right), \quad (4.24a)
\]

\[
\frac{\rho}{\lambda} \ddot{u}_2 = \nabla^D_2 \nabla^D_2 u_2 = \frac{1}{c_2} \left( \frac{u_{2,2}}{c_2} \right), \quad (4.24b)
\]

\[
\frac{I}{\psi + \epsilon} \ddot{\phi}_3 = \nabla^D_1 \nabla^D_1 \phi_3 + \nabla^D_2 \nabla^D_2 \phi_3 = \left[ \frac{1}{c_1} \left( \frac{\phi_{3,1}}{c_1} \right)_{,1} + \frac{1}{c_2} \left( \frac{\phi_{3,2}}{c_2} \right)_{,2} \right] \quad (4.24c)
\]

Clearly, the the waves involved in the dynamics of $u_1$ and $u_2$ are of dilatational type. As a result, the modal analysis for these two variables is no different from the one presented in section 4.3.1. The celerity of the wave reduces to $\vartheta_d = \sqrt{\frac{\lambda}{\rho}}$, because $\beta = 0$. Expressing $u_i = U_i e^{j\omega t}$, we obtain

\[
U_i = C_1 f_1(x_i, k) + C_2 f_2(x_i, k) \quad i = 1, 2 \quad (4.25a)
\]

\[
k = \frac{\omega}{\vartheta_d} \quad (4.25b)
\]

$f_1$ and $f_2$ being the general fractal harmonic functions presented in Eq. (4.13). Concerning $\phi_3$, we introduce the space-time separation of variables where we have $\phi_3 \equiv \Phi_3(x_1, x_2) e^{j\omega t}$. The modal function, $\Phi_3$, satisfies the 2d fractal Helmholtz equation, which can be solved as explained in the Appendix B if the spatial decomposition $\Phi_3 \equiv F(x_1) G(x_2)$ is performed. Setting the corresponding wave speed $\vartheta_r = \sqrt{\frac{\psi + \epsilon}{I}}$ and $k = \frac{\omega}{\vartheta_r}$, we obtain

\[
\frac{\nabla^D_1 \nabla^D_1 F}{F} + \frac{\nabla^D_2 \nabla^D_2 G}{G} + k^2 = 0 \quad (4.26)
\]

By further assigning $k^2 = k^2_1 + k^2_2$, the above equation can be decoupled where $F$ and $G$ each satisfy the 1d Helmholtz equation. Thus, the decomposed
eigenmodes are expressed as

\[ F(x_1, k_1) = M_1 f_1(x_1, k_1) + M_2 f_2(x_1, k_1) \] (4.27a)

\[ G(x_2, k_2) = N_1 f_1(x_2, k_2) + N_2 f_2(x_2, k_2) \] (4.27b)

All four constants, in addition to the two wavenumbers \((k_1 \text{ and } k_2)\), are BC dependent.

4.3.4 Out-of-plane problem

We now consider our second planar problem which is a generalization of the out-of-plane elasticity. No in-plane displacement or out-of-plane rotation is allowed. We consider the case in which the in-plane rotations depend on the planar coordinates \((x_1, x_2)\) while the out-of-plane displacement, \(u_3\), depends on \(x_3\) for a reason that will be identified later in the problem’s analysis. The kinematics thus far is

\[ u_1 \equiv 0 \quad u_2 \equiv 0 \quad u_3 \equiv u_3(x_1, x_2, x_3) \] (4.28a)

\[ \phi_1 \equiv \phi_1(x_1, x_2) \quad \phi_2 \equiv \phi_2(x_1, x_2) \quad \phi_3 \equiv 0 \] (4.28b)

The resulting strain and curvature tensors for this prescribed kinematics are

\[
\gamma_{ij} = \begin{bmatrix}
0 & 0 & \nabla^D_1 u_3 + \frac{\phi_2}{c_1} \\
0 & 0 & \nabla^D_2 u_3 - \frac{\phi_1}{c_2} \\
-\frac{\phi_2}{c_3} & \frac{\phi_1}{c_3} & \nabla^D_3 u_3
\end{bmatrix}
\]

\[
\kappa_{ij} = \begin{bmatrix}
\nabla^D_1 \phi_1 & \nabla^D_1 \phi_2 & 0 \\
\nabla^D_2 \phi_1 & \nabla^D_2 \phi_2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (4.29)
The non-zero components of the stress tensor expressed in terms of the unknown fields become

\[ \tau_{11} = \tau_{22} = \lambda \nabla_3^D u_3 \]  
(4.30a)

\[ \tau_{33} = (2\beta + \lambda) \nabla_3^D u_3 \]  
(4.30b)

\[ \tau_{13} = (\beta + \alpha)\gamma_{13} + (\beta - \alpha)\gamma_{31} \]  
(4.30c)

\[ \tau_{31} = (\beta - \alpha)\gamma_{13} + (\beta + \alpha)\gamma_{31} \]  
(4.30d)

\[ \tau_{23} = (\beta + \alpha)\gamma_{23} + (\beta - \alpha)\gamma_{32} \]  
(4.30e)

\[ \tau_{32} = (\beta - \alpha)\gamma_{23} + (\beta + \alpha)\gamma_{32} \]  
(4.30f)

and similarly for the couple-stress tensor, we have

\[ \mu_{11} = (2\psi + \eta) \nabla_1^D \phi_1 + \eta \nabla_2^D \phi_2 \]  
(4.31a)

\[ \mu_{22} = (2\psi + \eta) \nabla_2^D \phi_2 + \eta \nabla_1^D \phi_1 \]  
(4.31b)

\[ \mu_{33} = \eta(\nabla_1^D \phi_1 + \nabla_2^D \phi_2) \]  
(4.31c)

\[ \mu_{12} = (\psi + \epsilon) \nabla_1^D \phi_2 + (\psi - \epsilon) \nabla_2^D \phi_1 \]  
(4.31d)

\[ \mu_{21} = (\psi + \epsilon) \nabla_2^D \phi_1 + (\psi - \epsilon) \nabla_1^D \phi_2 \]  
(4.31e)

The resulting angular momentum equation for \( \phi_1 \) becomes

\[ I \ddot{\phi}_1 = \nabla_1^D \mu_{11} + \nabla_2^D \mu_{21} + \nabla_3^D \mu_{31} + \frac{\tau_{23}}{c_2} - \frac{\tau_{32}}{c_3} \]

\[ = (2\psi + \eta) \nabla_1^D \nabla_1^D \phi_1 + (\psi - \epsilon + \eta) \nabla_1^D \nabla_2^D \phi_2 + (\psi + \epsilon) \nabla_2^D \nabla_2^D \phi_1 \]  
(4.32)

\[ + (\beta + \alpha) \left( \frac{\gamma_{23}}{c_2} - \frac{\gamma_{32}}{c_3} \right) + (\beta - \alpha) \left( \frac{\gamma_{32}}{c_2} - \frac{\gamma_{23}}{c_3} \right) \]
The above equation contains the term $c_3$ which depends on $x_3$. Prescribing $\phi_1$ to solely depend on the in-plane coordinates requires the $c_3$ term to be eliminated and this can be realized by setting $\alpha = \beta = 0$. In addition, the full decoupling of the mathematical model (where $\phi_2$ vanishes from the $\phi_1$ equation and vice versa) necessitates the condition $\psi - \epsilon + \eta = 0$. Under such conditions, we have for $\phi_1$ and $\phi_2$

$$I\ddot{\phi}_1 = (\eta + 2\psi) \left[ \nabla_1^D \nabla_1^D \phi_1 + \nabla_2^D \nabla_2^D \phi_1 \right]$$

$$= (\eta + 2\psi) \left[ \frac{1}{c_1} \left( \frac{\phi_{1,1}}{c_1} \right)_{,1} + \frac{1}{c_2} \left( \frac{\phi_{1,2}}{c_2} \right)_{,2} \right] \quad (4.33a)$$

$$I\ddot{\phi}_2 = (\eta + 2\psi) \left[ \frac{1}{c_2} \left( \frac{\phi_{2,1}}{c_1} \right)_{,2} + \frac{1}{c_1} \left( \frac{\phi_{2,2}}{c_2} \right)_{,1} \right] \quad (4.33b)$$

Concerning $u_3$, after applying all the necessary assignments, its linear momentum equation becomes

$$\rho \ddot{u}_3 = \nabla_3^D \tau_{33} = \lambda \nabla_3^D \nabla_3^D u_3 \quad (4.34)$$

At this point, it becomes clear why $u_3$ is set to depend on $x_3$. If this were not the case, $u_3$ would not possess any meaningful dynamics in this problem.

Having crafted the mathematical setting for this problem, we now present its analytical solution. The procedure to solve for the fields is realized by considering modal analysis. As noticed, $\phi_1$ and $\phi_2$ obey the same equation and thus, they must admit the same general solution. The Fourier decomposition sets $\phi_1 = e^{j\omega t} \Phi_1(x_1, x_2) = e^{j\omega t} F(x_1)G(x_2)$, where $\Phi_1$ satisfies the 2d fractal Helmholtz equation

$$\nabla_1^D \nabla_1^D \Phi_1 + \nabla_2^D \nabla_2^D \Phi_1 + k^2 \Phi_1 = 0 \quad (4.35)$$
More explicitly, $F$ and $G$ each satisfies the 1d fractal Helmholtz equation

$$\nabla_1^D \nabla_1^D F + k_1^2 F = 0 \quad (4.36a)$$

$$\nabla_2^D \nabla_2^D G + k_2^2 G = 0 \quad (4.36b)$$

$$k^2 = k_1^2 + k_2^2 = \left( \frac{\omega}{\vartheta_t} \right)^2 \quad (4.36c)$$

$$\vartheta_t = \sqrt{\frac{\eta + 2\psi}{I}} \quad (4.36d)$$

As a result, the well known solution for $F$ and $G$ is constructed using the fractal harmonic functions $f_1$ and $f_2$

$$F(x_1) = C_1 f_1 (x_1, k_1) + C_2 f_2 (x_1, k_1) \quad (4.37a)$$

$$G(x_2) = C_3 f_1 (x_2, k_2) + C_4 f_2 (x_2, k_2) \quad (4.37b)$$

The eigenvalue solutions to $k_1$ and $k_2$, in addition to that of constants, $C_1$, $C_2$, $C_3$, and $C_4$ are determined through the application of the BC. Note that the torsional wave speed $\vartheta_t$ which was introduced in the second problem, characterizes the out-of-plane waves corresponding to microrotations.

The out-of-plane displacement obeys the “dilatational” 1d fractal wave equation, thus its corresponding modal problem admits the fractal harmonic functions as the general solution. Mathematically, setting $u_3 (x_i, t) = U_3 (x_i) e^{j\omega t}$, $U_3 (x_i)$ satisfies the 1d fractal Helmholtz equation in $x_3$, where

$$U_3 (x_i) = C_1 (x_1, x_2) f_1 (x_3, k) + C_2 (x_1, x_2) f_2 (x_3, k) \quad (4.38a)$$

$$k = \frac{\omega}{\vartheta_d} \quad \vartheta_d = \sqrt{\frac{\lambda}{\rho}} \quad (4.38b)$$
Note that $C_1$ and $C_2$ are allowed to depend on $x_1$ and $x_2$, because $u_3$, as prescribed for this problem, depends in the general case on all the coordinates. Here $C_1$ and $C_2$ are determined through the application of the BC.

4.4 Numerical Solutions

Having developed the full field equations for the generalized problem and considered particular cases of known closed form solutions, our goal is now to construct a valid and accurate computational tool capable of handling arbitrary initial boundary value (IBV) problems unsolvable by analytic methods. Due to formulation restriction, the numerical tool is limited to treat box shaped domains in Cartesian systems. Given this restraint and the considerable size of such a 3d problem, the most efficient method to design the numerical solver would be to implement a direct discretization method for the field equations and adopt the explicit scheme for the time march. As such, the finite difference method is applied with discrete variables $u_i(x_k, t_n)$ and $\phi_i(x_k, t_n)$ which are assigned on uniform meshes in space and time. The second order accurate central difference discretization approximates all the spatial derivatives while the first order accurate (Euler’s forward method) treats the acceleration terms (27). The resulting numerical scheme is conditionally stable and its parallel implementation for the case of expensive problems is straightforward and highly efficient.

The validation of the numerical solver is achieved by simulating the four previously discussed problems where the transient response for modal excitation is evaluated. The first modal excitation is employed for all problems and a strong match between the numerical and exact solution is observed. We now discuss the numerical simulations and show the computed time histories
Figure 4.2: Transient solution for the dilatational wave problem. The exact solution is shown in solid black and the numerical is shown in coloured dots.

For the variables of interest and the 3d contour plots of the variable fields in the entire numerical domain.

For the first problem, we set $\lambda = \beta = 1$, and the micropolar constants whose values are irrelevant to the solution are all set to one. The density and the inertia terms are set to one as well. These values result in having the frequencies of first mode oscillation $\omega_1 = \omega_2 = 5.441$ and $\omega_3 = 3.514$. We used 1000 time steps to span two periods of $u_1$ oscillation, which implies that $\Delta t \approx 2.31 \times 10^{-3}$. The grid is uniform with $\Delta x_1 = \Delta x_2 = \Delta x_3 = 10^{-2}$. The transient solutions for the displacement field are shown in Figure 4.4 and the corresponding contour plots for the same problem are shown in Figure 4.4.

For the second problem, we set $\psi = \eta = 1$ in addition to the condition $\alpha = \beta = 0$. All other constants are irrelevant to the solution. The density and the inertia terms are set to one for simplicity. The resulting oscillation frequencies of the first mode are $\omega_1 = \omega_2 = 5.441$ and $\omega_3 = 3.514$. We used 1000 time steps to span two periods of $\phi_1$ oscillation, which implies that $\Delta t \approx 2.31 \times 10^{-3}$. The grid is identical to that of the first problem. The
Figure 4.3: First problem contour plots for $u_i$ at different instants.
Figure 4.4: Transient solution for the torsional wave problem. The exact solution is shown in solid black and the numerical is shown in coloured dots.

transient solution for the microrotation field is shown in Figure 4.4 and the corresponding contour plots for the same problem are shown in Figure 4.5.

For the in-plane problem, we set the stiffness constants $\psi = \epsilon = \lambda = 1$ and set $\alpha = \beta = 0$. The remaining terms, irrelevant to the problem, assumed the value of one. The density and the inertia terms are set to one for simplicity. The resulting frequencies for the first modal excitation of $u_1$ and $u_2$ are $\omega_1 = \omega_2 = 3.141$, while that of $\phi_3$ for the first mode is $\omega_3 = 6.283$. We used 900 time steps to span two periods of $\phi_3$ oscillation, which implies that $\Delta t \approx 2.222 \times 10^{-3}$. The grid is identical to that of all previous problems. The time history of the transient solution is shown in Figure 4.4 and the corresponding contour plots are shown in Figure 4.7.

In the simulation of the out-of-plane problem, we set $\psi = \eta = 1$ and $\epsilon = 2$ to satisfy $\psi + \eta - \epsilon = 0$. In addition, we set $\alpha = \beta = 0$. The Lamé’s constant $\lambda$ can assume any value and was set to one. The density and the inertia terms are set to one for simplicity. The resulting frequencies for the first modal excitation of $\phi_1$ and $\phi_2$ are $\omega_1 = \omega_2 = 7.695$, while that of $u_3$ for
Figure 4.5: Second problem contour plots for $\phi_i$ at different times.
Figure 4.6: Transient solution for the in-plane problem. The exact solution is shown in solid black and the numerical is shown in coloured dots.

the first mode is $\omega_3 = 2.029$. We used 1000 time steps to span two periods of $\phi_1$ (and $\phi_2$) oscillation, which implies that $\Delta t \approx 1.633 \times 10^{-3}$. The grid is identical to that of all previous problems. The transient solution for the variables of interest is shown in Figure 4.4 and the corresponding contour plots are shown in Figure 4.9.
Figure 4.7: In-plane problem contour plots for $u_1$, $u_2$, and $\phi_3$ at different times.
Figure 4.8: Transient solution for the out-of-plane. The exact solution is shown in solid black and the numerical is shown in coloured dots.
(a) $\phi_1$, $t = 0.0$
(b) $\phi_2$, $t = 0.0$
(c) $u_3$, $t = 0.0$
(d) $\phi_1$, $t = 0.98$
(e) $\phi_2$, $t = 0.98$
(f) $u_3$, $t = 0.98$
(g) $\phi_1$, $t = 1.63$
(h) $\phi_2$, $t = 1.63$
(i) $u_3$, $t = 1.63$

Figure 4.9: Out-of-plane problem contour plots for $\phi_1$, $\phi_2$, and $u_3$ at different times.
In this chapter, we drift from mechanics of fractal media to discuss the free vibration of elastic spherical structures in the presence of an externally unbounded continuous acoustic medium. In this vibration, damping associated with the radiation of energy from the confined solid medium to the surrounding acoustic medium is observed. The main objective of this work is to evaluate the coupled system response (solid displacement and acoustic pressure) and characterizing the acoustic radiation damping by relating it to the media properties. Acoustic damping is demonstrated for two problems: the thin spherical shell and the solid sphere. The mathematical approach followed in solving these coupled problems is based on the Laplace transform method. The linear under-damped harmonic oscillator is used as the reference model for damping estimation. The damping evaluation is performed in frequency as well as in time domains; both investigations lead to identical damping factor expressions.

5.1 Background

Coupled fluid-solid interaction (FSI) problems are in general complex. They require advanced computational methods to handle them. Multiple studies
have previously investigated the vibration of structures while interacting with a surrounding acoustic medium (30; 31; 32; 33; 34). Experimental investigations were also conducted on lightweight aerospace structures to estimate their acoustic radiation damping under a wide band of frequency excitations (35). During this interaction, the structure’s energy is continuously radiated into the surrounding fluid medium, resulting in a damped structure’s response. In (36), a general expression for the total loss factor exhibited in the vibration of a solid body when immersed in a fluid medium is given as

$$\eta_t = \eta_{struc} + \eta_{aero} + \eta_{rad}$$  \hspace{1cm} (5.1)$$

The total loss factor $\eta_t$, consists of three components: the structural loss factor $\eta_{struc}$, which represents damping due to the material properties of the structure (e.g., viscoelasticity), the aerodynamic loss factor $\eta_{aero}$, which is associated with the presence of a non-zero mean flow over the structure, and finally, the radiation loss factor $\eta_{rad}$, the focus of our work, which is related to the radiation of sound as a consequence of the structure’s vibration. Suppressing the structural loss by assuming a non-dissipative material model and eliminating the aerodynamic loss by considering a perturbing flow in a quiescent fluid medium, the acoustic radiation becomes the sole source of damping pertaining to the problem. Obviously, when the structure is vibrating in vacuum, no acoustic radiation loss is encountered and thus, $\eta_{rad} = 0$. In many situations, acoustic radiation effects can be reasonably neglected; however, in some cases, particularly for thin lightweight structures, acoustic damping can be an order of magnitude higher than its structural counterpart (35).

The generalized acoustic-structure interaction problem, whose schematic
layout is shown in Figure 5.1, is comprised of a solid medium $\Omega_s$ that is wholly immersed in an acoustic medium $\Omega_a$. For our work, we adopt the simplest possible models for both $\Omega_s$ and $\Omega_a$ in hope of making analytical solutions feasible. Therefore, the small-strain Hookean model whose elastodynamics is governed by the Navier equation (25), and the inviscid compressible fluid in which the acoustic wave propagation is described by the well-known wave equation (17), are applied in $\Omega_s$ and $\Omega_a$ respectively. The two media have a common interface $\Gamma_{in}$ on which relevant boundary conditions (BC) are maintained to complete the problem’s description. As a physical constraint, no interpenetration is allowed through $\Gamma_{in}$, thus the normal displacement of the structural medium must equate its fluid counterpart; this ensures a continuous displacement field throughout. In addition, the equilibrium on the common surface requires balancing the solid normal traction to the acoustic pressure. Finally, at the infinite boundary $\Gamma_\infty$, the non-reflection (wave absorption) condition is applied to emulate the absence of any reflecting surface within the acoustic medium at far field. In summary, the governing equations that constitute the mathematical core of the general coupled problem are presented as follows

\begin{equation}
\frac{1}{1 - 2\nu} u_{j,ji} + u_{i,jj} = \frac{2(1 + \nu)\rho_s}{E} \ddot{u}_i \quad \text{in } \Omega_s \quad (5.2a)
\end{equation}

\begin{equation}
p_{kk} = \frac{1}{\vartheta^2_a} \ddot{p} \quad \text{in } \Omega_a \quad (5.2b)
\end{equation}

\begin{equation}
\ddot{u}_k n_k = -\frac{1}{\rho_a} p_{,j} n_j \quad \text{on } \Gamma_{in} \quad (5.2c)
\end{equation}

\begin{equation}
\sigma_{ij} n_j = -p n_i \quad \text{on } \Gamma_{in} \quad (5.2d)
\end{equation}

\begin{equation}
\lim_{|r| \to \infty} |r| \left( \frac{\partial p}{\partial r} + \frac{1}{\vartheta_a} \frac{\partial p}{\partial t} \right) = 0 \quad \text{on } \Gamma_\infty \quad (5.2e)
\end{equation}
where $E$, $\nu$, and $\rho_s$ are respectively the Young’s modulus, Poisson’s ratio, and mass density of the solid medium, while $\vartheta_a$ and $\rho_a$ are respectively the speed of sound and mass density corresponding to the acoustic medium. The above model was applied in (37) where an FSI problem is solved computationally.

The unknowns of Eq. (5.2) are, to a great importance, the solid displacement $u$, and less importantly, the acoustic pressure $p$. The main objective of our work is to analytically solve the elastodynamic-acoustic interaction problem described by Eq. (5.2) for two structures: the thin spherical shell and the solid sphere. In both problems, acoustic damping is demonstrated and a closed-form expression for the damping factor is obtained revealing its dependence on the media properties, problem geometry, and modal wavenumbers. The obtained damping factor expression is further verified via the damping extraction algorithms applied on transient solutions.

The major significance of this research is the achievement of the exact solutions for both the displacement and the pressure fields while solving a single coupled problem. These solutions constitute a trustworthy validation means for complex FSI numerical models, and to a less extent, experimental procedures. In previous works, the acoustic-structure interaction problem was
considered in the framework where the dynamics is triggered by an external source of excitation (e.g., applied step load on the structure as in (32) or pressure pulse in the acoustic medium as in (33)). In our work however, we consider the situation of self-excitation, where the initially deformed structure is the sole source of energy pertaining to the coupled system. In such a case, the structure’s response has no choice but to decay continuously with time due to the lack of any external energy source that could compensate for the radiated acoustic energy. As a result, under self-excitation, the coupled system reveals its intrinsic damping characteristics allowing for a correlation between the natural response and the acoustic damping behaviour to be established. The mathematical analysis of this correlation leads to the qualitative evaluation of the damping factor, the topic that constitutes the real novelty of this research since no such evaluation has been conducted before.

In this work, we will first briefly overview the various damping extraction methods associated with the transient analysis of the linear under-damped harmonic oscillator. Next, we analyse the thin shell problem, which is characterized by its mathematical simplicity with regard to solution formulation and damping estimation. Then, we discuss the more challenging solid sphere problem in which we present the radial mode of oscillation and its corresponding energy functions, to later solve the coupled problem and verify the damping expression. Finally, we conduct a qualitative comparison between the radiation damping resistances of our problems and those of some previously analysed problems, to draw meaningful conclusions about general acoustic damping estimation.
5.2 Damping Extraction Methods

The coupled elastodynamic-acoustic problem is linear as depicted in Eq. (5.2), therefore, the damping behaviour of the coupled system is expected to mimic that of the simplest linear dynamic model: the ideal under-damped harmonic oscillator. This model is widely explored in (38) and in other references. Its natural vibration is governed by

\[ \ddot{u} + 2\zeta \omega_n \dot{u} + \omega_n^2 u = 0 \]  

which admits a normalized displacement solution \( u(t) \) given by

\[ u(t) = \exp(-\zeta \omega_n t) \left[ \cos(\omega_d t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_d t) \right] \] 

The total energy stored in the system, \( e(t) > 0 \), is the sum of kinetic and potential energies. It is expressed in normalized form as

\[ e(t) = u^2 + \frac{\dot{u}^2}{\omega_n^2} \]

\[ = \exp(-2\zeta \omega_n t) \left[ 1 - \frac{\zeta^2 \cos(2\omega_d t)}{1 - \zeta^2} \right] + \frac{\zeta \sin(2\omega_d t)}{\sqrt{1 - \zeta^2}} \] 

where the the damped frequency \( \omega_d \) is defined as

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \] 

In account of the above analysis, mathematical procedures for evaluating the damping property of a dynamic system are established. The log decrement method explained in (39), is a statistical algorithm that extracts a characteristic damping ratio from the transient response, portraying the
Figure 5.2: Transient response for the under-damped harmonic oscillator; labelled are the extrema points in (a) the displacement response and the inflection points in (b) the energy response

intrinsic dissipation of a damped oscillator. Mathematically, for a typical under-damped oscillatory displacement response, consistent with Eq. (5.4) and shown in Fig. 5.2(a), the algorithm predicts an average displacement-based damping ratio, $\bar{\zeta}_{\text{dis}}$, computed as follows

$$\bar{\zeta}_{\text{dis}} = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\log \left( \frac{|u_{n+1}|}{|u_n|} \right)}{\sqrt{\log^2 \left( \frac{|u_{n+1}|}{|u_n|} \right) + \pi^2}}$$

(5.7)

where $u_n$ is the $n$th extremum value of $u(t)$, and $N$ is the total number of all successive extrema used in the evaluation. For the algorithm to generate a meaningful damping ratio, the displacement peak values must steadily decay with time, i.e. $|u_{n+1}| < |u_n|$ for all $n$. In some cases, this condition fails (as will be shown in the solid sphere problem), rendering the displacement-based log decrement method erroneous. By necessity to overcome this failure, we extend the applicability of the log decrement method into the energy re-
response, and a corresponding energy-based algorithm is developed. Considering the energy response expressed in Eq. (5.5) and plotted in figure 5.2(b), we observe a non-increasing or “staircase” behaviour that is typical for this type of dynamic systems. The interesting points to consider for this algorithm are those where the response flats out, i.e. inflection points on which the first and second time derivatives vanish. The average energy-based damping ratio, \( \bar{\zeta}_{en} \), can be evaluated using the following formula

\[
\bar{\zeta}_{en} = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\log \left( \frac{e_{n+1}}{e_n} \right)}{\sqrt{\log^2 \left( \frac{e_{n+1}}{e_n} \right) + 4\pi^2}}
\]  

(5.8)

This algorithm requires the identification of \( N \) consecutive inflection points \( e_n \), the first one occurs at initial time. The obvious condition for the algorithm’s success is the steady decay of energy, i.e. \( e_{n+1} < e_n \) for all \( n \), and this is always satisfied for a damped system subject to natural excitations. Hence, the energy-based log decrement method supersedes its displacement-based counterpart by possessing an unrestricted applicability.

In addition to damping extraction procedures based on transient analysis, methods based on the frequency (Laplace) response are applied in this work too. Since our analysis is primarily conducted in the frequency domain, the frequency-based damping extraction method generates a meaningful estimate for the damping ratio bypassing all errors involved in the Laplace transform inversion. In addition, this method expresses the damping ratio in closed form by explicitly revealing its physical dependence on the problem’s governing parameters. In comparison, the log decrement method simply provides a “numerical” value for the damping ratio without unveiling its mathematical structure. In our work, we achieve the damping characterization by first
applying the frequency-based method to obtain the closed-form expression of the damping ratio, and then verifying it through log-decrement methods. The mathematical procedure of the frequency method is presented as follows. The Laplace form of \( u(t) \) is expressed as

\[
\hat{U}(s) = \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]  

(5.9)

The Bode plot of this expression is shown in Fig. 5.3. The response peaks near the natural frequency where

\[
\omega_p \approx \omega_n \left(1 - 4\zeta^4\right)
\]  

(5.10a)

\[
\left|\hat{U}\right|_p \approx \frac{1 + 2\zeta^2}{2\zeta \omega_n}
\]  

(5.10b)

Inverting Eq. (5.10b) to solve for \( \zeta \), we obtain

\[
\zeta \approx \frac{1}{2\sqrt{\left|\hat{U}\right|_p^2 \omega_n^2 - 1}}
\]  

(5.11)

The above equations are simplified from a more complex form where an approximation for small \( \zeta \) is appreciated. Acoustic radiation problems are in general lightly damped; thus the approximation is valid and the analysis is accurate.

### 5.3 Pulsating Thin Spherical Shell

In this section, we present the first and simplest coupled problem: radial pulsation of a thin spherical shell. Consider the elastic thin spherical shell of mean radius \( R \) and thickness \( b \) where \( b \ll R \). The shell’s inner core is void.
and its center resides on the origin of the spherical coordinate system. The spherical symmetry is achieved by considering a uniform and wholly radial displacement field throughout the shell surface, thus \( u \equiv u(t) \). This symmetry extends to the surrounding acoustic part, where \( p \) spatially depends on the radial distance only, i.e. \( p \equiv p(r, t) \). Initially, the acoustic medium is quiescent and the shell is deformed by \( u_0 \). The governing simplified elastodynamic equation describing the radial oscillation of the shell under this prescription is given as (also applied in (32))

\[
\frac{\text{d}^2 u}{\text{d}t^2} + \left( \frac{\vartheta_s}{R} \right)^2 u = -\frac{p_{\text{out}}}{b \rho_s} \tag{5.12}
\]

where \( p_{\text{out}} \) is the uniform acoustic pressure applied on the shell’s outer surface, i.e. \( p(R, t) \). The structural wave speed is defined as

\[
\vartheta_s = \sqrt{\frac{2E}{\rho_s(1 - \nu)}} \tag{5.13}
\]
On the acoustic side, the acoustic wave equation (D’Alembertian form) expressed in spherical coordinates reduces to

\[ r^2 \frac{\partial^2 p}{\partial r^2} + 2r \frac{\partial p}{\partial r} - r^2 \frac{\partial^2 p}{\partial \vartheta^2} \frac{\partial}{\partial t^2} = 0 \quad r \geq R \quad (5.14) \]

On the shell surface, the no-interpenetration condition for small shell radial displacement is expressed as

\[ \frac{d^2 u}{dt^2} = -\frac{1}{\rho_a} \frac{\partial p}{\partial r} \bigg|_{r=R} \quad (5.15) \]

And finally, at far field, the Sommerfeld radiation condition becomes

\[ \lim_{r \to \infty} r \left( \frac{\partial p}{\partial r} + \frac{1}{\vartheta_a} \frac{\partial p}{\partial t} \right) = 0 \quad (5.16) \]

Equations (6.12a), (5.14), (6.12c), and (5.16) fundamentally describe the coupled problem. The inversion into the Laplace domain is applied in the following mathematical analysis leading to solving for the shell’s response.

When the wave equation is transformed into the Laplace domain, we obtain the 0th order spherical Bessel equation in \( r \) admitting the spherical Hankel functions of first and second kind as fundamental solutions. Thus

\[ \hat{P}(r, s) = C(s)h_0^{(1)} \left( -\frac{isr}{\vartheta_a} \right) + D(s)h_0^{(2)} \left( -\frac{isr}{\vartheta_a} \right) \quad r \geq R \quad (5.17) \]

where \( C(s) \) and \( D(s) \) are determined by applying the necessary BC and \( i = \sqrt{-1} \). For the Sommerfeld condition in (5.16) to be satisfied, the first kind Hankel function \( h_0^{(1)} \) must not appear in the pressure general solution, this is achieved by setting \( C(s) \equiv 0 \). The reader is referred to the appendix A4 of (17) or the appendix of (18) to better understand the asymptotic
behaviour of Hankel functions and their derivatives. Substituting for \( \hat{P} \) and its derivative in the Laplace versions of Eq. (6.12a) and Eq. (6.12c) and then taking the ratio of these two equations to eliminate \( D(s) \), we obtain an expression for \( \hat{U} \) given by

\[
\hat{U}(s) = \frac{s^2 + s\left(\frac{\partial_a}{R} + \frac{\partial_a \rho_s}{\rho_s}\right)}{s^3 + s^2\left(\frac{\partial_a}{R} + \frac{\partial_a \rho_s}{\rho_s}\right) + s \frac{\varphi^2}{R^2} + \frac{\varphi_0^2 s}{R^2}}
\]  

(5.18)

If we normalize the displacement with \( u_0 \), time with \( R / \vartheta_s \) and consequently the frequency (and Laplace variable, \( s \)) with \( \vartheta_s \), the above expression in normalized form becomes

\[
\hat{U}(s) = \frac{s^2 + s\left(\vartheta_s \vartheta_s + 2 \zeta_0\right)}{s^3 + s^2\left(\vartheta_s \vartheta_s + 2 \zeta_0\right) + s + \vartheta_s}
\]  

(5.19)

where \( \zeta_0 \) is defined as

\[
\zeta_0 = \frac{1}{2} \frac{\rho_s \vartheta_s R}{\vartheta_s b}
\]  

(5.20)

Applying the frequency method to extract the damping expression \( \zeta_f \), we obtain

\[
\zeta_f = \frac{\zeta_0}{\sqrt{1 + 4\zeta_0 \vartheta_s}}
\]  

(5.21)

In real applications, \( \vartheta_s, \zeta_0 \ll 1 \), thus \( \zeta_0 \) constitutes a first order approximation for \( \zeta_f \). Note that \( \zeta_0 \) is inversely proportional to \( \rho_s \) and \( b \), confirming that acoustic damping intensifies for thin lightweight structures.

Substituting the expression of \( \hat{U} \) into the Laplace form of Eq. (6.12c) to solve for \( D(s) \), the acoustic pressure (which can be normalized by the char-
characteristic pressure \( \rho_a \partial_a \partial_s \frac{u_0}{R} \) is then obtained as follows

\[
\frac{\hat{P}(r, s)}{\rho_a \partial_a \partial_s \frac{u_0}{R}} = -\frac{R}{r} \frac{s \exp\left[-\frac{\partial_s}{\partial_a} (\frac{r}{R} - 1)s\right]}{s^3 + s^2 \left(\frac{\partial_s}{\partial_a} + 2\zeta_0\right) + s + \frac{\partial_a}{\partial_s}}
\]  

(5.22)

Note that both pressure and displacement solutions possess the same characteristic equation (denominator of their Laplace form), thus they exhibit the same dynamic behaviour, in particular, damping behaviour. The Laplace inversion of the displacement and pressure expressions into the time domain is performed via the method of partial fractions (consult chapter 1 of (40) and appendix B.2 of (25) for understanding the inversion technique). Plots of the displacement and pressure solutions are shown in Fig. 5.4 and Fig. 5.5 respectively.

Concerning the pressure response, we mark two observations. First, a period of silence i.e. \( p = 0 \) occurs everywhere in the acoustic domain; its duration increases linearly with the distance from the shell’s surface according
Figure 5.5: Transient solution for the acoustic pressure at different radial locations for $\zeta_0 = 0.025$ and $\frac{\vartheta_a}{\vartheta_s} = 0.1$

to

$$T_{\text{silence}} = \frac{r - R}{\vartheta_a} \quad (5.23)$$

Second, the maximum amplitude of the acoustic pressure decays as we move away from the shell’s surface to eventually become zero at infinity. The displacement simulations are conducted at fixed density ratio $\frac{\rho_a}{\rho_s} = 0.05$ and fixed geometry $\frac{R}{b} = 10$, but variable wave speed ratio $\frac{\vartheta_a}{\vartheta_s} = 0.1, 0.2,$ and $0.4$ resulting in three different values for $\zeta_0$ and consequently, three cases 1, 2, and 3 respectively. It is shown that as the ratio $\frac{\vartheta_a}{\vartheta_s}$ decreases, the relative error between the exact damping value (obtained by log decrement method) and the closed form value (obtained by frequency method) decreases, and both values approach $\zeta_0$ corroborating the validity of the asymptotic expression of Eq. (5.21) at low ratio of wave speeds. The results of the three cases are presented in Table 5.1.
Table 5.1: Results of the damping factors (in %) for all three cases. Note the strong similarity at low damping.

<table>
<thead>
<tr>
<th></th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_0$ %</td>
<td>2.500</td>
<td>5.000</td>
<td>10.000</td>
</tr>
<tr>
<td>$\zeta_f$ %</td>
<td>2.487</td>
<td>4.903</td>
<td>9.285</td>
</tr>
<tr>
<td>$\bar{\zeta}_{\text{dis}}$ %</td>
<td>2.466</td>
<td>4.842</td>
<td>8.560</td>
</tr>
<tr>
<td>Rel. err. %</td>
<td>0.851</td>
<td>1.260</td>
<td>8.470</td>
</tr>
</tbody>
</table>

5.4 Vibrating Solid Sphere

In this section, we demonstrate acoustic damping in the vibrating solid sphere problem which is in concept not different from that of the thin spherical shell, but mathematically more complex. The complexity arises because of the presence of modes of vibrations for the solid sphere whose elastodynamics is described by a partial differential equation instead of an ordinary one as is the case for the thin shell. Our first insight about this problem is that the acoustic damping will be mode dependent. Indeed, the closed form expression of the damping factor, along with the results of the transient simulations, corroborate this conjecture. Spherical symmetry will again be enforced to simplify this three-dimensional problem whereby the displacement and the pressure will be solely dependent on the radial distance. Before discussing the coupled problem, we will briefly present the solid sphere radial vibration in vacuum in which we overview the modal analysis and determine the mode shapes and their corresponding natural frequencies. We also introduce the energy calculation which will be applied in the damping evaluation procedures.
5.4.1 Elastodynamic overview

Consider the elastic solid sphere of radius $R$, centered at the origin of the spherical coordinate system. We consider radial modes of vibration, thus the transcendental and the azimuthal displacements are suppressed to zero ($u_\theta \equiv 0$, $u_\phi \equiv 0$, and $u_r \equiv u(r, t)$). Equations listed in appendix A.9.2 of (25) help understanding the following analysis. The spherical version of the Navier equation relevant to our problem is

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} - 2u = r^2 \frac{\ddot{u}}{\vartheta_s^2}$$  \hspace{1cm} (5.24)

in which the wave speed $\vartheta_s$ is defined as

$$\vartheta_s = \sqrt{\frac{E(1-\nu)}{\rho_s(1+\nu)(1-2\nu)}}$$  \hspace{1cm} (5.25)

To find the mode shapes $u_m$, we set $u(r, t) = u_m(r)e^{i\omega_m t}$. Substituting this form into Eq. (5.24), the following eigenvalue problem is obtained

$$r^2 \frac{d^2 u_m}{dr^2} + 2r \frac{du_m}{dr} + u_m \left[ \left( \frac{r\omega_m}{\vartheta_s} \right)^2 - 2 \right] = 0$$  \hspace{1cm} (5.26)

This is a first order spherical Bessel equation admitting the spherical Bessel functions of first and second kind as general solutions thus

$$u_m(r) = C_1 j_1 \left( \frac{r\omega_m}{\vartheta_s} \right) + C_2 y_1 \left( \frac{r\omega_m}{\vartheta_s} \right)$$  \hspace{1cm} (5.27)

The BC relevant to this problem are: finite displacement at the origin ($r = 0$) and traction-free outer outer surface, i.e. $\sigma_{rr}|_{R} = 0$. The first BC forces $C_2$ to vanish because $y_1$ is singular at zero. For simplicity, we set $C_1 = 1$ and
we define the normalized modal wave number $\delta_m$ as

$$\delta_m = \frac{\omega_m R}{\vartheta_s} \quad (5.28)$$

Hence, the mode shape becomes

$$u_m(r) = j_1 \left( \delta_m \frac{r}{R} \right) \quad (5.29)$$

Define a modified Poisson’s ratio $\beta$ as follows

$$\beta = \frac{4\nu - 2}{1 - \nu} \quad -2 < \beta < 0 \quad (5.30)$$

The application of the second BC results in the following transcendental equation for $\delta_m$

$$\tan \delta_m = \frac{\beta \delta_m}{\delta_m^2 + \beta} \quad (5.31)$$

This nonlinear equation is solved numerically via a root finding algorithm (e.g., bisection) to determine the eigenvalues of $\delta_m$. A meaningful asymptotic
approximation for $\delta_m$ at large integer $m$ is given by $\delta_m \approx m\pi + \frac{\beta}{m\pi}$. The plot in Fig. 5.6 shows the first four modal functions corresponding to $\beta = -1$ or $\nu = \frac{1}{3}$.

### 5.4.2 Energy calculations

We now introduce the energy calculations corresponding to the radial deformation of this spherical body. We encourage the reader to refer to appendix A.8 and A.9.2 of (25) to better understand the mathematical analysis. For the general case, the stored strain energy per unit volume of an elastic body is defined as

$$e_p = \int \sigma_{ij} \epsilon_{ij}$$

In spherical problems, and for the case of radial displacement field depending solely on $r$, the expression for $e_p$ reduces to

$$e_p(r, t) = \rho_s \frac{\partial^2 s^2}{2} \left[\left(1 + \frac{\beta}{2}\right) \epsilon_{ii} \epsilon_{jj} - \frac{\beta}{2} \epsilon_{ij} \epsilon_{ij}\right]$$

Clearly, this energy form is space dependent. In order to observe global effects within the entire sphere (example: energy decay), we introduce the volume average strain energy defined as follows

$$\bar{e}_p(t) = \frac{1}{V} \int_V e_p(r, t) dV = \frac{3}{R^3} \int_0^R e_p(r, t) r^2 dr$$

In a similar approach, the kinetic energy per unit volume is defined as

$$e_k(r, t) = \frac{\rho_s}{2} \left(\frac{\partial u}{\partial t}\right)^2$$
and for the same reason (spatial dependency), we introduce the volume average kinetic energy defined as

\[
\bar{e}_k(t) = \frac{1}{V} \int_V e_k(r,t) \, dV = \frac{3}{R^3} \int_0^R e_k(r,t) r^2 \, dr
\]  

(5.36)

Finally, we define the normalized volume average total energy stored in the solid sphere by summing the strain and kinetic energy and normalizing the resulting sum with its initial value. As a result we have

\[
\bar{e}_t = \frac{\bar{e}_p + \bar{e}_k}{\bar{e}_p|_{t=0} + \bar{e}_k|_{t=0}}
\]

(5.37)

For free vibrations in vacuum, the energy form of (5.37) remains constant (identically one) perpetually. In such a case, an exchange between kinetic and strain energy occurs without any external dissipation or addition of energy. However, in the coupled situation where an interaction between the sphere and its surrounding fluid medium takes place, the total energy decays due to acoustic radiation. The understanding of this decay helps characterizing the acoustic damping, the topic of our investigation.

5.4.3 Coupled problem formulation

In this section, we consider the coupled solid sphere problem, where an acoustic medium fills the entire space surrounding the solid sphere. The goal is to evaluate the acoustic damping corresponding to each mode of vibration. Thus, a single mode is excited in each case study, and the resulting damping factor is recorded. In brief, the trend to obtain a verified equation for the acoustic damping ratio is explained in these steps:
1. solve the IBVP (single-mode excitation) and obtain the Laplace form of the displacement and acoustic pressure

2. apply the frequency method to extract the closed form expression of the damping ratio

3. invert the Laplace form of the displacement into time domain

4. obtain the velocity and strain fields to solve for the total energy

5. apply the log decrement methods and verify the results with the expression obtained in step 2

For the structural domain, the IBVP is formulated by considering Eq. (5.24) with modified BC on the sphere’s outer surface. Initially, the sphere is deformed in accordance with the mode of interest; thus the modal function \( u_m(r) \) appears in the Laplace form of Eq. (5.24), which is an inhomogeneous spherical Bessel equation in \( r \) expressed as

\[
r^2 \frac{\partial^2 \hat{U}}{\partial r^2} + 2r \frac{\partial \hat{U}}{\partial r} + \hat{U} \left[ \left( \frac{isr}{s} \right)^2 - 2 \right] = -\frac{sr^2}{\partial_s^2} u_m(r) \tag{5.38}
\]

The full solution of \( \hat{U} \) consists of the homogeneous part (spherical Bessel function) in addition to the particular one imposed by the right-hand side term. The full solution, satisfying the finite displacement BC at the sphere’s centre, admits the form

\[
\hat{U}(r, s) = C(s) j_1 \left( \frac{isr}{s} \right) + u_m(r) \frac{s}{s^2 + \omega_m^2} \tag{5.39}
\]

Concerning the acoustic medium, the general solution for the pressure is identical to that of the pulsating thin shell problem. This is intuitively explained by observing that the acoustic wave propagation is indifferent to
the inner core of the spherical domain whether it is void or filled. As a matter of fact, Eq. (5.14) and Eq. (5.16) are independent from any solid influence. Therefore, the acoustic pressure’s general solution is given by

$$\hat{P}(r, s) = D(s) h_0^{(2)} \left( \frac{-isr}{\vartheta_a} \right)$$  \hspace{1cm} (5.40)

Equations (5.39) and (5.40) contain two unknown constants $C(s)$ and $D(s)$; they are determined by applying the coupled BC on the common interface, the sphere’s outer surface. The no-interpenetration BC is no different from that of the thin shell problem expressed in Eq. (6.12c). Rewriting it in Laplace domain, we obtain

$$\rho_a \left[ s^2 \hat{U}(R, s) - s u_m(R) \right] = -\frac{\partial \hat{P}}{\partial r} |_R$$  \hspace{1cm} (5.41)

The traction BC balances the radial stress with the acoustic pressure. This BC is expressed in (a) time and (b) Laplace domain as follows

$$\sigma_{rr}|_R = -p|_R$$  \hspace{1cm} (5.42a)

$$\rho_s \vartheta_s^2 \left[ \frac{\partial \hat{U}}{\partial r} |_R + \frac{2\nu}{(1-\nu)R} \hat{U}(R, s) \right] = -\hat{P}(R, s)$$  \hspace{1cm} (5.42b)

In account of Eq. (5.41) and Eq. (5.42b), we can solve for $C(s)$ and $D(s)$ after an extensive algebraic work. Hence the Laplace form of the sphere’s displacement and the acoustic pressure are obtained.

We introduce the dimensionless parameter $q \equiv \frac{\rho_a \vartheta_a}{\rho_s \vartheta_s}$, and we note that $q \ll 1$ in most real applications. This parameter appears in the expression of the damping coefficient as will be shown. We simplify the expressions of the coupled solution by first normalizing the Laplace variable $s$ with $\omega_m$ and the
displacement with \( u_m(R) \), and second, by substituting the spherical Bessel functions of complex argument with their equivalent hyperbolic functions to obtain the solution of the displacement field as follows

\[
\hat{U}(r, s) = \frac{s}{s^2 + 1} \left[ \frac{s}{U_1(s)} \hat{U}_2(r, s) + \frac{u_m(r)}{u_m(R)} \right] \quad (5.43a)
\]

\[
\hat{U}_{1,\text{num}}(s) = q \delta_m^2 [\delta_m s \tanh(\delta_m s)]
\]

\[
\hat{U}_{1,\text{den}}(s) = q \left( \delta_m s \right)^3 + \beta (\delta_m s)^2 + \beta \frac{\partial s}{\partial s} \delta_m s + \tanh(\delta_m s) \left[ (\delta_m s)^3 + \left( \frac{\partial s}{\partial s} - q \right) (\delta_m s)^2 - \beta \delta_m s - \beta \frac{\partial s}{\partial s} \right]
\]

\[
\hat{U}_2(r, s) = \frac{R}{\delta_m s \cosh(\frac{R}{s} \delta_m s) - \frac{R}{s} \sinh(\frac{R}{s} \delta_m s)} \quad (5.43d)
\]

and that of the acoustic pressure field as

\[
\hat{P}(R, s) = \frac{\rho_a \partial_a \delta_m u_m(R)}{R} \left[ \frac{s}{s^2 + 1} \left( \hat{U}_1 + 1 \right) - 1 \right] \quad (5.44a)
\]

\[
\hat{P}(r, s) = \frac{R}{r} \hat{P}(R, s) \exp \left[ -\delta_m \frac{s}{\delta_m} \left( \frac{R}{R} - 1 \right) s \right] \quad (5.44b)
\]

Clearly, these Laplace expressions are not at all familiar; their inversion technique will be discussed in the next subsection.

5.4.4 Damping evaluation

The frequency method is applied using the Laplace form of the displacement presented in Eq. (5.43). On the outer surface \((r = R)\), \(\hat{U}_2\) becomes 1 and the expression of \(\hat{U}\) reduces to

\[
\hat{U}(R, s) = \frac{s}{s^2 + 1} \left[ \hat{U}(s) + 1 \right] \quad (5.45)
\]
Figure 5.7: Bode plot of the sphere’s outer surface displacement for the first two modes.

The amplitude of $\hat{U}(R,s)$ is plotted in Fig. 5.7. Note that this frequency response is behaviourally identical to that of the harmonic oscillator shown in Fig. 5.3. The major peak of the response occurs at slightly below the natural frequency. The value of the peak increases as we go higher in modes, hinting to damping attenuation for higher modes. In addition, second and higher modes exhibit secondary peaks occurring far from the frequency of excitation; these peaks are irrelevant to damping evaluation but affect the transient response as will be realized. Solving analytically for the peak frequency and the corresponding response amplitude is not feasible due to the mathematical complexity involved in the $\hat{U}(R,s)$ expression. We will approximate the peak of the amplitude response by the value occurring at the natural frequency, in other words

$$\left|\hat{U}_p\right| \approx \left| \lim_{s \to i} \hat{U}(R,s) \right|$$

$$= \left| \frac{\delta_m^2 + \beta^2 + 3\beta}{2q\delta_m} - i \left(1 + \frac{\vartheta}{\varphi_s} \frac{\delta_m^2 + \beta^2 + 3\beta}{2q\delta_m^2}\right) \right|$$

(5.46)
The damping value can be then obtained using Eq. (5.11), thus

$$\zeta_{m,f} \simeq \frac{q\delta_m}{\beta^2 + 3\beta \sqrt{1 + \left( \frac{\vartheta_a}{\delta_m \vartheta_s} \right)^2 + \frac{q\vartheta_s}{\delta_m + \beta^2 + 3\beta \vartheta_s}}}, \quad (5.47)$$

Note that the square root term in Eq. (5.47) can be safely disregarded for small \( \vartheta_a \) and high modes (large \( \delta_m \)). In spite of the approximation applied to obtain the final form of \( \zeta_{m,f} \), the latter still meaningfully evaluates the acoustic damping as is confirmed by the transient analysis results shown in Table 5.2.

The displacement \( u(r, t) \) is obtained by inverting the Laplace expression of Eq. (5.43). As such, we have

$$u(r, t) = \cos(t) * u_1(t) * u_2(r, t) + \frac{u_m(r)}{u_m(R)} \cos(t) \quad (5.48)$$

where * refers to the convolution operator. On the outer surface, the displacement simplifies to

$$u(R, t) = \cos(t) * u_1(t) + \cos(t) \quad (5.49)$$

The Laplace inversion is performed using the Bromwich method explained in appendix B.2 of (25) and chapter 2 of (40). In the general case, we have

$$u(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \exp(st) \hat{U}(s) \, ds = \sum_{k=1}^{N_p} \text{Res} \left( \hat{U}(s) , s_k \right) \exp(s_k t) \quad (5.50)$$

where \( s_k \) is the \( k \)th pole of \( \hat{U}(s) \) and \( N_p \) is the total number of poles. In this problem, \( \hat{U}_1(s) \) has infinitely many poles (one is real and the remaining are complex conjugates), thus if the sum in Eq. (5.50) were to converge, it can
be truncated. Asymptotic analysis shows that the complex part of the poles increases indefinitely rendering their residues insignificant. We can therefore safely disregard the contribution of these large poles and consider the first pole pairs with the smallest complex part. We have indeed considered the first fifty conjugate pairs of poles. The same procedure applies to $\hat{U}_2(r, s)$. The poles were obtained numerically via the two-dimensional bisection method (a root finding algorithm of assured convergence). The convolution integral is performed numerically using the cubic splines method. The displacement response at the outer surface for the first three modes at three different values of $q$ is shown in Fig. 5.8. For real applications, the solid is primarily metallic and the fluid is ambient air; thus a $q$ value of $10^{-4} \sim 10^{-3}$ is reasonable. For undersea applications, $10^{-1}$ would be a realistic value for $q$. But in this case, the viscous dissipation in water especially at low speed (low Reynolds number) would intensify, spoiling the major assumption of its negligence when evaluating acoustic damping. In any case, the system is simulated at $q = \{10^{-3}, 10^{-2}, 10^{-1}\}$ in order to test the validity of the $\zeta_{m,f}$ expression over a wide range of $q$. In all cases, we fixed $\frac{\omega_a}{\omega_s} = 0.1$. The transient solution of the acoustic pressure at three spots in the fluid domain is shown in Fig. 5.9.

The transient response due to the first modal excitation reveals damping characteristics similar to those of the harmonic oscillator where the magnitude of the extrema points decay steadily with time. However, for the second and higher modes, this steady decay property is lost. This observation is justified by the presence of the secondary (minor) peaks in the frequency response shown in Fig. 5.7. The presence of these minor peaks points to a slight contribution into the transient response from an additional frequency, the issue that leads to an intermittent perturbation of the steady
Figure 5.8: Transient solutions of the sphere’s outer surface normalized displacement
Figure 5.9: Transient solution of the normalized acoustic pressure at three different radial locations for the first modal excitation, $q = 10^{-2}$, $\frac{\dot{u}}{u_e} = 0.1$

decay of the values of the extrema points. Being the case, the displacement version of the log decrement method fails to predict the damping ratio for other than the first mode; the energy version comes into effect to fulfil this prediction. The strain field, along with the velocity field are obtained from the displacement solution via direct numerical differentiation with respect to space and time. The strain and the kinetic energy are consequently evaluated as explained in Eq. (5.34) and Eq. (5.36). The total energy is then obtained for the first three modes at the three chosen values of $q$ as shown in Fig. 5.10. The energy-based log decrement method is finally applied and the extracted damping coefficients are listed in Table 5.2. Strong agreement between the various coefficients is observed particularly at small $q$ and low modes.

5.5 Radiation Resistance Investigation

In this section, we conduct a qualitative comparison with regard to acoustic damping characteristics between the coupled systems discussed in our work and some other previously analysed ones. Indeed, a significant resemblance
Figure 5.10: Transient solution of the sphere’s volume averaged normalized total energy
Table 5.2: Summarized damping evaluation results. Clear matching is noted at low $q$

<table>
<thead>
<tr>
<th>Mode</th>
<th>$q$</th>
<th>$\zeta_t %$</th>
<th>$\zeta_{dis} %$</th>
<th>$\zeta_{en} %$</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>$10^{-3}$</td>
<td>0.049</td>
<td>0.049</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>$10^{-2}$</td>
<td>0.496</td>
<td>0.491</td>
<td>0.494</td>
</tr>
<tr>
<td></td>
<td>$10^{-1}$</td>
<td>4.942</td>
<td>4.816</td>
<td>4.849</td>
</tr>
<tr>
<td>second</td>
<td>$10^{-3}$</td>
<td>0.017</td>
<td>NA</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td>$10^{-2}$</td>
<td>0.173</td>
<td>NA</td>
<td>0.171</td>
</tr>
<tr>
<td></td>
<td>$10^{-1}$</td>
<td>1.726</td>
<td>NA</td>
<td>1.730</td>
</tr>
<tr>
<td>third</td>
<td>$10^{-3}$</td>
<td>0.010</td>
<td>NA</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>$10^{-2}$</td>
<td>0.110</td>
<td>NA</td>
<td>0.108</td>
</tr>
<tr>
<td></td>
<td>$10^{-1}$</td>
<td>1.098</td>
<td>NA</td>
<td>1.191</td>
</tr>
</tbody>
</table>

is noticed between the findings, validating the analysis performed in this work and corroborating the dependence of acoustic damping on the physical properties of the coupled system. The key parameter on which the following analysis is centred is the acoustic radiation resistance $R_{rad}$, which is nothing but the equivalent of the damper in the under-damped harmonic oscillator.

In (41), an expression for the acoustic radiation resistance of a one-sided flat plate subject to excitation at frequency $f$ higher than a certain critical frequency $f_c$, is given by

$$R_{rad,\text{plate}} = \frac{\rho_a v_a A}{\sqrt{1 - \frac{f_c}{f}}} \quad \text{for} \quad f > f_c$$  \hspace{1cm} (5.51)

For our problems, the radiation resistance is obtained through the following evaluation

$$R_{rad} = 2m\zeta\omega$$  \hspace{1cm} (5.52)

Applying Eq. (5.52) on the spherical systems, we obtain after simplifying the
expressions of $\zeta$ by suppressing higher order terms

\begin{align}
R_{\text{rad,shell}} &= \rho_a \vartheta_a A \quad (5.53a) \\
R_{\text{rad,sphere}} &= \frac{2}{3} \rho_a \vartheta_a A \quad (5.53b)
\end{align}

where $A$ is the surface area in contact with the acoustic medium. The direct dependence of the resistance value on the $\rho_a \vartheta_a A$ factor is noticed in the spherical problems as it is the case with the flat plate one. As such, we conjecture that acoustic damping in both natural and forced excitations for general structures is dominated by this factor. For the case of forced excitations, frequency effects arise but can be lumped in a secondary proportionality factor as shown in Eq. (5.51).
In this chapter, we continue the study of elastodynamic-acoustic interaction in which we incorporate the isotropic fractal model in the fluid medium. In order to formulate the acoustic solution, it is necessary to derive the Sommerfeld condition for this fractal model. The thin shell problem is considered and radiation damping is demonstrated. The mathematical procedure of Chapter 5 (Laplace transform method) is applied to obtain the coupled response. In addition to the classical physical and geometric properties of the media, the Hausdorff dimension of the fractal model affects the damping coefficient of the acoustic radiation.

6.1 Sommerfeld Condition

Our focus in this section, is on deriving the Sommerfeld condition involved in the wave propagation in infinite isotropic fractal media. The approach we follow to derive the formulae is identical to that applied in the continuous case. The difference is purely mathematical rising from the altered form of the wave equation under scrutiny.

Consider the infinite isotropic fractal domain $W_D$ that is hollow from inside with inner boundary $S_{in}$ and infinite boundary $S_\infty$. For simplicity, we assume
the entire boundary \( S_T = S_{in} \cup S_\infty \) to be continuous for a reason that will be identified later. We recall from Chapter 2, the fractal Helmholtz equation to be

\[
\nabla^2_D u + k^2 u = 0
\]  

(6.1)

The general solutions to this equation are the fractal radial harmonic functions of the first and second kind \( F^{(1)}_\nu \) and \( F^{(2)}_\nu \). For the ease of the analysis, we introduce the fractal Hankel functions of first and second kind \( G^{(1)}_\nu \) and \( G^{(2)}_\nu \) defined as \( i = \sqrt{-1} \)

\[
G^{(1)}_\nu = F^{(1)}_\nu + iF^{(2)}_\nu \quad \text{(6.2a)}
\]

\[
G^{(2)}_\nu = F^{(1)}_\nu - iF^{(2)}_\nu \quad \text{(6.2b)}
\]

or more explicitly,

\[
G^{(1)}_\nu (r, k, D) = r^{\frac{D-4}{2}} H^{(1)}_\nu \left( \frac{\lambda kr^{D-2}}{D-2} \right) \quad \text{(6.3a)}
\]

\[
G^{(2)}_\nu (r, k, D) = r^{\frac{D-4}{2}} H^{(2)}_\nu \left( \frac{\lambda kr^{D-2}}{D-2} \right) \quad \text{(6.3b)}
\]

where \( H^{(1)}_\nu \) and \( H^{(2)}_\nu \) are the conventional Hankel functions of first and second kind respectively. Recall Eq. (2.26) of Chapter 2 to better understand the mathematical significance of the terms in the above equations.

Consider the vector field \( \vec{A} = u \nabla_D v \) where \( u \) and \( v \) are two scalar functions, we have

\[
\text{Div}_D \left( \vec{A} \right) = \nabla_D u \cdot \nabla_D v + u \nabla^2_D v
\]  

(6.4)
Recall that Gauss’s theorem for this fractal domain is given as

$$\int_{S_T} \vec{A} \cdot d\vec{S} = \int_{W_D} \text{Div}_D \left( \vec{A} \right) dV_D$$ \hspace{1cm} (6.5)$$

Substituting for $\vec{A}$ and $\text{Div}_D \left( \vec{A} \right)$ in the above equation, we obtain the fractal Green’s first identity

$$\int_{W_D} \left[ \nabla_D u \cdot \nabla_D v + u \nabla_D^2 v \right] dV_D = \int_{S_T} u \nabla_D v \cdot d\vec{S}$$ \hspace{1cm} (6.6)$$

Writing the above equation with $u$ and $v$ interchanged and then subtracting, we obtain Green’s symmetric identity

$$\int_{W_D} \left[ u \nabla_D^2 v - u \nabla_D^2 v \right] dV_D = \int_{S_T} \left[ u \nabla_D v - v \nabla_D u \right] \cdot d\vec{S}$$ \hspace{1cm} (6.7)$$

If $u$ and $v$ satisfy the fractal Helmholtz equation, the singularity present at the origin ($r = 0$) alters the volume integral in such a way that the solution at a given point $P$ in the domain can be evaluated using the surface integral through the following relation

$$-4\pi u \left( P \right) = \int_{S_T} \frac{1}{c_V} \frac{\partial r}{\partial n} \left[ \frac{\partial v}{\partial r} v - v \frac{\partial u}{\partial r} \right] dS$$ \hspace{1cm} (6.8)$$

The Sommerfeld condition states that infinity can never be a source of energy, it must always be an energy sink. Therefore, the solution within the domain cannot physically depend on the interaction at infinity. As a result, the integral over the infinite surface must vanish from the above equation.
is possible if and only if the integrand is set identically to zero. Thus

$$\lim_{r \to \infty} \frac{1}{c \nu} \frac{\partial}{\partial n} \left[ u \frac{\partial v}{\partial r} - v \frac{\partial u}{\partial r} \right] = 0 \quad (6.9)$$

must be enforced for any function \( v \) satisfying the fractal Helmholtz equation. We chose \( v \equiv G^{(1)}_\nu \), and after an extensive algebraic work involving asymptotic approximations for Hankel functions and their derivatives at large arguments, we come up with the following condition

$$\lim_{r \to \infty} r^{4-D} \left[ \frac{\partial u}{\partial r} - \frac{i \lambda k}{r^{D}} u \right] = 0 \quad (6.10)$$

Our first remark is that for the case \( D = 3 \), the Sommerfeld condition for 3d continua is recovered. In addition, we detected that the above condition can be expressed in terms of the fractal derivative in the form of

$$\lim_{r \to \infty} r \left[ \nabla^D u - i k u \right] = 0 \quad (6.11)$$

which further corroborates the mathematical consistency whereby our model collapses to the continuous one once the fractal derivative is replaced with the conventional one, and the reverse is also true.

Now that the Sommerfeld condition has been established, the consideration of coupled problems is feasible, particularly when it comes to identifying the physically meaningful acoustic solution.

### 6.2 Thin Shell Vibration

In this section, we consider the first problem we have treated in the previous chapter while introducing the isotropic fractal model which was thoroughly
investigated in Chapter 2. The study of this ideal problem is of great relevance when it comes to understanding the interaction of fractal media with their surroundings which could be of continuous or even fractal nature. A clear example of this interaction is the human brain problem. The brain has a very complex structure which is composed of constitutively heterogeneous parts. The heterogeneity is even extended to geometric properties whereby the cerebro-spinal fluid (CSF) is known to have a fractal nature, whereby the outer skull is a continuous elastic solid. Therefore, the understanding of elastodynamics in fractal media, although necessary, stays futile unless incorporated in multi-physics problems where interactions, similar to those occurring in real applications, are investigated. Our goal of considering this coupled problem is to first disseminate the applicability of the isotropic fractal model we have explored, and second, to promote the general understanding of an acoustic-solid problem, thoroughly discussed in continuum mechanics but not in fractal mechanics.

The schematic layout of the problem is shown in Fig. 6.1. The radial
symmetry is not affected by the fractal setting, and the kinematics is identical
to that of the continuous problem. The governing equations, which should
be straightforward by now, are stated as

\[ \frac{d^2 u}{dt^2} + \left( \frac{\vartheta_s}{R} \right)^2 u = -\frac{p_{\text{out}}}{b\varrho_s} \]  
(6.12a)

\[ \frac{\partial^2 p}{\partial t^2} = \left( \frac{\vartheta_a}{\lambda} \right)^2 r^{4-2D} \left[ (5 - D) \frac{\partial p}{\partial r} + r^2 \frac{\partial^2 p}{\partial r^2} \right] \quad r > R \]  
(6.12b)

\[ \frac{d^2 u}{dt^2} = -\frac{1}{\rho_a} \nabla^D p \big|_{r=R} = -\frac{1}{\rho_a} \frac{1}{c_V} \frac{\partial p}{\partial r} \big|_{r=R} \]  
(6.12c)

Transforming into the Laplace domain, Eq. (6.12b) becomes the fractal
Helmholtz equation with unknown \( \hat{P}(r, s) \),

\[ \nabla^D_\phi \hat{P} - \left( \frac{s}{\vartheta_a} \right)^2 \hat{P} = 0 \]  
(6.13)

The general solution to this equation is

\[ \hat{P}(r, s) = C(s) G^{(1)}_{\nu} \left( r, -\frac{i s}{\vartheta_a}, D \right) + D(s) G^{(2)}_{\nu} \left( r, -\frac{i s}{\vartheta_a}, D \right) \]  
(6.14)

in which \( \nu = \frac{4-D}{2(D-2)} \) because the propagated wave is purely radial. The
corresponding Sommerfeld condition becomes

\[ \lim_{r \to \infty} r^{4-D} \left[ \frac{\partial \hat{P}}{\partial r} - \frac{\lambda s}{\vartheta_a r^{3-D}} \hat{P} \right] = 0 \]  
(6.15)

The satisfaction of the Sommerfeld condition requires the vanishing of \( G^{(1)}_{\nu} \)
from the general solution and this can be achieved only if we set \( C(s) \equiv 0 \).

On account of the Laplace forms of Equations (6.12a) and (6.12c), we
produce the following algebraic relation involving no unknown but $\hat{U}$

$$\frac{s^2 \hat{U} - u(0) + \left(\frac{\theta}{\rho_a} \right)^2}{s^2 \hat{U} - u(0)} = \frac{\rho_a c_v \hat{P}(R, s)}{b \rho_s \frac{\partial \hat{P}}{\partial r}|_{r=R}} = \frac{\rho_a \frac{\partial}{\partial \nu} G^{(2)}_{\nu}(R, -\frac{i \nu}{\rho_a}, D)}{i s b \rho_s G^{(2)}_{\nu+1}(R, -\frac{i \nu}{\rho_a}, D)}$$

(6.16)

Using the above equation, we can solve for $\hat{U}(s)$. Transforming the expression back to the time domain is not straightforward, it requires finding the denominator poles which can be performed only using a complex root-finding algorithm. Nevertheless, we can identify the behaviour of the response at early times by considering an asymptotic form of the Laplace solution at large $s$. In particular, if we consider the following limit

$$\lim_{s \to \infty} \frac{G^{(2)}_{\nu}(R, -\frac{i \nu}{\rho_a}, D)}{G^{(2)}_{\nu+1}(R, -\frac{i \nu}{\rho_a}, D)} \approx -i \left(1 + \frac{\partial_a}{\lambda s R^{D-2}}\right),$$

the above expression can be considerably simplified to a linear system response. In brief, the asymptotic solution for the displacement at early times become

$$\hat{U}(s) \approx \frac{s^2 + 2 \zeta_0 s + \theta}{s^4 + 2 \zeta_0 s^3 + s^2(\theta + 1)}$$

(6.17)

in which

$$\zeta_0 = \frac{1}{2} \frac{\rho_a}{\rho_s} \frac{\partial_a R}{\partial_s b}$$

(6.18)

$$\theta = 2 \zeta_0 \frac{\partial_a}{\partial_s} \frac{1}{\lambda R^{D-3}}$$

(6.19)

As noticed, $\zeta_0$ is independent of any fractal effects, while $\theta$ is, and for $D = 3$, fractal effects are suppressed from the solution. Most importantly, we confirmed that fractal effects are influential to shaping the shell displacement and their negligence renders the solution erroneous.
7.1 Contributions

In this research work, we explored the topic of fractal mechanics by formulating a field theory in which the fundamental conservation laws in mechanics are derived in fractal setting. We applied the dimensional regularization on the fractal model to seek a homogenized continuous model which can be treated mathematically like any other problem in continuum mechanics. The main focus of our research was the topic of elastodynamics and wave propagation in which we employed both analytical and numerical procedures in pursuing the investigation. The remarkable consistency observed in the results demonstrates the applicability of both approaches to solving the problems.

In exploring the wave propagation in isotopic fractal media, we achieved the modal decomposition in the spherical coordinate system, and we introduced the fractal radial harmonic functions of first and second kind as general solutions to the radial mode equation. We treated a BVP problem analytically and developed an FEM solver in which transient simulations of arbitrary BC can be conducted.

For the elastodynamics in anisotropic fractal solids, we first explored the propagation of dilatational waves in the constitutive setting of Hooke’s law. We realized the modal decoupling and solved the universal modal equation...
(fractal Helmholtz equation) in terms of fractal harmonic functions. The latter were applied in solving three eigenvalue problems of different BC. We have also implemented a computational method to analyse these problems, whereby an FEM solver was developed and validated, and the transient response corresponding to modal excitations was evaluated. Recognizing the necessity to utilize a non-classical elasticity theory to explore the generalized problem, a 3d elastodynamic micropolar fractal model is developed. For this model, the analytical approach is feasible only for those problems whose kinematics is prescribed in such a way as to decouple the field equations. Four of such problems, which are of eigenvalue type, were analysed using modal decomposition. The pure dilatational and the pure torsional wave propagation problems resulted in the same modal equation, the 1d fractal Helmholtz equation. The in-plane and out-of-plane problems required the solution of the 2d fractal Helmholtz equation. We then constructed a numerical solver based on the FDM, and, for all four problems we conducted transient simulations for the first modal excitations.

Having developed and validated ideal models by which the wave propagation in isotropic (fluid-like) fractal media and anisotropic fractal solids can be simulated, the trend to pursue an investigation about the acoustic-solid interaction in fractals becomes feasible. The complexity of such a problem mandates its discussion in the continuum first where the mathematical procedures applied in deriving the solution are better conceived. Indeed, we considered two spherical structural-acoustic models and demonstrated the phenomenon of acoustic radiation damping. Here, the under-damped linear harmonic oscillator accurately depicts the energy dissipation in the coupled system. The damping coefficients obtained by various damping extraction methods revealed the dependence of acoustic damping on the physical proper-
ties of the media (mass densities and speed of sound) in addition to being inversely proportional to the wavenumber of the mode under excitation. Beside its significance in evaluating damping in elastodynamic-acoustic interaction, this work represents a useful reference in validating complex multi-physics simulations. For the fractal setting, we substituted the fluid continuum with our isotropic fractal model and simulated the thin shell problem. The reproduction of the Sommefeld radiation condition in fractal media was first realized to distinguish the physically acceptable acoustic solution. Fractal effects appeared in the coupled response of the solid part and acoustic damping now further depends on $D$ in addition to the other parameters depicted in the continuous simulation.

Finally, this research has accomplished a considerable advancement in the currently active exploration of the mechanics of fractal media. Its significance lies in (i) the rigorous mathematical approach in formulating the conservation laws and deriving the field equations for general problems, and (ii) verification of the numerical procedures to solve such problems. Upon the findings achieved in this work, new horizons in fractal mechanics involving real life applications, can be pursued.

7.2 Future Trends

The overall achievements of this work, mainly the establishment of field theories to characterize the elastodynamics of fractal media and its demonstration on idealized problems, opens the gate for new horizons in fractal mechanics, mainly, the pursuit of problem solving in real life applications. Indeed, the essence on which fractal mechanics is founded, is its capability to treat certain problems that cannot be investigated using conventional continuum mechan-
ics. In the following, we present some of these problems, highlighting the applicability of the field theory we introduced in conducting the analysis.

As discussed in Chapter 1, the biological environment is abundant in fractal patterns. In this regard, the simulation of the human brain when subject to shocks (impact loading) is a meaningful illustration of fractal mechanics utilization. The human brain has a constitutively heterogeneous structure, necessitating a multi-disciplinary modelling process to describe the problem. But most importantly, the brain is known to have a very irregular geometry, thus the incorporation of fractal models accounting for this irregularity, enhances the overall effectiveness of the simulation. The discussion of the multi-physics idealized problem in Chapter 6 where a single fractal model is adopted, indicates the influence of the fractal property onto the coupled response. This finding, despite being preliminary, hints to the necessity of incorporating fractal models in conducting the real brain simulation whereby the resulting enriched analysis portrays a more accurate dynamic behavior for this biological system.

An additional application for the field theories pertaining to these fractal models is the simulation of waves in underground structures. This type of media is composed of random and irregularly shaped rocks; it exhibits a fractal nature within a certain scale range. Explosions in these structures (e.g. mining activities or weaponry accidents) trigger a blast wave which could lead to catastrophic failure of rock layers inside these cavities. The understanding of this elastodynamic problem is best realized within the framework of fractal mechanics, in particular the micropolar anisotropic fractal solid model.

Finally, the elastodynamic analysis discussed in this work finds application in fractal geometry whereby the determining of the Hausdorff dimension $D$, can be achieved using techniques relying on wave propagation. In the
modal analysis we have conducted on both isotropic and anisotropic fractal models, it is shown that the modal frequency depends on $D$. Being the case, performing the revere analysis in which the response of a fractal body due to a prescribed excitation is examined in the frequency domain, unveils the inherent properties of the system, mainly the Hausdorff dimension. This procedure constitutes a novel technique in evaluating $D$, which to date is determined only through statistical methods relying on advanced imaging.
The spherical fractal Helmholtz equation with variable whose independent variable the radial distance \( r \), is given as

\[
r : r^2 F'' + (5 - D) r F' + \left[ (\lambda k)^2 r^{2D-4} - l(l + 1) \right] F = 0 \tag{A.1}
\]

A smart transformation to eliminate the non-integer power is to apply the following transformation: \( y = r^{D-2} \). The derivatives become

\[
F' = \frac{dF}{dr} \longrightarrow (D - 2) \frac{dF}{dy} r^{D-3}
\]
\[
F'' = \frac{d^2F}{dr^2} \longrightarrow (D - 2)^2 r^{D-3} \left[ r^{D-3} \frac{d^2F}{dy^2} + \frac{1}{r} \frac{D - 3}{D - 2} \frac{dF}{dy} \right] \tag{A.2}
\]

substituting the new derivatives into Eq. (A.1), we obtain the following equation

\[
y^2 (D - 2)^2 F'' + 2 (D - 2)^2 y F' + \left[ (\lambda k)^2 y^2 - l(l + 1) \right] F = 0 \tag{A.3}
\]

We propose a solution to the above equation of the form \( F(y) = y^\sigma G(y) \). The resulting ODE for \( G \) becomes

\[
y^2 (D - 2)^2 G'' + \frac{2}{D - 2} \left[ \sigma (D - 2) + 1 \right] y G' + \left[ \frac{2\sigma}{D - 2} + \sigma (\sigma - 1) + \frac{(\lambda k)^2 y^2 - l(l + 1)}{(D - 2)^2} \right] G = 0 \tag{A.4}
\]
Setting $\frac{2}{D-2} [\sigma (D-2) + 1] = 1$, thus $\sigma = \frac{D-4}{2(D-2)}$, we obtain a Bessel equation given as

$$y^2 G'' + y G' + \left[ \frac{(\lambda k)^2 y^2}{(D-2)^2} - \nu^2 \right] G = 0 \quad (A.5)$$

where $\nu = \sqrt{(4-D)^2 + 4l(l+1)}$. The general solutions for this equation are the Bessel functions of the first and second kind of order $\nu$, thus

$$G_1 (y) = J_\nu \left( \frac{\lambda k y}{D-2} \right)$$

$$G_2 (y) = Y_\nu \left( \frac{\lambda k y}{D-2} \right) \quad (A.6)$$

Expressing the solution of $F (r)$, we have

$$F^{(1)}_\nu (r, k, D) = r^{\frac{D-4}{2}} J_\nu \left( \frac{\lambda k r^{D-2}}{D-2} \right) \quad (A.7a)$$

$$F^{(2)}_\nu (r, k, D) = r^{\frac{D-4}{2}} Y_\nu \left( \frac{\lambda k r^{D-2}}{D-2} \right) \quad (A.7b)$$
The 1d fractal Helmholtz equation is an ordinary differential equation (ODE) expressed in (a) compact and (b) expanded forms as follows

\[ \nabla^D \nabla^D F + k^2 F = 0 \tag{B.1a} \]

\[ \frac{d^2 F}{dx^2} - \frac{1}{c} \frac{d F}{dx} \frac{dc}{dx} + k^2 c^2 F = 0, \quad c = D (L - x)^{D-1} \tag{B.1b} \]

An obvious transformation to conceal the \( L \) term is first applied, it is given as

\[ y = L - x \tag{B.2} \]

the resulting ODE in terms of \( y \) becomes

\[ y^2 \frac{d^2 F}{dy^2} - (D - 1) y \frac{dF}{dy} + k^2 D^2 y^{2D} F(y) = 0 \tag{B.3} \]

The presence of the non-integer power in the last term deters a quick insight for a solution form. At this point, two approaches can be followed to determine the solution. The first, which is less likely to be noticed, requires the application of the following transformation \( z = y^s \). The derivatives are
transformed as follows

\[
\frac{dF}{dy} \rightarrow s \frac{dF}{dz}^{(1-\frac{1}{\alpha})} \\
\frac{d^2F}{dy^2} \rightarrow s^2 z^{(2-\frac{2}{\alpha})} \frac{d^2F}{dz^2} + s (s-1) \frac{dF}{dz}^{(1-\frac{1}{\alpha})}
\]

(B.4)

and the resulting equation in \(z\) becomes

\[
s^2 z^2 \frac{d^2F}{dz^2} + sz (s-D) \frac{dF}{dz} + k^2 D^2 z^{\frac{2D}{D}} F = 0
\]

(B.5)

When setting \(s = D\), the equation reduces to a one of harmonic type in \(z\), which admits a general solution given by

\[
F(z) = C_1 \cos (kz) + C_2 \sin (kz)
\]

(B.6)

Rewriting the solution in terms of the original variable \(x\), it becomes

\[
F(x) = C_1 \cos \left[ k (L-x)^D \right] + C_2 \sin \left[ k (L-x)^D \right]
\]

(B.7)

We denote the two components of the general solution as the fractal harmonic functions \(F_1(x, k)\) and \(F_2(x, k)\) respectively.

In case the second transformation cannot be easily noticed, there exists an alternative approach in which we seek a series solution to Eq. (B.3) of the form

\[
F(y) = \sum_{n=0}^{\infty} a_n y^{(n\alpha+\beta)}
\]

(B.8)

where \(\alpha\) and \(\beta\) are two independent constants, in general, non-integer. The attempt to have a series solution with pure integer powers fails because of the presence of a non-integer power in the ODE itself. The substitution of
this series form into the ODE produces the following relations

\[ \alpha = 2D \quad (B.9a) \]

\[ \beta = \begin{cases} 
0 & \text{or} \\
D & 
\end{cases} \quad (B.9b) \]

For the first case, \( \beta_1 = 0 \), the recurrence relation for \( a_n \) is given as

\[ a_n = -\frac{k^2}{2n(2n-1)}a_{n-1} \quad n \geq 1 \quad (B.10) \]

from which the general form of the \( a_n \) term is deduced to be

\[ a_n = \frac{(-1)^n k^{2n} \Gamma \left( \frac{1}{2} \right)}{4^n n! \Gamma \left( n + \frac{1}{2} \right)} a_0 \quad (B.11) \]

As a result, the first homogeneous solution, \( F_1 \), in a series form becomes

\[ F_1 (y) = a_0 \Gamma \left( \frac{1}{2} \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma \left( n + \frac{1}{2} \right)} \left( \frac{ky^D}{2} \right)^{2n} \quad (B.12) \]

Recognizing that the series form of the Bessel function of first kind and order \( \nu \), \( J_\nu (y) \), is given as (18)

\[ J_\nu (y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma (n + 1 + \nu)} \left( \frac{y}{2} \right)^{2n + \nu} \quad (B.13) \]

The similarity of the forms in equations (B.12) and (B.13) permits the compaction of the first solution to \( F_1 (y) \simeq \sqrt{ky^D} J_{-0.5} (ky^D) \).

For the second case, \( \beta_2 = D \), the following recurrence relation for \( a_n \) is
obtained
\[ a_n = -\frac{k^2}{2n(2n+1)}a_{n-1} \quad n \geq 1 \quad (B.14) \]

which leads to a general form of \( a_n \) given as
\[ a_n = \frac{(-1)^n k^{2n} \Gamma \left( \frac{1}{2} \right)}{4^n n! \Gamma \left( n + \frac{3}{2} \right)} a_0 \quad (B.15) \]

Consequently, the second homogeneous solution, \( F_2 \), admits the following series form
\[ F_2 (y) = a_0 \Gamma \left( \frac{1}{2} \right) y^D \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma \left( n + \frac{3}{2} \right)} \left( \frac{ky^D}{2} \right)^{2n} \quad (B.16) \]

Similarly to the first solution, the series form of \( F_2 \) is matched to that of Eq. (B.13), resulting in having \( F_2 (y) \simeq \sqrt{ky^D} J_{0.5} (ky^D) \). In conclusion, the general solution is
\[ F (y) = \sqrt{ky^D} \left[ J_{-0.5} (ky^D) + J_{0.5} (ky^D) \right] \quad (B.17) \]

We here note that the above solution is independently achieved in chap. 6 of where a more general ODE is treated in term of Bessel functions.

In the following steps, we will bridge the general solutions from their Bessel function form to their equivalent harmonic form. For this proof, we rely on two important identities concerning Bessel functions. The first, is the Bessel integral given as
\[ J_{\nu} (y) = \int_0^y (y^2 - \tau^2)^{\nu-0.5} \cos \tau d\tau \]
\[ 2^{\nu-1} \Gamma \left( \nu + \frac{1}{2} \right) \sqrt{\pi} y^\nu \quad (B.18) \]
and the second, is a recurrence relation defined as

\[ J_{\nu-1}(y) + J_{\nu+1}(y) = \frac{2\nu}{y} J_{\nu}(y) \tag{B.19} \]

From Eq. (B.18) we obtain \( J_{0.5}(y) = \sqrt{\frac{2}{\pi}} \frac{\sin y}{\sqrt{y}} \) and \( J_{1.5}(y) = \sqrt{\frac{2}{\pi}} \frac{\sin y - y \cos y}{y \sqrt{y}} \).

Incorporating the above relations into Eq. (B.19), we obtain \( J_{-0.5}(y) = \sqrt{\frac{2}{\pi}} \frac{\cos y}{\sqrt{y}} \).

Substituting the harmonic form of \( J_{-0.5} \) and \( J_{0.5} \) into Eq. (B.17), the solution form of Eq. (B.7) is reproduced.

The 2d fractal Helmholtz equation is expressed as

\[ \nabla_1^D \nabla_1^D F + \nabla_2^D \nabla_2^D F + k^2 F = 0 \tag{B.20} \]

By setting \( F(x_1, x_2) = G(x_1) \cdot H(x_2) \) and \( k^2 = k_1^2 + k_2^2 \), a separation of variables is achieved. The outcome is two independent 1d fractal Helmholtz equations for \( G \) and \( H \) with eigenvalues \( k_1 \) and \( k_2 \) respectively. Figure B.1 show the contour plots for some modes on the \((x_1,x_2)\) section (Sierpiński carpet), where homogeneous Dirichlet BCs are applied on the entire boundary. The first mode was excited for \( \phi_3 \) in the in-plane problem, and for \( \phi_1, \phi_2 \) in the out-of-plane problem. In addition, the mode shapes are considered along the \((x_1,x_3)\) surface, their corresponding contour plots are shown in Figure B.2.
Figure B.1: Mode shapes corresponding to solutions of the 2d fractal Helmholtz equation where \( L_1 = L_2 = 1 \), \( D_1 = D_2 = \frac{1}{3} \left( \ln \frac{18}{\ln 3} \right) \).
Figure B.2: Mode shapes corresponding to solutions of the 2d fractal Helmholtz equation where $L_1 = 1$, $L_3 = 2$ and $D_1 = \frac{1}{3} \ln \frac{18}{\ln 3}$, $D_3 = \frac{\ln 2}{\ln 3}$.
REFERENCES


