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GRADUALLY VARIED FLOW IN UNIFORM CHANNELS
ON MILD SLOPES

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UNIVERSITY OF ILLINOIS BULLETIN

Volume 50, Number 28; November, 1952. Published seven times each month by the University of Illinois. Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Office of Publication, 358 Administration Building, Urbana, Ill.
ABSTRACT

The theory of varied flow was first postulated in a complete and comprehensive manner by J. M. Belanger. The fundamental principles were so well covered that little has since been added to modify the differential equation of gradually varied flow in its original form. All subsequent investigations have been confined chiefly to the methods of solution of the differential equation by direct, approximate, and graphical integration.

In this bulletin three well-known methods of integration are discussed with special reference to their advantages and limitations, and a new method is introduced. Recent investigations have tended to favor direct integration in order to dispense with the cumbersome succession of steps required by the approximate and graphical methods. Following the current trend, a more convenient procedure is introduced in which limitations on the existing methods of direct integration are removed, but embodying at the same time all their advantages. The practical application of the proposed method to the determination of surface profiles of flow in various channels is given. In the special cases of the rectangular and circular channels charts have been prepared to facilitate the profile computations.

The validity of the proposed method is determined by comparing the theoretical results with actual profile observation under various conditions of flow in experimental flumes. The comparison of test data is also extended to the theoretical results computed by the more widely accepted methods.
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I. INTRODUCTION

The study of the hydraulics of flow in open channels is not always subject to analysis in an exact or rigorous manner, because of the large number of fundamental variables involved. Under these circumstances it becomes necessary to consider only the effect of those variables having a predominant influence upon the flow. Indeed, the consideration of predominant forces not only simplifies the solution of many practical problems, but also permits the use of models in the analysis of flow characteristics of prototype hydraulic structures. That the results obtained from such simplification cannot be regarded as better than approximate is evident, but frequently the limitations imposed by this simplification are compatible with the desired accuracy of results in practice.

In the case of steady flow in open channels the forces present are those due to gravity, inertia, viscosity, elasticity, and surface tension, the last two being inconsequential except in small models. Gravity forces are generally taken into consideration, while on the other hand either the viscosity or inertia forces may be neglected in the solution of many practical problems in open-channel flow. These problems involve either one of two broad classes of motion: rapidly varied flow in which gravity and inertia are the predominant forces; and gradually varied flow, in which the flow pattern is determined primarily by gravity and viscosity forces. Rapidly varied flow is usually associated with rapid or abrupt changes in the depth of flow occurring in a comparatively short distance, such as the hydraulic drop or the hydraulic jump. Gradually varied flow, on the other hand, is characterized by the absence of a disrupted surface over a considerable reach of channel, the changes of depth being so gradual that the flow may be considered essentially parallel. Most of the problems in gradually varied flow are concerned with the determination of surface profiles of flow in both natural and uniform channels. The present work is confined mainly to this phase of open-channel flow.

I. Purpose and Scope of Research

The purpose of this research was to develop a method of computing surface profiles of gradually varied flow in uniform channels whereby the results could be obtained more directly
than are possible with the existing methods. To determine the accuracy of the proposed method, profiles of flow were recorded in experimental channels of rectangular, trapezoidal, and circular cross-sections under various conditions of flow. The test data are compared with the results computed by the proposed method and also by other methods that are in use. This investigation was confined mainly to backwater and downdrop curves in uniform channels on mild slopes, inasmuch as they form the more important classes of surface profiles encountered in practice.

2. Notation

The nomenclature used in this bulletin conforms essentially to that prepared by the Committee of the American Standards Association on Letter Symbols for Hydraulics, approved in January, 1942. The symbols, defined when they first appear, are summarized below for reference.

\[ A: \text{ total cross-section area} \]
\[ b: \text{ width of channel} \]
\[ b_w: \text{ width of water surface} \]
\[ C, C_0: \text{ coefficient of flow (Chezy)} \]
\[ C_0: \text{ exponent in Mononobe’s functions} \]
\[ C_m: \text{ constant in the proposed varied-flow equation to account for nonuniform velocity distribution} \]
\[ C_1: \text{ a parameter in the proposed varied-flow equation} \]
\[ D: \text{ diameter of circular conduit} \]
\[ E: \text{ energy per unit weight} \]
\[ E_m: \text{ mean energy per unit weight} \]
\[ E_s: \text{ specific energy} \]
\[ G: \frac{Q^2}{gA^3} \left( \frac{K}{K_0} \right)^2 b_w \]
\[ g: \text{ gravitational acceleration} \]
\[ h: \text{ potential head} = \gamma + Z \]
\[ K: \text{ conveyance of a cross-section} \]
\[ k: \text{ a constant in the proposed varied-flow equation to account for eddy losses} \]
\[ l: \text{ exponent in the wetted perimeter-depth relationship} \]
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$M$: exponent in the area-depth relationship
$N$: hydraulic exponent
$n$: coefficient of roughness (Kutter and Manning)
$p$: wetted perimeter of cross-sectional area
$Q$: discharge rate of flow
$R$: hydraulic radius of cross-sectional area
$r$: ratio of depths, $y/y_0$
$r_c$: ratio, $y_c/y_0$
$S$: slope of energy grade line
$S_c$: critical slope
$S_f$: friction slope
$S_0$: slope of channel bed, $\tan \alpha_0$
$V$: mean velocity, $Q/A$
$x$: horizontal length
$y$: vertical depth from bottom of channel
$y_c$: Belanger critical depth
$y_0$: depth for uniform flow
$Z$: elevation above datum
$\alpha_0$: angle of an inclined channel bed measured from the horizontal
$B$: Bakhmeteff varied-flow function
$\Delta$: used with $x$, $y$, $r$, $B$ to indicate the difference between two particular values
$\phi$: function of
$\Phi_B$: Bresse backwater function
$\Phi_1$, $\Phi_2$: Mononobe backwater function
$\Psi_1$, $\Psi_2$: Mononobe dropdown function

Except for the classification of surface profiles, the subscripts 1 and 2 are used to refer, respectively, to upstream and downstream sections; critical conditions are designated by the subscript $c$, and uniform conditions by the subscript $0$. 
II. STEADY FLOW IN OPEN CHANNELS

3. Uniform Flow and Varied Flow

Flow in a channel is called uniform when certain elements related to the flow and the channel remain constant from section to section. These elements are the depth of flow, the cross-sectional area of flow, the velocity of flow and the slope of the channel. Strictly speaking then, uniform flow can take place only in prismatic channels of constant slope. Although varied or nonuniform flow is the rule in nature, it can also be made to take place in prismatic channels.

This bulletin deals primarily with certain phenomena connected with varied flow. Varied flow must also be distinguished from variable flow. In varied flow the velocity and depth, while varying from section to section, do not vary with time. In variable flow, however, these elements vary with time as in the case of waves. There are two main types of varied flow - one in which the depth of the water increases gradually downstream, and the other in which the depth of the water decreases gradually downstream. The first type gives rise to a surface profile curve which is called a backwater curve. The second type results in a dropdown curve.

It will sometimes happen that the conditions of flow are such that there may be a sudden break in the surface of the backwater curve. We then get what is called a "hydraulic jump." Similarly a sudden break in the surface of a dropdown curve gives rise to a "hydraulic drop."

4. Energy Relations in Varied Flow

In order to derive the equation of gradually varied flow, it is first necessary to review the various energy relations existing in uniform channels during the gradually varied flow. Consider a liquid flowing in a prismatic channel at an angle \( \alpha_0 \) and having a constant slope \( S_0 = \tan \alpha_0 \) (Fig. 1). Let it be assumed that the velocity at any particular section is uniform over the cross section. Then, at any cross section, if \( E_m \) equals the mean energy per unit weight of liquid,

\[
E_m = h + \frac{v^2}{2g}
\]
where \( h = y + Z \) and \( \nu^2/2g \) represents the average kinetic energy of the stream.

Differentiating with respect to \( x \), the displacement coordinate,

\[
\frac{dE_m}{dx} = \frac{dh}{dx} + \frac{d}{dx} \left( \frac{\nu^2}{2g} \right)
\]

(2)

Consider the flow to be uniform; i.e., \( V \) is constant. Then Eq. 2 becomes

\[
\frac{dE_m}{dx} = \frac{dh}{dx}
\]

(3)

Equation 3 states that in uniform flow, the rate of change of energy is equal to the rate of change of the surface above the datum. It should be remembered that \( dh/dx \) will be numerically negative, since the change of energy in the downstream direction is negative. Since in uniform flow, \( \frac{dh}{dx} = -S_0 \), the slope of the channel,

\[
\frac{dE_m}{dx} = -S_0
\]

(4)

From Eq. 4 it may be seen that in uniform flow the work done by gravity is just sufficient to overcome the resistances of the channel.

In uniform flow the liquid flows at a constant depth for a given rate of discharge \( Q \). This "normal depth" is designated by the symbol \( y_0 \). It should be remembered that for a given channel and a given discharge, \( y_0 \) is one of the characteristics of the channel.

Returning to Eq. 2 for the general case of nonuniform flow, \( V \) will be a function of \( x \). At any depth of flow
\[ V^2 = \left( \frac{Q}{A} \right)^2 \]

where \( A \) is a function of \( y \) and \( Q \) is constant. Differentiating with respect to \( x \)

\[
\frac{d}{dx} \left( V^2 \right) = Q^2 \frac{d}{dx} \left( \frac{1}{A^2} \right) = -2 \frac{Q^2}{A^3} \frac{dA}{dy} \frac{dy}{dx}
\]

(5)

Substituting Eq. 5 into Eq. 2 gives

\[
\frac{dE}{dx} = \frac{dh}{dx} - \frac{Q^2}{gA^3} \frac{dA}{dy} \frac{dy}{dx}
\]

(6)

To evaluate \( dA/dy \), consider Fig. 2, the cross-section of a prismatic channel with a surface width \( b_w \). Clearly,

\[
dA = b_w \, dy \quad \text{or} \quad \frac{dA}{dy} = b_w
\]

(7)

Substitution of Eq. 7 into Eq. 6 yields

\[
\frac{dE_a}{dx} = \frac{dh}{dx} - b \frac{Q^2}{gA^3} \frac{dy}{dx}
\]

(8)

Next, \( dh/dx \) can be evaluated in terms of \( y \) and \( x \) if in Fig. 3 the slope is considered so flat that \( \sin \alpha_0 \) can be assumed equal to the slope \( S_0 \) and \( \cos \alpha_0 \) can be assumed unity, then

\[
\frac{dh}{dx} = S_0 dx - dy
\]

(9)

Noting that \( dh/dx \) is negative, if Eq. 9 is substituted into Eq. 8
It is now necessary to find an expression for the resistance loss in varied flow \( \frac{dE_m}{dx} \), in order to solve Eq. 10.

5. Evaluation of Resistance Losses in Varied Flow

While it is probably true that the resistance losses of varied flow follow a somewhat different law than that of uniform flow, in most practical cases where the change in depth is gradual, it is close to the truth if the following assumption is made: the rate of resistance losses in varied flow at any section of depth \( y \) is identical with the rate of loss which would take place if the flow in the same channel were uniform with the same discharge and at the same depth. Uniform flow under these conditions can take place only in a channel whose bottom slope is different from the bottom slope of the channel in which the gradually varied flow is occurring. The bottom slope of the uniform flow channel which will give the desired rate of loss due to frictional resistances can be obtained from any empirical equation of uniform flow. This resistance slope \( S_f \) can be obtained, for example, from the Manning formula\(^{(1)}\) in which the foot is used as the unit length.

\[
S_f = \left( \frac{n}{1.486} \right)^2 \left( \frac{V}{R^{2/3}} \right)^2 = \left( \frac{n}{1.486} \right)^2 \frac{Q^2}{A^2 R^{4/3}} \tag{11a}
\]

In this case the frictional losses \( S_f \) are inversely proportional to some function of the depth of flow. For developmental purposes Eq. 11a is used in the form...
\[ S_f = \frac{Q^2}{K^2} \] (11b)

where \( K \) is called the "conveyance" of the channel and is a function of depth. The meaning of Eq. 11b should be clearly understood. It states that for a given channel with a given discharge there is a certain slope \( S_f \) which is just sufficient to overcome the resistances. The bottom slope \( S_0 \) of the gradually varied flow channel may be greater or less than \( S_f \). The frictional resistance \( S_f \) is numerically equal to \( \frac{dE_m}{dx} \) but is different in sign, since the former is a rate of loss of energy whereas the latter is expressed as a rate of change of energy. Using the relationships expressed in Eqs. 10 and 11, the equation of gradually varied flow in uniform channels can be formulated.
III. EQUATION OF GRADUALLY VARIED FLOW IN UNIFORM CHANNELS

6. Formulation of the Equation of Varied Flow

Substituting \(-S_f\) from Eq. 11 for \(dE_m/dx\) in Eq. 10 gives

\[
-S_f = -S_0 + \frac{dy}{dx} - b_w \frac{Q^2}{gA^3} \frac{dy}{dx}
\]

(12a)

or transposing

\[
\frac{dy}{dx} = \frac{S_0 - S_f}{(1 - \frac{b w Q^2}{gA^3})}
\]

(12b)

and

\[
dx = \frac{(1 - \frac{b w Q^2}{gA^3})}{S_0 - S_f} \, dy
\]

(13)

Equation 13 expresses the change in \(x\) in terms of the change in \(y\) and functions of \(y\) and is the required differential equation for nonuniform flow in prismatic channels. Although Eq. 13 is a first-order differential equation with variables \(x\) and \(y\) separated, its solution cannot be obtained by quadrature. For this reason various simplifying assumptions have to be made, and different authorities have suggested different assumptions, more or less useful from a practical standpoint. Before describing these solutions it is necessary to consider briefly the limitations of Eq. 13.

7. Limitations of Applicability of the Varied-Flow Equation

The equation of gradually varied flow was derived from certain simplifying assumptions; therefore, it may not be valid beyond the premises upon which the derivation is based. These assumptions are as follows:

(a) Steady flow in uniform channels on flat slopes
(b) Hydrostatic distribution of pressure intensity over a normal section
(c) Uniform distribution of velocity across the entire section of flow.

The applicability of Eq. 13 is necessarily limited to flat slopes because in the derivation\(^\text{[2]}\), \(\sin \alpha_0\) is assumed to be equal to \(\tan \alpha_0 (S_0)\) and \(\cos \alpha_0\) and \(\cos^2 \alpha_0\) are assumed to be unity. In other words, the depth of flow was assumed to be the same whether the vertical or normal direction was used. Similarly, the distance along the bottom of the inclined channel was assumed to be the same as its horizontal projection. For a channel slope of 1:10 the value of \(\cos^2 \alpha_0\) is 0.99 resulting in an error of only 1 percent.

Strictly speaking, hydrostatic distribution of pressure intensity is attained only when the normal component of acceleration is zero. This condition imposes the restrictions that the stream lines be neither divergent nor curvilinear\(^\text{[3]}\). For gradually varied flow in uniform channels, these conditions are essentially met since the changes in depth are so gradual that the flow may be considered substantially parallel. If the flow involves rapid changes in depth or direction, the acceleration effects cannot be neglected and the equation of gradually varied flow is not applicable.

The assumption of uniform velocity distribution generally involves little error, particularly in cases where the velocity head is small compared with the depth, as is the case in gradually varied flow on mild slopes. This is due to the fact that the velocity-correction factor is actually very close to unity\(^\text{[4]}\). Values of this factor under various conditions have been found to vary between 1.01 and 1.12, averaging about 1.05. If the velocity distribution deviates substantially from unity, then the velocity-correction factor must be considered in the derivation of the gradually varied-flow equation.

8. Specific Energy, Critical Flow, and Critical Slope

In order to obtain a better concept of the relationship between \(S_0\) and \(S_f\), Bernoulli's equation for expressing the conditions between two cross-sections of a stream, 1 and 2, may be stated in the form

\[
(P.E.)_1 + (K.E.)_1 = (P.E.)_2 + (K.E.)_2 + \text{(losses)}_{1-2} \quad (14)
\]

In the equation, \(P.E.\) stands for the potential energy head, \(h\); \(K.E.\) for the kinetic energy head, \(V^2/2g\); and the losses are assumed to be equal to the frictional head losses between sections 1 and 2. Transposing
(P.E.)_1 - (P.E.)_2 - (losses)_1-2 = (K.E.)_2 - (K.E.)_1 \quad (15)

For a length of channel \( dx \), we may write this relationship as

\[
S_0 \, dx - S_f \, dx = (K.E.)_2 - (K.E.)_1
\]

Equation 16 indicates that if \( S_0 \) is greater than \( S_f \), the work done by gravity is greater than that required to overcome friction and that some energy will be added to the stream. If, however, \( S_f \) is greater than \( S_0 \), some energy will be withdrawn from the store of energy in the stream. The former condition will be represented by a rise of the water surface downstream and the latter by a fall of the water surface downstream.

These relations can be made clearer by the concept of "specific energy" of the flow, \( E_s \). If we refer to the bed of the stream as datum, the specific energy of the stream at any section is:

\[
E_s = y + \frac{V^2}{2g}
\]

where \( y \) is the depth of flow at the section. The distinction between Eqs. 1 and 17 is important. Equation 1 is a measure of the mean energy of the stream above any datum; Eq. 17 is a measure of the specific energy of the stream where the bottom of the channel is used as the datum. In uniform flow, \( E_s \) is constant. In varied flow, \( E_s \) will change from section to section.

For \( dE_s/dx \) positive, the specific energy increases downstream. This means that the work of gravity exceeds that required to overcome resistance losses, and the excess adds to the specific energy. If the work of gravity is insufficient to overcome the frictional resistances, part of the specific energy is used to make up the difference.

By plotting \( E_s \) (Eq. 17) against \( y \) it may be shown that for any channel with a given discharge, there is a depth \( y \) for which \( E_s \) is a minimum. This depth is called the critical depth, \( y_c \). A given discharge \( Q \) may flow in a channel at various depths, \( y \). For each value of \( y \) there corresponds a value of the specific energy given by Eq. 17. But under no circumstances will the content of specific energy fall below a minimum value of \( E_s \), and this minimum is attained at the critical depth, \( y_c \). Substituting \( Q/A \) for \( V \) in Eq. 17, differentiating with respect to \( y \), and equating to zero gives

\[
\frac{dE_s}{dy} = 1 - \frac{Q^2}{gA^3} \frac{dA}{dy} = 0
\]
Since \( \frac{dA}{dy} = b_w \) (Eq. 7),

\[
\frac{dE_s}{dy} = 1 - \frac{b_w Q^2}{g A^3} = 0
\]

(18)

or

\[
\frac{A^3}{b_w} = \frac{Q^2}{g}
\]

(19)

In other words, the depth of flow for a given discharge is critical for the conditions expressed in Eq. 19. For a given discharge, the slope that will make the flow critical at the normal depth is called the critical slope and is designated \( S_c \).

9. Classification of Surface Profiles

Longitudinal profiles of flow in open channels may be conveniently classified according to the bed slope \( S_0 \) and the depth of flow \( y \) in the following manner.\(^5\)

<table>
<thead>
<tr>
<th>Slope</th>
<th>Designation</th>
<th>Relation of Depth, ( y ), to Uniform and Critical Depths</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adverse, ( S_0 &lt; 0 )</td>
<td>A₂</td>
<td>( y_0 &gt; y &gt; y_c )</td>
</tr>
<tr>
<td></td>
<td>A₃</td>
<td>( y_0 &gt; y_c &gt; y )</td>
</tr>
<tr>
<td>Horizontal, ( S_0 = 0 )</td>
<td>H₂</td>
<td>( y_0 &gt; y &gt; y_c )</td>
</tr>
<tr>
<td></td>
<td>H₃</td>
<td>( y_0 &gt; y_c &gt; y )</td>
</tr>
<tr>
<td>Mild, ( S_0 &lt; S_c &gt; 0 )</td>
<td>M₁</td>
<td>( y &gt; y_0 &gt; y_c )</td>
</tr>
<tr>
<td></td>
<td>M₂</td>
<td>( y_0 &gt; y &gt; y_c )</td>
</tr>
<tr>
<td></td>
<td>M₃</td>
<td>( y_0 &gt; y_c &gt; y )</td>
</tr>
<tr>
<td>Critical, ( S_0 = S_c &gt; 0 )</td>
<td>C₁</td>
<td>( y &gt; y_c = y_0 )</td>
</tr>
<tr>
<td></td>
<td>C₃</td>
<td>( y_0 = y_c &gt; y )</td>
</tr>
<tr>
<td>Steep, ( S_0 &gt; S_c &gt; 0 )</td>
<td>S₁</td>
<td>( y &gt; y_c &gt; y_0 )</td>
</tr>
<tr>
<td></td>
<td>S₂</td>
<td>( y_c &gt; y &gt; y_0 )</td>
</tr>
<tr>
<td></td>
<td>S₃</td>
<td>( y_c &gt; y_0 &gt; y )</td>
</tr>
</tbody>
</table>
If $S_0 < 0$, the channel is termed adverse $A$; if $S_0 = 0$, it is horizontal $H$; and if $S_0 > 0$, it is called sustaining and is further classified by the degree of slope. Uniform motion in open channels of sustaining slopes may proceed in the tranquil, critical, or rapid state depending on whether the uniform depth of flow $y_0$ is greater than, equal to, or less than the critical depth, $y_c$. Sustaining slopes producing uniform motion in the tranquil, critical, and rapid states are said to be, respectively mild $M$, critical $C$, and steep $S$.

![Fig. 4. Longitudinal Profiles of $M_1$ and $M_2$ Curves](image)

If the actual depth of flow $y$ lies above both the uniform-depth and critical-depth lines drawn parallel to the bottom of the channel, it is of type 1; if between these lines it is of type 2; and if below both lines, it is of type 3. The twelve types of surface profile are outlined in Table I.

Of these surface profiles, the more common types are those on mild slopes $M_1$ and $M_2$ (See Fig. 4). The former refers to the portion of the water surface lying above the uniform depth, and the latter to that lying between the uniform and critical depths. The $M_1$ curve, commonly known as the "backwater" curve, is produced when the downstream depth of flow is raised above the uniform depth by a dam or other obstruction. The $M_2$, or "dropout" curve, results when the downstream depth is lowered below the uniform depth. The upstream and downstream limits of the $M_1$ and $M_2$ curves are reached when $y = y_0$, $y = \infty$; and $y = y_0$, $y = y_c$, respectively. The conditions of flow existing at these limits may be found by consideration of the gradually varied-flow equation. That is:
\[
\frac{dy}{dx} = \frac{S_0 - S_f}{(1 - \frac{b_x Q^2}{gA^3})} \quad (12b)
\]

At the upstream limit, \( y = y_0 \); the friction slope \( S_f \) and that of the channel \( S_0 \) are identical; therefore, the left side of Eq. 12b becomes zero. This condition indicates that both the \( M_1 \) and \( M_2 \) curves approach this limit asymptotically. Theoretically, the length of these curves extends to infinity. For practical purposes, however, these curves may be assumed to end at a depth slightly above (for the \( M_1 \) curve) or below (for the \( M_2 \) curve) the uniform depth.

At the downstream limit for the \( M_1 \) curve, \( y = \infty \), the friction slope \( S_f \) becomes zero, while the denominator of the right side of Eq. 12b has a magnitude of unity. The left side of the equation, consequently, is equal to \( S_0 \). Thus, the \( M_1 \) curve approaches the downstream limit horizontally.

At the downstream limit of the \( M_2 \) curve, \( y = y_c \), the denominator of the right side of the equation becomes zero, so that the left side approaches negative infinity as a limit. Consequently, the \( M_2 \) curve should end downward in a vertical direction.

The shapes of the \( M_1 \)- and \( M_2 \)-types of surface profile are shown in Fig. 4 in greatly reduced horizontal scale. It should be emphasized that the limiting conditions serve only the useful purpose of establishing the general shape of the different types of surface curves. Before many of the limits are reached, the assumption of parallel flow may no longer be valid. Equation 13, then, cannot be correctly applied to investigate the existing state of flow under those conditions.
IV. INTEGRATION OF THE GRADUALLY VARIED-FLOW EQUATION: EXISTING METHODS

10. General Review

The theory of varied flow was first postulated in a complete and comprehensive manner by J. N. Belanger. His paper, published in 1828, "Essai sur la solution numérique de quelques problèmes, relatives au mouvement permanent des eaux courantes," contains, in part, the general differential equation for gradually varied flow and a method of solution by successive approximations. The fundamental principles were so well covered that little has since been added to modify the differential equation in its original form. All subsequent investigations have been confined mainly to the methods of solution of the gradually varied flow equation by direct, approximate, and graphical integration.

The earlier attempts to obtain an analytical solution by direct integration of the differential equation were restricted to flow in channels of special form. The case of a rectangular channel of great width was treated by Dupuit (1848) and in a somewhat different manner by Ruhlmann (1880), both ignoring the effect due to changes of velocity. The same case was presented in complete form by Bresse (1860) and subsequently by Grashof. These methods of integration were based on the Chezy equation with a constant coefficient in evaluating the friction slope. More recent solutions, based on a variable coefficient in the Chezy equation, were due to Schaffernak (1913), Ehrenberger (1914), Baticle (1921), Kozeny (1928), Schoklitsch (1920), and Gunder (1943). The case of a broad parabolic channel was treated by Tolkmitt (1898). The first attempt to arrive at a solution suitable for any type of cross-section was begun by Bakhmeteff (1932), and subsequently presented in a more complete form by Mononobe (1936).

In the solution of the differential equation for gradually varied flow by approximate integration, the channel is first divided into several reaches and the depth at the end of each reach determined by a trial-and-error procedure. A more direct solution was given by Husted (1924), in which the lengths of successive reaches were found for assumed increments of depth. Since the results are obtained by a series of successive computations, this method is generally known as the "step method."
The gradually varied-flow equation may also be solved by graphical integration.\(^{(11,\,12)}\) The graphical methods involve plotting suitable variables in rectangular coordinates in such a manner that the distance along the channel is given by the area under the curve. The area is found by means of a planimeter.

All three methods, direct, approximate and graphical integration, are used to a varying extent in computing surface profiles of flow in uniform channels.

II. Integration of the Equation of Varied Flow

Several methods, more or less exact, are available for the integration of Eq. 13. Only the more important ones are described briefly here. These are the methods of graphical integration, the step method, and the direct integration methods of Bakhmeteff and Mononobe. For discussing and comparing these methods it is convenient to express Eq. 13 in a different form. From Eq. 13

\[
\left(1 - \frac{b_0 Q^2}{g A_3^3}\right) dx = \frac{dy}{S_0 - S_f} \\
S_0 dx = \frac{dy}{1 - \frac{S_f}{S_0}} \tag{20}
\]

But

\[
S_f = \frac{Q^2}{A^2 C^2 R^{4/3}} \\
S_0 = \frac{Q^2}{A_0^2 C_0^2 R_0^{4/3}}
\]

where \(R\) is the hydraulic radius at normal depth \(y\) and \(R_0\) is the hydraulic radius at any depth of flow \(y_0\). By letting

\[
ACR^{2/3} = K \\
A_0 C_0 R_0^{2/3} = K_0
\]

where \(K\) is called the conveyance of the channel, then \(S_f\) and \(S_0\) become
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\[ S_f = \frac{Q^2}{K^2} \]

\[ S_0 = \frac{Q^2}{K_0^2} \]

And so

\[ \frac{S_f}{S_0} = \frac{K_0^2}{K^2} \] (21)

Next, let

\[ G = \frac{b_w Q^2}{gA^3} \left( \frac{k}{K_0} \right)^2 \] (22)

Substituting Eqs. 21 and 22 into Eq. 20 gives

\[ S_0 \, dx = \frac{(1-G) \left( \frac{K_0}{K} \right)^2}{1 - \left( \frac{K_0}{K} \right)^2} \, dy \] (23)

Equation 23 is a useful form of Eq. 13 and is used as the basis for discussing the various methods of integrating the equation of varied flow.

a. Graphical Integration.

This may be called an exact method, since no simplifying assumptions are involved. Equation 23 may be written in the form

\[ S_0 \, dx = dy + \frac{1 - G}{\left( \frac{K}{K_0} \right)^2 - 1} \, dy \] (24)

Integrating Eq. 24 between two sections \( x_1 \) and \( x_2 \) at which the depths are \( y_1 \) and \( y_2 \) gives

\[ \int_{x_1}^{x_2} dx = \frac{1}{S_0} \left[ \int_{y_1}^{y_2} dy + \int_{y_1}^{y_2} \frac{1 - G}{\left( \frac{K}{K_0} \right)^2 - 1} \, dy \right] \]

or

\[ \Delta x = x_2 - x_1 = \frac{1}{S_0} \left[ (y_2 - y_1) + \int_{y_1}^{y_2} \frac{1 - G}{\left( \frac{K}{K_0} \right)^2 - 1} \, dy \right] \] (25)
To evaluate the integral in Eq. 25, the reciprocal of \( \frac{K^2}{K_0^2} - 1 \) is plotted as a function of \((1-G)y\) in Fig. 5.

For any known values of \(y_1\) and \(y_2\) the shaded area in Fig. 5, obtained by a planimeter or analytically, gives the value of this integral and the distance \(\Delta x\) may be calculated. By assuming various successive values of \(y\), the required curve is obtained to any degree of accuracy desired.

![Fig. 5. Solution of Gradually Varied Flow Equation by Graphical Integration](image)

b. The Step Method

This method of integration is widely used and is the simplest to perform. Equation 13 may be written

\[
\Delta x = \frac{(1 - \frac{b_wQ^2}{gA^3})}{S_0 - S_f} \Delta y
\]

(26)

Provided \(\Delta y\) is small, an approximation of \((1 - \frac{b_wQ^2}{gA^3})\) and \(S_f\) can be obtained by evaluating these quantities as the mean of their values at the two extremes of \(\Delta x\). In other words, if \(y_1\) is the depth at \(x_1\), and \(y_2\) is the depth at \(x_1 + \Delta x\), then these quantities can be evaluated by using a value of \(y = (y_1 + y_2)/2\). Since \(\Delta y = y_2 - y_1\), the right side of Eq. 26 is completely known, and the length of the reach \(\Delta x\) can be determined. By taking successive increments of \(\Delta y\), the complete surface profile may be plotted. This method gives surprisingly accurate results.
c. Methods of Direct Integration

The Method of Bakhmeteff. Bakhmeteff developed a method for the integration of the varied-flow equation based on the use of a "varied-flow function" derived from two assumptions:

(1) The conveyance of a channel \( K = ACR^2/3 \), may be approximated within a limited range of depths by the relationship

\[
K^2 = A^2C^2R^4/3 = \text{a constant} \times y^N
\]  

(27a)

so that

\[
\left( \frac{K}{K_0} \right)^2 = \frac{A^2C^2R^4/3}{A_0^2C_0^2R_0^4/3} = \left( \frac{y}{y_0} \right)^N
\]  

(27b)

where \( N \) is called the "hydraulic exponent."

(2) \( G = \frac{b_aQ^2}{gA^3} \left( \frac{K}{K_0} \right)^2 \) and changes very slowly in any practical case so that for a reasonably short range of integration, it may be approximated by an average constant value.

Using these assumptions and setting \( y/y_0 = r \) and \( dy = y_0 \, dr \), Eq. 24 can be expanded and integrated from a section where \( x = x_1, y = y_1 \) to a section where \( x = x_2, y = y_2 \), resulting in

\[
\Delta x = x_2 - x_1 = \frac{y_0}{S_0} \left[ (r_2 - r_1) - (1-G) \int_{r_1}^{r_2} \frac{dr}{1-r^N} \right]
\]  

(28)

The integral \( \int_{r_1}^{r_2} \frac{dr}{1-r^N} \) is designated as \( B(N,r) \), the Bakhmeteff form as

\[
\Delta x = x_2 - x_1 = \frac{y_0}{S_0} \left\{ (r_2 - r_1) - (1-G) [B(N,r_2) - B(N,r_1)] \right\}
\]  

(29)

The values of the varied-flow function covering a wide range of values of \( N \) and \( r \) have been computed and presented in tabular form to facilitate practical profile computations. In order to determine the surface profile in any given case using Eq. 29, the following step procedure is used:

1. Determine the class of curve by evaluating \( y_c \) and \( y_0 \)
2. Determine the value of the "hydraulic exponent" \( N \) by means of a logarithmic plot of Eq. 27a.
3. Plot the value of \( (1 - G) \) against \( y \) so that the average
value of \((1 - G)\) between the assumed depths of flow may be evaluated

(4) obtain the proper value of \(B(N,r)\) from the tables for the terminal values of \(r\)

(5) substitute the required values in Eq. 29 and compute the values of \(\Delta x\).

The Method of Mononobe.\(^7\) Mononobe developed a method for integrating the varied-flow equation based on the assumption that both the area \(A\) and the wetted perimeter \(p_w\) may be expressed as monomial functions of the depth \(y\). That is:

\[
p_w = \text{a constant} \times y^l
\]

\[
A^2 = \text{a constant} \times y^M
\]

In Eq. 30 and 31 the exponents \(l\) and \(M\) will generally vary slightly with the depth \(y\); in these cases, however, their average values in the range of depths under consideration may be used. Mononobe has shown that the above empirical relationships are applicable to many types of cross-sections in common use. Using these relationships and Manning's friction-slope formula (Eq 11b) it can be shown that

\[
\frac{S_f}{S_0} = \left(\frac{y}{y_0}\right)^{4/3l - 5/3M}
\]

By substituting this relationship and other relationships derived from the assumptions in Eq. 32, Mononobe developed the following equation of gradually varied flow:

\[
\Delta x = x_2 - x_1 = \frac{y_0}{S_0} \left[ \int_r^r \frac{C_0}{r^{C_0-1}} \, dr - \frac{1}{2} \int_r^r \frac{C_0}{r^{C_0-1}} \, dr \right]
\]

in which \(r = y/y_0\), and \(C_0 = 5/3M - 4/3l\). Mononobe further simplified Eq. 30 by using

\[
\Phi_1(r) = \int_{1.001}^r \frac{C_0}{r^{C_0-1}} \, dr
\]

\[
\Phi_2(r) = \int_{1.001}^r \frac{C_0^{-M-1}}{r^{C_0-1}} \, dr
\]

These are referred to as Mononobe's "backwater functions." Upon substitution, Eq. 30 reduces to
\[ \Delta x = x_2 - x_1 = \frac{y_0}{S_0} \left( \Phi_1(r_2) - \Phi_1(r_1) \right) - \frac{V_0^2 M}{2gy_0} \left[ \Phi_2(r_2) - \Phi_2(r_1) \right] \] (35)

The value of \( \Phi_2 \) depends only on the values of \( C_0 \) and \( r \), whereas the value of \( \Phi_2 \) is a function of \( C_0, M \) and \( r \). The values of the backwater functions have been computed for various values of \( C_0, M \) and \( r \) and are available in graphical form.\(^7\) These charts are for use only in the case of backwater curves since the values of \( \Phi_1 \) and \( \Phi_2 \) are given for corresponding values of \( r \) greater than unity.

For convenience in integration in the cases of the dropdown curve, the variable is taken to be \( \frac{1}{r} \) instead of \( r \). Accordingly, Eq. 32 is modified to

\[ \Delta x = \frac{y_0}{S_0} \int_{1/r_1}^{1/r_2} \frac{(1/r)^2}{C_0-1} d(1/r) - \frac{V_0^2 M}{2gy_0} \int_{1/r_1}^{1/r_2} \frac{(1/r)^M-1}{C_0-1} d(1/r) \] (36)

and Mononobe’s dropdown functions are defined by

\[ \psi_1\left(\frac{1}{r}\right) = \int_{1.001}^{\frac{1}{r}} \frac{(1/r)^2}{C_0-1} d(1/r) \] (37)

\[ \psi_2\left(\frac{1}{r}\right) = \int_{1.001}^{\frac{1}{r}} \frac{(1/r)^{M-1}}{C_0-1} d(1/r) \] (38)

so that Eq. 36 may be written as

\[ \Delta x = \frac{y_0}{S_0} \left[ \psi_1\left(\frac{1}{r_2}\right) - \psi_1\left(\frac{1}{r_1}\right) \right] - \frac{V_0^2 M}{2gy_0} \left[ \psi_2\left(\frac{1}{r_2}\right) - \psi_2\left(\frac{1}{r_1}\right) \right] \] (39)

As is the case with the backwater functions, the dropdown function \( \psi_1 \) depends only on the value of \( C_0 \) and \( \frac{1}{r} \), whereas \( \psi_2 \) is a function of \( C_0, M \) and \( \frac{1}{r} \). The values of the dropdown functions corresponding to various values of \( C_0, M \) and \( \frac{1}{r} \), are given by Mononobe in graphical form.\(^{18}\)
In applying Mononobe's method to profile computations it will be found that logarithmic plots of the wetted perimeter against depth do not approximate straight lines very well; therefore, an error in the results may be introduced by assuming the wetted perimeter to be a monomial function of the depth. The greatest drawback in this method, however, lies in the inherent error due to interpolation and the difficulty in using the charts to find the values of the dropdown and the backwater functions. It will be noted from Eqs. 35 and 39 that the values of these functions are multiplied by the term $\gamma_0/S_0$ in computing the distance along the channel $\Delta x$. The value of $\gamma_0/S_0$ is generally large, so that a slight error due to interpolation may result in an appreciable error in the computed distance along the channel. For this reason, it has been suggested that the values of the varied-flow functions be given to the fourth decimal place in order to keep the error due to interpolation to a minimum. The graphical charts prepared by Mononobe are drawn on such a scale that the values of the backwater and dropdown functions can be read only to the second decimal place. It would appear that unless the charts are drawn to a much larger scale, the presentation of the values of these functions in tabular form is to be preferred.
V. INTEGRATION OF THE GRADUALLY VARIED-FLOW EQUATION:
PROPOSED METHOD

12. Assumptions

In the foregoing discussion it is pointed out that assumptions are necessary to simplify the solution of the gradually varied-flow equation by direct integration. The equation is integrated, in each case, between the range of depths in which the simplifying assumptions are valid. Thus, the number of steps required in the integration procedure for the whole range of depths under consideration will depend on the limiting conditions of each set of assumptions. For instance in the "step method," the assumption that the mean friction slope is equal to the mean of the friction slopes at the ends of a reach is valid only for a small increment of depth. Accordingly, a comparatively large number of steps is required in the computation procedure. In the method of Bakhmeteff the empirical relationship between the conveyance and depth holds for a wide range of depths; therefore, for those cases involving insignificant effect due to velocity head changes, the computations may generally be made in one step. If the effect of velocity head change is not negligible on the other hand, the limitation imposed by assuming $G$ to be constant makes it necessary to confine the computations to small increments of depth. It is evident that the assumptions made must conform to the actual conditions over a wide range of limits if the method were to yield reasonably accurate results without recourse to a large number of intermediate steps.

In the light of the studies made by Bakhmeteff$^7$ and Morinobu$^7$, it appears that both the area and the conveyance can be expressed as monomial functions of the depth over a considerable range for the channel sections in common use. These assumptions are made in the proposed method of integrating the gradually varied-flow equation, namely

$$K^2 = \text{a constant} \times y^N \quad (27a)$$

$$A^2 = \text{a constant} \times y^M \quad (31)$$

Equations 27a and 31 are only approximate; some cross-sections satisfy these empirical relationships more closely than others.
However, even in channels not well adapted to the monomial approximations over their entire range, the surface curves usually cover only a limited range so that the actual variation of the hydraulic elements conforms closely to these empirical relationships. In most cases it will be possible to choose values of the constants and the exponents that will make Eqs. 27a and 31 express the values of $K$ and $A$ with sufficient accuracy over a wide range of depths. If greater accuracy in results is desired, the whole range of depths under consideration may be subdivided to determine the values of the exponents separately for each interval. However, this procedure will hardly be required in making practical profile computations.

It will be noted that the values of the exponents $N$ and $M$ are characteristics only of the given cross-section. As such, they can be determined once and for all for a given channel. Hence, a saving of time as well as labor of computations may be effected in those problems in which a large number of surface curves are to be computed for the same channel under a wide variation of discharge.

13. Derivation of Proposed Equation

The differential equation of gradually varied flow may be solved with the aid of the simplifying assumptions by starting from Eq. 13; i.e.,

$$
\frac{b Q^2}{gA^3} \frac{dA}{S_0 - S_f} \frac{dA}{S_0 - S_f} \frac{dy}{dy} = \frac{\frac{Q^2}{2g} d\left(\frac{1}{A^2}\right)}{S_0 - S_f}
$$

Multiplying both sides of the equation by $S_0/y_0$ gives

$$
\frac{S_0}{y_0} \frac{dx}{dx} = \frac{\frac{dy}{y_0}}{1 - \frac{S_f}{S_0}} + \frac{\frac{Q^2}{2g y_0} d\left(\frac{1}{A^2}\right)}{1 - \frac{S_f}{S_0}}
$$

By means of Eq. 31 and 27

$$
\frac{S_f}{S_0} = \left(\frac{K}{K}\right)^2 = \left(\frac{y_0}{y}\right)^N
$$
Therefore, Eq. 40 becomes

\[
\frac{S_0}{y_0} \, dx = \frac{\frac{d}{y}}{1 - \left(\frac{y_0}{y}\right)^N} + \frac{\frac{Q^2}{2gy_0} \, d\left(\frac{1}{A^2}\right)}{1 - \left(\frac{y_0}{y}\right)^N} \tag{41}
\]

Since

\[
\frac{d}{y_0} \left(\frac{y}{y_0}\right)^N = \left[1 + \frac{\frac{y}{y_0}}{1 - \left(\frac{y_0}{y}\right)^N}\right] d\left(\frac{y}{y_0}\right)
\]

and

\[
\frac{Q^2}{2gy_0} \, d\left(\frac{1}{A^2}\right) = \frac{Q^2}{2gy_0} \, d\left(\frac{A_0^2}{A^2}\right) = \frac{\frac{V_0^2}{2gy_0}}{1 - \left(\frac{y_0}{y}\right)^N}
\]

substituting into Eq. 41 gives

\[
\frac{S_0}{y_0} \, dx = \left[1 + \frac{1}{\left(\frac{y}{y_0}\right)^N - 1}\right] d\left(\frac{y}{y_0}\right) + \frac{\frac{V_0^2}{2gy_0} \, d\left(\frac{A_0^2}{A^2}\right)}{1 - \left(\frac{y_0}{y}\right)^N} \tag{42a}
\]

and integration yields

\[
\frac{S_0}{y_0} \, x = \frac{y}{y_0} + \int \frac{d\left(\frac{y}{y_0}\right)}{\left(\frac{y}{y_0}\right)^N - 1} + \frac{\frac{V_0^2}{2gy_0} \, d\left(\frac{A_0^2}{A^2}\right)}{1 - \left(\frac{y_0}{y}\right)^N} + \text{constant} \tag{42b}
\]
Designating as before $y/y_0 = r$ and $B(N,r) = \int \frac{dr}{1 - r^N}$ Eq. 42b becomes

$$\frac{S_0}{y_0} x = r - B(N,r) + \frac{V_0^2 A_0^2}{2gy_0} \int \frac{d\left(\frac{A_0^2}{A^2}\right)}{1 - \left(\frac{y_0}{y}\right)^N} + \text{constant} \quad (43)$$

The integral on the right side of Eq. 43 may be reduced to the form of the Bakhmeteff varied-flow function with the aid of Eq. 31, from which it can be deduced that

$$\left(\frac{y_0}{y}\right)^N = \left[\left(\frac{y_0}{y}\right)^M\right]^{N/M} = \left[\frac{A_0^2}{A^2}\right]^{N/M}$$

Therefore

$$\frac{d\left(\frac{A_0^2}{A^2}\right)}{A^2} = \frac{d\left(\frac{A_0^2}{A^2}\right)}{1 - \left(\frac{y_0}{y}\right)^N} \quad (44)$$

Now, the integral on the right side of Eq. 44 is in exactly the same form as Bakhmeteff's varied-flow function as defined previously. In this case the exponent is $N/M$ and the variable is $A_0^2/A^2$, therefore

$$\int \frac{d\left(\frac{A_0^2}{A^2}\right)}{1 - \left(\frac{A_0^2}{A^2}\right)^{N/M}} = B\left(\frac{N A_0^2}{M^{' A^2}}\right) \quad (45)$$

and Eq. 43 becomes

$$x = \frac{y_0}{S_0} [r - B(N,r) + \frac{V_0^2}{2gy_0} B\left(\frac{N A_0^2}{M^{' A^2}}\right)] \quad (46)$$

in which the constant of integration has been intentionally omitted because Eq. 46 is used only to determine the difference in the values of $x$ between any two given sections.

For the downstream section at which $r = r_2$, and $A = A_2$,

$$x_2 = \frac{y_0}{S_0} [r_2 - B(N,r_2) + \frac{V_0^2}{2gy_0} B\left(\frac{N A_0^2}{M^{' A^2}}\right)] \quad (47a)$$
and for the upstream section at which \( r = r_1 \), and \( A = A_1 \),

\[
x_1 = \frac{y_0}{S_0} \left[ r_1 - B(N, r_1) + \frac{V_0^2}{2\gamma_0} B\left(\frac{N}{M}, \frac{A_0^2}{A_1^2}\right) \right]
\] (47b)

Using the \( \Delta \) notation and subtracting gives

\[
\Delta x = x_2 - x_1 = \frac{y_0}{S_0} [\Delta r - \Delta B(N, r) + \frac{V_0^2}{2\gamma_0} \Delta B\left(\frac{N}{M}, \frac{A_0^2}{A^2}\right)]
\] (48)

In applying Eq. 48 to practical profile computations, it is necessary to know the values of both of these varied-flow functions. Bakhmeteff has computed the first function for various values of \( r \) in the range of \( N \) lying between 2.8 and 5.4. \(^{(3)}\) For the common types of practical channel sections the value of the exponent \( N/M \) generally lies between 1.0 and 2.0 which is less than the range covered by Bakhmeteff's tables. Hence a new set of tables has to be computed for evaluating the second varied-flow function in this range of \( N/M \), and for appropriate values of the variable \( A_0^2/A^2 \). This has been done with the results presented in Tables 2 and 3 in which the values of the function are given for sufficiently small intervals of both the variable and the exponent to permit interpolation. The values of the varied-flow function for the range of \( A_0^2/A^2 \) between 0 and 0.990 are given in Table 2, and for the range between 1.01 and 20 in Table 3. In Table 3 all the values shown have been increased by 3 to prevent the values from changing signs. This procedure is permissible since the relative values of the functions, which are used in the computations, remain unaffected.

The proposed method, as well as that of Mononobe, differs from Bakhmeteff's method essentially in the treatment of the velocity head changes. In Bakhmeteff's method the factor \( G \), reflecting these changes, is assumed to be constant. The limits of integration should, therefore, be confined to small increments of depth inasmuch as the factor \( G \) is a variable with respect to the depth. In the proposed method \( G \) is, in effect, regarded as a function of the depth so that the integration limits may be generally taken over the entire range of depths considered. For these cases, the proposed method will afford a more direct computation procedure whereby the results can be obtained without recourse to successive steps.

The integral embodying the effect of changes of velocity head in the proposed method (Eq. 45) is reduced to the form of the
varied-flow function defined by Bakhmeteff. This form is preferred in order to avoid the necessary adoption of a new term, as is done in the method of Mononobe. In fact, both Mononobe's backwater and dropdown functions are reducible to the same form as Bakhmeteff's varied-flow function. It seems that little, if anything, is gained by defining them in their present form, especially when the resulting integrals $\Phi_2$ and $\Psi_2$ are expressed in terms of three parameters. The integral in Eq. 45, on the other hand, is dependent upon only two parameters, $N/M$ and $A_0^2/A^2$.

### Table 2

Values of the Varied Flow Function $B\left(\frac{N/A_0^2}{M/A^2}\right)$: $\frac{A_0^2}{A^2} < 1$

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A_0^{-2}/A^2; consequently, Tables 2 and 3 representing the values of this integral are more simple to use than the graphical charts prepared by Mononobe\(^{(7)}\) for the $\Phi_2$ and $\psi_2$ functions.

14. Computation Procedure

In applying Eq. 48 to profile computations, it should be borne in mind that the downstream conditions constitute the upper limit of integration. The computations should be made with strict observation to sign. If the location of the depths is assumed correctly for the upper and lower limits, the distance between any two sections $\Delta x$ will always appear as positive.*

To compute the surface profile of flow in a given channel when the rate of discharge is known, the following procedure may be used:

a. Determine, from the given conditions, the points of control from which the computations may start, and the type of surface curve that will develop.

b. Determine the values of the exponents $N$ and $M$ in Eqs. 27a and 31 for the entire range of depths covered by the surface curve. This can be done algebraically by substituting into these equations a pair of simultaneous values of the depths at the ends of the range under consideration and the corresponding values of the conveyance or the area. It can also be done graphically by plotting on logarithmic scale the conveyance and area as ordinates and the corresponding values of the depth as abscissas. A straight line is then drawn to follow the points as closely as possible. The slope of this line is one-half the value of the exponent. The algebraic method corresponds to fitting a straight line through the initial and final points on the logarithmic plot; it has the disadvantage of not revealing the degree of approximation for the intermediate points. For this reason, the graphical method is more likely to give the average values of the exponents by following all the points as closely as possible.

c. Compute the distance between any two given sections according to Eq. 48. The necessary parameters in finding the values of the two varied-flow functions are $(N, r)$, and $\frac{A_0^{-2}}{M', \frac{A^2}{A^2}}$.

The value of $N$ will generally be between 2.8 and 4.0 and for this range the tables by Bakhmeteff\(^{(9)}\) should be consulted. For the range of the exponent $N/M$ between 1.0 and 2.0, Table 2 or 3 may be used, depending on the value of the variable. In the event

*While either $x_1$ or both $x_2$ and $x_1$ may have negative signs, $\Delta x = x_2 - x_1$ will always be positive.
Table 3
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*All values in this table have been increased by 3 to prevent the numbers from being negative.
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</tr>
<tr>
<td>14.50</td>
<td>0.397</td>
<td>1.414</td>
<td>1.881</td>
<td>2.318</td>
<td>2.620</td>
<td>3.069</td>
<td></td>
</tr>
<tr>
<td>15.00</td>
<td>0.361</td>
<td>1.393</td>
<td>1.866</td>
<td>2.309</td>
<td>2.615</td>
<td>3.067</td>
<td></td>
</tr>
<tr>
<td>15.50</td>
<td>0.326</td>
<td>1.374</td>
<td>1.854</td>
<td>2.301</td>
<td>2.608</td>
<td>3.065</td>
<td></td>
</tr>
<tr>
<td>16.00</td>
<td>0.292</td>
<td>1.356</td>
<td>1.840</td>
<td>2.292</td>
<td>2.602</td>
<td>3.063</td>
<td></td>
</tr>
<tr>
<td>16.50</td>
<td>0.259</td>
<td>1.335</td>
<td>1.827</td>
<td>2.285</td>
<td>2.598</td>
<td>3.061</td>
<td></td>
</tr>
<tr>
<td>17.00</td>
<td>0.227</td>
<td>1.319</td>
<td>1.816</td>
<td>2.277</td>
<td>2.591</td>
<td>3.059</td>
<td></td>
</tr>
<tr>
<td>17.50</td>
<td>0.197</td>
<td>1.301</td>
<td>1.804</td>
<td>2.270</td>
<td>2.588</td>
<td>3.057</td>
<td></td>
</tr>
<tr>
<td>18.00</td>
<td>0.167</td>
<td>1.286</td>
<td>1.794</td>
<td>2.263</td>
<td>2.581</td>
<td>3.056</td>
<td></td>
</tr>
<tr>
<td>19.00</td>
<td>0.110</td>
<td>1.254</td>
<td>1.773</td>
<td>2.251</td>
<td>2.574</td>
<td>3.053</td>
<td></td>
</tr>
<tr>
<td>20.00</td>
<td>0.056</td>
<td>1.224</td>
<td>1.753</td>
<td>2.239</td>
<td>2.566</td>
<td>3.050</td>
<td></td>
</tr>
</tbody>
</table>
that either the exponent or the variable lies between the values shown in these tables, the value of the varied-flow function may be found by straight-line interpolation.

This computation procedure is illustrated by the following examples.

a. Example 1.

Let it be desired to find the distance along a rectangular 10-ft-wide channel between the depths \( y_2 = 7.0 \) ft and \( y_1 = 4.5 \) ft when the rate of discharge is 136.0 cu ft per sec. The slope of the channel \( S_0 \) is 0.0004, and the value of \( n \), the roughness factor in Manning's equation, may be taken as 0.015.

The uniform depth, \( y_0 \), computed from Manning's equation is 4.0 ft for the given rate of discharge. The critical depth \( y_c \) is computed to be 1.79 ft. The profile of flow, therefore, belongs to the case of backwater, or the \( M_1 \)-type of surface curve.

<table>
<thead>
<tr>
<th>Depth (ft)</th>
<th>Area (ft)</th>
<th>Wetted Perimeter</th>
<th>Hydraulic Radius</th>
<th>( R^2/3 )</th>
<th>Conveyance</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>20</td>
<td>14.0</td>
<td>1.43</td>
<td>1.28</td>
<td>2,540</td>
</tr>
<tr>
<td>3.0</td>
<td>30</td>
<td>16.0</td>
<td>1.87</td>
<td>1.52</td>
<td>4,530</td>
</tr>
<tr>
<td>4.0</td>
<td>40</td>
<td>18.0</td>
<td>2.22</td>
<td>1.71</td>
<td>6,800</td>
</tr>
<tr>
<td>5.0</td>
<td>50</td>
<td>20.0</td>
<td>2.50</td>
<td>1.85</td>
<td>9,190</td>
</tr>
<tr>
<td>6.0</td>
<td>60</td>
<td>22.0</td>
<td>2.72</td>
<td>1.96</td>
<td>11,630</td>
</tr>
<tr>
<td>7.0</td>
<td>70</td>
<td>24.0</td>
<td>2.92</td>
<td>2.05</td>
<td>14,200</td>
</tr>
<tr>
<td>8.0</td>
<td>80</td>
<td>26.0</td>
<td>3.08</td>
<td>2.13</td>
<td>16,900</td>
</tr>
</tbody>
</table>

The hydraulic properties of the 10-ft rectangular channel are shown in Table 4, in which the conveyance \( K \) is computed according to Eq. 11a. The value of the hydraulic exponent \( N \) is found from the logarithmic plot of conveyance-depth shown in Fig. 6. The slope of the line is 1.40; therefore the hydraulic exponent \( N \) has a value of 2.80. The value of \( M \) in Eq. 31 is 2.0, since the area is directly proportional to the depth for channels of rectangular cross-section.

The given data are

\[
\frac{y_0}{S_0} = \frac{4.0}{0.0004} = 10,000
\]

\[
\frac{V_0^2}{2gy_0} = \frac{(3.4)^2}{64.4 \times 4} = 0.045
\]
Therefore

\[
\begin{align*}
    r_1 &= \frac{y_1}{y_0} = \frac{4.5}{4.0} = 1.125 \\
    r_2 &= \frac{y_2}{y_0} = \frac{7.0}{4.0} = 1.750
\end{align*}
\]

Therefore

\[
\Delta r = r_2 - r_1 = 0.625
\]

From the tables prepared by Bakhmeteff,

\[
\begin{align*}
    B(2.8,1.125) &= 0.705 \\
    B(2.8,1.75) &= 0.212
\end{align*}
\]

hence

\[
\Delta B(N,r) = -0.493
\]

The parameters necessary for using Table 2 are

\[\text{Fig. 6. Logarithmic Plot of Conveyance } K \text{ Against Depth } y\]

for the 10-ft Rectangular Channel
\[
\frac{N}{M} = \frac{2.8}{2.0} = 1.40
\]
\[
\frac{A_0^2}{A_1^2} = \frac{40^2}{45^2} = 0.79
\]
\[
\frac{A_0^2}{A_2^2} = \frac{40^2}{70^2} = 0.33
\]

from which

\[
B(1.4, 0.79) = 1.273
\]
\[
B(1.4, 0.33) = 0.365
\]

Therefore

\[
\Delta B = \frac{N}{M} = \frac{A_0^2}{A_2^2} = -0.908
\]

Substituting these values into Eq. 48 gives

\[
\Delta x = x_2 - x_1 = 10,000 \left[ 0.625 + 0.493 - 0.045 (0.908) \right]
\]
\[
= 10,770 \text{ ft}
\]

b. Example 2.

Determine the distance along the rectangular channel in the previous example between the depth \(y_2 = 2.0 \text{ ft}\) and \(y_1 = 3.6 \text{ ft}\). The profile of flow in this case obviously belongs to the dropdown, or \(M_2\)-type of surface. The given data are

\[
\frac{y_0}{S_0} = 10,000
\]
\[
\frac{V_0^2}{2gy_0} = 0.045
\]
\[
r_1 = \frac{y_1}{y_0} = \frac{3.6}{4.0} = 0.90
\]
\[
r_2 = \frac{y_2}{y_0} = \frac{2.0}{4.0} = 0.50
\]

Therefore

\[
\Delta r = r_2 - r_1 = -0.40
\]
From Bakhmeteff's tables,
\[ B(2.8, 0.90) = 1.253 \]
\[ B(2.8, 0.50) = 0.521 \]

hence
\[ \Delta B(N, r) = -0.732 \]

The parameters necessary for using Table 3 are
\[ \frac{A_0^2}{A_1^2} = \frac{40^2}{36^2} = 1.24 \]
\[ \frac{A_0^2}{A_2^2} = \frac{40^2}{20^2} = 4.0 \]
\[ \frac{N}{M} = 1.40 \]

from which
\[ B(1.4, 1.24) = 4.246 \]
\[ B(1.4, 4.00) = 2.692 \]

Therefore
\[ \Delta B\left(\frac{-A_0^2}{M A^2}\right) = -1.554 \]

Substituting these values into Eq. 48 gives
\[ \Delta x = x_2 - x_1 = 10,000 \left[ -0.40 + 0.732 - 0.045 \times (1.554) \right] \]
\[ = 2620 \text{ ft} \]

c. Example 3.

Let it be desired to determine the extent of the dropdown curve in a 10-ft circular conduit \((n = 0.015)\) laid on a slope of 0.001 when the rate of flow is 305 cu ft per sec. The upper limit of the curve may be taken as 0.95 of the uniform depth.

The uniform and critical depths are first computed for the given rate of flow and are found to be 6.0 ft and 4.11 ft respectively. The values of the exponents \(N\) and \(M\) may be determined according to Eqs. 27a and 31 as follows:
\[ \left( \frac{K}{K_0} \right)^2 = \left( \frac{\gamma}{\gamma_0} \right)^N ; \quad \left( \frac{A}{A_0} \right)^2 = \left( \frac{\gamma}{\gamma_0} \right)^M \]
Taking logarithms of both sides of the equations yields

\[
N = 2 \frac{\log(K)}{\log(y - y_0)} ; M = 2 \frac{\log(A)}{\log(y - y_0)}
\]

The exponents may be found by substituting into these equations a pair of simultaneous values of the depth at the ends of the range under consideration and the corresponding values of the conveyance or area.

In the example above when \( y = 0.95 \times 6.0 = 5.7 \text{ ft} \)

\[
K = 8900; \quad A = 46.20
\]

and when \( y_0 = 4.11 \text{ ft} \)

\[
K_0 = 5060; \quad A_0 = 30.42
\]

Therefore

\[
N = 2 \frac{\log(8900)}{\log(5.7 - 4.11)} = 3.50
\]

\[
M = 2 \frac{\log(46.2)}{\log(30.42 - 4.11)} = 2.58
\]

and

\[
\frac{N}{M} = 1.355
\]

The required distance along the channel may be determined according to the scheme of computations in the following table. Applying the results in the table to Eq. 48 yields,

\[
\Delta x = x_2 - x_1 = \frac{6.0}{0.001} \left[ (0.685 - 0.95) - (0.733 - 1.369) \right]
\]

\[
+ \frac{(6.2)^2}{64.4 \times 6} \left( 2.986 - 4.65 \right)
\]

\[
= 1230 \text{ ft}
\]
Computations for Dropdown Curve in 10-ft Circular Channel

Upstream Section, \( y_1 = 5.7 \) ft

\[
r_1 = \frac{y_1}{y_0} = 0.95
\]

\[
B(N, r_1) = 1.369
\]

\[
\frac{A_0^2}{A_1^2} = \frac{(49.2)^2}{(46.2)^2} = 1.14
\]

\[
B\left(\frac{N A_0^2}{M A_1^2}\right) = 4.650
\]

Downstream Section, \( y_2 = 4.11 \) ft

\[
r_2 = \frac{y_2}{y_0} = 0.685
\]

\[
B(N, r_2) = 0.733
\]

\[
\frac{A_0^2}{A_2^2} = \frac{(49.2)^2}{(30.42)^2} = 2.63
\]

\[
B\left(\frac{N A_0^2}{M A_2^2}\right) = 2.986
\]

\* Example 4. \*

Supposing the 10-ft circular channel in the previous example is preceded by a chute in which the flow of 305 cu ft per sec takes place at a uniform depth of 2.0 ft, what will be the length of the surface curve along the channel?

The depth of flow at the junction between the circular channel and the chute will be taken as 2.0 ft. Since the flow must cross the critical depth of 4.11 ft to attain a uniform depth of 6.0 ft in the circular conduit, the surface curve will end in a hydraulic jump. The problem, then, is to determine the length of the \( M_3 \)-type surface curve that lies between the limiting depths of 2.0 ft and the lower conjugate depth of 2.7 ft at the foot of the jump. The computations are shown in the following table, assuming the same values for \( N \) and \( M \) as in Example 3.

Substituting these results into Eq. 48 gives

Computations for the Length of the \( M_3 \)-Type Surface Curve in the 10-ft Circular Channel

Upstream Section, \( y_1 = 2.0 \) ft

\[
r_1 = \frac{y_1}{y_0} = 0.333
\]

\[
B(N, r_1) = 0.334
\]

\[
\frac{A_0^2}{A_1^2} = \frac{(49.2)^2}{(11.18)^2} = 19.60
\]

\[
B\left(\frac{N A_0^2}{M A_1^2}\right) = 1.846
\]
The application of the proposed method to profile computations has been illustrated by solving examples involving the three types of surface curves of flow in uniform channels on mild slopes. In all the foregoing computations the velocity distribution is assumed to be unity, and no account is taken of the eddy losses accompanying retarded flow as in the case of the backwater and $M_3$-type of surface profile. Equation 48 may be modified to include the effect of nonuniform velocity distribution as well as the eddy losses, which are usually estimated as some percentage of the change of velocity head between adjacent sections. The modified form of the equation, following the procedure in deriving Eq. 48, becomes

$$
\Delta x = \frac{y_0}{S_0} \left[ (\Delta r - \Delta B(N, r)) + (C_m - k) \frac{V_0^2}{2gy_0} \frac{N A_0^2}{M A_2^2} \right] 
$$

in which the correction factor for eddy losses $k$ and the momentum factor $C_m$ are assumed to be constant from section to section. It should be noted that these factors tend to offset each other due to their opposite sign; the net effect, generally, is to prolong the length of the $M_1$ curves and shorten the $M_3$-type of surface curves. For instance, using values of $C_m = 1.05$ and $k = 0.10$ (assumed) in the solution of Example 1 would yield a result of 10,870 ft instead of 10,770 ft; and in Example 4, the length of the $M_3$ curve would be 174 ft instead of 186 ft. These results show only a small deviation from those computed by means
of Eq. 48. In the event, however, that the corrections for non-uniform velocity distribution and eddy losses are large enough to be significant, the computations may be made in accordance with Eq. 49.

15. Special Charts

In the foregoing examples the distance along the channel is computed between two sections of known depths for a given rate of flow. In other types of problems it may be required to determine the depth of flow at the end of a given reach of channel. When $\Delta x$ or $y$ is to be computed over a wide variation of the discharge, much of the subsequent labor of computation can be avoided by preparing an auxiliary curve for the solution of the gradually varied-flow equation. This can be done by plotting $x^2/y_0$ against $\frac{S_0}{y_0}$ according to Eq. 46:

$$x \frac{S_0}{y_0} = r - B(N,r) - \frac{V_0^2}{2gy_0} B\left(\frac{N}{M}, \frac{A_0^2}{A^2}\right)$$

$$= r - B(N,r) - \frac{V_0^2}{2gy_0} B\left(\frac{N}{M}, r^{-M}\right)$$

In the above equation the values of the exponents $N$ and $M$ are determined by the given channel section; therefore, it is possible to compute the values of $x \frac{S_0}{y_0}$ corresponding to the various values of $\frac{V_0^2}{2gy_0}$ and $r$ for the channel under consideration.

Such a plot is shown in Figs. 8a and 10a, in which $x \frac{S_0}{y_0}$ is plotted against the variable $r$ and a parameter $C_1$ for various values of the exponents $N$ and $M$. It is shown in the following discussion that the particular values of the exponents used for these figures correspond, respectively, to the special cases of channels with rectangular and circular cross-sections.

*From Eq. 31

$$\frac{A_0^2}{A^2} = \left(\frac{\gamma_0}{y_0}\right)^{-M} = r^{-M}$$
a. Rectangular Channels.

For channels of rectangular cross-sections the area is directly proportional to the depth; therefore, the exponent $M$ has a value of 2.0 according to Eq. 31. The hydraulic exponent $N$ may be determined in accordance with Eqs. 11a and 27a.

$$K = \frac{1.486}{n} AR^{2/3} = \text{a constant} \times y^{N/2}$$

Thus it can be seen that the value of $N$ can be determined either by a logarithmic plot of the conveyance against depth as explained previously, or of the product $AR^{2/3}$ against depth as shown in Fig. 9. In the latter case, the roughness factor $n$ need not be taken into consideration in computing $N$.

<table>
<thead>
<tr>
<th>Table 5</th>
<th>Values of the Exponents M and N for Rectangular Channels of Various Widths</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width, $b$ (ft)</td>
<td>Hydraulic Radius, $R$ (ft)</td>
</tr>
<tr>
<td>$b &lt;&lt; y$</td>
<td>$R \approx \frac{b}{2}$</td>
</tr>
<tr>
<td>5.0</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>$R = \frac{by}{b + 2y}$</td>
</tr>
<tr>
<td>20.0</td>
<td></td>
</tr>
<tr>
<td>30.0</td>
<td></td>
</tr>
<tr>
<td>$b &gt;&gt; y$</td>
<td>$R \approx y$</td>
</tr>
</tbody>
</table>

For rectangular channels of broad cross-section the hydraulic radius $R$ is roughly proportional to the depth. Therefore, the product $AR^{2/3}$ varies approximately as the $5/3$ power of the depth, in which case $N$ is equal to 3.33. The minimum value of $N$ is attained in rectangular channels whose width is small compared to the depth, for which the hydraulic radius remains relatively constant. Consequently, $N$ is equal to 2.0 in the case of rectangular channels with deep and narrow cross-section. For the intermediate cases $N$ lies between these limiting values.

Values of $N$ determined from a logarithmic plot of the product $AR^{2/3}$ against depth $y$ for the various widths of channel are shown in Table 5. The average value of $N$ is found in each case for the range of depths stated.
From the results in Table 5 it can be seen that the exponents \( N \) and \( M \) for a broad rectangular channel are, respectively, 3.33 and 2.0. Using these particular values of the exponents, the values of \( \frac{xS_0}{y_0} \) calculated from Eq. 46 are shown in Table 6 for various values of \( r > 1 \) and for the parameter \( C_1 \). The results are also presented graphically in Fig. 7, in which the value of \( \frac{xS_0}{y_0} \) is separated into two parts, \([r - B(N,r)]\) and \(C_1B\left(\frac{N}{M}, r^{-M}\right)\), for convenience in plotting.

Similarly the values of \( \frac{xS_0}{y_0} \) for \( N \) and \( M \) equal to 2.8 and 2.0, corresponding to an average width channel, are computed in Table 7 and plotted in Fig. 8. The values for \( r > 1 \) are in Fig. 8a where again the components \([r - B(N,r)]\) and \(C_1B\left(\frac{N}{M}, r^{-M}\right)\) are used for convenience. In Fig. 8b are the values for \( r < 1 \). The application of Fig. 8a and 8b is recommended for rectangular channels between 10 and 30 ft wide since the hydraulic exponent in this range deviates only slightly from the average value of 2.8 used in the computations.

b. Circular Channels.

In the case of circular channels the exponents \( M \) and \( N \) may be found by a logarithmic plot of the area and the product \( AR^{2/3} \), respectively, against the depth. This has been done in Fig. 9 for 5, 10, and 20-ft circular conduits in a range of depths between 0.2 and 0.7 of the diameter. The plot shows the slope of the lines to be remarkably consistent with one another, the exponent \( N \) having an average value of 3.60 and \( M \), 2.56. Applying these values of the exponents to Eq. 46, it is possible to construct the auxiliary curve for solving the equation of gradually varied flow in circular channels. Table 8 shows the computations of \( \frac{xS_0}{y_0} \) for various values of the parameter \( C_1 \) and variable \( r \).

The results for \( r > 1 \) are plotted in Fig. 10a in which the value of \( \frac{xS_0}{y_0} \) is again separated into the two parts. Figure 10b shows the curves for \( r < 1 \). The curves in Fig. 10a and 10b may be used to determine the surface profiles of flow in circular channels between 5.0 and 20 ft in diameter.
In applying the charts to profile determinations it should be noted that the curves possess several characteristic features. First, all the variables appear as ratios; the resulting plot is therefore dimensionless. Second, the use of the curves is not confined to any one channel roughness since the exponents $N$ and $M$ are determined only by the shape of the channel section. Third, the curves for $r < 1.0$ (Figs. 8b, 10b) possess a maximum point corresponding to the value of $r_c$ at the critical depth. Since this point constitutes the lower limit of the dropdown curve, the portion of the auxiliary curve comprising values of $r$ greater than $r_c$ is to be used in connection with the $M_2$-type surface curve; the remaining part of the curve, less than $r_c$, is to be used for the $M_3$-type surface curve. In the event that a given value of $x \frac{S_0}{y_0}$ is satisfied by more than one value of $r$, the required result can be obtained by considering the physical conditions of flow.

16. Practical Application of the Charts

The special charts will be found to effect a saving of time and labor of computations in those problems involving a wide variation of discharge, as shown in the following examples.

a. Example 5

A natural channel, whose cross-section may be taken as a broad rectangle, is laid on an average slope of 0.0004. Uniform flow is attained at a depth of 4.0 ft when the discharge is found to be 12.0 cu ft per sec per unit width of channel. Find the distance between two sections, 6 and 9 ft deep respectively, when the discharge is 15 cu ft per sec per unit width, assuming: (a) the effect of nonuniform velocity distribution and eddy losses to be negligible; (b) the effect of velocity head changes to be negligible; and (c) $C_m = 1.10$, $k = 0.20$.

The uniform depth for $q = 15.0$ cu ft per sec may be found as follows:

$$\frac{Q}{Q_0} = \frac{q}{q_0} = \frac{K S_0^{\frac{1}{4}}}{K_0 S_0^{\frac{1}{4}}} = \left(\frac{y}{y_0}\right)^{5/3}$$

Therefore

$$y_0 = y \left(\frac{q_0}{q}\right)^{3/5} = 4 \left(\frac{15}{12}\right)^{3/5} = 4.57 \text{ ft}$$

Hence

$$\frac{V_0^2}{2gy_0} = \frac{1}{2g} \frac{Q^2}{b^2 y_0^3} \frac{1}{2g} \frac{q^2}{y_0^3} = 0.0365$$
<table>
<thead>
<tr>
<th>( r )</th>
<th>( r^{-M} )</th>
<th>( B(N, r) )</th>
<th>( \frac{N}{M} \cdot r^{-M} )</th>
<th>( \frac{S_0}{y_0} )</th>
<th>( \frac{S_0}{y_0} )</th>
<th>( \frac{S_0}{y_0} )</th>
<th>( \frac{S_0}{y_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.080</td>
<td>0.857</td>
<td>0.619</td>
<td>1.425</td>
<td>0.029</td>
<td>0.490</td>
<td>0.057</td>
<td>0.518</td>
</tr>
<tr>
<td>1.100</td>
<td>0.826</td>
<td>0.559</td>
<td>1.287</td>
<td>0.026</td>
<td>0.567</td>
<td>0.051</td>
<td>0.592</td>
</tr>
<tr>
<td>1.150</td>
<td>0.756</td>
<td>0.452</td>
<td>1.068</td>
<td>0.021</td>
<td>0.719</td>
<td>0.043</td>
<td>0.741</td>
</tr>
<tr>
<td>1.241</td>
<td>0.650</td>
<td>0.339</td>
<td>0.826</td>
<td>0.017</td>
<td>0.919</td>
<td>0.033</td>
<td>0.935</td>
</tr>
<tr>
<td>1.414</td>
<td>0.500</td>
<td>0.224</td>
<td>0.574</td>
<td>0.011</td>
<td>1.201</td>
<td>0.023</td>
<td>1.213</td>
</tr>
<tr>
<td>1.581</td>
<td>0.400</td>
<td>0.164</td>
<td>0.437</td>
<td>0.009</td>
<td>1.426</td>
<td>0.017</td>
<td>1.434</td>
</tr>
<tr>
<td>1.826</td>
<td>0.300</td>
<td>0.113</td>
<td>0.318</td>
<td>0.006</td>
<td>1.719</td>
<td>0.013</td>
<td>1.726</td>
</tr>
<tr>
<td>2.000</td>
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<td>0.259</td>
<td>0.005</td>
<td>1.915</td>
<td>0.010</td>
<td>1.920</td>
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<tr>
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<td>0.071</td>
<td>0.211</td>
<td>0.004</td>
<td>2.133</td>
<td>0.008</td>
<td>2.137</td>
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<td>0.095</td>
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<td>3.225</td>
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Fig. 7. Graphical Solution of the Equation of Gradually Varied Flow in Broad Rectangular Channels
### Table 7

**Values of \( x \frac{S_0}{y_0} \) for Various Values of \( r \) and \( C_1 \):**

Rectangular Channels of Finite Width-\( M = 2.0 \), \( N = 2.8 \), \( \frac{N}{M} = 1.4 \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( r^{-M} )</th>
<th>( B(N, r) )</th>
<th>( B(N, r^{-M}) )</th>
<th>( C_1 )</th>
<th>( C_1 B(N, r^{-M}) )</th>
<th>( \frac{S_0}{y_0} )</th>
<th>( C_1 B(N, r^{-M}) )</th>
<th>( \frac{S_0}{y_0} )</th>
<th>( C_1 B(N, r^{-M}) )</th>
<th>( \frac{S_0}{y_0} )</th>
<th>( C_1 B(N, r^{-M}) )</th>
<th>( \frac{S_0}{y_0} )</th>
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<tr>
<td>0.990</td>
<td>1.020</td>
<td>2.106</td>
<td>5.992</td>
<td>0.120</td>
<td>-0.996</td>
<td>0.240</td>
<td>-0.876</td>
<td>0.360</td>
<td>-0.756</td>
<td>0.479</td>
<td>-0.637</td>
<td>0.599</td>
</tr>
<tr>
<td>0.980</td>
<td>1.040</td>
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<td>5.499</td>
<td>0.110</td>
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<td>0.220</td>
<td>-0.655</td>
<td>0.330</td>
<td>-0.545</td>
<td>0.440</td>
<td>-0.435</td>
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<td>5.113</td>
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<td>0.205</td>
<td>-0.494</td>
<td>0.307</td>
<td>-0.392</td>
<td>0.409</td>
<td>-0.290</td>
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<td>0.087</td>
<td>-0.308</td>
<td>0.175</td>
<td>-0.220</td>
<td>0.262</td>
<td>-0.133</td>
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<td>1.078</td>
<td>3.896</td>
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<td>0.156</td>
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<td>0.039</td>
<td>0.081</td>
<td>0.079</td>
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<td>0.160</td>
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<tr>
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<td>0.047</td>
<td>0.550</td>
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<td>0.573</td>
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<td>2.924</td>
<td>0.005</td>
<td>2.927</td>
<td>0.007</td>
<td>2.929</td>
<td>0.009</td>
<td>2.931</td>
<td>0.011</td>
</tr>
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</table>
Fig. 8. Graphical Solution of the Equation of

\[ r - B(3.6, r) = \frac{S_0}{y_0} \]

\[ (N=3.60, M=2.56, r \geq 1) \]
Gradually Varied Flow in Rectangular Channels

$N = 3.6, \quad M = 2.56, \quad r < 1$
The distance along the channel may be found with the aid of Fig. 8. The parameters required in using the chart are \( r \) and \( C_1 \). In Example 5(a),

\[
C_1 = \frac{V_0^2}{2gy_0} = 0.0365
\]

\[
r_1 = \frac{y_1}{y_0} = \frac{6}{4.57} = 1.31
\]

Applying these values of the parameters to Fig. 8 yields

\[
\begin{align*}
r - B(N, r) &= 1.030 \\
C_1B(N, r^{-M}) &= 0.025
\end{align*}
\]

Fig. 9. Logarithmic Plot of the Area \( A \) and of the Product \( AR^{2/3} \) Against Depth \( y \) for Circular Channels
<table>
<thead>
<tr>
<th>( r )</th>
<th>( r^{-M} )</th>
<th>( B(N, r) )</th>
<th>( B(N, r)^{-M} )</th>
<th>( 0.04 )</th>
<th>( 0.06 )</th>
<th>( 0.08 )</th>
<th>( 0.10 )</th>
<th>( 0.12 )</th>
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<tr>
<td>0.992</td>
<td>1.020</td>
<td>1.904</td>
<td>5.992</td>
<td>0.240 &amp; -0.672</td>
<td>0.360 &amp; -0.552</td>
<td>0.479 &amp; -0.433</td>
<td>0.599 &amp; -0.313</td>
<td>0.719 &amp; -0.193</td>
</tr>
<tr>
<td>0.985</td>
<td>1.040</td>
<td>1.699</td>
<td>5.499</td>
<td>0.220 &amp; -0.494</td>
<td>0.330 &amp; -0.384</td>
<td>0.440 &amp; -0.274</td>
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<td>0.660 &amp; -0.054</td>
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<td>1.547</td>
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<td>0.307 &amp; -0.266</td>
<td>0.409 &amp; -0.164</td>
<td>0.511 &amp; -0.062</td>
<td>0.614 &amp; 0.041</td>
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<td>4.367</td>
<td>0.175 &amp; -0.151</td>
<td>0.262 &amp; -0.064</td>
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<td>0.437 &amp; 0.111</td>
<td>0.524 &amp; 0.198</td>
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<tr>
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<td>3.896</td>
<td>0.156 &amp; -0.040</td>
<td>0.234 &amp; 0.038</td>
<td>0.312 &amp; 0.116</td>
<td>0.390 &amp; 0.194</td>
<td>0.468 &amp; 0.272</td>
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<tr>
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<td>0.290 &amp; 0.153</td>
<td>0.363 &amp; 0.226</td>
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</tr>
<tr>
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<td>0.834</td>
<td>2.827</td>
<td>0.131 &amp; 0.054</td>
<td>0.197 &amp; 0.120</td>
<td>0.265 &amp; 0.186</td>
<td>0.329 &amp; 0.252</td>
<td>0.394 &amp; 0.317</td>
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<td>0.161 &amp; 0.142</td>
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<td>0.269 &amp; 0.250</td>
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<td>0.225 &amp; 0.221</td>
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<td>0.094 &amp; 0.556</td>
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<td>0.070 &amp; 0.738</td>
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</table>

Table 8

Values of \( x \frac{S_0}{\gamma_0} \) for Various Values of \( r \) and \( C_1 \): Circular Channels - \( M = 2.56, N = 3.6, \frac{M}{N} = 1.4 \)
Fig. 10. Graphical Solution of the Equation of $x \frac{S}{y_0} = r - B(2.8, r) + C_B(1.4, r^2)$ 
$N = 2.8, M = 2.0, r > 1$
Gradually Varied Flow in Circular Channels
Therefore
\[ \frac{S_0}{y_0} = 1.030 + 0.025 = 1.055 \]

Similarly, at the downstream section where \( r_2 = \frac{y_2}{y_0} = 1.97 \),
\[ \frac{x_2}{y_0} = 1.878 + 0.008 = 1.886 \]

Whence
\[ \Delta x \frac{S_0}{y_0} = x_2 \frac{S_0}{y_0} - x_1 \frac{S_0}{y_0} = 1.886 - 1.055 = 0.831 \]
or
\[ \Delta x = \frac{y_0}{S_0} (0.831) = 9500 \text{ ft} \]

For the case of negligible velocity head changes, the integral embodying the kinetic energy changes; i.e., \( B^M_N, r^{-M} \), is omitted from the final equation (Eq. 48). In Example 5b, therefore, \( C_1 = 0 \) and
\[ \Delta x = \frac{y_0}{S_0} [\Delta r - \Delta B(3.33, r)] \]
\[ = 11,430 (1.878 - 1.030) = 9690 \text{ ft} \]

For Example 5c in which the effects due to non-uniform velocity distribution and eddy losses are not negligible, the modified equation of gradually varied flow should be used. Referring to Eq. 49 it can be seen that
\[ C_1 = (C_m - k) \frac{V_0^2}{2gy_0} = (1.10 - 0.20) (0.0365) = 0.033 \]

At the upstream section where \( r_1 = 1.31 \),
\[ r - B(N, r) = 1.030 \]
\[ C_1 B^N_M, r^{-M} = 0.023 \]

At the downstream section where \( r_2 = 1.97 \)
\[ r - B(N, r) = 1.878 \]
\[ C_1 B^N_M, r^{-M} = 0.0075 \]
Therefore
\[ \Delta x = \frac{y_0}{S_0} \left( 1.878 + 0.0075 - 1.030 - 0.023 \right) = 9,520 \text{ ft} \]

**b. Example 6**

Suppose the rate of flow in the broad rectangular channel in the previous example is to be measured by two gaging stations 5000 ft apart. If the stage at the upstream gage is 5.0 ft and the fall is 1.5 ft as indicated by the downstream gage, what is the rate of flow in the channel? Assume \( C_m = 1.10 \) and \( k = 0.20 \).

The problem consists of selecting a discharge such that the fall between the two sections will be equal to 1.5 ft under the given conditions. This can be done by trial, or by graphical interpolation from a plot of the fall against assumed rates of flow as follows:

The maximum fall between the two sections is attained when the surface is level, for which

\[ q = 0 \]
\[ y_2 = y_1 + 5000 S_0 \]
\[ y_2 - y_1 = 2.0 \text{ ft} \]

The minimum fall occurs where the flow is uniform, i.e.,

\[ y_1 = y_2 = 5.0 \text{ ft} \]
\[ q = 12 \left( \frac{5}{4} \right)^{5/3} = 17.4 \text{ cfs} \]

Since the fall of 1.5 ft lies between the maximum and minimum conditions, the corresponding flow must assume an intermediate value between zero and 17.4 cu ft per sec. To plot the fall-discharge curve, the computations may proceed by taking a series

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( q )</th>
<th>( y_0 )</th>
<th>( C_1 )</th>
<th>( r_1 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( r_2 )</th>
<th>( y_2 )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ft</td>
<td>cfs/ft</td>
<td>ft</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5.0</td>
<td>7.46</td>
<td>3.0</td>
<td>0.029</td>
<td>1.67</td>
<td>1.450</td>
<td>0.667</td>
<td>2.207</td>
<td>2.27</td>
<td>6.81</td>
</tr>
<tr>
<td>5.0</td>
<td>9.60</td>
<td>3.5</td>
<td>0.030</td>
<td>1.43</td>
<td>1.236</td>
<td>0.571</td>
<td>1.807</td>
<td>1.90</td>
<td>6.65</td>
</tr>
<tr>
<td>5.0</td>
<td>12.00</td>
<td>4.0</td>
<td>0.032</td>
<td>1.25</td>
<td>0.951</td>
<td>0.500</td>
<td>1.451</td>
<td>1.60</td>
<td>6.40</td>
</tr>
<tr>
<td>5.0</td>
<td>14.70</td>
<td>4.5</td>
<td>0.033</td>
<td>1.11</td>
<td>0.623</td>
<td>0.444</td>
<td>1.067</td>
<td>1.32</td>
<td>5.94</td>
</tr>
<tr>
<td>5.0</td>
<td>17.40</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
of intermediate flows and finding the corresponding fall, the upstream depth \( y_1 \) being held constant at 5.0 ft. This is done in the preceding table.

From these results it is apparent that the flow corresponding to a fall of 1.5 ft lies between 9.60 and 12.0 cu ft per sec. By graphical interpolation from the fall-discharge curve in Fig. 11 it is found to be 11.5 cu ft per sec.

If it is desired to establish the stage fall-discharge relationship for the channel similar curves may be drawn, each corresponding to a fixed depth at the upstream section. From such a plot it is possible to determine the discharge in the channel when the upstream gage reading, together with the fall, is given.

c. Example 7.

Water is fed from an impounding reservoir to a storage reservoir by gravity flow through the rectangular channel described in Example 1. The length of the channel \( \Delta x \) is 5000 ft, the slope \( S_0 \) is 0.0004. Assuming the depth at the outlet of the impounding reservoir to stand relatively constant at 5.0 ft, construct the delivery curve of the channel for various depths at the inlet to the storage reservoir.

The delivery curve (see Fig. 12a) has several special points. At \( a \) there is no flow, since the water surface is level. The corresponding depth at the lower section is

\[
y_2 = 5.0 + (0.0004) 5000 = 7.0 \text{ ft}
\]

At point \( b \), corresponding to \( y_2 = 5.0 \text{ ft} \), the flow is uniform and the rate of discharge is
At point \( c \), corresponding to \( y_2 = y_c \), the delivery is a maximum; no additional flow can be obtained by further lowering \( y_2 \) below the critical depth \( y_c \). Therefore, for the portion of the curve lying between points \( a \) and \( b \), corresponding to rates of flow below 183.8 cu ft per sec, the surface profile belongs to the \( M_1 \)-type for which Fig. 8a is applicable. At higher rates of flow the surface profile assumes the \( M_2 \) curve, for which Fig. 8b may be used.

The computation procedure is illustrated by first determining a particular point on the delivery curve, lying between points \( a \) and \( b \). For this portion of the curve the rate of discharge is less than 183.8 cu ft per sec.

Assume \( Q = 136 \) cu ft per sec, for which

\[
\begin{align*}
\gamma_0 &= 4.0 \text{ ft} \\
C_1 &= \frac{V_0^2}{2gy_0} = 0.045 \\
r_1 &= \frac{y_1}{y_0} = \frac{5}{4} = 1.25
\end{align*}
\]

The value of \( x \frac{S_0}{y_0} \), read from Fig. 8a, is 0.793. Therefore

\[
x_1 = 0.793 \frac{y_0}{S_0} = 7930 \text{ ft}
\]

Consequently

\[
x_2 = \Delta x + x_1 = 5000 + 7930 = 12,930 \text{ ft}
\]

so that,

\[
x_2 \frac{S_0}{y_0} = 12,930 (0.0001) = 1.293
\]

From Fig. 8a, \( r_2 \) is found to be 1.55. Hence, the depth at the lower section is

\[
\gamma_2 = 1.55 (4.0) = 6.20 \text{ ft}
\]

For a point on the delivery curve between points \( b \) and \( c \), the rate of flow is greater than 183.8 cu ft per sec. Assume \( Q = 204 \) cu ft per sec, for which
Fig. 12a. Delivery of a Canal; Fig. 12b. Delivery of the 10-ft Rectangular Channel in Example 7

\[ y_0 = 5.4 \text{ ft} \]

\[ C_1 = \frac{V_0^2}{2gy_0} = 0.041 \]

\[ r_1 = \frac{y_1}{y_0} = \frac{5.0}{5.4} = 0.925 \]

The corresponding value of \( \frac{x}{y_0} \) from Fig. 8b is -0.26. Therefore,

\[ x_1 = -0.26 \left( \frac{5.4}{0.0004} \right) = -3510 \text{ ft} \]

Consequently,

\[ x_2 = \Delta x + x_1 = 5000 - 3510 = 1490 \text{ ft} \]

and,

\[ x_2 \frac{S_0}{y_0} = 1490 \left( \frac{0.0004}{5.4} \right) = 0.11 \]

From Fig. 8b it is seen that for \( C_1 = 0.041 \), the maximum value that \( \frac{x}{y_0} \) may reach is 0.093, corresponding to \( r_c = 0.433 \), there being no value of \( r \) which would satisfy \( x \frac{S_0}{y_0} = 0.11 \). The physical meaning of this result is that the \( M_2 \) curve corresponding to \( y_0 = 5.4 \text{ ft} \) and \( Q = 204 \text{ cu ft per sec} \) is shorter than the length of the channel. In other words, the delivery \( Q = 204 \text{ cu ft per sec} \) is in excess of \( Q_{\text{max}} \).

To find the value of the maximum delivery, which obviously lies between 204 cu ft per sec and the uniform flow of 183.8 cu ft
per sec, it is necessary to determine the values of \( y_1 \) for several intermediate rates of flow, making in each case, \( y_2 = y_0 \). The discharge, which results in \( y_1 = 5.0 \) ft is the maximum delivery. The method of computation is shown in the following table, from which the maximum discharge is found to be 202 cu ft per sec.

**Computation of Maximum Delivery**

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( y_0 )</th>
<th>( C_1 )</th>
<th>( y_c )</th>
<th>( r_2 )</th>
<th>( x_2 y_0 )</th>
<th>( x_0 )</th>
<th>( x_1 y_0 )</th>
<th>( r_1 )</th>
<th>( y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>assumed</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td></td>
</tr>
<tr>
<td>188</td>
<td>5.1</td>
<td>0.042</td>
<td>2.22</td>
<td>0.435</td>
<td>0.094</td>
<td>0.393</td>
<td>-0.299</td>
<td>0.936</td>
<td>4.77</td>
</tr>
<tr>
<td>194</td>
<td>5.2</td>
<td>0.042</td>
<td>2.27</td>
<td>0.436</td>
<td>0.094</td>
<td>0.385</td>
<td>-0.291</td>
<td>0.935</td>
<td>4.86</td>
</tr>
<tr>
<td>202</td>
<td>5.36</td>
<td>0.041</td>
<td>2.34</td>
<td>0.436</td>
<td>0.092</td>
<td>0.374</td>
<td>-0.282</td>
<td>0.932</td>
<td>5.00</td>
</tr>
</tbody>
</table>

The computations of \( y_2 \) for different rates of flow in the range between the maximum delivery and \( Q = 70.0 \) cu ft per sec are tabulated below. Assuming \( y_1 \) to remain constant at 5.0 ft, the delivery of the channel for various depths \( y_2 \) at the lower section is plotted in Fig. 12b.

**Computation of the Delivery**

\( (y_1 = 5.0 \) ft)\

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( y_c )</th>
<th>( y_0 )</th>
<th>( C_1 )</th>
<th>( r_1 )</th>
<th>( x_1 y_0 )</th>
<th>( x_0 )</th>
<th>( x_2 y_0 )</th>
<th>( r_2 )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>cfs</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td>ft</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>7.00</td>
</tr>
<tr>
<td>70.0</td>
<td>1.15</td>
<td>2.5</td>
<td>0.049</td>
<td>2.00</td>
<td>1.845</td>
<td>0.800</td>
<td>2.645</td>
<td>2.73</td>
<td>6.81</td>
</tr>
<tr>
<td>112.5</td>
<td>1.58</td>
<td>3.5</td>
<td>0.046</td>
<td>1.43</td>
<td>1.114</td>
<td>0.571</td>
<td>1.685</td>
<td>1.87</td>
<td>6.53</td>
</tr>
<tr>
<td>136.0</td>
<td>1.79</td>
<td>4.0</td>
<td>0.045</td>
<td>1.25</td>
<td>0.793</td>
<td>0.500</td>
<td>1.293</td>
<td>1.55</td>
<td>6.20</td>
</tr>
<tr>
<td>160.0</td>
<td>2.00</td>
<td>4.5</td>
<td>0.044</td>
<td>1.11</td>
<td>0.427</td>
<td>0.445</td>
<td>0.872</td>
<td>1.29</td>
<td>6.80</td>
</tr>
<tr>
<td>183.8</td>
<td>2.20</td>
<td>5.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5.00</td>
</tr>
<tr>
<td>188.0</td>
<td>2.22</td>
<td>5.1</td>
<td>0.042</td>
<td>0.980</td>
<td>-0.644</td>
<td>0.393</td>
<td>-0.251</td>
<td>0.925</td>
<td>4.71</td>
</tr>
<tr>
<td>194.0</td>
<td>2.27</td>
<td>5.2</td>
<td>0.042</td>
<td>0.961</td>
<td>-0.440</td>
<td>0.385</td>
<td>-0.055</td>
<td>0.695</td>
<td>3.61</td>
</tr>
<tr>
<td>202.0</td>
<td>2.34</td>
<td>5.36</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.34</td>
</tr>
</tbody>
</table>

The delivery curve in this example is obtained under the assumption that the depth at the upper section \( y_1 \) remains fixed. In the general case, however, both the depths \( y_1 \) and \( y_2 \) will vary. The problem, then, is to determine the rate of flow in the channel for any possible combination of depths at the given sections. This can be done by computing as before a series of
delivery curves, each corresponding to a fixed value of the depth at one of the given sections. The final results may be represented graphically on a three-dimensional plot, showing the relationship between the fixed depth, fall and discharge.

d. Example 8.

The flow in a 10-ft circular conduit \((S_0 = 0.001, n = 0.015)\) is 305 cu ft per sec. The depth at a certain section is found to be 5.70 ft. Compute the depth at a section 1230 ft downstream from the first section.

This problem is obviously the reverse of Example 3. The uniform and critical depths are respectively \(y_0 = 6.0\) ft and \(y_C = 4.11\) ft. The other data are

\[
C_1 = \frac{y_0^2}{2gy_0} = 0.10
\]

\[
r_1 = \frac{5.7}{6.0} = 0.95
\]

Using these results in Fig. 10b, \(x = \frac{S_0}{y_0}\) is found to be 0.05. Therefore

\[
x_1 = \frac{y_0}{S_0} (0.05) = 300 \text{ ft}
\]

\[
x_2 = 300 + 1230 = 1530 \text{ ft}
\]

and

\[
x_2 \frac{S_0}{y_0} = 1530 \left(\frac{0.001}{6}\right) = 0.255
\]

From Fig. 10b it is found that for \(C_1 = 0.10, x = \frac{S_0}{y_0}\) is satisfied by two values of \(r\); viz, 0.72 and 0.63. The corresponding depths are, respectively, 4.32 ft and 3.78 ft. Evidently the required depth is 4.32 ft since the dropdown curve cannot cross the critical depth of 4.11 ft to reach a depth of 3.78 ft without passing through a hydraulic drop, a condition to which the gradually varied-flow equation no longer applies.

It should be noted that these results are obtained with the aid of Fig. 10b, which is based on \(N = 3.60\) and \(M = 2.56\). If the actual values of the exponents for the 10-ft circular channel are used \((N = 3.50, M = 2.58)\), the required depth should come out the same as in Example 3; viz, 4.11 ft. The deviation of results in
using the prepared chart is roughly 5.0 percent, which falls within the degree of accuracy warranted by most practical profile computations.

17. Summary

The application of the proposed method to profile determinations has been clearly illustrated by solving several typical examples. Special charts based on the proposed method were drawn to facilitate the solution of the equation of gradually varied flow in rectangular and circular channels. These charts are best adapted to the determination of the delivery in a canal, thereby affecting a great saving of time, as well as the labor of computation, over the graphical or the step methods. The computations, from which the special charts were prepared, are also given. Using these computations the charts should be reproduced to a more usable scale. In the event that the actual exponents for the given channel vary slightly from those used in preparing Figs. 7, 8, and 10, the charts may be applied to yield at least a first approximation, as shown in Example 8. For channels of other cross-sections similar charts should be prepared according to the particular values of the exponents.
VI. LABORATORY INVESTIGATIONS

18. Purpose and Scope

The purpose of the laboratory investigation was twofold: to determine the deviation of the actual surface profiles from the theoretical; and to obtain data on actual surface profiles as the basis for comparison between the proposed method and the various methods of computation that are in common use.

In the laboratory the actual surface profiles of flow in experimental channels of rectangular, trapezoidal and circular cross-sections were recorded for different rates of flow. The channels were all set on a sustaining slope of 0.001 due to the limited capacities of the pumps used. Only the backwater (\(M_1\)-type surface curve) and the dropdown (\(M_2\)-type surface curve) were observed, inasmuch as they are the more common types of surface curves encountered in practice. The length of the backwater curve was limited by the depth for uniform flow and the height of the side walls of the experimental channels; therefore, a low rate of flow was used in the backwater experiments to provide a wider range of depths covered by the \(M_1\) curves. Similarly, a high rate of flow was used in the dropdown experiments since, in this case, the length of the \(M_2\) curve is confined to the range between the critical depth and that for uniform flow. For this reason, each surface curve was observed only for the rate of flow most suitable under the limitations of the experimental equipment. The different rates of discharge used in the experiments are given in Tables 10-12 and ranged between 0.176 cu ft per sec and 0.425 cu ft per sec.

19. Apparatus

In the arrangement shown in Fig. 13 two 2-in. centrifugal pumps were used to lift water from the sump into the constant-head tank (A) from which the water flowed by gravity into the weir box (B) through a 4-in. connecting pipe. The rate of flow was controlled by means of a 4-in. gate valve, the excess water being returned to the sump from the constant-head tank through a 3-in. drain. The water flowed over a 90-deg \(V\)-notch weir in the weir box, and thence fell freely into the entrance tank (C). A hanging baffle was provided in the entrance tank to cut down the
Fig. 13. Plan and Elevation of Experimental Flume
inlet turbulence. The depth of flow in the experimental channel (D) was controlled by adjusting the opening of the tail-water gate in the outlet box (E), from which the water returned to the sump.

Two motor-driven centrifugal pumps were used. The combined capacity of the pumps was approximately 200 gal per min under a 20-ft head. The two 2-in. discharge pipes from these pumps met in a 4-in. riser pipe connected to the bottom of the constant-head tank.

The constant-head tank consisted of a 50-gallon drum placed concentrically inside a larger cylindrical tank, 30 in. in diameter. The rim of the drum, approximately 6.0 ft around, served to maintain practically a constant head for the rates of flow used in the experiments. The excess water overflowed the constant-head tank into the large tank and returned to the sump through a 3-in. drain. To regulate the rate of flow, a 4-in. gate valve was installed in the pipe connecting the constant-head tank to the weir box.

The weir box was constructed as a rectangular tank 2 x 2.5 x 4 ft. A gravel-filled baffle was placed at the inlet end of the weir box to reduce the entrance turbulence of the incoming flow. The tank was separated into two compartments by a 90-deg V-notch weir which was used for measuring the rate of discharge. The location of the weir plate, made of 1/8-in. black iron, and the dimensions of the V-notch weir are shown in Fig. 13. The rate of discharge corresponding to a given head on the weir was read from a calibration curve. The head on the V-notch was measured by a point gage set above the stilling well which was connected to the side of the weir box at a point 14 in. upstream from the weir. A 14-in. rectangular opening extending the full depth of the weir was cut in the downstream side to permit a free fall of water into the entrance tank.

The entrance tank consisted of a 30-in. drum, in which a diametrical wooden partition extending 2.5 ft into the tank was fastened perpendicular to the direction of flow. The partition served as a hanging baffle to reduce the inlet turbulence as the water entered the experimental channel.

The depth of flow in the channel was controlled by adjusting the tail-water gate in the outlet box which was mounted at the downstream end of the channel. The tail-water gate consisted of three vertical vanes made of wood with short pieces of sheet metal projecting from the edges as guide vanes. The wooden vanes were pivoted at the bottom of the outlet box. By turning the vanes about their vertical axes the area of the opening was adjusted to produce the desired depths of flow in the channel.
A view of the experimental channel and the outlet box with the vanes in position is shown in Fig. 14.

Point gages were used to measure the depths of flow along the channel and the head on the 90-deg V-notch weir. These gages (Fig. 15) were made of 3/8-in. round steel rods, 15 in. long, and threaded for a distance of 12 in. The unthreaded portion was tapered to a point; the top of the threaded portion was shaped and fitted with a small circular handle for turning the gage. One side of the threaded stem was flattened to record the number of the threads from the handle. The base of the gage, a 3/4 x 1
Fig. 15. Point Gage

x 6-in. block of steel, was tapped and fastened rigidly to a wooden frame across the top of the channel. The face of the base was marked with ten equal divisions along the diameter of the rod. By counting the number of threads on the rod above the base and the fraction of a thread on the base-scale, it was possible to take readings to the nearest one-tenth of a thread, that is, to 0.006 in. The depth of water was found by taking the difference of the gage readings at the water surface and some other reference point.

The experimental channel itself was located above the sump, the elevation and plan of which are also shown in Fig. 13. It rested on a framework of two 2 x 6-in. beams, spaced 5 in. apart with 2 x 4-in. spacers. Side braces were used to hold the channel sections rigidly in place. The beams were supported every 8 ft by 4 x 4-in. posts, which were braced laterally and longitudinally with 1 x 6-in. planks. A series of holes were drilled in the posts through which a 1/2-in. round iron pipe was inserted cross-wise between each set of supports. A 4 x 4-in. crosspiece was placed on top of the iron pipe. The channel and the supporting beams rested on sets of screw jacks, which in turn rested on the cross-pieces. The method of support is shown in
Fig. 13. By moving the iron pipe vertically along the series of holes in the posts, the channel could be set on any desired slope. The final adjustments were made with the jack screws which had a maximum lift of 2 in. A side view of one of the experimental channels and the supports is given in Fig. 16.

Three types of channel sections - rectangular, trapezoidal and circular - were used in the tests. The rectangular channel was made of 14-gage black iron, bent into 8 x 8-in. sides; the trapezoidal section was made from the same channel by bending the vertical sides 70 deg with the horizontal. These channels were fabricated in lengths of 4 ft and 8 ft. The joints of the separate sections were made water-tight by caulking them with plasticine clay. The total length of the channels was 40 ft; however, to avoid the error caused by turbulence at the inlet and outlet of the channel, only the 24-ft center section was used in observing the surface curves in the tests.

The circular channel was made of 24-gage black iron in 10-ft sections. An additional framework was used to support the circular channel. It consisted of rectangular boards, spaced 2½ ft apart, each of which had a 10-in. diameter semicircle cut out of the top. Rectangular slots, 1 x ½ in., were cut at the quarter points of the semicircle and longitudinal braces were inserted through these slots along the entire length of the channel. The framework rested on the 2-in. beams, with spacers fitted in

![Fig. 16. Side View of Experimental Channel and Supports](image-url)
between to prevent any horizontal movement. The sheet iron was
rolled by hand in the laboratory into the framework to form the
circular channel. Because of the inadequacy of the method used,
kinks appeared in several places along the channel. The final
dimensions of the channel were determined by blocking the ends of
a horizontal section, into which known volumes of water were
added and the corresponding depths accurately determined by means
of the point gages. The average area of the cross-section at a
given depth was found by dividing the corresponding volume of
water by the length of the section. The results, as well as the
actual values of the wetted perimeter, are given in Table 9. The
total length of the circular channel was 40 ft, of which only
the 22.5-ft center section was used as the effective length in
observing the surface profiles.

The dimensions and the hydraulic properties of the experimen-
tal channels are shown in Table 9. In determining the surface
roughness of the channels, the uniform depths corresponding
to four different rates of discharge were observed in each chan-
nel, the results being tabulated below. The value of \( n \), the
roughness factor, is computed by substituting the given data into
Manning's equation. In making subsequent profile computations
for the various channels the average value of \( n \), given in the
last column of Table 9 is used.

20. Experimental Procedure

Each channel was set on a sustaining slope of 0.001. This was
done by letting water stand in the entire channel and adjusting
the jack screws until the depth of water, measured with the point
gages, increased uniformly in the downstream direction. Since
the point gages were spaced every 8 ft, the increment of depth
between the gages was adjusted to 0.008 ft for the desired slope.

The depth of water in the channel was measured by lowering the
gages until the point barely touched the surface. The difference
between the gage reading at the water surface and the invert of
the channel was converted to the depth by a conversion factor,
which was determined by comparison with a standard gage. The
head on the weir was found in a similar manner, except that in
this case the zero reading was taken at the crest of the V-notch
weir. The rate of discharge used in the tests was found from a
calibration curve for the recorded head on the weir, which was
checked before and after each run.

The laboratory procedure used in determining the surface
curves followed essentially the same technique introduced by
Mononobe.\(^1\,^8\) That is, the entire surface curve was obtained by
conducting a series of profile determinations, the initial depth
in a new run being adjusted approximately to the final depth
Table 9
Hydraulic Properties of Experimental Channels

<table>
<thead>
<tr>
<th>Depth, y (ft)</th>
<th>Area, $A$ (sq ft)</th>
<th>Wetted Perimeter, $p_w$ (ft)</th>
<th>Radii, $R$ (ft)</th>
<th>Conveyance, $K$</th>
<th>$G$</th>
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</thead>
<tbody>
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</tbody>
</table>

**Rectangular Channel**
Slope, $S_0 = 0.001$; Roughness, $n = 0.0092$

| 0.24 | 0.160 | 1.147 | 0.140 | 6.952 | 0.246 |
| 0.28 | 0.187 | 1.227 | 0.152 | 8.597 | 0.234 |
| 0.32 | 0.213 | 1.307 | 0.163 | 10.273| 0.228 |
| 0.36 | 0.240 | 1.387 | 0.173 | 12.041| 0.219 |
| 0.40 | 0.267 | 1.467 | 0.182 | 13.861| 0.211 |
| 0.44 | 0.293 | 1.547 | 0.189 | 15.608| 0.200 |
| 0.48 | 0.320 | 1.627 | 0.197 | 17.506| 0.194 |

**Trapezoidal Channel**
Slope, $S_0 = 0.001$; Roughness, $n = 0.0106$

| 0.24 | 0.181 | 1.177 | 0.154 | 7.287 | 0.235 |
| 0.28 | 0.215 | 1.263 | 0.171 | 9.298 | 0.235 |
| 0.32 | 0.251 | 1.348 | 0.186 | 11.460| 0.233 |
| 0.36 | 0.287 | 1.433 | 0.200 | 13.770| 0.231 |
| 0.40 | 0.325 | 1.518 | 0.214 | 16.279| 0.229 |
| 0.44 | 0.364 | 1.603 | 0.227 | 18.957| 0.229 |
| 0.48 | 0.404 | 1.688 | 0.239 | 21.794| 0.228 |
| 0.52 | 0.445 | 1.773 | 0.251 | 24.834| 0.228 |
| 0.56 | 0.488 | 1.858 | 0.262 | 27.966| 0.223 |

**Circular Channel**
Slope, $S_0 = 0.001$; Roughness, $n = 0.0102$

| 0.099 | 0.030 | 0.508 | 0.060 | 0.669 | 0.228 |
| 0.134 | 0.048 | 0.596 | 0.081 | 1.311 | 0.254 |
| 0.166 | 0.065 | 0.680 | 0.096 | 1.983 | 0.255 |
| 0.202 | 0.088 | 0.773 | 0.114 | 3.008 | 0.260 |
| 0.235 | 0.110 | 0.849 | 0.130 | 4.103 | 0.268 |
| 0.268 | 0.132 | 0.919 | 0.144 | 5.273 | 0.268 |
| 0.299 | 0.155 | 0.987 | 0.157 | 6.573 | 0.273 |
| 0.328 | 0.177 | 1.052 | 0.168 | 7.855 | 0.273 |
| 0.357 | 0.199 | 1.109 | 0.179 | 9.214 | 0.273 |
| 0.387 | 0.221 | 1.193 | 0.185 | 10.464| 0.265 |
| 0.414 | 0.243 | 1.240 | 0.196 | 11.952| 0.265 |

in the run before. By connecting the series of profiles thus obtained, it was possible to extend the total length of the surface curve beyond the limit confined to the effective length of the experimental channels. The applicability of this experimental procedure has been well substantiated by tests conducted by Mitchell and Barron.\(^{13}\)

In the profile tests the flow was first adjusted to uniform conditions by manipulating the tail-water gate. The opening of
the gate was then changed by making a small turn of the vanes to establish nonuniform flow in the channel. The surface profile was recorded by four point-gages, spaced 8 ft apart for the rectangular and trapezoidal channels and 7.5 ft apart for the circular channels. The readings were taken only after the flow had become steady. In the succeeding runs the upstream depth was adjusted to duplicate the downstream depth in the run before. This procedure was repeated until the flow reached a depth slightly smaller than the height of the channel in the backwater experiments, and larger than the critical depth in the dropdown experiments. In the circular channel the flow was too turbulent to permit accurate readings of the gages at the smaller depths.

Table 10
Comparison of Test Data in Rectangular Channel with Results Computed by Various Methods

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</tr>
<tr>
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<tr>
<td></td>
<td>$S_0 = 0.001$ $Q = 0.346$ cu ft/sec $y_0 = 3.97$ in.</td>
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<td>3.03</td>
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To reduce observational error the initial depth in the succeeding run was made slightly larger (dropout) or smaller (backwater) than the final depth in the preceding run.

21. Results and Discussion

The results of the profile determinations in the various channels are presented in Tables 10, 11 and 12, in which the observed depths are given for the corresponding distances along the channel. The test data are also presented in graphical form by plotting the elevation above an arbitrary datum against the distance. The datum, indicated in Fig. 17, is arbitrarily chosen at a distance 5 in. below the origin 0, referring to the initial section from which the observed profiles are recorded. Therefore the elevation corresponding to a given depth \( y \) at a distance \( x \) from the origin may be computed according to

\[
E_l = 5 + y - S_0 x
\]

<table>
<thead>
<tr>
<th>Table II</th>
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<tbody>
<tr>
<td>Comparison of Test Data in Trapezoidal Channel with Results Computed by Various Methods</td>
</tr>
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<tbody>
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<tr>
<td>( S_0 = 0.001 )</td>
<td>( Q = 0.205 \text{ cu ft/sec} )</td>
<td>( y_0 = 2.74 \text{ in.} )</td>
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<td>4.99</td>
<td>4.98</td>
<td>4.93</td>
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</table>

| Dropdown    |
| \( S_0 = 0.001 \) | \( Q = 0.425 \text{ cu ft/sec} \) | \( y_0 = 4.34 \text{ in.} \) |
| 16          | 4.10    | 4.08          | 4.09    | 4.09           | 4.08        |
| 32          | 4.04    | 4.04          | 4.05    | 4.05           | 4.04        |
| 48          | 3.98    | 3.98          | 3.97    | 3.97           | 3.98        |
| 64          | 3.90    | 3.91          | 3.88    | 3.88           | 3.91        |
| 80          | 3.77    | 3.82          | 3.78    | 3.76           | 3.78        |
| 96          | 3.63    | 3.67          | 3.68    | 3.49           | 3.54        |
| 112         | 3.26    | 3.41          | 3.36    | 3.00           | 3.13        |
The elevations, computed from the observed depths, are plotted against the corresponding distance from the origin; the data are shown as small circles in the diagrams. Backwater measurements in the rectangular, trapezoidal and circular channels are presented in Figs. 18a,b - 20a,b; corresponding dropdown measurements in the various channels are shown in Figs. 18c,d - 20c,d. It must be emphasized that these figures are drawn to a much distorted scale, so that a slight experimental error may result in a noticeable deviation from the general trend of the data.

The results, obtained from the profile determinations in the various experimental channels, are subject to the following sources of error:

(1) Observation. In the dropdown experiments the surface of

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the water became too turbulent to permit precise measurements of
the depth. The fluctuations of the water surface were more pro-
nounced in the circular channel. In addition, when the water
flowed past the joints of the channel sections a rippling motion
of the surface was produced which was carried some distance
beyond the joints.

(2) Point Gages. The point gages were made in the laboratory.
Because of the slight bending of the rods it was impossible
to move them vertically without some side deflection of the
point. However, since the side motion was limited, it is not
believed that the depth would be greatly affected as a result of
the slightly inclined positions of the rods.

(3) Structural Defects. The circular channel was rolled by
hand, giving rise to kinks in several places. The crown of the
invert was displaced from the longitudinal axis where the kinks
were so formed that the location of the zero reading of some of
the gages could not be accurately determined.

A slight deformation of the channel sections at the ends made
it difficult to align the joints to a smooth surface. A small
amount of leakage took place at some of the clay-filled joints,
but the losses were not appreciable compared to the rates of flow
used in the tests.

After the channel was set to the desired slope, the deflection
of the section between beam supports was checked. This was done
by finding the depth at midspan below the level water surface.
The results indicated that the assumption of uniform slope
resulted in but a slight error.

Theoretically, the observed data should lie on a continuous
curve. It can be seen from Figs. 18-20 that the trend of the
actual data was generally along a smooth curve. The deviation of
some of the points, somewhat exaggerated in these figures due to
the distorted scale used, is probably caused primarily by the
observational error. The extent of experimental error, however, is obviously fairly small in view of the close agreement between the observed and theoretical results as presented in Tables 10, 11 and 12.

22. Comparison of Actual and Theoretical Profiles

The test data are compared with results computed by the graphical, step, and Bakhmeteff methods, as well as the method proposed in this bulletin. The theoretical curves were obtained by computing the distance along the channel between known sections. In each case the initial section was taken at the origin 0 in Fig. 17, and the computations proceeded by small increments of depth. The results of the various methods of computation are plotted in Figs. 18-20 for easy comparison with the test data. The backwater curves in the various channels are shown separately for the several methods. Those computed by the proposed and graphical methods are shown in Figs. 18a, 19a and 20a; those by the step and Bakhmeteff methods, in Figs. 18b, 19b and 20b. Similarly the dropdown curves in the various channels for the proposed and graphical methods are given in Figs. 18c, 19c and 20c; those for the step and Bakhmeteff methods, in Figs. 18d, 19d and 20d. The elevations above the arbitrary datum are read directly from the backwater and dropdown curves for various distances along the channel. The results are also presented in Tables 10, 11 and 12, in which the depths shown are computed from the corresponding elevations according to Fig. 17.

The method of Mononobe is omitted from these tables because his charts\(^7\) are not suitable for computing short sections such as those used in the observed profiles. For this purpose the charts should be redrawn to such a scale as to be compatible with the required degree of accuracy of the results. This could not be done, because the original values of the varied-flow functions for the preparation of the charts are not included in Mononobe’s paper.

Tables 10, 11 and 12 show that the discrepancy between the observed profiles and the theoretical curves in the various channels is reasonably small. The actual deviation is somewhat exaggerated in Figs. 18-20 as a result of the distorted scale used in plotting the curves. In addition to experimental error the discrepancy between observed and theoretical profiles may be due to any of the following causes:

1. Incorrect assumptions in computation procedure. An error may result from using average values of the exponents, \(N\) and \(M\), and the factor \(G\), in the proposed and Bakhmeteff methods; and from using the evaluation of the mean friction slope as the
Fig. 18. Backwater and Dropdown Measurements in Rectangular Channel: Comparison of Test Results with Results Computed by Various Methods
average of the slopes at the ends of a section in the step
method. The error, however, is likely to be small because in the
first two methods the values of the exponents and $G$ are relatively
constant in the range of depths covered by the observed
profiles (see Table 9) while in the third method a large number of
steps is used to give better accuracy.

(2) Erroneous assumptions in derivation of varied-flow equa-
tion. In solving the differential equation of gradually varied
flow by graphical integration no assumption is made in the integ-
ration procedure; therefore any discrepancy between the results
of the graphical method and the test data, other than that due to
experimental error, may be taken as evidence of the fallacy of
the assumptions made in deriving the varied-flow equation. These
assumptions are:

(a) hydrostatic distribution of pressure intensity
(b) uniform velocity distribution; i.e., $C_n = 1.0$
(c) losses confined mainly to friction, computed
as if the flow were uniform under identical
conditions of depth, discharge, and boundary
roughness.

Of these three assumptions the first is inconsequential except
in the immediate vicinity of the critical section. It seems
unlikely that this assumption would lead to any appreciable error
inasmuch as the dropdown measurements were not extended to the
critical depth. The error in the results is probably due pri-
marily to the second two assumptions.

The effects of the second assumption may be determined by a
consideration of the gradually varied-flow Eq. 13:

$$\frac{1 - \frac{bQ^2}{gA^3}}{S_0 - S_f} \, dx = \frac{C_n \frac{bQ^2}{gA^3}}{S_0 - S_f} \, dy$$  \hspace{1cm} (50)$$

The effect of taking velocity distribution into consideration is
to increase the value of the second term on the right. It will
be noted that the first and second terms are always opposite in
sign, the latter being negative, for the $M_1$- and $M_2$-types of
surface profiles. Therefore the effect of nonuniform velocity
distribution is to shorten the distance along the channel between
two given depths. It follows then that in the case of the back-
water profile the actual curve will lie above the theoretical
one, resulting in depths larger than those computed by the graph-
ical method; while for the dropdown curve the actual profile will
fall below the theoretical resulting in actual depths which are
Fig. 19. Backwater and Dropdown Measurements in Trapezoidal Channel: Comparison of Test Results with Results Computed by Various Methods
smaller than those computed by assuming uniform distribution of velocity (i.e., \( C_m = 1.0 \)).

Finally, the assumption that the losses in gradually varied flow are the same as for uniform flow may lead to erroneous results. This is due to the fact that the flow in the laboratory tests is subject to an additional loss from turbulence, caused by the water flowing past the joints of the channel sections. The effect of this additional loss is to increase the friction slope \( S_f \) in Eq. 50, with the results that the difference between the two slopes, \( S_f \) and \( S_0 \), becomes smaller in the case of the backwater and larger in the case of the dropdown curves. Consequently, the distance along the channel between two given sections will be increased for the backwater and reduced for the dropdown profiles. In either case the actual profile will lie below the theoretical curve, yielding depths smaller than those computed by neglecting turbulent losses.

Thus it can be seen that in the backwater case the effects of nonuniform velocity distribution and turbulence tend to offset each other. The net result will depend upon which of these effects is predominant. For the backwater determination in the circular channel (Fig. 20a), the test data fall consistently below the theoretical curve computed by the graphical method. This is probably due to the losses caused by turbulence, a factor which is omitted in the theoretical profile computations. For the rectangular and trapezoidal channels (Figs. 18 and 19a), some of the actual data are greater than the theoretical values. This discrepancy may be due to the effect of the nonuniform distribution of the velocity, which is neglected in the graphical method.

In the dropdown case the effects both of nonuniform velocity distribution and turbulence tend to shorten the actual profile from the theoretical curve. Therefore the observed depth at a given section should be smaller than that computed by the graphical method. This can be seen from the dropdown measurements in the trapezoidal and circular channels, shown in Tables 11 and 12 (or Figs. 19c,d and 20c,d). The greater discrepancy in results for the dropdown curves may be caused, partly at least, by the effects of nonuniform distribution of velocity and turbulent losses, both of which produce errors of similar sign.

It should be emphasized that the maximum deviation of the test data from the theoretical values computed by the graphical method is reasonably small, (of the order of 0.12 in.). This close agreement may be taken as evidence that the assumptions in the gradually varied-flow equation did not result in any substantial discrepancy between the theoretical and the observed profiles.
Fig. 20. Backwater and Dropdown Measurements in Circular Channel:
Comparison of Test Results with Results Computed by Various Methods
The comparison between the experimental data and the results of the graphical, step, proposed, and Bakhmeteff methods, given in Tables 10, 11 and 12, show the degree of accuracy of the computations in that order. Both the graphical and the step methods are likely to yield more precise results because of the large number of intermediate steps taken in computing the surface profiles. In the proposed and Bakhmeteff methods the results are not affected by the number of intermediate steps, which are used only to compute the elevations along the channel. In fact, these methods tend to become less accurate when applied to short reaches of channel, since in these cases a corresponding error due to interpolation of the varied-flow functions yields a wider discrepancy of results. The deviation in the proposed method is probably caused by interpolation; however, the error is reasonably small in view of the close agreement between the actual profiles and those computed by the proposed method. In general, there is only a slight difference between the results of the proposed and Bakhmeteff methods. This is due to the fact that the observed profiles are confined to a limited range of depths, in which the actual variation of the factor $G$ is small. Under these conditions it makes little difference whether $G$ is assumed to be constant as in Bakhmeteff's method, or considered as a variable with depth as in the proposed method.
VII. CONCLUSIONS

The accuracy of the proposed method is determined by comparing the computed surface curves with the observed profiles in various experimental channels of rectangular, trapezoidal, and circular cross-sections. The test data are also compared with the results computed by the other existing methods. The method of Mononobe is omitted because his graphical charts cannot be read with the desired degree of accuracy. From the results of the experimental investigations, the following conclusions are drawn:

1. No substantial error is introduced by the assumptions made in the gradually varied-flow equation as indicated by the close agreement between the observed profiles and those computed by the graphical method.

2. The surface curves computed by the proposed method agree reasonably well with the observed profiles so that it may be expected that the proposed method can be applied to practical computations with reasonable accuracy.

3. The computations of surface profiles by the graphical, step, proposed, and Bakhmeteff methods yield only slightly different results, the order of accuracy being the same as the order of the above list.

It must be emphasized that the steps used in making the computations were so small that the graphical and step methods yielded results which were more accurate than those obtained by the proposed method. In practice, however, the proposed method will be decidedly more convenient to use inasmuch as the computations need not proceed by successive increments. The results of the experiments indicate that the precision of the proposed method falls well within the degree of refinement warranted in most practical profile computations.
VIII. SUMMARY

The basic assumptions leading to the equation of gradually varied flow are discussed at some length. The simplification is based upon the one-dimensional method of analysis, in which the flow is considered as being essentially parallel. The differential equation for gradually varied flow is derived from the simplified equation of momentum, in contrast to the more common procedure of deriving it from energy considerations. The difference between the two forms of the equation is explained. It is concluded that the form derived from the momentum principle is more consistent with the empirical expressions for the evaluation of the friction slope since the latter expressions are essentially momentum equations.

The solution of the differential equation for gradually varied flow by graphical, approximate and direct integration is discussed with regard to the advantages and limitations of each method. The choice of the method to be used is decided by the type of problem under investigation. Theoretically, the graphical method should yield results of the greatest accuracy inasmuch as no assumption is made in the integration procedure. The work involved in this method, however, is quite tedious so that it has not been widely used. In many practical problems the method of approximate integration or the step method may be used with sufficient accuracy. The step method is limited by the fact that computations must be made by small increments of depth in order to satisfy the assumptions made in evaluating the mean friction slope within each interval. The objection is not serious, however, in solving the type of problems in which only a few surface curves have to be computed and elevations are needed all along the channel. In another type of problem, on the other hand, such as the determination of the delivery in a channel connecting two reservoirs, a large number of surface curves are required for a larger number of discharge rates. Under these circumstances, the use of the direct integration methods will prove to be of decided advantage in the time and labor of computations saved.

The progress of the various direct integration methods represents the trend in developing a solution of the gradually varied-flow equation to dispense with the large number of intermediate steps required by approximate integration. The earlier attempts
are incomplete in that either the effect due to velocity changes is neglected or the friction slope is evaluated by means of the Chezy equation with a constant coefficient, the use of which has been outmoded. Moreover, these methods are based on some idealized channel cross-section of simple geometric shapes. These simplifications may lead to a substantial deviation between the computed and observed profiles, as shown by the experiments conducted by Mononobe. For closer agreement between the theoretical and actual results, the actual shape of the channel must be considered, and the friction slope should be evaluated by a more suitable friction formula than that of Chezy.

By using an empirical relationship between the conveyance and the depth, Bakhmeteff was able to derive a more general solution in which the effect due to the shape of the channel may be taken into consideration, and the friction slope may be evaluated by any suitable friction formula. His method is convenient when applied to the problems in which the effect of velocity head changes is insignificant. Otherwise, the correction that must be made to take the velocity head changes into account necessarily destroys the convenience of his method. In the method of Mononobe the computation procedure is more direct; unfortunately it is marred by the difficulty of using his graphical charts and the inherent approximations that are likely to result due to interpolation of the varied-flow functions.

These difficulties are avoided in the proposed method which is, in effect, an extension of Bakhmeteff's method to cover those cases involving significant effects due to velocity head changes. The integral, embodying these changes, is reduced to the form of the varied-flow function defined by Bakhmeteff. The values of the integral have been computed over a wide range, sufficient to meet the conditions encountered in most practical profile computations. The convenience of the proposed method is illustrated by solving several typical examples.
APPENDIX: BIBLIOGRAPHY


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