ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

PRODUCTION NOTE

University of Illinois at
Urbana-Champaign Library
Stresses around Mine Openings in Some Simple Geologic Structures

by
R. D. Caudle
G. B. Clark
Stresses around Mine Openings in Some Simple Geologic Structures

by

R. D. Caudle
FORMERLY RESEARCH ASSISTANT IN MINING ENGINEERING

G. B. Clark
FORMERLY PROFESSOR OF MINING ENGINEERING

ENGINEERING EXPERIMENT STATION BULLETIN NO. 430
CONTENTS

I. INTRODUCTION

II. INITIAL STRESSES AND STRESS CONCENTRATIONS
IN THE EARTH'S CRUST

III. STRESSES AROUND OPENINGS IN SOLID HOMOGENEOUS
MATERIALS

1. Early Underground Stress Analysis
2. Theory of Elasticity Applied to Underground Mine Structures
3. Photoelasticity Applied to Underground Mine Structures
4. Stress Distribution around a Single Opening
   a. Circular Openings
   b. Elliptical Openings
   c. Ovaloidal Openings
   d. Rectangular Openings
   e. Summary
5. Stress Distribution around Multiple Openings
   a. Circular Openings
   b. Ovaloidal Openings
   c. Rectangular Openings
   d. Summary

IV. STRESSES IN SIMPLE STRATIFIED ROOFS

6. Centrifugal Testing to Simulate Stresses Occurring in
   Rock Beams Underground
7. Comparison of Stresses in Rock Beams for Three Types of
   Loads by means of the Theory of Elasticity

V. MATHEMATICAL ANALYSIS OF STRESSES IN SIMPLE
ROOF STRATA

8. Simple Beams
   a. Uniform Load
   b. Loaded by Own Weight
   c. Centrifugal Loading
9. Restrained Beams
10. Summary

VI. SUMMARY AND CONCLUSIONS

VII. BIBLIOGRAPHY
<table>
<thead>
<tr>
<th>FIGURES</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Assumed States of Stress in the Earth at a Great Distance from any Disturbing Influence</td>
<td>9</td>
</tr>
<tr>
<td>2. Pressure Dome and Stress Trajectories around a Drift</td>
<td>12</td>
</tr>
<tr>
<td>3. Tangential Stresses for a Circular Cylindrical Opening in a Semi-Infinite Mass as Affected by Increasing Depth</td>
<td>14</td>
</tr>
<tr>
<td>4. Uniform Compressive Stresses in an Infinite Plate at a Great Distance from any Disturbing Influences</td>
<td>15</td>
</tr>
<tr>
<td>5. Areal Distribution of Radial Stress along the Horizontal and Vertical Axes of Symmetry for a Circular Hole in an Infinite Plate</td>
<td>15</td>
</tr>
<tr>
<td>6. Areal Distribution of Tangential Stress along the Horizontal and Vertical Axes of Symmetry for a Circular Hole in an Infinite Plate</td>
<td>15</td>
</tr>
<tr>
<td>7. Tangential Stress Concentration on the Boundary of a Circular Opening in an Infinite Plate</td>
<td>16</td>
</tr>
<tr>
<td>8. Stress Concentration on the Boundary of an Ellipse at the Major and Minor Axes as the Height-to-Width Ratio Varies</td>
<td>17</td>
</tr>
<tr>
<td>9. Tangential Stress Concentration on the Boundary of an Ovaloidal Opening (Square with Semicircles Attached to Opposite Ends)</td>
<td>18</td>
</tr>
<tr>
<td>10. Maximum Stress Concentration as a Function of Height-to-Width Ratio for Ovaloidal Opening — Unidirectional Stress Field</td>
<td>18</td>
</tr>
<tr>
<td>11. Stress Concentration at End of Axis Perpendicular to the Direction of Applied Stress as a Function of Height-to-Width Ratio — Unidirectional Stress Field</td>
<td>18</td>
</tr>
<tr>
<td>12. Maximum Stress Concentration as a Function of Height-to-Width Ratio for Rectangular Openings Having Slightly Rounded Corners — Unidirectional Stress Field</td>
<td>19</td>
</tr>
<tr>
<td>13. Tangential Stress Concentration on the Boundary of a Square Opening in an Infinite Plate for the Three Initial States of Stress</td>
<td>20</td>
</tr>
<tr>
<td>14. Effect of Shape of Opening on Maximum Stress Concentration — Unidirectional Stress Field</td>
<td>20</td>
</tr>
<tr>
<td>15. Comparison of Critical Compressive Tangential Stress for Rectangles and Ellipse</td>
<td>21</td>
</tr>
<tr>
<td>16. Comparison of Critical Compressive Tangential Stress for Rectangles and Ellipse under Conditions of Hydrostatic Pressure</td>
<td>22</td>
</tr>
<tr>
<td>17. Stress Concentration as a Function of the Ratio of Opening Width to Pillar Width in an Applied Stress Field Perpendicular to Line of Centers</td>
<td>23</td>
</tr>
<tr>
<td>18. Distribution of Shear Stress in Pillar Formed by Two Circular Holes — Applied Stress Field Perpendicular to Line of Centers</td>
<td>23</td>
</tr>
<tr>
<td>19. Shear Stress Distribution in Central Pillars — Plate Containing Five Circular Openings in an Applied Stress Field Perpendicular to Line of Centers</td>
<td>24</td>
</tr>
</tbody>
</table>
FIGURES (Continued)

20. Relation between Maximum Stress Concentration and Number of Pillars for Ratio of Opening Width to Pillar Width of 4.0
21. Maximum Stress Concentration in Pillars Formed by Circular Openings as a Function of the Ratio Opening Width to Pillar Width in an Applied Stress Field Perpendicular to Line of Centers
22. Stress Concentration as a Function of Opening Width to Pillar Width Ratio in an Applied Stress Field Perpendicular to Line of Centers
23. Shear Stress Distribution in Central Pillars — Plate Containing Five Ovaloidal Openings (Height-to-Width Ratio = 0.5) in an Applied Stress Field Perpendicular to Line of Centers
24. Maximum Stress Concentration as a Function of Percent of Recovery for Pillars Formed by Five Openings
25. Comparison of the Empirical Equation and the Experimental Data
26. Simple Beams Showing Method of Support, Loads and Coordinate Systems
27. Restrained Beams Showing Loads, Restraints and Coordinate Systems
28. Isoclinics and Stress Trajectories

TABLES

1. Critical Values of Tangential Stress on an Elliptical Boundary
2. Ovaloidal Openings — Unidirectional Stress Field
3. Stress Concentration for Ovaloids — Hydrostatic Stress Field
4. Critical Compressive Tangential Stress for a Pair of Circular Holes
5. Stress Concentration for a Plate Containing Two Circular Openings
6. Stress Concentration for a Plate Containing Three Circular Openings
7. Stress Concentration for a Plate Containing Five Circular Openings
8. Stress Concentration for a Plate Containing Two Ovaloids
9. Stress Concentration for a Plate Containing Five Ovaloids
10. Stress Concentration for a Plate Containing Five Ovaloidal Openings
11. Compilation of Stress Equations for Simple Beam Loaded by Three Type Loads
12. Compilation of Analogous Stress Equations for Simple Beam Loaded by Three Type Loads
13. Compilation of Stress Equations for Restrained Beam Loaded by Three Type Loads
14. Compilation of Analogous Stress Equations for Restrained Beam Loaded by Three Type Loads
This page is intentionally blank.
I. INTRODUCTION

The problem of accurately determining the stresses which exist in rocks in the earth's crust has long been of interest to engineers and geologists. Many mining problems are directly concerned with the stresses which may cause mine openings to collapse during the course of their usage. This Bulletin is concerned with two phases of occurrence of rock stresses: (1) the stresses existing in the rock before the introduction of mine openings, and (2) the altered rock stresses due to the introduction of mine openings. The development of the investigation of rock stresses has to the present been limited to simple geological structures.

There have been many hypotheses formulated to explain the pre-stressed state of the earth's crust and its causes. Unfortunately, no one has been able to measure the initial earth stresses without changing them in the process, and thus invalidating the results. At present, only a qualitative evaluation has been made of the initial earth stresses from observation of the effect of stresses upon actual mine openings, and upon geologic structures.

It has been the general practice to date for the design of mine structures (width, height, and contour of mine openings, and the size and shape of pillars) to be determined upon an empirical basis. This is true largely because (1) the effects of making an underground mine opening upon the pre-existing stresses within the surrounding rock have not been understood, (2) the concept of pre-existing stresses has been expressed in an inexact manner, and (3) the varying effects of complex geologic conditions have not been determined. Because of lack of knowledge of these three important points, mine structures have been designed from formulae for which there is no complete justification (except whether they succeed or fail). To insure the stability of the mine structures designed in this manner, it has been necessary to apply safety factors of such a magnitude that the original method of solution becomes of questionable value.

Research concerned with the stresses around mine openings may be classified in three general categories: (1) theoretical studies of a purely mathematical character, (2) studies of models intended to duplicate the stress conditions existing in the prototype mine opening, and (3) observation and measurement of the stress conditions in an actual mine opening.

Theoretical studies differ widely in the basic assumptions made about the physical properties of the rock itself. This controversial issue has yet to be resolved completely. Thus, some solutions of problems in underground stress analysis assume that rock is elastic, homogeneous, and isotropic in character; others assume that rock possesses plastic, viscous, elasticoviscous properties, or a combination of them. Recently there has been a trend toward the application of soil mechanics to underground mining problems. As yet, there has not been conclusive evidence presented to indicate which of the particular methods of solution has the greatest applicability. In this survey a review is made of the theoretical studies which have been based upon the assumption that rock surrounding the mine opening is elastic and isotropic in character. Such an approach was chosen since most of the preliminary stress studies have been from this viewpoint, and it furnishes a starting point from which studies involving more complex physical characteristics of rock may be considered.

Experimentation with full-scale rock structures such as those which are found in underground mine openings has, with few exceptions, proven too impractical and too costly to be worthwhile. If exact results are to be obtained, it is usually more practical to conduct them in a laboratory where test conditions can be closely controlled. Inasmuch as full-scale models of mine structures cannot be constructed in the laboratory, however, the most feasible approach to the problem is to resort to the use of small scale models. In general, model studies are divided into two categories, those involving photoelastic principles, and those concerned with testing models made of rock from the prototype. In photoelastic studies a model of the prototype is made from a suitable transparent material. This model is placed in an instrument (polariscope) and examined...
while under a stress similar to that applied to the prototype. The stress concentrations which are disclosed by the polarized light are then compared with those in the prototype. In the second method, models made of rock from the prototype are stressed in a manner which will give an approximation of the stresses applied to the prototype; the stresses developed within the model are measured with strain gages, or the model is stressed to the failure point. Several methods of applying stress to a rock model have been attempted. One of the most successful methods has been that of applying a centrifugal force to the model in order to simulate stresses due to the model's own weight. This method of model study is explained in Chapter V, which contains a theoretical analysis of some of the simpler problems that are applicable to centrifugal testing.

It should be made clear that none of the methods for solution of underground stresses explains all stress phenomena observed because of the lack of accurate knowledge of the physical properties of rocks under field conditions, and the great complexity of these conditions due to inhomogeneity of the rock, geologic discontinuities, and many other factors.

It is the purpose of this paper to present an analysis of underground stresses which may be applied to certain simple geologic structures to clarify some points of fundamental research which have previously been neglected, and to indicate further possible applications of this research. In addition, it is hoped that the review of literature concerned with underground stress analysis from the viewpoint of elasticity, photoelasticity, and centrifugal testing will prove beneficial to others as a basis for more research.
II. INITIAL STRESSES AND STRESS CONCENTRATIONS IN THE EARTH’S CRUST

Several scientists have made interpretations of the initial stresses existing underground before a mine opening has been introduced. In general, these stresses are known to be influenced primarily by the weight of the overlying material, the relation of the opening of the rock masses around it (depth of overburden, etc.), geologic discontinuities (faulting, bedding planes, etc.), and the physical characteristics of the surrounding rock.

A reasonable hypothesis for the initial stresses existing in underground rock before a mine opening has been introduced was used by Mindlin in 1939. It was assumed that stresses within the earth at different depths may be approximated by one of three states of pressure, as shown in Fig. 1:

- **Hydrostatic pressure**
- **Laterally restrained**
- **No lateral restraint**

Fig. 1. Assumed States of Stress in the Earth at a Great Distance from any Disturbing Influence

They are (1) initially hydrostatic stresses acting on each unit of the solid, the state of materials at depths greater than those now mined, (2) initial lateral restraint during the application of the gravitational field, an approximation of the forces acting at an intermediate depth within the earth, and (3) no initial lateral restraint on a unit of the solid, the state of some materials in the immediate vicinity of the surface. These cases represent the range of variation of earth stresses. The actual initial stress condition existing underground before a mine opening is introduced generally lies between the two extremes. For this reason, these three conditions have been widely used in solutions by photoelastic and elastic analytical methods.

From observation of the phenomena which occur in all types of underground mine workings, it is apparent that these three conditions are not always sufficient to embrace all possible initial earth stresses. Beyl has pointed out that the effect of overbridging beds and the lateral transmission of stress, which occurs in pulverulent matter, partly relieve the excavations of vertical stress caused by the weight of the rock. In some cases erosion relieves some of the vertical stress while the lateral pressure remains constant. Furthermore, where work is carried on at great depth, lateral pressures are sometimes observed which are higher than expected for that depth. Even in areas of regular horizontal stratification, one can notice traces of horizontal thrusts. This is an indication of appreciable orogenetic (mountain building) pressure in a horizontal direction. Beyl lists another pressure with a thermic origin. This pressure is multilateral due to the exothermic transformation of peat into lignite and coal, the intrusion of magma, or the occurrence of metamorphism.

Beyl obtains the state of stress in the rock mass at the surface and at depth by the superposition of three fields of pressure: (1) a horizontal force due to orogenetic pressure which is often the largest component, (2) a vertical force resulting from the weight and representing part of the weight of overlying deposits, (3) a hydrostatic pressure equal in all directions. He has made use of these concepts to explain on a qualitative basis some of the conditions observed in rock formations under high

* Parenthesized superscripts refer to correspondingly numbered entries in the Bibliography.
pressures. Unfortunately solutions based upon these suppositions require a knowledge of the orogenetic pressure and the vertical forces transferred by the layers of rock to lower layers (Beyl's theory of the effect of overbridging beds) which is not available at the present time. For that reason, in the study of this problem use is made of the three conditions postulated by Mindlin which cover most of the circumstances encountered except when lateral pressures are greater than vertical pressures. For this latter case, it is only necessary to superimpose a horizontal stress of the desired magnitude upon the stresses given for the case of no lateral restraint. The difficulty remaining is in determining the required magnitude of the horizontal stress. If a method can be devised for measuring the pre-existing stresses in the earth's crust without disturbing them, the possibilities of expressing those stresses mathematically with a higher degree of accuracy will be greatly increased.
The immediate purpose of stress analysis of underground mine structures is essentially twofold: (1) to obtain a concept of the effect of the size and shape of a single mine opening upon the stresses existing initially (before the opening was introduced) within the surrounding rock, and (2) to determine the effect of a group of mine openings upon the stresses existing initially within the rock as their size, shape, number, and relative positions are varied. The ultimate purpose is to apply the results of these analyses to achieve more economical mining operations.

Stress analyses of underground mine openings for many simple cases have been performed by pure mathematical analysis and an analysis of models in the laboratory. The results of these analyses cannot always be applied directly to obtain quantitative results for the general underground mining problem because they are solutions of problems which were chosen for the simplicity of the concepts and mathematics involved. They are specific cases, and their field of application is limited. It is this aspect of stress analysis which necessitates the development of a mode of investigation in which the theoretical and model studies are checked or supplemented by field studies and experimental laboratory work, or vice versa. In this manner, the fullest benefit of the laboratory work as well as the field work may be obtained.

1. Early Underground Stress Analysis

With early investigators the rock surrounding mine openings was assumed to approximate some solvable, fundamental structural unit, primarily because of the manner in which failure was observed to occur at the mine opening. Many early investigations were centered upon the observation that a dome-shaped space forms around certain collapsing underground openings. The rock in the top of the original opening fails, leaving a dome-shaped structure or opening which apparently re-establishes equilibrium. In most instances this attempt to correlate underground structures with some structural unit was only an approximation, primarily because it was based on the assumption that the rock at some distance from the mine opening had no effect upon the stresses in its immediate vicinity.

During the period 1881 to 1885, two investigators, Fayol and Rizha, proposed theories with similar content. These theories were the forerunners of the "dome" theories. The mine opening was assumed to be surrounded by a roughly spherical shell within which the rock was loaded by its own weight. Thus, the rock within the dome dropped when the pull of gravity exceeded the cohesion with the surrounding rock.

In 1935, the dome theory was extended in an article by Dinsdale. In essence, he assumed an egg-shaped pressure ring surrounding the mine opening, and within this ring the hanging wall was separated from the external rock by shearing action and rested upon the supports within the opening. Figure 2 represents a cross-section of the mine opening, illustrating the so-called dome. The conclusions reached by Dinsdale were that the pressure on the immediate ribs and supports within the opening is small in comparison to the stresses a short distance from the sides of the opening. Dinsdale shows, in a simple static analysis, that the height of the dome increases with depth; thus, the pressure on the supports is proportional to depth. The value of this analysis is doubtful since the basis for the assumptions leading to its solution are subject to question.

A general criticism of the dome theory has been given by Shoemaker. His objections are: (1) the occurrence of the pressure dome itself is assumed but not explained; (2) the theory takes no account of forces outside of the dome acting on the rock within the dome; (3) the magnitude of forces assumed is not sufficient to account for the observed effects; and (4) the theory is based on the assumption that the rock is stressed within its elastic range. (This last assumption can be justified in many cases, however.)

In similar manner, various other theories have been presented which treat layers of rock immediately overlying the mine opening as beams loaded
individually by their own weight. These have differed only in the manner in which the beams were considered to be restrained and in the manner of their failure. One of the most advanced hypotheses of this type was published by George S. Rice in 1923; he assumed that, when failure of the beam occurs, inward shearing at the ends causes the formation of a dome-shaped space. These concepts are subject to criticism since the beams were assumed to be loaded only by their own weight, and no consideration was made of an external load on them.

Theoretical research upon the problem of underground mine structures has advanced rapidly in the last fifteen years, particularly with the application of the theory of elasticity and photoelasticity to the problem.

2. Theory of Elasticity Applied to Underground Mine Structures

A solution of the problem of the distribution of stress around a mine opening by the theory of elasticity requires some basic generalizations and assumptions; it involves solving a stress function for the problem which is related to the existing stress conditions by the boundary stresses and stress-strain relationships. The assumptions to affect a solution are (1) the rock is assumed to be of a homogeneous isotropic nature; (2) the mine opening is approximated by a definite geometrical figure; (3) the mine opening is assumed to be horizontal throughout its length and very long in comparison to its cross-section; (4) the stresses along the length of the opening are assumed to be uniform; (5) the underground mine is assumed to consist of an opening or a series of openings in an infinite or semi-infinite (bounded only by the earth’s surface) mass; and (6) the stresses encountered lie within the elastic limits of the materials. These assumptions are necessary both to permit the application of the theory of elasticity to the problem and to simplify the mathematics of the analysis. They necessarily cause a loss of generality and narrow the field of application of the results, but, they give an approximation of the stresses which may be expected under conditions approaching the ideal case. In addition, they form a basis for a more advanced theory in which it is not necessary to make such confining assumptions. A justification of these assumptions has been given by several authors. Duvall in particular has made a complete study of them.

3. Photoelasticity Applied to Underground Mine Structures

In the case of simple ideal problems, the theoretical approach is perhaps the most satisfactory since it provides an exact solution to the problem. In most instances, however, underground openings do not have simple boundaries. They are often of large number, and are arranged in a manner which is difficult to analyze mathematically. It is in the approximate solution of these more difficult problems that photoelasticity has its best application.

The photoelastic method of stress analysis is based upon the principle that, when certain transparent materials are stressed, their optical properties undergo changes which can be measured and related quantitatively to the stress state. Furthermore a model stressed in this manner presents a picture of the stresses in a prototype, regardless of the difference in the elastic constants possessed by model and prototype, if the elastic constants are the same throughout the body and if the body consists of one continuous piece (loading must be similar, of course).

The assumptions in applying the photoelastic method to stresses around mine openings are: (1) those made for the solution by the theory of elasticity, and (2) the postulate that the stresses...
about a mine opening may be approximated by the stresses about a similarly shaped opening in a plate under the same load as exists on a cross-section of the mine opening. The error involved in making the latter assumption is very negligible for depths greater than about 2.5 times the diameter of the hole, as shown by Panek.\(^{(15)}\)

In general, a solution by the photoelastic method requires: (1) selecting a suitable transparent material and making from it a model of the structural number under investigation; (2) loading the model in a way similar to the loading of the prototype and measuring the resulting optical effects in a polariscope; (3) translating the optical measurements into terms of stress and interpreting the latter by means of the fundamental theory of elasticity; and (4) transforming the stress distribution on the model to the analogous stress system for the prototype. For a more complete discussion of the photoelastic method, Frocht\(^{(7)}\) has compiled a two-volume work upon this subject.

Results obtained by use of a photoelastic model serve as a solution for the prototype, because it can be shown mathematically that for a prototype opening at a considerable distance from the surface the prototype is approximately in a state of plane stress. Thus, the model stresses are for practical purposes directly proportional to the corresponding prototype stresses, because the mathematical solution of the two problems is identical if certain negligible terms are omitted. In model and in prototype, the stress distribution depends only upon the shapes of the openings and their orientation in respect to the initial stresses. Therefore, the scale ratio (ratio of a linear model dimension to the corresponding linear prototype dimension) may be chosen at will. The only requirement is that the model be geometrically similar to its prototype.\(^{(15)}\)

The photoelastic method is not an exact method of solution, since it is subject to the errors inherent in experimental analysis. It has been shown by Duvall that these errors do not cause a difference from the theoretical value in the simple cases of more than 6 percent.\(^{(5)}\)

The primary objective in most early analyses was the determination of the stress distribution around a tunnel or shaft, or in pillars or arches. Only recently has attention been directed toward determination of the stresses in a mine as a unit, that is, the stress concentrations due to a number of underground openings.

4. Stress Distribution around a Single Opening

As a logical starting point, the problem of a single underground opening and its effects upon the stresses in the surrounding rock will be considered.

The effect of making openings of different shapes upon the stresses existing in rock masses before the openings are made is of fundamental interest. A series of geometrical shapes has been chosen by various investigators which are relatively simple to solve mathematically and which also approximate certain typical underground openings. Horizontal cylindrical mine openings, with circular, elliptical, ovaloidal, and rectangular cross-sections are considered in that order.

Solutions of these problems have three immediate objectives: (1) to determine the effect of the different shapes upon the stress concentrations at the boundaries of the openings for different states of initial stress in the rock; (2) to determine the shape best suited (smallest stress concentration induced in the surrounding rock) for each of the initial stress conditions within the earth; and (3) to determine approximately the stress which exists around actual mine openings.

a. Circular Openings

A complete work on the stresses existing around a tunnel was published by Mindlin in 1939.\(^{(13)}\) By means of the theory of elasticity he solved the problem of stresses around a horizontal cylindrical hole of circular cross-section in a semi-infinite elastic solid stressed by gravity. He assumed that stresses within the earth at different depths may be approximated by three states of pressure which existed before the opening was made, as is shown in Figure 1. The problem is one of mathematical complexity. By introduction of a bipolar coordinate system,\(^{(7)}\) it is greatly simplified, and an exact solution of the classical elasticity equations can be obtained. The length of the tunnel is considered to be large in comparison to its diameter. This and the fact that the body force is uniform permits the treatment of the problem as one in plane strain.

Panek\(^{(15)}\) has made a further development in Mindlin's analysis and its application. The zone of stress caused by the introduction of an opening is confined to a small area about the opening, and the maximum tensile and compressive stresses occur on the boundary of the opening. In general, the back (roof) and floor of the opening are in tension and the ribs (walls) are in compression, with the exception that, when the lateral initial earth pres-
sure is greater than about one-half the vertical pressure, the tangential stress on the entire boundary of the opening is compressive. The critical tensile stress seldom exceeds in magnitude the initial vertical pressure and is only slightly affected by the size of opening, but it is highly sensitive to the ratio of lateral to vertical initial pressures. Thus, the value of Poisson's ratio for rock is very important in the intermediate case, since it determines the lateral pressure.

The critical values (on the horizontal and vertical diameters of the circle at the boundary) of the tangential stress are shown in Fig. 3 for the three different pressure states. In these cases, the initial vertical rock pressure is taken to have a magnitude of \(-1\), i.e., it is compressive. It is apparent from these sets of curves that there is little change in magnitude of the stress concentration factor for a depth of hole-to-hole diameter ratio \((d/h)\) of greater than 2.5. This does not mean that the actual stress does not increase with depth; it only indicates that the stress concentration factor remains constant.

It can also be seen from Fig. 3 that, for a hole at a considerable distance from the surface, the size of the opening has little effect upon the critical stresses. The tangential stress is shown as a dimensionless ratio and may be converted to a stress in pounds per square inch by multiplying by \(1.2d\), where \(d\) is depth from the surface in feet. This is equivalent to assuming a specific gravity of 2.77 for the rock encountered in the homogeneous mass.

In his analysis, Panek has compared the stresses as obtained by Mindlin with stresses in three cases analogous to those solved by Mindlin. The latter stresses were obtained by assuming the circular opening to exist in a plate in a uniform-stress field. This presents the problem as one of plane stress rather than plane strain, simplifying it considerably, as will be shown.

The initial stresses, applied to the edges of a plate, were:

(a) Uniform compressive stresses \(S_y = S_x\).

(b) Uniform compressive stress \(S_y\); a uniform compressive stress \(S_x = \frac{\nu}{1 - \nu} S_y\).

(c) Uniform compressive stress \(S_y\); \(S_x = 0\).

Figure 4 shows the three cases of initial stress assumed by Panek, in a manner analogous to Fig. 1.
The solution of the stresses around a circular opening for these particular stress fields is not difficult. By use of polar coordinates with origin at the center of the opening, a solution for the radial, tangential and shearing stresses may be obtained for the three cases. The stresses for these simplified cases are independent of the size of the hole, and are also independent of the elastic moduli of the material. Figures 5 and 6 show that the zone of disturbance is very definitely localized in the neighborhood of the opening, that is, within a distance of three times the radius of the center of the circular opening.

As is the case in the analysis by Mindlin, the critical stresses occur on the edge of the hole on
the vertical and horizontal diameters, and they act parallel to the boundary of the hole. Figure 7 illustrates the variation in tangential stress around the perimeter of the opening for the three different cases. If it is desired to find the shearing stress acting at any point upon the boundary, it is merely one-half the tangential stress. One can show that the maximum shearing stress also varies from point to point, reaching its greatest value on the edge of the hole.

![Diagram of stress concentration around an elliptical boundary](image)

When the opening is far from the surface (for large d/h), the values of the tangential stress for the three respective cases, as determined by Mindlin's analysis, are found to approach the corresponding values as given by Panek for the uniform stress field. Thus, it is apparent that the much simpler solution may be used to determine the stress concentration due to a circular opening when it is far from the surface of the rock mass (when the roof of the hole is at a distance below the surface equal to more than twice the hole diameter). This is a very important conclusion, because it greatly simplifies the testing of experimental models, such as plastic plate models for photoelastic studies. The use of uniform-stress fields permits the solution of problems, such as the introduction of openings of geometrical shape other than circles, which would be difficult and tedious to solve by the theory of elasticity.

The problem of a circular opening in a uniform stress field has also been solved photoelastically by various authors, and the results have been found quite comparable to the theoretical results. In most instances, the case of a plate in a vertical uniform stress field with no lateral restraint has been solved, and from these, the other two cases have been obtained by superposition. In addition, numerous problems of more complex nature have been solved by use of circular openings in a plate, using photoelastic methods. An example is the case of a circular opening very close to a free boundary.

b. Elliptical Openings

The problem of stresses around an elliptical opening introduced in a uniform stress field has been solved by C. E. Inglis. The solution was effected by the use of curvilinear coordinates, and the complete method of solution is given in his paper. Table 1 indicates the variation in the critical boundary stresses with a change in the ratio of the major to minor axes (w/h), and with a change in the angle of inclination (δ) of the major axis of the ellipse with a horizontal plane and with the magnitude of the initial horizontal stress (S).

A number of important facts may be established from the data given for an elliptical opening:

1. In the case of hydrostatic pressure, critical compressive stress is always on the major axis, being independent of δ, and the entire boundary is in compression. The minimum stress occurs on the minor axis.
2. For initial horizontal pressure (S,) less than the initial vertical pressure (S,), the critical compressive stress tends to occur at the sides of the opening and the critical tensile stress at the top and bottom.
3. For a given S, and w/h, the critical compressive stress is greatest at δ = 0 deg and smallest at δ = 90 deg. (Critical comp-

---

Table 1

<table>
<thead>
<tr>
<th>w/h</th>
<th>δ</th>
<th>S,=0</th>
<th>Critical Value of S,</th>
<th>S,=S,/3</th>
<th>S,=S,</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.0</td>
<td>0.0</td>
<td>0</td>
<td>-2.7</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0.53</td>
<td>0</td>
<td>-4.0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.53</td>
<td>-0.25</td>
<td>0</td>
<td>-4.0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.53</td>
<td>-0.25</td>
<td>0</td>
<td>-4.0</td>
</tr>
</tbody>
</table>

Parentheses indicate that the value is not actually a critical one.

w/h = the ratio of the major to minor axis
δ = the angle of inclination of the major axis with the horizontal

---

---
pressive stress increases linearly with increasing $w/h$ when $\delta$ is small.

Some interesting results may be obtained by altering the shape of the ellipse. Starting with a circular shape and elongating the horizontal diameter, the rib compression increases, and the back stress becomes more tensile, never exceeding $+1$ however. Starting with a circular shape and elongating the vertical diameter, the rib compression decreases slightly and the back stress becomes more compressive for most values of $S_x$. These facts are illustrated by Fig. 8.

![Fig. 8. Stress Concentration on the Boundary of an Ellipse at the Major and Minor Axes as the Height-to-Width Ratio Varies](image)

Variation of the lateral pressure $S_x$ has little effect on the critical rib stress, but is a major factor in determining the critical back stress. The critical stresses are at a minimum when the maximum rib stress is equal to the maximum back stress. This is true only when $\delta = 90$ deg, or when $w/h = 1$ (a circle). By use of the equation determining the tangential stresses around an elliptical opening, and the fact that the minimum critical stresses occur at $\delta = 90$ deg, it can be shown that the shape of the ellipse for which the critical stresses are a minimum is given by $w/h = S_y/S_x$, from which it can be seen that the larger $S_y/S_x$, the more elongated the ellipse. When the shape of the ellipse is given by the above formula, the ellipse having its major axis vertical, it can also be shown that the magnitude of minimum critical stress is given by $S_z = S_x + S_y$, a compression equal to the sum of the initial vertical and horizontal earth pressures.

c. Ovaloidal Openings

The case of an ovaloid hole in a uniformly loaded plate has been investigated by Martin Greenspan. He considered the case of an ovaloid which was a square with a semi-circle erected on each of two opposite sides. An exact solution of this problem was obtained by approximating a true ovaloid with a figure which could be represented by a much simpler set of parametric equations. Greenspan considered the problem of a plate in a state of generalized plane stress, the stress at points remote from the hole having the constant normal components in the horizontal direction $\sigma_x = S_x$ and in the vertical direction $\sigma_y = S_y$; the constant shearing stress was $\tau_{xy} = T_{xy}$. By making use of a curvilinear coordinate system, an equation for the tangential stress about the opening was obtained. As examples of this solution, Greenspan gives two complete calculations, one considering a tension applied to the plate as parallel to the long axis of the ovaloid, and the other considering a tension applied parallel to the short axis as ovaloid. To apply Greenspan's results to a mine opening it is necessary to compare his solution for the stresses in a plate with an ovaloid-shape hole, which is essentially in a state of plane stress, with the case of stresses around a horizontal cylindrical opening with an ovaloid cross-section in a semi-infinite mass, which is essentially in a state of plane strain. As discussed previously, this comparison is justified, as the difference in magnitudes of the solutions for plane stress and plane strain is negligible. Greenspan's solution for tension may be applied to mining problems where the initial stresses are compressive by changing the signs of appropriate terms in the stress equation.

It is also possible, by the principle of superposition, to superimpose one of the results obtained upon the other and obtain the stress conditions existing around the ovaloid opening for the three initial states of stress which were investigated in the case of circular and elliptical openings. Figure 9 illustrates the tangential stress existing around a square with semi-circular ends and with the major axis vertical or horizontal, respectively, for the three initial states of stress.

Duvall tested by the photoelastic method a series of ovaloidal openings with axes parallel and perpendicular to an applied unidirectional stress field; he determined the maximum stress concentration on the boundary, as well as the stress concentrations at the ends of each axis.

Table 2 gives the results of these tests. As the height-to-width ratio decreases, the maximum stress concentration increases without limit. (See Figure 10.) The experimental points were obtained by means of the photoelastic method.

The relation between the height-to-width ratio and the stress concentration on the horizontal axis
of an ovaloid at its boundary is given in Fig. 11, as compared to corresponding data for an elliptical opening. From a comparison of Figs. 10 and 11, it can be seen that the maximum stress concentration at the boundary of an ovaloid does not necessarily exist on the horizontal axis of the ovaloid, and may differ considerably from the stress concentration on the horizontal axis.

The stress concentration produced on the ends of the vertical axis of an ovaloid in a unidirectional stress field is approximately equal in magnitude, but opposite in sign, to the applied stress and is practically independent of the height-to-width ratio.

From data obtained for ovaloidal openings in thin plates subjected to a unidirectional stress field, the boundary stresses for a hydrostatic stress field were obtained by algebraic addition of stresses. Table 3 indicates the stress concentration at the ends of the major and minor axes of an ovaloid in a hydrostatic stress field. The stress concentrations

<table>
<thead>
<tr>
<th>Height of Opening (in.)</th>
<th>Width of Opening (in.)</th>
<th>Ratio of $h/w$</th>
<th>Stress Concentration at End of Major Axis</th>
<th>Stress Concentration at End of Minor Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.388</td>
<td>1.504</td>
<td>0.258</td>
<td>-0.93</td>
<td>5.35</td>
</tr>
<tr>
<td>0.389</td>
<td>1.141</td>
<td>0.341</td>
<td>-1.07</td>
<td>4.75</td>
</tr>
<tr>
<td>0.394</td>
<td>1.500</td>
<td>0.399</td>
<td>-1.01</td>
<td>4.14</td>
</tr>
<tr>
<td>0.396</td>
<td>761</td>
<td>0.312</td>
<td>-0.92</td>
<td>4.02</td>
</tr>
<tr>
<td>0.401</td>
<td>761</td>
<td>1.000</td>
<td>-1.09</td>
<td>3.17</td>
</tr>
<tr>
<td>0.767</td>
<td>767</td>
<td>1.000</td>
<td>-1.14</td>
<td>3.05</td>
</tr>
<tr>
<td>1.511</td>
<td>764</td>
<td>1.98</td>
<td>-1.20</td>
<td>1.90</td>
</tr>
<tr>
<td>2.262</td>
<td>762</td>
<td>2.96</td>
<td>-1.17</td>
<td>2.40</td>
</tr>
<tr>
<td>2.512</td>
<td>765</td>
<td>3.29</td>
<td>-1.28</td>
<td>1.65</td>
</tr>
<tr>
<td>3.015</td>
<td>765</td>
<td>3.94</td>
<td>-1.17</td>
<td>2.83</td>
</tr>
</tbody>
</table>

* Computed from unidirectional stress field data.

Table 3 is after Duvall.
are compressive at the ends of both axes of the ovaloid regardless of the major to minor axis ratio. As the major to minor axis ratio increases, the stress concentration at the end of the major axis increases, and that at the end of the minor axis decreases in a like manner.

d. Rectangular Openings

A study has been made of the influence of a rectangular opening — with a short dimension \( h \), a long dimension \( w \), and a small fillet of radius \( r \) at the corners — upon the stresses in a plate subjected to the three aforementioned states of initial stress.

Panek\(^{13}\) and Duvall\(^{15}\) have furnished considerable information, largely from photoelastic experiment, about the stress concentrations existing around rectangles with side dimensions of varying proportion. Primarily, interest lies in the effect upon the maximum stress concentration of varying the long-to-short dimension ratio \( (w/h) \), the radius of curvature-to-short dimension ratio \( (r/h) \) or the angle \( \delta \) which the long dimension \( w \) makes with the horizontal.

Consider an opening of rectangular cross-section in a plate situated in a two-dimensional stress field, where \( S_y \) is the initial vertical stress and \( S_x \) is the initial horizontal stress. \( S_y \) is permitted to possess all values between zero and \( S_y \), so that the forms of initial stress previously discussed are amply covered. In general, there will be a tensile boundary tangential stress on the top and bottom of the opening. With the major dimension of the rectangle horizontal, the critical tensile stress occurs at the center of the back and floor. As the angle of the major dimension of the rectangle with the horizontal increases, the critical compressive stress moves from the intersection of the fillets with the vertical sides to the lateral fillets, and the critical tensile stress shifts to the highest and lowest both occurring at or close to the ribs. After a 90-deg rotation, the critical stresses are again tensile at the middle of the top and bottom and compressive at the ribs at the fillets.

In the case of \( S_y \) near zero (no lateral restraint) and between 30 and 80 deg, the critical tensile stress becomes excessively high; under other conditions it does not exceed +1. The value of the critical tensile stress is sensitive to the value of \( S_y \), decreasing as the ratio of \( S_x/S_y \) increases. In general, for values of \( S_x \) greater than \( S_y/2 \), the stress on the boundary of the rectangle becomes wholly compressive in nature.

The critical compressive stress increases linearly with increasing \( w/h \), and is not altered to any extent by increase in the lateral pressure for small values of \( \delta \). For values of \( \delta \) less than 80 deg, the critical tensile stress tends to decrease slightly with an increase in \( w/h \).

Figure 12 gives the relation between the maximum stress concentration and the height-to-width ratio \( (h/w) \) for a rectangular opening in a plate placed in a unidirectional stress field. The maximum stress concentration increases without limit as \( h/w \) decreases, but the rate of increase is not as great as for ovaloidal or elliptical openings.

As the fillet radius decreases from \( r/h = 1/4 \) to \( r/h = 1/12 \), the critical compressive stress increases; the stress at \( r/h = 1/12 \) is only 1.5 times greater than for \( r/h = 1/4 \). The smaller the radius of curvature of the fillet, the nearer the critical compressive stress is located toward the center of the fillet.

Figure 13 illustrates the tangential boundary stresses occurring on the perimeter of a square (a special case of a rectangle) for the three types of initial stress, from a mathematical development by Greenspan.\(^9\) It is interesting to note that the maximum compressive stress on the boundary occurs at the filleted corner, and that for the case of hydrostatic pressure the maximum stress concentration has a value of 4.57 as compared to a circle under the same conditions. As the initial horizontal stress decreases, the back stress becomes more tensile but is never greater than about +0.8.
No lateral restraint
(Poisson's ratio for rock is taken to be 0.25 in this illustration.)

**Fig. 73.** Tangential Stress Concentration on the Boundary of a Square Opening in an Infinite Plate for the Three Initial States of Stress

**e. Summary**

In order to apply the results obtained to the mining of long single openings in underground mines where the rock formations approach a homogeneous elastic medium, it is necessary to compare the stress concentrations for the different shapes of opening for each of the three initial states of rock pressure.

**No Lateral Restraint.** In the case of a mine opening introduced in a unidirectional force field (no lateral restraint), two important facts can be determined: (1) To reduce the maximum stress concentration around an opening having a height-to-width ratio greater than unity, the opening should approximate an ellipse. (2) To reduce the maximum stress concentration around an opening having height-to-width ratio less than unity, the opening should approximate a rectangle with rounded corners. Figure 14 gives a comparison of stress concentrations for different shape openings in a unidirectional field. Stress concentration decreases with an increase in $h/w$. It is observed that when $h/w = 1$, an ellipse becomes a circle and a rectangle becomes a square. A rectangle with $r/h = 1/2$ is an ovaloid; and when $h/w = 1$, this also becomes a circle. Figure 15 illustrates the shape of opening which is most favorable for a given condition of major-to-minor axis ratio and angle of inclination ($\theta$) of the major axis with the horizontal

where $S_y = -1$ and $S_x = S_y/3$ ($s_i$ for $S_x = 0$ differs very little from $s_i$ for $S_x = S_y/3$). It can be seen that this information agrees with that given in Fig. 14.

The two important factors that cause high stress concentrations around openings in a unidirectional stress field are a height-to-width ratio less than unity and sharp corners on the horizontal axis of the hole.

The maximum stress concentration is related to the maximum compressive stress which exists in the vicinity of the mine opening, in general found in some portion of the immediate ribs. It is necessary to choose a shape of opening such that the compressive stresses induced are not sufficient to cause crushing of the ribs or any other type of failure of the opening.

A stress concentration occurs at the middle of the back and floor which is tensile in nature. It is approximately equal in magnitude to the initial vertical pressure for the different shapes of openings. This stress may cause roof failures in many instances where the strength of the roof as a unit is less than the tensile stress concentration. In the choice of openings, the shape which induces the least critical compressive stress has been described. This choice is based not only on the qualification that the critical compressive stresses be minimized, but also on the fact that the shape which induces the least critical compressive stress generally induces the least critical tensile stress. The least
critical compressive stress induced by an opening in a unidirectional stress field occurs in the case of an ellipse with $h/w$ greater than 5.

Laterally Restrained. In choosing the most desirable shape of opening to be introduced into a laterally restrained plate in a uniform stress field, we assume that $S_x = S_y/3$. This is equivalent to assuming that $v = 0.25$ for the rock in the vicinity of the hole, since $S_x = \frac{1}{1-v} S_y$ in the case of a laterally restrained plate.

Actually, the tangential stress pattern around an opening in a laterally restrained plate does not differ greatly from the tangential stress for a plate without lateral restraint, with the exception that the magnitude of the tensile stress is decreased markedly.

The trends of the critical compressive stress with the height-to-width ratio are plotted in Fig. 15. Ovaloids (rectangles where $r/h = \frac{1}{2}$) have not been included. It can be seen by interpolation, noting that an ellipse and ovaloid are equivalent when $w = h$, that for height-to-width ratios less than about 1, the ovaloids (rectangles with maximum radius fillets) are the preferred shape. For $h/w$ between 1 and 4, the most favorable shape is that of an ellipse. For $h/w$ greater than 4, ovaloids are again the most favorable shape. The lowest possible critical compressive stress is induced by an ellipse $h/w = 3$.

Hydrostatic Pressure. Under conditions of hydrostatic pressure the ovaloid induces the smallest critical stress throughout the range of $h/w$. Next best shapes are an ellipse for $h/w$ less than 2, and a rectangle with large radius fillet for $h/w$ greater than 2. This is shown in Fig. 16. The least possible critical compressive stress is induced by an opening of circular cross section.

5. Stress Distribution around Multiple Openings

There are many instances underground where mine openings occur sufficiently close together that the introduction of one opening affects the stress concentrations around another. This condition is of primary importance, because stress concentrations are increased when two or more openings are in close proximity. In the case of multiple openings, interest is again centered upon the points of maximum stress concentration, as well as the stress distribution in pillars formed by two or more openings, the relationships between stress concentrations, and the size and shape of pillars.

In a specific application to mining operations, the results from a study of multiple openings may be used to estimate the stress distribution in pillars in order to show what factors cause high local
stresses and how local stresses may be reduced by
the proper design of mine openings.

To date, there have been only a few solutions
of the problems of stress distribution around mul-
tiple openings. Solving these problems by the theory
of elasticity involves very complex equations even
for the most simple geometric shapes; thus only a
few have been made. The photoelastic method,
however, lends itself readily to the solution of these
problems.

In addition to the simplifying assumptions
necessary for the solution of single opening prob-
lems by the theory of elasticity or by photoelastic
methods, the further assumption is made for mul-
tiple openings that each pillar between openings is
uniform in width and height and is long in compari-
sion to its width and height. In these problems, as in
those solved previously, solutions were obtained
assuming a condition of plane stress. This produced
results which are approximately equal to the actual
conditions existing around mine openings which are
in a state of plane strain. (The results differ only
by infinitesimals.) With these assumptions the stress
distribution in pillars or around one of the openings
in underground mines can be determined by an
elastic or photoelastic study of the stresses existing
around two or more openings in wide, thin plates.

Only two states of initial stress (pressure)
within the earth will be considered here. They are
that of (1) hydrostatic pressure, and (2) no lateral
restraint. These two states represent the usual
limits of stress expected underground and were
chosen because the stresses existing within the
earth where mine openings are introduced should
occur somewhere between these extremes.

a. Circular Openings

The first problem considered is that of two cir-
cular openings and the pillar left between them.
Ling(12) solved this problem analytically. He de-
termined the critical compressive stresses within
the pillar (on horizontal diameters at the boundary
of the holes). For a ratio of pillar width to open-
ing height $P/h = 2$, it can be seen that these
stresses differ little from those for a single opening.
The differences in $s_\gamma$ between small $P/h$ and large
$P/h$ is less than 1 (see Table 4). The critical com-
pressive stresses range between $-3.26$ and $-2.99$

![Fig. 16. Comparison of Critical Compressive Tangential Stress for Rectangles and Ellipse under Conditions of Hydrostatic Pressure](image)

### Table 4

<table>
<thead>
<tr>
<th>$P/h$</th>
<th>Critical $s_\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2$</td>
<td>$-3.26$</td>
</tr>
<tr>
<td>$1$</td>
<td>$-3.02$</td>
</tr>
<tr>
<td>$2$</td>
<td>$-2.99$</td>
</tr>
<tr>
<td>$4$</td>
<td>$-3.00$</td>
</tr>
<tr>
<td>$6$</td>
<td>$-2.89$</td>
</tr>
</tbody>
</table>

$P/h = \text{the ratio of pillar width to opening height}$

Applied stress field is perpendicular to the line of centers of the holes.

Table 4 is after Panek.

Duvall(6) solved the problem of two circular
openings which were placed in a unidirectional
stress field supplied perpendicular to the line of
centers of the two holes, as indicated in Fig. 17.
The stress concentrations were determined photo-
elastically at points A, B and C, as the opening-to-
pillar width ratio was varied. The stress concen-
tration at position B (pillar rib) proves to be the
maximum within the plate. The stresses at A and
B are tangent to the boundary of the opening and
are compressive in nature. The stress at position C
is tangent to the boundary, equal in magnitude to
the applied load ($S_y$), and of opposite sign to the
applied load. An increase in the ratio of opening-to-
pillar width causes an increase of stress concentra-
tion at A and B. The stress concentration is always
greater at B than at A, and the rate of increase
with the increase of opening-to-pillar width is also
greater at B than at A. Table 5 lists the results of
photoelastic studies of the plate with two circular
openings. These results compare favorably with
those given by theoretical analysis. In addition,
they show that for $P/h$ less than $1/2$, a width of
Table 5

<table>
<thead>
<tr>
<th>Ratio of Opening Width to Pillar Width</th>
<th>Stress Concentration at Positions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>0.330</td>
<td>2.97</td>
</tr>
<tr>
<td>0.670</td>
<td>3.06</td>
</tr>
<tr>
<td>1.330</td>
<td>3.13</td>
</tr>
<tr>
<td>2.670</td>
<td>3.27</td>
</tr>
<tr>
<td>4.74</td>
<td>3.42</td>
</tr>
</tbody>
</table>

Applied stress field is perpendicular to the line of centers of the holes. Table 5 is after Duvall.

opening-to-pillar width ratio of greater than 2, the stress concentration at the pillar rib increases very rapidly.

A plot of the shear stress concentration against the distance in from the boundary of the pillar is shown graphically in Fig. 18 for varying ratios of opening-to-pillar width. The distribution of shear stress in the pillar becomes more nearly uniform as the opening-to-width ratio becomes large. This indicates that for a small pillar formed by two large openings, the average stress in the pillar is almost as large as the maximum stress.

The next problem considered is that of three circular openings in a plate (centers lying on a straight line). Duvall solved this problem using photelastic methods for the case of no lateral restraint. A load was applied in the same manner as for the problem of two circular openings, and the stress concentrations determined at points A, B, C, D and E (see Fig. 17) as the ratio of opening width to pillar width was varied. Table 6 gives the data pertaining to this series of models. The stress concentration at position C (inner ribs of pillars) is the maximum which occurs within the plate. The stresses at A, B and C have the same sign as the
applied stress, that is, compression when the applied stress is compressive. The stresses at D and E have the opposite sign to the applied stress, that is, tension when the applied stress is compression.

Figure 17 shows the relationship between the opening-to-pillar width ratio and the stress concentration at positions A, B and C for the three circular openings. The stress concentrations at C are not only largest, but also show the greatest increase with increase in the opening-to-pillar width ratio. Stress concentrations at C range from 3.21 for opening-to-pillar width ratio of 1.08, to 5.12 for opening-to-pillar width ratio of 5.12. This is a sizeable increase over the stresses induced by two circular openings.

Capper also studied the problem of three circular holes in a plate subjected to a stress perpendicular to their line of centers. His solutions were obtained by photoelastic methods. The case of a plate whose edge was at a distance of two radii from the center of the outside hole was investigated. Green solved a similar problem theoretically in which he assumed an infinitely wide plate. Duvall's photoelastic solutions utilized plates where the plate width was such that the distance from the edge of the plate to the center of the outside hole was equal to or greater than four times the radius of the outside opening. The three solutions agree if the differences in edge conditions are taken into account.

Another case which has been studied is that of five circular holes forming four pillars (see Fig. 17). The problem has been solved by Duvall. The load was applied perpendicular to the line of centers, and the stress concentrations at positions A, B, C, D, and E were determined photoelastically. Results are given in Table 7. The stress concentrations produced on the boundary of the circles at the ends of the diameters perpendicular to the line of centers of the openings are approximately unity; they are of opposite sign to the applied stress, that is, tension occurs when the applied stress is compression. Figure 17 shows the relationship between the opening-to-pillar width ratio and the stress concentrations at positions A, B, and E. The difference between the stress concentration for positions E, D, C, and B is small, but between B and A it is large. Thus for a large number of openings the maximum stress concentrations in all but the outermost pillars is nearly uniform, being less in the pillars near the side walls than in the central pillars.

The distribution of shearing stress through the central pillars at the line of centers, Fig. 19, indicates that the maximum stress concentration produced in the pillars does not increase as rapidly as the average pillar stress with an increase in room-to-pillar width. Therefore the average stress in the pillar is very close to the maximum stress for large ratios of room-to-pillar width.

The theoretical stress distribution has been studied by Howland for an infinitely wide plate containing an infinite row of circular holes, the plate being subjected to a uniform stress perpendicular to the lines of centers of the holes (no

<table>
<thead>
<tr>
<th>Ratio of Opening Width to Pillar</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.08</td>
<td>3.21</td>
<td>3.21</td>
<td>3.21</td>
<td>-0.81</td>
<td>-1.02</td>
</tr>
<tr>
<td>2.22</td>
<td>3.47</td>
<td>3.56</td>
<td>3.74</td>
<td>-0.93</td>
<td>-1.03</td>
</tr>
<tr>
<td>2.61</td>
<td>3.60</td>
<td>3.88</td>
<td>4.08</td>
<td>-0.94</td>
<td>-1.02</td>
</tr>
<tr>
<td>4.76</td>
<td>3.80</td>
<td>4.92</td>
<td>5.12</td>
<td>-1.05</td>
<td>-1.03</td>
</tr>
</tbody>
</table>

Table 6: Stress Concentration for a Plate Containing Three Circular Openings (Solution by Photoelasticity)

Table 7: Stress Concentration for a Plate Containing Five Circular Openings (Solution by Photoelasticity)

<table>
<thead>
<tr>
<th>Ratio of Opening Width to Pillar</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.07</td>
<td>3.29</td>
<td>3.29</td>
<td>3.29</td>
<td>3.29</td>
<td>3.29</td>
</tr>
<tr>
<td>2.21</td>
<td>3.63</td>
<td>3.72</td>
<td>3.89</td>
<td>4.03</td>
<td>4.06</td>
</tr>
<tr>
<td>2.96</td>
<td>3.93</td>
<td>4.04</td>
<td>4.09</td>
<td>4.12</td>
<td>4.14</td>
</tr>
<tr>
<td>4.35</td>
<td>4.96</td>
<td>5.12</td>
<td>5.22</td>
<td>5.24</td>
<td>5.27</td>
</tr>
</tbody>
</table>

Applied stress field is perpendicular to the line of centers of the holes. Table 7 is after Duvall.
lateral restraint). (10) Duvall's experimental photoelastic analysis of the stress distribution for five circular holes in a plate with edges a finite distance (4 times the radius of a hole) from the center of the outside hole, which was done under similar conditions, was found to be in close agreement with Howland's theoretical study. Howland found the stress concentration at the end of the horizontal

hole diameters to be 3.24 when the ratio of opening width to pillar width was 1.0, and Duvall found that the stress concentration in the center pillars for five openings was 3.29 for opening-to-pillar width ratio of 1.07. This agreement indicates that five circular holes approach the case of a row of an infinite number of holes in an infinite plate.

By plotting the maximum stress concentration against the number of pillars for a given ratio of opening-to-pillar width, a curve is obtained as shown in Fig. 20. From the curve it is apparent that for five openings the maximum stress concentration in the pillars has reached an asymptotic value and the addition of more pillars would not increase the maximum stress concentration appreciably. Figure 21 indicates that the stress concentration in the central pillars for five openings approaches the average stress concentration in the pillars for opening-to-pillar width ratios greater than 4.

b. Ovaloidal Openings

The stresses around two ovaloidal openings (height-to-width ratio of openings equal to 0.5) were investigated by Duvall. (6) A load was applied perpendicular to the line of centers of the openings as shown in Fig. 22, and the stress concentrations produced were measured by means of the photoelastic method. Table 8 gives the data for this series of tests. The stress at position C is of opposite sign to the applied stress (that is, tension occurs if the applied stress is compression) and is somewhat less than the applied stress. The stresses at positions A and B are of the same sign as the applied stress. The stress concentration at point B is the maximum produced on the boundary of the openings. The relation between the opening-to-pillar width ratio and the stress concentrations at positions A and B is given in Fig. 22. For small open-
as in the previous experiments, and the stress concentrations produced were evaluated photoelastically. Figure 22 indicates the manner of loading and positions A, B, C, D and E where stress concentrations were determined. Table 9 gives the results of the tests. The stresses at positions A, B, C, D and E have the same sign as the applied stress. The stresses on the boundary at the vertical axis of the openings are of slightly less magnitude and of opposite sign to the applied stress. Figure 23 illustrates the relationship between the opening-to-pillar width ratio and the stress concentrations at positions A, B, and E. The values of stress concentrations at positions C and D lie between those of the stress concentrations at B and E; thus they have been omitted. The maximum stress concentration occurs at position E, ranging from 4.17 for an opening-to-pillar width ratio of 1.03, to 6.10 for an opening-to-pillar width ratio of 4.28. This is considerably higher than for the case of five circular holes in the same situation.

The shear stress distribution through the central pillars along the line of centers of the openings was determined and plotted as a function of the distance from the edge of the pillar for the different opening-to-pillar width ratios. These curves (Fig. 23) show that stress distribution in the pillars becomes more nearly uniform as the opening-to-pillar width ratio becomes larger; this was seen in the case of circular openings.

In the case of a plate containing five ovaloidal openings (height-to-width ratio equal to 2.0), the
load was applied as in previous experiments, and the stress concentrations produced at positions A, B, and C (see Fig. 22) were determined photoelastically. At position C the stress is tangent to the boundary, it is of slightly less magnitude than the applied stress, but of opposite sign. The stresses at points A and B are tangent to the boundary and of the same sign as the applied stress. Maximum stress concentration occurs at A. The relations between the stress concentrations and the opening-to-pillar width ratio are shown in Fig. 22 and in Table 10. It is interesting to note that the stress concentrations for five ovaloidal openings with height-to-width ratio of 2.0 are less than the corresponding stress concentrations for five circular openings when both are in a unidirectional stress field.

c. Rectangular Openings
Panek performed an extensive series of experiments with plates containing two rectangular openings with corner fillets. The ratio of the major to minor axis \( R/h \), the ratio of the radius of fillets to the minor axis \( r/h \), the angle of inclination of the major axis with the horizontal \( \theta \), and the ratio of the pillar width to the minor axis \( P/h \) were all varied to determine their effects upon the stress concentrations around the openings; the photoelastic method was used.

It was found that the pillar width had comparatively little effect on the critical tensile stress. Decreasing the pillar width from \( P/h = \infty \) to \( P/h = 1 \) adds a critical compressive stress concentration about 1.5 to the original stress concentration, when \( R/h = 3 \). The greater the \( R/h \), the greater this increase in stress. \( R/h \) is the most important factor controlling the critical pillar stress when \( P/h \) is greater than 1. The stresses in the roof and floor of the openings appear to be little affected by either \( R/h \) or \( P/h \). Hence, if the lateral pressure is small and the roof fails in tension, there will be practically no advantage to be gained by decreasing the roof span. Panek observed that the smaller the \( P/h \), the more rapid the increase of \( \sigma_t \) with \( R/h \). When \( P/h \) is less than 1, the critical stress is influenced at least as much by the pillar width as by the room width; for pillars several times as high as they are wide, \( P/h \) is probably more important than \( R/h \). Rectangles with large radius fillets were more satisfactory than those with small radius fillets.

d. Summary
It has been shown by investigators that the stress concentrations induced by a number of openings are greater than those induced by a single opening of a similar geometrical shape. In general, the stress concentrations increase with the addition of each hole until about five holes are in the plate. Then the stress concentrations remain almost constant with the addition of more holes. It can be said that the stress conditions existing in the central pillars approximate those existing in the pillars formed by an infinite number of holes.

As the opening-to-pillar width ratio increases, the average stress concentration in the pillars increases at a faster rate than does the maximum stress concentration. This indicates that the average stress approaches the value of the maximum stress concentration within the pillar. At about 75 percent recovery (removal of material), the stress...
concentration and average stress within the pillar are nearly equal.

In most instances a tensile stress is produced in the roof and floor of the mine openings, approximately of the magnitude of the applied stress. With the application of lateral confining pressures, these tensile stresses diminish very rapidly.

In an ore body, where pillars are all of the same height, the pillars in the center of the stopes are indicated (theoretically) as under more stress than those near the side walls of the stope. The maximum stress concentration in pillars is on or near the rib of the pillar, indicating that failure should occur first at the surface of the pillar.

For a ratio of pillar width to pillar height not less than one, it is more advantageous to decrease the long cross-sectional dimension of the openings rather than to widen the pillars if critical pillar stress is too high. If the ratio of the pillar width to pillar height is less than one, pillar width should be increased and possibly room width decreased if critical pillar stress is too high.

Duvall gives a very simple derivation of the stresses existing in the pillars formed by the introduction of an infinite number of holes in a plate of infinite extent. Holes were all of equal size (any geometrical shape), equally spaced. The derivation is given here in its entirety since it illustrates a simple, direct approach to a difficult problem. If the plate is stressed perpendicularly to the line of centers of the holes, so that the average stress in the plate at a great distance from the row of holes is \( S_o \), then the load on any one pillar is given by

\[
L_p = S_o d (W_o + W_p) \tag{1}
\]

where

- \( L_p \) = load supported by one pillar
- \( S_o \) = average stress in plate at a distance from row of holes
- \( W_o \) = width of opening
- \( W_p \) = width of pillar
- \( t \) = thickness of plate

If a uniform stress distribution in each pillar is presumed so that the average stress within the pillar is given by \( S_n \), then the load on any pillar is given also by

\[
L_p = S_n t W_p \tag{2}
\]

From Eqs. 1 and 2 the following relation can be derived

\[
\frac{S_n}{S_o} = 1 + \frac{W_o}{W_p} \tag{3}
\]

where \( S_n/S_o \) is the stress concentration for the average stress in the pillars. This is approximately equal to the maximum stress concentration where the ratio of the opening-to-pillar width is large.

Insufficient data were obtained to compare the different shapes of openings for the hydrostatic loading. In the case where the initial earth stress condition is one of no lateral restraint, a comparison of the maximum stress concentration produced in pillars formed by five ovaloids having a height-to-width ratio of 0.5, five ovaloids having a height-to-width ratio of 2.0, and five circles is of value. The maximum stress concentration for each of these three shapes has been plotted as a function of the percent mined area or percent recovery in Fig. 24. Also shown is a plot of Eq. 3 which gives the average stress concentration for an infinite number of pillars. Figure 24 shows that for less than 50 percent recovery the maximum stress concentration is not greatly affected by the percent recovery, but for recoveries greater than 50 percent the maximum stress concentration increases more rapidly with an increase in the percent recovery. The three curves for the circular and ovaloidal openings are similar in shape; thus the effect of percent recovery upon the maximum stress concentration in pillars is independent of the shape of the openings. The order of preference for underground mine openings with regard to stress concentrations is (1) ovaloids with height-to-width ratio of 2.0, (2) circles, and (3) ovaloids with height-to-width ratio of 0.5.

From the experimental data contained in Fig. 24, the following equation was derived: \(^{(6)}\)

\[
K = S + 0.09 \left[ \left( \frac{100}{100 - R} \right)^2 - 1 \right] \tag{4}
\]
where

\[ K = \text{maximum stress concentration in pillars} \]

\[ S = \text{maximum stress concentration around a single opening} \]

\[ R = \text{percent recovery} \]

Equation 4 can be made to fit data for openings of any shape by choosing the proper value of the maximum stress concentration. For example, in an unidirectional stress field (no lateral restraint) \( S = 3.0 \) for a circle, \( S = 3.9 \) for ovaloid having a height-to-width ratio of 0.5, and \( S = 2.64 \) for ovaloid having a height-to-width ratio of 2.0. Comparison of the experimental data and results obtained from the empirical equation are given in Fig. 25. Curves are drawn from the equation; points are from the experimental data.

Equation 3 gives good results for percent recoveries greater than 75 percent, but Eq. 4 gives a better approximation over a wider range.
IV. STRESSES IN SIMPLE STRATIFIED ROOFS

One of the principal difficulties in determining the stress distribution around actual mine openings lies in attempting to evaluate the effect of complex geologic conditions upon the stresses. In the previous chapter, some solutions for the stress conditions existing around mine workings in solid homogeneous materials stressed within their elastic limit were given. It is apparent, however, that fissures, faults, bedding planes, intrusions, metamorphism of the rock, or any features which destroy the continuous, homogeneous nature of the rock, alter the stresses existing around mine openings. Solutions taken from the previous chapter and applied under these conditions would thus be in error because of the discontinuities introduced by these geologic conditions.

In general, these geologic factors are not easily analyzed. The strengthening or weakening effects which they have upon rock and the included mine openings have not been determined except for a few field correlations which have generally been of local nature. There is a definite need for widespread field correlations of the different geologic factors to determine the effect of each individual factor upon the strength of rock and the stresses around mine openings. This would simplify any attempt to express the effect of a specific geologic condition upon the stress around a mine opening quantitatively.

Some very simple geologic conditions have been approximated in the laboratory and their effect upon the stress distribution around mine openings has been evaluated. One example is the case of mine openings existing in stratified materials which are uninterrupted by faults or fissures, and the beds of which are homogeneous in character. Solutions of this problem have been obtained by the use of rock models taken from the site of the actual mine opening, and more recently, solutions for some simple specific cases were obtained using the theory of elasticity.

6. Centrifugal Testing to Simulate Stresses Occurring in Rock Beams Underground

An examination of Fig. 2, which was used to illustrate the dome theory, indicates that the intermediate roof of the opening, which is rectangular, may be considered as a beam or group of beams in layers loaded by the material above it and by its own weight. The material overlying the roof beams applies a variable weight to the beams according to its rigidity and other properties. In stratified materials, the roof of a rectangular mine opening may be considered as a beam loaded by the overlying material and restrained between the pillars and the overlying material (degree of restraint is also dependent upon the amount and character of overlying material).

One method of determining the stresses in these rock beams has been to make small-scale models from the rock and apply a load similar to that applied to the prototype. To be of benefit, it must be possible, by the laws of similitude, to calculate from the model what the results will be in the prototype. Thus, if the scale is one to ten, and the roof of the model deflects .01 in., how much will the prototype deflect? The principle of similitude states that if the ratio of the linear dimensions of the model to the prototype is known, and if the ratio of the weight of the model to the weight of the prototype is equal to the same amount, all other properties of the two bodies being equivalent, the model will behave in a manner similar to the prototype. Stated differently, the model will behave like the prototype if the density or effective weight of the model is increased in the same proportion that its linear dimensions are decreased. Bucky(2) published a paper in 1931 in which he described a centrifuge of his own design which could be used to increase the effective weight of the rock model in the same proportion that its linear dimensions were decreased. A scale model built of the same material as the prototype was rotated in the centrifuge at such a speed that it exerted a force against the end plate of the centrifuge equal to its weight multiplied by the number of times it had been reduced in scale, giving results in the model similar to those in the prototype. The stresses in the model approximate those in the prototype due to a gravitational field if the model is subjected to an acceleration in the
centrifuge equal (in units of gravity) inversely to
the scale multiplier of the model (stress = force per
unit area = density x acceleration x linear dimen-
sion). Density is the same constant for both the
prototype and the model. The linear measurement
has been reduced by the ratio stated previously, so
that the acceleration must be increased by the same
ratio to keep the stress in model and prototype the
same. The number of times the effective weight of
the model is increased is called the Model Ratio.
The model dimension times the Model Ratio de-
termines the dimension of the prototype that will
behave in the same manner as the model being
tested.\(^\text{(20)}\)

In order for this to be a successful method of
testing, certain desirable characteristics are re-
quired of the rock models. These are (1) the model
should be an exact scalar representation of the pro-
totyple; (2) the rock should be the same material
as the prototype; (3) the model should be loaded
in the same manner as the prototype; and (4) fa-
cilities must be available to observe and record the
effects of the loading upon the model. The rock
model tested in the centrifuge is loaded as nearly
like the prototype as possible, since loads on rock
beams due to their own weight and the weight of
overlying materials are duplicated to a close ap-
proximation in the centrifuge. (See Chapter V). To
observe the specimen in the centrifuge it is neces-
sary to use stroboscopic light.

The use of the centrifuge employed by Bucky
has certain distinct advantages over other methods
of testing: (1) it is a source of information for
mine openings stressed to failure; (2) it enables the
user to obtain solutions in a short period of time;
(3) it may be applied to many underground prob-
lems; and (4) it closely duplicates, in the model,
the loading of the prototype. It also appears to
have several disadvantages: (1) the grain structure
of the rock may have little effect on the stress in
the prototype, but the grains are not reduced in
the model so that they become larger in relative
size and might possibly have some effect upon the
stress in the model; (2) rock being heterogeneous
in character, it would appear difficult to obtain a
specimen of rock for a model which would exhibit
the over-all properties of the prototype; (3) obtain-
ing the correct amount of restraint at the ends of
the beams is a problem; and (4) with the method
outlined, it is possible to determine only certain
maximum stresses which occur at the time of
failure. The over-all stress pattern obtainable by
other methods is not easily obtained by centrifugal
testing, particularly in the elastic range. This is
important because a knowledge of the stress con-
centrations below the elastic limit permits the de-
signing of structures in which stresses are below the
elastic limit. Some attempts have been made to
rectify this difficulty by using photoelastic models
in the centrifuge, but the stresses were difficult to
evaluate, due to the photographic problems in-
volved. (The use of stroboscopic light is required.)
A possible source of error in the first centrifuge
used by Bucky was the fact that the lever arm of
the centrifuge was only 8 to 8\(\frac{1}{4}\) in. long. For speci-
mens more than a few inches in depth, this means
that the variation in effective weight of a specimen
is considerably over its depth. In the case of a
single beam this condition would not lead to diffi-
culty since the error is small, but in the case of
multiple beams it appears that larger error might
be introduced. The centrifuge now in use by Bucky
has a 2-ft lever arm so that the variation of cen-
trifugal force over the depth of a beam and its
related load is not very large in comparison to the
magnitude of the centrifugal forces.

It was found in a series of experiments by
Bucky that below a certain thickness the beams
failed by tensile fracture of the lower center sec-
tion, but for greater thickness of beams an arch-
like piece of rock separated from the center of the
beam, leaving the remainder of the beam standing.
Bucky explained this latter phenomena as a natural
arching of the rock similar to that which occurs
underground in the roof of the openings where the
stresses are very high.

In general the procedure has been to cut single-
beam models from the rock to be tested and run
them to failure in the centrifuge. This determines
their relative strengths. The beams simulating the
prototypes are then tested in the centrifuge and
also stressed to failure. It is comparatively simple
to determine the load on the model beam simulating
the prototype; and utilizing the previously de-
termined strengths, it is possible to determine the
maximum length of beam which can be used for a
particular load.

The centrifuge, however, is a very costly piece
of equipment, and it is questionable whether it
would be used if a simpler way could be found for
obtaining the desired results.

A new manner of using the centrifuge has been
introduced by the applied Physics Branch of the
Strain gages are attached to the rock beams being tested to determine the actual strain up to the point of failure. This procedure should help in the determination of the stress distribution in a rock model in the centrifuge.

7. Comparison of Stresses in Rock Beams for Three Types of Loads by means of the Theory of Elasticity

The problem of stresses in stratified roof rock also lends itself to solution by the theory of elasticity. In the instances where the roof rock is assumed to consist of only a single stratum, the mathematics are not too involved. For a larger number of beams, the solutions become more complex. In the solutions given here, interest was centered around three questions: (1) Is it possible to approximate the stress existing in a rock roof by assuming that the load is equivalent to a uniform load on the top of the beam? (2) Does centrifugal testing give an accurate measurement of the stress within the roof members? (3) Is there enough difference between the stresses obtained by centrifugal testing and by applying a uniform load to warrant the use of a centrifuge? In addition, it was desired to obtain the fundamental concepts necessary to determine just what effect the presence of stratified layers over a mine working have on the stresses in the mine roof. This subject is treated in the next chapter as far as it has been developed.
V. MATHEMATICAL ANALYSIS OF STRESSES IN SIMPLE ROOF STRATA

The previous four chapters present a summary of many important cases of stress analysis applied to underground mine openings, accounts of which may be found in the literature. With the exception of a uniformly loaded simple beam, no fundamental stress analyses appear to have been made of the stresses existing in sections of mine roofs which may be considered as beams.

Where the dimensions of flat-lying openings underground are of proper magnitude, the roof may be considered to be composed of a plate. A very close approximation to this can be obtained by assuming that a section of the roof may be considered as a beam. Bucky made this assumption in much of his laboratory testing in the centrifuge, but stressed the beams to failure rather than analyzing the stress distribution on a fundamental basis to compare it with the prototype.

The following analyses (with the exception noted above) were made by the authors to show the stress distribution which occurs in simple and restrained beams under the three different conditions which might be encountered underground or in laboratory testing, i.e., (1) uniformly loaded, (2) loaded by own weight, and (3) centrifugally loaded. The mathematical development presented should be of value in clarifying previous research results as well as furnishing basic information for future investigations.

B. Simple Beams

Only in rare instances can the roof of an opening be represented by a simple supported beam. However, the stress analysis of this case serves as a basis for the solution of other problems directly concerned with mining. The fundamental assumptions made previously for solutions by the theory of elasticity must be made also for the following.

a. Uniform Load

The solution of the stress distribution in a beam subjected to a uniformly distributed load is treated by Timoshenko, whose notation is employed herein. It is assumed, together with the assumptions of elastic behavior indicated earlier, that a small rectangular beam can be taken as being representative of a section of a mine roof, that is, a solution in plane stress may be used to closely approximate the condition of plane strain.

Figure 26 illustrates the method of loading the beam and the coordinate system used. The stresses in a beam under uniform load are as follows:

\[
\begin{align*}
\sigma_x &= \frac{3}{4} \frac{qy}{4e^3} \left( \frac{2y^2}{3} - x^2 \right) + \frac{3}{4} \frac{qy}{e^3} \left( l^2 - \frac{2x^2}{5} \right) \\
\sigma_y &= -\frac{3qy}{4e^3} \left( \frac{y^2}{3} - e^2 \right) - \frac{q}{2} \\
\tau_{xy} &= -\frac{3qx}{4e^3} \left( e^2 - y^2 \right)
\end{align*}
\]

It can be clearly seen that the maximum tensile stress \(\sigma_x\) occurs at the midpoint of the beam in the lowermost fibers. It is important to note that both the magnitude and distribution of the stress is independent of the elastic constants of the material.

b. Loaded by Own Weight

Solutions of the stress distribution in a beam loaded by its own weight (weight per unit volume = \(\rho g\)) involve the simultaneous solution of the differential equations of equilibrium.

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g &= 0
\end{align*}
\]

and the compatibility equation for plane stress,

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -(1 + \nu) \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right)
\]

which must also satisfy the boundary conditions. The equilibrium equations are satisfied if the stresses conform to the following equations:

\[
\sigma_x = \frac{\partial^2 \phi}{\partial y^2}
\]
Substituting Eqs. 11, 12 and 13 back into Eq. 10 one determines a new equation:

\[
\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0
\] (14)

which is known as the differential stress function. Solutions of Eq. 14 which satisfy the boundary conditions are unique solutions and are the solutions being sought. It is found that the solution for a beam loaded by its own weight may be considered as the sum of four polynomials:

\[
\phi = \phi_2 + \phi_3 + \phi_4 + \phi_5
\] (15)

where

\[
\phi_2 = \frac{a_2}{2} x^2 + b_2 xy + \frac{c_2}{2} y^2
\] (16)

\[
\phi_3 = \frac{a_3}{3.2} x^3 + b_3 x^2 y + \frac{c_3}{2} xy^2 + \frac{d_3}{3.2} y^3
\] (17)

\[
\phi_4 = \frac{a_4}{4.3} x^4 + b_4 y^3 + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3.2} xy^3
\] (18)

\[
\phi_5 = \frac{a_5}{5.4} x^5 + b_5 x^4 y + \frac{c_5}{3.2} x^3 y^2 + \frac{d_5}{3.2} x^2 y^3
\] (19)

In order for Eq. 14 to be satisfied by Eqs. 18 and 19 it is necessary that

\[
e_4 = -(2c_4 + a_4)
\] (20)

\[
e_5 = -(2c_5 + 3a_5)
\] (21)

\[
f_5 = -\frac{1}{3} (b_5 + 2b_5)
\] (22)

Writing \(\sigma_y, \sigma_y, \text{ and } \tau_{xy}\) in terms of the stress function \(\phi\), they can be evaluated at the boundaries for the following boundary conditions:

\[
\begin{align*}
\sigma_y &= 0 & \sigma_y &= 0 & \text{at } y = c \\
\sigma_y &= 0 & \sigma_y &= 0 & \text{at } y = -c \\
\tau_{xy} &= 0 & \tau_{xy} &= 0 & \text{at } x = l \\
\tau_{xy} &= 0 & \tau_{xy} &= 0 & \text{at } x = -l \\
\int_{-c}^{c} \sigma_y dy &= 0 & \int_{-c}^{c} \sigma_y dy &= 0 \\
\int_{-c}^{c} \tau_{xy} dy &= -2cpgl & \int_{-c}^{c} \tau_{xy} dy &= 2cpgl
\end{align*}
\] (23)

Evaluating the normal and shearing stresses at the boundaries, the following are obtained

\[
\begin{align*}
\sigma_x &= \frac{3pgy}{2c^2} (l^2 - x^2) + \frac{pgy}{c^3} \left(y^2 - \frac{3c^2}{5}\right) \\
\sigma_y &= \frac{pgy}{2c^2} (c^2 - y^2) \\
\tau_{xy} &= -\frac{3pgx}{2c^2} (c^2 - y^2)
\end{align*}
\] (24) (25)

It is interesting to note that the \(\sigma_y\) force is very small, being zero at the bottom and top and also at the center of the beam. In the upper half of the beam \(\sigma_y\) is compressive, and in the bottom of the beam \(\sigma_y\) is tensile. Figure 26 shows the manner of loading and the coordinate axes.

c. Centrifugal Loading

The case of a beam loaded centrifugally must be treated in a somewhat different manner to the case...
of a beam loaded by its own weight due to the fact that the load (body force \( Y \)) is a function of \( y \), so
\[
Y = -\frac{\partial V}{\partial y} = \rho_0^2(y + r_o) \tag{26}
\]

Where
- \( \omega \) is the angular velocity
- \( r_o \) is the radius of the midpoint of the beam
- \( V \) is the body force potential
\[
V = -\frac{\rho_0^2}{2}(y + r_o)^2 \tag{27}
\]

The compatibility equation for plane stress (Eq. 10) becomes
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -(1 + \nu) \rho_0^2 \tag{28}
\]

It is found that the equilibrium equations are satisfied, if the stresses are made to conform to the following
\[
\sigma_x = \frac{\partial \phi}{\partial y} + V = \frac{\partial^2 \phi}{\partial y^2} - \frac{\rho_0^2}{2} (y + r_o)^2 \tag{29}
\]
\[
\sigma_y = \frac{\partial \phi}{\partial x} + V = \frac{\partial^2 \phi}{\partial x^2} - \frac{\rho_0^2}{2} (y + r_o)^2 \tag{30}
\]
\[
\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{31}
\]

Substituting Eqs. 29, 30, and 31 into Eq. 28 determines the differential stress function,
\[
\frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2} = -(1 + \nu) \rho_0^2 \tag{32}
\]
which is found to have the solution
\[
\phi = \phi_0 + \phi_1 + \phi_2 + \frac{(1-\nu)}{4 \cdot 3} \rho_0^2 d \phi(y + r_o)^4 \tag{33}
\]
where \( d \phi \) is an arbitrary constant to be determined. It is found that in order for Eq. 33 to satisfy the differential stress function, \( d \phi \) must be equal to \( \frac{1}{2} \).

At the boundaries the following relationships hold:
\[
\begin{align*}
\text{at } y = c & \quad \frac{\partial \phi}{\partial y} = 0 \quad \sigma_y = 0 \quad \tau_{xy} = 0 \\
\text{at } y = -c & \quad \sigma_y = 0 \quad \tau_{xy} = 0 \\
\text{at } x = l & \quad \int_y^c \sigma_x dy = 0 \quad \int_y^c \tau_{xy} dy = 0 \\
\text{at } x = -l & \quad \int_y^c \sigma_x dy = 0 \quad \int_y^c \tau_{xy} dy = 0 \\
\int_y^c \sigma_2 y dy = 0 & \quad \int_y^c \tau_{xy} dy = -2 \rho_0^2 r_o l_c \\
\int_y^c \tau_{xy} dy = 0 & \quad \int_y^c \tau_{xy} dy = 2 \rho_0^2 r_o l_c
\end{align*}
\]

Writing the stresses in terms of the stress function and the body force potential, they can be evaluated at the boundaries, and the following equations for stress throughout the body are obtained:
\[
\sigma_x = \frac{3\rho_0^2 r_o}{2c^2} (b^2 - x^2) + \frac{\rho_0^2 r_o \psi}{c^3} \left( y^2 - \frac{3c^2}{5} \right)
\]
\[
\sigma_y = \frac{\rho_0^2}{2} \left( \frac{c^2}{3} - y^2 \right) \tag{34}
\]
\[
\tau_{xy} = -\frac{3\rho_0^2 r_o x}{2c^3} (c^2 - y^2) \tag{35}
\]

It can be seen that these equations for stress are quite similar to those obtained for a uniform load and a gravitational load, with the exception of the final term in \( \sigma_x \). The stress \( \sigma_y \) is found to be very small in comparison to the maximum \( \sigma_x \), as would be expected. The equations of stress for the simple beam under the three load types are given in Tables 11 and 12 for comparison.

9. Restrained Beams

The investigation of simple beams which has been given provides a basis for the examination of stresses in more complicated beam structures. The next step is the consideration of a beam loaded by the same three types of loads as were used for simple beams, but for those with restrained ends. This condition occurs over every rectangular mine opening if the stresses are not so high that the ends of beams at the pillar are stressed beyond their elastic limit. Beams under thick cover may be thought of as completely restrained, that is, the end of the beam may be considered to lie at the edge of the pillar; whereas beams under light cover may be considered to be only partially restrained, that is, the end of the beam lies somewhere over the interior of the pillar. Under some conditions, except for degree of restraint, both beams may be considered fully restrained with the lengths dependent upon the degree of restraint imposed by the overburden (Fig. 27). The investigation may be carried through by two different methods. In either method, the assumption is made (supported by Saint Venant's Principle) that the stress \( \sigma_x \) at the ends of the beam may be replaced by the sum of two stresses—a uniform normal stress and a stress due to a coupled applied to the ends of the beam—without altering stresses from those actually ex-
### Table 11
Compilation of Stress Equations for Simple Beam Loaded by Three Type Loads

<table>
<thead>
<tr>
<th>No.</th>
<th>Total Load</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \tau_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Uniform Load</td>
<td>( \frac{3qy}{4c^3} (l^2 - x^2) + \frac{qy}{2c^3} \left( y^2 - \frac{3c^2}{5} \right) )</td>
<td>( \frac{qy}{4c^3} (3c^2 - y^2) - \frac{q}{2} )</td>
<td>( -\frac{3qx}{4c^3} (c^2 - y^2) )</td>
</tr>
<tr>
<td>1B</td>
<td>Loaded by Own Weight</td>
<td>( \frac{3pq}{4c^2} (l^2 - x^2) + \frac{pqy}{2c^3} (y^2 - \frac{3c^2}{5}) )</td>
<td>( \frac{pqy}{2c^2} (c^2 - y^2) )</td>
<td>( -\frac{3pqy}{2c^2} (c^2 - y^2) )</td>
</tr>
<tr>
<td>1C</td>
<td>Centrifugal Loading</td>
<td>( \frac{3pq^2r}{2c^2} (l^2 - x^2) + \frac{pq^2r}{c^2} \left( y^2 - \frac{3c^2}{5} \right) )</td>
<td>( \frac{pq^2r}{2c^2} (c^2 - y^2) \left( \frac{ry}{c^2} + 1 \right) )</td>
<td>( -\frac{3pq^2r}{2c^2} (c^2 - y^2) )</td>
</tr>
</tbody>
</table>

\[ \text{where } \sigma_x = \sigma_x' + \sigma_x'' \]

### Table 12
Compilation of Analogous Stress Equations for Simple Beam Loaded by Three Type Loads

<table>
<thead>
<tr>
<th>No.</th>
<th>Total Load</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \tau_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Uniform Load</td>
<td>( \frac{3qy}{4c^3} (l^2 - x^2) + \frac{qy}{2c^3} \left( y^2 - \frac{3c^2}{5} \right) )</td>
<td>( \frac{qy}{4c^3} (3c^2 - y^2) - \frac{q}{2} )</td>
<td>( -\frac{3qx}{4c^3} (c^2 - y^2) )</td>
</tr>
<tr>
<td>1B</td>
<td>Loaded by Own Weight</td>
<td>( \frac{3pq}{4c^2} (l^2 - x^2) + \frac{pqy}{2c^3} (y^2 - \frac{3c^2}{5}) )</td>
<td>( \frac{pqy}{2c^2} (c^2 - y^2) )</td>
<td>( -\frac{3pqy}{2c^2} (c^2 - y^2) )</td>
</tr>
<tr>
<td>1C</td>
<td>Centrifugal Loading</td>
<td>( \frac{3pq^2r}{2c^2} (l^2 - x^2) + \frac{pq^2r}{c^2} \left( y^2 - \frac{3c^2}{5} \right) )</td>
<td>( \frac{pq^2r}{2c^2} (c^2 - y^2) \left( \frac{ry}{c^2} + 1 \right) )</td>
<td>( -\frac{3pq^2r}{2c^2} (c^2 - y^2) )</td>
</tr>
</tbody>
</table>

\[ \text{where } \sigma_x' = \sigma'_{x} \text{ and } \sigma_x'' = \sigma''_{x} \]

**Set 2cpp equal to the load } q \text{ to obtain a comparison of the magnitude of stresses.}

**Set 2cpp equal to the load } q \text{ to obtain a comparison of the magnitude of stresses.}

Existing in a restrained beam at distances a short distance away from the ends. This is merely another way of stating that in order to insure that the ends of the beam are completely restrained, we must apply a uniform normal stress \((r)\) to the ends of the beam, so that there is no horizontal movement of the ends. A moment \((M)\) must also be applied so that there is no rotation of the vertical fibers of the beam at the center point of the ends.

Following the first method, it is then necessary to add the following boundary conditions

\[
\int_{-l}^{l} \sigma_x dy = 2cr \quad (37)
\]

\[
\int_{-l}^{l} \sigma_y dy = M \quad (38)
\]

to the boundary conditions for the previous beams and solve for the stress equations in a manner similar to that previously described. The stresses are evaluated at the boundaries, taking account of the two new boundary conditions, and an expression for \(\sigma_x\) involving \(r\) and \(M\) is determined.

\[
\sigma_x = r - \frac{3My}{2c^3} + \sigma_x' \quad (39)
\]

where \(\sigma_x'\) is the stress given for the respective loads for a simple beam. It may also be shown that the new boundary conditions for \(\sigma_y\) have no effect upon \(\sigma_x\) or \(\tau_{xy}\), so that their values are those given previously for simple beams.

Equation 39 provides a basis for the second method of investigation, that is, superposition of stresses. By superimposing upon \(\sigma_x\) a uniform stress \(r\) and a bending stress \(-3M/2c^3\), Eq. 39 is obtained. This method of superposition has been explained by Timoshenko.

In either method of investigation use of only the stress relationship involves indeterminate values. From a study of elementary stress solutions it is apparent that the only means of solution remaining is one involving strain relationships.
Using Timoshenko’s notation, the following strain relationships are properties of restrained beams which can be used to determine the unknowns \( r \) and \( M \).

- \( u = 0 \) (no horizontal displacements) \( (x = 0) \)
- \( u = 0 \) (no horizontal displacements) \( (x = 0) \)
- \( \frac{\partial u}{\partial y} = 0 \) (vertical fibers remain vertical) \( (x = 0) \)
- \( \frac{\partial v}{\partial y} = 0 \) (vertical fibers remain vertical) \( (x = 0) \)
- \( v = 0 \) (no vertical displacement at) \( (x = 0) \)

For the case of plane stress the following relations give the strains in terms of the stresses. These equations are employed to obtain the solution of the problem. They are:

\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \tau_{xy} \quad (45)
\]

\[
\frac{\partial u}{\partial x} = \frac{\sigma_x - \nu\sigma_y}{E} \quad (46)
\]

\[
\frac{\partial v}{\partial y} = \frac{\sigma_y - \nu\sigma_x}{E} \quad (47)
\]

Substituting the values of \( \sigma_x \), \( \sigma_y \), and \( \tau_{xy} \) into the above equations, and integrating, expressions in \( u \) and \( v \) containing the terms \( r \) and \( M \) are obtained. Use is made of the strain relationships given previously to determine arbitrary constants of integration and the quantities \( r \) and \( M \). The values for \( r \) and \( M \) when substituted into the stress Eq. 30 give the results for three different load types shown in Table 13.

10. Summary

It has been previously noted that the stress \( \sigma_y \) induced in simple beams is very small in comparison to the stress \( \sigma_x \) at the center of the beam. The stress \( \sigma_y \) is also dependent on the length of the beam.

In order to determine whether the stresses in simple beams loaded by three load types are comparable, the load per unit length of beam was made equal in all three cases. The loads, \( 2cpg \) and \( 2cpr \), were set equal to the load per unit length \( q \) and the equations for the three cases were determined in terms of \( q \), as is shown in Table 12. The shearing stresses in all three cases are equal. The \( \sigma_x \) stresses for a uniform load and for a beam loaded by its own weight are found to be the same, but the stress \( \sigma_y \) for the case of centrifugal loading differs from the other two cases by an additional term. For beams of normal \( l/c \) ratio, it is seen that this term is negligible in comparison to the other terms, and thus does not influence the \( \sigma_y \) forces measurably. Therefore, \( \sigma_y \) may be said to be equal for the three cases without introducing any measurable error. This is not true of \( \sigma_y \), however; \( \sigma_y \) is small in the case of the centrifugal loading and in that of a beam loaded by its own weight, in both cases being only a small fraction of the load per unit length of beam. In the case of a beam loaded by a uniform load, \( \sigma_y \) ranges from a value equal to the
Table 13

Compilation of Stress Equations for Restrained Beam Loaded by Three Type Loads

<table>
<thead>
<tr>
<th>No.</th>
<th>Total Load</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \tau_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>11A</td>
<td>Uniform Load</td>
<td>( \sigma_x = \frac{3qy}{4c^2} (z^2 - x^2) + \frac{qy}{2c^2} \left( y^2 - \frac{3z^2}{5} \right) )</td>
<td>( \sigma_y = -\frac{q}{2} )</td>
<td>( \tau_{xy} = -\frac{3qx}{4c^2} (c^2 - y^2) )</td>
</tr>
<tr>
<td>11B</td>
<td>Loaded by Own Weight</td>
<td>( \sigma_x = \frac{3qy}{2c^2} (z^2 - x^2) + \frac{qy}{c^2} \left( y^2 - \frac{3z^2}{5} \right) )</td>
<td>( \sigma_y = \frac{qy}{2c^2} (c^2 - y^2) )</td>
<td>( \tau_{xy} = -\frac{3qy}{2c^2} (c^2 - y^2) )</td>
</tr>
<tr>
<td>11C</td>
<td>Centrifugal Loading</td>
<td>( \sigma_x = \frac{3\rho_o \omega^2 r y}{2c^2} (z^2 - x^2) + \frac{\rho_o \omega^2 r y}{c^2} \left( y^2 - \frac{3z^2}{5} \right) )</td>
<td>( \sigma_y = -\frac{\rho_o \omega^2}{2} \left( c^2 - y^2 \right) \left( \frac{r y}{c^2} + 1 \right) )</td>
<td>( \tau_{xy} = -\frac{3\rho_o \omega^2 r x}{2c^2} (c^2 - y^2) )</td>
</tr>
</tbody>
</table>

The 28. They are sketched using the isoclinics. (Each stress trajectory crosses an individual isoclinic at an angle with the horizontal which the isoclinic represents.) That is, a stress trajectory crossing the 40-deg isoclinic must cross that isoclinic at an angle of 40 deg with the horizontal. It can be seen that the stress trajectories for the three types of loads are also almost identical.

In the case of restrained beams, it can be seen that \( \sigma_y \) and \( \tau_{xy} \) remain the same as in the case of the simple beams. See Table 13. The additional terms which have been superimposed upon the original \( \sigma_x \) equations for the simple beams to obtain the \( \sigma_x \) stresses are different for the three different types of loading. To determine the magnitude of this difference, it is necessary to assume that the load per unit length for each of the three beams is the same, as was done in the case of simple beams. Substituting the values of the centrifugal load and gravitational load equal to the uniform load (per unit length of beam) into the stress equations for the respective beams, the stress equations for the three types of load are obtained considering an equal load on the three beams. These equations are shown in Table 14. A beam loaded centrifugally and a beam loaded by its own weight differ in \( \sigma_x \) only in an additional term in the equation for centrifugal loading. This term is so small as to be negligible. Similarly, the \( \sigma_y \) stresses differ in the case of uniform loading from the other two cases by a term, \( -q/2 \). Isoclinics and stress trajectories

load per unit length of beam at the top of the beam to a value of zero stress at the bottom surface of the beam.

The \( \sigma_y \) stresses are not large enough in any of the three cases to be directly responsible for failure of a beam because as the \( \sigma_y \) forces are increased to a critical value the \( \sigma_x \) forces have already exceeded this value due to their greater initial value, and because the relationship of the stresses is directly proportional. It is of interest to determine whether the difference in \( \sigma_y \) for the three cases has any effect upon the direction of the stresses in the body and the general distribution of stress throughout the body. Isoclinics were drawn in Fig. 28 for the three types of load. Along each isoclinic the direction of the principal stresses are a constant. That is, along each isoclinic the direction of the stresses in the body and the difference in the case of uniform loading from the other two cases by a term, -q/2. Isoclinics and stress trajectories

It can be seen that the isoclinics for the three cases are almost identical.

Stress trajectories, lines along which the direction of the principal stresses are tangent to the lines, are of interest. These have been shown in Fig. 28. They are sketched using the isoclinics. (Each stress trajectory crosses an individual isoclinic at an angle with the horizontal which the isoclinic represents.) That is, a stress trajectory crossing the 40-deg isoclinic must cross that isoclinic at an angle of 40 deg with the horizontal. It can be seen that the stress trajectories for the three types of loads are also almost identical.

In the case of restrained beams, it can be seen that \( \sigma_y \) and \( \tau_{xy} \) remain the same as in the case of the simple beams. See Table 13. The additional terms which have been superimposed upon the original \( \sigma_x \) equations for the simple beams to obtain the \( \sigma_x \) stresses are different for the three different types of loading. To determine the magnitude of this difference, it is necessary to assume that the load per unit length for each of the three beams is the same, as was done in the case of simple beams. Substituting the values of the centrifugal load and gravitational load equal to the uniform load (per unit length of beam) into the stress equations for the respective beams, the stress equations for the three types of load are obtained considering an equal load on the three beams. These equations are shown in Table 14. A beam loaded centrifugally and a beam loaded by its own weight differ in \( \sigma_x \) only in an additional term in the equation for centrifugal loading. This term is so small as to be negligible. Similarly, the \( \sigma_y \) stresses differ in the case of uniform loading from the other two cases by a term, \( -q/2 \). Isoclinics and stress trajectories
Simple beam under uniform load, \( q = 1 \), \( c = 3/4 \) and \( l = 5 \)

Simple beam under centrifugal loading or loaded by its own weight, \( 2cg = 2cp \left( r_0 + y \right) = 1 \), \( c = 3/4 \), \( l = 5 \), \( r_0 = 10 \) and \( v = 0.25 \)

Restrained beam under uniform load, centrifugal load or gravitational load, \( q = 2cp = 2cp \left( r_0 + y \right) = 1 \), \( c = 3/4 \), \( l = 5 \), \( r_0 = 10 \) and \( v = 0.25 \)

Table 14
Compilation of Analogous Stress Equations for Restrained Beam Loaded by Three Type Loads

<table>
<thead>
<tr>
<th>No.</th>
<th>Total Load</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \tau_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>11A</td>
<td>Uniform Load</td>
<td>( \frac{3qy}{4c^3} (r^2-x^2) + \frac{qy}{2c^3} \left( y^2 - \frac{3c^2}{5} \right) )</td>
<td>( \frac{qy}{4c^3} (3c^2 - y^2) ) ( - \frac{q}{2} )</td>
<td>( \frac{3qx}{4c^3} (c^2 - y^2) )</td>
</tr>
<tr>
<td>11B*</td>
<td>Loaded by Own Weight</td>
<td>( \frac{3qy}{4c^3} (r^2-x^2) - \frac{qy}{2c^3} \left( y^2 - \frac{3c^2}{5} \right) )</td>
<td>( \frac{qy}{4c^3} (c^2 - y^2) )</td>
<td>( \frac{3qx}{4c^3} (c^2 - y^2) )</td>
</tr>
<tr>
<td>11C†</td>
<td>Centrifugal Loading</td>
<td>( \frac{3qy}{4c^3} (r^2-x^2) + \frac{qy}{2c^3} \left( y^2 - \frac{3c^2}{5} \right) ) + ( \frac{qy}{4cr^6} \left( \frac{c^2}{3} - y^2 \right) )</td>
<td>( \frac{qy}{4c^3} \left( \frac{1 - c^2}{r_0 y} \right) (c^2 - y^2) )</td>
<td>( \frac{3qx}{4c^3} (c^2 - y^2) )</td>
</tr>
</tbody>
</table>

* Set \( 2cpq \) equal to the load \( q \) to obtain a comparison of the magnitude of stresses.
† Set \( 2cp_qc^2 \) equal to the load \( q \) to obtain a comparison of the magnitude of stresses.
were drawn for the three beams with the different load types and it was found that no appreciable difference in isoclinics or stress trajectories exists for restrained beams loaded by uniform load, gravitational load, or centrifugal load. Figure 28 illustrates the isoclinics and stress trajectories in beams of this type (close approximation).

In summarizing, it may be stated that the critical stresses within both simple and restrained beams loaded by (1) uniform loads, (2) gravitational loads, or (3) centrifugal loads are the same, with the exception of certain terms of orders which may be neglected without appreciable error. Thus it is possible to approximate the stresses in a roof beam of rock in a mine, either by centrifugal loading or by uniform loading, if it may be considered as a simple beam or a restrained beam, and if the load is due to its own weight or a uniform load, or a combination of the two. If testing is proposed for beams of this type, it appears that it would be simpler to load the beams with a uniform load than to resort to the more complicated and expensive method of centrifugal testing where the tests are to be made within the elastic limits of the material.

It is not possible to make an evaluation of the three types of loads for multiple beams of different materials or a single beam loaded by overlying materials from the data obtained above because the load transmitted from one beam to another is not necessarily uniform. In fact there may be no load transmitted between layers in certain cases underground, so it would appear that uniform loading would not be applicable to this case. From the material observed concerning the experimental results of centrifugal testing, that method would appear preferable for this type of problem.
VI. SUMMARY AND CONCLUSIONS

The design of underground mining structures has been limited by a lack of knowledge of (1) the stresses existing in rock underground before the introduction of a mine opening, (2) the effect of the introduction of a mine opening upon the pre-existing rock stresses, (3) the effect of complex geologic conditions, and (4) the physical properties of rock in situ.

It has been shown that the stresses existing within rock underground may generally be thought of as intermediate between the stresses which would exist if there were no lateral restraint and those where hydrostatic pressures are found. In a few exceptional cases, it has been shown that the horizontal pressures may be greater than the vertical pressures, and compensation must be made for this fact.

The effect of introduction of mine openings upon the stresses in rock underground has been solved for some of the simplest geologic conditions—homogeneous, isotropic, elastic material. The opening best suited for a mine opening (in regard to stress introduced) is either (1) a circle, for hydrostatic pressures; (2) an ellipse with ratio of height-to-width equal to 3, for an opening in material which has been laterally restrained; or (3) an ellipse with height-to-width ratio greater than 5, for a mine opening in ground which has no lateral restraint.

In general, mines are composed of numerous adjacent openings underground rather than a single opening. The stresses in the pillars are found to increase with the addition of openings, until about five openings lie parallel to one another. Additional openings have no effect on the critical compressive stress in the pillars. The maximum stress concentration occurs in the centermost pillar on or near the rib of the pillar. If pillars are taller than they are wide, it is often more advantageous to decrease the long cross-sectional dimension of the openings than to widen the pillars, if critical stress is very high, or pillar width may be increased and possibly room width decreased.

The effect of many geologic features, such as faults, fissures, and metamorphism, upon the stresses induced around mine openings has not been evaluated. Certain simple geologic structural problems, more specifically the stresses in stratified rock, may be solved by laboratory and analytical methods. It has been shown in Chapter V how the problem of stresses in roof rock, where it acts as a simple or completely restrained beam, may be solved by the theory of elasticity when the rock is stressed within its elastic limit; and the results of these solutions are given in Tables 11 and 13. In addition, numerous other geologic conditions may lend themselves to solutions by photoelastic methods, or by testing of rock models in a centrifuge.

The problem of measuring the physical properties of undisturbed rock in place remains to be solved.
VII. BIBLIOGRAPHY


This page is intentionally blank.
This page is intentionally blank.
The Engineering Experiment Station was established by act of the University of Illinois Board of Trustees on December 8, 1903. Its purpose is to conduct engineering investigations that are important to the industrial interests of the state.

The management of the Station is vested in an Executive Staff composed of the Director, the Associate Director, the heads of the departments in the College of Engineering, the professor in charge of Chemical Engineering, and the Director of Engineering Information and Publications. This staff is responsible for establishing the general policies governing the work of the Station. All members of the College of Engineering teaching staff are encouraged to engage in the scientific research of the Station.

To make the results of its investigations available to the public, the Station publishes a series of bulletins. Occasionally it publishes circulars which may contain timely information compiled from various sources not readily accessible to the Station clientele or may contain important information obtained during the investigation of a particular research project but not having a direct bearing on it. A few reprints of articles appearing in the technical press and written by members of the staff are also published.

In ordering copies of these publications reference should be made to the Engineering Experiment Station Bulletin, Circular, or Reprint Series number which is at the upper left hand corner on the cover. Address

THE ENGINEERING EXPERIMENT STATION

UNIVERSITY OF ILLINOIS

URBANA, ILLINOIS