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TESTS OF THE FATIGUE STRENGTH OF STEAM TURBINE BLADE SHAPES

A REPORT OF AN INVESTIGATION

CONDUCTED BY

THE ENGINEERING EXPERIMENT STATION

IN COOPERATION WITH

THE ALLIS-CHALMERS MANUFACTURING COMPANY

BY

HERBERT F. MOORE
STUART W. LYON
NORVILLE J. ALLEMAN

BULLETIN No. 183
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BULLETIN No. 183

OCTOBER, 1928

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Research Assistant in Engineering Materials

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ENGINEERING EXPERIMENT STATION
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<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>5</td>
</tr>
<tr>
<td>1. General Purpose of Tests</td>
<td>5</td>
</tr>
<tr>
<td>2. Acknowledgments</td>
<td>5</td>
</tr>
<tr>
<td>II. MATERIALS, SPECIMENS, APPARATUS, AND METHODS OF TESTING</td>
<td>6</td>
</tr>
<tr>
<td>3. Materials and Specimens</td>
<td>6</td>
</tr>
<tr>
<td>4. Testing Apparatus and Methods</td>
<td>7</td>
</tr>
<tr>
<td>5. Determination of Unit Stress in Specimens</td>
<td>14</td>
</tr>
<tr>
<td>6. Test Data</td>
<td>15</td>
</tr>
<tr>
<td>III. RESULTS AND CONCLUSIONS</td>
<td>15</td>
</tr>
<tr>
<td>7. Results and Conclusions</td>
<td>15</td>
</tr>
<tr>
<td>APPENDIX A. A BRIEF DISCUSSION OF THE GENERAL THEORY OF THE FLEXURE OF</td>
<td>21</td>
</tr>
<tr>
<td>STEAM TURBINE BLADES, WITH SPECIAL APPLICATION TO THE FLEXURE OF</td>
<td></td>
</tr>
<tr>
<td>STEAM TURBINE BLADES</td>
<td></td>
</tr>
<tr>
<td>1. Introductory</td>
<td>21</td>
</tr>
<tr>
<td>2. Product of Inertia</td>
<td>22</td>
</tr>
<tr>
<td>3. Determination of Product of Inertia for any Area</td>
<td>23</td>
</tr>
<tr>
<td>4. General Formula for Flexure</td>
<td>24</td>
</tr>
<tr>
<td>5. Application of Theory to Turbine Blades</td>
<td>28</td>
</tr>
<tr>
<td>6. Illustrative Computation of ( Z_{\text{min}} ) for ( 3/4 )-inch</td>
<td>34</td>
</tr>
<tr>
<td>Cupronickel Blading</td>
<td></td>
</tr>
</tbody>
</table>
LIST OF FIGURES

<table>
<thead>
<tr>
<th>NO.</th>
<th>DESCRIPTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Micrographs of Metals in Blading</td>
<td>6</td>
</tr>
<tr>
<td>2.</td>
<td>Cross-sections of Blading</td>
<td>6</td>
</tr>
<tr>
<td>3.</td>
<td>Testing Machine for Fatigue Tests</td>
<td>8</td>
</tr>
<tr>
<td>4.</td>
<td>Working Drawing of Chuck for Specimen of Blading</td>
<td>9</td>
</tr>
<tr>
<td>5.</td>
<td>Chuck for Specimen of Blading</td>
<td>10</td>
</tr>
<tr>
<td>7.</td>
<td>$S-N$ Diagrams for $\frac{3}{8}$-inch and 1-inch Cupro-nickel Blading</td>
<td>16</td>
</tr>
<tr>
<td>8.</td>
<td>$S-N$ Diagrams for $\frac{3}{4}$-inch and 1-inch Monel Metal Blading</td>
<td>17</td>
</tr>
<tr>
<td>9.</td>
<td>$S-N$ Diagrams for $\frac{3}{4}$-inch and 1-inch Cyclops Metal Blading</td>
<td>20</td>
</tr>
<tr>
<td>10.</td>
<td>Diagram for Moments and Product of Inertia of Irregular Section.</td>
<td>24</td>
</tr>
<tr>
<td>11.</td>
<td>Diagram for Moments and Neutral Axis of Irregular Section under Oblique Flexure.</td>
<td>25</td>
</tr>
<tr>
<td>12.</td>
<td>Selection of $X$ Axis for Specimen of Blading</td>
<td>28</td>
</tr>
<tr>
<td>13.</td>
<td>Cross-section of Specimen of Blading (enlarged eight times)</td>
<td>30</td>
</tr>
<tr>
<td>14.</td>
<td>Diagram for Graphical Determination of Moment of Inertia of Irregular Shaped Area.</td>
<td>31</td>
</tr>
<tr>
<td>15.</td>
<td>Diagram for Graphical Solution for Moment of Inertia of Specimen of Blading.</td>
<td>32</td>
</tr>
<tr>
<td>16.</td>
<td>Moment of Inertia Diagrams for Specimen of Blading, Taken about Three Different Axes.</td>
<td>33</td>
</tr>
</tbody>
</table>

LIST OF TABLES

<table>
<thead>
<tr>
<th>NO.</th>
<th>DESCRIPTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Chemical Analyses of Turbine Blade Materials</td>
<td>12</td>
</tr>
<tr>
<td>2.</td>
<td>Test Data for Fatigue Tests of Turbine Blading</td>
<td>18</td>
</tr>
<tr>
<td>3.</td>
<td>Endurance Limits for Turbine Blading</td>
<td>21</td>
</tr>
<tr>
<td>4.</td>
<td>Comparison of Estimated Endurance Limit of Material with Endurance Limit Developed in Fatigue Tests of Drawn Blading.</td>
<td>22</td>
</tr>
<tr>
<td>5.</td>
<td>Properties of Cross-sections of Turbine Blading Tested.</td>
<td>35</td>
</tr>
</tbody>
</table>
TESTS OF THE FATIGUE STRENGTH OF STEAM TURBINE BLADE SHAPES

I. INTRODUCTION

1. General Purpose of Tests.—In studying the strength of any machine or structural member the problem must be considered from at least two viewpoints: (1) the number, direction, and magnitude of the stresses set up in the part under conditions of service, and (2) the strength of the part to resist those stresses. A determination of the stresses set up in service involves a study of many parts in service, and of the varying conditions of service; it is, in general, a problem of study “in the field” rather than in the laboratory. The study of the strength of a machine part to resist stresses of various kinds can be made, in many cases, in the laboratory.

The tests herein reported constitute a study of the strength of steam turbine blade shapes of several different metals to resist many cycles of reversed bending stress; they throw no light on the stresses such blades would be called on to withstand in service. It is, however, known that in service steam turbine blades are subjected to many cycles of bending stress superimposed on a steady stress, due to centrifugal force.

The specimens for the tests herein reported were cut from bars of metal having a cross-section of the same shape and size as the turbine blade. They were tested as rotating beams, so that during a revolution the specimen was subjected to a transverse bending moment which changed direction through a complete circle. The results of the tests are believed to throw some light on the question as to whether turbine blades under reversed flexure will withstand stresses as high as those indicated by fatigue tests of specially prepared specimens of the metal in the blades.

2. Acknowledgments.—Acknowledgment is made to the ALLIS-CHALMERS MANUFACTURING COMPANY, whose financial support made the tests possible, and especially to Mr. J. FLETCHER HARPER, Research Engineer, Mr. R. S. MACPHERRAN, Chief Chemist, and Mr. E. H. BROWN, Assistant Manager, Steam Turbine Department, for their helpful suggestions, and for the data they furnished concerning the chemical composition and the metallographic structure of the material in the blade shapes tested.
These tests have been a part of the work of the Engineering Experiment Station of the University of Illinois, of which Dean M. S. Ketchum is the director, and of the Department of Theoretical and Applied Mechanics, of which Prof. M. L. Enger is the head.

II. Materials, Specimens, Apparatus, and Methods of Testing

3. Materials and Specimens.—The specimens furnished were made from three different metals: (1) monel metal, (2) a copper-nickel alloy known as cupro-nickel, having a content of approximately 80 per cent copper and 20 per cent nickel, and (3) cyclops metal, known as No. 17. Table 1 gives the chemical composition of the various metals, and Fig. 1 shows micrographs of the metal.
FIG. 1. MICROGRAPHS OF METALS IN BLADING

\[\frac{3}{4}\text{-in. Cyclops Steel (x 100)}\]

\[1\text{-in. Cyclops Steel (x 100)}\]

\[\frac{3}{4}\text{-in. Cupro-Nickel (x 100)}\]

\[1\text{-in. Cupro-Nickel (x 100)}\]
Fig. 1 (Continued). Micrographs of Metals in Blading
THE FATIGUE STRENGTH OF STEAM TURBINE BLADE SHAPES

in the different blade sections. The material was furnished by the Allis-Chalmers Manufacturing Company in the form of drawn bars about 3 feet long, with a cross-section the shape of that of the turbine blade. Two different sizes of blading were tested for each metal, and these sizes are designated by the nominal width of the blading from tip to tip as ¾-inch and 1-inch. The cross-sections for blading of different metals were not quite the same. Figure 2 shows the cross-sections of the different bladings furnished. The specimens tested in reversed flexure consisted of pieces of blading 4½ inches long cut from the bars. All specimens were tested "as received," with no further heat treatment, and with the edges of the blades left as fabricated without further smoothing or polishing.

4. Testing Apparatus and Methods.—The testing machines used were of the rotating-beam type, in which the specimen acts as a beam on which weights are hung which set up a bending moment in the specimen. The weights are hung from bearings in which the specimen rotates, and as it does so the stress on any longitudinal "fiber" changes from compression to tension and back again during each revolution.

Figure 3 shows one of the three testing machines used,—all three were alike. The specimen S is held in chucks C₁ and C₂. These chucks are mounted in ball bearings B₁, B₂, B₃, and B₄. From B₂ and B₃ weights W are hung which set up a constant and determinable bending moment over the central part of the length of the specimen S. The chucks and specimen are rotated by means of the drive pulley P. A revolution counter R gives the number of cycles of stress applied. When the specimen breaks, the bearings B₂, B₃, and the attached weights fall and, in falling, open a switch, thereby stopping the motor which drives the machine.

Figure 4 shows in detail the construction of the chucks and of the special jaws for one of the turbine blades. Figure 5 is from a photograph of a pair of chucks and jaws. The jaws were made of bronze, and the edges where they bore on the specimen were slightly rounded, so that cutting action was reduced to a minimum. For each lot of blading tested a special pair of jaws was necessary. The jaw for each lot of blading was so shaped that the longitudinal axis through the center of gravity of the cross-section of the specimen was nominally coincident with the axis of rotation of the chucks.

Even with jaws carefully designed for blading the slight irregularities in blading as furnished make perfect centering impossible, and some eccentricity of motion of specimen, with consequent inertia forces, results. Moreover, since the specimens of blading are
FIG. 3. TESTING MACHINE FOR FATIGUE TESTS
stiffer in flexure when the tips are in a vertical plane than when they are in a horizontal plane there would be some vibration even for a perfectly centered specimen. For imperfect centering the vibration is approximately a harmonic motion with one cycle of vibration per revolution; for varying stiffness of specimen the vibration is approximately a harmonic motion with two cycles of vibration per revolution. These two cases will be called one-cycle and two-cycle vibration, respectively. The resultant vibration of the specimen is a combination of the one-cycle and the two-cycle vibration, the relative effect of the two types of vibration varying with the loading.

A rough estimate of the maximum inertia forces set up may be made as follows: By taking a specimen which seemed to show as much vibration as did any observed, there was obtained a record
with deflection as ordinates and time as abscissas, using a lever deflectometer recording deflection on a piece of smoked glass given uniform motion. During the test the machine was run at normal speed (130 r. p. m.). A record of vibration of one center bearing of the machine was made, and also a record of vibration of the weights suspended, by a coil spring, from the bearings (see Fig. 3a). The record was magnified by projection with a stereopticon, and the projected image was traced on a sheet of cardboard. Figure 6 shows the record of the vibration of the bearing, which had an amplitude of 0.09 inch. The record of vibration of the weights was similar in form and had one-third the amplitude of that of the bearing (that is, 0.03 inch). In Fig. 6 there have been drawn curves for one-cycle and two-cycle harmonic motion, using the same scale as that used for the curve of actual vibration, and with each harmonic having the same amplitude* as that of the actual observed vibration. The period of the two-cycle harmonic is half that and the period of the one-cycle harmonic is the same as that of the period of a complete cycle for the observed vibration. It is a simple matter to compute the inertia forces set up if the vibration followed either the two-cycle or the one-cycle harmonic. Now the acceleration for any point on a time-space curve is approximately proportional to the reciprocal of the radius of curvature of the space-time curve at that point. By direct measurement the minimum radius for the curve of actual vibration was found at $R_s$, Fig. 6, and the min-

*The fixing of the amplitude and the periods of the imaginary harmonic vibrations at the values given is a matter of convenience. Any amplitude and any period could have been used so long as the scale of Fig. 6 was not changed.
THE FATIGUE STRENGTH OF STEAM TURBINE BLADE SHAPES

Fig. 6. Vibration Diagram for Central Bearings of Testing Machine.
<table>
<thead>
<tr>
<th>Metal</th>
<th>Blading</th>
<th>Copper</th>
<th>Nickel</th>
<th>Iron</th>
<th>Manganese</th>
<th>Carbon</th>
<th>Silicon</th>
<th>Chromium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cupro-Nickel</td>
<td>¾ inch</td>
<td>79.29</td>
<td>20.24</td>
<td>0.36</td>
<td>0.20</td>
<td>....</td>
<td>....</td>
<td>....</td>
</tr>
<tr>
<td></td>
<td>1 inch</td>
<td>79.82</td>
<td>19.64</td>
<td>0.27</td>
<td>0.22</td>
<td>....</td>
<td>....</td>
<td>....</td>
</tr>
<tr>
<td>Monel</td>
<td>¾ inch</td>
<td>27.18</td>
<td>68.32</td>
<td>2.20</td>
<td>1.93</td>
<td>0.15</td>
<td>....</td>
<td>....</td>
</tr>
<tr>
<td></td>
<td>1 inch</td>
<td>26.78</td>
<td>68.77</td>
<td>2.28</td>
<td>1.77</td>
<td>0.15</td>
<td>....</td>
<td>....</td>
</tr>
<tr>
<td>Cyclops</td>
<td>¾ inch</td>
<td>....</td>
<td>18.97</td>
<td>diff.</td>
<td>0.63</td>
<td>0.46</td>
<td>1.14</td>
<td>7.67</td>
</tr>
<tr>
<td></td>
<td>1 inch</td>
<td>....</td>
<td>19.25</td>
<td>diff.</td>
<td>0.67</td>
<td>0.49</td>
<td>0.86</td>
<td>7.36</td>
</tr>
</tbody>
</table>

**TABLE 1**

**CHEMICAL ANALYSES OF TURBINE BLADE MATERIALS**

Analyses made in the laboratories of the Allis-Chalmers Manufacturing Company.
imimum radius for the two-cycle harmonic vibration curve was found at \( r \). By direct measurement \( r/R = 0.69/0.46 = 1.5 \). The maximum inertia force in the actual vibration may then be estimated as 1.5 times the maximum inertia force for two-cycle harmonic vibration with the same amplitude.

Let \( W_1 \) denote the weight of the hangers and bearings and \( F_1 \) the inertia force due to their vibration; let \( W_2 \) denote the weight suspended from the coil spring (see Fig. 3a) and \( F_2 \) the corresponding inertia force; let \( W \) and \( F \) represent the total weight on the specimen and the inertia force respectively. Let \( 2r_1 \) denote the amplitude of vibration of the bearing and \( 2r_2 \) that of the weights, \( \omega \) the angular velocity in radians per second, and \( g \) the acceleration due to gravitation, (32.2 feet per second per second).

Then for two-cycle harmonic vibration (light line diagram in Fig. 6)

\[
r_1 = \frac{0.09}{2 \times 12} = 0.00375 \text{ ft.}
\]

\[
\omega (2 \text{ cycles per rev.}) = 2\left(\frac{130 \times 2\pi}{60}\right) = 27.2 \text{ radians per sec.}
\]

\[
r_2 = \frac{0.03}{2 \times 12} = 0.00125 \text{ ft.}
\]

\[
W_1 = 0.40W \\
W_2 = 0.60W
\]

\[
F_1 = \frac{W_1}{g} r_1 \omega^2 = W_1 \left(\frac{0.00375 \times 27.2^2}{32.2}\right) = 0.086 \ W_1
\]

\[
= 0.086 \times 0.4 \ W = 0.0344 \ W
\]

\[
F_2 = \frac{W_2}{g} r_2 \omega^2 = W_2 \left(\frac{0.00125 \times 27.2^2}{32.2}\right) = 0.028 \ W_2
\]

\[
= 0.028 \times 0.6 \ W = 0.0168 \ W
\]

\[
F = F_1 + F_2 = 0.0512 \ W
\]

However, the maximum inertia force for the actual vibration as given by the heavy-line diagram in Fig. 6 is 1.5 times the maximum inertia force for two-cycle harmonic vibration, or \( 1.5 \times 0.0512 \ W = 0.077 \ W \). That is, the inertia forces due to vibration may have increased the effect of the weights by as much as 7.7 per cent at certain portions of the revolution. Just how the maximum
unit stress, and how much the range of stress during a revolution were affected during each test could have been determined only by a complex analysis of the vibration in each test. However, it seems reasonably certain that any error in estimated endurance limit, due to vibration of specimen, is not greater than 7.7 per cent.

5. Determination of Unit Stress in Specimens.—The common formula for flexural stress is:

\[ S = \frac{M}{I/c} \]

in which

- \( M \) denotes the bending moment, in inch-pounds
- \( I \) denotes the moment of inertia of the cross-section about the neutral axis, in inches
- \( c \) denotes the distance from the neutral axis to the most distant fiber of the cross-section, in inches
- \( S \) denotes the maximum unit stress set up, in lb. per sq. in.

The neutral axis passes through the center of gravity of the cross-section, and is assumed to be perpendicular to the plane of the bending moment. This assumption, and consequently the common flexure formula, holds good only when the bending moment is applied in the direction of a "principal axis" of the cross-section. A principal axis for an area is one for which the moment of inertia is either a maximum or a minimum. The most common example of a principal axis is an axis of symmetry or an axis perpendicular to an axis of symmetry. Most cases of flexure of common structural parts involve bending in a direction perpendicular to or coincident with an axis of symmetry of cross-section (rectangular, circular, I-shaped, T-shaped, and channel sections, but not angle bars nor Z-bars).

As shown in Fig. 2 the cross-sections of the blading tested had no axes of symmetry, and, moreover, the specimens are subjected to bending in all directions perpendicular to the longitudinal axis of the specimen. Hence the common flexure formula cannot be used to determine the maximum unit stress during a cycle of stress. Recourse is had to the general theory of flexure for bars of uniform cross-section, developed by Mohr and by Land* by the aid of which it is

possible to determine a value which can be used in place of \( I/c \) in the common flexure formula. In this bulletin this value is called the \textit{generalized section modulus}, and is denoted by the letter \( Z \).

The detailed derivation of a value of \( Z \) for one of the specimens tested, a summary of values for all sections tested, and a brief mathematical discussion of the general theory of flexure of a straight bar of uniform cross-section, with special application to the flexure of turbine blades, is given in the appendix to this bulletin.

6. Test Data.—The fatigue tests of the specimens of turbine blading were carried out in a manner similar to that used for ordinary fatigue tests.* A specimen was given a load which, it was estimated, would cause fracture after a few thousands of cycles of stress, and this test was carried to destruction. Then a specimen of the same lot was tested with a smaller load, with a longer “run” before fracture. This program was carried out until the load was so reduced that a specimen ran out to 5 000 000 or more cycles of stress without fracture.

The results of this series of tests were plotted on logarithmic coordinate paper, using values of maximum unit stress \( S \) as ordinates and values of the number of cycles for fracture \( N \) as abscissas. Such a diagram is called an \( S-N \) diagram, and the endurance limit for the lot of specimens tested is taken as the unit stress corresponding to the horizontal line which the \( S-N \) diagram approaches as \( N \) increases. The location of this horizontal line for these tests of turbine blading was a matter of estimation rather than of precise determination, as may be seen from an inspection of the \( S-N \) diagrams, which are shown in Figs. 7-9 inclusive. Table 2 gives the test data of the fatigue tests.

III. Results and Conclusions

7. Results and Conclusions.—During the progress of any test occasional examinations were made for the appearance of fatigue cracks. In many cases such cracks were detected before they had spread to complete failure. In all cases observed the cracks started at one of the tips of the cross-section, usually at the thicker tip. This is in accord with the location of the maximum unit stress under flexure. In most cases the fracture occurred about \( \frac{1}{4} \) inch along the specimen from one of the jaws, but in a few cases it occurred

near the center of the length of the specimen. This would seem to indicate that there was some, but not very much, localization of stress at the edges of the jaws.

The ideal way to compare the fatigue strength of a machine part with the fatigue strength of the material in that part is to compare an endurance limit determined by tests of specimens of the part itself with the endurance limit determined by tests of carefully shaped test specimens cut from specimens of the machine part. In the case of the turbine blading this was not feasible on account of the small size of the specimens which could be cut from the blading, and especially in view of the desirability of getting specimens of material from the thin tips of the blading where fatigue failure

Fig. 7. S-N Diagrams for 3/4-inch and 1-inch Cupro-Nickel Blading
started. However, an estimate of the fatigue strength of the material was made, based on the relation, noted by many experimenters, that the fatigue strength is, in general, proportional to hardness as determined by a ball-impression test.* The turbine blading specimens were too thin to use Brinell tests, and accordingly Rockwell "B" hardness tests† were made, especially on portions of the blading near the tips. Using the relation determined by Petrenko‡ the Rockwell hardness numbers were translated into equivalent Brinell numbers. From the test results in the files of the Fatigue of Metals Investigation at the University of Illinois the ratio of endurance limit to Brinell number was found to be about 250 for cyclops

†In the Rockwell "B" test the impression is made by a steel ball 1/16 inch in diameter, pressed against the specimen with a force of 100 kilograms.
### Table 2
**TEST DATA FOR FATIGUE TESTS OF TURBINE BLADING**

**Constants of Testing Machines:**
- **Machine No. 1,** $S = \frac{2W}{Z_{\text{min}}}$
  - $S$ denotes the maximum unit stress set up in the specimen during one revolution, lb. per sq. in.
- **Machine No. 2,** $S = 1.71 \frac{W}{Z_{\text{min}}}$
  - $W$ denotes the total weight hung on the specimen, pounds
- **Machine No. 3,** $S = 1.75 \frac{W}{Z_{\text{min}}}$
  - $Z_{\text{min}}$ denotes the minimum value of the generalized section modulus of the blading, in$^3$

<table>
<thead>
<tr>
<th>Testing Load Unit Stress</th>
<th>Cycles for Fracture</th>
<th>Fracture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine No.</td>
<td>lb.</td>
<td>lb. per sq. in.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$S$</td>
</tr>
<tr>
<td>1</td>
<td>39.0</td>
<td>50 700</td>
</tr>
<tr>
<td>2</td>
<td>31.6</td>
<td>46 000</td>
</tr>
<tr>
<td>3</td>
<td>25.6</td>
<td>54 400</td>
</tr>
<tr>
<td>1</td>
<td>19.2</td>
<td>27 900</td>
</tr>
<tr>
<td>2</td>
<td>16.0</td>
<td>23 200</td>
</tr>
<tr>
<td>3</td>
<td>14.6</td>
<td>21 200</td>
</tr>
<tr>
<td>1</td>
<td>13.0</td>
<td>18 900</td>
</tr>
<tr>
<td>2</td>
<td>12.2</td>
<td>17 700</td>
</tr>
<tr>
<td>3</td>
<td>26.2</td>
<td>33 400</td>
</tr>
<tr>
<td>1</td>
<td>23.6</td>
<td>29 900</td>
</tr>
<tr>
<td>2</td>
<td>22.7</td>
<td>25 900</td>
</tr>
<tr>
<td>3</td>
<td>22.7</td>
<td>25 900</td>
</tr>
<tr>
<td>1</td>
<td>21.8</td>
<td>27 800</td>
</tr>
<tr>
<td>2</td>
<td>21.0</td>
<td>26 600</td>
</tr>
<tr>
<td>3</td>
<td>19.2</td>
<td>24 400</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>35.0</td>
<td>35 600</td>
</tr>
<tr>
<td>2</td>
<td>32.7</td>
<td>30 400</td>
</tr>
<tr>
<td>3</td>
<td>34.4</td>
<td>28 400</td>
</tr>
<tr>
<td>1</td>
<td>54.4</td>
<td>26 400</td>
</tr>
<tr>
<td>2</td>
<td>48.2</td>
<td>23 400</td>
</tr>
<tr>
<td>3</td>
<td>70.3</td>
<td>40 000</td>
</tr>
<tr>
<td>1</td>
<td>43.0</td>
<td>24 400</td>
</tr>
<tr>
<td>2</td>
<td>40.4</td>
<td>23 000</td>
</tr>
<tr>
<td>3</td>
<td>36.8</td>
<td>20 900</td>
</tr>
<tr>
<td>1</td>
<td>94.0</td>
<td>46 800</td>
</tr>
<tr>
<td>2</td>
<td>62.8</td>
<td>31 200</td>
</tr>
<tr>
<td>3</td>
<td>56.3</td>
<td>28 100</td>
</tr>
<tr>
<td>1</td>
<td>54.3</td>
<td>27 000</td>
</tr>
<tr>
<td>2</td>
<td>52.2</td>
<td>26 000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>61.4</td>
<td>130 000</td>
</tr>
<tr>
<td>2</td>
<td>50.6</td>
<td>108 000</td>
</tr>
<tr>
<td>3</td>
<td>41.4</td>
<td>87 700</td>
</tr>
<tr>
<td>1</td>
<td>31.4</td>
<td>66 600</td>
</tr>
<tr>
<td>2</td>
<td>27.4</td>
<td>58 200</td>
</tr>
<tr>
<td>3</td>
<td>26.7</td>
<td>56 800</td>
</tr>
<tr>
<td>1</td>
<td>25.5</td>
<td>54 300</td>
</tr>
<tr>
<td>2</td>
<td>24.4</td>
<td>51 900</td>
</tr>
<tr>
<td>3</td>
<td>23.6</td>
<td>50 100</td>
</tr>
</tbody>
</table>

**Cupro-nickel, 3/4-inch Blading;** $Z_{\text{min}} = 0.001375$ in$^3$

**Cupro-nickel, 1-inch Blading;** $Z_{\text{min}} = 0.00352$ in$^3$

**Monel Metal 3/4-inch Blading;** $Z_{\text{min}} = 0.000943$ in$^3$
### Table 2 (Concluded)

<table>
<thead>
<tr>
<th>Testing Machine No.</th>
<th>Load W</th>
<th>Unit Stress S per sq. in.</th>
<th>Cycles for Fracture N</th>
<th>Fracture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23.2</td>
<td>49 400</td>
<td>338 700</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>22.1</td>
<td>47 000</td>
<td>376 500</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>20.5</td>
<td>45 600</td>
<td>85 800</td>
<td>Central, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>19.1</td>
<td>40 500</td>
<td>501 500</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>18.2</td>
<td>38 800</td>
<td>3 784 700</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>1</td>
<td>17.5</td>
<td>37 200</td>
<td>4 008 900</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>3</td>
<td>22.7</td>
<td>42 200</td>
<td>6 653 300</td>
<td>Specimen did not fail</td>
</tr>
</tbody>
</table>

Monel Metal, 1-inch Blading; $Z_{min} = 0.00279$ in$^3$

<table>
<thead>
<tr>
<th>Testing Machine No.</th>
<th>Load W</th>
<th>Unit Stress S per sq. in.</th>
<th>Cycles for Fracture N</th>
<th>Fracture</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>115.0</td>
<td>70 300</td>
<td>45 000</td>
<td>About ¼-in. from jaw</td>
</tr>
<tr>
<td>2</td>
<td>104.5</td>
<td>64 400</td>
<td>215 600</td>
<td>Central, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>98.2</td>
<td>60 200</td>
<td>215 600</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>88.0</td>
<td>55 900</td>
<td>116 900</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>88.0</td>
<td>55 900</td>
<td>116 900</td>
<td>Near jaws and also central, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>83.7</td>
<td>51 300</td>
<td>215 500</td>
<td>Central, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>71.5</td>
<td>43 800</td>
<td>604 000</td>
<td>Central, thin tip</td>
</tr>
<tr>
<td>2</td>
<td>62.7</td>
<td>38 400</td>
<td>1 002 000</td>
<td>About third point of span, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>60.6</td>
<td>37 200</td>
<td>525 000</td>
<td>About ¼-in. from jaw, complete fracture</td>
</tr>
<tr>
<td>2</td>
<td>60.6</td>
<td>37 200</td>
<td>6 744 300</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>2</td>
<td>58.5</td>
<td>35 800</td>
<td>6 090 300</td>
<td>Specimen did not fail</td>
</tr>
</tbody>
</table>

Cyclops Metal, $\frac{3}{4}$-inch Blading; $Z_{min} = 0.00118$ in$^3$

<table>
<thead>
<tr>
<th>Testing Machine No.</th>
<th>Load W</th>
<th>Unit Stress S per sq. in.</th>
<th>Cycles for Fracture N</th>
<th>Fracture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>54.4</td>
<td>92 300</td>
<td>9 600</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>39.0</td>
<td>68 400</td>
<td>37 200</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>36.7</td>
<td>62 400</td>
<td>99 400</td>
<td>Central, complete fracture</td>
</tr>
<tr>
<td>1</td>
<td>31.4</td>
<td>53 300</td>
<td>120 800</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>27.4</td>
<td>46 700</td>
<td>312 700</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>27.0</td>
<td>46 000</td>
<td>795 600</td>
<td>About ¼-in. from jaw, complete fracture</td>
</tr>
<tr>
<td>1</td>
<td>26.2</td>
<td>44 600</td>
<td>1 477 800</td>
<td>About ¼-in. from jaw, complete fracture</td>
</tr>
<tr>
<td>1</td>
<td>25.2</td>
<td>42 600</td>
<td>2 260 000</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>3</td>
<td>28.8</td>
<td>43 000</td>
<td>419 100</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>3</td>
<td>27.1</td>
<td>40 300</td>
<td>1 012 300</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>3</td>
<td>27.1</td>
<td>40 300</td>
<td>6 546 900</td>
<td>Specimen did not fail</td>
</tr>
</tbody>
</table>

Cyclops Metal, 1-inch Blading; $Z_{min} = 0.00350$ in$^3$

<table>
<thead>
<tr>
<th>Testing Machine No.</th>
<th>Load W</th>
<th>Unit Stress S per sq. in.</th>
<th>Cycles for Fracture N</th>
<th>Fracture</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>134.9</td>
<td>65 800</td>
<td>50 000</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>118.6</td>
<td>57 900</td>
<td>58 000</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>108.5</td>
<td>53 000</td>
<td>78 900</td>
<td>About ¼-in. from jaw, both tips</td>
</tr>
<tr>
<td>2</td>
<td>104.5</td>
<td>51 000</td>
<td>596 500</td>
<td>Central, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>101.0</td>
<td>49 300</td>
<td>113 000</td>
<td>Central, thin tip</td>
</tr>
<tr>
<td>2</td>
<td>99.5</td>
<td>47 300</td>
<td>347 200</td>
<td>Central, thick tip</td>
</tr>
<tr>
<td>2</td>
<td>96.2</td>
<td>47 000</td>
<td>104 700</td>
<td>Central, thin tip</td>
</tr>
<tr>
<td>2</td>
<td>92.0</td>
<td>45 000</td>
<td>6 388 400</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>2</td>
<td>83.7</td>
<td>40 900</td>
<td>9 975 700</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>1</td>
<td>105.0</td>
<td>60 000</td>
<td>35 100</td>
<td>Retest of unbroken specimen</td>
</tr>
<tr>
<td>1</td>
<td>96.5</td>
<td>55 200</td>
<td>31 700</td>
<td>About ¼-in. from jaw, thick tip</td>
</tr>
<tr>
<td>1</td>
<td>73.8</td>
<td>42 200</td>
<td>6 900 600</td>
<td>Specimen did not fail</td>
</tr>
<tr>
<td>1</td>
<td>60.2</td>
<td>34 400</td>
<td>7 207 000</td>
<td>Specimen did not fail</td>
</tr>
</tbody>
</table>
metal, 190 for monel metal, and 166 for a copper-nickel alloy not dissimilar to that used in the cupro-nickel blading specimens.

Table 3 gives the values of endurance limit determined by fatigue tests of the specimens of blading, and also the estimated endurance limit as determined from the Rockwell “B” hardness tests.

In drawing conclusions from these tests it should be clearly recognized that three factors make such conclusions less quantitatively precise than conclusions drawn from carefully made fatigue tests of specially shaped specimens.

(1) Owing to the necessity of slow speed operation of the testing machines the $S-N$ diagrams are not so satisfactory as they would have been if it had been possible to apply more cycles of stress in the time available.

(2) The graphical methods used in determining maximum unit stress are less precise than the computation methods used for computing unit stress in rotating-beam specimens of circular cross-section.

(3) The method of estimating the endurance limit of the material is a rather roundabout one. It seemed to be the only
method available owing to the thinness of the blading material. However, it is to be noted that in every case more than 90 per cent of the estimated endurance limit of the material was developed in the blading specimens tested as fabricated, and the general conclusion seems justified that steam turbine blades carefully socketed may be expected to develop nearly the full fatigue strength of the material from which they are made. This conclusion is in harmony with general experience with blades in service, as reported by the Allis-Chalmers engineers.

APPENDIX A

A BRIEF DISCUSSION OF THE GENERAL THEORY OF THE FLEXURE OF A STRAIGHT BAR OF UNIFORM CROSS-SECTION, WITH SPECIAL APPLICATION TO THE FLEXURE OF STEAM TURBINE BLADES

1. Introductory.—The common flexure formula is

\[ S = \frac{M}{I/c} \]

in which

- \( S \) denotes the maximum unit stress, in lb. per sq. in.
- \( M \) denotes the bending moment, in inch-pounds
- \( I \) denotes the moment of inertia of the cross-section of the piece about a neutral axis passing through the center of gravity of the cross-section perpendicular to the plane of bending, in inches
- \( c \) denotes the distance from the neutral axis to the extreme point on the outline of the cross-section, in inches
### Table 4

**Comparison of Estimated Endurance Limit of Material with Endurance Limit Developed in Fatigue Tests of Drawn Blading**

<table>
<thead>
<tr>
<th>Material</th>
<th>Average Equivalent Brinell Number</th>
<th>Estimated Endurance Limit for Material (A)</th>
<th>Average Endurance Limit for Tests of Blading (B)</th>
<th>Ratio (B : (A))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monel Metal</td>
<td>212</td>
<td>40 200*</td>
<td>37 000</td>
<td>0.92</td>
</tr>
<tr>
<td>Cupro-nickel Alloy</td>
<td>117</td>
<td>40 400†</td>
<td>20 000</td>
<td>1.03</td>
</tr>
<tr>
<td>Cyclops Metal</td>
<td>171</td>
<td>42 800‡</td>
<td>42 500</td>
<td>0.99</td>
</tr>
</tbody>
</table>

*190. (Brinell Hardness Number)
†166. (Brinell Hardness Number)
‡250. (Brinell Hardness Number)

This common formula is true only for beams in which the direction of the moment (strictly speaking, the direction of the trace of the plane of bending) is either along a principal axis of the cross-section or perpendicular thereto. In general, the area of any cross-section has two principal axes at right angles to each other, about one of which the moment of inertia is a maximum and about the other of which it is a minimum. If the cross-section has an axis of symmetry, this axis is one of the principal axes. The usefulness of the common flexure formula depends on the fact that for most beams in engineering practice the direction of the bending moment is either along an axis of symmetry or is perpendicular to such an axis. This is true for most I-beams, T-beams, channels, rectangular beams, and circular beams; it is *not* true for angles loaded parallel to one leg, for Z-bars, nor for any shape with an oblique bending moment (note: a circular beam cannot be given an oblique bending moment) and for many cases of oblique bending moment the maximum unit stress is higher than the value given by the common flexure formula.

For symmetrical sections loaded in a plane of symmetry, or perpendicular thereto the neutral axis passes through the center of gravity of the cross-section perpendicular to the plane of bending. For the general case of flexure, the neutral axis passes through the center of gravity of the cross-section, but in general, is *not* at right angles to the plane of the loads.

2. **Product of Inertia.**—To develop the general theory of flexure it is necessary to consider one new property of the area of the cross-section of the flexural member—the *product of inertia*. The
moment of inertia of an area about an axis is the summation of the products of each elementary area and the square of its distance from the given axis, or, expressed in symbols,

\[ I = \Sigma y^2 \Delta A \]

For any area, \( I \) is never zero and is always plus.

The product of inertia of an area about any pair of axes at right angles to each other is equal to the summation of the products of each elementary area and the value of its coördinates with respect to those axes, or expressed in symbols:

\[ J = \Sigma x y \Delta A \]

in which \( J \) denotes the product of inertia with respect to the \( X \) and \( Y \) axes.

Since either \( x \) or \( y \) may be either plus or minus it is evident that for certain axes the product of inertia may be negative, and for one pair, zero. If one of the axes of reference is an axis of symmetry of the cross-section, \( J \) equals zero; for every elementary area on the positive side of an axis of symmetry there is a companion elementary area on the negative side. The product of inertia of an area is measured in the same units as is the moment of inertia—inches^4.

3. Determination of Product of Inertia for any Area.—The value of \( J \) for any area about any given \( X \) and \( Y \) axes may be determined by finding the values of \( I \) (\( I_x \) and \( I_y \)) about the given axes and about a third axis, all axes passing through the center of gravity of the area. For irregular areas, such as turbine blade cross-sections, these three values of \( I \) may be found by graphical means, which will be described in a later paragraph.

In the irregular area shown in Fig. 10 let \( I_x \) denote the moment of inertia about the \( X \) axis, \( I_y \) the moment of inertia about the \( Y \) axis, and \( I_u \) the moment of inertia about any third axis, called the \( U \) axis in this case^*; \( J_{xy} \) denotes the product of inertia about the \( X \) and \( Y \) axes, and \( \theta \) denotes the angle between the \( X \) axis and the \( U \) axis.

Then

\[
I_u = \Sigma v^2 \Delta A = \Sigma (PB - AB)^2 \Delta A \\
= \Sigma (PB - OD \sin \theta)^2 \Delta A = \Sigma (y \cos \theta - x \sin \theta)^2 \Delta A \\
= \Sigma (y^2 \cos^2 \theta + x^2 \sin^2 \theta - 2xy \sin \theta \cos \theta) \Delta A \\
= \Sigma y^2 \cos^2 \theta \Delta A + x^2 \sin^2 \theta \Delta A - 2xy \sin \theta \cos \theta \Delta A \\
= I_x \cos^2 \theta + I_y \sin^2 \theta - 2J_{xy} \sin \theta \cos \theta
\]

\(^*\)This method of determining \( J_{xy} \) is based on the treatment given on p. 198 of Seely and Ensign, "Analytical Mechanics for Engineers."
Whence \( 2J_{xy} \sin \theta \cos \theta = I_x \cos^2 \theta + I_y \sin^2 \theta - I_u \)

\[
J_{xy} = \frac{I_x \cos^2 \theta + I_y \sin^2 \theta - I_u}{2 \sin \theta \cos \theta}
\]

If \( \theta = 45^\circ \), \( \sin \theta = \cos \theta = \frac{1}{\sqrt{2}} \), and for this particular case, denoting \( I_u \) by \( I_{45} \)

\[
J_{xy} = \frac{I_x}{2} + \frac{I_y}{2} - I_{45}
\]

(1)

4. General Formula for Flexure.*—Let Fig. 11 represent the cross-section of a straight bar subjected to a bending moment \( M \) acting in the direction shown (\( MO \) is the trace of the plane of the bending moment on the plane of the cross-section). \( NN \) is the neutral axis passing through the center of gravity of the cross-section.

By Hooke’s law and the law of conservation of plane sections the unit stress along any line \( QP \), perpendicular to \( NN \), varies so that

\[
\frac{S_{\text{max}}}{S} = \frac{\nu'}{\nu}
\]

*The development here presented follows the general lines of that given by L. J. Johnson in Trans. A. S. C. E. Vol. LVI, p. 16 (1906) which, in turn, is based on the work of Mohr and of Land.
in which \( S_{\text{max}} \) is the unit stress at the extreme point \( P' \) of the line, 
\( S \) denotes the unit stress at any point \( P \) in the line, and \( v \) and \( v' \) are 
the respective distances from \( NN \) of the points \( P \) and \( P' \). If \( \alpha \) is the 
angle between \( NN \) and \( XX \), then

\[
S = S_{\text{max}} \frac{v}{v'} = \frac{S_{\text{max}}}{v'} (y \cos \alpha - x \sin \alpha)
\]  
(2)

(Note: In Fig. 11 \( x \) is negative)

But equilibrium of moments is established if

\[
\Sigma x \ S \ \Delta A = M \cos \lambda
\]

and

\[
\Sigma y \ S \ \Delta A = M \sin \lambda
\]

in which \( x \) and \( y \) are the coordinates of any point \( P \), \( S \) is the unit 
stress at that point, \( M \) is the bending moment at the given cross-
section, and \( \lambda \) is the angle between the direction of the bending 
moment and the \( X \) axis.
Substituting the value of $S$ from (2)

$$\frac{S_{\text{max}}}{\nu'} \left( \Sigma x y \cos a \Delta A - \Sigma x^2 \sin a \Delta A \right) = M \cos \lambda$$

$$\frac{S_{\text{max}}}{\nu'} \left( J_{xy} \cos a - I_y \sin a \right) = M \cos \lambda$$

$$\frac{S_{\text{max}}}{\nu'} \left( \Sigma y^2 \cos a \Delta A - \Sigma x y \sin a \Delta A \right) = M \sin \lambda$$

$$\frac{S_{\text{max}}}{\nu'} \left( I_x \cos a - J_{xy} \sin a \right) = M \sin \lambda$$

(3)

(4)

Dividing (4) by (3)

$$\frac{I_x \cos a - J_{xy} \sin a}{J_{xy} \cos a - I_y \sin a} = \tan \lambda = \frac{I_x - J_{xy} \tan \alpha}{J_{xy} - I_y \tan \alpha}$$

whence

$$\tan \alpha = \frac{I_x - J_{xy} \tan \lambda}{J_{xy} - I_y \tan \lambda}$$

(5)

From (4)

$$\frac{S_{\text{max}}}{\nu'} = \frac{M \sin \lambda}{I_x \cos a - J_{xy} \sin a}$$

(6)

From (2) and (6)

$$S = \frac{M \sin \lambda (y \cos a - x \sin a)}{I_x \cos a - J_{xy} \sin a} = \frac{M (y - x \tan \alpha) \sin \lambda}{I_x - J_{xy} \tan \alpha}$$

Substituting the value of $\tan \alpha$ from (5)

$$S = \frac{M \left( y - x \frac{I_x - x J_{xy} \tan \lambda}{J_{xy} - I_y \tan \lambda} \right) \sin \lambda}{I_x - \frac{I_x J_{xy} - J_{xy}^2 \tan \lambda}{J_{xy} - I_y \tan \lambda}}$$

$$= \frac{M (y J_{xy} - y I_y \tan \lambda - x I_x + x J_{xy} \tan \lambda) \sin \lambda}{I_x J_{xy} - I_x I_y \tan \lambda - I_x J_{xy} + J_{xy}^2 \tan \lambda}$$

Dividing both numerator and denominator by $\tan \lambda$
THE FATIGUE STRENGTH OF STEAM TURBINE BLADE SHAPES

\[ S = \frac{M (y J_{xy} - y I_y \tan \lambda - x I_x + x J_{xy} \tan \lambda) \cos \lambda}{J_{xy}^2 - I_x I_y} \]

\[ = \frac{M [y (J_{xy} \cos \lambda - I_y \sin \lambda) + x (J_{xy} \sin \lambda - I_x \cos \lambda)]}{J_{xy}^2 - I_x I_y} \]

From this equation may be obtained a value for \( Z \), the generalized section modulus of the cross-section—a quantity such that if \( M \) be divided by \( Z \) there results the unit stress at the point whose coordinates are \( x \) and \( y \).

\[ Z = \frac{I_x I_y - J_{xy}^2}{(I_y \sin \lambda - J_{xy} \cos \lambda) y + (I_x \cos \lambda - J_{xy} \sin \lambda) x} \quad (7) \]

Usually the point of extreme fiber stress, that is the point on the outline of a cross-section for which the distance from the neutral axis is a maximum, can be determined by inspection. However, in a turbine blade the actual point of extreme stress would change slightly, as \( \lambda \) changes, and it cannot be told at once which tip will receive the most severe stress as the specimen blade revolves in the testing machine (see Fig. 3); hence it is necessary to determine minimum \( Z \) for both tips. However, the point of extreme fiber stress will vary but little in location on the blade, and without serious error it may be assumed that as \( \lambda \) changes the point on either tip most distant from the neutral axis remains at the point located by the coordinates \( x \) and \( y \).

In determining stress values for the blades tested the \( X \) axis was located as a line through the center of gravity of the cross-section parallel to a line tangent to the tips of the blade. This is shown in Fig. 12, which also shows the location of the \( x \) and \( y \) coordinates for extreme points.

As \( \lambda \) varies the maximum unit stress will become a maximum when \( Z \) is a minimum, or when \( 1/Z \) is a maximum; that is when

\[ \frac{d}{d \lambda} \left( \frac{1}{Z} \right) = 0 \]

\[ \frac{d}{d \lambda} \left[ \frac{(I_y \sin \lambda - J_{xy} \cos \lambda) y + (I_x \cos \lambda - J_{xy} \sin \lambda) x}{I_x I_y - J_{xy}^2} \right] = 0 \]

that is when

\[ y I_y \cos \lambda + y J_{xy} \sin \lambda - x I_x \sin \lambda - x J_{xy} \cos \lambda = 0 \]
that is when
\[(y I_y - x J_{xy}) \cos \lambda = (x I_x - y J_{xy}) \sin \lambda\]
\[\tan \lambda = \frac{y I_y - x J_{xy}}{x I_x - y J_{xy}}\]  
(8)
for a maximum unit stress.

5. Application of Theory to Turbine Blades.—In determining for any given turbine blade the maximum unit stress set up by a given bending moment as the direction of the moment changes it is necessary first of all to determine \(I_x, I_y,\) and \(J_{xy}\) for the \(X\) and \(Y\) axes for the cross-section of the turbine blade. The \(X\) axis is arbitrarily selected as indicated in the foregoing paragraph (and also in Fig. 12) and the \(Y\) axis is taken perpendicular to the \(X\) axis through the center of gravity of the cross-section. The shape of the turbine blade does not lend itself to mathematical analysis and graphical methods must be used for determining the moments of inertia.

The size of cross-section of the turbine blades tested was too small for accurate graphical calculation by direct reproduction of the outline of the cross-section, so the following method was used. A piece of turbine blading was cut square across, and from the edges of the piece all burrs were removed. The piece of blading being used as a punch, a hole the shape of the cross-section was cut out of a piece of black paper, a sheet of lead being placed under the paper. Then this hole was photographed, using a magnification of approximately 10 times. Measurements of the width and depth of the photograph were checked against micrometer measurements of the blading to make sure that no appreciable distortion had occurred due to optical error in the camera lens, or to unequal shrinkage of the photographic film. Figure 13 is from the enlarged photograph of the cross-section of the \(3/4\)-inch cupro-nickel blading.
The center of gravity of the cross-section was determined by tracing the outline of the enlarged photograph on cardboard, cutting out the shape, and then suspending from threads hung successively at different points, the center of gravity being located by the intersection of the lines of the suspending threads.

The following method* was used to determine $I_x$, $I_y$, and $I_{45}$ ($I$ about an axis at 45 degrees with the $X$ axis). In Fig. 14 the area whose moment of inertia is to be determined is the outer figure $A$, and the axis of reference is $XX$. Draw any line, such as $X'X'$, parallel to and at the distance $h$ from $XX$. Draw any chord, such as $BB$, parallel to $XX$, and project the points $B,B$ on $X'X'$ thereby locating the points $C,C$. Connect the points $C,C$ to any point $P$ on $XX$ by lines intersecting $BB$ at points $D,D$. Project the points on $X'X'$ thereby locating the points $E,E$. Connect the points $E,E$ to $P$ by lines cutting the line $BB$ in points $F,F$. This construction may be repeated for a number of chords parallel to $BB$, thus obtaining a number of pairs of points corresponding to $F,F$. Connect all such points by a smooth curve outlining the shaded area $A'$. The moment of inertia of area $A$ about $XX$ is given by the product of the shaded area $A'$ and $h^2$.

$$ I_x = A'h^2 $$

$A'$ may be measured by the use of a planimeter, or by other methods such as Simpson's rule or Durand's rule.

The proof of this method is as follows:

Denoting $BB$ by $x$, $DD$ by $x'$, and $FF$ by $x''$ we have from the similar triangles $PFF$ and $PEE$,

$$ x''/y = x'/h $$

and from similar triangles $PDD$ and $PCC$,

$$ x'/y = x/h $$

From these two equations

$$ xy^2 = h^2 x'' $$

The moment of inertia of $A$ with respect to $XX$ is given by the equation

$$ I = \Sigma y^2 \Delta A = \Sigma y^2 x \Delta y = h^2 \Sigma x'' \Delta y = h^2 A'. $$

Figure 15 shows the application of the method to the determination of the moment of inertia of the $\frac{3}{4}$-inch cupro-nickel blad-

*This method is the one given on page 203 of Seely and Ensign, "Analytical Mechanics for Engineers."
ing about a horizontal axis, when the tips of the blading are in a vertical line. Figure 15 (a) shows the determination for a chord in the upper part of the cross-section, and Fig. 15 (b) shows the determination for a chord in the lower part. In all cases of determining moments of inertia of blading $XX$ (or $YY$ as in Fig. 15) was taken through the center of gravity of the cross-section and $X'X'$ (or $Y'Y'$ as in Fig. 15) was drawn tangent to one of the tips ($X'X'$ is tangent to both tips when the tips are in a horizontal line).

Figure 16 shows the areas $A$ and $A'$ for the $\frac{3}{4}$-inch cupronickel blading, for tips vertical ($I_y$ in Formula (1) on page 24), for tips horizontal ($I_x$ in Formula (1) on page 24), and for tips at 45 degrees with the $X$ axis ($I_{45}$ in Formula (1) on page 24).

Having determined $I_x$, $I_y$, and $I_{45}$ for any cross-section

$$J_{xy} = I_x/2 + I_y/2 - I_{45}$$

Then from Equation (8), page 28, and the extreme points of the cross-section being assumed as always located by the coordinates $x$ (or $x'$) and $y$, Fig. 12, the value of $\tan \lambda$ to give a maximum unit stress is determined. From a table of trigonometric functions
values of sin λ and cos λ are then taken, these values are inserted in Equation (7), page 27, and the value of the minimum generalized section modulus, $Z_{\text{min}}$, is thus determined. In doing this it is necessary to compute values of $Z_{\text{min}}$, regarding each tip successively as...
containing the extreme point, that is, to compute the value of $Z_{\text{min}}$
using as coordinates both the point $x,y$ (Fig. 12) and the point $x', y$.

The moment set up by the testing machine is $Wa/2$ in which $W$
denotes the load hung on the specimen, and $a$ denotes the distance from $B_1$
to $B_2$ (or $B_3$ to $B_4$) in Fig. 3. Then the maximum unit stress set up during one revolution of the testing machine is

$$\frac{Wa/2}{Z_{\text{min}}}$$

and the range of stress is from this magnitude in compression to the same magnitude in tension.

Table 5 gives values of $I_x$, $I_y$, $J_{xy}$, $x$, $y$, and $Z_{\text{min}}$ for the different lots of blading tested.

6. Illustrative Computation of $Z_{\text{min}}$ for 3/4-inch Cupro-nickel Blading.—Figures 15 and 16 give enlarged cross-sections for blading and diagrams for determining $I_y$, $I_x$, and $I_{45}$. The linear magnification for the enlarged cross-section is 10.04 times.

For the extreme points of the cross-section (Fig. 12) $x = -0.350$
in. for thick tip or $x = +0.513$ in. for thin tip, and $y = +0.135$ in.
for either tip.

From Fig. 16 (a) $I_y = A'_y h^2_y = \frac{1.362}{(10.04)^2} \times \frac{(5.13)^2}{(10.04)^2} = 0.003528$ in.$^4$\
From Fig. 16 (b) $I_x = A'_z h^2_z = \frac{1.058}{(10.04)^2} \times \frac{(1.35)^2}{(10.04)^2} = 0.000198$ in.$^4$\
From Fig. 16 (c) $I_{45} = A'_{45} h^2_{45} = \frac{1.630}{(10.04)^2} \times \frac{(3.39)^2}{(10.04)^2} = 0.001844$ in.$^4$\

$J_{xy} = I_y/2 + I_x/2 - I_{45} = \frac{0.003528}{2} + \frac{0.000198}{2} - 0.001844$\
$= 0.000015$ in.$^4$

Considering the thick tip as containing the extreme point, for maximum unit stress during a revolution, $x = -0.350$

$$\tan \lambda = \frac{y I_y - x J_{xy}}{x I_x - y J_{xy}} = \frac{0.135 \times 0.003528 - (-0.350) 0.000015}{(-0.350) 0.000198 - 0.135 \times 0.000015}$$
$$= -7.03$$

From trigonometric tables $\lambda = 98^\circ 6'$; $\sin \lambda = 0.9900$; $\cos \lambda = -0.1409$;

$$Z_{\text{min}} = \frac{I_x I_y - J_{xy}^2}{y (I_y \sin \lambda - J_{xy} \cos \lambda) + x (I_x \cos \lambda - J_{xy} \sin \lambda)}$$

*Figure 16 is somewhat reduced from the original from which computations were made.*
<table>
<thead>
<tr>
<th>Metal</th>
<th>Blade</th>
<th>Coordinates of Points Farthest from Neutral Axis</th>
<th>Moments of Inertia</th>
<th>Product of Inertia</th>
<th>Angle between X Axis and Direction of Bending Moment</th>
<th>Minimum Generalized Section Modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( x ) in.</td>
<td>( y ) in.</td>
<td>( I_x ) in(^4)</td>
<td>( I_y ) in(^4)</td>
<td>( J_{xy} ) in(^4)</td>
</tr>
<tr>
<td>Cupro-nickel</td>
<td>( \frac{3}{4} )-in.</td>
<td>0.513</td>
<td>0.135</td>
<td>0.003658</td>
<td>0.0001988</td>
<td>0.000015</td>
</tr>
<tr>
<td></td>
<td>1-in.</td>
<td>-0.350</td>
<td>0.191</td>
<td>0.012093</td>
<td>0.0008648</td>
<td>0.000012</td>
</tr>
<tr>
<td>Monel</td>
<td>( \frac{3}{4} )-in.</td>
<td>0.523</td>
<td>0.138</td>
<td>0.002808</td>
<td>0.0001920</td>
<td>-0.000224</td>
</tr>
<tr>
<td></td>
<td>1-in.</td>
<td>-0.350</td>
<td>0.181</td>
<td>0.011722</td>
<td>0.0005129</td>
<td>0.000042</td>
</tr>
<tr>
<td>Cyclops</td>
<td>( \frac{3}{4} )-in.</td>
<td>0.505</td>
<td>0.139</td>
<td>0.003274</td>
<td>0.0001981</td>
<td>0.000176</td>
</tr>
<tr>
<td></td>
<td>1-in.</td>
<td>-0.350</td>
<td>0.195</td>
<td>0.012388</td>
<td>0.0007746</td>
<td>0.000484</td>
</tr>
</tbody>
</table>

*Value to be used; the two values of \( Z_{min} \) indicate the values for giving the unit-stress for each tip. The tip corresponding to the value marked with an * is the most stressed tip. For the "thick" tip of a blade the value of \( z \) is —; for the "thin" tip the value of \( z \) is +. See Fig. 12.
\[ Z_{\text{min}} = \frac{0.0001898 \times 0.003528 - (0.000015)^2}{0.135 [0.003528 \times 0.99 - 0.000015 (-0.1409)] + (-0.350) [0.0001898 (-0.1409) - 0.000015 \times 0.99]} = 0.001375 \text{ in.}^3 \]

Considering the thin tip as containing the extreme point, for maximum unit stress during a revolution, \( x = +0.513 \)
\[
\tan \lambda = \frac{0.135 \times 0.003528 - 0.513 \times 0.000015}{0.513 \times 0.0001898 - 0.135 \times 0.000015} = 4.915
\]

From trigonometric tables, \( \lambda = 78^\circ \, 30' \); \( \sin \lambda = 0.9799; \cos \lambda = 0.1994 \).

\[
Z_{\text{min}} = \frac{0.0001898 \times 0.003528 - (0.000015)^2}{0.135 [0.003528 \times 0.9799 - 0.000015 \times 0.1994] + 0.513 [0.0001898 \times 0.1994 - 0.000015 \times 0.9799]} = 0.001400 \text{ in.}^3
\]

The smaller value, \( Z_{\text{min}} = 0.001375 \text{ in.}^3 \) should, of course, be used.
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