ON VERTEX DEGREES, GRAPH DECOMPOSITION, AND CIRCULAR CHROMATIC RAMSEY NUMBER

BY

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DISSERTATION

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Abstract

In this thesis, we study extremal problems about vertex degrees and a variant of Ramsey number of graphs, and also structural problems about graph decomposition.

In a list $(d_1, \ldots, d_n)$ of positive integers, let r and s denote the largest and smallest entries. A list is *gap-free* if each integer between r and s is present. In Chapter 2, we prove that a gap-free list with even sum is graphic if it has at least $r + \frac{r+s+1}{2s}$ terms. With no restriction on gaps, length at least $\frac{(r+s+1)^2}{4s}$ suffices, as proved by Zverovich and Zverovich. Both bounds are sharp within 1. When the gaps between consecutive terms are bounded by g, we prove a more general length threshold that includes both of these results. As a tool, we prove that if a positive list d with even sum has no repeated entries other than r and s (and the length exceeds r), then to prove that d is graphic it suffices to check only the ℓth Erdős–Gallai inequality, where $\ell = \max\{k\colon\ d_k \geq k\}$.

For outerplanar graphs on n vertices, we determine the maximum number of vertices of degree at least k. For $k = 4$ (and $n \geq 7$), the answer is $n - 4$. For $k = 5$ (and $n \geq 4$), the answer is $\left\lfloor \frac{2n-8}{3} \right\rfloor$ (except one less when $n \equiv 1 \mod 6$). For $k \geq 6$ (and $n \geq k + 2$), the answer is $\left\lfloor \frac{n-6}{k-4} \right\rfloor$. As a tool, we determine the maximum sum of the degrees of s vertices. We also determine the maximum sum of the degrees of the vertices with degree at least k.

A T-*decomposition* of a graph G is a decomposition of G into isomorphic copies of T. Let T be a tree with m edges. In Chapter 3, we extend the ideas of Snevily and Avgustinovitch to prove the existence of T-decompositions for more $2m$-regular graphs and m-regular bipartite graphs. In particular, for $r_1, \ldots, r_k$ with $\sum_{i=1}^{k} r_i = m$, we seek sufficient conditions for

every cartesian product of graphs $G_1, \ldots, G_k$ with G_i being $2r_i$-regular for all i to have a T-decomposition. One sufficient condition is the existence of a k-edge-coloring of T with r_i edges of color i such that every path in T uses some color once or twice. Another sufficient condition is that $r_i \le \lceil \frac{m+1}{2} \rceil$ for all i and $m/k < 4$.

Finally, in Chapter 4, we introduce the *circular chromatic Ramsey number* $R_{\chi_c}(\mathcal{F}, \mathcal{G})$ as the infimum of the circular chromatic numbers $\chi_c(H)$ of graphs H such that every red/blue edge-coloring of H yields a red copy of a graph in $\mathcal{F}$ or a blue copy of a graph in $\mathcal{G}$. We prove $R_{\chi_c}(K_3, K_3) = 6$ and $R_{\chi_c}(K_3, K_4) = 9$. Also, if $2 < \chi_c(G) \le 5/2$, then $R_{\chi_c}(G, G) = 4$. Furthermore, no graph has circular chromatic Ramsey number between 4 and 5. Also, with $R_{\chi_c}(z) = \inf\{R_{\chi_c}(G)\colon\ \chi_c(G) \ge z\}$, we prove $R_{\chi_c}(k) \le k(k-1)$ for $k \in \mathbb{N} - \{1\}$.

To my parents

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Table of Contents

Chapter 1

Introduction

Graphs are structures that can represent the relations between objects, such as individuals, computers, network devices, molecules, etc. Graph theory, the study of graphs, has applications in computer science, biology, chemistry, etc. In this thesis, we focus on several topics in graph theory involving vertex degrees, graph decomposition, and parameter Ramsey theory. Terms not defined in this initial description of our results can be found in Section 1.4 where reviews basic terminology in graph theory.

The *degree list* of a graph is the list of all vertex degrees in the graph. The degree list can be determined easily from the graph. Given list it is harder to determine whether or not there is a graph that has that list as its degree list; such a graph *realizes* the list, and the list is then called *graphic*. Zverovich and Zverovich [43] proved that if the length of a list is at least $\frac{(r+s+1)^2}{4s}$ where r and s denote the largest and smallest entries, respectively, then the list is graphic. In Chapter 2, we give a more general length threshold that includes this result. This is joint work with Michael D. Barrus, Stephen G. Hartke, and Douglas B. West.

In Chapter 2, we also prove a result about vertex degrees in outerplanar graphs. Among outerplanar graphs on n vertices, we determine the maximum number of vertices of degree at least k. For $k = 4$ (and $n \geq 7$), the answer is $n - 4$. For $k = 5$ (and $n \geq 4$), the answer is $\left\lfloor \frac{2n-8}{3} \right\rfloor$ (except one less when $n \equiv 1 \mod 6$). For $k \geq 6$ (and $n \geq k + 2$), the answer is $\left\lfloor \frac{n-6}{k-4} \right\rfloor$. We also determine the maximum sum of the degrees of s vertices in an n-vertex outerplanar graph and the maximum sum of the degrees of the vertices with degree at least k. This is joint work with Douglas B. West.

In Chapter 3, we consider problems in graph decomposition. Ringel [33] famously conjectured that K_{2m+1} decomposes into $2m + 1$ copies of any tree with m edges. Ringel's Conjecture is implied by many stronger conjectures, and one by Häggkvist [18] states that any $2m$-regular graph decomposes into copies of any tree T with m edges; such a decomposition is a *T-decomposition*. Graham and Häggkvist [18] also conjectured that any m-regular bipartite graph has a T-decomposition. We enlarge the family of $2m$-regular graphs found by Snevily [36] and the family of m-regular bipartite graphs found by Avgustinovitch [2] that are known to be true for the conjectures. This is joint work with Alexandr V. Kostochka and Douglas B. West.

Finally, in Chapter 4, we study a variant of graph Ramsey Theory. Classical graph Ramsey problems consider the 2-edge-colorings of a complete graph H called the host graph and ask whether a monochromatic copy of some target graph G must be present; we say that *H forces G*, written $H \to G$, if every 2-edge-coloring of H yields a monochromatic copy of G. The minimum number of vertices in a complete graph that forces G is the *Ramsey number* of G, written $R(G)$. Parameter Ramsey problems consider more general host graphs than complete graphs, and the minimum value of the parameter ρ among host graphs that force G is the *ρ-Ramsey number* of G, written $R_\rho(G)$. Thus the classical graph Ramsey number is the "order" Ramsey number. We consider the circular chromatic Ramsey number, written R_{χ_c}, and determine the values of $R_{\chi_c}(K_3), R_{\chi_c}(K_4)$, and $R_{\chi_c}(C_k)$ for all k. More generally, given two target graphs F and G, we ask whether $H \to (F, G)$, which means every red/blue edge-coloring of H yields either a red copy of F or a blue copy of G. We also determine the value $R_{\chi_c}(K_3, K_4)$. This is joint work with Claude Tardif, Douglas B. West, and Xuding Zhu.

1.1 Vertex Degrees

Recall that a list of integers is *graphic* if it is the list of vertex degrees for some graph (with no loops or multiple edges). Since the order of integers in the list does not affect the realizability, we may restrict out attention to nonincreasing lists of positive integers; we write a list d as $(d_1, \ldots, d_n)$ with $d_1 \geq \cdots \geq d_n$. Graphic lists have many characterizations. Havel [21] and Hakimi [19] independently proved a recursive characterization that d is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element d_1 and subtracting 1 from its d_1 next largest elements. There are also non-recursive characterizations; the most famous is due to Erdős and Gallai [9]. They proved in 1960 that a list d is graphic if and only if it has even sum and satisfies $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for each integer k with $1 \leq k \leq n$. In fact, it suffices to check the first l of these inequalities, where $l = \max\{k \colon\ d_k \geq k\}$. The values d_i satisfying $d_1 > d_i > d_n$ are called *internal values*. We prove in Chapter 2 that if a list has no repeated internal values (and $n > r$), then it suffices to check only the lth inequality. The conclusion also holds when there is exactly one instance of two consecutive equal internal values, and this is sharp.

Caro and West [6] discussed the realizability of *packed lists*; lists in which all but one value between the largest and smallest entries have the same multiplicity in the list. Such a list is *gap-free*, since every internal value appears in the list. It is natural to ask whether sufficiently long gap-free lists are graphic. We prove that a gap-free list with even sum having maximum r and minimum s is graphic if its length is at least $r + \frac{r+s+1}{2s}$. With no restriction on gaps, an even-summed list with maximum r and minimum s is graphic if its length is at least $\frac{(r+s+1)^2}{4s}$, as proved by Zverovich and Zverovich [43]. Both bounds are sharp. Let the *gap* at i in a list d be $d_i - d_{i+1}$. For lists where the gaps between consecutive values are restricted to be at most g, we also prove a more general length threshold for being graphic that includes both of these results.

Our other results about vertex degrees involve outerplanar graphs. Erdős and Griggs [9] asked for the minimum, over n-vertex planar graphs, of the number of vertices with degree less than k. For $k \leq 6$, the answer follows from results of Grünbaum and Motzkin [17]. West and Will [40] determined the answer for $k \geq 12$, obtained the best lower bounds for $7 \leq k \leq 11$, and provided constructions achieving those bounds for infinitely many n when $7 \leq k \leq 10$. Griggs and Lin [15] independently found the same lower bounds for $7 \leq k \leq 10$ and gave constructions achieving the lower bounds when $7 \leq k \leq 11$ for *all* sufficiently large n.

We also study the analogous question for outerplanar graphs, expressed in terms of large-degree vertices. That is, we determine $\beta_k(n)$, the maximum number of vertices with degree at least k in an n-vertex outerplanar graph. In second part of Chapter 2, we show $\beta_2(n) = n$ and $\beta_3(n) = n - 2$. For sufficiently large n, we prove $\beta_4(n) = n - 4$ and $\beta_5(n) = \left\lfloor \frac{2n-8}{3} \right\rfloor$, and $\beta_k(n) = \left\lfloor \frac{n-6}{n-4} \right\rfloor$ for $k \geq 6$.

1.2 Decomposition of Graphs into Trees

Our next result considers a problem in graph decomposition. A *decomposition* of a graph G is a set of pairwise edge-disjoint subgraphs of G whose union is G. When each subgraph in the decomposition is isomorphic to a fixed graph H, the decomposition is an *H-decomposition* of G. Ringel [33] conjectured that the complete graph K_{2m+1} has a T-decomposition whenever T is a tree with m edges. This conjecture is still unsolved. Rosa [35] proved that Ringel's Conjecture is implied by the stronger Graceful Tree Conjecture, which states that every tree has a graceful labeling. A *graceful labeling* of a graph G with m edges is a function $f\colon V(G) \to \{0, \ldots, m\}$ such that distinct vertices receive distinct numbers and $\{|f(u) - f(v)|\colon uv \in E(G)\} = [m]$. Independent of its connection to Ringel's Conjecture, the Graceful Tree Conjecture has become famous in its own right. Ringel's Conjecture is also a special case of other conjectures. For example, Häggkvist [18] conjectured that every $2m$-regular graph has a T-decomposition. Analogously, Graham and Häggkvist [18] also conjectured that every m-regular bipartite graph has a T-decomposition. Snevily [36] proved the two latter conjectures for some special classes of graphs, and Avgustinovich [2] obtained results on decompositions of bipartite graphs into copies of T that occur as induced subgraphs. In Chapter 3, we combine and extend these ideas to the graphs that are cartesian products of $2m$-regular graphs and to the graphs that are cartesian products of m-regular bipartite graphs. Our theorem yields the earlier results as corollaries.

As a tool, we focus on special edge-colorings of trees. Let T be a tree with m edges, and let r be a nondecreasing k-tuple with sum m. An edge-coloring of T is *r-exact* if it has exactly r_i edges of color i for $1 \leq i \leq k$. An edge-coloring of T is *q-good* if every color appears at most q times on every path (such a path is *q-bounded*). An r-exact edge-coloring of T is *weakly 2-good* if every path in T is either 2-bounded or has a color appearing only on a 3-edge subpath whose two internal vertices have degree 2 in T. We prove that if $r_k \leq \lceil \frac{m+1}{2} \rceil$ and $m/k < 4$, then T has a weakly 2-good r-exact edge-coloring. We also prove that if T has a weakly 2-good r-exact edge-coloring, then any product of simple regular graphs with degrees $2r_1, \ldots, 2r_k$ has a T-decomposition. Consequently, if $r_k \leq \lceil \frac{m+1}{2} \rceil$ and $m/k < 4$, then any product of regular graphs with degrees $2r_1, \ldots, 2r_k$ has a T-decomposition whenever T is a tree with m edges.

1.3 Circular Chromatic Ramsey Number

Given families $\mathcal{F}$ and $\mathcal{G}$ of graphs, the classical *Ramsey number* $R(\mathcal{F}, \mathcal{G})$ is the minimum number of vertices in a graph H such that every red/blue edge-coloring of H yields a red copy of a graph in $\mathcal{F}$ or a blue copy of a graph in $\mathcal{G}$. That is, $R(\mathcal{F}, \mathcal{G}) = \min\{|V(H)|\colon\ H \to (\mathcal{F}, \mathcal{G})\}$. The *chromatic Ramsey number* $R_c(\mathcal{F}, \mathcal{G})$ is $\min\{\chi(H)\colon\ H \to (\mathcal{F}, \mathcal{G})\}$. In Chapter 4, we introduce the circular chromatic Ramsey number. A (p,q)-coloring of a graph G is a map $f\colon V(G) \to \mathbb{Z}_p$ such that the colors on adjacent vertices differ by at least q. The *circular chromatic number* of G, written $\chi_c(G)$, is $\inf\{p/q\colon G \text{ has a } (p,q)\text{-coloring}\}$. The *circular chromatic Ramsey number* $R_{\chi_c}(\mathcal{F},\mathcal{G})$ is then $\inf\{\chi_c(H)\colon H \to (\mathcal{F},\mathcal{G})\}$.

A *homomorphic image* of a graph G is obtained by collapsing independent sets of vertices into single vertices (satisfying the preservation of edges); extra copies of resulting edges are deleted. Let $\mathrm{Hom}(G)$ denote the family of minimal homomorphic images of G. For a family $\mathcal{G}$ of graphs, let $\mathrm{Hom}(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} \mathrm{Hom}(G)$. Burr, Erdős, and Lovász [5] proved $R_\chi(\mathcal{F},\mathcal{G}) = R(\mathrm{Hom}(\mathcal{F}),\mathrm{Hom}(\mathcal{G}))$. Their argument applies also to R_{χ_c}, yielding $R_{\chi_c}(\mathcal{F},\mathcal{G}) = \inf\{p/q\colon K_{p:q} \to (\mathrm{Hom}(\mathcal{F}),\mathrm{Hom}(\mathcal{G}))\}$. Using this as a tool, we prove $R_{\chi_c}(K_3,K_3) = 6$ and $R_{\chi_c}(K_3,K_4) = 9$. Also, we show $R_{\chi_c}(C_5,C_5) = 4$. This yields more generally that if $2 < \chi_c(G) \le 5/2$, then $R_{\chi_c}(G,G) = 4$. From the characterization of $R_\chi(G)$ in [5], it follows that no graph has circular chromatic Ramsey number between 4 and 5. In fact, we have not found any non-integer circular chromatic Ramsey number. As a small step toward answering this question, we show $9/2 \le R_{\chi_c}(C_3,C_5) \le 5$.

Finally, we consider a question analogous to the conjecture by Burr, Erdős, and Lovász [5] that $R_\chi(k) = (k-1)^2+1$, where $R_\chi(k) = \min\{R_\chi(G)\colon \chi(G) = k\}$. This conjecture was proved by Zhu [42]. We define $R_{\chi_c}(z) = \inf\{R_{\chi_c}(G)\colon \chi_c(G) \ge z\}$ and prove $R_{\chi_c}(k) \le k(k-1)$ for $k \in \mathbb{N} - \{1\}$. Note that $k(k-1) > (k-1)^2+1$ for $k \ge 3$. The value of $R_{\chi_c}(k)$ may possibly exceed the value of $R_\chi(k)$ because graphs with $\chi_c(G) = k$ are rare among these with $\chi(G) = k$.

1.4 Definitions and Notation

In this section, we give background definitions and notation about graphs. A *graph* G consists of two sets $V(G)$ and $E(G)$, where $V(G)$ is the *vertex set* and $E(G)$ is the *edge set* of G, respectively. Each element of $V(G)$ is a *vertex* of G, and each element of $E(G)$ is an unordered pair of distinct vertices, called an *edge*. The *order* of a graph is the size of its vertex set, and the *size* of a graph is the size of its edge set. In this thesis, all graphs are assumed to have finite order and size.

In writing edges of a graph, we use uv to denote an edge $\{u, v\}$, and we refer to u and v as *endpoints* of the edge. If uv is an edge, then u and v are *adjacent* and the edge uv is *incident* to u and v. The *degree* of a vertex v in G, written $d_G(v)$, is the number of edges incident to v; we use $d(v)$ if G is understood. A graph is *regular* if its vertices all have the same degree.

Let G be an n-vertex graph, with vertices $v_1, \ldots, v_n$ indexed in nonincreasing order of degrees. The *degree list* of G is the list $(d(v_1), \ldots, d(v_n))$. If G has m edges and realizes $(d_1, \ldots, d_n)$, then since each edge is incident to its two endpoints, we have the well-known *Degree-Sum Formula*, which states that

$$\sum_{i=1}^{n} d_i = 2m.$$

The *cartesian product* of graphs G and H, written $G \square H$, is the graph with vertex set $\{(u, v)\colon\ u \in V(G), v \in V(H)\}$ such that $(u, v)(u', v')$ is an edge if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. A *subgraph* of a graph G is a graph H, written $H \subseteq G$, such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and we may say that G *contains* H or H is *contained in* G. A graph parameter ρ is *monotone* if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$. A subgraph H of G is a *1-factor* if $d_H(v) = 1$ for all $v \in V(G)$.

A *path* is a graph whose vertices can be ordered so that two vertices are consecutive in the list if and only if they form an edge. A *cycle* consists of a path plus an edge joining its first and last vertices. For a graph having a cycle, the *girth* is the number of edges in a shortest cycle in the graph. A *complete graph* is a graph whose vertices are pairwise adjacent. A graph G is *bipartite* if $V(G)$ is the union of two disjoint sets X and Y and each edge in G has endpoints in both X and Y; the sets X and Y are *partite sets* of G. A *complete bipartite graph* is a bipartite graph such that every vertex is adjacent to every vertex in the other partite set. We use K_n, P_n, C_n for the complete graph, path, and cycle on n vertices, and we use $K_{m,n}$ for the complete bipartite graph with partite sets of sizes m and n.

A graph is *connected* if any two vertices in it are contained in some path. A *tree* is a connected graph containing no cycle. The *leaves* of a tree are the vertices with degree 1, and the *pendant edges* are the edges incident to a leaf.

A graph G is *planar* if it can be drawn in the plane so that the edges intersect only at their endpoints; such a drawing is an *embedding* of G; a *plane graph* is a particular embedding of a planar graph. The *faces* of a plane graph are the maximal regions disjoint from the edges. Each finite plane graph has exactly one unbounded face; other faces are bounded. A planar graph is *outerplanar* if it has an embedding with every vertex on the boundary of the unbounded face; an *outerplane graph* is such an embedding.

A *k-coloring* of a graph G is a map $f\colon V(G) \to \{1, \dots, k\}$, where we call $1, \dots, k$ *colors*; a coloring is *proper* if adjacent vertices are assigned distinct colors. The *chromatic number* of G, written $\chi(G)$, is the least k such that G has a proper k-coloring. Similarly, a k-edge-coloring of a graph G is a map $f\colon E(G) \to \{1, \dots, k\}$. An k-edge-coloring of G can be interpreted as a decomposition of G into k edge-disjoint subgraphs; each edge lies in exactly one of the subgraphs.

A subgraph G of an edge-colored graph is *monochromatic* if all edges in G have the same color. We say that H *forces* G, written $H \to G$, if every edge-coloring of H yields a monochromatic copy of G. The *Ramsey number* of a graph G, written $R(G)$, is the least n such that $K_n \to G$.

The set $\{1, \dots, k\}$ is denoted $[k]$. A *relation* R on a set X is a subset of the cartesian product $X \times X$; we also write xRy for $(x, y) \in R$. A *partial order* on X is a relation that is *reflexive* (xRx for all x), *antisymmetric* (xRy and yRx imply $x = y$), and *transitive* (xRy and yRz imply xRz). A *partially ordered set* (or *poset*) P is a set with a partial order on it. In this case we write $x \leq_P y$ (or simply $x \leq y$ when P is understood) for xRy. In a poset P, an element is *maximal* if no other element is greater than it. If $x < y$ and there is no z with $x < z < y$, then y *covers* x in P. An *ideal* in P is a subset I such that $x \in I$ and $y < x$ imply $y \in I$.

Chapter 2

Vertex Degrees

2.1 Introduction

This chapter is based on joint work with Michael Barrus, Stephen Hartke, and Douglas West, appearing in [3], and joint work with Douglas West, appearing in [25]. A list of integers is *graphic* if it is the list of vertex degrees for some graph (with no loops or multiple edges). We consider only nonincreasing positive lists, writing a list d as $(d_1, \ldots, d_n)$ with $d_1 \geq \cdots \geq d_n$. In Sections 2.2–2.4, we study extremal problems about when the length of a list with certain properties forces the list to be graphic. In Sections 2.5–Section 2.7, we study extremal problems about the elements in degree lists of outerplanar graphs.

A graphic list is *gap-free* if it has entries with all values between the largest entry r and the smallest entry s; it is *even-summed* if $\sum_{i=1}^{n} d_i$ is even. We define the *gap at i* in a list d to be $d_i - d_{i+1}$. A list with $r = s$ is graphic if it has even sum and $n > r$ (realized using edge-disjoint spanning cycles or 1-factors, depending on whether n is odd or even). The same conclusion also holds when $r - s = 1$ (see [39]). Among even-summed lists with largest entry r, smallest entry s, and all gaps at most g, we seek the smallest n such that every such list with length at least n is graphic; we determine it within 1.

Graphic lists have many characterizations. Erdős and Gallai [9] proved in 1960 that a list d is graphic if and only if it has even sum and satisfies $L_k(d) \leq R_k(d)$ for each integer k with $1 \leq k \leq n$, where $L_k(d) = \sum_{i=1}^{k} d_i$ and $R_k(d) = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$. In fact, Zverovich and Zverovich [43] proved that it suffices to check the first ℓ of these inequalities, where $\ell = \max\{k\colon\ d_k \ge k\}$ (see [20, 30] for a slightly weaker statement). Eggleton [7] (and later Tripathi and Vijay [37]) proved the stronger statement that it suffices to check only the inequalities for the last index having each value in the list. Values between r and s are *internal values*. As a tool in our argument, we prove in Section 2.3 that if a list has no repeated internal values (and $n > r$), then it suffices to check only the ℓth inequality. The conclusion also holds when there is exactly one instance of two consecutive equal internal values, and this is sharp.

We approach our problem by finding a length threshold for lists to satisfy the Erdős–Gallai inequalities, which we henceforth call the *E-G inequalities*. We use the *Aigner–Triesch method*. In Section 2.2, we describe the use of this method to prove sufficiency of conditions for lists to be graphic. The method introduces an order relation P on the set of lists with fixed sum and reduces the problem to proving that lists that are maximal in P among those satisfying the condition are graphic. We further reduce the problem by comparing these maximal lists from the sets with various sums, reducing sufficiency to the study of certain key lists.

Let $D_n(r, s, g)$ denote the set of nonincreasing nonnegative integer lists with length n, largest entry r, smallest entry s, and all gaps at most g. The case $g = 0$ requires only $n > r$, as mentioned in the first paragraph, so we henceforth restrict to $g > 0$ and consider only $r > s$. The first and last nonzero gaps in a list are *external gaps* (they may be at the same position); gaps between them are *internal gaps*. A list in $D_n(r, s, g)$ is *g-uniform* if all internal and external gaps except possibly the last one equal g. We show in Section 2.2 that to prove sufficiency of the length threshold for lists with gaps at most g, it suffices to prove it sufficient for g-uniform lists. The resulting sharpness examples are g-uniform.

Call a list *feasible* if it satisfies the E-G inequalities. In terms of r, s, and g, we obtain a sharp threshold $h(r, s, g)$ such that when $n \ge h(r, s, g)$, every list in $D_n(r, s, g)$ is feasible; the argument that proves the threshold sufficient also constructs an infeasible list when n is smaller. The general expression for $h(r,s,g)$ is obtained in Theorem 2.4.2. When g divides $r-s$, the formula for $h(r,s,g)$ simplifies to

$$\frac{1}{s}\left(\left\lfloor\frac{(r+s)^2}{4}\right\rfloor+\left\lceil\frac{r+s}{2}\right\rceil-\frac{g}{2}\left\lfloor\frac{(r-s)^2}{2g^2}\right\rfloor\right).$$

Depending on the parameters r,s,g, the infeasible list given for $n=\lceil h(r,s,g)\rceil-1$ may have odd sum. In this case, the length threshold sufficient for even-summed lists with specified maximum, minimum, and bound on gaps to be graphic may be smaller by 1 than the threshold we give for feasibility. We show that the difference between the threshold lengths for feasible lists and graphic lists is never more than 1, and we present a family with $g=1$ where the thresholds do differ by 1.

The extreme cases for g hold particular interest. When $g=1$, the threshold for feasibility reduces to $r+\frac{r+s+\epsilon}{2s}$ (linear in r), where $\epsilon=0$ if $r+s$ is even and $\epsilon=1$ if $r+s$ is odd. This is the most severe restriction on gaps. The other end of the spectrum is $g=r-s$, which means that no gap restriction is imposed. Here the threshold reduces to $\frac{(r+s+1)^2-\epsilon'}{4s}$ (quadratic in r), where $\epsilon'=0$ if $r+s$ is odd and $\epsilon'=1$ if $r+s$ is even. Zverovich and Zverovich [43] showed that $\frac{(r+s+1)^2}{4s}$ suffices.

2.2 Aigner–Triesch Method

By the Erdős–Gallai Theorem, a list is graphic if and only if it is feasible and has even sum. In light of this theorem, using the Aigner–Triesch method [1] to show sufficiency of conditions for feasibility will also give sufficient conditions for lists to be graphic. This allows us to ignore the parity of the degree sum in applying this method. Let R be a family of lists (for example, the graphic lists or the feasible lists). The Aigner–Triesch method for proving that a condition Q is sufficient for membership in R consists of three steps:

1. Define a poset P on the set of lists (usually with fixed sum) and show that the elements of P belonging to R form an ideal (a downward-closed set) in P.
2. Determine the maximal elements of P among those satisfying Q.
3. Prove that these maximal elements are in R.

For lists with fixed sum, an order relation often used in applying the Aigner–Triesch method is the *dominance order*, which puts $d \le d'$ if $\sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} d'_i$ for all k (trailing terms are assumed to be 0). For the dominance order on a set of lists with fixed sum, the proof of Step 1 when R is the family of graphic lists is immediate and is used in [1]; we present the corresponding argument in Lemma 2.2.1 for the family of feasible lists.

When Step 1 holds for a given poset, it also holds for any subposet. Let P_m be the dominance order on nonincreasing nonnegative integer lists with sum m. After proving Step 1 for P_m, we will consider subposets of the form $P_{m,n,r,s,g}$, fixing the sum m, length n, largest entry r, positive smallest entry s, and bound g on all gaps. Since g is only a bound on the largest gap, these subposets are not disjoint. Nevertheless, Step 1 will hold for each such subposet. The condition Q we want to prove sufficient is a lower bound on the length n; therefore, in $P_{m,n,r,s,g}$ all lists or no lists satisfy Q. For Step 2, we prove that $P_{m,n,r,s,g}$ has a unique maximal element. We then show that the maximal element of $P_{m,n,r,s,g}$ is feasible when the length threshold in terms of the parameters r, s, and g is satisfied. To do this, we compare the maximal elements for distinct values of m, thus reducing the problem to showing feasibility for the maximal element of certain key subposets.

We begin with Step 1 for P_m. All lists are nonincreasing. *Shifting a unit from i to j* in a list d produces another nonincreasing list d' that agrees with d in all positions except i and j, and in those positions $d'_i = d_i - 1$ and $d'_j = d_j + 1$ (see Figure 2.1). The unit is shifted *later* if $i < j$, otherwise *earlier*. In a poset, an element x *covers* an element y if $y < x$ and there is no element z such that $y < z < x$. To prove Step 1, we show (1) if d covers d' in P_m, then one unit can be shifted later in d to obtain d', and (2) shifting one unit later preserves feasibility.

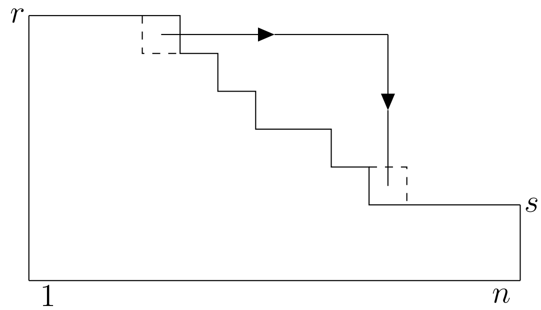


Figure 2.1: Shifting a unit later.

Lemma 2.2.1. *If d covers d' in P_m and d is feasible, then d' is feasible.*

Proof. We obtain d' from d by shifting a unit later. Let i and j be the first and last positions where d and d' differ. Since $d > d'$, we have $d_i > d'_i$ and $d_j < d'_j$. Choose p and q to minimize $q - p$ such that $i \leq p < q \leq j$ and $d_p > d'_p$ and $d_q < d'_q$. Form $\hat{d}$ by shifting a unit from position p to position q in d. Since $d > \hat{d} \geq d'$ in P_m, we obtain $\hat{d} = d'$.

Hence it suffices to prove that shifting a unit later preserves feasibility. Suppose that d' is obtained from d by shifting a unit from i to j with $i < j$. We compare the kth E-G inequalities for d and d'. Since we shifted a unit later, $L_k(d') \leq L_k(d)$.

The only position that can contribute less to $R_k(d')$ than to $R_k(d)$ (smaller by 1) is position i, and it does so only when $d_i \leq k < i$. Since we shifted later from i to j, also $d_j < d_i \leq k < j$, and hence in this case position j contributes more to $R_k(d')$ than to $R_k(d)$. Thus $R_k(d') \geq R_k(d)$ for all k, and feasibility of d implies feasibility of d'. □

Lemma 2.2.2. *The poset $P_{m,n,r,s,g}$ has a unique maximal element. In it, there is at most one internal gap that is less than g, and if there is such a gap it is $g - 1$.*

Proof. Let d be a maximal element of $P_{m,n,r,s,g}$. If d has at least two internal gaps that are less than g, then let i and j be the positions of the first and last such gaps. Form d' by shifting a unit from $j+1$ to i (earlier). The gaps at i and j increase by 1, but they are still at most g. The gaps at $i-1$ and $j+1$ decrease by 1, but the choice of i and j implies that they were external or were equal to g before the shift. In either case, the list d' is nonincreasing and belongs to $P_{m,n,r,s,g}$. Since $d' > d$, this is a contradiction.

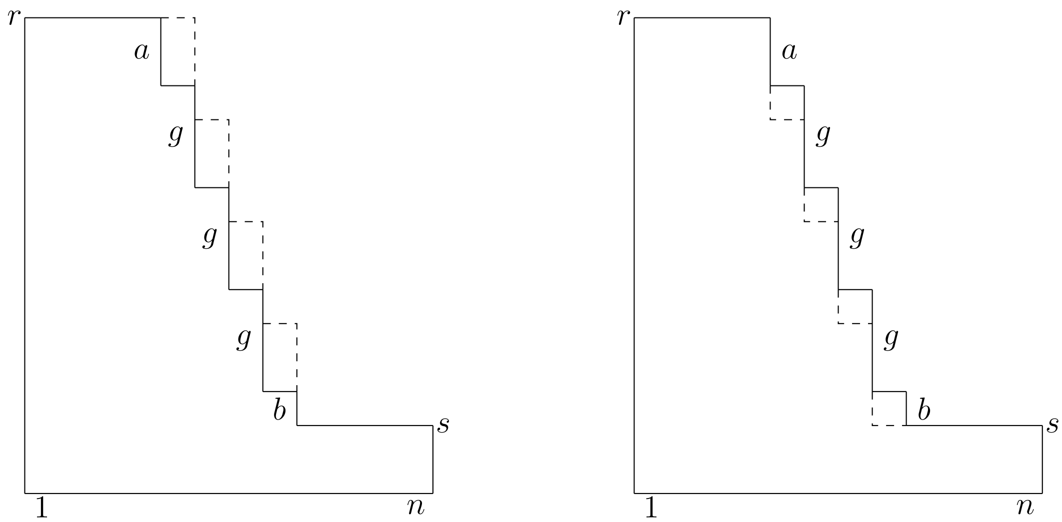
Hence at most one internal gap is less than g, say at j. If it is less than $g-1$, then define d' by shifting a unit from $j+1$ to j. The gap at j grows by 2, and the gaps at $j-1$ and $j+1$ (which were external or equal to g) are smaller by 1. Again $d' \in P_{m,n,r,s,g}$ and $d' > d$.

It remains to prove that only one element of $P_{m,n,r,s,g}$ can have the properties obtained above for all maximal elements. Suppose that d and d' are distinct maximal elements of $P_{m,n,r,s,g}$. Let i be the first index where d and d' differ, named so that $d'_i < d_i$. Let k be the last index such that $d'_k > s$. Since $d'_i < d_i \le r$, the first nonzero gap in d' (the external gap) occurs before i. For j with $i \le j \le k$, using the properties of internal gaps shown above, we conclude that $d'_j \le d'_i - (j-i)g + 1$. Meanwhile, $d_j \ge d_i - (j-i)g$; therefore, $d'_j \le d_j$. For $j > k$, again $d'_j = s \le d_j$. These inequalities imply that the sum of terms in d exceeds the sum in d', which contradicts $d, d' \in P_{m,n,r,s,g}$. □

Including the external gaps, the unique maximal element of $P_{m,n,r,s,g}$ has at most three gaps that are less than g. We next reduce the problem of proving that the length condition suffices for feasibility to proving it for g-uniform lists. Recall that a list in $P_{m,n,r,s,g}$ is g-uniform if every nonzero gap except possibly the last equals g.

Definition 2.2.3. From a list $d \in D_n(r,s,g)$ with internal gaps equal g except perhaps for one $g-1$, we define g-uniform lists $d^+, d^- \in D_n(r,s,g)$. Let $d^+ = d^- = d$ when d has at most one nonzero gap. Otherwise, let the external gaps in d be a and later b, and let $c = a+b$.

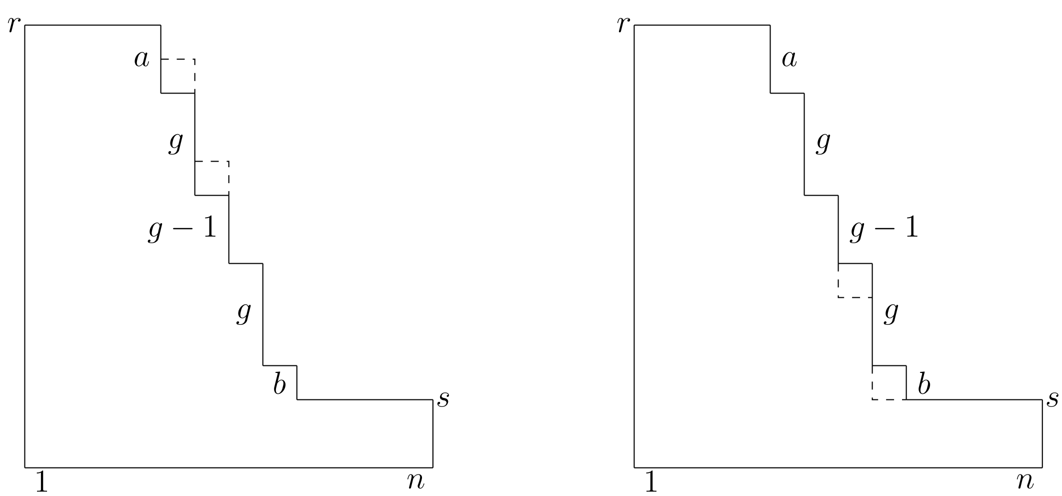
If every internal gap in d is g, then define d^+ from d by adding a to each d_i such that $r > d_i > s$, except that when $c > g$ also add $c-g$ to the first copy of s. Define d^- from d by subtracting $g-a$ from each d_i such that $r > d_i > s$, except that when $c < g$ subtract only b from the last entry before the first copy of s, making it equal to s. See Figure 2.2.



(a) d^+ (b) d^-

Figure 2.2: d^+ and d^- when every internal gap is g.

If d has an internal gap of $g - 1$ at some position j, then first form $\hat{d}^+$ by adding 1 to each d_i such that $i \leq j$ and $r > d_i$, and form $\hat{d}^-$ by subtracting 1 from each d_i such $i > j$ and $d_i > s$. Now all internal gaps in $\hat{d}^+$ and $\hat{d}^-$ equal g. Form d^+ from $\hat{d}^+$ in the way that d^+ is formed from d above, and form d^- from $\hat{d}^-$ in the way that d^- is formed from d above. See Figure 2.3



(a) $\hat{d}^+$ (b) $\hat{d}^-$

Figure 2.3: $\hat{d}^+$ and $\hat{d}^-$ when some internal gap is $g - 1$.

These lists are illustrated below, with $c = a + b$.

list	case	change	values					
d	no $g-1$		$\ldots, r,$	$r-a,$	$r-a-g, \ldots,$	$s+b+g,$	$s+b,$	$s, \ldots$
d^+	$a+b \leq g$	$+a$	$\ldots, r,$	$r,$	$r-g, \ldots,$	$s+c+g,$	$s+c,$	$s, \ldots$
d^+	$a+b > g$	exception	$\ldots, r,$	$r,$	$r-g, \ldots,$	$s+c+g,$	$s+c,$	$s+c-g, \ldots$
d^-	$a+b \geq g$	$-(g-a)$	$\ldots, r,$	$r-g,$	$r-2g, \ldots,$	$s+c,$	$s+c-g,$	$s, \ldots$
d^-	$a+b < g$	exception	$\ldots, r,$	$r-g,$	$r-2g, \ldots,$	$s+c,$	$s,$	$s, \ldots$
d	$g-1$ at j		$\ldots, r,$	$r-a,$	$r-a-g, \ldots,$	$s+b+g,$	$s+b,$	$s, \ldots$
$\hat{d}^+$		$+1$ or 0	$\ldots, r,$	$r-a+1,$	$r-a+1-g, \ldots,$	$s+b+g,$	$s+b,$	$s, \ldots$
$\hat{d}^-$		0 or -1	$\ldots, r,$	$r-a,$	$r-a-g, \ldots,$	$s+b+g-1,$	$s+b-1,$	$s, \ldots$

The display above shows the construction of d^+ and d^- term-by-term. When every internal gap in d equals g, explicitly d^+ and d^- are g-uniform (the last nonzero gap is $a+b$ or $a+b-g$). When d has one internal gap equal to $g-1$, the intermediate lists $\hat{d}^+$ and $\hat{d}^-$ eliminate that difficulty. The subsequent changes are as applied previously to a list with no such gap, so again the final lists d^+ and d^- are g-uniform.

Our requirement that $a = g$ for a g-uniform list is asymmetric. The proofs could be carried through with $b = g$ instead. Choosing one alternative for the definition of g-uniform reduces the set of lists we need to test to prove the length threshold.

Lemma 2.2.4. *Let d be the maximal element of $P_{m,n,r,s,g}$. If the g-uniform lists d^+ and d^- formed from d in Definition 2.2.3 are both feasible, then d is feasible.*

Proof. Suppose first that every internal gap of d is g. For each k, we compare $L_k(d)$ and $R_k(d)$ using $L_k(d^+) \leq R_k(d^+)$ and $L_k(d^-) \leq R_k(d^-)$. If $d_k = r$, then $L_k(d) = L_k(d^-) \leq R_k(d^-) \leq R_k(d)$. If $d_k = s$, then $R_k(d) = R_k(d^+) \geq L_k(d^+) \geq L_k(d)$.

Hence we may assume $s < d_k < r$. Let $t = |\{i\colon\ d_k \le d_i < r\}|$; each index i counted here is at most k, since each internal gap is nonzero. Let $t' = |\{i\colon\ i > k \text{ and } s < d_i < k\}|$. Note that t and t' count disjoint sets of indices. If $i > k$ and $d_i \ge k$, then the index i is not counted by t or t'.

We compare contributions to the kth E-G inequality. Since $d_k > s$, the computation of $L_k(d^+)$ is not affected by the exception when $a + b > g$. The computation of $L_k(d^-)$ is affected only when d_k is the last term before the first s and $b < g - a$; in that case the difference is smaller by $g - a - b$, and we can incorporate this by writing an inequality.

$$L_k(d^+) = L_k(d) + ta; \qquad L_k(d^-) \ge L_k(d) - t(g-a);$$
$$R_k(d^+) \le R_k(d) + (t' + \epsilon^+)a; \qquad R_k(d^-) \le R_k(d) - (t' - \epsilon^-)(g-a).$$

To handle the exceptions in Definition 2.2.3, we set (ϵ^+, ϵ^-) to $(1,0)$ if $a + b > g$, to $(0,1)$ if $a + b < g$, and to $(0,0)$ if $a + b = g$. Before considering that, the reason for the inequality bounding $R_k(d^+)$ is that $\min\{k, d_i^+\}$ is taken before contributing to R_k. For $R_k(d^-)$, the contribution from each index counted by t' decreases by $g - b$, and an entry with $i > k$ and $d_i \ge k$ may contribute less to $R_k(d^-)$ than to $R_k(d)$. For the exceptions, if $a + b > g$, then $R_k(d^+)$ is larger by $a + b - g$, which is at most a. If $a + b < g$, then $R_k(d^-)$ is larger by $g - a - b$, which is less than $g - a$.

These computations and the feasibility of d^+ and d^- yield two upper bounds on $L_k(d)$:

$$\begin{array}{ccccccc} L_k(d) & = & L_k(d^+) - ta & \le & R_k(d^+) - ta & \le & R_k(d) + (t' - t + \epsilon^+)a; \\ L_k(d) & \le & L_k(d^-) + t(g-a) & \le & R_k(d^-) + t(g-a) & \le & R_k(d) + (t - t' + \epsilon^-)(g-a). \end{array}$$

Both inequalities bound $L_k(d)$, and always one of the additive constants is nonpositive, since only one of $\{\epsilon^+, \epsilon^-\}$ can be positive. Hence $L_k(d) \le R_k(d)$, and we conclude that d is feasible.

Now suppose that every internal gap of d is g except for one $g - 1$ at position j. Recall that $\hat{d}^+$ is formed from d by adding 1 to each d_i such that $i \le j$ and $r > d_i$. Let us call the process in Definition 2.2.3 that produces a g-uniform list by augmenting some entries by a (and maybe augmenting the first s by $a+b-g$) the *augmentation procedure*. By definition, we obtain d^+ from $\hat{d}^+$ using the augmentation procedure. We note that the *same* list results from $\hat{d}^-$ via the augmentation procedure. Similarly, the same list d^- is obtained from both $\hat{d}^-$ and $\hat{d}^+$. Since we are given that d^+ and d^- are feasible, the preceding argument implies that $\hat{d}^+$ and $\hat{d}^-$ are feasible.

We now compare d with $\hat{d}^+$ and $\hat{d}^-$ to show that d is feasible. If $k \le j$, then $L_k(d) = L_k(\hat{d}^-) \le R_k(\hat{d}^-) \le R_k(d)$. If $k \ge j$, then $R_k(d) = R_k(\hat{d}^+) \ge L_k(\hat{d}^+) \ge L_k(d)$. Thus $L_k(d) \le R_k(d)$ for all k, as desired. □

Although d^+ and d^- generally have different sum from d, they have the same length, maximum, minimum, and bound on gaps. Hence each satisfies the desired length threshold if and only if d does. We conclude that if satisfying the length threshold suffices to make a g-uniform list feasible, then it is also sufficient in the larger family $D_n(r, s, g)$.

2.3 Lists without Internal Repetitions

To simplify our study of g-uniform lists, we reduce the problem of checking feasibility to checking the $\ell(d)$th E-G inequality, where $\ell(d) = \max\{k\colon\, d_k \ge k\}$. In fact, we prove that for every list having at most one internal gap equal to 0, it suffices to check only the $\ell(d)$th E-G inequality. Furthermore, this result is sharp.

Example 2.3.1. For $j \ge 3$, let $d = (2j, 2j-1, \ldots, j+1, j, j, j, 1^{(j[j-3]/2)})$. The initial portion is a strictly decreasing list of j terms before the double repetition. Thus $d_j = j+1$ and $d_{j+1} = j$, so $\ell(d) = j$. We have $L_j(d) = j(3j+1)/2$ and $R_j(d) = j(j-1) + 3j + j(j-3)/2$; equality holds. Nevertheless, $L_{j-1}(d) = (3j^2 - j - 2)/2$ and $R_{j-1}(d) = (j-1)(j-2) + 4(j-1) + j(j-3)/2 = (3j^2 - j - 2)/2 - 1$, so the list is not feasible.

Similarly, for $j \geq 5$, let $d = (2j-1, \ldots, j+1, j, j, j-1, j-1, 1^{(j[j-5]/2+2)})$. Now there are $j-1$ terms before the first repetition, so again $\ell(d) = j$. Now $L_j(d) = j(3j-1)/2$ and $R_j(d) = j(j-1) + 3j - 2 + j(j-5)/2 + 2$; equality holds. However, $L_{j-2}(d) = R_{j-2}(d) + 1$, so the list is not feasible. □

Recall that in general d is feasible if and only if $L_k(d) \leq R_k(d)$ for $1 \leq k \leq \ell(d)$ [43].

Theorem 2.3.2. *Let d be a nonincreasing integer list of length n with largest entry r and smallest entry s, such that $n > r$ and $d_{i+1} = d_i$ for at most one index with $d_i \notin \{r, s\}$. If $L_{\ell(d)} \leq R_{\ell(d)}$, then d is feasible.*

Proof. We reduce $L_k(d) \leq R_k(d)$ for $k \leq \ell(d)$ to $L_{\ell(d)} \leq R_{\ell(d)}$. If $k \leq s$, then $L_k(d) \leq kr \leq k(n-1) = k(k-1) + (n-k)k = R_k(d)$, since $\min\{k, d_i\} = k$ for all i. Hence it suffices to show that $L_{k+1}(d) \leq R_{k+1}(d)$ implies $L_k(d) \leq R_k(d)$ for k with $s < k < \ell(d)$.

Let j be the last index such that $d_j > k$; since $k < \ell(d)$, we have $d_{k+1} > k$, and hence $j > k$. If $j > r$, then we use $\min\{d_i, k\} = k$ for $i \leq j$ to compute

$$L_k(d) \leq kr \leq k(j-1) = k(k-1) + k(j-k) \leq R_k(d).$$

Now consider $j \leq r$. By the choice of j, $\sum_{i=k+2}^{n} \min\{k+1, d_i\} = j-k-1+\sum_{i=k+2}^{n} \min\{k, d_i\}$. Thus

$$R_k(d) = R_{k+1}(d) - 2k + \min\{k, d_{k+1}\} - (j-k-1),$$

which simplifies to $R_k(d) = R_{k+1}(d) - j + 1$. Therefore,

$$L_k(d) = L_{k+1}(d) - d_{k+1} \leq R_{k+1}(d) - d_{k+1} = R_k(d) + j - d_{k+1} - 1.$$

If $d_{k+1} = r$, then we have $L_k(d) \leq R_k(d)$ since $j \leq r$. If $d_{k+1} < r$, then since $d_j > k > s$, the gaps from $k+1$ through $j-1$ are nonzero, except possibly for one. Hence $d_{k+1} - d_j \geq j - (k+1) - 1$, and thus $j - 1 - d_{k+1} \leq k + 1 - d_j \leq 0$, which yields $L_k(d) \leq R_k(d)$. □

2.4 The Length Threshold

In Section 2.2, we reduced feasibility of the maximal element in $P_{m,n,r,s,g}$ to showing that two "nearby" g-uniform lists having the same length but different sum are feasible. Proving feasibility for g-uniform lists in $D_n(r, s, g)$ implies that all lists in $D_n(r, s, g)$ are feasible. Since g-uniform lists have no internal repetitions, Theorem 2.3.2 implies that for feasibility of a g-uniform list d, it suffices to check only the $\ell(d)$th E-G inequality. In this section, we obtain a sharp threshold $h(r, s, g)$ such that if $n \geq h(r, s, g)$, then the $\ell(d)$th inequality for a g-uniform list d in $D_n(r, s, g)$ does hold.

Working backward from the first copy of s, the number of steps to reach the last copy of r in a g-uniform list is $\lceil (r - s)/g \rceil$. With $x + 1$ being the number of copies of r and y being the number of copies of s, we thus have $n = x + y + z$, where $z = \lceil (r - s)/g \rceil$.

To start the proof, we eliminate easy cases for the value of $\ell(d)$. Recall that the reduction to the $\ell(d)$th inequality (Theorem 2.3.2) requires $n > r$, which is equivalent to the condition $L_1(d) \leq R_1(d)$.

Lemma 2.4.1. *For $n > r$, if $d \in D_n(r, s, g)$, then d is feasible unless $x < \ell(d) \leq n - y$.*

Proof. If $x \geq r$, then $d_{r+1} = r$ and $\ell(d) = r$. Since it suffices to prove the kth inequality, where $k \leq \ell(d) = r$, we have $L_k(d) = kr = k(k-1)+(r+1-k)k \leq k(k-1)+(n-k)k \leq R_k(d)$, since $\min\{k, r\} = k$. Hence d is feasible unless $x < r$. In this case $d_{x+1} = r > x$, which yields $\ell(d) > x$.

As we remarked in proving Theorem 2.3.2, the kth E-G inequality holds whenever $k \leq s$ (if $n > r$). Hence we have feasibility unless $\ell(d) > s$, which requires $d_{s+1} > s$. Hence the number of copies of s is less than $n - s$; that is, $y < n - s$. Since $d_{n-y+1} = s < n - y + 1$, we have $\ell(d) \leq n - y$. □

We remark that the conditions of Lemma 2.4.1 cannot be weakened when $n = r + 1$ and $r > s$, since the lists $(r^{(r)}, r - 1)$ and $(r^{(s+1)}, s^{(r-s)})$ are not feasible. We can now obtain the length threshold for feasibility.

Theorem 2.4.2. *Given $r, s, g \in \mathbb{N}$ with $r > s$, let $z = \lceil (r-s)/g \rceil$ and $b = r - s - g(z-1)$. If $n \geq h(r, s, g)$, then every list in $D_n(r, s, g)$ is feasible, where*

$$h(r, s, g) = \frac{1}{s}\left(\left\lfloor \frac{(r+s)^2}{4} \right\rfloor + \left\lceil \frac{r+s}{2} \right\rceil - b\left\lfloor \frac{z}{2} \right\rfloor + \frac{gz}{2} - \frac{g}{2}\left\lceil \frac{z^2}{2} \right\rceil\right).$$

Furthermore, the bound is sharp; $D_n(r, s, g)$ has an infeasible list when $n = \lceil h(r, s, g) \rceil - 1$.

Proof. By Lemma 2.2.4, it suffices to determine the threshold on n so that the g-uniform lists in $D_n(r, s, g)$ are feasible. There are z nonzero gaps, and only the last can fail to be g; it equals b. For g-uniform lists, we have reduced the checking of feasibility to checking the $\ell(d)$th E-G inequality (by Theorem 2.3.2). By Lemma 2.4.1, we may assume that $x < \ell(d) \leq n - y$, where d has $x+1$ copies of r and y copies of s.

Given the parameters n, r, s, g, a g-uniform list in $D_n(r, s, g)$ is completely determined by specifying x; hence specifying x also determines $\ell(d)$. We henceforth abbreviate $\ell(d)$ to ℓ and think of ℓ, L_ℓ, and R_ℓ as functions of x. Our proof is in three steps: we find the value of ℓ such that the ℓth E-G inequality is hardest to satisfy (meaning that if that one holds then they all hold), determine the value of x that yields that value of ℓ, and finally determine the threshold length where that inequality holds.

To facilitate the explanation of the argument, we illustrate the critical situation in Figure 2.4; the height of the ith column is d_i. The data is $(r, s, g) = (19, 3, 5)$, which produces $(z, b) = (4, 1)$. The critical choices are $(\ell, x) = (12, 10)$, and the threshold for n is 26.

Step 1: *For each n, the inequality $L_\ell \leq R_\ell$ is hardest to satisfy when $\ell = \lceil (r+s)/2 \rceil$.* As noted, Lemma 2.4.1 allows us to assume that $x < \ell \leq n - y$ (ignore for now that ℓ is drawn as $x + \lceil z/2 \rceil$ in Figure 2.4). By the definition of ℓ, we have $d_i \geq \ell$ for $i \leq \ell$ and $d_i \leq \ell$ for $i > \ell$. Thus L_ℓ is the area of the diagram in and above the gray box, while R_ℓ is the area of the diagram in and to the right of the gray box. The list is determined by choosing

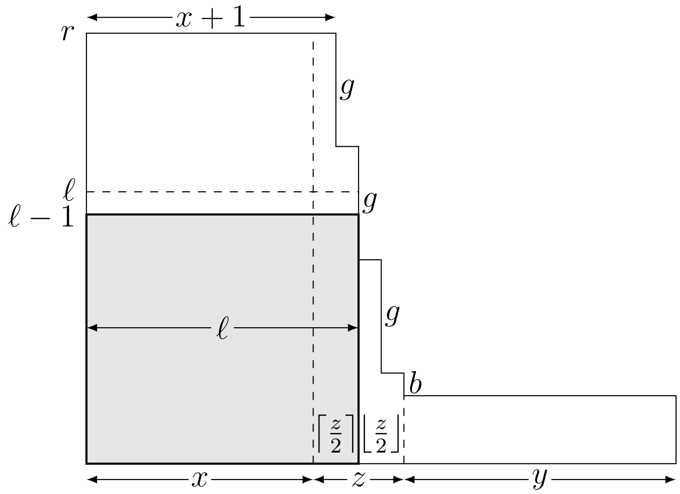


Figure 2.4: A g-uniform list (at the threshold)

x; changing x moves the staircase and also moves ℓ. We study the change in $R_\ell - L_\ell$ as x changes.

Both L_ℓ and R_ℓ count the area of the gray box; the difference cancels it. The remainder of L_ℓ is an $(r-\ell+1)$-by-ℓ rectangle with an arithmetic sum missing. The remainder of R_ℓ is an y-by-s rectangle plus an arithmetic sum. That is,

$$R_\ell - L_\ell = ys + (x+z-\ell)(s+b) + g\binom{x+z-\ell}{2} - (r-\ell+1)\ell + g\binom{\ell-x}{2}. \qquad (2.1)$$

To prove that $R_\ell - L_\ell$ is minimized when x is chosen to make $\ell = \lceil (r+s)/2 \rceil$, we begin with a formula for ℓ in terms of x. Under the condition $x < \ell \le n-y$, the value of ℓ is the largest i such that $r - g(i-x-1) \ge i$. This simplifies to $i \le \frac{r+g(x+1)}{g+1}$, and thus $\ell = \left\lfloor \frac{r+g(x+1)}{g+1} \right\rfloor$. Note that when x increases by 1, usually ℓ increases by 1, but when $r + g(x+1) \equiv 0 \mod (g+1)$, the value of ℓ is the same for x and $x+1$.

When increasing x by 1 also increases ℓ, the only terms in the formula in (2.1) that change are $-(r-\ell+1)\ell$ and ys (since n and z are fixed, $x+y$ is fixed). Hence the change is $2\ell - r - s$.

When $\ell(x+1) = \ell(x)$, the other terms change. Since $x + z - \ell$ increases and $\ell - x$ decreases, the change is $-s + (s + b) + g(x + z - \ell) - g(\ell - x - 1)$, which simplifies to $b + g(z + 2x - 2\ell + 1)$. Since $zg = r - s - b + g$, we can rewrite this as $r - s + 2g(x - \ell + 1)$. The condition $\ell(x+1) = \ell(x)$ occurs when $r + g(x+1) \equiv 0 \mod (g+1)$, so $\ell = \frac{r+g(x+1)}{g+1}$, and $x + 1 - \ell = (\ell - r)/g$. Thus again the change is $2\ell - r - s$.

When x is small, $R_\ell - L_\ell$ decreases as x increases until ℓ reaches $\lceil (r+s)/2 \rceil$. Thereafter, $R_\ell - L_\ell$ increases as x continues to increase. Hence for fixed n all g-uniform lists are feasible if and only if the list obtained by choosing x to produce $\ell = \lceil (r+s)/2 \rceil$ is feasible.

Step 2: *Setting* $x = \lceil (r+s)/2 \rceil - \lceil z/2 \rceil$ *yields* $\ell(x) = \lceil (r+s)/2 \rceil$. Since ℓ increases by 0 or 1 as x increases, some choice of x produces $\ell(x) = \lceil (r+s)/2 \rceil$. Let $\lambda = \lceil (r+s)/2 \rceil$, and set $x = \lambda - \lceil z/2 \rceil$. We show that $d_\lambda \geq \lambda$ and $d_{\lambda+1} < \lambda + 1$ for this choice of x. Recall that $d_i = r - g(i - x - 1)$ for $x < i \leq x + z$. Since $\lambda = x + \lceil z/2 \rceil$ and $gz = r - s + g - b$,

$$\begin{aligned} d_\lambda &= r - g(\lceil z/2 \rceil - 1) \geq r - g(z-1)/2 \\ &= r - (r - s + g - b)/2 + g/2 = (r+s)/2 + b/2 \geq \lambda. \end{aligned}$$

Similarly, if $\lambda + 1 \leq x + z$, then

$$\begin{aligned} d_{\lambda+1} &= r - g \lceil z/2 \rceil \leq r - gz/2 \\ &= r - (r - s + g - b)/2 = (r+s)/2 - (g-b)/2 < \lambda + 1. \end{aligned}$$

The inequality $\lambda + 1 \leq x + z$ fails if and only if $z = 1$. In this case, $r - s = g$ and $\lambda = x + 1$. Since $r > s$, we have $x = \lambda - 1 = \lceil (r+s)/2 \rceil - 1 \geq s$. Hence $d_{\lambda+1} = s \leq x < x + 2 = \lambda + 1$. In both cases, we obtain $d_{\lambda+1} < \lambda + 1$. We have therefore shown that $\ell = \lambda = \lceil (r+s)/2 \rceil$.

Step 3: $n \geq h(r,s,g)$ *is sufficient for feasibility.* Having reduced the problem to studying the unique g-uniform list of length n with $x = \lceil (r+s)/2 \rceil - \lceil z/2 \rceil$ and $\ell = \lceil (r+s)/2 \rceil$ (see

Figure 2.4), it suffices to determine the threshold on n such that $R_\ell - L_\ell \geq 0$. We simplify (2.1) using $y = n - x - z = n - \ell + \lfloor z/2 \rfloor$, $r - \ell = \lfloor (r-s)/2 \rfloor$, and $\ell - x = \lceil z/2 \rceil$ to obtain

$$\begin{aligned} R_\ell - L_\ell &= (n - \ell - \left\lfloor \frac{z}{2} \right\rfloor)s + \left\lfloor \frac{z}{2} \right\rfloor (s+b) + g\binom{\lfloor z/2 \rfloor}{2} - \left(\left\lfloor \frac{r-s}{2} \right\rfloor + 1\right)\left\lceil \frac{r+s}{2} \right\rceil + g\binom{\lceil z/2 \rceil}{2} \\ &= ns - \left\lceil \frac{r+s}{2} \right\rceil \left(\left\lfloor \frac{r+s}{2} \right\rfloor + 1\right) + b\left\lfloor \frac{z}{2} \right\rfloor - \frac{gz}{2} + \frac{g}{2}\left(\left\lfloor \frac{z}{2} \right\rfloor \left\lfloor \frac{z}{2} \right\rfloor + \left\lceil \frac{z}{2} \right\rceil \left\lceil \frac{z}{2} \right\rceil\right) \\ &= ns - \left\lfloor \frac{(r+s)^2}{4} \right\rfloor - \left\lceil \frac{r+s}{2} \right\rceil + b\left\lfloor \frac{z}{2} \right\rfloor - \frac{gz}{2} + \frac{g}{2}\left\lceil \frac{z^2}{2} \right\rceil \qquad = ns - sh(r,s,g). \end{aligned}$$

We conclude that if $n \geq h(r,s,g)$, then every list in $D_n(r,s,g)$ is feasible.

Step 4: *The bound is sharp; that is, when $n = \lceil h(r,s,g) \rceil$ there is an infeasible list in $D_{n-1}(r,s,g)$.* Since $z = \lceil (r-s)/g \rceil$, always $z \leq r - s$. With x set to $\lceil (r+s)/2 \rceil - \lceil z/2 \rceil$, we thus have $x + z = \lceil (r+s)/2 \rceil + \lfloor z/2 \rfloor \leq r$.

Since every list in $D_n(r,s,g)$ is feasible when $n = \lceil h(r,s,g) \rceil$, the first E-G inequality requires $\lceil h(r,s,g) \rceil \geq r + 1$. Furthermore, all lists with length at most r are infeasible. Therefore, we may assume that $\lceil h(r,s,g) \rceil \geq r + 2$. Since $x + z \leq r$, the key g-uniform list at the threshold length has at least two copies of s. Hence the list obtained by deleting one copy of s belongs to $D_{n-1}(r,s,g)$ and is infeasible. □

The expression for the threshold length simplifies when $g \mid (r-s)$, in which case $b = g$.

Corollary 2.4.3. *For $g \mid (r-s)$, the threshold length for feasibility of all lists in $D_n(r,s,g)$ is*

$$\frac{1}{s}\left(\left\lfloor \frac{(r+s)^2}{4} \right\rfloor + \left\lceil \frac{r+s}{2} \right\rceil - \frac{g}{2}\left\lfloor \frac{(r-s)^2}{2g^2} \right\rfloor\right).$$

When $g = 1$, this simplifies to $r + \frac{r+s+\epsilon}{2s}$, with $\epsilon = 0$ for even $r+s$ and $\epsilon = 1$ for odd $r+s$. If $g = r - s$ (and hence there is no gap constraint), then it simplifies to $\frac{(r+s+1)^2 - \epsilon'}{4s}$, where $\epsilon' = 0$ if $r+s$ is odd and $\epsilon' = 1$ if $r+s$ is even. Furthermore, the thresholds are sharp.

Proof. For the first statement, set $b = g$ and $z = (r-s)/g$. For the second and third, set

$g = 1$ or $z = 1$. Sharpness was proved in greater generality in Theorem 2.4.2. □

Finally, we return to our motivating question about the threshold length for even-summed lists to be graphic. It may happen that our infeasible list with $n = \lceil h(r, s, g)\rceil - 1$ has odd sum. If all infeasible lists with that length have odd sum, then the threshold will be $\lceil h(r, s, g)\rceil - 1$. Before exhibiting a family where this occurs, we show that the threshold never declines by more than 1.

Theorem 2.4.4. *For all r, s, g, the least n such that all even-summed lists in $D_n(r, s, g)$ are graphic is $\lceil h(r, s, g)\rceil$ or $\lceil h(r, s, g)\rceil - 1$.*

Proof. Let $m = \lceil h(r, s, g)\rceil$. Since every list of length r fails the first E-G inequality, having length $m - 2$ be sufficient for even-summed lists to be graphic requires $\lceil h(r, s, g)\rceil \geq r + 3$. Since we have noted that $x + z \leq r$ at the key value of x, the key g-uniform list with length m has at least three copies of s.

Deleting one copy of s yields an infeasible list d in $D_{m-1}(r, s, g)$, meaning that $L_\ell(d) - R_\ell(d) \geq 1$. Deleting another copy of s still retains a copy of s and hence yields an infeasible list d' in $D_{m-2}(r, s, g)$ with $L_\ell(d') - R_\ell(d') \geq s + 1 \geq 2$.

Since $z = \lceil (r - s)/g\rceil \leq r - s$, we have $x = \lceil (r + s)/2\rceil - \lceil z/2\rceil \geq \lceil (r + s)/2\rceil - \lceil (r - s)/2\rceil = s$. Hence there are more than s copies of r in d'. If d' has even sum, then d' is the desired infeasible list. If d' has odd sum, then we decrease the last copy of r by 1 to obtain a infeasible list with length $\lceil h(r, s, g)\rceil - 2$ that has even sum. □

Determining when the threshold length for feasibility of even-summed lists in $D_n(r, s, g)$ equals $\lceil h(r, s, g)\rceil$ is messy, because attention must be paid to the exact value of $L_\ell - R_\ell$ at the key value of x. In lieu of discussing that, we close with an example of a family where the length threshold for even-summed lists to be graphic is less than the threshold for feasibility. We will use the Havel–Hakimi Theorem [21, 19], which states that a list d with even sum is graphic if and only if the list d' obtained from it by deleting a largest entry k and subtracting 1 from k largest remaining entries is graphic. Call that operation a *Havel–Hakimi step.*

Theorem 2.4.5. *For $n \geq \lceil h(r,1,1) \rceil - 1$, every even-summed list in $D_n(r,1,1)$ is graphic.*

Proof. From Corollary 2.4.3, $h(r,1,1) = \lfloor 3r/2 \rfloor + 1$. Nevertheless, we prove that already length $\lfloor 3r/2 \rfloor$ is sufficient. (Theorem 2.4.4 implies that no smaller length is sufficient.)

Consider $d \in D_n(r,1,1)$ with even sum. We use induction on r. For $r \leq 2$, lists with even sum and length at least $\lfloor 3r/2 \rfloor$ are graphic. Now consider $r > 2$. Let d' be the list obtained by applying a Havel-Hakimi step to d. Note that $d'_1 \in \{r, r-1, r-2\}$ and that d' is gap-free. If $d'_1 = r$, then $d_{r+1} = r$ and the first computation in Lemma 2.4.1 shows that d is graphic. Note that d has at least $r-2$ distinct values between r and 1. Hence if $d'_1 = r-2$, then $d_2 = r-1$ and $d' \in D_{n'}(r-2,1,1)$ with $n' \in \{n-1, n-2, n-3\}$. Since $n' \geq n-3 \geq \lfloor \frac{3r}{2} \rfloor - 3 = \left\lfloor \frac{3(r-2)}{2} \right\rfloor$, the induction hypothesis implies that d' is graphic.

Now suppose $d'_1 = r-1$, so $d_2 = r$ or $d_{r+2} = r-1$. If $d_{r+2} = r-1$, then $d' \in D_{n-1}(r-1,1,1)$. By the induction hypothesis, d' is graphic.

Finally, suppose $d'_1 = r-1$ and $d_2 = r$. Since d is gap-free, $d_r \geq 2$. Hence $d' \in D_{n'}(r-1,1,1)$ with $n' \in \{n-1, n-2\}$. If $n' = n-1$, then $n' = n-1 \geq \lfloor \frac{3r}{2} \rfloor - 1 \geq \left\lfloor \frac{3(r-1)}{2} \right\rfloor$, and the induction hypothesis applies. If $n' = n-2$, then $d_r = 2$ and $d_{r+1} = 1$, and hence $d = (r, r, r-1, ..., 2, 1, 1^{(\lfloor r/2 \rfloor - 1)})$. If r is odd, then the degree sum of d is $2r + \binom{r}{2} + (r-1)/2 - 1$, which is odd, so this case does not occur. If r is even, then

$$n' = n-2 \geq \left\lfloor \frac{3r}{2} \right\rfloor - 2 = \left\lfloor \frac{3(r-1)-1}{2} \right\rfloor = \left\lfloor \frac{3(r-1)}{2} \right\rfloor,$$

and again the induction hypothesis applies. □

2.5 A Problem on Outerplanar Graphs

In the rest of Chapter 2, we study another extremal problem about vertex degrees involving outerplanar graphs. For $n > k > 0$, Erdős and Griggs [9] asked for the minimum, over n-vertex planar graphs, of the number of vertices with degree less than k. For $k \leq 6$, the optimal values follow from results of Grünbaum and Motzkin [17]. West and Will [40] determined the optimal values for $k \geq 12$, obtained the best lower bounds for $7 \leq k \leq 11$, and provided constructions achieving those bounds for infinitely many n when $7 \leq k \leq 10$. Griggs and Lin [15] independently found the same lower bounds for $7 \leq k \leq 10$ and gave constructions achieving the lower bounds when $7 \leq k \leq 11$ for *all* sufficiently large n.

We study the analogous question for outerplanar graphs, expressed in terms of large-degree vertices. Let $\beta_k(n)$ be the maximum, over n-vertex outerplanar graphs, of the number of vertices having degree at least k. For $k \leq 2$, the problem is trivial; $\beta_k(n) = n$, achieved by a cycle (or by any maximal outerplanar graph).

When $k \in \{3, 4\}$, the square of a path shows that $\beta_3(n) \geq n - 2$ and $\beta_4(n) \geq n - 4$. Since every outerplanar graph with $n \geq 2$ has at least two vertices of degree at most 2, $\beta_3(n) = n - 2$. We will prove $\beta_4(n) = n - 4$ when $n \geq 7$ (Theorem 2.6.6). For $k = 5$ and $n \geq 4$, we prove $\beta_k(n) = \lfloor 2(n-4)/3 \rfloor$, except one less when $n \equiv 1 \mod 6$ (Theorem 2.6.5). For $k \geq 6$ and $n \geq k + 2$, we prove $\beta_k(n) = \lfloor (n-6)/(k-4) \rfloor$ (Theorem 2.7.4).

We close this introduction with a general upper bound that is optimal for $k = 5$ when $n \not\equiv 1 \mod 6$. In Section 2.6.5 we improve the upper bound by 1 when $n \equiv 1 \mod 6$ and provide the general construction that meets the bound; these ideas also give the upper bound for $k = 4$. In Section 2.7 we solve the problem for $k \geq 6$. The bounds in [40] were obtained by first solving a related problem, which here corresponds to maximizing the sum of the degrees of the vertices with degree at least k. We use this approach in Section 2.7 to prove the upper bound on $\beta_k(n)$ when $k \geq 6$.

Adding edges does not decrease the number of vertices with degree at least k, so an n-vertex outerplanar graph with $\beta_k(n)$ vertices of degree at least k must be a maximal outerplanar graph, which we abbreviate to *MOP*. For a MOP with n vertices, let β be the number of vertices having degree at least k, and let n_2 be the number of vertices having degree 2. A MOP with n vertices has $2n-3$ edges, so summing the vertex degrees yields

$$2n_2 + 3(n - n_2 - \beta) + k\beta \leq 4n - 6. \tag{2.2}$$

This inequality simplifies to $(k-3)\beta \leq n + n_2 - 6$. Using $n_2 \leq n - \beta$ then yields $\beta_k(n) \leq \lfloor 2(n-3)/(k-2) \rfloor$. To improve the bound, we need a structural lemma.

Lemma 2.5.1. *Let G be an n-vertex MOP with external cycle C. If $n \geq 4$, then G has two vertices with degree in $\{3, 4\}$ that are not consecutive along C.*

Proof. We use induction on n. Note that G contains $n-3$ chords of C. If every chord lies in a triangle with two external edges, then $n \leq 6$ and $\Delta(G) \leq 4$, and the two neighbors of a vertex of degree 2 are the desired vertices. This case includes the MOPs for $n \in \{4, 5\}$.

Otherwise, a chord xy not in a triangle with two external edges splits G into two MOPs with at least four vertices, each with x and y consecutive along its external cycle. By the induction hypothesis, each has a vertex with degree 3 or 4 outside $\{x, y\}$. In G, those two vertices retain their degrees, and they are separated along C by x and y. □

Corollary 2.5.2. *If $k \geq 5$ and $n \geq 4$, then $\beta_k(n) \leq \lfloor 2(n-4)/(k-2) \rfloor$.*

Proof. Lemma 2.5.1 yields $n - n_2 - \beta \geq 2$, and hence $n_2 \leq n - \beta - 2$. Substituting this improved inequality into the inequality $(k-3)\beta \leq n + n_2 - 6$ that follows from (1) yields $\beta_k(n) \leq 2(n-4)/(k-2)$. □

Corollary 2.5.2 gives us a target to aim for in the construction for $k = 5$. For $k \geq 6$ we will need further improvement of the upper bound. The argument of Corollary 2.5.2 is not valid for $k = 4$, since vertices of degree 4 are counted by β. The upper bound $\beta_4(n) \le n - 4$ (for $n \ge 7$) will come as a byproduct of ideas in the next section.

2.6 The Solution for $k = 5$

We begin with the construction. Let $\langle v_1, \ldots, v_k \rangle$ and $[v_1, \ldots, v_k]$ denote a path and a cycle with vertices $v_1, \ldots, v_k$ in order, respectively. Let B be the graph formed from the cycle $[v, u, x, w, y, z]$ by adding the path $\langle u, w, v, y \rangle$. (see Figure 2.5). The reason for naming the vertices in this way is that we will create copies of B in a large graph by adding the vertices in the order u, v, w, x, y, z.

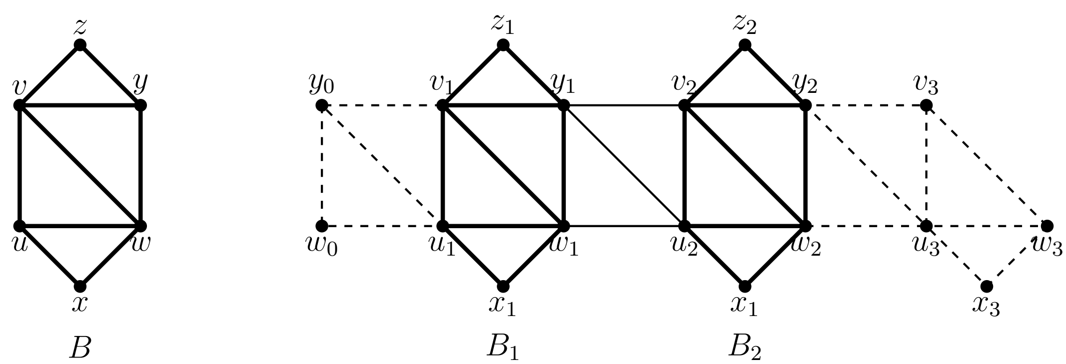


Figure 2.5: The graph B and its use in constructing F.

To facilitate discussion, define a *j-vertex* to be a vertex of degree j, and define a *j^+-vertex* to be a vertex of degree at least j.

Lemma 2.6.1. *If $n \ge 4$, then* $\beta_5(n) \ge \begin{cases} \lfloor 2(n-5)/3 \rfloor & \text{if } n \equiv 1 \mod 6, \\ \lfloor 2(n-4)/3 \rfloor & \text{otherwise.} \end{cases}$

Proof. Begin at $n = 2$ with one edge having endpoints w_0 and y_0. Add vertices one by one, always adding a vertex adjacent to two earlier neighboring external vertices; the result is always a MOP. For $6q - 3 \le n \le 6q + 2$, add the vertices $u_q, v_q, w_q, x_q, y_q, z_q$ in order. The added vertex is the center of a 3-vertex path; the paths added are successively $\langle w_{q-1}, u_q, y_{q-1} \rangle$,

$\langle y_{q-1}, v_q, u_q\rangle$, $\langle u_q, w_q, v_q\rangle$, $\langle u_q, x_q, w_q\rangle$, $\langle v_q, y_q, w_q\rangle$, and $\langle v_q, z_q, y_q\rangle$. After z_q is added, the subgraph induced by $\{u_q, v_q, w_q, x_q, y_q, z_q\}$ is isomorphic to B; call it B_q (see Figure 2.5).

When $n = 6$, the addition of x_1 augments u_1 to degree 5 (the first such vertex). The second occurs at v_1 when z_1 is added to reach eight vertices. This agrees with the claimed values for n from 4 through 8. Subsequently, addition of u_q, v_q, x_q, z_q raises $w_{q-1}, y_{q-1}, u_q, v_q$, respectively, to degree 5. Addition of w_q and y_q does not introduce a 5-vertex, and x_q and z_q never exceed degree 2.

When $n = 6 \cdot 2 - 4$, we have two 5-vertices; note that $2 = \lfloor 2(8-4)/3 \rfloor$. For $6q - 3 \leq n \leq 6q + 2$, the values required by the stated formula for the number of 5^+-vertices are $4q-5, 4q-4, 4q-4, 4q-3, 4q-3, 4q-2$, respectively. The induction hypothesis for an induction on q states that when $n = 6q - 4 = 6(q-1) + 2$, the graph has $4(q-1) - 2$ vertices of degree 5. Starting from this point, we augmented one vertex to degree 5 when n is congruent to each of $\{-3, -2, 0, 2\}$ modulo 6, matching the formula. □

The lower bound in Lemma 2.6.1 and upper bound in Corollary 2.5.2 are equal except when $n \equiv 1 \mod 6$. In this case we improve the upper bound by showing that there is no outerplanar graph having the vertex degrees required to achieve equality in the upper bound. The construction shows $\beta_5(n) \geq \lfloor 2(n-5)/3 \rfloor$. We used the existence of two vertices with degree 3 or 4 to improve the upper bound from $\lfloor 2(n-3)/3 \rfloor$ to $\lfloor 2(n-4)/3 \rfloor$, which differs from $\lfloor 2(n-5)/3 \rfloor$ by 1 when $n \equiv 1 \mod 6$. We begin by showing that slightly stronger hypotheses further reduce the bound.

Lemma 2.6.2. *If G is a MOP having a 6^+-vertex, or a 3-vertex and a 4-vertex, or two 4-vertices, or at least three 3-vertices, then G has at most $\lfloor (2n-9)/3 \rfloor$ vertices of degree at least 5.*

Proof. We have proved that $\beta \leq (2n-8)/3$. If we cannot improve the upper bound to $(2n-9)/3$, then equality must hold in all the inequalities that produced the upper bound

$(2n-8)/3$. Thus $n_2 = n - \beta - 2$, which forbids a third vertex with degree 3 or 4. Also, the sum of the degrees must equal $2n_2 + 3(n - n_2 - \beta) + 5\beta$ (see (1)). This requires that the two vertices of degree 3 or 4 both have degree 3 and that no vertex has degree at least 6. □

Let T be the subgraph of the dual graph of G induced by the vertices corresponding to bounded faces of G; we call T the *dual tree* of G. Since G is a MOP, T is a tree. A triangular face in G having j edges on the external cycle corresponds to a $(3-j)$-vertex in T. The next lemma will reduce the proof of the theorem to the case where T is a special type of tree.

Lemma 2.6.3. *If the neighbor of a leaf t has degree 2 in the dual tree T, then G has a 3-vertex on the triangle corresponding to t. If two leaves in T have a common neighbor, then the common vertex of the corresponding triangles in G is a 4-vertex in G.*

Proof. Let x, y, z be the vertices of the triangle in G corresponding to t, with x having degree 2 in G. The neighboring triangle t' raises the degree of y and z to 3. If t' has degree 2 in T, then in G only one of $\{y, z\}$ can gain another incident edge.

Two leaves in T having a common neighbor $\hat{t}$ correspond to two triangles in G having a common vertex z. The vertex $\hat{t}$ in T corresponds to a triangle in G that shares an edge with each of them. No further edges besides the four in these triangles are incident to z. □

A triangle in a MOP is *internal* if none of its edges lie on the external cycle.

Lemma 2.6.4. *In a MOP with n vertices, let n_2 be the number of 2-vertices and t be the number of internal triangles. If $n \geq 4$, then $t = n_2 - 2$.*

Proof. Let T be the dual tree. Note that T has $n-2$ vertices, of which n_1 have degree 1, t have degree 3, and the rest have degree 2. By counting the edges in terms of degrees, $2(n-3) = n_2 + 3t + 2(n - 2 - n_2 - t)$, which yields $n_2 - 2 = t$. □

Theorem 2.6.5. *If $n \geq 4$, then* $\beta_5(n) = \begin{cases} \lfloor 2(n-5)/3 \rfloor & \textit{if } n \equiv 1 \mod 6, \\ \lfloor 2(n-4)/3 \rfloor & \textit{otherwise.} \end{cases}$

Proof. It suffices to prove the upper bound when $n \equiv 1 \mod 6$. Let $n = 6q + 1$. Corollary 2.5.2 yields $\beta \le 4q-2$, and we want to improve this to $4q-3$, which equals $\lfloor (2n-9)/3 \rfloor$. If the upper bound cannot be improved, then by the computation in the proof of Lemma 2.6.2 we may assume that G has exactly two 3-vertices, exactly $4q - 2$ vertices of degree 5, and exactly $2q + 1$ vertices of degree 2.

A MOP with n vertices has $n - 2$ bounded faces, so T has $n - 2$ vertices. Since G has $2q+1$ vertices of degree 2, there are $2q+1$ leaves in T; by Lemma 2.6.4, T has $2q-1$ vertices of degree 3. The remaining $2q - 1$ vertices of T have degree 2.

A *caterpillar* is a tree such that deleting all the leaves yields a path, called its *spine*. A tree that is not a caterpillar contains as a subtree the graph Y obtained by subdividing each edge of the star $K_{1,3}$. If $Y \subseteq T$, then consider longest paths in T starting from the central vertex v of Y along each of the three incident edges. Each such path reaches a leaf. By Lemma 2.6.3, each such path generates a vertex of degree 3 or 4 in G. Since G has at most two such vertices, T is a caterpillar. Furthermore, since G has no 4-vertices, Lemma 2.6.3 implies that each endpoint of the spine of T has degree 2 in T.

Consider vertices $a, b, c \in V(T)$ such that $ab, bc \in E(T)$. The corresponding three triangles in G have a common vertex x. If a and c are 3-vertices in T, then x has degree 6 in G. Hence no two 3-vertices in T have a common neighbor. This implies that along the spine of T (which starts and ends with 2-vertices), there are at most two consecutive 3-vertices, and non-consecutive 3-vertices are separated by at least two 2-vertices.

In particular, every run of 3-vertices has at most two vertices, every run of 2-vertices has at least two vertices (except possibly the runs at the ends), and the number of runs of 2-vertices is one more than the number of runs of 3-vertices. The only way this can produce the same number of 2-vertices and 3-vertices is $2, 3, 3, 2, 2, 3, 3, \ldots, 2, 2, 3, 3, 2$. However, in this configuration the number of vertices of each type is even and cannot equal $2q - 1$.

We have proved that no outerplanar graph has the required vertex degrees. $\square$

Theorem 2.6.6. *If $n \geq 7$, then $\beta_4(n) = n - 4$.*

Proof. When the dual tree T is a path, the graph G has two 2-vertices, two 3-vertices, and $n-4$ vertices of degree 4; this proves the lower bound. For the upper bound, since leaves of T correspond to triangles in G having 2-vertices, we may assume that T has at most three leaves. If T has only two leaves, then the neighbor of each has degree 2 in T, and Lemma 2.6.3 provides two 3-vertices in G, matching the construction.

If T has exactly three leaves, then T has one 3-vertex and at least four other vertices, since T has $n-2$ vertices and $n \geq 7$. Hence at least one leaf in T has a neighbor of degree 2, and Lemma 2.6.3 provides one 3-vertex in addition to the three 2-vertices in G. (In fact, this case yields another construction having exactly $n-4$ vertices with degree at least 4.) □

2.7 The Solution for $k \geq 6$

Trivially, $\beta_k(n) = 1$ when $k = n - 1$, which does not satisfy the general formula. We restrict our attention to $n \geq k + 2$ and begin with the construction. Fix k with $k \geq 6$.

Form a graph B' from B in Section 2.6 by respectively replacing edges yz and ux with paths P and Q, each having $k-6$ internal vertices. Make v adjacent to all of $V(P)$ and w adjacent to all of $V(Q)$ (see Figure 2.6). Since P and Q have $k-4$ vertices each, B' has $2k-6$ vertices; also, B' is a MOP. Its vertices v and w of maximum degree have degree $k-2$.

Lemma 2.7.1. *If $n \geq k+2$, then $\beta_k(n) \geq \left\lfloor \dfrac{n-6}{k-4} \right\rfloor$.*

Proof. Letting $n = 2(k-4)q + 6 + r$ with $0 \leq r < 2(k-4)$, the claim is equivalent to $\beta_k(n) \geq 2q$ when $0 \leq r < k-4$ and $\beta_k(n) \geq 2q+1$ when $k-4 \leq r < 2(k-4)$.

Let F' be the union of q copies $B'_1, \ldots, B'_q$ of B', modifying the names of vertices in B'_i by adding the subscript i and taking $y_i = v_{i+1}$ and $w_i = u_{i+1}$ for $i \geq 1$ (see the solid graph in Figure 2.6). Note that F' is a MOP with $2 + 2q(k-4)$ vertices, and the $2q-2$ vertices

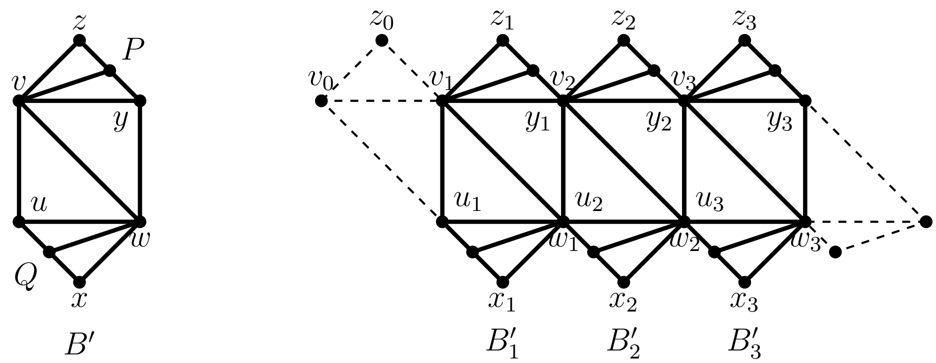


Figure 2.6: The graph B' and its use in constructing F'.

that lie in two copies of B' have degree k. To obtain the lower bound, we will add vertices one by one as in Lemma 2.6.1, but we will start with the appended vertices z_0 and v_0 in order to raise v_1 to degree k quickly.

Each vertex when added will be a 2-vertex appended to an edge of the external cycle, so we always have a MOP. Begin with the triangle on $\{v_1, z_0, v_0\}$. Add u_1, w_1, y_1 and all of P in B'_1 in order. We now have $k + 1$ vertices, and v_1 has degree k. There remains only one k-vertex as we add the rest of Q to complete B'_1, at which point w_1 has degree $k - 2$.

Having completed B'_q, we have $2q(k - 4) + 4$ vertices (including v_0 and z_0), of which $2q-1$ vertices have degree k (including v_1). We show that adding $V(B'_{q+1})$ in the right order produces two more k-vertices at the right times.

With two more vertices, $n = 2q(k-4)+6$ and $r = 0$, and the number of k-vertices should be $2q$. Add w_{q+1} and then the first vertex of Q from B'_{q+1}, raising the degree of w_q to k.

The next k-vertex should arrive when $n = 2q(k-4)+6+(k-4)$, with $r = k-4$. Adding the $k - 4$ vertices from y_{q+1} to z_{q+1} along P in B'_{q+1} raises the degree of y_q from 4 to k. Finally, we add the rest of Q to complete B'_{q+1}. No k-vertex appears, but the degree of w_{q+1} rises to $k - 2$ to be ready for the next iteration. □

In order to prove the upper bound, we consider a related problem. Let $D(n, s)$ be the maximum sum of the degrees of s vertices in an n-vertex outerplanar graph. Of course, these will be s vertices of largest degrees, and the maximum will be achieved by a MOP. We will obtain an upper bound on $D(n, s)$ by considering the structure of the subgraph induced by the vertices with largest degrees. Subsequently, we will apply the bound on $D(n, s)$ to prove that $\beta_k(n) \le \lfloor \frac{n-6}{k-4} \rfloor$ when $k \ge 6$.

Lemma 2.7.2. *Fix s with $1 \le s \le n$. If G is a MOP in which the sum of the degrees of some s vertices is $D(n, s)$, then each set of s vertices with largest degrees in G induces a MOP.*

Proof. Let C be the external cycle in an outerplanar embedding of G, with vertices $v_1, \ldots, v_n$ in order. Let S be a set of s vertices with largest degrees.

We show first that the outer boundary of the subgraph induced by some such set S is a cycle. For $x \in S$, let y be the next vertex of S along C. Let P be the path from x to y along C. If x is not adjacent to y, then let u be the last vertex of P adjacent to x. Since G is a MOP, by the choice of u there is a triangle containing xu whose third vertex is not on P; let z be its third vertex. Let v be the next vertex of P after u. Let U be the set of neighbors of u not on the x, u-subpath of P; note that $z \in U$. Replace the edges from u to U with edges from x to $U \cup \{v\}$. Each edge moved increases the degree of x, and hence the sum of the s largest degrees does not decrease. (If we removed an edge from a neighbor of x that is in S and its degree is no longer among the s largest, then it was replaced by a vertex of the same degree; thus we have increased the sum of the s largest degrees, which contradicts the choice of G.)

The last neighbor of x is now farther along P. When it reaches y the sum of the m largest degrees increases. Since we started with a MOP maximizing this sum, the edge xy must have been present initially.

We have shown that the outer boundary of $G[S]$ is a cycle. Every bounded face is a face of G, since there are no vertices of G inside it. Hence $G[S]$ is a MOP. □

In fact, when $s < n-1$ there is always a unique set of s vertices with largest degrees in a graph maximizing the sum of those degrees.

Theorem 2.7.3. *The maximum value $D(n,s)$ of the sum of s vertex degrees in an n-vertex outerplanar graph is given by*

$$D(n,s) = \begin{cases} n-1 & \text{if } s = 1, \\ n-6+4s & \text{if } s < n/2, \\ 2n-6+2s & \text{if } s \geq n/2. \end{cases}$$

Proof. Let G be a MOP in which some set S of s vertices has degree-sum $D(n,s)$. If $s = 1$, then $n-1$ is clearly an upper bound, achieved by a star. For $s \geq 2$, let G be a MOP achieving the maximum; we know that $G[S]$ is also a MOP and hence has $2s-3$ edges. The question then becomes how the remaining $n-s$ vertices can be added to produce the maximum sum of the degrees in S.

Consider an outerplanar embedding of G. The subgraph induced by S is also an outerplanar embedding of S. Since $G[S]$ has $2s-3$ edges, the outer boundary of the subgraph is a cycle. In the embedding of G, no vertex of $V(G)-S$ appears inside this cycle. Also, vertices outside S can be adjacent to only two vertices of S, and they can be adjacent to two only if those two are consecutive on the outer boundary of $G[S]$. This implies that at most s vertices of $V(G)-S$ can have two neighbors in S, and the rest have at most one neighbor in S. Furthermore, the vertices outside S can be added to achieve this bound.

If $s \geq n/2$, then we add $2(n-s)$ to the degree-sum within $G[S]$, obtaining $D(n,s) = 2n-6+2s$. If $s \leq n/2$, then we add $2s + 1(n-2s)$, obtaining $D(n,s) = n-6+4s$. □

Theorem 2.7.4. *If $k \geq 6$, then $\beta_k(n) \leq \left\lfloor \frac{n-6}{k-4} \right\rfloor$.*

Proof. In an extremal graph, the $\beta_k(n)$ vertices with degree at least k have the largest degrees. With $s = \beta_k(n)$, we have $sk \leq D(n,s)$. Using the bound obtained in Theorem 2.7.3, we have

$$sk \leq \begin{cases} n-6+4s & \text{if } s < n/2, \\ 2n-6+2s & \text{if } s \geq n/2. \end{cases}$$

If $k \geq 6$ and $s \geq n/2$, then

$$6s \leq ks \leq 2n-6+2s \leq 6s-6.$$

Hence $k \geq 6$ implies $s < n/2$, and therefore $ks \leq n-6+4s$, which simplifies to $s \leq \frac{n-6}{k-4}$. □

Finally, we consider the maximum sum of the degrees of the vertices with degree at least k. Essentially, the point is that we cannot increase this sum by using fewer than $\beta_k(n)$ vertices with degrees larger than k.

Corollary 2.7.5. *For $k \geq 6$, the maximum sum of the degrees of the vertices with degree at least k in an n-vertex outerplanar graph is $n-6+4\left\lfloor \frac{n-6}{k-4} \right\rfloor$.*

Proof. Let G be an n-vertex outerplanar graph, and let $S = \{v \in V(G)\colon\ d(v) \geq k\}$. Let $s = |S|$. Since these are the vertices of largest degree, $\sum_{v \in S} d(v) \leq D(n,s)$.

For $k \geq 6$, since $D(n,s)$ is monotone increasing in s, we obtain a bound on the sum by using the bound on $\beta_k(n)$ obtained in Theorem 2.7.4. Since $\beta_k(n) < n/2$, Theorem 2.7.3 yields $\sum_{v \in S} d(v) \leq D(n,s) \leq D(n, \beta_k(n)) = n-6+4\left\lfloor \frac{n-6}{k-4} \right\rfloor$.

We show that a modification of the construction in Lemma 2.7.1 achieves equality in the bound. When $n = (k-4)q' + 6$ for some integer q', let G be the graph constructed in Lemma 2.7.1, having $\beta_k(n)$ vertices of degree at least k; here $\beta_k(n) = q'$. All q' of these vertices have degree exactly k, so the sum of their degrees is $q'k$, equaling the upper bound here. For each increase in n over the next $k-5$ vertices, adding one vertex of degree 2 can increase the degree-sum of these vertices by 1, again equaling the upper bound here. When the $(k-4)$th addition is reached, start over with the construction from Lemma 2.7.1 for the new value of q'. □

Remark 2.7.6. Similar analysis solves the problem of maximizing the sum of the degrees of the vertices with degree at least 5. We remark that the maximum value when $n \not\equiv 1 \mod 6$ is $2n - 8 + 4\beta_k(n)$, which is smaller by 2 than the former in terms of $\beta_k(n)$ when $k \geq 6$. Writing the expression as $2(n - \beta_k(n)) - 2 + (4\beta_k(n) - 6)$, we begin with degree-sum $4\beta_k - 6$ within the MOP H induced by the vertices of degree at least 5. Since $\beta_5(n)$ is roughly $2n/3$, we should be able to augment the sum of the degrees by 2 for each of the remaining $n - \beta_k(n)$ vertices. However, H has at least two vertices of degree 2. For each such vertex, raising its degree to 5 requires one of the added vertices to contribute only one instead of 2. With this adjustment, the improved upper bound meets the construction. When $n \equiv 1 \mod 6$, there are additional technicalities we leave to the reader.

Chapter 3

Decomposition of Graphs into Trees

3.1 Introduction

This chapter is based on joint work with Alexandr Kostochka, and Douglas West, appearing in [23]. Ringel [33] conjectured that for every tree T with m edges, the complete graph K_{2m+1} decomposes into copies of T, meaning that the edges of K_{2m+1} can be partitioned into classes forming copies of T. Such a partition is a *T-decomposition*. Häggkvist [18] conjectured more generally that every $2m$-regular graph has a T-decomposition. Graham and Häggkvist [18] conjectured that every m-regular bipartite graph has a T-decomposition. The restriction to bipartite graphs for T-decomposition of m-regular graphs is due to the elementary observation that an m-regular graph decomposes into copies of $K_{1,m}$ if and only if it is bipartite.

In this chapter we broaden the classes of instances where the conjectures of [18] are known to hold. We begin by reviewing earlier results on these problems.

Theorem 3.1.1. (Snevily [36]) *Let T be a tree with m edges. If G is $2m$-regular and has girth greater than the diameter of T, then G has a T-decomposition.*

Häggkvist [18] stated without proof the stronger result that girth at least $\operatorname{diam} T$ suffices.

Theorem 3.1.2. (Snevily [36]) *If T is a tree with m edges, and G is the cartesian product of m cycles, then G has a T-decomposition.*

Theorem 3.1.3. (Snevily [36]) *If T is a tree with m edges, and G is the cartesian product of a $2l$-cycle and $m-2$ copies of K_2, then G has a T-decomposition.*

The special case of Theorem 3.1.3 with $l = 2$ and $m \geq 2$ is the m-dimensional hypercube; this case was solved earlier by Fink [11]. Fink also showed that the trees in the decomposition could be required to be induced subgraphs.

Our theorem yields various strengthenings of these results.

Corollary 3.1.4. *Let T be an edge-colored tree such that every path P in T uses some color that appears on at most q edges of P. If the color classes have sizes $r_1, \ldots, r_k$, and G is the cartesian product of regular graphs of degrees $2r_1, \ldots, 2r_k$, each having girth greater than q, then G has a T-decomposition.*

For $r_1 = \cdots = r_k = 1$, Corollary 3.1.4 yields Theorem 3.1.2. For $k = 1$ and general $r_1 = m$, it becomes Theorem 3.1.1. In this case ($k = 1$ and no cartesian products), there has been some work on decompositions into special trees.

Theorem 3.1.5 (Kouider and Lonc [29])**.** *For $m \leq 2g - 3$, every $2m$-regular graph with girth at least g decomposes into paths of length m.*

Theorem 3.1.5 strengthens Theorem 3.1.1 for the special case of paths. We will use the case $m = 3$ of their technically stronger version of Theorem 3.1.5 in giving an application of our theorem. Meanwhile, what our Theorem 3.2.1 says for $2m$-regular graphs is the following, which essentially is implicit in Snevily's proof of Theorem 3.1.1. (Neither of Corollary 3.1.6 and Theorem 3.1.5 implies the other.)

Corollary 3.1.6. *Let T be a tree with m edges, and let G be a $2m$-regular graph. If G has a 2-factorization such that every cycle consisting of edges from distinct 2-factors has length greater than the diameter of T, then G has a T-decomposition.*

Finally, for cartesian products of bipartite graphs our theorem yields the following, which becomes Theorem 3.1.3 when $r_1 = 2$ and $r_2 = \cdots = r_k = 1$ and the factors are connected.

Corollary 3.1.7. *Let T be an edge-colored tree such that every path P in T uses some color that appears on at most three edges of P. If the color classes have sizes $r_1, \ldots, r_k$, and G is the cartesian product of regular bipartite graphs of degrees $r_1, \ldots, r_k$, then G has a T-decomposition.*

Snevily [36] proved his results by seeking more structure in the decompositions. He labeled $V(T)$ and required each vertex of G to appear with distinct labels in the copies of T incident to it. Avgustinovich [2] obtained results on decompositions of bipartite graphs into induced copies of T by considering labels on the edges of T. We combine and extend these ideas to give a general sufficient condition in Theorem 3.2.1 for the existence of a T-decomposition of G when G is a $2m$-regular cartesian product of regular graphs with even degree. (There is an analogous result for m-regular cartesian products of regular bipartite graphs, but we leave discussion of that to Section 3.2.)

We employ Avgustinovich's edge-labeling idea in the sense of coloring the edges of T. When G is the cartesian product of $G_1, \ldots, G_k$ and G_i is $2r_i$-regular, with $\sum r_i = m$, we give color i to r_i edges in T. The existence of a suitable edge-coloring guarantees the decomposition. As in Snevily's results, we guarantee a decomposition having a stronger property to facilitate the inductive proof. Each vertex appears in $m + 1$ copies of T, once representing each of the $m + 1$ vertices in a numbering of $V(T)$.

As suggested in Corollary 3.1.6, our general sufficient condition in Theorem 3.2.1 permits more delicate interaction between the edge-coloring of T and chosen 2-factorizations of $G_1, \ldots, G_k$, rather than just imposing girth requirements on $G_1, \ldots, G_k$. Girth requirements are one way to ensure that the hypotheses of Theorem 3.2.1 hold. In Sections 3.3–3.5, we study conditions on r to guarantee that T has an edge-coloring of the type needed to guarantee (via Theorem 3.2.1) that a T-decomposition will exist regardless of the girth or choice of 2-factorizations in $G_1, \ldots, G_k$. To make this precise, we introduce some terminology.

Definition 3.1.8. Given a k-tuple r with sum m, an edge-coloring of a tree with m edges is *r-exact* if it has r_i edges of color i, for $1 \leq i \leq k$. We always index the multiplicities so that $r_1 \leq \cdots \leq r_k$. An edge-coloring of a tree T is *q-good* if every path in T has some color appearing on it that appears at most q times on it (such a path is *q-bounded*).

Corollary 3.1.4 states that if T has a 2-good r-exact edge-coloring, then every product of simple regular graphs with degrees $2r_1, \ldots, 2r_k$ has a T-decomposition. (Similarly, when each G_i is bipartite and r_i-regular, one seeks a 3-good r-exact edge-coloring, since the product has girth at least 4.)

When $r_1 \geq 3$ (and hence $m/k \geq 3$), the path P_m has no 2-good r-exact edge-coloring. Nevertheless, we will study circumstances with $m/k < 4$ under which a coloring that guarantees T-decompositions exists.

A tree T is *special* if it has a vertex x such that every component of $T - x$ has at most two edges. Large special trees are very far from paths. In Section 3.3, we discuss when special trees have 2-good r-exact edge-colorings.

In Section 3.4 we introduce a weaker restriction on edge-colorings. An edge-coloring of T is *weakly 2-good* if every path in T is either 2-bounded or has a color appearing only on a 3-edge subpath whose two internal vertices have degree 2 in T. Using a result of Kouider and Lonc [29] on decomposition of regular graphs, we apply our general condition in Theorem 3.2.1 to prove that if T has a weakly 2-good r-exact edge-coloring, then again every cartesian product of regular graphs with degrees $2r_1, \ldots, 2r_k$ has a T-decomposition.

By using the results on 2-good edge-colorings of special trees, we show that $m/k < 4$ and $r_k \leq \lceil \frac{m+1}{2} \rceil$ together guarantee weakly 2-good r-exact edge-colorings of all trees with m edges. Certain cases in our inductive proof of this result require splitting the list r into two lists with sum r to which the induction hypothesis can be applied. In particular, one needs each list in the split to have sufficiently many nonzero terms. The splittability results are of interest on their own. They are the most difficult technical results of the paper, so we postpone their proofs to Section 3.5.

3.2 The General Decomposition Theorem

Let G be the cartesian product of regular graphs $G_1, \ldots, G_k$. The product decomposes naturally into copies of $G_1, \ldots, G_k$, which yields a natural k-coloring of $E(G)$ by giving color i to the edges whose endpoints differ in the ith coordinate (this *coordinate coloring* forms copies of G_i). To produce a T-decomposition of G, we similarly color $E(T)$ with k colors, and the inductive proof will produce a decomposition in which for each i the edges of color i in each copy of T belong to copies of G_i in the coordinate coloring of G. Thus the sizes $r_1, \ldots, r_k$ of the color classes in T must be proportional to the sizes of $G_1, \ldots, G_k$.

We require further structure for the coloring and the decomposition. Our approach works in two settings: either each G_i is a $2r_i$-regular graph, or each G_i is an r_i-regular bipartite graph. In each case, we use a factorization $\mathbf{F}_i$ of each G_i. In the non-bipartite case, $\mathbf{F}_i$ is a 2-factorization, guaranteed to exist by Petersen's Theorem [32]. In the bipartite case, $\mathbf{F}_i$ is a 1-factorization, guaranteed to exist by the Marriage Theorem of Frobenius and König [28]. In both cases, $\mathbf{F}_i$ consists of r_i factors. Given a one-to-one correspondence between $\mathbf{F}_i$ and the set of edges with color i in T, our T-decomposition of G embeds each edge of T with color i along an edge arising from the corresponding factor in $\mathbf{F}_i$.

Theorem 3.2.1. *Let T be a tree with m edges. Let r be a nondecreasing k-tuple with sum m. Color $E(T)$ so that r_i edges have color i. Let G be the cartesian product of multigraphs $G_1, \ldots, G_k$, where*

Case 1: each G_i is an r_i-regular bipartite multigraph, or

Case 2: each G_i is a $2r_i$-regular multigraph.

In Case j, for each i let $\mathbf{F}_i$ be a j-factorization of G_i, and establish a one-to-one correspondence that pairs each edge of color i in T with one factor in $\mathbf{F}_i$. If every path P in T has an edge of some color i such that G_i has no cycle with edges in distinct $\mathbf{F}_i$-classes all corresponding to edges of P, then G has a T-decomposition.

Proof. The proofs for both Cases are very similar, so we work with "Case j", where $j \in \{1, 2\}$. As described above, the coordinate coloring gives color i to each edge of G whose endpoints differ in coordinate i in the cartesian product. Furthermore, the j-factorizations $\mathbf{F}_1, \ldots, \mathbf{F}_k$ yield a *canonical j-factorization* of G by decomposing each copy of G_i according to $\mathbf{F}_i$ and combining these decompositions. Thus each edge of T corresponds to a j-factor of G.

We prove a stronger result by induction on m. We produce a T-decomposition such that in each copy of T, each edge e is embedded as an edge of the j-factor in G corresponding to e. Furthermore, each vertex of G represents distinct vertices of T in the copies of T using it in the decomposition. More precisely, in Case 2 each vertex of G appears in $m + 1$ copies of T, once as each vertex of T. In Case 1, with T having partite sets X' and Y', and G having partite sets X and Y, each vertex of X appears in $|X'|$ copies of T, once as each vertex of X', and similarly for Y and Y'.

For $m = 1$, the claim is immediate. In Case 1, G consists of isolated edges that can be labeled as desired. In Case 2, follow the cycles in the single 2-factor, labeling each edge in order with the two leaves of T.

For $m > 1$, let u be a leaf of T, with neighbor v, and let $T' = T - u$. By symmetry, we may assume that uv has color k in the coloring of $E(T)$. Let H be the j-factor of G_k in $\mathbf{F}_k$ that corresponds to uv.

Let G' be the graph obtained by deleting $E(H)$ from all copies of G_k in the product. Thus G' is the cartesian product of $G_k - E(H)$ with all of $G_1, \ldots, G_{k-1}$ (when $k = 1$, this degenerates to $G = G_k$ and $G' = G - E(H)$). Since the paths in T' are contained in

T, deleting $E(H)$ leaves j-factorizations that satisfy the hypotheses for G'. Consider the T'-decomposition of G' provided by applying the induction hypothesis to G'.

In Case 1, we may assume by symmetry that $v \in X'$, and for each $w \in X$ we let wy be the edge incident to w in H. In Case 2, for each $w \in V(G)$ we let y be the vertex following w on the cycle through w in H (along a consistent orientation of the cycle).

We extend the copy $\hat{T}$ of T' having v at w by adding the edge wy. To see that y is not already in $\hat{T}$, suppose that it is, and let P be the path from w to y in $\hat{T}$. The edges of a single color i along P correspond to distinct j-factors in $\mathbf{F}_i$. The edge wy in color k corresponds to a different j-factor in $\mathbf{F}_k$ from the others in color k along P. Together, P and wy complete a cycle C in G. If color i appears on C, then C collapses to a nontrivial closed trail in G_i using edges from different j-factors in $\mathbf{F}_i$. This closed trail contains a cycle in G_i through distinct j-factors. This statement holds for every color that appears on P, which contradicts the hypothesis about paths in T.

Hence $y \notin V(\hat{T})$, and the extensions are copies of T. Furthermore, the required stronger statements about the placement of edges and vertices in the decomposition are preserved. □

There is no obvious common generalization of Cases 1 and 2.

Example 3.2.2. If G is the cartesian product of a $2r$-regular graph C and an s-regular bipartite graph B, one would seek a T-decomposition of G, where T has $r+s$ edges. When $C = K_3$ and $B = K_{3,3}$, we have $r = 1$ and $s = 3$, but the product has 45 edges, and 45 is not divisible by 4. □

In the rest of this section, we study paths. We begin with a simple way to guarantee q-good edge-colorings.

Definition 3.2.3. A k-tuple r is *greedily q-good* if $r_i \le q\left(1 + \sum_{j<i} r_j\right)$ for all i.

Corollary 3.2.4. *Let T be a path with m edges, and let r be a k-tuple of positive integers with sum m. Let G be the cartesian product of graphs $G_1, \ldots, G_k$. If each G_i is $2r_i$-regular* *and r is greedily 2-good, then G has a T-decomposition. If each G_i is bipartite and r is greedily 3-good, then G has a T-decomposition.*

Proof. Since a vacuous sum is 0, we have $r_1 \leq 2$ in the first case and $r_1 \leq 3$ in the second.

Consider the first statement. By Corollary 3.1.4, it suffices to partition $E(T)$ into color classes of sizes $r_1, \ldots, r_k$ such that each subpath uses a color that appears at most twice on it, since each G_i has girth at least 3.

Starting with r_1 copies of 1, we inductively produce a list of colors in order for the edges. To add copies of i, insert at most two copies of i in each space between entries of the previous list. Since $r_i \leq 2(1 + \sum_{j=1}^{i-1} r_j)$, there is enough room to do this.

To complete the proof, observe that on every subpath, the smallest label appears at most twice. This holds because a path with three copies of i on it must have an smaller label on some internal edge.

Since bipartite graphs have girth at least 4, the analogous argument works for the second statement, using Corollary 3.1.7. □

Lemma 3.2.5. *If $m/k < q+1$, then r is greedily q-good, and hence P_m has a q-good r-exact edge-coloring.*

Proof. If $\sum_{j \leq i} r_j \geq i(q+1)$ for some i, then $r_j \geq q+1$ for $j \geq i$, since r is nondecreasing. Hence $m \geq \sum_{j=1}^{k} r_j \geq k(q+1)$, which contradicts $m < k(q+1)$. Therefore, we have $\sum_{j \leq i} r_j < i(q+1)$ for each i. Also, $i - 1 \leq \sum_{j<i} r_j$, so

$$r_i + \sum_{j<i} r_j < i(q+1) \leq q + 1 + (q+1) \sum_{j<i} r_j,$$

which simplifies to $r_i \leq q(1 + \sum_{j<i} r_j)$. □

Being greedily 2-good is not a necessary condition for P_m to have a 2-good r-exact edge-coloring. For example, when $r = (2, 26, 26, 26)$, still there is a 2-good r-exact edge-coloring of P_{81}. On the other hand, Lemma 3.2.5 is sharp: some lists satisfying $m/k < q+1$ are not greedily $(q-1)$-good, and the ratio m/k needed to guarantee q-good r-exact colorings for general trees must be much smaller.

Example 3.2.6. Define r by $r_i = 1$ for $1 \le i \le k-1$ and $r_k = qk$. Since $\sum r_i = (q+1)k-1$, the ratio condition holds, but $r_k = qk > (q-1)k = (q-1)(1+\sum_{i<k} r_i)$. Hence r is not greedily $(q-1)$-good.

Similarly, if $r_1 = q$ and $r_i = q+1$ for $2 \le i \le k$, then $\sum r_i = (q+1)k-1$, but $r_1 > q-1$. Again r is not greedily $(q-1)$-good.

Now consider a tree T having one central vertex of degree $k+1$ that is a common endpoint of $k+1$ paths of length $\lceil (q+1)/2 \rceil$. Thus $m = (k+1)\lceil (q+1)/2 \rceil$, so m/k is just over half of $q+1$. Let $r_i = 1$ for $i < k$ and $r_k = m-k+1$. Every r-exact edge-coloring leaves two branches completely in color k, forming a monochromatic path of length at least $q+1$. □

Example 3.2.6 suggests that general trees are much more difficult to handle than paths.

3.3 2-Good Edge-Colorings of Special Trees

We now restrict our attention to Case 2: G is the cartesian product of $G_1, \ldots, G_k$, where each G_i is $2r_i$-regular. Let $r = (r_1, \ldots, r_k)$, indexed in nondecreasing order, and let $m = \sum r_i$. If the factors are simple graphs, then every cycle contains at least three edges. In this case, if T has a 2-good r-exact edge-coloring, then Theorem 3.2.1 implies that G has a T-decomposition. Thus it is natural to ask (1) when does a tree have such an edge-coloring, and (2) are there weaker conditions than 2-good edge-coloring for T that guarantee a T-decomposition of G?

For simplicity, we always assume that T has m edges and r is a nondecreasing list of k positive integers with sum m. Let $\ell(v)$ be the number of leaf neighbors of a vertex v in T.

Proposition 3.3.1. *If T has a 2-good r-exact edge-coloring, then $r_k \le m - d(v) + \max\{\ell(v), 1\}$ for all $v \in V(T)$. In addition, only when $\ell(v) = 0$ and the components of $T - v$ are all stars.*

Proof. Given a 2-good r-exact edge-coloring of T, let E_k be the set of edges having color k. Fix $v \in V(T)$. Let F be the set of edges incident to v and F' be the subset of F consisting of edges incident to leaves of T. If $|E_k \cap F| \le 1$, then $r_k \le m - d(v) + 1$, since otherwise there is a path of length 3 in color k. If $|E_k \cap F| \ge 2$, then the edges in $E_k - F$ are not incident to any edge in $E_k \cap F$. Each edge of $E_k \cap (F - F')$ is incident to at least one edge that is not incident to v and does not lie in E_k (see Figure 3.1). Thus $|E_k - F'| = |E_k - F| + |E_k \cap (F - F')| \le m - d(v)$. Since $|E_k \cap F'| \le \ell(v)$, we have $|E_k| = |E_k - F'| + |E_k \cap F'| \le m - d(v) + \ell(v)$.

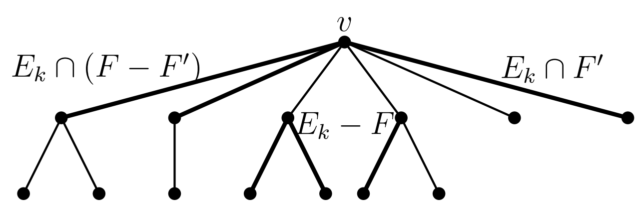


Figure 3.1: An example of tree T for Proposition 3.3.1.

If $r_k > m - d(v) + \ell(v)$, then $\ell(v) = 0$ and $r_k = m - d(v) + 1$, which requires that $|E_k \cap F| = 1$ and all edges not incident to v have color k. Therefore, every component of $T - v$ has no 3-edge path and hence is a star. □

This suggests the question of when this condition is sufficient. Unfortunately, it is not sufficient even for trees with diameter 4.

Example 3.3.2. Given $a \ge 3$ and $b \ge 2$, let $T_{a,b}$ be the tree having a vertex u with $d(x) = a$ such that every component of $T - u$ is a star with b edges whose center is adjacent to x. Consider $r = (r_1, r_2)$ with $a + b \le r_1 \le m/2$ and $r_1 + r_2 = m$. Note that $\ell(x) = 0$ and $m = a + ab$. Note also that always $r_2 \le m - d(x) - b$, so the necessary condition holds.

We show that $T_{a,b}$ has 2-good r-exact edge-colorings only when $r_1 - a$ is a multiple of $b-1$. Suppose $T_{a,b}$ has a 2-good r-exact edge-coloring. Let F be the set of edges incident to x, and let i be the number of edges in F having color 1.

Since $r_2 \geq m/2 \geq a+b$, we have $r_2 > a$, and therefore $i \geq 1$. If $i = 1$, then at least two edges in F have color 2, since $a \geq 3$. The pendant edges incident to an edge in F of color 2 now must have color 1; otherwise we get a monochromatic path of length 3. Hence $r_1 \geq 1 + (a-1)b$. Since $r_1 \leq m/2$, we have $1 + (a-1)b \leq (a+ab)/2$, which simplifies to $(a-2)(b-1) \leq 0$, a contradiction. If $i \geq 2$, then the pendant edges incident to an edge of color 1 have color 2. If $i \geq a-1$, then $r_1 \leq a-1+b$, which contradicts $r_1 \geq a+b$. Therefore, at least two edges in F have color 2.

In the remaining case, $2 \leq i \leq a-2$. Now the color of every pendant edge differs from the color of the edge in F incident to it. Hence $r_1 = a - i + ib$. Consequently, if $a+b \leq r_1 \leq m/2$ and $r_1 - a$ is not a multiple of $b-1$, then $T_{a,b}$ with $a \geq 3$ and $b \geq 2$ has no 2-good r-exact edge-coloring. For the degenerate case $b = 1$, a 2-good r-exact edge-coloring exists only when $r_1 \in \{a-1, a\}$. □

Nevertheless, the condition is sufficient for a special family of trees with diameter 4.

Definition 3.3.3. Given the nondecreasing list r of length k, define a function $c_r : [m] \to [k]$ by letting $c_r(t)$ be the least index h such that $t \leq \sum_{i \leq h} r_i$.

Lemma 3.3.4. *Let T be a tree consisting of paths of length at most 2 having a common endpoint x. If $r_k \leq m - d(v) + \max\{\ell(v), 1\}$ for all $v \in V(T)$, then T has a 2-good r-exact edge-coloring.*

Proof. Note that $m - d(v) + \max\{\ell(v), 1\}$ is minimized when $v = x$. Note also that $m = 2d(x) - \ell(x)$. Index the edges as $e_1, \ldots, e_m$ so that the first $d(x) - \ell(x)$ edges are the non-pendant edges incident to x, the next $\ell(x)$ edges are the pendant edges incident to x, and the last $m - d(x)$ edges are the edges not incident to x, with e_t incident to $e_{t-d(x)}$ for

$d(x) < t \le m$. Let the color assigned to edge e_t be $c_r(t)$. By construction, this coloring is r-exact; we claim that also it is 2-good.

Suppose that P is a monochromatic 3-edge path in this coloring. Let e_t be the edge in P with least index. Note that $1 \le t \le d(x) - \max\{\ell(x), 1\}$. It follows that $e_{t'}$ gets color $c_r(t)$ for all t' with $t \le t' \le t + d(v)$, and hence $r_{c_r(t)} \ge d(x) + 1$. If also $c_r(t) < k$, then $r_{c_r(t)} + r_k > 2d(x) \ge m$. Thus $c_r(t) = k$, so $e_{t'}$ gets color k for $t \le t' \le m$. Therefore, $r_k \ge m - t + 1 \ge m - d(x) + \max\{\ell(x), 1\} + 1$, a contradiction. □

Example 3.3.2 shows that when $a \ge 3$ and $b \ge 2$, the condition $r_k \le m - d(v) + \max\{\ell(v), 1\}$ for all v is not sufficient for $T_{a,b}$ to have a 2-good r-exact edge-coloring. Lemma 3.3.4 includes the degenerate case of $T_{a,b}$ when $b = 1$. We next consider a generalization of $T_{a,2}$. A *special tree* is a tree T having a special vertex x such that every component of $T - x$ has at most two edges. Although the condition on r_k in Lemmas 3.3.4 is not sufficient to guarantee 2-good r-exact edge-colorings for special trees (as in Example 3.3.2 with $b = 2$), we will prove in Lemma 3.3.7 that it does suffice for special trees when also $m/k < 4$ and $m \ge 8$. We first prove a lemma about a special subclass of special trees.

Lemma 3.3.5. *Let T be a tree consisting of d_1 paths of length 1, d_2 paths of length 2, and d_3 paths of length 3 having a common endpoint x. Let $j = c_r(d_2 + d_3)$. If $r_j \le d(x) + \sum_{i<j} r_i$ when $c_r(d_3) = j$, or $r_j \le m - d(x) + \ell(x)$ when $c_r(d_3) < j$, then T has a 2-good r-exact edge-coloring such that on each path with endpoint x the edge incident to x gets a color distinct from the colors assigned to the other edges of that path.*

Proof. Consider the multiset U consisting of r_i copies of color i for $1 \le i \le k$; note that U has size $d_1 + 2d_2 + 3d_3$. Let S be a multiset consisting of $d_2 + d_3$ smallest elements of U (since $j = c_r(d_2 + d_3)$, they are all at most j), and let $R = U - S$. We will partition U into multisets assigned to the components of $T - x$ (we just call them "sets"). A component of $T - x$ having p vertices gets a set of size p to be used on its edges and the edge joining it to x. We form the sets of size 1, then size 3, then size 2. First let d_1 smallest elements of R be the sets of size 1. Next iteratively associate a smallest remaining element of S with two smallest remaining elements of R; do this d_3 times. Finally, associate a smallest remaining element of S with a smallest remaining element of R.

This procedure creates the desired sets if in each set the smallest element occurs only once, which holds by construction when the smallest element is less than j. Since the smallest element in sets of size at least 2 comes from S and is always at most j, it suffices to show that when the smallest element is j there is no other j in the set. We bound the allowed multiplicity of j in two cases.

Case 1: $c_r(d_3) = j$. In this case, shown in Figure 3.2 at most d_1 copies of j form sets of size 1. In the step forming sets of size 3, at most two copies of j remaining in R are associated with each element of S that is less than j (there are $\sum_{i<j} r_i$ of them). We need that at most one copy of j in S (there are $|S| - \sum_{i<j} r_i$ of them) and no copy of j in R appears in each set of size 2 and in each other set of size 3. Hence it is necessary and sufficient to have $r_j \le d_1 + 2\sum_{i<j} r_i + |S| - \sum_{i<j} r_i$. This is equivalent to the hypothesis, since $|S| = d_2 + d_3$ and $d(x) = d_1 + d_2 + d_3$.

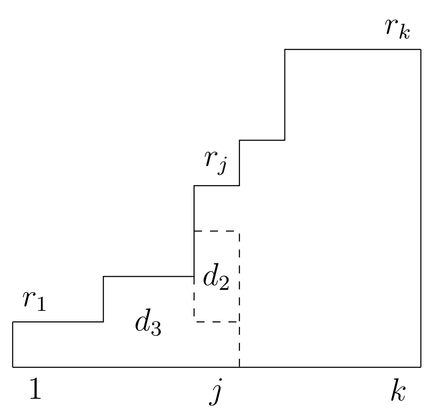


Figure 3.2: Case 1 in Lemma 3.3.5.

Case 2: $c_r(d_3) < j$. In this case, shown in Figure 3.3 we need that at most two copies of j in R appear in each set of size 3 formed, and at most one copy of j appears in each set of size 2. Hence it is necessary and sufficient to have $r_j \leq d_1 + 2d_3 + d_2$. Since $\ell(x) = d_1$ and $m = d_1 + 2d_2 + 3d_3$, the required inequality is equivalent to that given in the hypothesis.

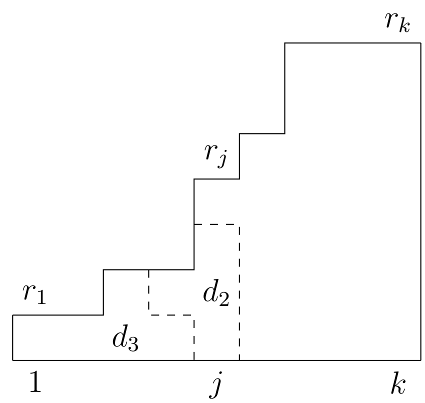


Figure 3.3: Case 2 in Lemma 3.3.5.

□

Remark 3.3.6. In Corollary 3.2.4, the condition $r_i \leq 2(1 + \sum_{j<i} r_i)$ (for all i) is shown to be sufficient for a path to have a 2-good r-exact edge-coloring. Lemma 3.2.5 shows that if $m/k < 3$, then that condition always holds, and hence a 2-good r-exact edge-coloring of the path exists. However, when $m/k < 3$ is changed to $m/k < 4$, the full path is not 2-bounded when $m \geq 6$ and $r_i = 3$ for all i.

For special trees with $m \geq 8$, the condition $m/k < 4$ suffices as long as r_k is not too big.

Lemma 3.3.7. *Let T be a special tree with m edges, where $m \geq 8$. If $r_k \leq m - d(v) + \max\{\ell(v), 1\}$ for all $v \in V(T)$, and $m/k < 4$, then T has a 2-good r-exact edge-coloring.*

Proof. Let x be the special vertex of T. Note that $\min_v\{m - d(v) + \max\{\ell(v), 1\}\} = m - d(x) + \max\{\ell(x), 1\}$. Form T' by replacing each copy of $K_{1,3}$ in T that has x as a leaf with a copy of P_4 having x as a leaf. Lemma 3.3.5 will apply to give an edge coloring of T'. For the copies of $K_{1,3}$ replaced with paths, we assign the edge incident to x the same color as in T', and assign the other edges the remaining colors. The resulting edge-coloring is 2-good and r-exact. To apply Lemma 3.3.5, it suffices to show that the inequalities in the hypothesis of Lemma 3.3.5 are satisfied for T'.

For T', we have $r_k \leq m - d(v) + \max\{\ell(v), 1\}$ for all v if and only if the inequality holds for $v = x$. Note that T' also has m edges, and $d_T(x) = d_{T'}(x)$ and $\ell_T(x) = \ell_{T'}(x)$. Given d_1, d_2, and d_3 defined as in Lemma 3.3.5 for T', let $j = c_r(d_2 + d_3)$. Note that $d_1 = \ell(x)$. In the case $c_r(d_3) < j$, since $r_k \leq m - d(x) + \max\{\ell(x), 1\}$ is given, it follows that $r_i \leq m - d(x) + \max\{\ell(x), 1\}$ for all i. Hence Lemma 3.3.5 applies unless $r_j = m - d(x) + 1 > m - d(x) + \ell(x)$, which requires $\ell(x) = d_1 = 0$. If $j < k$, then $2d_2 + 3d_3 = m \geq r_j + r_k = 2(m - d(x) + 2) = 2d_2 + 4d_3 + 2$, a contradiction. If $j = k$, then assigning color k to the edges not incident to x, and the rest colors to the rest edges obtain a 2-good r-exact edge-coloring of T'. Therefore we need only consider the case $c_r(d_3) = j$.

Let $L = \sum_{i<j} r_i$. Note that $L < d_3$, since $c_r(d_3) = j$; we need $r_j \leq d(x) + L$ to apply Lemma 3.3.5. Suppose $r_j \geq d(x) + L + 1$. Since r is nondecreasing, $r_i \geq r_j$ for $i \geq j$, and hence

$$m = \sum_i r_i \geq L + \sum_{i \geq j} (d(x) + L + 1) \geq L + (k - j + 1)(d(x) + L + 1). \qquad (3.1)$$

If $j \leq k - 2$, then $m \geq L + 3(d(x) + L + 1) > 3d(x)$, a contradiction since $m = 3d_3 + 2d_2 + d_1 \leq 3(d_3 + d_2 + d_1) = 3d(x)$. If $j = k$, then since $L < d_3$ and $r_j \leq m - d(x) + \ell(x)$, we have

$$m = L + r_k < d_3 + m - d(x) + \ell(x) = 3d_3 + d_2 + d_1 \leq m,$$

a contradiction. Therefore, $j = k - 1$.

Substituting $k - j = 1$ into (3.1) yields $m \geq 2d(x) + 3L + 2$. Using also $m \leq 3d(x)$ obtains $d(x) \geq 3L + 2$, and hence $m \geq 9L + 6$. On the other hand, since $j \leq L + 1$ and $m < 4k$, we have $m < 4k = 4(j + 1) \leq 4L + 8$. Thus $9L + 6 \leq m < 4L + 8$, which implies $L = 0$, and so $6 \leq m < 8$, a contradiction since $m \geq 8$. □

Applying Corollary 3.1.4, we have the following corollary of Theorem 3.4.2.

Corollary 3.3.8. *Let T be a special tree with m edges, where $m > 8$. Let G be the cartesian* *product of $G_1, \dots, G_k$, where each G_i is either an r_i-regular bipartite graph or a $2r_i$-regular graph. If $r_k \le m - d(v) + \ell(v)$ for all $v \in V(T)$, and $m/k < 4$, then G has a T-decomposition.*

3.4 Weakly 2-Good Edge-Coloring of General Trees

As mentioned, Corollary 3.3.8 fails for general trees, since the conditions $r_k \le m - d(v) + \ell(v)$ and $m/k < 4$ are not sufficient for paths of length at least 6 to have a 2-good r-exact edge-coloring. However, existence of a 2-good r-exact edge-coloring in T is not a necessary condition for G to have a T-decomposition, so there should be a condition *weaker* than the this that still suffices for G to have a T-decomposition.

Definition 3.4.1. A 3-bounded edge-colored path in T is *weakly 2-bounded* if either it is 2-bounded or it has a color appearing only on a 3-edge subpath whose two internal vertices have degree 2 in T. An edge-coloring of T is *weakly 2-good* if every path is weakly 2-bounded.

Theorem 3.4.2. *Let T be a special tree with m edges. If $r_k \le m - d(v) + \max\{\ell(v), 1\}$ for all $v \in V(T)$, and $m/k < 4$, then T has a weakly 2-good r-exact edge-coloring.*

Proof. Note that T has a 2-good r-exact edge-coloring by Lemma 3.3.7 when $m \ge 8$. When $m \le 7$, assign colors $1, \dots, k$ in order to the edges in the increasing order of the distance from the special vertex. This yields a weakly 2-good edge-coloring of T. □

We use a result of Kouider and Lonc [29] to show in Theorem 3.4.4 that the existence of a weakly 2-good r-exact edge-coloring of T guarantees a T-decomposition in G. We will show later that such an edge-coloring exists in any tree with m edges, including a path, if $r_k \le \lceil \frac{m+1}{2} \rceil$ and $m/k < 4$.

Theorem 3.4.3. (Kouider and Lonc [29]) *Each $2m$-regular graph G with girth at least $(m+3)/2$ has a P_{m+1}-decomposition with the property that each vertex of G occurs as an endpoint in exactly two of the copies of P_{m+1}.*

Theorem 3.4.4. *Let r be a list of positive integers with sum m. Let T be a tree with m edges, and let G be the cartesian product of graphs $G_1, \ldots, G_k$, where G_i is $2r_i$-regular, for all i. If T has a weakly 2-good r-exact edge-coloring, then G has a T-decomposition.*

Proof. For each i, let $\mathbf{F_i}$ be a 2-factorization of G_i. Consider a bijection that pairs each edge of color i in T with a 2-factor in $\mathbf{F_i}$. In a weakly 2-good edge-coloring f of T, the internal vertices of each monochromatic 3-edge path in T have degree 2 in T. Let T' be the tree obtained from T by shrinking each monochromatic 3-edge path to an edge having the same endpoints and the same color. Let $E'(T')$ be the set of edges in T' that arise by shrinking monochromatic 3-edge paths. Let f' be the edge-coloring of T' that arises from f by shrinking these paths. We claim that f' is 2-good.

Each path P in T' corresponds to a path Q in T. Since f is weakly 2-good, Q is either 2-bounded or has a color appearing only on a 3-edge subpath whose internal vertices have degree 2 in T. Since every monochromatic 3-edge path in T is shrunk to an edge in T', the corresponding path P in T' is 2-bounded. Hence f' is 2-good.

The edges of a monochromatic 3-edge path in T of color i correspond to three 2-factors in $\mathbf{F_i}$ that together form a 6-regular subgraph H in G_i. Consider the P_4-decomposition guaranteed by Theorem 3.4.3 (note that H always has girth at least $(m+3)/2$ when $m = 3$). For each copy of P_4 in the decomposition of H, delete the edges and add an edge joining the endpoints of the copy. By the property that each vertex of H occurs as an endpoint exactly twice in the decomposition, the resulting object H' is a 2-regular loopless multigraph. Obtain G_i' from G_i by replacing H with the resulting H' for each 3-edge path with color i in f. After doing this for all i, let G' be the cartesian product of $G_1', \ldots, G_k'$.

Since T' has a 2-good edge-coloring, G' has a T'-decomposition, by Theorem 3.2.1. We extend each copy of T' to a copy of T, yielding a T-decomposition of G. For each $e \in E'(T')$, replace the edge in each copy of T' that represents e with a 3-edge path having the same endpoints, yielding a copy of T, since this is the reverse of how T' was obtained from T.

Each edge e in $E'(T')$ corresponds to a 2-factor forming a copy of H' in G' whose edges appear as e in distinct copies of T'. This copy of H' arose from a copy of H in G with each edge in the copy of H' corresponding to a 3-edge path in the copy of H. Thus the 3-edge paths in all copies of T that represent the 3-edge path corresponding to e decompose the copies of H, and the copies of T form a T-decomposition of G. □

By Lemma 3.2.5 and Example 3.2.6, the condition $m/k < 4$ suffices for paths to have r-exact edge-colorings that are 3-good, but not 2-good. However, the condition $m/k < 4$ does suffice for a weakly 2-good r-exact edge-coloring.

Lemma 3.4.5. *If $m/k < 4$, then P_{m+1} has a weakly 2-good r-exact edge-coloring.*

Proof. We use induction on k. If $k = 1$, then $m \leq 3$, and giving all edges the same color is weakly 2-good. Consider $k > 1$. Always $r_1 \leq 3$. If $m < r_1 + 4(k-1)$, then split P into a subpath P' with $m - r_1$ edges and a subpath P'' with r_1 edges. Assign color 1 to the r_1 edges of P''. Since $k-1$ colors remain for P', which has fewer than $4(k-1)$ edges, by the induction hypothesis P' has a weakly 2-good r'-exact edge-coloring. Since P' and P'' use disjoint sets of colors, the full edge-coloring is weakly 2-good.

If $m \geq r_1 + 4(k-1)$, then $r_k \geq \frac{m-r_1}{k-1} \geq 4$. Split P_{m+1} into a subpath P' with $m-4$ edges and a subpath P'' of length 4. Let r' be the list $r_2, \ldots, r_k - (4-r_1)$. Since $(m-4)/(k-1) < 4$, the induction hypothesis implies that P' has a weakly 2-good r'-exact edge-coloring. For the remaining four edges, assign r_1 edges color 1 and $4 - r_1$ edges color k, with color k not being assigned to the edge incident to P'. The full edge-coloring is weakly 2-good, since $r_1 \leq 3$. □

In the proof of Lemma 3.4.5, we split P_{m+1} into two paths colored using an appropriate "split" of r into two lists. The next lemma discusses such numerical splits in more generality and helps in showing that $r_k \leq \lceil \frac{m+1}{2} \rceil$ and $m/k < 4$ together are sufficient for any tree to have a weakly 2-good r-exact edge-coloring. The *essential mean* of a list is the average of its nonzero terms. A list with sum m is *half-bounded* if every term is at most $\lceil \frac{m+1}{2} \rceil$, and it is *nearly half-bounded* if every term is at most $\left\lfloor \frac{m+3}{2} \right\rfloor$. A *split* of a nonnegative k-tuple r consists of two nonnegative k-tuples r' and r'' such that $r'_i + r''_i = r_i$ for $1 \le i \le k$. The two lemmas below will be used to prove our main theorem. The proofs are somewhat technical, so we postpone them to Section 3.5.

Lemma 3.4.6. *Let r be a nearly half-bounded list with sum m.*

a) If $0 < m' < m$, then r splits into half-bounded lists r' and r'' having essential means at most m/k and sums m' and $m - m'$, respectively.

b) Let $b = m - k \left\lfloor \frac{m}{k} \right\rfloor$. If $3k \le m < \left\lfloor \frac{m}{k} \right\rfloor (k+1)$, then for m' with $b < m' < m - b$ the essential means can be required to be less than $\lfloor m/k \rfloor$.

Let a *nontrivial star* be a star with at least one edge, and let a *penultimate edge* in a tree be an edge whose deletion leaves a component that is a nontrivial star.

Lemma 3.4.7. *Let T be a tree with m edges.*

a) If T is not a special tree, then T has an edge e whose deletion leaves components T' and T'' such that T' is a special tree with at least three edges whose vertex incident to e can designated as the special vertex.

b) If T is neither a path nor a star, then T has an edge e whose deletion leaves components T' and T'' such that T' is a nontrivial star and $T'' + e$ is not a path.

Proof. For a longest path in T, let $(1, b, c)$ be the degrees of the first three vertices. Choose P to be a longest path that lexicographically maximizes $(1, b, c)$. Let z, y, x, w be the first four vertices of P in order (T is not a star).

(a) If $d_T(y) \ge 4$, then since the component of $T - xy$ containing z is a star (and hence a special tree) with at least three edges, the edge xy suffices. If $d_T(y) = 3$, then by the choice of P, all neighbors of x other than w have degree at most 3 in T. The component of $T - wx$ containing z is a special tree, and hence wx suffices. Since $d_T(y) \ge 2$, we may henceforth assume $d_T(y) = 2$.

If $d_T(x) \geq 3$, then by the choice of P every neighbor of x other than y has degree at most 2, since $d_T(y) = 2$. The component of $T - wx$ containing z is a special tree with at least three edges, and again wx suffices. The remaining case is $d_T(y) = d_T(y) = 2$. Since $d_T(x) \geq 2$, we now need only consider $d_T(x) = 2$. By the choice of P, every components of $T - w$ except one is isomorphic to a path of length 2. Since T is not a special tree, w has a neighbor v on P other than x. Thus the component of $T - vw$ containing z is a special tree with at least three edges, and hence vw suffices.

(b) The edge xy suffices unless the component of $T - xy$ not containing z is a path P' starting with x. Since T is not a star, P' has length at least 1. Let e be the edge of T incident to the last edge of P'. The component of $T - e$ not containing z is P_2, a nontrivial star. Since T is not a path, the component of $T - xy$ containing z is a star with at least two edges, and adding e completes a subgraph that is not a path. □

Our main result in this section gives numerical conditions on r to imply that every cartesian product of regular graphs with degrees $2r_1, \ldots, 2r_k$ has a T-decomposition when T is any tree with $\sum r_i$ edges.

Theorem 3.4.8. *Let T be a tree with m edges. If $r_k \leq \lceil \frac{m+1}{2} \rceil$ and $m/k < 4$, then T has a weakly 2-good r-exact edge-coloring.*

Proof. We use induction on m. If $m \leq 7$, then T is either a special tree or a path. Consider $m \geq 8$, and thus $k \geq 3$. If T is a special tree or a path, then T has a weakly 2-good edge-coloring, by Lemma 3.4.5 and Theorem 3.4.2. Thus we may assume that T is neither a special tree nor a path. Since at most one term in r equals $\lceil \frac{m+1}{2} \rceil$, if there are two largest terms, then they are less than $\lceil \frac{m+1}{2} \rceil$. Since $m/k < 4$, we have $r_1 \in \{1, 2, 3\}$.

Case 1: $r_1 = 1$. Since the list $(r_2, \ldots, r_k)$ has sum $m - 1$, and $r_k \leq \lceil \frac{m+1}{2} \rceil = \left\lfloor \frac{(m-1)+3}{2} \right\rfloor$, the list is nearly half-bounded. Since T is not a special tree and $m \geq 8$, Lemma 3.4.7a yields an edge e whose deletion leaves components T' and T'' such that both components have at least three edges. (If T'' does not have three edges, then it and e can be added to T', making T a special tree).

If $4k-3 \leq m \leq 4k-1$, then $4(k-1) \leq m-1 \leq 4(k-1)+2$. It follows that $(m-1)-(k-1)\left\lfloor \frac{m-1}{k-1} \right\rfloor \leq 2$, and hence Lemma 3.4.6b applies, since $|E(T')| > 2$ and $|E(T'')| > 2$. Hence the list $r_2, \ldots, r_k$ splits into half-bounded lists r' and r'' with sums $|E(T')|$ and $|E(T'')|$, respectively, and both of their essential means are less than 4. If $m \leq 4k-4$, then $m-1 \leq 4(k-1)-1$, and hence by Lemma 3.4.6a the list $r_2, \ldots, r_k$ splits into half-bounded lists r' and r'' with sums $|E(T')|$ and $|E(T'')|$, respectively, and both of their essential means are less than 4.

Therefore, in either case, the list $r_2, \ldots, r_k$ splits into half-bounded lists r' and r'' with sums $|E(T')|$ and $|E(T'')|$ such that both essential means are less than 4. Assign color 1 to the edge e, and apply the induction hypothesis to both T' and T'' to obtain weakly 2-good edge-colorings. The full edge-coloring is weakly 2-good.

Case 2: $r_1 = 2$. Here the list $(r_1 - 1, r_2, \ldots, r_k)$ is nearly half-bounded and has sum $m-1$, which is at most $4k-2$. Since $m \geq 8$, there is an edge e of T whose deletion leaves nontrivial components T' and T''. By Lemma 3.4.6a, the list $r_1 - 1, r_2, \ldots, r_k$ splits into half-bounded lists r' and r'' with sums $|E(T')|$ and $|E(T'')|$ whose their essential means are less than 4. Assign color 1 to e, and apply the induction hypothesis to both T' and T'' to obtain a weakly 2-good edge-colorings. The full edge-coloring is weakly 2-good.

Case 3: $r_1 = 3$. Consider the list $(r_2, \ldots, r_k)$. Since $m - 3k \leq k-1$, we have $m - 3k \leq \frac{m-3k}{2} + \frac{k-1}{2} = \frac{m+1}{2} - k - 1$. Also, since $r_i \geq r_1 = 3$, we have $r_k = m - \sum_{i=1}^{k-1} r_i \leq m - 3(k-1)$. Hence $r_k \leq m - 3k + 3 \leq \frac{m+1}{2} - k + 2 \leq \left\lfloor \frac{(m-3)+3}{2} \right\rfloor$, since $k \geq 3$. Therefore, $(r_2, \ldots, r_k)$ is nearly half-bounded.

Since T is not a path or a star, by Lemma 3.4.7b it has an edge e whose deletion leaves components T' and T'' such that $T'' + e$ is not a path. Let e_1 and e_2 be two pendant edges of $T'' + e$ other than e. Since the list $r_2, \ldots, r_k$ is nearly half-bounded and has sum $m-3$, which is at most $4(k-1)$, by Lemma 3.4.6a it splits into half-bounded lists r' and r'' with sums $|E(T')|-2$ and $|E(T'')|$ whose essential means are less than 4. Assign color 1 to all of $\{e, e_1, e_2\}$, and apply the induction hypothesis to the two remaining trees to obtain weakly 2-good edge-colorings. The full edge-coloring is weakly 2-good. $\square$

3.5 List Splittability

In this section, we prove the lemmas about splitting lists that were used in Section 3.4. Recall that a *split* of the k-tuple r consists of two nonnegative k-tuples r' and r'' such that $r'_i + r''_i = r_i$ for $1 \le i \le k$. Given m' with $0 < m' < m = \sum_{i=1}^{k} r_i$, let $m'' = m - m'$. We will first give sufficient conditions for a split of r into half-bounded lists r' and r'' with sums m' and m'', respectively, such that both r' and r'' have at least certain numbers of nonzero terms. We apply this in Lemma 3.5.3 to show that if r is nearly half-bounded (meaning $r_k \le \lfloor \frac{m+3}{2} \rfloor$), then r splits into half-bounded lists r' and r'' having essential means at most the essential mean m/k of r. Under additional hypotheses, for most values of m' the essential means of r' and r'' can also be required to be less than $\lfloor m/k \rfloor$. We state the first lemma using x instead of m' because we will also apply it in the complementary situation where $x = m''$.

Lemma 3.5.1. *Let r be a nearly half-bounded list with sum m. For integer x with $0 < x < m$, let $t_i = \min\{r_i, \lceil \frac{m-x+1}{2} \rceil\}$ for $1 \le i \le k$. Let $j = \max\{i\colon\, r_i \le \lceil \frac{m-x+1}{2} \rceil\}$. For integer y with $0 \le y < k$, let S be a subset of $\{1, \ldots, j\}$ having size $\max\{0, (y+1)-(k-j)\}$. Let $s_i = r_i - t_i + 1$ for $i \in S$ and $s_i = r_i - t_i$ for $i \notin S$. If (1) $y+1 \le x$ and (2) either $2y+1 \le x$ or $2(k-y) \ge m-x$, then $\sum_{i=1}^{k} s_i \le x$. Furthermore, s has at least $y+1$ nonzero terms.*

Proof. Note that $r_i - t_i = \max\{0, r_i - \lceil \frac{m-x+1}{2} \rceil\}$; in particular $r_i - t_i = 0$ and $s_i = 1$ for

$i \in S$. (See Figure 3.4.) By the choice of j,

$$\sum_{i=1}^{k} s_i = |S| + \sum_{i>j} \left(r_i - \left\lceil \frac{m-x+1}{2} \right\rceil \right).$$

If $j = k$, then $\sum_{i=1}^{k} s_i = |S| = y + 1 \leq x$. If $j \leq k-2$, then $\sum_{i>j} \left\lceil \frac{m-x+1}{2} \right\rceil \geq 2 \left\lceil \frac{m-x+1}{2} \right\rceil$. Since $|S| \leq j$ and $\sum_{i>j} r_i \leq m - j$, we have $\sum_{i=1}^{k} s_i \leq j + m - j - (m - x + 1) < x$. If $j = k - 1$, then $\sum_{i=1}^{k} s_i = y + r_k - \left\lceil \frac{m-x+1}{2} \right\rceil$. Since $r_k \leq \left\lfloor \frac{m+3}{2} \right\rfloor$, in the case $2y + 1 \leq x$ we have $\sum_{i=1}^{k} s_i \leq \left\lfloor \frac{x-1}{2} \right\rfloor + \left\lfloor \frac{m+3}{2} \right\rfloor - \left\lceil \frac{m-x+1}{2} \right\rceil \leq x$. Since also $r_k \leq m - k + 1$, in the case $2(k - y) \geq m - x$ we have

$$\begin{aligned} \sum_{i=1}^{k} s_i &\leq y + m - k + 1 - \left\lceil \frac{m-x+1}{2} \right\rceil \\ &= y - k + 1 + \left\lfloor \frac{m-x-1}{2} \right\rfloor + x \\ &\leq y - k + 1 + k - y - 1 + x = x. \end{aligned}$$

To count the nonzero terms in s, note that if $i > j$, then $s_i > 0$. If $i \leq j$, then $s_i > 0$ for $i \in S$. Hence s has at least $|S| + k - j$ nonzero terms, which is at least $y + 1$. □

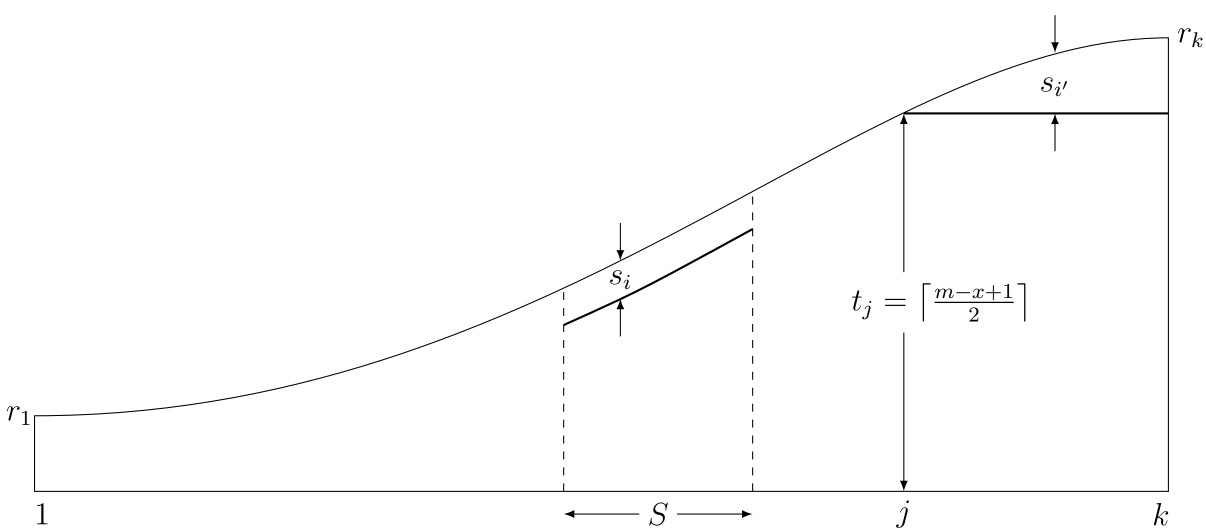


Figure 3.4: Illustration for lists s and t, and set S in Lemma 3.5.1.

When comparing lists of the same length, an expression like $r' \leq t'$ means $r'_i \leq t'_i$ for all i. Our plan is as follows.

Remark 3.5.2. We will first define t' and t'' as instances of the list t in Lemma 3.5.1 with $x = m - m'$ and $x = m'$, respectively. By the definition of t, any list r' satisfying $r' \leq t'$ and $\sum_{i=1}^{k} r'_i = m'$ is half-bounded, and similarly for r''.

We will next obtain lists s' and s'' as instances of the list s in Lemma 3.5.1 such that $s' \leq r - s''$ and $\sum_{i=1}^{k} s'_i \leq m' \leq \sum_{i=1}^{k}(r_i - s''_i)$. Given such lists, we produce r' by starting with s' and augmenting elements of the list, while keeping the ith element at most $r_i - s''_i$, until we reach sum m'. Since $r' \geq s'$, the list r' has at least as many nonzero terms as s'. Similarly, since $r - r'' = r' \leq r - s''$ implies $r'' \geq s''$, the list r'' has at least as many nonzero terms as s''. Furthermore, s' and s'' will be defined so that $r - s'' \leq t'$ and $r - s' \leq t''$. It then follows that $r' \leq r - s'' \leq t'$ and $r'' = r - r' \leq r - s' \leq t''$. Hence r' and r'' are half-bounded.

It remains to obtain such lists s' and s'' having sufficiently many nonzero terms (to make the essential means small). We will do this using special sets S' and S'' in the manner in which s is defined from t in Lemma 3.5.1. We will need to ensure that the specifications of S' and S'' do not prevent $s' \leq r - s''$. See Figure 3.5.

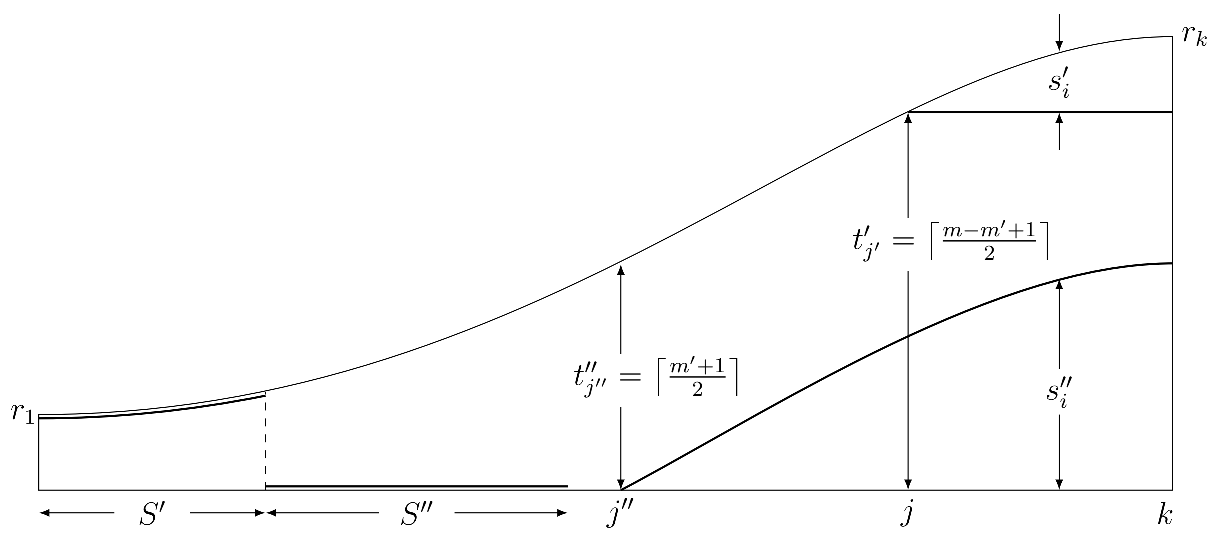


Figure 3.5: Illustration for lists s', s'', t', t'' and sets S', S'' in Remark 3.5.2.

Lemma 3.5.3. *Let r be a nearly half-bounded list with sum m.*

a) If $0 < m' < m$, then r splits into half-bounded lists r' and r'' with sums m' and m'' having essential means at most m/k.

b) Let $b = m - k\left\lfloor \frac{m}{k} \right\rfloor$. If $3k \le m \le \left\lfloor \frac{m}{k} \right\rfloor (k+1)$ and $b < m' < m - b$, then the essential means of r' and r'' can be required to be less than $\left\lfloor \frac{m}{k} \right\rfloor$.

Proof. We will define parameters k' and k'' and construct lists s' and s'' with $k'+1$ nonzero terms and $k''+1$ nonzero terms, respectively, such that $s' \le r - s''$ and $\sum s'_i \le m' \le \sum(r_i - s''_i)$. The lists s' and s'' will be instances of s obtained from r as in Lemma 3.5.1, using parameters x and y and an appropriate set S. For s', we use $x = m'$ and $y = k'$. For s'', we use $x = m - m'$ and $y = k''$. We let S' and S'' denote the sets to be used as S in determining s' and s'', respectively. Similarly, let j' and j'' denote the index j computed in the two instances. Let $k' = \left\lfloor \frac{m'}{a} \right\rfloor$ and $k'' = k - k' - \delta$, where a and δ will be defined differently for part (a) and part (b). In both cases, $k' \le \frac{m'}{a} < k' + 1$.

(a) Since the conclusion is obvious if $r_i = 1$ for all i, we assume $r_k \ge 2$. We set $a = m/k$ and $\delta = 1$, so $k'' = k - k' - 1$. To see that having lists s' and s'' as described above suffices, note that any list r' with $r' \ge s'$ and sum m' has at least $k'+1$ nonzero terms, and hence has essential mean at most $\frac{m'}{k'+1}$, which is less than a by the choice of k'. Similarly, any list r'' with $r'' \ge s''$ and sum $m - m'$ has essential mean at most $\frac{m-m'}{k-k'}$, which is at most a since $mk' \le m'k$. As noted in Remark 3.5.2, $s' \le r - s''$ and $\sum s'_i \le m' \le \sum(r_i - s''_i)$ allows us to obtain such r' and r'' by iteratively augmenting terms.

To apply Lemma 3.5.1, we need to define S' and S'' appropriately. Each choice of y (k' or k'', respectively) must be less than k. We have $k' = \left\lfloor \frac{m'k}{m} \right\rfloor < k$ and $k'' = k - k' - 1 < k$. Since we want S' to be a set of size $\max\{0, (k'+1) - (k - j')\}$, let $S' = \{k - k', \ldots, j'\}$; this set is empty if $k' + 1 \le k - j'$. Similarly, since $(k''+1) - (k - j'') = j'' - k'$ when $y = k'' = k - k' - 1$, we need $|S''| = \max\{0, j'' - k'\}$. We set $S'' = \{1, \ldots, j'' - k'\}$, except $S'' = \{1, \ldots, j'' - k' - 1\} \cup \{j''\}$ when $j'' = k$. As in Lemma 3.5.1, $s'_i = 1$ for $i \in S'$

and $s'_i = \max\left\{0, r_i - \left\lceil \frac{m-m'+1}{2} \right\rceil\right\}$ for $i \notin S'$. Similarly, $r_i - s''_i = r_i - 1$ for $i \in S''$ and $r_i - s''_i = \min\left\{r_i, \left\lceil \frac{m'+1}{2} \right\rceil\right\} \geq 1$ for $i \notin S''$.

We need to show $s' \leq r - s''$. Since $r_i \leq r_k \leq \left\lfloor \frac{m+3}{2} \right\rfloor \leq \left\lceil \frac{m-m'+1}{2} \right\rceil + \left\lceil \frac{m'+1}{2} \right\rceil$, we have $s'_i \leq r_i - s''_i$ when $i \notin S'$ and $i \notin S''$. The cases when i is in one of S' and S'' are immediate. When $i \in S' \cap S''$, we need $r_i \geq 2$. We have $i \in S' \cap S''$ only when $j'' = k$ and $i = k$; now $r_k \geq 2$ suffices.

To show $\sum s'_i \leq m' \leq \sum(r_i - s''_i)$, we apply Lemma 3.5.1 twice. First consider $\sum s'_i \leq m'$. Recall that $a = m/k$. Note that $a > 1$, since $r_k \geq 2$. Now $y = k' \leq m'/a < m' = x$. When $a > 2$, we have $2y = 2k' \leq 2m'/a < m' = x$. When $a \leq 2$, since $\frac{m-m'}{k-k'} \leq a \leq 2$, we have $2(k-y) = 2(k-k') \geq m - m' = m - x$. Hence the hypotheses of Lemma 3.5.1 hold, and we conclude $\sum s'_i \leq m'$.

To prove $m' \leq \sum(r_i - s''_i)$, we show $\sum s''_i \leq m - m'$. In this application of Lemma 3.5.1, we have $y = k''$ and $x = m - m'$. Since $k' \geq m'/a$ and $k \leq m$, we have $y+1 = k-k' \leq k(1-\frac{m'}{m}) = \frac{k}{m}(m-m') \leq x$. When $m'/k' \geq 2$, we have $2y+1 < 2(k-k') \leq \frac{m'}{k'}(k-k') \leq m - m' = x$. When $m'/k' < 2$, we have $2(k-y) > 2k' > m' = m - x$. Hence again the hypotheses of Lemma 3.5.1 hold, and we conclude $\sum s''_i \leq m - m'$.

(b) We set $a = \left\lfloor \frac{m}{k} \right\rfloor$. As in part (a), any list r' with $r' \geq s'$ and sum m' has essential mean less than a. If $m - m' < a(k-k')$, then $k - k'$ nonzero terms are enough for r'' to have essential mean less than a. Otherwise $m - m' < a(k - k' + 1)$ (since $m' \geq ak'$), and then $k - k' + 1$ nonzero terms are enough. Hence we set $k'' = k - k' - \delta$, where $\delta = 1$ if $m - m' < a(k-k')$ and $\delta = 0$ if $m - m' \geq a(k-k')$. Again we need y (k' or k'', respectively) to be less than k. We have $k' = \left\lfloor \frac{m}{a} \right\rfloor < \left\lfloor \frac{m-b}{a} \right\rfloor < k$ and $k'' = k - k' - \delta < k$.

We define S' and S'' as follows. As in part (a), let $S' = \{k-k', \ldots, j'\}$. For S'', we set $S'' = \{1, \ldots, j'' - k' + 1\}$, except $S'' = \{1, \ldots, j'' - k'\} \cup \{j''\}$ when $j'' = k - 1$ and $S'' = \{1, \ldots, j''-k'-1\} \cup \{j''-1, j''\}$ when $j'' = k$. Again, we need $r_i \geq 2$. We have $i \in S' \cap S''$ only when $j'' \geq k-1$, in which case $S' \cap S'' \subseteq \{k-1, k\}$. If $k = 1$, then $r_k = m \geq 3k = 3$. If $k \geq 2$, then since $r_k \le \left\lfloor \frac{m+3}{2} \right\rfloor$ and $m \ge 3k$ we have $r_{k-1} \ge m - r_k - (k-2) \ge \left\lceil \frac{m-3}{2} \right\rceil - (k-2) \ge 2$.

Now we show $\sum s'_i \le m'$ and $\sum s''_i \le m - m'$ by applying Lemma 3.5.1 twice. To confirm the hypotheses of Lemma 3.5.1, it suffices to show $2y+1 \le x$ when $(y, x) = (k', m')$ and when $(y, x) = (k'', m - m')$. Since $a \ge 3$, we have $2k' + 1 = 2\left\lfloor \frac{m'}{a} \right\rfloor + 1 \le m'$. Hence we conclude $\sum s'_i \le m'$. Now consider $\sum s''_i \le m - m'$. When $\delta = 0$, we have $2k'' + 1 = 2(k - k') + 1$ and $m - m' \ge a(k - k')$. Since $a \ge 3$ and $k' < k$, we have $2(k - k') + 1 \le a(k - k') \le m - m'$. When $\delta = 1$, we have $2k'' + 1 = 2(k - k') - 1$ and $m - m' > a(k - k' - 1)$. Note that $2(k - k') - 1 \le a(k - k' - 1)$ is equivalent to $0 < (a - 2)(k - k')$, which again holds since $a \ge 3$ and $k' < k$. □

Remark 3.5.4. In the proof of Lemma 3.5.3(a) the essential mean of r' is actually less than the essential mean m/k of r.

In Lemma 3.5.3(b), the condition $m \ge 3k$ can be relaxed to $m \ge 2k$, but then $r_{k-1} \ge 2$ needs to be required, since $r_{k-1} = 1$ can happen when $a = 2$. However, the proof needs more case analysis and we do not need this strengthening. Since no list has essential mean less than 1, the condition $2k \le m$ cannot be relaxed more. With $m \ge 2k$, the condition $r_{k-1} \ge 2$ cannot be relaxed more, as shown by the list $(1, \ldots, 1, k+1)$, where k is even. When $m' = k$, the list has no split consisting of half-bounded lists with essential means less than 2.

Chapter 4

Circular Chromatic Ramsey Number

4.1 Introduction

This chapter is based on joint work with Claude Tardif, Douglas West, and Xuding Zhu, appearing in [24]. Ramsey proved in 1932 that given $p, q \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then every red/blue edge-coloring of K_n yields a red copy of K_p or a blue copy of K_q. The *(classical) Ramsey number* $R(p, q)$ is the least such n_0. In particular, when $p = q$ this means that every edge-coloring of a sufficiently large complete graph contains a monochromatic copy of K_p. We can generalize the problem by asking for monochromatic copies of graphs other than complete graphs. Given graphs F and G, the *(graph) Ramsey number* $R(F, G)$ is the least n such that every red/blue edge-coloring of K_n yields a red copy of F or a blue copy of G. Since a monochromatic copy of K_p contains a monochromatic copy of every p-vertex graph, $R(F, G) \leq R(p, q)$, where $p = |V(F)|$ and $q = |V(G)|$. The inequality may be strict, since forcing (F, G) may be easier than forcing (K_p, K_q). For example, every 2-edge-coloring of K_3 yields a monochromatic copy of P_3, even though $R(3, 3) = 6$.

We can further generalize the problem by coloring the edges of graphs other than complete graphs and generalizing the targets to graph families. Given families $\mathcal{F}$ and $\mathcal{G}$ of graphs, we seek a *host graph* H such that every red/blue edge-coloring of H yields a red copy of a graph in $\mathcal{F}$ or a blue copy of a graph in $\mathcal{G}$; we then write $H \to (\mathcal{F}, \mathcal{G})$ and say that H *forces* $(\mathcal{F}, \mathcal{G})$. The graph Ramsey number $R(F, G)$ is then $\min\{|V(K_n)|\colon K_n \to (\{F\}, \{G\})\}$. In fact, $R(F, G) = \min\{|V(H)|\colon H \to (\{F\}, \{G\})\}$, since every supergraph of a graph forcing

$(\mathcal{F},\mathcal{G})$ also forces $(\mathcal{F},\mathcal{G})$. Instead of minimizing the number of vertices in the host graphs, we can consider any monotone graph parameter. For any monotone graph parameter ρ, the ρ-Ramsey number of $(\mathcal{F},\mathcal{G})$, written $R_\rho(\mathcal{F},\mathcal{G})$, is $\inf\{\rho(H)\colon H \to (\mathcal{F},\mathcal{G})\}$. As mentioned, classically ρ was the number of vertices; the notion has also been studied with ρ being the the clique number [12, 31], the chromatic number [5, 41], the number of edges (yielding the *size Ramsey number*) [8, 34], and the maximum degree [26, 27, 22]. In this chapter, we study the circular chromatic Ramsey number, which arises when $\rho = \chi_c$; we will define the circular chromatic number of a graph shortly.

Given positive integers p and q with $p \geq 2q$, the *generalized complete graph* $K_{p:q}$ has vertex set $\{v_i\colon 0 \leq i \leq p-1\}$ and edge set $\{v_iv_j\colon q \leq |i-j| \leq p-q\}$; note that $K_{p:1}$ is the complete graph K_p. A *homomorphism* of a graph G to a graph H maps $V(G)$ into $V(H)$ such that adjacent vertices of G are mapped to adjacent vertices of H; the graph H is a *homomorphic image* of G. It is well known that if $p'/q' \geq p/q$, then $K_{p':q'}$ is a homomorphic image of $K_{p:q}$. If $p/q = p'/q'$ with $p \neq p'$, then $K_{p:q}$ and $K_{p':q'}$ are different graphs, but they are *homomorphically equivalent*, meaning that each admits a homomorphism into the other. Let $\mathrm{Hom}(G)$ denote the family of homomorphic images of G. When G admits a homomorphism into H, we say that G is *H-colorable*; when $H = K_{p:q}$, we abbreviate to *(p,q)-colorable*. The *circular chromatic number* of a graph G, written $\chi_c(G)$, is the infimum of all p/q such that G is (p,q)-colorable; thus by the definition $\chi_c(G) \leq \chi(G)$. It is well known [4, 38] that in fact $\chi_c(G)$ is rational, and hence the "infimum" can be replaced with "minimum". Also, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ [4, 38], which yields $\chi(G) = \lceil \chi_c(G) \rceil$, and so the circular chromatic number is a refinement of the chromatic number.

Letting $\rho = \chi_c$, we study the *circular chromatic Ramsey number* $R_{\chi_c}(\mathcal{F},\mathcal{G})$, defined by

$$R_{\chi_c}(\mathcal{F},\mathcal{G}) = \inf\{\chi_c(H)\colon H \to (\mathcal{F},\mathcal{G})\}.$$

Because χ_c may have any rational value at least 2, we need the infimum here; the minimum may not exist. We first introduce two monotonicity properties for R_{χ_c}.

Lemma 4.1.1. *If $H \to (\mathcal{F}, \mathcal{G})$ and there is a homomorphism from H to H', then $H' \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$.*

Proof. Let ϕ be a homomorphism from H to H'. Consider any red/blue edge-coloring f of H'. Lift the coloring f back to H, giving each edge in H the color of its image under ϕ. In the resulting edge-coloring of H, the red graph in $\mathcal{F}$ or blue graph in $\mathcal{G}$ forced by H maps under ϕ to a graph in $\mathrm{Hom}(\mathcal{F})$ or $\mathrm{Hom}(\mathcal{G})$ that is monochromatic under f. Hence $H' \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. □

Lemma 4.1.2. *If each graph in the family $\mathcal{F}'$ is a homomorphic image of some graph in the family $\mathcal{F}$, and similarly for $\mathcal{G}'$ and $\mathcal{G}$, then $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \le R_{\chi_c}(\mathcal{F}', \mathcal{G}')$.*

Proof. By the hypothesis, $\mathrm{Hom}(\mathcal{F}') \subseteq \mathrm{Hom}(\mathcal{F})$ and $\mathrm{Hom}(\mathcal{G}') \subseteq \mathrm{Hom}(\mathcal{G})$. Therefore, $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}'), \mathrm{Hom}(\mathcal{G}'))$ implies $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. Hence

$$\begin{aligned} R_{\chi_c}(\mathcal{F}, \mathcal{G}) &= \inf\{p/q\colon\ K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\} \\ &\le \inf\{p/q\colon\ K_{p:q} \to (\mathrm{Hom}(\mathcal{F}'), \mathrm{Hom}(\mathcal{G}'))\} = R_{\chi_c}(\mathcal{F}', \mathcal{G}'). \end{aligned}$$ □

Similar to $\chi(H) - 1 < \chi_c(H) \le \chi(H)$, we obtain the following bounds on $R_{\chi_c}(\mathcal{F}, \mathcal{G})$.

Proposition 4.1.3. *$R_\chi(\mathcal{F}, \mathcal{G}) - 1 \le R_{\chi_c}(\mathcal{F}, \mathcal{G}) \le R_\chi(\mathcal{F}, \mathcal{G})$.*

Proof. Since $\chi_c(H) \le \chi(H)$ when $H \to (\mathcal{F}, \mathcal{G})$, we have $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \le R_\chi(\mathcal{F}, \mathcal{G})$. For the other inequality, if $k \le R_{\chi_c}(\mathcal{F}, \mathcal{G}) < k+1$ for some integer k, then $H \to (\mathcal{F}, \mathcal{G})$ for some graph H with $k \le \chi_c(H) < k+1$. Since $k \le \chi(H) \le k+1$, we have $R_\chi(\mathcal{F}, \mathcal{G}) \le k+1$, and hence $R_\chi(\mathcal{F}, \mathcal{G}) \le R_{\chi_c}(\mathcal{F}, \mathcal{G}) + 1$. □

Early work for the circular chromatic number studied when the upper or lower bound in $\chi(H) < \chi_c(H) \leq \chi(H)$ holds with equality. It is also interesting to ask the same question for the bounds on $R_{\chi_c}(\mathcal{F}, \mathcal{G})$ in Proposition 4.1.3. Another related question is whether $R_{\chi_c}(\mathcal{G}, \mathcal{H})$ ever fails to be an integer.

We next introduce a useful result. A homomorphic image of a graph G is obtained by collapsing independent sets of vertices into single vertices (satisfying the preservation of edges); extra copies of resulting edges are deleted. Conversely, if H contains a homomorphic image of an n-vertex graph G, then $H[n]$ contains G, where $H[n]$ is obtained from H by expanding each vertex into an independent set of size n (and each edge into a copy of the complete bipartite graph $K_{n,n}$). Since $H[n]$ admits a homomorphism into H and homomorphisms compose, $\chi_c(H[n]) \leq \chi_c(H)$. By monotonicity, equality holds.

For a family $\mathcal{G}$ of graphs, let $\mathrm{Hom}(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} \mathrm{Hom}(G)$. Burr, Erdős, and Lovász [5] proved

$$R_\chi(\mathcal{F}, \mathcal{G}) = R(\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G})) = \inf\{n\colon\ K_n \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}.$$

Their idea in [5] applies also to R_{χ_c}, yielding the following result.

Theorem 4.1.4. $R_{\chi_c}(\mathcal{F}, \mathcal{G}) = \inf\{p/q\colon\ K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}$.

Proof. Suppose that a graph H with $\chi_c(H) = p/q$ forces $(\mathcal{F}, \mathcal{G})$. Since, H is a homomorphism to $K_{p:q}$, by Lemma 4.1.1 we have $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. Thus $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \geq \inf\{p/q\colon\ K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}$.

For the reverse inequality, suppose $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. Let $H = K_{p:q}[n]$, where n is the largest number of vertices among graphs in $\mathcal{F}$ or $\mathcal{G}$. When N is sufficiently large, every 2-edge-coloring of $H[N]$ contains a copy of H in which each copy of $K_{n,n}$ corresponding to a single edge of $K_{p:q}$ is monochromatic. This is proved by iterating the bipartite version of Ramsey's Theorem, as in [5]. In the corresponding 2-edge-coloring of $K_{p:q}$, there is a red copy of some graph in $\mathrm{Hom}(\mathcal{F})$ or a blue copy of some graph in $\mathrm{Hom}(\mathcal{G})$, since $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. In the 2-edge-coloring of H, this subgraph expands into a monochromatic copy of the corresponding graph in $\mathcal{F}$ or $\mathcal{G}$ with the right color. Hence $H[N] \to (\mathcal{F}, \mathcal{G})$. Since $\chi_c(H[N]) = p/q$, we have $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \le \inf\{p/q\colon\ K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}$. □

For simplicity, when $\mathcal{F} = \{F\}$ and $\mathcal{G} = \{G\}$ we write $H \to (F, G)$ for $H \to (\mathcal{F}, \mathcal{G})$. Also, we write $H \to G$ for $H \to (G, G)$; note that always $R_\chi(G) - 1 \le R_{\chi_c}(G) \le R_\chi(G)$. Theorem 4.1.2 yields $R_{\chi_c}(G) \le R_{\chi_c}(K_{p:q})$ when $\chi_c(G) = p/q$, so studying $R_{\chi_c}(K_{p:q})$ is of interest to give upper bounds on $R_{\chi_c}(G)$, especially when $\chi_c(G)$ is small. In Section 4.2, we compute circular chromatic Ramsey numbers for small complete graphs.

Section 4.3 presents the proof of $R_{\chi_c}(C_5) = 4$ as Theorem 4.3.4. This yields the following corollary.

Corollary 4.1.5. $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \le 4$ *whenever* $\mathcal{F}$ *and* $\mathcal{G}$ *each contain some graph with circular chromatic number at most* $5/2$.

Proof. Set $\mathcal{F}' = \mathcal{G}' = \{C_5\}$ in Lemma 4.1.2. □

Every graph with chromatic number at most 4 decomposes into two bipartite graphs. Hence $R_{\chi_c}(G) \ge 4$ when G is non-bipartite. From the characterization of $R_\chi(G)$ in [5], it follows that the chromatic Ramsey number of any 3-chromatic graph G is 5 or 6, and it is 5 if and only if G admits a homomorphism to C_5, meaning $\chi_c(G) \le 5/2$. Hence $2 < \chi_c(G) \le 5/2$ implies $R_{\chi_c}(G) = 4$, from Corollary 4.1.5. In particular, R_{χ_c} and R_χ are different.

The next topic of interest is to determine $R_{\chi_c}(G)$ when $5/2 < \chi_c(G) \le 3$. Note that $R_\chi(G) = 6$ for such G. Since $R_\chi(G) - 1 \le R_{\chi_c}(G) \le R_\chi(G)$, if $5/2 < \chi_c(G) < 3$ then $5 \le R_{\chi_c}(G) \le 6$, which implies that there are no circular chromatic Ramsey numbers between 4 and 5. In fact, we have not found any non-integer circular chromatic Ramsey number. As a small piece of information about this question, we show $9/2 \le R_{\chi_c}(C_3, C_5) \le 5$ in Section 4.3.

Nevertheless, the values of $R_{\chi_c}(G)$ vary exponentially when $\chi_c(G)$ is fixed. Let $R_{\chi_c}(z) = \inf\{R_{\chi_c}(G)\colon\ \chi_c(G) \ge z\}$ and $R_\chi(k) = \inf\{R_\chi(G)\colon\ \chi(G) = k\}$. Zhu [42] proved the conjecture from [5] that $R_\chi(k) = (k-1)^2 + 1$. The definitions then yield $R_{\chi_c}(k) \le k^2 + 1$, and in fact we prove in Section 4.4 that $R_{\chi_c}(k) \le k(k-1)$. On the other hand, $R_{\chi_c}(K_k, K_k) \ge R_\chi(K_k, K_k) - 1 = R(k,k) - 1 > k^{1/2}2^k$, using the well-known lower bound on classical Ramsey numbers. In particular, $R_{\chi_c}(G)$ is not determined by $\chi_c(G)$.

4.2 Complete Graphs

In this section, we determine $R_{\chi_c}(K_3)$ and $R_{\chi_c}(K_3, K_4)$. Let the *length* of an edge v_iv_j in $K_{p:q}$ be $\min\{|i-j|, p-|i-j|\}$. Recall that $R_\chi(G) - 1 \le R_{\chi_c}(G) \le R_\chi(G)$.

Theorem 4.2.1. *$R_{\chi_c}(K_3) = 6$.*

Proof. Since $R_\chi(K_3) = 6$, we have $R_{\chi_c}(K_3) \le 6$. For the lower bound, it suffices to find for all q a 2-edge-coloring of $K_{6q-1:q}$ having no monochromatic triangle. Assign red to the edges with lengths $q, \dots, 2q-1$, and assign blue to the edges with lengths $2q, \dots, \lfloor \frac{6q-1}{2} \rfloor$.

The lengths of the edges of a triangle in $K_{6q-1:q}$ can be named a, b, c so that $a+b+c = 6q-1$ or $a+b = c$. For red edges of lengths a, b, c, we have $a+b+c \le 6q-3 < 6q-1$ and $a+b \ge 2q > c$. For blue edges of these lengths, we have $a+b+c \ge 6q > 6q-1$ and $a+b \ge 4q > c$. With a contradiction in each case, there is no monochromatic triangle. □

Theorem 4.2.2. *$R_{\chi_c}(K_3, K_4) = 9$.*

Proof. Since $R_\chi(K_3, K_4) = 9$, we have $R_{\chi_c}(K_3, K_4) \le 9$. For the lower bound, it suffices to find for all q a 2-edge-coloring of $K_{9q-1:q}$ having no red triangle and no blue 4-clique. Assign red to all edges with lengths $q, \dots, 2q-1$ and $4q, \dots, \lfloor \frac{9q-1}{2} \rfloor$, and assign blue to the edges with lengths $2q, \dots, 4q-1$.

The lengths of the three edges of a triangle in $K_{9q-1:q}$ can be indexed as a, b, c with $a \leq b \leq c$ so that $a + b + c = 9q - 1$ or $a + b = c \leq (9q - 1)/2$. If $c \leq 2q - 1$, then the sum is at most $6q - 3$, while $a + b > c$. If $b \geq 4q$, then again $a + b > c$. Finally, if $c \geq 4q$ and $b < 2q$, then $a + b < c$ and $a + b + c < 8.5q$. Hence there is no red triangle.

Consider the four vertices of a 4-clique in $K_{9q-1:q}$ in cyclic order of indices. The edge joining two opposite vertices u and v forms triangles with each of the two remaining vertices, x and y. In one of these triangles, the length of uv is the sum of the lengths of the other two edges. However, the sum of the lengths of two blue edges exceeds the lengths of all blue edges. Hence there is no blue 4-clique. $\square$

Determining $R_{\chi_c}(K_4)$ or $R_{\chi_c}(K_3, K_5)$ is more difficult. Since $R(K_s, K_t) \leq R(K_{s-1}, K_t) + R(K_s, K_{t-1})$ (one less if the summands are both even), we have $R_{\chi_c}(K_4) \leq R(K_4, K_4) \leq 9 + 9 \leq 18$, and $R_{\chi_c}(K_3, K_5) \leq 5 + 9 = 14$.

Greenwood and Gleason [14] showed $R(K_4) = 18$ by coloring K_{17} with no monochromatic 4-clique. We call the host $K_{17:1}$ because the coloring has cyclic symmetry, constant on edges of fixed length. Edges of lengths $1, 2, 4, 8$ are red; those of lengths $2, 5, 6, 7$ are blue. There are monochromatic 4-cycles, but then the diagonals do not both have that color.

To show $R_{\chi_c}(K_4) = 18$, if the edges of $K_{18q-1:q}$ are colored similarly, then following the constructions for the theorems above the edges of lengths $jq, \ldots, (j+1)q - 1$ are red for $j \in \{1, 2, 4, 8\}$ and blue for $j \in \{3, 5, 6, 7\}$. Since the maximum length is $9q - 1$, all the edges are colored. Unfortunately, when $q = 2$ this coloring of $K_{35:2}$ creates a blue 4-clique having an outer 4-cycle with edges of lengths $10, 10, 10, 5$ and diagonals of length 15. This suggests trying to show $R_{\chi_c}(K_4) = 17$ by showing $K_{17q+1:q} \to K_4$ for $q > 1$, but we have not been able to do this.

Similar difficulties arise with $R_{\chi_c}(K_3, K_5)$, where $R_{\chi}(K_3, K_5) = R(K_3, K_5) = 14$ is known. The cyclic coloring of K_{13} with red on edges of lengths 1 and 5 and blue on the rest yields no red triangle or blue 4-clique. However, giving red to edges of lengths in $\{2, 3, 10, 11\}$ in $K_{27:2}$

and blue to the rest yields a blue copy of K_5 but in fact a blue copy of K_6, with distances along the outer cycle alternating between 4 and 5. Again this suggests $R_{\chi_c}(K_3, K_5) < R_\chi(K_3, K_5)$.

4.3 Odd Cycles

Our main task in this section is to show $R_{\chi_c}(C_5) = 4$. Since C_3 and C_5 are homomorphic images of C_5, we accomplish this by proving for $q \geq 1$ that every 2-edge-coloring of $K_{4q+1:q}$ has a monochromatic 3-cycle or a monochromatic 5-cycle.

Let $K^-_{p:q}$ denote $K_{p:q} - v_0 v_q$ (deleting a shortest edge). In this graph we call the endpoints of the edge that was deleted the *special pair*. Let a $3,5$-*free coloring* of a graph G be a 2-edge-coloring having no monochromatic 3-cycle or 5-cycle. We start with two basic lemmas.

Lemma 4.3.1. *Every $3,5$-free coloring of $K^-_{5:1}$ has monochromatic paths of length 2 in both colors joining v_0 and v_4, the endpoints of the missing edge.*

Proof. Consider a $3,5$-free coloring. There are nine edges; let red be the larger class. Each color class must be bipartite.

There are five or six red edges, since the maximum number of edges in a bipartite subgraph of K_5 is 6, achieved only by $K_{2,3}$. To have at least five red edges, the partite sets of the red graph must have sizes 2 and 3. Hence the red graph is $K_{2,3}$ with at most one edge deleted.

Since the blue graph must not contain a triangle, the partite set of size 3 must contain $\{v_0, v_4\}$. Now there is a blue path joining them through the third vertex of that part. There is a red path joining them via the other partite set, because there are two such possible paths and at most one edge was deleted from $K_{2,3}$ to form the red graph. □

Lemma 4.3.2. *In $V(K^-_{4q+1:q})$, let $S = \{v_0, v_q, v_{2q+1}, v_{3q+1}\}$ and $T = \{v_0, v_q, v_{2q}, v_{3q+1}\}$. Both $K^-_{4q+1:q} - S$ and $K^-_{4q+1:q} - T$ are isomorphic to $K^-_{4(q-1)+1:q-1}$, with $\{v_{q+1}, v_{2q}\}$ being the special pair when S is deleted and $\{v_{2q+1}, v_{3q}\}$ being the special pair when T is deleted.*

Proof. The vertices of S or T are spaced by $q, q, q, q+1$ (in cyclic order) along the indexing. Hence when S or T is deleted, any two vertices at least $q-1$ steps apart in the new indexing were separated by a deleted vertex and hence were at least q steps apart in the old indexing, except the pair $\{v_{q+1}, v_{2q}\}$ in the first case and the pair $\{v_{2q+1}, v_{3q}\}$ in the second case. Hence the edges are those of $K^-_{4(q-1)+1:q-1}$, with the special pairs as specified. □

Our approach is to prove inductively for $q \geq 1$ that every $3,5$-free coloring of $K^-_{4q+1:q}$ yields paths of length 2 in both colors joining the vertices of the special pair. Lemma 4.3.1 shows this for $q = 1$. When $q = 2$, the special pairs in the two resulting subgraphs $K^-_{9:2} - S$ and $K^-_{9:2} - T$ in Lemma 4.3.2 are $\{v_3, v_4\}$ and $\{v_5, v_6\}$. In the inductive proof of the main theorem, we will combine Lemma 4.3.2 with the following technical result about these two pairs in $K^-_{9:2}$. We write a path or cycle with vertices $v_1, \ldots, v_n$ in order as $\langle v_1, \ldots, v_n \rangle$ or $[v_1, \ldots, v_n]$, respectively.

Lemma 4.3.3. *Any $3,5$-free coloring of $K^-_{9:2} - \{v_1, v_8\}$ having no monochromatic v_3, v_4-path or v_5, v_6-path of length 3 has monochromatic v_0, v_2-paths of length 2 in both colors.*

Proof. Let $G' = K^-_{9:2} - \{v_1, v_8\}$ and $G = G' - v_0$, shown in bold in Figure 4.1. Let G_r and G_b be the red and blue color classes of G under the given $3,5$-free coloring. Since G has only six vertices, G_r and G_b are bipartite. We prove first that v_3 and v_4 are in the same partite set in each of G_r and G_b, as are v_5 and v_6. By symmetry, it suffices to forbid v_3 and v_4 being in opposite parts in G_r.

By hypothesis there is no red v_3, v_4-path of length 3, so being in opposite parts requires a spanning v_3, v_4-path P in G_r. After v_3, the next vertex u must be one of $\{v_5, v_6, v_7\}$. In each case, we obtain a contradiction. If $u = v_5$, then $P = \langle v_3, v_5, v_7, v_2, v_6, v_4 \rangle$, but then $\langle v_5, v_7, v_2, v_6 \rangle$ is a forbidden red v_5, v_6-path of length 3. If $u = v_6$, then $P = \langle v_3, v_6, v_2, v_5, v_7, v_4 \rangle$. To avoid completing red odd cycles with edges of P, both v_3v_7 and v_7v_2 must be blue. Now there are v_3, v_2-paths of length 2 in both colors, and one extends along v_2v_4 to complete a

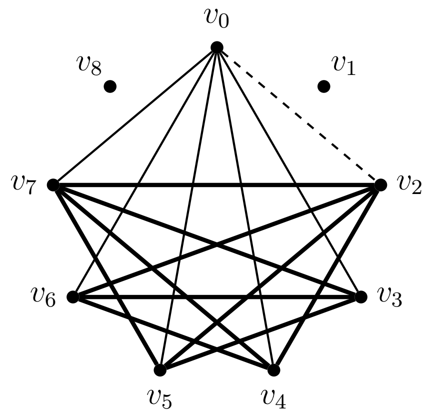


Figure 4.1: The graphs G' and G in Lemma 4.3.3

monochromatic v_3, v_4-path of length 3. If $u = v_7$, then $P = \langle v_3, v_7, v_5, v_2, v_6, v_4\rangle$. To avoid completing red odd cycles with edges of P, all of $\{v_7v_2, v_2v_4, v_4v_7\}$ must be blue, which completes a blue 3-cycle.

Now the pairs $\{v_3, v_4\}$ and $\{v_5, v_6\}$ each lie in one partite set in both G_r and G_b. Since $\{v_2, v_4, v_6\}$ and $\{v_3, v_5, v_7\}$ form triangles, putting all of v_3, v_4, v_5, v_6 into the same part in G_r or G_b forces v_2 and v_7 into the other part. Similarly, since $\{v_2, v_7\}$ cannot lie in the same part with v_4 or v_5, putting $\{v_3, v_4\}$ and $\{v_5, v_6\}$ into opposite parts forces v_2 and v_7 into opposite parts. Hence each of the resulting bipartitions R and B of the indices has three possibilities: (3456|27), (347|562), and (342|567). Since the edges within a partite set get the other color, each choice for R restricts the choice for B. Since the two subgraphs cannot have the same bipartition, by symmetry there remain three cases. In each case we study G' to obtain the monochromatic v_0, v_2-paths of length 2 in both colors.

Case 1: $R = (3456|27)$, $B = (347|562)$. If v_0v_7 is red, then avoiding $[v_0, v_7, v_4]$ in red makes v_0v_4 blue. Now avoiding $[v_0, v_4, v_6, v_3, v_5]$ in blue makes v_0v_5 red, so $\langle v_0, v_5, v_2\rangle$ is red. Avoiding $[v_4, v_7, v_0, v_5, v_2]$ in red makes v_4v_2 blue, so $\langle v_0, v_4, v_2\rangle$ is blue.

If v_0v_7 is blue, then $\langle v_0, v_7, v_2\rangle$ is blue. Avoiding $\langle v_0, v_5, v_2\rangle$ and $\langle v_0, v_6, v_2\rangle$ in red would make v_0v_5 and v_0v_6 blue. Avoiding $[v_0, v_4, v_6]$ in blue then makes v_0v_4 red. Avoiding

$[v_2, v_7, v_0, v_6, v_4]$ in blue makes v_4v_2 red, and now $\langle v_0, v_4, v_2\rangle$ is red.

Case 2: $R = (3456|27)$, $B = (342|567)$. If v_0v_7 is red, then avoiding $[v_0, v_7, v_5]$ in red makes v_0v_5 blue. Now avoiding $[v_0, v_5, v_3, v_6, v_4]$ in blue makes v_0v_4 red, and hence $\langle v_0, v_4, v_2\rangle$ is red. Avoiding $[v_5, v_7, v_0, v_4, v_2]$ in red makes v_5v_2 blue, so $\langle v_0, v_5, v_2\rangle$ is blue.

If v_0v_7 is blue, then $\langle v_0, v_7, v_2\rangle$ is blue. Avoiding $\langle v_0, v_4, v_2\rangle$ in red would make v_0v_4 blue, and then avoiding $[v_0, v_4, v_6]$ in blue makes v_0v_6 red. Avoiding $[v_2, v_7, v_0, v_4, v_6]$ in blue makes v_2v_6 red, and now $\langle v_0, v_6, v_2\rangle$ is red.

Case 3: $R = (342|567)$, $B = (347|562)$. If v_0v_7 is red, then avoiding $[v_0, v_7, v_4]$ in red makes v_0v_4 blue, so $\langle v_0, v_4, v_2\rangle$ is blue. Now avoiding $[v_0, v_4, v_2, v_7, v_5]$ in blue makes v_2v_7 or v_0v_5 red, so $\langle v_0, v_7, v_2\rangle$ or $\langle v_0, v_5, v_2\rangle$ is red.

If v_0v_7 is blue, then avoiding $[v_0, v_7, v_5]$ in blue makes v_0v_5 red, so $\langle v_0, v_5, v_2\rangle$ is red. Avoiding $[v_0, v_5, v_2, v_7, v_4]$ in red makes v_2v_7 or v_0v_4 blue, so $\langle v_0, v_7, v_2\rangle$ or $\langle v_0, v_4, v_2\rangle$ is blue.

□

We can now complete the proof of the main result of this section.

Theorem 4.3.4. *$R_{\chi_c}(C_5) = 4$.*

Proof. It suffices to show $K_{4q+1:q} \to \{C_3, C_5\}$ for $q \geq 1$. We use induction on q to prove that every $3,5$-free coloring of $K^-_{4q+1:q}$ contains monochromatic v_0, v_q-paths of length 2 in both colors. Adding the edge v_0v_q then completes a monochromatic triangle. Lemma 4.3.1 proves the case $q = 1$.

For $q > 1$, let $G = K^-_{4q+1:q}$. Consider a $3,5$-free coloring of G. Let $S = \{v_0, v_q, v_{2q+1}, v_{3q+1}\}$ and $T = \{v_0, v_q, v_{2q}, v_{3q+1}\}$. By Lemma 4.3.2, both $G - S$ and $G - T$ are isomorphic to $K^-_{4(q-1)+1:q-1}$, with special pairs $\{v_{q+1}, v_{2q}\}$ and $\{v_{2q+1}, v_{3q}\}$, respectively. By the induction hypothesis, there are monochromatic v_{q+1}, v_{2q}-paths and v_{2q+1}, v_{3q}-paths of length 2 in both colors. A monochromatic v_{q+1}, v_{2q}-path or v_{2q+1}, v_{3q}-path of length 3 in G would thus complete a monochromatic closed odd walk of length 5, which would yield a monochromatic

3-cycle or 5-cycle, so there is no such path for either pair.

Now consider the subgraph of G induced by $\{v_0, v_q, v_{q+1}, v_{2q}, v_{2q+1}, v_{3q}, v_{3q+1}\}$. This subgraph is isomorphic to $K_{9:2}^- - \{v_1, v_8\}$, with the vertices representing $v_0, v_2, v_3, v_4, v_5, v_6, v_7$ in order. By Lemma 4.3.3, there are monochromatic v_0, v_q-paths of length 2 in both colors. □

We close this section by proving $9/2 \leq R_{\chi_c}(C_3, C_5) \leq 5$.

Theorem 4.3.5. $9/2 \leq R_{\chi_c}(C_3, C_5) \leq 5$

Proof. For the upper bound, it suffices to show that $K_{5:1} \rightarrow (\mathrm{Hom}(C_3), \mathrm{Hom}(C_5))$. Note that $\mathrm{Hom}(C_5) = \{C_3, C_5\}$. It is well known that the only red/blue-coloring of K_5 having no monochromatic triangle has monochromatic 5-cycles in both colors.

For the lower bound, we show that the red/blue-coloring of $E(K_{9:2})$ in Figure 4.2 contains no red copy of C_3 and no blue copy of C_3 or C_5. To describe the coloring, let G_r and G_b denote the red subgraph and blue subgraph, respectively. The graph G_r has the edges v_0v_i for $3 \leq i \leq 6$ and all edges joining $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7, v_8\}$ except $\{v_4v_6, v_6v_3, v_3v_5\}$ (i.e., the edges joining vertices whose edges to v_0 are red). The graph G_b contains all the remaining edges.

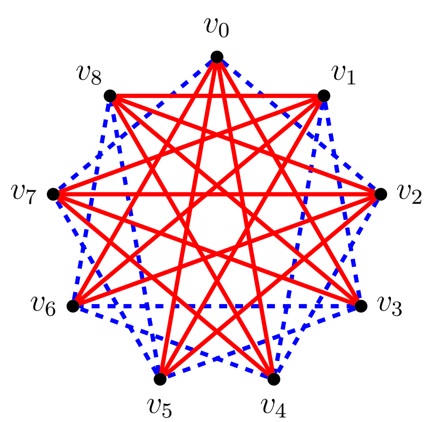


Figure 4.2: Coloring for Theorem 4.3.5; solid edges are red.

Note that $G_r - v_0$ is bipartite. Also, since all edges among neighbors of v_0 are blue, v_0 lies in no red triangle. Hence G_r has no red triangle.

Since $G_b - v_0$ is bipartite with partite sets $\{v_1, v_2, v_5, v_6\}$ and $\{v_3, v_4, v_7, v_8\}$, every blue odd cycle visits v_0. Note that G_b has all edges of length 2 except v_1v_8 and all edges of length 3 except v_1v_4, v_3v_6, v_5v_8. Since v_0 and its two neighbors (v_2 and v_7) have degree 2 in G_b (forming the path $\langle v_4, v_2, v_0, v_7, v_5\rangle$) and v_4v_5 is not an edge, there is no blue odd cycle of length at most 5. □

Remark 4.3.6. Dan Cranston showed $R_{\chi_c}(C_3, C_5) \geq 14/3$, and Douglas West showed $R_{\chi_c}(C_3, C_7) \leq 9/2$.

4.4 Minimizing $R_{\chi_c}(z)$

We defined $R_{\chi_c}(z) = \inf\{R_{\chi_c}(G)\colon\ \chi_c(G) \geq z\}$ and $R_\chi(k) = \inf\{R_\chi(G)\colon\ \chi(G) = k\}$, in Section 4.1. Since Zhu [42] proved the conjecture from [5] that $R_\chi(k) = (k-1)^2 + 1$, we obtain

$$\begin{aligned} R_{\chi_c}(k) &= \inf\{R_{\chi_c}(G)\colon\ \chi_c(G) \geq k\} \leq \inf\{R_\chi(G)\colon\ \chi_c(G) \geq k\} \\ &\leq \inf\{R_\chi(G)\colon\ \chi(G) \geq k+1\} \\ &\leq \inf\{R_\chi(G)\colon\ \chi(G) = k+1\} = R_\chi(k+1) = k^2 + 1. \end{aligned}$$

The argument of Lemma 4.4.2 explains why we use $\chi_c(G) \geq z$ instead of $\chi_c(G) = z$ in defining $R_{\chi_c}(z)$. We prove the stronger inequality $R_{\chi_c}(k) \leq k(k-1)$, following the method of [42]. The *fractional chromatic number* of a graph G, written $\chi_f(G)$, is the linear programming relaxation of the chromatic number. That is, $\chi_f(G)$ is the minimum sum of weights on the independent sets in G such that each vertex belongs to independent sets with total weight at least 1. A (p, q)-coloring of G provides such a weighting with total weight p/q, so $\chi_f(G) \leq \chi_c(G)$. The *direct product* of graphs G and H, written $G \times H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$

and $vv' \in E(H)$. Zhu [42] proved the following theorem.

Theorem 4.4.1 (Zhu [42])**.** $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$.

Iterating the product yields $\chi_f(G_1 \times \cdots \times G_t) = \min\{\chi_f(G_1), \ldots, \chi_f(G_t)\}$.

Lemma 4.4.2. *If every 2-edge-coloring of H contains a monochromatic subgraph with fractional chromatic number at least z, then there exists a graph G with $\chi_f(G) \geq z$ such that $H \to \mathrm{Hom}(G)$.*

Proof. The number of 2-edge-colorings of H is $2^{|E(H)|}$. Let G_i be a graph with fractional chromatic number at least z that occurs as a monochromatic subgraph in the ith coloring. Let $G = G_1 \times \cdots \times G_{2^{|E(H)|}}$. Each G_i is a homomorphic image of G, obtained by mapping the independent sets having a fixed value in the ith coordinate into the corresponding vertices in G_i. Hence $H \to \mathrm{Hom}(G)$, by construction. □

Lemma 4.4.3. *If every 2-edge-coloring of $K_{p:q}$ contains a monochromatic subgraph with fractional chromatic number at least z, then $R_{\chi_c}(z) \leq p/q$; that is, there exists a graph G with $\chi_c(G) \geq z$ and $R_{\chi_c}(G) \leq p/q$.*

Proof. By Lemma 4.4.2, there exist a graph G with $\chi_f(G) \geq z$ and $K_{p:q} \to \mathrm{Hom}(G)$. Since $\chi_c(G) \geq \chi_f(G)$ and $R_{\chi_c}(G) = \inf\{p/q\colon\ K_{p:q} \to \mathrm{Hom}(G)\}$, the claim follows. □

To complete the desired result, we study edge-coloring of $K_{k(k-1)}$.

Theorem 4.4.4. $R_{\chi_c}(k) \leq k(k-1)$ *for* $k \in \mathbb{N} - \{1\}$.

Proof. Let G_R and G_B be the spanning subgraphs formed by the color classes in a red/blue edge-coloring of $K_{k(k-1)}$, respectively. If G_R has a clique of size k, then $\chi_f(G_R) \geq k$; otherwise, the complement G_B of G_R has independence number at most $k-1$, and then $\chi_f(G_B) \geq k(k-1)/(k-1) = k$. Since $K_{k(k-1)} = K_{k(k-1):1}$, the claim follows from Lemma 4.4.3. □

In Section 4.3, we proved that $R_{\chi_c}(z) = 4$ for $2 < z \leq 5/2$. Theorem 4.4.4 includes $R_{\chi_c}(3) \leq 6$, which also follows directly from $R(K_3, K_3) = 6$. Nevertheless, it seems likely that this easy upper bound can be improved. Since there is no circular Ramsey number between 4 and 5, it follows that $R_{\chi_c}(z) \geq 5$ for $z > 5/2$. This suggests the following question.

Question 4.4.5. Is it true that $R_{\chi_c}(z) = 5$ for $5/2 < z \leq 3$?

A positive answer would follow from showing for $q \geq 1$ that every red/blue edge-coloring of $K_{5q+1:q}$ contains a monochromatic subgraph with fractional chromatic number at least 3.

Remark 4.4.6. A step in this direction would be to show that every 2-edge-coloring of $K_{11:2}$ contains a monochromatic subgraph with fractional chromatic number bounded below by some value z with $z > 5/2$, perhaps $8/3$ or even $11/4$. Such a result would yield $R_{\chi_c}(z) \leq 5.5$.

Possibly also we can determine or bound R_{χ_c} for one or more of the graphs in Figure 4.3.

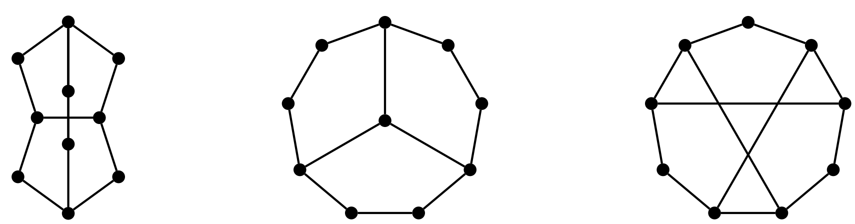


Figure 4.3: Graphs suggested for Remark 4.4.6

References

[1] M. Aigner and E. Triesch, Realizability and uniqueness in graphs, *Discrete Math.* **136** (1994), 103–113.

[2] S. V. Avgustinovich, On the partition of a set of edges of a graph into isomorphic trees, (Russian) *Diskretn. Anal. Issled. Oper. Ser. 1*, **4** (1997), 3–5, 115.

[3] M. D. Barrus, S. G. Hartke, K. F. Jao, and D. B. West, Length thresholds for graphic lists given fixed largest and smallest entries and bounded gaps, *Discrete Math.* **312** (2012), 1494–1501.

[4] J. A. Bondy and P. Hell, A note on the star chromatic number. *J. Graph Theory* **14** (1990), 479–482.

[5] S. Burr, P. Erdős, and L. Lovász, On graphs of Ramsey type, *Ars Combinatoria* **1** (1976), 167–190.

[6] Y. Caro and D. B. West, Repetition number of graphs, *Electron. J. Combin.* **16** (2009), no. 1, Research Paper 7.

[7] R. B. Eggleton, Graphic sequences and graphic polynomials: a report. *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday)*, Vol. 1, pp. 385–392. *Colloq. Math. Soc. Janos Bolyai*, **10** (North-Holland, Amsterdam, 1975).

[8] P. Erdős, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, The size Ramsey number, *Period. Math. Hungar.* **9** (1978), 145–161.

[9] P. Erdős and T. Gallai, Graphs with prescribed degree of vertices (Hungarian), *Mat. Lapok* **11** (1960), 264–274.

[10] P. Erdős and J. R. Griggs, Problem Session, Workshop on Planar Graphs, DIMACS, Rutgers Univ., 1991.

[11] J. F. Fink, On the decomposition of n-cubes into isomorphic trees, *J. Graph Theory* **14** (1990), 405–411.

[12] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.* **18** (1970), 19–24.

[13] R. L. Graham, B. L. Rothchild and J. H. Spencer, Ramsey Theory, *John Wiley & Sons*, 1990.

[14] R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1–7.

[15] J. R. Griggs and Y.-C. Lin, Planar graphs with few vertices of small degree. *Discrete Math.* **143** (1995), 47–70.

[16] J. R. Griggs and Y.-C. Lin, The maximum sum of degrees above a threshold in planar graphs. *Discrete Math.* **169** (1997), 233–243.

[17] B. Grünbaum and T. S. Motzkin, The number of hexagons and the simplicity of geodesics on certain polyhedra. *Canad. J. Math.* **15** (1963), 744–751.

[18] R. Häggkvist, Decompositions of complete bipartite graphs, in *Surveys in Combinatorics, 1989* (J. Siemons, ed.) *London Math. Soc. Lect. Notes* **141**, Cambridge Univ. Press (1989), 115–147.

[19] S. L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph, *SIAM J. Appl. Math.* **10** (1962), 496–506.

[20] P. L. Hammer, T. Ibaraki, and B. Simeone, Threshold sequences, *SIAM J. Alg. Disc. Methods* **2** (1981), 39–49.

[21] V. Havel, A remark on the existence of finite graphs, *Casopis Pest. Mat.* **80** (1955), 477–480.

[22] P. Horn, K. G. Milans, and V. Rödl, Degree Ramsey number of blowups of trees, submitted.

[23] K. F. Jao, A. V. Kostochka, and D. B. West, Decomposition of cartesian products of regular graphs into isomorphic trees, submitted.

[24] K. F. Jao, C. Tardif, D. B. West, and X. Zhu, Circular chromatic Ramsey number, in preparation.

[25] K. F. Jao, and D. B. West, Vertex degrees in outerplanar graphs, *J. Comb. Math. & Comb. Computing* **82** (2012), 229–239.

[26] T. Jiang, K.G. Milans, and D.B. West, Degree Ramsey number for cycles and blowups of trees, *Europ. J. Combin.* **34** (2013), 414–423.

[27] W.B. Kinnersley, K.G. Milans, and D.B. West, Degree Ramsey numbers of graphs, *Combin., Probab., Comput.* **21** (2012), 229–253.

[28] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. Math. Ann. **77** (1916), 453–465.

[29] M. Kouider and Z. Lonc, Path decompositions and perfect path double covers, *Australas. J. Combin.* **19** (1999), 261–274.

[30] S. Y. R. Li, Graphic sequences with unique realization, *J. Combinatorial Theory Ser. B* **19** (1975), 42–68.

[31] J. Nešetřil and V. Rödl, The Ramsey property for graphs with forbidden complete subgraphs, *J. Combinatorial Theory Ser. B* **20** (1976), no. 3, 243–249.

[32] J. Petersen, Die Theorie der regulären Graphen, *Acta Math.* **15** (1891), 193–220.

[33] G. Ringel, Problem 25, *Theory of Graphs and Its Applications (Proc. Symp. Smolenice 1963)*, (Czech. Acad. Sci., 1964), 162.

[34] V. Rödl and E. Szemerédi, On size Ramsey numbers of graphs with bounded maximum degree, *Combinatorica* **20** (2000), 257–262.

[35] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris., (1967) 349–355.

[36] H. Snevily, *Combinatorics of finite sets*, PhD Thesis, University of Illinois, 1991.

[37] A. Tripathi, and S. Vijay, A note on a theorem of Erdős and Gallai. *Discrete Math.* **265** (2003), 417–420.

[38] A. Vince, Star chromatic number, *J. Graph Theory* **12** (1988), 551–559.

[39] D. B. West, Exercise 1.3.57, *Introduction to Graph Theory, 2nd ed.*, (Prentice-Hall, 2001), 52.

[40] D. B. West and T. G. Will, Vertex degrees in planar graphs, in *Planar graphs (New Brunswick, NJ, 1991)*, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **9** (Amer. Math. Soc., 1993), 139–149.

[41] X. Zhu, Chromatic Ramsey numbers, *Discrete Math.* **190** (1998), 215–222.

[42] X. Zhu, The fractional version of Hedetniemi's conjecture is true, *Europ. J. Combin.* **32** (2011), 1168–1175.

[43] I. E. Zverovich and V. E. Zverovich, Contributions to the theory of graphic sequences, *Discrete Math.* **105** (1992), 293–303.