THE ANALYTIC AND ASYMPTOTIC BEHAVIORS OF VORTICES

BY

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DISSERTATION

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Abstract

We study vortex equations with a parameter s on smooth vector bundles E over compact Kähler manifolds M. For each s, we invoke techniques in [Br] by turning vortex equations into the elliptic partial differential equations considered in [K-W] and obtain a family of solutions. Our results show that away from a singular set, such a family exhibit well controlled convergent behaviors, leading us to prove conjectures posed by Baptista in [Ba] concerning dynamic behaviors of vortices. These results are published in [Li].

We also analyze the analytic singularities on the singular set. The analytic singularities of the PDE's reflect topological inconsistencies as $s \to \infty$. On the second part of the thesis, we form a modification of the limiting objects, leading to a phenomenon of energy concentration known as the "bubbling". We briefly survey the established bubbling results in literature.

To the memory of my beloved grandparents

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Notation

Unless otherwise stated, the following notation is reserved for certain frequently used definitions.

(M,ω): A compact Kähler manifold of complex dimension n with Kähler form ω.

(Σ,ω): A Riemann surface $(n=1)$ with Kähler form ω.

E: A smooth vector bundle over (M,ω) of rank m and degree r.

L: A smooth line bundle over (M,ω) of degree r.

$W^{k,p}$: The space of functions with finite Sobolev k,p norms on M (Σ).

$\mathcal{C}^j$: The space of functions with finite Sobolev j,∞ norms on M (Σ).

$\mathcal{C}^\infty$: The space of smooth functions on M (Σ).

$\Omega^0(E)$ $\big(\Omega^0(L)\big)$: The space of smooth global sections of the vector (line) bundle E (L) over M (Σ).

Ω^p: The space of smooth p-forms on M (Σ).

$\Omega^p(E)$ $\big(\Omega^p(L)\big)$: The space of smooth E- $(L$-$)$ valued p-forms of the vector (line) bundle E (L) over M (Σ).

$\Omega^{p,q}$: The space of smooth p,q-forms, decomposed with respect to a fixed complex structure of $M(\Sigma)$.

$\Omega^{p,q}(E)$ $\big(\Omega^{p,q}(L)\big)$: The space of smooth p,q-forms with values in E (L), decomposed with respect to a fixed complex structure of $M(\Sigma)$.

$W^{k,p}_q$: The space of q-forms on M with finite Sobolev k,p norms.

$W^{k,p}_q(E)$ $((\Omega^p_{k,p}(L)))$: The space of E-$(L$-$)$ valued q-forms on M with finite Sobolev k,p norms.

$\mathcal{H}$: The space of Hermitian structures of the vector (line) bundle E (L) over M (L).

$\mathcal{A}$: The space of connections of the vector (line) bundle E (L) over M (L).

$\mathcal{A}(H)$: The space of unitary connections with respect to H.

$\mathcal{G}$: The H unitary gauge group of the vector (line) bundle E (L) over M (Σ).

$\mathcal{G}_{\mathbb{C}}$: The complex gauge group of the vector (line) bundle E (L) over M (Σ).

$\mathcal{G}^{k,p}$: The space of gauges with finite Sobolev k, p norms.

$\nu_k(s, \tau)$: The $\mathcal{G}$-gauge classes of solutions to the vortex equations with parameters s and τ.

$\nu_{k,0}(s, \tau)$: The open subset of $\nu_k(s, \tau)$ where the k sections do not vanish simultaneously.

Chapter 1

Introduction

Vortex equations have appeared in various forms, settings, contexts, and levels of generality. They are generally minimizing equations to certain gauge invariant functionals on bundles over smooth manifolds. Given a Kähler manifold (M, ω), and a Hermitian vector bundle (E, H) over it, the functional considered in this thesis depend on a unitary connection $D \in \mathcal{A}(H)$, k smooth sections $\phi = (\phi_1, \ldots, \phi_k) \in \Omega^0(E) \times \ldots \times \Omega^0(E)$, and parameters s, τ. We refer to it as the Yang-Mills-Higgs functional, given by

$$YMH_{\tau,s}(D,\phi) := \frac{1}{2s^2} \left|\left|F_D\right|\right|^2_{L^2} + \sum_{i=1}^{k} \left|\left|D\phi_i\right|\right|^2_{L^2} + \frac{s^2}{4} \left|\left|\sum_{i=1}^{k} |\phi_i|^2_H - \tau I\right|\right|^2_{L^2}. \tag{1.1}$$

These equations originate from physical problems of finding equilibrium state in a configuration space. For $\phi = 0$ and $s, \tau = 1$, (1.1) degenerates into the classical pure Yang-Mills functional, whose minimizers are precisely the connections D with harmonic curvatures with respect to the Hodge Laplacian. An early occurrence of $YMH_{\tau,s}$ can be found in Ginzburg and Landau's description of the free energy of superconducting materials, which depends on the external electromagnetic field strength and the state function of certain electron pairs known as the "Cooper pairs". Finding the equilibrium state of the material amounts to minimizing the free energy. See [J-T] for the complete descriptions. The minimizing data of $YMH_{\tau,s}$ are called the vortices. Bradlow has derived the characterizing equations for minimizers for $YMH_{\tau,s}$ in [Br] (see Chapter 3 for more details):

$$\begin{cases} F_D^{(0,2)} = 0 \\ D^{(0,1)}\phi = 0 \\ \sqrt{-1}\Lambda F_D + \frac{s^2}{2}(\sum_{i=1}^k |\phi_i|_H^2 - \tau) = 0. \end{cases} \tag{1.2}$$

The equations in (1.2) are invariant under unitary gauges, and we denote the gauge classes of solutions by $\nu_k(s,\tau)$. Explicit descriptions of moduli spaces $\nu_1(1,\tau)$ are available in [J-T], when $M = \mathbb{C}$ and in [Br], when $M = \Sigma$, a closed Riemann surface. We provide a brief summary of results from [Br] in Chapter 3. For general k, explicit descriptions of $\nu_k(1,\tau)$ are available in [B-D-W]. We will show that these descriptions remain valid for $\nu_k(s,\tau)$ with essentially no modification when $s < \infty$. However, some modification is necessary when $s = \infty$.

The key to establish these descriptions is to employ techniques in [Br] of constructing an equivalency between solving the vortex equations (1.2) and solving certain elliptic PDEs, whose unique solutions are guaranteed by analytic tools developed in [K-W]. Our main achievement is to analyze the techniques in [K-W] further and construct uniformity of solutions corresponding to vortices whose k sections do not vanish simultaneously.

The uniformity also leads to certain convergent properties, leading us to solve a conjecture posed by Baptista in [Ba] regarding the evolution of metrics. In [M], [M-N], and [S], it was pointed out that a naturally defined L^2 metrics on the moduli spaces of solutions to (1.2) (see Chapter 6 for the precise definition) gives rises to a good approximation of the scattering of fields in certain configuration space. The physical observation motivates one to describe the mathematical behaviors of the L^2 metrics and its asymptotic behaviors as $s \to \infty$. In [Ba], it was conjectured that as $s \to \infty$, the naturally defined L^2 metrics evolve to a familiar one. We prove this conjecture affirmatively in Chapter 6.

Singularities of our analysis arise when the sections possess common zeros. The singularities are reflected by topological inconsistencies of (1.2) as $s \to \infty$, a classical phenomenon known as "bubbling". We will describe these precisely in both analytic and topological points of view, and attempt to form modifications. These modifications force us to adjust the initial topological settings. These discussions form the second part of the thesis.

At the end, we provide a brief survey of established results by [C-G-R-S],[O],[Wo],[X], and [Z] on bubbling phenomenon on more general vortex equations known as the "symplectic vortex equations". These results take place on much greater level of generality, making our discussions in Chapter 7 special cases.

1.1 Statements of the Results

For each $s > 0$, it is a classical result from [K-W] that given $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}^+$, and a non-positive smooth function h, the partial differential equation defined on a compact Riemannian manifold (M, g):

$$\Delta\varphi_s = c(s) - s^2 h e^{\varphi_s},$$

where $c(s) = c_1 - s^2 c_2$, admits a unique solution in $W^{2,p}$, for all p large enough. One of our main results is to establish a uniform bound and convergent behaviors of the family of solutions when h is strictly negative.

Theorem 1.1.1 (Main Theorem)**.** *On a compact Riemannian manifold M without boundary, let c_1 be any constant, c_2 any positive constant, and h any negative smooth function. Let $c(s) = c_1 - c_2 s^2$, for each s large enough, the unique solutions $\varphi_s \in C^\infty$ for the equations*

$$\Delta\varphi_s = c(s) - s^2 h e^{\varphi_s}.$$

are uniformly bounded in $W^{l,p}$ for all $l \in \mathbb{N}$ and $p \in [1, \infty]$. Moreover, in the limit $s \to \infty$,

φ_s converges smoothly to

$$\varphi_\infty = \log\left(\frac{c_2}{-h}\right),$$

the unique solution to

$$he^{\varphi_\infty} + c_2 = 0.$$

Explicit descriptions of $\nu_k(s,\tau)$ have been established in [Br] and [B-D-W]. We are particularly interested in the open subset $\nu_{k,0}(s,\tau)$, the open subset of $\nu_k(s,\tau)$ consisting of those k sections that do not vanish simultaneously. For each s, there is a bijection

$$\Phi_s : Hol_r(\Sigma, \mathbb{CP}^{k-1}) \to \nu_{k,0}(s).$$

See section 3 for the details of the correspondence. Here, $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ is the space of degree r holomorphic maps from Σ to $\mathbb{CP}^{k-1}$. With Theorem 1.1.1 above, we are able to control the family of correspondences Φ_s in suitable topologies. Such a control can be applied to prove a conjecture on dynamics of vortices posed in [Ba]. On the space $\nu_{k,0}(s,\tau)$ defined above, there is a naturally defined L^2 metric, or the kinetic energy of variations of vortices:

$$g_s((\dot{A}_s, \dot{\phi}_s), (\dot{A}_s, \dot{\phi}_s)) = \int_\Sigma \frac{1}{4s^2}\dot{A}_s \wedge *_\Sigma \dot{A}_s + < \dot{\phi}_s, \dot{\phi}_s >_H vol_\Sigma, \tag{1.3}$$

where $(\dot{A}_s, \dot{\phi}_s) \in T_{[D_s,\phi_s]}\nu_{k,0}(s,\tau)$ is the infinitesimal variation of vortices (with certain orthogonality restrictions). On the other hand, there is a classical defined L^2 metric, denoted by $< \cdot, \cdot >_{L^2}$, on $Hol_r(\Sigma, \mathbb{CP}^{k-1})$. It is natural to ask whether the correspondences

$$\Phi_s : (Hol_r(\Sigma, \mathbb{CP}^{k-1}), < \cdot, \cdot >_{L^2}) \to (\nu_{k,0}(s), g_s)$$

are isometries in any sense. Baptista has conjectured the affirmative answer to this question when $s = \infty$. In Chapter 6, we prove the conjecture using the Main Theorem 1.1.1 and other analytic tools. The precise statement is the displayed below. See Chapter 6 for precise definitions and settings.

Proposition 1.1.2 (Precise Baptista's Conjecture)**.** *Equipping* $\mathbb{CP}^{k-1}$ *with the Fubini-Study metric, the sequence of metrics* g_s *on* $\nu_{k,0}(s)$ *given by* (6.1) *Cheeger-Gromov converges smoothly to a multiple of the ordinary* L^2 *metric* $< \cdot, \cdot >_{L^2}$ *on* $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ *given by* (6.2)*. The family of diffeomorphisms are given by* Φ_s*.*

The two results are published on [Li].

The description of the entire moduli space $\nu_k(s, \tau)$ is available in literature such as [B-D-W], for s finite and infinite. For finite s, the classical description of $\nu_k(s, \tau)$ is compatible with our description of open set $\nu_{k,0}(s, \tau)$ and some obvious boundary component. However, the space $\nu_k(\infty, \tau)$ from classical literature does not quite fit the into limiting picture of our main results. We supply an explanation to account for the discrepancy, which is mainly due the bubbling of energies of the vortices near the singularities arisen from the common zeros of the vortices (or the "boundary component" of the moduli space $\nu_k(s, \tau)$.

Chapter 2

Preliminaries

2.1 Definitions

We need to properly introduce the definitions of all the mathematical objects used from complex differential geometry as well as their most basic properties. All the definitions are defined for both vector bundles E and line bundles L, even though we will only list the ones for E. Most definitions are summarized from [K].

- Holomorphic Structures

Holomorphic structures on a complex vector bundle $E \to M$ are complex structures on the total space E. It is given by a $\bar{\partial}$ operator on $\Omega^0(E)$.

Definition 2.1.1 (Holomorphic Structures)**.** A holomorphic structure E is a $\mathbb{C}$-linear operator

$$\bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E),$$

satisfying Leibiniz rule.

Definition 2.1.2 (Holomorphic Sections)**.** A section $s \in \Omega^0(E)$ is a holomorphic section if $\bar{\partial} s = 0$.

- <u>Connections</u>

There are three equivalent ways to define connections (See, for example, [DK]). We adapt the point of view from differential geometry.

Definition 2.1.3 (Connections)**.** Given a rank r smooth vector bundle E over an n-dimensional manifold M, a connection D on $\Omega^0(E)$ is a $\mathbb{C}$-linear map, sending a smooth section to a E-valued one form, satisfying the Leibiniz rule. That is, a linear map

$$D : \Omega^0(E) \to \Omega^1(E)$$

over $\mathbb{C}$ so that

$$D(f\sigma) = \sigma df + fD\sigma \qquad \forall f \in \mathcal{C}^\infty, \quad \sigma \in \Omega^0(E).$$

Inductively, we define connections on $\Omega^p(E)$ to be a $\mathbb{C}$-linear map

$$D : \Omega^p(E) \to \Omega^{p+1}(E),$$

by setting

$$D(\sigma\phi) = (D\sigma) \wedge \phi + \sigma d\phi \qquad \forall \sigma \in \Omega^0(E), \quad \phi \in \Omega^p.$$

For $s \in \Omega^p(E)$, Ds is called the covariant derivative of s.

The space of connections is denoted by $\mathcal{A}$.

The $\mathbb{C}$-linearity of the connection D makes it uniquely characterized by a matrix of locally defined one forms.

Definition 2.1.4 (Connection One Forms)**.** Let $s = (s_1, \ldots, s_r)$ be a local frame of E over an open set $U \subseteq M$, $s_i \in \Omega^0|_U$, and we have

$$Ds_i = \sum_j s_j \omega_i^j, \qquad \omega_i^j \in \Omega_U^1.$$

The matrix $A = (\omega_i^j)$ is called the *connection 1-form* of D.

There is an obvious notion of directional derivative for a connection D:

Definition 2.1.5 (Directional Covariant Derivative)**.** Given $s \in \Omega^0(E)$, and $X \in \Omega^0(TM)$, the covariant derivative of s in the direction of X is given by

$$D_X s = (Ds)(X) \in \Omega^0(E).$$

- Full Covariant Derivative

When the base manifold M is equipped with a Riemannian metric g, it determines a unique connection ∇^M compatible with g and the smooth structure of M, the so called Levi Civita connection. We may define the notion of full covariant derivative on $\Omega^q(E)$, accounting contributions from both D and ∇^M.

Definition 2.1.6 (Full Covariant Derivative)**.** Let ∇^M be the unique Levi Civita connection with respect to g, and D a connection on E. For $\alpha \in \Omega^q(E)$, the full covariant derivative of α, $\nabla\alpha \in \Omega^{q+1}(E)$, acts on $k+1$ vector fields $X_0, \ldots, X_k$ by

$$\nabla\alpha(X_0, \ldots, X_k) = D_{X_0}(\alpha(X_0, \ldots, X_k)) - \alpha(\nabla^M_{X_0} X_1, X_2, \ldots, X_k) - \ldots - \alpha(X_1, \ldots, X_{k-1}, \nabla^M_{X_0} X_k) \tag{2.1}$$

Of course, it makes sense to apply ∇ to a E-valued differential q form multiple times, and we denote ∇^j for that purpose. If $\{\partial_j\}$ is a local frame of TM, we denote ∇_j to be the directional derivative of the full covariant derivative in the direction $X_0 = \partial_j$. For a multi-index $j = (j_1, \ldots, j_l) \in \mathbb{N}^l$, we follow the convention to denote $\nabla_j = \nabla_{j_1} \ldots \nabla_{j_l}$.

- <u>Curvatures</u>

Definition 2.1.7 (Curvature of a Connection)**.** The curvature F_D of a connection D is defined as

$$F_D := D^2 = D \circ D : \Omega^0(E) \to \Omega^2(E).$$

Given a local frame $s = (s_1, \ldots, s_r)$ of E over U, we can write out the coordinate representation of F_D:

$$F_D s_i = \sum_j s_j \theta_i^j \qquad \theta_i^j \in \Omega_U^2.$$

Following Definition 2.1.4, one can readily verify that

$$\theta_i^j = \sum_k \omega_k^j \wedge \omega_i^k + d\omega_i^j.$$

In the matrix notation, we denote

$$\Theta = (\theta_i^j),$$

and

$$\Theta = dA + A \wedge A.$$

The wedge product between matrices is exactly the same as the multiplication of matrices of scalars, where we multiply differential forms with wedge product.

Definition 2.1.8. The matrix Θ is called the curvature two form of the connection D.

Unlike connections, the curvature operator is linear over $\mathcal{C}^\infty$ (see [K]):

$$F_D(f\sigma) = fF_D(\sigma), \qquad \forall \sigma \in \Omega^0(E), \quad f \in \Omega^0.$$

Therefore, we may regard F_D as an element of $\Omega^2(End(E))$.

• Hermitian Structures

A Hermitian structure on a vector bundle E is the complexified version of Riemannian structure. Precisely, we have

Definition 2.1.9 (Hermitian Structure)**.** Given a complex vector bundle E over a smooth n-manifold M, a Hermitian structure is a section of the dual bundle of $E \otimes \bar{E}$, or an element of $\Omega^0(E \otimes \bar{E})^*$, satisfying the following conditions for all $\psi, \eta \in \Omega^0(E)$:

- $H(\psi, \eta)$ is $\mathbb{C}-$ linear in ψ,
- $H(\psi, \eta) = \overline{H(\eta, \psi)}$,
- $H(\psi, \psi) > 0 \qquad \forall \psi \neq 0$,
- The function defined by $f(x) = H_x(\psi_x, \eta_x)$ is smooth.

The local trivialization of E over an open subset U provides coordinate expressions of a Hermitian structure H. Let $s_U = (s_1, \ldots, s_r)$ again be the local frame of E over U. The fourth condition in Definition 2.1.9 yields a collection of smooth functions defined by

$$H_{i\bar{j}}(x) = H_x(s_i(x), s_j(x)), \qquad i, j \in 1, \ldots, m$$

over U. With respect to this frame, we have a positive definite, Hermitian matrix given by

$$H_U = (H_{i\bar{j}})$$

at every point on U. We denote (E, H) for a vector bundle E endowed with a Hermitian structure H.

When imposing certain relations, Hermitian structures control connections on E quite dominantly.

Definition 2.1.10 (H-Unitary Connection)**.** A connection D on E is called H-unitary, if it satisfies

$$d(H(\psi, \eta)) = h(D\psi, \eta) + h(\psi, D\eta), \qquad \forall \psi, \eta \in \Omega^0(E).$$

Similar to the unique existence of Levi Civita connection, when smooth structure of the tangent bundle is given, unitary connection exists uniquely when a holomorphic structure $\bar{\partial}$ is fixed on E. Explicitly, if d' is the $(1, 0)$ component of the exterior derivative of d with respect to $\bar{\partial}$ (i.e. the components spanned by local frames in the kernel of $\bar{\partial}$), the Hermitian structure $(H_{i\bar{j}})$ and the unitary connection forms (ω_i^j) are related by

$$d' H_{i\bar{j}} = \sum_a \omega_i^a H_{a\bar{j}}. \tag{2.2}$$

For a given Hermitian structure H, we denote $\mathcal{A}(H)$ the space of all H-unitary connections. Clearly, it is in one-to-one correspondence with the space of integrable holomorphic structures.

We use the notation D_H for a connection that is H-unitary, and the corresponding curvature is denoted by F_H.

- Gauge Actions

This thesis focuses heavily on the elliptic analysis of gauge actions of several differential geometric objects on a complex vector or line bundle. We must properly define the notion of gauge groups and their actions on relevant spaces. We will introduce definitions from the differential geometric point of view summarized from [DK].

Definition 2.1.11 (Gauge Groups)**.** Given a vector bundle E over Riemannian manifold M, the gauge group $\mathcal{G}_E$ is the set of automorphisms of E preserving the fibers and covering the identity mapping.

The gauge group induces several actions on various objects. The most natural one is the action on $\Omega^0(E)$ simply by applying the automorphism:

Definition 2.1.12 (Gauge Action on $\Omega^0(E)$)**.** Given $u \in \mathcal{G}_E$, it acts on a section $s \in \Omega^0(E)$ by

$$u^* s(x) = u(s(x)),$$

for all $x \in M$.

$\mathcal{G}_E$ acts on the space of holomorphic structures by conjugation:

Definition 2.1.13 (Gauge Action on Holomorphic Structures)**.** Given $\bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E)$ and $u \in \mathcal{G}_E$, we have

$$u^* \bar{\partial}(s) = u\bar{\partial}\left(u^{-1}s\right).$$

On the space of connections $\mathcal{A}$, it also acts by conjugation:

Definition 2.1.14 (Gauge Action on $\mathcal{A}$)**.** Given $u \in \mathcal{G}_E$, it acts on $D \in \mathcal{A}$ by

$$u^* D(s) = uD(u^{-1}s),$$

for all $s \in \Omega^0(E)$.

Using the Lebiniz rule introduced in Definition 2.1.3, we can observe the corresponding gauge action on connection one forms, which is identical to the change of representation with respect to different local frames on the overlaps:

Definition 2.1.15 (Gauge Action on Connection One Forms)**.** Given $u \in \mathcal{G}_E$, it acts on a connection one form A by

$$u^*A(s) = A(s) + u^{-1}(s)du,$$

for all $s \in \Omega^0(E)$.

One can readily compute the action of $\mathcal{G}_E$ on the curvature forms.

Definition 2.1.16. Given $u \in \mathcal{G}_E$, it acts on a curvature form θ by

$$u^*\theta(s) = u^{-1}\theta u(s),$$

for all $s \in \Omega^0(E)$.

In another words, curvature forms transform tensorially under bundle automorphism and may be viewed as a section of $\Omega^2(EndE)$ with structure group $\mathcal{G}_E$.

On a complex vector bundle of rank r, we may view a gauge $u \in \mathcal{G}_E$ locally as a smooth map from a trivializing cover U to $GL(2r, \mathbb{R})$. If the bundle is equipped with additional structure, it is natural to restrict the image of u to certain subgroups of $GL(2r, \mathbb{R})$ in order to preserve the structure. The gauge group $\mathcal{G}_{\mathbb{C}}$ consists of gauges that preserve the complex structure of the vector bundle E, and therefore may be locally viewed as a smooth map from U to $GL(r, \mathbb{C})$. When a Hermitian structure H is given on E, it endows an inner product structure on the vector space $\mathbb{C}^r$, and we may further require the gauges to preserve this inner product. These gauges therefore locally take values in $U(r)$, the unitary group, and we denote it by $\mathcal{G}$, the unitary gauge group with respect to the Hermitian structure H.

• Some Topological Invariants Derived from Curvatures

It is a well known fact that curvature form Θ of a vector bundle (E, D) over M generates characteristic classes in $H^{2i}(M, \mathbb{R})$, called Chern classes, invariant of connections D. We state the facts here. See, for example [K], for complete derivations.

Given any connection D, and its corresponding curvature F_D with its curvature form Θ, we consider the following expression:

$$det\left(I_r - \frac{1}{2\pi i}\Theta\right) = 1 + \sum_{j=1}^{r} f_k(\Theta).$$

Here, each f_k is a homogeneous polynomial of degree k, invariant under $GL(r, \mathbb{C})$ action by conjugation. One can readily show that

$$f_k(\Theta) = \frac{(-1)^k}{(2\pi i)^k k!} \sum \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} \theta^{i_1}_{j_1} \wedge \ldots \wedge \theta^{i_k}_{j_k}.$$

Our particular focus lies on $k = 1$, where $f_1(\Theta)$ is essentially the trace of θ:

$$f_1(\Theta) = -\frac{1}{2\pi i} \sum_j \theta^j_j.$$

It is a classical result that $f_1(\Theta)$ is closed, and the cohomology class it represents in $H^2(M, \mathbb{Z})$ is independent of the connection D. It is therefore a topological invariant of the vector bundle E, denoted by $c_1(E)$, the first Chern class of E. In fact, for each $k \in \{1, \ldots, r\}$, $f_k(\Theta)$ produces a class in $H^{2k}(M, \mathbb{Z})$ that is independent of the connection D, and they are referred to as the k^{th} Chern class of E. Their independence from connections and Hermitian structures is a vital feature for our analysis in the thesis when we try to obtain some uniformity over a family of connections and Hermitian metrics.

A very important topological invariant for a vector bundle is its degree.

Definition 2.1.17 (Degree of Bundle)**.** The degree of a vector bundle E over a Kähler manifold (M, ω) is

$$deg(E) = \int_M c_1(E) \wedge w^{n-1}.$$

It is a classical fact that when $M = \Sigma$, a Riemann surface, the integral above is an integer.

2.2 Analytic Preliminaries

Many Sobolev type estimates and embeddings are extensively used in this thesis. We list them here for readers' references. Most results are excerpted from [W]. Throughout this section, all the functions and Sobolev norms are defined on a compact domain $U \subset \mathbb{R}^n$ unless otherwise specified. As it will be shown in the next section, all estimates and inequalities of Sobolev norms on U generalize to Sobolev norms on vector valued functions and differential forms on general compact Riemannian manifolds (M, g).

Theorem 2.2.1 (Sobolev Embeddings and Estimates). *Let* $j < k \in \mathbb{N}$ *and* $1 \leq p, q < \infty$, *we have he following statements:*

(i)If $k - \frac{n}{p} \geq j - \frac{n}{q}$, *the inclusion*

$$W^{k,p} \hookrightarrow W^{j,p}$$

is continuous. That is, for some constant $C > 0$,

$$\|\alpha\|_{W^{j,q}} \leq C \|\alpha\|_{W^{k,p}} ,$$

for all $\alpha \in W^{j,q}$.

(ii) If $k - \frac{n}{p} > j - \frac{n}{q}$, *then the inclusion above is compact. That is, a bounded sequence in* $W^{k,p}$ *contains a convergent subsequence in* $W^{j,q}$.

(iii) If $k - \frac{n}{p} > j$, *the inclusion*

$$W^{k,p} \hookrightarrow \mathcal{C}^j$$

is continuous. That is, for some $C > 0$,

$$\|\alpha\|_{W^{j,\infty}} \leq C \|\alpha\|_{W^{k,p}} ,$$

for all $\alpha \in W^{k,p}$.

Moreover, the inclusion is compact.

A consequence of these estimates is the estimate of Sobolev norm of product.

Lemma 2.2.2 (Product Inequality). *Let $l \in \mathbb{N}_0$ and $1 \leq p, r, s < \infty$ be real numbers such that*

$$r, s \geq p \qquad \text{and} \qquad \frac{1}{r} + \frac{1}{s} < \frac{k}{n} + \frac{1}{p}$$

or

$$r, s > p \qquad \text{and} \qquad \frac{1}{r} + \frac{1}{s} \leq \frac{k}{n} + \frac{1}{p}.$$

Then, there is a constant $C > 0$ so that for all $f \in W^{k,r}$ and $g \in W^{k,s}$, we have $fg \in W^{k,p}$ and

$$\|fg\|_{W^{k,p}} \leq C \|f\|_{W^{k,r}} \|g\|_{W^{k,s}} .$$

Much of the analysis in the thesis concern the elliptic operators between Banach spaces. We recall the definition of elliptic operators.

Definition 2.2.3. Given two Banach spaces B_1 and B_2, and a linear operator between them

$$L : B_1 \to B_2.$$

L is called elliptic if there exists $C > 0$ such that

$$\|L\xi\|_{B_2} \geq C \|\xi\|_{B_1},$$

for all $\xi \in B_1$.

Elliptic operators are well known for their good controls of regularities on the solutions. We will make extensive use of the following Calderon-Zygmund type L^p-elliptic regularity:

Proposition 2.2.4 (L^p-elliptic regularity)**.** *If L is a smooth elliptic operator then for any $p > 1$:*

$$\|u\|_{W^{k+2,p}} \leq C\left(\|L(u)\|_{W^{k,p}} + \|u\|_{L^p}\right).$$

In particular, if $u \in L^p$ is a weak solution of $L(u) = f$ with $f \in L^p$ for $p \geq 2$, then $u \in W^{2,p}$.

We also recall the following classical theorem in functional analysis.

Theorem 2.2.5 (Banach-Alaoglu Theorem)**.** *[W]*

Every bounded sequence in $W^{k,p}$ has a weakly convergent subsequence in the same topology.

A classical fact on functional analysis will be useful for obtaining bounds on connection forms in Uhlenbeck gauge in Chapter 7.

Lemma 2.2.6 (Bijectivity of Perturbation)**.** *(cf. Lemma E.4 in [W]) Let $T, S : X \to Z$ be bounded linear operators between Banach spaces. Suppose that T is bijective and $\|T^{-1}\|\|S\| < 1$. Then the perturbed operator $T + S : X \to Z$ is also bijective, and*

$$\left\|(T+S)^{-1}\right\| \leq \frac{\|T^{-1}\|}{1 - \|T^{-1}\|\|S\|},$$

$$\left\|(T+S)^{-1} - T^{-1}\right\| \leq \frac{\|T^{-1}\|^2\|S\|}{1 - \|T^{-1}\|\|S\|}.$$

Finally, we note a straightforward analytic fact:

Lemma 2.2.7 (Convergence of Powers)**.** *Given two families of functions* $\{f_s\}$ *and* $\{g_s\}$ *on* $U \subset \mathbb{R}^n$*, such that* $\{g_s\}$ *are uniformly bounded in* L^p *for all p, and*

$$\lim_{s\to\infty} \|f_s - g_s\|_{L^p} = 0 \ \ \forall\, p,$$

we have

$$\lim_{s\to\infty} \left\|f_s^N - g_s^N\right\|_{L^p} = 0 \ \ \forall\, p.$$

Here $N \in \mathbb{N}$ *is arbitrary.*

Proof. The lemma follows from induction on N. The case $N = 0$ is the assumption of the lemma. Suppose that

$$\lim_{s\to\infty} \left\|f_s^j - g_s^j\right\|_{L^p} = 0 \ \ \forall\, p, \forall\, j < N.$$

Since g_s's are uniformly bounded in all L^p, straightforward application of Hölder inequality implies that g_s^K's are uniformly bounded in all L^p for arbitrary $K \in \mathbb{N}$. The inductive hypothesis above then implies the uniform bound of f_s^j in L^p:

$$\left\|f_s^j\right\|_{L^p} \leq C \ \ \forall\, p, \ \ \forall\, s \ \ , \forall\, j < N.$$

Further application of Hölder inequality shows that

$$\left\|f_s^j g_s^K\right\|_{L^p} \leq C \ \ \forall\, p, \ \ \forall\, s \ \ , \forall\, j < N.$$

One finally observes that

$$\left|f_s^N - g_s^N\right|^p = |f_s - g_s|^p \left|f_s^{N-1} + f_s^{N-2} g_s + \ldots + g_s^{N-1}\right|^p .$$

The bounded-ness conclusions above imply that the second term on the right is bounded in L^1. The first term on the right approaches zero in L^1 as $s \to \infty$ by assumption, and the conclusion of the lemma follows.

□

2.3 Inner Product and Sobolev Spaces of Differential Forms

In previous sections, we have listed all the relevant analytic techniques for our purposes. They are however only defined for functions on $\mathbb{R}^n$, and we wish to generalize them to differential forms with values in vector spaces and general compact Riemannian manifold (M, g). To start, we properly define inner product of differential q forms, complex or vector valued. Fix a Hermitian structure H on E, and a Riemannian metric g on M. Both of them give rise to unique metric compatible connections, D and ∇^M, with total covariant derivative defined in Definition 2.1.6. Let $\bar{g} = (g^{ij})$ denotes the inverse matrix of $g = (g_{ij})$.

Definition 2.3.1 (Pointwise Inner Product on Ω^q)**.** Given basis covectors $\alpha = dx^{i_1} \wedge \ldots \wedge dx^{i_q}$ and $\beta = dx^{j_1} \wedge \ldots \wedge dx^{j_q}$ of $\Lambda^q T^*M$, the pointwise inner product of them is defined by

$$< \alpha, \beta >_g := det\left(g^{i_l j_k}\right)_{1 \leq l,k \leq q}.$$

That is, the determinant of the minor of the matrix $\bar{g}$ formed by the intersection of $i_1, \ldots, i_q$ rows and $j_1, \ldots, j_q$ rows. The subscript g above is often omitted when no confusion arises. The definition above is then extended $\mathbb{C}$-linearly to vector-valued q forms.

Definition 2.3.2 (Pointwise Inner Product on $\Omega^q(E)$)**.** For $u = s\alpha, v = t\beta \in \Omega^q(E) = \Gamma(E \otimes \wedge^q T^*M)$, their inner product is defined by

$$< u, v >_{H,g} := < s, t >_H < \alpha, \beta >_g .$$

Again, the subscripts are often suppressed when no confusion arises.

When M is a Kähler manifold with the complex structure, the Kähler form ω and the complex structure J are related by

$$\omega(\cdot, \cdot) = g < J\cdot, \cdot > .$$

The complex structure of M complexifies TM and T^*M, turning them into Hermitian vector bundles with Hermitian structure g or $\bar{g}$. Any wedge product of T^*M splits into holomorphic part and anti-holomorphic part:

$$\bigwedge^K T^*M = \bigoplus_{r+q=K} \left(\bigwedge^r T^*M' \oplus \bigwedge^q T^*M'' \right).$$

Fix a local unitary (orthonormal) frame $dz^1, \ldots, dz^n$ of T^*M over U so that

$$g = \sum_{j=1}^{n} dz^j d\bar{z}^j,$$

and

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

With the complex structure available, we recall the notion of Hodge * operator. It takes any basis covector of $\Omega^{r,q}$, $\phi = dz^{i_1} \wedge \ldots \wedge dz^{i_r} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}$, into its orthogonal complement with a sign depending on the permutation of the indices, so that

$$\phi \wedge \overline{*\phi} = \frac{1}{n!} \omega^n := dvol,$$

the volume form with respect to ω. See, for example [K], for precise definitions of *. The

Hodge * operator defines a global inner product on $\Omega^{r,q}$.

Definition 2.3.3 (Global Inner Product on $\Omega^{r,q}$)**.** For $\phi, \psi \in \Omega^{r,q}$, we define the global inner product of them by

$$(\phi, \psi) = \int_M \phi \wedge \overline{*\psi}.$$

The L^p norm of $u = s\alpha \in \Omega^q(E)$ can now be naturally defined by

$$\|u\|_{L^p} = \left(\int_M |s|_H^p |\alpha|_g^p\right)^{\frac{1}{p}},$$

where $|\cdot|_H$ and $|\cdot|_g$ are the pointwise norm given by the pointwise inner products defined above. We denote d^*, D^*, and ∇^* to the adjoint operators of d, D, and ∇, respectively, with respect to the global inner product defined above. In particular, one can readily derive that

$$d^* = - * d*,$$

on a closed Kähler manifold M. The notation $L^p_q(E)$ will be used to define the space of E-valued sections with finite L^p norms. We are now ready to define

Definition 2.3.4 (Sobolev (k,p)- norm on $\Omega^q(E)$)**.** Given $\alpha \in \Omega^q(E)$, the Sobolev k, p norm is defined by

$$\|\alpha\|_{W_q^{k,p}} = \left(\sum_{j=0}^{k} \left\|\nabla^j \alpha\right\|_{L^p}^p\right)^{\frac{1}{p}}.$$

We then denote $W_q^{k,p}(E)$ to be the subspace of $\Omega^q(E)$ with finite $W_q^{k,p}$ norms.

Sobolev norms clearly depend on the Kähler form ω. One of the central themes of this thesis is to observe the adiabatic limit of vortices when ω is conformally scaled by a constant. For that purpose, we state the dependence of $\|\cdot\|_{W^{k,p}}$, defined on Ω^q, on the conformal factor s of ω. More precisely, we examine the Sobolev norms when z^j is replaced by sz^j and ω is consequentially replaced with $s^{2n}\omega$. To preserve the orthonormality, the metric g on Ω^q is scaled to $s^{-2q}g$. Plugging into the definition of L^p norms, we see that, for $\alpha \in \Omega^q$, we have the following lemma.

Lemma 2.3.5. *On an n-(complex) dimensional Kähler manifold (M, ω), the effect of conformal scaling by constant s, i.e. $\omega \to s^{2n}\omega$, on the Sobolev (k, p) norm is given by*

$$\|\alpha\|_{W_q^{k,p},s} = s^{\frac{2n-pq}{p}} \|\phi\|_{L^p} . \tag{2.3}$$

In particular, when scaling the Kähler form ω of a Riemann surface Σ by s, the L^2 norm of a section is scaled by s^2, that of a one form is scale invariant, and that of a two form is scaled by s^{-2}. Since the total covariant derivative is linear, one can immediately derive the formula of Sobolev $W^{k,p}$ norm upon rescaling:

$$\|\alpha\|_{W_q^{k,p},s} = \left(\sum_{j=0}^{k} s^{2n-p(q+j)} \left\|\nabla^j \alpha\right\|_{L^p}^p \right)^{\frac{1}{p}} . \tag{2.4}$$

It is also evident from Lemma 2.3.5 that the L^2 norms of curvature forms on a Riemannian four manifold are scale invariant, a classical fact from the Yang-Mills theory.

When the Riemannian manifold (M, g) is compact, there is a considerably more elementary interpretation of the Sobolev space of vector valued p forms. For a rank r vector bundle $E \to M$, let $M = \bigcup_{i=1}^N U_i$ be the finite cover, and $\Psi_i : \pi^{-1}(U_i) \underset{\sim}{\to} V_i \times \mathbb{R}^r$ be the local trivialization over U_i, where V_i is an open set in $\mathbb{R}^n$ identified with U_i via its coordinate patch. A section $\alpha \in \Omega(E)$ is then locally represented over U_i by a collection of functions

$$(\Psi_{i*}\alpha)_l : V_i \to \mathbb{R},$$

with $l = 1, \ldots, n$. On U_i, an inspection of the definition of Sobolev (k, p) norm (2.3.4) over U_i indicates that each summand $\|\nabla^j \alpha\|_{L^p}^p$ is a polynomial of connection forms over U_i, entries of the matrix g (or $\bar{g}$), and the $(\Psi_{i*}\alpha)_l$'s above. On the overlaps $U_i \bigcap U_{i'}$, the two quantities are related by transition functions. Since there are only finitely many covering open charts, we only have a total of finitely many local coordinate representations of connection forms, Riemannian metrics, and transition functions. All of them are smooth and independent of α. They therefore possess uniform L^∞ bounds up to in each C^k, and the Sobolev norm defined in Definition 2.3.4 is equivalent to

$$\left(\sum_{i=1}^{N}\sum_{l=1}^{r}\|\Psi_{i*}\alpha\|_{W_q^{k,p}(V_i)}\right)^{\frac{1}{p}}. \tag{2.5}$$

Therefore, Sobolev norms on a compact base manifold can be made equivalent to the classical notions of Sobolev norms of functions, and therefore many estimate results in functional analysis generalize to Sobolev spaces of differential forms, complex or vector valued on general Riemannian manifold (M, g). In particular, all the estimates and convergent statements in section 2.2 carry over to general Sobolev spaces of forms.

A well known elliptic operator on a Riemannian manifold (M, g) is the Hodge Laplacian operator. It is given locally by

$$\Delta_g = \sum_{i,k} g^{ik}\nabla_i\nabla_k.$$

In Euclidean coordinate, it degenerates to

$$\Delta = \sum_i \frac{\partial^2}{\partial x_i^2},$$

which is the opposite of the Laplacian defined from geometric perspective $\Delta = d^*d + dd^*$.

The following inequality is useful for the bound of energy density:

Lemma 2.3.6 (Mean Value Inequality)**.** *Let $B_R(z_0)$ be a ball in $\mathbb{C}^n$ with radius R centered at z_0, and a nonnegative function $f \in C^2(B_R(z_0))$. Suppose there is a positive constant C' such that*

$$\Delta f \geq -C' f^2.$$

Then, there is a constant $C_R > 0$, inversely proportional to the volume of $B_R(z_0)$, such that

$$f(z_0) \leq C_R \int_{B_R(z_0)} f(z) dvol.$$

We shall also need the following Maximum principle.

Lemma 2.3.7. *For the elliptic operator $L = \Delta - k$, where k is any smooth positive function, the following is true: for any $p > dim M$, if $u \in W^{2,p}$ satisfies $Lu \geq 0$, then $u \leq 0$.*

Proof. By Morrey's inequality $u \in C^1(M)$. Therefore $w = \max\{0, u(x)\} \in W^{1,p}$. However, from our assumptions,

$$\begin{aligned} 0 &\leq \int_M wLudV = -\int_M (u\Delta w + wku)\, dV \\ &\leq -\int_M \left(w\Delta w + kw^2\right) dV = -\int_M \left(|\nabla w|^2 + kw^2\right) dV. \end{aligned}$$

Since $k > 0$, it is necessary that $w \equiv 0$, which implies that $u \leq 0$.

□

A classical analytic result for solving Neumann problem with boundary is essential to construct connection forms in certain gauges where elliptic estimates are possible. This result will be recalled in section 7.1.

Theorem 2.3.8 (Neumann Problem with Inhomogeneous Boundary Conditions)**.** *Consider a Riemannian manifold (M, g) with boundary ∂M. Let $f \in L^p$ and $g \in W^{1,p}(\partial M)$ such that*

$$g = G|_{\partial M}, \qquad \text{and} \qquad G \in W^{1,p}.$$

Then, the equation

$$\begin{cases} \Delta u = f & \text{on } M \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial M \end{cases}$$

has a solution in $W^{2,p}$ *if and only if*

$$\int_M f + \int_{\partial M} g = 0 \tag{2.6}$$

Moreover, the solution is unique up to an additive constant. (Here, Δ *is the Hodge Laplacian with respect to the metric* g*.)*

The following regularity result for solutions of Neumann problems will be useful.

Theorem 2.3.9 (Agmon, Douglis, Nirenberg)**.** *(cf. Theorem 3.2 in [W])*

Let $f \in W^{k,p}$ *and* $g \in W^{k+1,p}(\partial M)$ *for some* $k \in \mathbb{N}_0$*. Assume that a distribution* u *is a weak solution to the Neumann problems in Theorem 2.3.8 in the sense that*

$$< u, \Delta\psi >= \int_M f\psi + \int_{\partial M} g\psi,$$

for all $\psi \in \mathcal{C}^\infty$ *with zero derivative along the normal direction to* ∂M*. Then* $u \in W^{k+2,p}$ *and is a strong solution. Moreover, there exist constants* C, C' *so that for all* $u \in W^{k+2,p}$*,*

$$\|u\|_{W^{k+2,p}} \leq C\left(\|\Delta u\|_{W^{k,p}} + \left\|\frac{\partial u}{\partial \nu}\right\|_{W^{k+1,p}(\partial M)} + \|u\|_{W^{k+1,p}}\right),$$

$$\|u\|_{W^{k+2,p}} \leq C'\left(\|\Delta u\|_{W^{k,p}} + \left\|\frac{\partial u}{\partial \nu}\right\|_{W^{k+1,p}(\partial M)}\right).$$

We will need the following commutator relation between the Laplacian operator and covariant derivatives on a compact Riemannian manifold (M, g). See, for example [V] , for references. Let Riemannian curvature $R \in \Omega^2(End(TM))$ be the curvature operator of Levi Civita connection arisen from g defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In local frame field $\{\partial_i\}$, we have

$$R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l,$$

where R^l_{ijk} is a combination of terms from metric component functions (g_{ij}), and their derivatives up to second order. (Chirstoffel symbols)

From this, we can define the Ricci curvature tensor as follows:

$$R_{jk} = \sum_i R^i_{ijk}.$$

Moreover, the Laplacian operator defined by g is given locally by

$$\Delta_g = \sum_{i,k} g^{ik} \nabla_i \nabla_k,$$

where g^{ik}'s are the entries of the inverse matrix to $g = (g_{ik})$, and ∇^i is the covariant derivative along the i^{th} local frame. We will usually omit the subscript g if no confusion arises. The commutation relation we need is the following identity from [V]:

Lemma 2.3.10. *On a Riemannian manifold (M, g), and a smooth function $f \in C^\infty(M)$, we have,*

$$\Delta \nabla^2 f = \nabla^2 \Delta f + Rm * \nabla^2 f + \nabla Rm * \nabla f,$$

where Rm is the curvature tensor expressed as a $(0,4)$ tensor. $*$ is a heuristic symbol involving contractions of Rm with $\nabla^2 f$ and ∇RM with ∇f.

For higher order covariant derivatives, we can show, by a similar procedure as the one used in the lemma above, that:

Lemma 2.3.11. *One has:*

$$\Delta \nabla^l f = \nabla^l \Delta f + Q(R, f),$$

where $Q(R,f) = \sum_{j=0}^{l} \nabla^j Rm * \nabla^{l-j} f$, *is an expression involving the components of* Rm, f, *and their derivatives up to* l^{th} *order.*

We state a technical lemma for the weak compactness of gauge transformations. For k, p in the correct range for Sobolev type estimates, if a sequence of connection forms and another sequence of gauge transformed connection forms are both uniformly bounded in suitable Sobolev space, we expect some regularities on the sequence of gauges. Recall that a gauge $u \in \mathcal{G}$ acts on a connection form A by

$$u^* A = u^{-1} A u + u^{-1} du.$$

The following results are due to Uhlenbeck, and are excerpted from [W].

Theorem 2.3.12 (Weak Compactness of Gauges)**.** *Assume* $k \in \mathbb{N}$ *and* $p \in [1, \infty)$ *be such that* $kp > n$ *and* $p > \frac{n}{2}$. *Let* $\{A^\nu\} \subset W_1^{k-1,p}$ *and* $\{u^\nu\} \subset \mathcal{G}_{k,p}$ *be two sequences such that* $\|A^\nu\|_{W_1^{k-1,p}}$ *and* $\|u^{\nu *} A^\nu\|_{W_1^{k-1,p}}$ *are uniformly bounded. Then the follow statements hold:*

(i) On every trivializing domain $U_\alpha \subset M$, $\|(u_\alpha^\nu)^{-1} du_\alpha^\nu\|_{W_1^{k-1,p}}$ *are uniformly bounded.*

(ii) $\{u^\nu\}$ *possesses a subsequence convergent in* $\mathcal{C}^0$ *topology to some* $u^\infty \in \mathcal{G}^{k,p}$.

(iii) For $s \in [1, \infty)$ *satisfying* $\frac{1}{s} > \frac{1}{p} - \frac{1}{n}$, *the subsequence in (ii) can be chosen such that* $(u_\alpha^\nu)^{-1} du_\alpha^\nu$ *converges to* $(u_\alpha^\infty)^{-1} du_\alpha^\infty$ *in* $W_1^{k-1,s}$ *norm.*

The following elliptic estimates are important for constructions of Uhlenbeck's gauge (See section 7.1).

Theorem 2.3.13 (Elliptic Estimates for Uhlenbeck Connections)**.** *[W] For* $p \in (1, \infty)$, *there exists a constant* $C > 0$ *such that the following holds:*

(i) Suppose that $A \in W_1^{1,p}$ *satisfies* $*A|_{\partial M} = 0$ *and* $A|_{\partial M} = 0$*, then*

$$\|A\|_{W_1^{1,p}} \leq C\left(\|dA\|_{L^p} + \|d^*A\|_{L^p} + \|A\|_{L^p}\right).$$

(ii) Assume further that $H^1(M,\mathbb{R}) = 0$*, then*

$$\|A\|_{W_1^{1,p}} \leq C\left(\|dA\|_{L^p} + \|d^*A\|_{L^p}\right).$$

Chapter 3

Background and Established Results

3.1 Descriptions of $\nu_k(s,\tau)$ and $\nu_{k,0}(s,\tau)$

In this section, we provide sufficient descriptions of the relevant established results to be generalized in this thesis. Throughout this section, as well as the rest of the thesis, we will write the connection D and its connection form A interchangeably when no confusion arises. We recall the Yang-Mills-Higgs functional on $\mathcal{A}(H) \times \Omega^0(E) \times \ldots \times \Omega^0(E)$:

$$\begin{aligned} YMH_{\tau,s}(D,\phi) &:= \frac{1}{2s^2}\left\|F_D\right\|_{L^2}^2 + \sum_{i=1}^{k}\left\|D\phi_i\right\|_{L^2}^2 + \frac{s^2}{4}\left\|\sum_{i=1}^{k}|\phi_i|_H^2 - \tau I\right\|_{L^2}^2 \\ &= \int_M e_s(D,\phi)dvol_M, \end{aligned} \tag{3.1}$$

The integrand

$$e_s(D,A) := \frac{1}{2s^2}|F_D|^2 + \sum_{i=1}^{k}|D\phi_i|^2 + \frac{s^2}{4}\left|\sum_{i=1}^{k}|\phi_i|_H^2 - \tau I\right|^2$$

is the energy density with norms $|\cdot|$ defined for each complex or vector valued forms as in Definition 2.3.1 and Definition 2.3.2. In [Br], it was shown that through standard applications of Kähler identities, one may rewrite $YMH_{\tau,s}$ as:

$$
\begin{aligned}
& YMH_{\tau,s}(D,\phi) \\
= \ & \frac{4}{s^2}\left\|F_D^{(0,2)}\right\|_{L^2}^2 + 2\left\|D^{(0,1)}\phi\right\|_{L^2}^2 + \left\|\frac{\sqrt{-1}\Lambda F_D}{s} + \frac{s}{2}\left(\sum_{i=1}^{k}|\phi_i|_H^2 - \tau\right)\right\|_{L^2}^2 + \tau\int_\Sigma \sqrt{-1}\Lambda F_D
\end{aligned} \tag{3.2}
$$

The fourth term above is a topological invariant of the vector bundle and is therefore independent of the connection D and the sections ϕ. It is then clear that $YMH_{\tau,s}$ is minimized when all the non-negative terms are zero. This yields the minimizing equations for $YMH_{\tau,s}$ known as the s-vortex equations:

$$
\begin{cases}
F_D^{(0,2)} = 0 \\
D^{(0,1)}\phi = 0 \\
\sqrt{-1}\Lambda F_D + \frac{s^2}{2}(|\phi|_H^2 - \tau) = 0.
\end{cases} \tag{3.3}
$$

As s increases, the curvature term in the third equation in (3.3) becomes less and less dominating. Thus, at least formally, we have a set of vortex equations without curvature dependence at $s = \infty$ given by

$$
\begin{cases}
F_D^{(0,2)} = 0 \\
D^{(0,1)}\phi = 0 \\
|\phi|_H^2 - \tau = 0.
\end{cases} \tag{3.4}
$$

The solutions to (3.4) are clearly pairs of integrable connections D, and D-holomorphic sections ϕ with constant square H norm τ. Equations (3.3) and (3.4) above are both invariant under the unitary gauge $\mathcal{G}$ action defined in the preliminary section. Indeed, the H norm of ϕ is clearly invariant under unitary gauge. For the second equation above, we recall that a unitary gauge $e^{if} \in U(1)$ takes sections ϕ to $e^{if}\phi$ and the connection D to $e^{if}De^{-if}$. The second equation therefore remains valid and we may define the gauge classes of solutions:

Definition 3.1.1 (Moduli Spaces of Vortices)**.** For each k, s and τ, we define the moduli spaces of solutions

$$\nu_k(s,\tau) = \{[D,\phi] \in \mathcal{A}(H) \times \Omega^0(M,E) \mid (3.3) \text{ holds}\}/\mathcal{G},$$

and

$$\nu_k(\infty,\tau) = \{[D,\phi] \in \mathcal{A}(H) \times \Omega^0(M,E) \mid (3.4) \text{ holds}\}/\mathcal{G}.$$

We refer to the members of the collections above as vortices. Some important open subsets of them, on which k sections without common zeros, are of particular interest.

Definition 3.1.2 (Moduli Spaces of Non-Simultaneously Vanishing Vortices)**.**

$$\nu_{k,0}(s,\tau) = \{[D,(\phi_1,\cdots,\phi_k)] \in \nu_k(s,\tau) \mid \cap_{i=1}^k Z(\phi_i) = \emptyset\}.$$

From (3.4), we see that when $s=\infty$, it is not possible for k sections to have common zeros as the sum of their pointwise H norm is a positive constant. Therefore $\nu_k(\infty,\tau) = \nu_{k,0}(\infty,\tau)$ and there is no need for the additional definition. It is also clear that $\nu_{1,0}(s,\tau) = \emptyset$ for all s since a single section of a degree $r > 0$ line bundle must vanish precisely at r points, counting multiplicities.

$\nu_1(1,\tau)$ has been very explicitly described in various classical literatures. For $M = \mathbb{R}^2 \simeq \mathbb{C}$ and $E = L$, a line bundle, Jaffe and Taubes have identified $\nu_1(1,\tau)$ with the set of unordered r tuple of points on $\mathbb{C}$, or the symmetric r-product of $\mathbb{C}$ in [J-T]. Each tuple of unordered r points is identified with a unique gauge class of vortices vanishing precisely at those points. Since every line bundle over $\mathbb{R}^2$ is topologically trivial, the qualitative behaviors of vortices are not immediately revealing, although somewhat suggestive, of the topological features of the line bundles. For general base manifolds with nontrivial vector bundles over them, topological and algebraic structures of (E, H) become important. One of the remarkable results is the Hitchin-Kobayashi type correspondence between the stable vector bundles with a holomorphic section and gauge classes of solutions, established by Bradlow in [Br1]. The stability defined there is not only a condition on subsheaves of E, but also the parameter τ, and it is referred to as the τ-stability. In particular, for $(E, H) = (L, H)$, the stability conditions degenerate to a condition solely on τ. One immediate necessary condition of τ-stability for τ is, by integrating the third equation of (3.3) with $s = 1$, that $\tau \geq \frac{4\pi r}{vol\Sigma}$. In fact, in [Br], it was shown that the condition is also sufficient. Bradlow has worked out the following explicit description:

$$\nu_1(1,\tau) \simeq \begin{cases} \emptyset & ;\tau < \frac{4\pi r}{\mathrm{vol}\Sigma} \\ Jac^r\Sigma & ;\tau = \frac{4\pi r}{\mathrm{vol}\Sigma} \\ Sym^r\Sigma & ;\tau > \frac{4\pi r}{\mathrm{vol}\Sigma}. \end{cases} \tag{3.5}$$

Here, $Sym^r\Sigma$ is the space of unordered r tuple of points of Σ and $Jac^r\Sigma$ is the Jacobian torus of Σ parametrizing holomorphic structures of L. (See [Br]).

The parameter s does not alter the conclusion. We have seen that the effect of s^2 can be thought of as scaling the section ϕ and replacing τ by $s^2\tau$. This observation generalizes Bradlow's result in [Br] naturally:

$$\nu_1(s,\tau) = \begin{cases} \emptyset & ;s^2\tau < \frac{4\pi r}{\mathrm{vol}\Sigma} \\ Jac^r\Sigma & ;s^2\tau = \frac{4\pi r}{\mathrm{vol}\Sigma} \\ Sym^r\Sigma & ;s^2\tau > \frac{4\pi r}{\mathrm{vol}\Sigma}. \end{cases} \tag{3.6}$$

The crucial step to achieve this description is to switch perspective, from one in which we look for pairs (D, ϕ) on a bundle with fixed unitary structure, to one in which we look for a metric on a fixed holomorphic line bundle with a prescribed holomorphic section. In the second perspective, the analytic tools from [K-W] can be applied to solve for the special metrics. The equivalence of the two perspectives is given in [Br], which requires no modification for general values of s and k. We briefly summarize the constructions here.

Let $\mathcal{C}$ be the space of holomorphic structures of L, that is, the collection of $\bar{\partial}$ operators

$$\bar{\partial} : \Omega^0(L) \to \Omega^{0,1}(L)$$

satisfying Leibiniz rule. A classical fact from differential geometry is that given a Hermitian structure H, we have $\mathcal{A}(H) \simeq \mathcal{C}$. The original approach toward solving vortex equations is to fix a Hermitian structure H and consider the following space:

$$\mathcal{N}_k := \{(D, \phi) \in \mathcal{A}(H) \times \Omega^0(L) \times \ldots \times \Omega^0(L) \mid D^{0,1}\phi_i = 0 \ \forall i\}.$$

For a fixed H this space is bijective to

$$\{(\bar{\partial}, \phi) \in \mathcal{C} \times \Omega^0(L) \times \ldots \times \Omega^0(L) \mid \bar{\partial}\phi_i = 0 \ \forall i\} \tag{3.7}$$

The first approach requires one to find all pairs in $\mathcal{N}_k$ so that the third equation of the vortex equations (3.3) is satisfied.

Alternatively, we may start without fixing the Hermitian structure. The second description of $\mathcal{N}_k$ (3.7) continues to make sense, and we pick an arbitrary pair $(\bar{\partial}_L, \phi) \in \mathcal{N}_k$. This pair determines a unique connection D_K, and thus a unique curvature F_K, once a Hermitian metric K is chosen. We specifically choose K so that the third equation of (3.3) is satisfied with this metric, and the curvature it defines:

$$\sqrt{-1}\Lambda F_K + \frac{s^2}{2}\Big(\sum_{i=1}^{k} |\phi_i|_K^2 - \tau\Big) = 0.$$

Precisely, the alternative approach of the problem requires us to start with the space

$$\mathfrak{T}_k = \{(\bar{\partial}, \phi, K) \in \mathcal{C} \times \Omega^0(L) \times \ldots \times \Omega^0(L) \times \mathcal{H}\},$$

where $\mathcal{H}$ is the space of Hermitian structures of L. We fix the first two components, and the solvability statement states the unique existence of the corresponding third component.

Such an approach allows us to apply analytic techniques to solve the vortex equations. It is well known that any two Hermitian metrics are related by a positive, self-adjoint bundle endomorphism , i.e. by an element in the complex gauge group $\mathcal{G}_{\mathbb{C}}$. On a line bundle L, $End(L) \simeq L \otimes L^* \simeq \mathcal{O}_M$, so any two C^∞-Hermitian metrics on L, say H and K, are related by $K = f\, H$ with $f \in C^\infty(M)$ and $f = e^{2u} > 0$ for some $u \in C^\infty(M)$. Therefore, starting with a background metric H, finding the special metric K is equivalent to finding the unique function u_s satisfying certain s-dependent elliptic PDE determined by the third equation of (3.3).

This alternative approach is equivalent to the original one only if we are able to build a bijection between the two solution spaces, up to gauges. The gauge group for the alternative space is however not only $\mathcal{G}$ but rather $\mathcal{G}_{\mathbb{C}}$, the complex gauge group. It acts on $\mathfrak{T}_k$ by (recall the gauge actions defined in section 2.1)

$$g^*(\overline{\partial}_L, \phi, H) = (g \circ \overline{\partial}_L \circ g^{-1}, \phi g, He^{2u}).$$

Here, $g^* g = e^{2u_s}$ for a smooth real function u. Unlike the unitary gauge $\mathcal{G}$, this action does not necessarily preserve the H-norm of ϕ. We define

$$\mathcal{T}_k(s,\tau) = \{(\bar{\partial}_L, \phi, K) \in \mathfrak{T}_k\ ;\ (3.3) \text{ holds with metric } K\}/\mathcal{G}_{\mathbb{C}}. \tag{3.8}$$

We now exhibit the bijection between $\mathcal{T}_k(s,\tau)$ and $\nu_k(s,\tau)$.

Lemma 3.1.3. *[Br] There is a bijective correspondence between $\nu_k(s,\tau)$ and $\mathcal{T}_k(s,\tau)$.*

Proof. (Sketch) To define the forward map $P_s : \nu_k(s,\tau) \to \mathcal{T}_k(s,\tau)$, we take $[D, \phi] \in \nu_k(s,\tau)$.

The integrability of D implies that its anti-holomorphic part $D^{(0,1)}$ defines a holomorphic structure, and we define

$$P_s([D,\phi]) = [D^{0,1}, \phi, H],$$

where H is the background metric for which D is H-unitary. For the inverse map G_s, take $[\bar{\partial}_L, \phi, K] \in \mathcal{T}_k(s,\tau)$. The Hermitian metric K on L is related to H by $K = e^{2u}H$, and $g = e^u$ acts on holomorphic structure and sections as above. We define

$$G_s([\bar{\partial}_L, \phi, K]) = [D(g^*\bar{\partial}_L, H), \phi \circ g],$$

where $D(g^*\bar{\partial}_L, H)$ is the metric connection of H with holomorphic structure $g(\bar{\partial}_L)$. That the pair $(D(g(\bar{\partial}_L), H), \phi \circ g)$ solves the vortex equation (3.3) and that P_s and G_s are inverse to each other are proved in [Br]. □

The alternative perspective yields a much more intuitive understanding of Bradlow's description of $\nu_1(s,\tau)$ for large τ. An element $[z_1, \ldots, z_r] \in Sym^r\Sigma$ uniquely determines a pair $(\bar{\partial}_L, \phi)$ with $\bar{\partial}_L\phi = 0$, up to $\mathcal{G}_{\mathbb{C}}$ action, that vanishes precisely at these points. The identification

$$\mathcal{T}_1(1,\tau) \simeq Sym^r\Sigma$$

is achieved once we ensure that the third component K is uniquely determined by the first two, up to $\mathcal{G}_{\mathbb{C}}$. Such a claim can be verified by the unique solvability of the elliptic PDE derived from the vortex equations. The PDE to be satisfied is (see Theorem 4.1.2)

$$\Delta\varphi_s = -\left(\frac{1}{2}h\right)e^{\varphi_s} + c(s).$$

Here, h is a non-positive function determined by the background metric H and the initially given section ϕ, $c(s)$ is an s-dependent constant, and $\varphi_s = 2(u_s - \psi)$ with ψ a function independent of s. The unique solvability of the equation above is guaranteed by results in [K-W].

We wish to generalize Bradlow's descriptions to general k, assuming $\tau > \frac{4\pi r}{Vol\Sigma}$. The generalized descriptions are available in [B-D-W], where one starts with descriptions of $\nu_{k,0}(1,\tau)$. Sections in $\nu_{k,0}(1,\tau)$ naturally define a holomorphic map from Σ to $\mathbb{CP}^{k-1}$ of degree r by

$$p \mapsto [\phi_1(p), \ldots, \phi_k(p)].$$

The map is well defined since there is no common zeros. It is also independent of the choice of unitary gauge classes of $(A, \phi) \in \nu_{k,0}(1,\tau)$ since a different choice of representative multiplies each component of ϕ by the same nonvanishing function and therefore does not affect the values in $\mathbb{CP}^{k-1}$. Since each section mush vanishes r times, the map is of degree r. In fact, it has been shown in [B-D-W] that

Lemma 3.1.4. *There is a bijection*

$$\Phi_1 : Hol_r(\Sigma, \mathbb{CP}^{k-1}) \to \nu_{k,0}(1,\tau).$$

Naturally, we wish to construct a family of maps

$$\Phi_s : Hol_r(\Sigma, \mathbb{CP}^{k-1}) \to \nu_{k,0}(s,\tau).$$

to advance to the next level of generality. We have just described Φ_1^{-1}, which extends to Φ_s^{-1} without any modification. The forward map is more sophisticated, and will be constructed in the next chapter. The construction of Φ_s by itself however is not novel from [B-D-W]. Our main achievement is to show rather that this family is very well controlled as $s \to \infty$.

For the description of the entire moduli space $\nu_k(s,\tau)$, the general principal is to first recognize the common zeros of the k sections, an element in $Sym^l\Sigma$ for some $l \leq k$. Once the common zeros $\{z_1, \ldots, z_l\}$ (not necessarily distinct points) are located, we divide out the zeros near each z_j and form k sections that do not vanish simultaneously. Those sections are then identified with an element $\tilde{\phi} \in Hol_{r-l}(\Sigma, \mathbb{CP}^{k-1})$. Therefore, a vortex is labeled by an element in $Sym^r\Sigma$ and an element in $Hol_{r-l}(\Sigma, \mathbb{CP}^{k-1})$. Precisely, we consider the defining section $\sigma \in \mathcal{O}(V)$, where the divisor $V = \sum_{j=1}^{l} z_j \in Sym^l\Sigma$. The nonvanishing k sections are then $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_k) \in L \otimes \mathcal{O}(-D)$, so that

$$\phi = \hat{\phi} \otimes \sigma.$$

A vortex (D, ϕ) is therefore identified with $(V, \tilde{\phi})$. Doing this for each l, it was concluded in [B-D-W] that for large enough τ,

$$\nu_k(1, \tau) \simeq \bigsqcup_{l=0}^{r} Sym^l\Sigma \times Hol_{r-l}(\Sigma, \mathbb{CP}^{k-1}). \tag{3.9}$$

One notes that this is indeed a generalization of (3.5). For $k = 1$, $\mathbb{CP}^0$ consists of a single point, and $Hol_{r-l}(\Sigma, \mathbb{CP}^0)$ is empty except for $l = r$, where the set is a singleton and (3.5) is recovered. It is also clear that the open subset $\nu_{k,0}(s, \tau)$ corresponds exactly to the component with $l = 0$, which is a singleton. Therefore, we have an informal identification

$$Hol_r(\Sigma, \mathbb{CP}^{k-1}) \simeq \nu_{k,0}(s, \tau).$$

Our next topic is to describe the correspondence precisely.

Chapter 4

Generalizations of Established Results

4.1 Kazdan-Warner Equations

We now generalize the results described in previous chapters to general values of s. In fact, the parameter s has very little effect on the existence of Φ_s for each finite s. Since we are interested in the asymptotic behaviors as $s \to \infty$, the third possibility of (3.6) prevails and τ dependence becomes insignificant. We will therefore from now on assume $\tau = 1$ and write $\nu_k(s)$ (and $\nu_{k,0}(s)$) instead of $\nu_k(s, 1)$ (and $\nu_{k,0}(s, 1)$). We are therefore aiming to show

Lemma 4.1.1. *For all s with $s^2 \geq \frac{4\pi r}{Vol(\Sigma)}$, there is a bijection*

$$\Phi_s : Hol_r(\Sigma, \mathbb{CP}^{k-1}) \to \nu_{k,0}(s)(= \nu_k(\infty) if\ s = \infty).$$

Proof. The inverse map Φ_s^{-1} has been constructed in the previous section. For $s = \infty$, the lemma is clear from the vortex equation at infinity (3.4). We therefore assume that $s < \infty$.

Start with a holomorphic map $\tilde{\phi} \in Hol_r(\Sigma, \mathbb{CP}^{k-1})$. Let $L = \tilde{\phi}^* \mathcal{O}_{\mathbb{CP}^{k-1}}(1)$ be the pulled back line bundle of the anti-tautological bundle. It is endowed with sections $\phi = (\phi_1, \ldots, \phi_k)$ by pulling back linear sections $z_1, \ldots, z_k$ of $\mathcal{O}_{\mathbb{CP}^{k-1}}(1)$ via $\tilde{\phi}$. Precisely, we have $\phi_j = \tilde{\phi}^* z_j$. The map $\tilde{\phi}$ endows a holomorphic structure $\bar{\partial}_L$ on L, inherited from the natural complex structure on $\mathcal{O}_{\mathbb{CP}^{k-1}}(1)$, and a background metric H on L when a background metric is given on $\mathcal{O}_{\mathbb{CP}^{k-1}}$. We mimic Bradlow's arguments in [Br] to look for a special metric H_s, related to H by a complex gauge transformation $H_s = He^{2u_s}$, where u_s is a positive smooth function. The vortex equation (3.3) is to be satisfied if H is replaced by H_s. The triplet $[\bar{\partial}_L, \phi, H_s] \in \mathcal{T}_k(s)$ corresponds via Bradlow's identification to $[D_s, e^{u_s}\phi] \in \nu_{k,0}(s)$, where D_s is the unitary connection with respect to holomorphic structure $e^{u_s} \circ \bar{\partial}_L \circ e^{-u_s}$ and the Hermitian metric H, and we define

$$\Phi_s(\tilde{\phi}) = [D_s, e^{u_s}\phi]. \tag{4.1}$$

The lemma boils down to finding the function u_s, and therefore the following theorem:

Theorem 4.1.2 (Existence and Uniqueness of u_s)**.** *Fix a Hermitian line bundle* (L, H) *over Riemann surface* Σ*. On* $L \to \Sigma$ *we select a pair* $(\bar{\partial}_L, \phi)$ *so that* $\bar{\partial}_L \phi_i = 0\ \ \forall i$*. For all* $s \in \mathbb{R}$ *such that*

$$s^2\tau > \frac{4\pi r}{Vol\Sigma},$$

there exists a unique $u_s \in \mathcal{C}^\infty$ *so that*

$$(D_s, e^{u_s}\phi)$$

given immediately above solves the vortex equations (3.3)*.*

Proof. of the theorem

We prove this theorem from the alternative special metric approach toward the solution moduli space. If $H_s = g_s H$ one has:

$$\sqrt{-1}\Lambda F_{H_s} = \sqrt{-1}\Lambda F_H + \sqrt{-1}\Lambda\bar{\partial}(H^{-1}\partial H(g_s)).$$

Writing $g_s = e^{2u_s}$, we get

$$\sqrt{-1}\Lambda F_{H_s} = \sqrt{-1}\Lambda F_H + 2\sqrt{-1}\Lambda\bar{\partial}\partial u_s = \sqrt{-1}\Lambda F_H - \Delta_\omega u_s. \tag{4.2}$$

Here, Δ_ω is the Laplacian operator defined by Kähler class ω. From standard Kähler identities, we have

$$2\sqrt{-1}\Lambda\bar{\partial}\partial u_s = -\Delta u_s,$$

where we use "analyst's Laplacian" here. It is defined to that on Euclidean n-space $\omega = \sqrt{-1}\delta_{ij}dz^i \wedge d\bar{z}^j$,

$$\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j}.$$

We will omit the subscript if no confusion arises. Since $|\phi_i|^2_{H_s} = e^{2u_s}|\phi_i|^2_H \;\; \forall i$, it follows that we can rewrite the last equation in (3.3), with metric H replaced by H_s, as:

$$-\Delta u_s + \frac{s^2}{2}\sum_{i=1}^{k}|\phi_i|^2_H e^{2u_s} + \left(\sqrt{-1}\Lambda F_H - \frac{s^2}{2}\right) = 0. \tag{4.3}$$

If we normalize the Kähler metric so that $Vol_\omega(\Sigma) = 1$, we can define

$$\begin{aligned} c(s) := 2\int_\Sigma \left(\sqrt{-1}\Lambda F_H - \frac{s^2}{2}\right) dvol_\omega &= 2\int_\Sigma \sqrt{-1}\Lambda F_H \omega^n - \frac{s^2}{2}\, dvol_\omega \\ = 2\int_\Sigma \sqrt{-1}\Lambda F_H \omega^n - \frac{s^2}{2} &= 2c_1 - \frac{s^2}{2}, \end{aligned}$$

where $c_1 = \int_\Sigma \sqrt{-1}\Lambda F_H \omega^n$ is independent of s and H. Consider ψ, a solution to:

$$\Delta\psi = \left(\sqrt{-1}\Lambda F_H - \frac{\tau\, s^2}{2}\right) - \frac{c(s)}{2} = \sqrt{-1}\Lambda F_H - c_1,$$

which is clearly independent of s.

Setting $\varphi_s := 2(u_s - \psi)$, u_s satisfies (4.3) if and only if φ_s satisfies:

$$\Delta\varphi_s - \frac{s^2}{2}\left(\sum_{i=1}^{k}|\phi_i|^2_H\, e^{2\psi}\right)e^{\varphi_s} - c(s). \tag{4.4}$$

This is of the form:

$$\Delta\varphi_s = -\left(\frac{s^2}{2}h\right) e^{\varphi_s} + c(s), \tag{4.5}$$

with $h = -\sum_{i=1}^{k} |\phi_i|_H^2 e^{2\psi} < 0$ and $c(s) < 0$ (for large s). Equation (4.5) is of the form considered in [K-W], and solvability is summarized as

Theorem 4.1.3 (Solvability for the Elliptic PDE $\Delta u = -he^u + c$)**.** *[K-W] Let M be a compact Riemannian manifold with metric g. Given a constant $c \in \mathbb{R}$, and a smooth function h, consider the equation*

$$\Delta u = -he^u + c.$$

The solvability of this elliptic PDE is summarized as follows:

(i) If $c = 0$, a solution exists precisely when h changes sign and $\int_M h dvol_M < 0$.

(ii) If $c > 0$, a solution exists precisely when h is positive somewhere on M.

(iii) If $c < 0$, a solution exists (uniquely) precisely when $h \leq 0$ on M.

Our data clearly fall into the third category, for s big enough. We have thus established the existence of u_s. The uniqueness follows from elliptic regularity of solutions to (4.5)

□

□

This theorem establishes the fact that $Hol_r(\Sigma, \mathbb{CP}^{k-1}) \simeq \nu_{k,0}(s)$ for each s, and therefore all $\nu_{k,0}(s)$ are mutually bijective. After establishing the explicit topological descriptions of $\nu_{k,0}(s)$, we now aim to describe its geometric dependence on the parameter s, as well as establishing uniformity and convergent behaviors of the gauge functions u_s (or equivalently φ_s).

Chapter 5

Limiting Behaviors of Non-Simultaneously Vanishing Vortices

We now state our first main result. We have, for each large enough s, obtained the smooth function u_s that completes the correspondence between $\nu_{k,0}(s)$ and $Hol_r(\Sigma, \mathbb{CP}^{k-1})$. However, we have yet controlled the family $\{u_s\}$ in any reasonable norm. It is also desirable to conclude that $u_s \to u_\infty$ in suitable topology. Our main theorem answers this question affirmatively. The theorem is a general analytic result on a general compact Kähler manifold M without boundary, which is directly applicable to our setting. It can be studied independently from discussions in previous sections. For conveniences of applications, we use the same notations (with the exception that we replace $\frac{h}{2}$ with h) for the functions and constants that are to be applied toward the data in the proof of Theorem 4.1.2.

We are now ready to establish the results stated in Chapter 2, that is, the unique existence of solutions to the partial differential equations

$$\Delta\varphi_s = -s^2 h e^{\varphi_s} + c(s),$$

and their asymptotic behaviors with respect to s. We start with the uniqueness and existence of the solution to this equation for large enough s. For each fixed s, identical theorem and proof can be found in section 9 of [K-W]. However, certain functions and constants in the proof are specifically chosen to ensure uniformity over the parameter s and establish the bounded-ness property of solutions in the Main Theorem 5.0.5.

Theorem 5.0.4 (Existence and Uniqueness of φ_s)**.** *On a compact Riemannian manifold (M, g) without boundary, let c_1 be any constant, c_2 any positive constant, and h any negative* *smooth function. Let* $c(s) = c_1 - s^2 c_2$*, the partial differential equation*

$$\Delta\varphi_s = -s^2 h e^{\varphi_s} + c(s)$$

has a unique smooth solution for all s *large enough.*

Proof. We first establish the uniqueness, a consequence of the maximum principle. Fix $s \in \mathbb{R}$, suppose that φ_s^1 and φ_s^2 are solutions to the equation. By elliptic regularity, both functions are smooth, and so is their difference $\varphi_s^1 - \varphi_s^2$. If $\varphi_s^1 \neq \varphi_s^2$, without loss of generality, we may assume

$$\inf_M \{\varphi_s^1(x) - \varphi_s^2(x)\} < 0.$$

Since M is compact and $\varphi_s^1 - \varphi_s^2$ is smooth, the infimum must be attained at some point $x_0 \in M$. We have

$$\varphi_s^1(x_0) < \varphi_s^2(x_0).$$

It follows that

$$\Delta(\varphi_s^1 - \varphi_s^2)(x_0) = -s^2 h[e^{\varphi_s^1(x_0)} - e^{\varphi_s^2(x_0)}] < 0,$$

since $-h > 0$ and exponential functions are monotonically increasing. We have arrived at a contradiction since the Laplacian of a smooth function has to be nonnegative at the point of minimum value. Therefore, the solution for each s has to be unique.

Following [K-W], we show the existence of solutions by constructing a sub-solution $\varphi_{-,s}$ satisfying

$$\Delta\varphi_{-,s} - c(s) + s^2 h e^{\varphi_{-,s}} \geq 0,$$

and a super-solution $\varphi_{+,s}$ satisfying

$$\Delta\varphi_{+,s} - c(s) + s^2 h e^{\varphi_{+,s}} \leq 0.$$

We construct two functions, φ_- and φ_+, independent of s, satisfying the two inequalities above for all s. In the proof of the Main Theorem 5.0.5, we will construct s-dependent super and sub solutions for certain bounded-ness properties. Let $\kappa(x) := \max\{1, -h\} > 0$. Choose a real number α such that $\alpha\,\bar{\kappa} = -c_1$, where $\bar{\kappa}$ is the average value of κ over M. The function $\alpha\kappa + c_1$ is in L^p and it has zero average value. By the standard PDE theory we can solve:

$$\Delta w = \alpha\,\kappa + c_1,$$

with a unique solution $w \in W^{2,p}$. Choose a number λ such that $c_2 + he^{w-\lambda} > 0$. This is clearly possible by compactness of M, and the fact that $c_2 > 0$. We then set

$$\varphi_- = w - \lambda,$$

which is clearly in $W^{2,p}$, and compute:

$$\Delta\varphi_- - c(s) + +s^2 h\, e^{\varphi_-} = \alpha\kappa + c_1 + c_2 s^2 - c_1 + s^2\, h\, e^{w-\lambda} = \alpha\kappa + s^2 \left(c_2 + h\, e^{w-\lambda}\right).$$

The right hand side is clearly nonnegative for s large enough, so that φ_- ,which is independent of s, is indeed a sub-solution for all large s.

We now construct the super-solutions. These will be of the form

$$\varphi_+ = a\, v + b$$

for some suitable constants a and b, and where, setting $\bar{h} := \int_M h$, $v \in W^{2,p}$ is the unique solution to:

$$\Delta v = \bar{h} - h.$$

Since $h < 0$, we can find a large enough constant a and an appropriate constant b so that:

$$a\bar{h} < c_1.$$

After which we choose b so that:

$$h\, e^{av+b} + c_2 < 0.$$

With these choices, one verifies that

$$\begin{aligned}\Delta\varphi_+ - c(s) + s^2 h\, e^{\varphi_+} &= a(\bar{h} - h) - c_1 + s^2 c_2 + s^2 h e^{av+b}\\ &= \left(a\bar{h} - c_1\right) + s^2\left(h\, e^{av+b} + c_2\right) - ah.\end{aligned}$$

Since $\left(a\bar{h} - c_1\right) < 0$ and $h\, e^{av+b} + c_2 < 0$ by construction, for s large enough, we thus have,

$$\Delta\varphi_+ - c(s) + s^2 h\, e^{\varphi_+} \leq 0.$$

Noting that also φ_+ is independent of s, this concludes the constructions of the barriers φ_+ and φ_-.

We are now ready to solve the equation for each s. The solution will be the limit of certain iterative equations. Pick a constant $k > 0$ so that

$$k \geq \sup_M \kappa e^{\varphi_+},$$

and consider the family of iterations defined by

$$L_s = \Delta - s^2 kI.$$

Setting $\varphi_{0,s} := \varphi_+$, the uniform initial function for iterations for all s. Since $s^2k > 0$, L_s is invertible for each s, and we can therefore define the sequence $\{\varphi_{i,s}\}$ inductively by

$$\Delta\varphi_{i+1,s} - s^2 k\varphi_{i+1,s} = c(s) - s^2 k\varphi_{i,s} - s^2 he^{\varphi_{i,s}}.$$

That is, $\varphi_{i+1,s}$ is the unique solution to the equation

$$L_s(f) = c(s) - s^2 k\varphi_{i,s} - s^2 he^{\varphi_{i,s}}.$$

Since h is smooth, elliptic regularity ensures that φ_+ is smooth, which further ensures that $\varphi_{1,s}$, and therefore all $\varphi_{i,s}$, are smooth. We claim, as in [K-W], that for all i and s, we have the following monotonic and bounded-ness relation in i:

$$\varphi_- \leq \varphi_{i+1,s} \leq \varphi_{i,s} \leq \varphi_+ \tag{5.1}$$

This will be proved by induction using the maximum principle of L_s (See Lemma 2.3.7). For $i = 1$, we recall that

$$L_s(\varphi_+) = \Delta\varphi_+ - s^2 k\varphi_+ \leq c(s) - s^2 k\varphi_+ - s^2 e^{\varphi_+} = L_s(\varphi_{1,s}),$$

and therefore

$$L_s(\varphi_{1,s} - \varphi_+) \geq 0,$$

which implies $\varphi_{1,s} \leq \varphi_+$. Suppose now that $\varphi_{i,s} \leq \varphi_{i-1,s}$. Since $k > -he^{\varphi_+}$ by its definition, one can readily compute that

$$
\begin{aligned}
L_s(\varphi_{i+1,s} - \varphi_{i,s}) &\geq -s^2 h e^{\varphi_+} \left[e^{\varphi_{i,s}-\varphi_+} - e^{\varphi_{i-1,s}-\varphi_+} - (\varphi_{i,s} - \varphi_+) + (\varphi_{i-1,s} - \varphi_+) \right] \\
&= -s^2 h e^{\varphi_+} \left[F(\varphi_{i,s} - \varphi_+) - F(\varphi_{i-1,s} - \varphi_+) \right], \qquad (5.2)
\end{aligned}
$$

where

$$F(x) = e^x - x.$$

$F(x)$ is a decreasing function when $x \leq 0$ since

$$F'(x) = e^x - 1 \leq 0 \ \ \forall x \leq 0.$$

Since $\varphi_{i,s} - \varphi_+ \leq \varphi_{i-1,s} - \varphi_+ \leq 0$ by inductive hypothesis, we conclude that

$$[F(\varphi_{i,s} - \varphi_+) - F(\varphi_{i-1,s} - \varphi_+)] \geq 0,$$

making the right hand side of (5.2) positive (recall that $-h > 0$). This concludes the inductive step $\varphi_{i+1,s} \leq \varphi_{i,s}$ by the maximum principle of L_s. We finally show that

$$\varphi_- \leq \varphi_{i,s} \ \ \forall i.$$

This will again be shown by induction. To show that $\varphi_- \leq \varphi_+$ we suppose the contrary, that

$$\inf_M \{\varphi_+(x) - \varphi_-(x)\} < 0.$$

Since $\varphi_+ - \varphi_-$ is smooth and M is compact, the infimum must be attained at some point $x_0 \in M$. Therefore,

$$\Delta(\varphi_+ - \varphi_-)(x_0) \leq -s^2 h(e^{\varphi_+(x_0)} - e^{\varphi_-(x_0)}) < 0.$$

This is a contradiction since the Laplacian of a smooth function must be nonnegative at the minimum. We conclude that $\varphi_- \leq \varphi_+$. Now suppose that $\varphi_- \leq \varphi_{i,s}$. Identical computations as in (5.2) yield

$$L_s(\varphi_- - \varphi_{i+1,s}) \geq -s^2 h e^{\varphi_+} \left[F(\varphi_- - \varphi_+) - F(\varphi_{i,s} - \varphi_+)\right],$$

where $F(x) = e^x - x$ as above. Since $\varphi_- - \varphi_+ \leq \varphi_{i,s} - \varphi_+ \leq 0$ by inductive hypothesis, we again have $F(\varphi_- - \varphi_+) - F(\varphi_{i,s} - \varphi_+) \geq 0$ and therefore have established the inductive statement. The monotonicity relation (5.1) is established.

Next, we wish to show that for each s, $\varphi_{i,s}$ uniformly converge to a smooth function φ_s. This is a replica of argument from [K-W]. Recall the Sobolev inequality for $p > dim(M)$ and $\gamma \in (0,1)$:

$$\|\varphi_{i,s}\|_{C^{1,\gamma}} \leq C \|\varphi_{i,s}\|_{W^{2,p}}, \tag{5.3}$$

and the Calderon-Zygmund L^p elliptic regularity for L_s (Proposition 2.2.4):

$$\|\varphi_{i,s}\|_{W^{2,p}} \leq C \|L_s(\varphi_{i,s})\|_{L^p} + \|\varphi_{i,s}\|_{L^p}.$$

Also recall that

$$\|L_s(\varphi_{i,s})\|_{L^p} = \left\|c(s) - s^2 k \varphi_{i-1,s} - s^2 h e^{\varphi_{i-1,s}}\right\|_{L^p}. \tag{5.4}$$

For a fixed s, the right hand side of (5.4) is uniformly bounded. Together with (5.3), this implies that $\varphi_{i,s}$ and their first derivatives are uniformly bounded in L^∞. By the Theorem of Arzelo-Ascoli, $\varphi_{i,s}$ possesses a subsequence uniformly converging to a function φ_s as $s \to \infty$. The monotonicity of $\varphi_{i,s}$ in i implies that the subsequence is actually the entire sequence.

Moreover, the L^p regularity shows that

$$\|\varphi_{i+1,s} - \varphi_{j+1,s}\|_{W^{2,p}} \leq C \left(\left\| s^2 h \right\|_{L^p} \|e^{\varphi_{i,s}} - e^{\varphi_{j,s}}\|_{L^\infty} + \|k\|_{L^p} \|\varphi_{i,s} - \varphi_{j,s}\|_{L^\infty} \right).$$

For a fixed s, the sequence $\{\varphi_{i,s}\}_i$ converges in L^∞, making the right hand side of the inequality above Cauchy. The sequence $\{\varphi_{i,s}\}_i$ is therefore strongly Cauchy in $W^{2,p}$, and therefore strongly convergent. We have arrived at the conclusion that $\varphi_{i,s}$ converges to φ_s, a solution to the equation

$$\Delta\varphi_s = c(s) - s^2 h e^{\varphi_s}.$$

Since $\varphi_s \in W^{2,p}$, elliptic regularity further ensures that φ_s is smooth. This completes the proof of existence and uniqueness of the solutions to the equation.

□

We now state the Main Theorem, on the bounded-ness and convergence of φ_s . Once again, this theorem is a general analytic result which directly applies to the data in Theorem 4.1.2. The functions and constants here need not be related to our initial geometric and topological data. We nevertheless use the same notations for the convenience of application and comparison.

Theorem 5.0.5 (Main Theorem)**.** *On a compact Riemannian manifold M without boundary, let c_1 be any constant, c_2 any positive constant, and h any negative smooth function. Let $c(s) = c_1 - c_2 s^2$, for each s large enough, the unique solutions $\varphi_s \in C^\infty$ for the equations*

$$\Delta\varphi_s = c(s) - s^2 h e^{\varphi_s}.$$

are uniformly bounded in $W^{l,p}$ for all $l \in \mathbb{N}$ and $p \in [1, \infty]$. Moreover, in the limit $s \to \infty$, φ_s converges smoothly to

$$\varphi_\infty = \log\left(\frac{c_2}{-h}\right),$$

the unique solution to

$$he^{\varphi_\infty} + c_2 = 0.$$

Proof. We will choose a special set of super and sub solutions, now s dependent, so that

$$c(s) - s^2 h e^{\varphi_s}$$

are uniformly bounded in L^∞. This provides a uniform bound for $\|\varphi_s\|_{W^{2,p}}$ for all p.

Since the function h is smooth and does not vanish, the function $\log(-h)$ is smooth and therefore uniformly bounded on the compact manifold M. Consequentially, there exists then a constant $K > 0$ so that

$$\Delta(-\log(-h)) + K \geq 0,$$

and

$$\Delta(-\log(-h)) - K \leq 0.$$

For s large enough so that $-K - c(s) \geq 0$, we define

$$\varphi_{-,s} = \log\left(\frac{-K - c(s)}{-s^2 h}\right). \tag{5.5}$$

and

$$\varphi_{+,s} = \log\left(\frac{K - c(s)}{-s^2 h}\right). \tag{5.6}$$

Even though these functions now depend on s, they clearly remain to be uniformly bounded. Moreover, we now have, for all s,

$$c(s) - s^2 h e^{\varphi_{-,s}} = -K,$$

and

$$c(s) - s^2 h e^{\varphi_{+,s}} = K.$$

One can easily see that

$$\Delta\varphi_{-,s} = \Delta\varphi_{+,s} = \Delta(-\log(-h)),$$

since $-\log(-h)$ is the only non constant part of their definitions on M. By our choice of K, we have

$$\Delta\varphi_{-,s} - c(s) + s^2 h e^{\varphi_{-,s}} \geq 0.$$

and

$$\Delta\varphi_{+,s} - c(s) + s^2 h e^{\varphi_{+,s}} \leq 0,$$

verifying that they are indeed sub and super solutions. The same argument as in the proof of Theorem 5.0.4 confirms that

$$\varphi_{-,s} \leq \varphi_{+,s},$$

and

$$\varphi_{-,s} \leq \varphi_s \leq \varphi_{+,s} \tag{5.7}$$

for all s. The functions $\varphi_{-,s}$ and $\varphi_{+,s}$ are again uniformly bounded over s. In fact, one can observe that

$$\varphi_{+,s} - \varphi_{-,s} = \log\left(\frac{K - c_1 + c_2 s^2}{-K - c_1 + c_2 s^2}\right) \to 0$$

in L^∞ as $s \to \infty$. With the bounded-ness condition (5.7), we immediately conclude that

$$\varphi_\infty = \lim_{s\to\infty} \varphi_s = \lim_{s\to\infty} \varphi_{+,s} = \lim_{s\to\infty} \varphi_{-,s} = \log\left(\frac{-c_2}{h}\right),$$

in L^∞.

To show the convergence in general $W^{l,p}$, we first consider a family of approximated solutions that converge smoothly to φ_∞ as $s \to \infty$. Consider

$$v_s := \log\left(\frac{\Delta\left(-\log(-h)\right) - c(s)}{-s^2 h}\right) \tag{5.8}$$

Since the function inside the logarithm converges smoothly to $\frac{-c_2}{h}$, it is clear that

$$v_s \to \varphi_\infty$$

smoothly as $s \to \infty$. In fact, since all v_s are uniformly bounded, we have, for all $N \in \mathbb{N}$,

$$v_s^N \to \varphi_\infty^N,$$

smoothly as $s \to \infty$. These functions v_s are approximated solutions to the PDE in the following sense:

$$\Delta v_s = c(s) - s^2 h e^{v_s} + E_s,$$

where

$$E_s = \Delta \log\left(\frac{\Delta\left(-\log(-h)\right) - c(s)}{s^2}\right). \tag{5.9}$$

Without the h in the denominator, the function

$$\log\left(\frac{\Delta\left(-\log(-h)\right) - c(s)}{s^2}\right)$$

converge smoothly to a constant as $s \to \infty$ and therefore it is clear that $E_s \to 0$ smoothly as $s \to \infty$.

The convergence statement of the theorem then follows from the lemma below:

Lemma 5.0.6. *For all $l \in \mathbb{N}$, we have, with v_s and φ_s defined in this theorem, that*

$$\lim_{s\to\infty} \|\varphi_s - v_s\|_{W^{l,\infty}} = 0.$$

Proof. (*of the Lemma*)

We perform induction on l. The base case $l = 0$ has been established, as both v_s and φ_s converge uniformly to φ_∞ as $s \to \infty$. Before we establish the inductive step, we first make the following crucial claim.

Claim:

$$\lim_{s\to\infty} \left\|s^2\left(e^{\varphi_s} - e^{v_s}\right)\right\|_{L^\infty} = 0 \tag{5.10}$$

To verify the claim, we start with the difference of the equations satisfied by φ_s and v_s:

$$\Delta\left(\varphi_s - v_s\right) = -s^2 h e^{\varphi_s} + s^2 h e^{v_s} - E_s \tag{5.11}$$

For each s, since the function $\varphi_s - v_s$ is smooth on the compact manifold M, there is a point $x_s \in M$ such that

$$\varphi_s(x_s) - v_s(x_s) = \sup_{x\in M}\{\varphi_s(x) - v_s(x)\}.$$

The Laplacian of $\varphi_s - v_s$ must be non-positive at x_s, and we have

$$0 \geq \Delta\left(\varphi_s - v_s\right)(x_s) = -s^2 h(x_s) e^{\varphi_s(x_s)} + s^2 h(x_s) e^{v_s(x_s)} - E_s(x_s).$$

It follows that, for all $x \in M$,

$$\begin{aligned} E_s(x_s) &\geq -s^2 h(x_s) e^{v_s(x_s)} \left[e^{\varphi_s(x_s) - v_s(x_s)} - 1\right] \\ &\geq -s^2 h(x_s) e^{v_s(x_s)} \left[e^{\varphi_s(x) - v_s(x)} - 1\right] \\ &= -s^2 h(x_s) e^{v_s(x_s)} e^{-v_s(x)} \left[e^{\varphi_s(x)} - e^{v_s(x)}\right] \end{aligned} \tag{5.12}$$

The second inequality follows from the choice of x_s:

$$\varphi_s(x_s) - v_s(x_s) \geq \varphi_s(x) - v_s(x) \ \ \forall x \in M.$$

Since the exponential function is monotonically increasing, and that $-s^2 h(x_s) e^{v_s(x_s)} \geq 0$, the inequality follows. We therefore arrive at the conclusion

$$s^2 \left[e^{\varphi_s(x)} - e^{v_s(x)}\right] \leq E_s(x_s) \frac{e^{-v_s(x_s)} e^{v_s(x)}}{-h(x_s)}. \tag{5.13}$$

Since v_s is uniformly convergent, thus bounded, and $h(x_s) \neq 0$, the fractional term is uniformly bounded. Since $E_s \to 0$ uniformly, the upper bound we have just obtained decays to 0 uniformly.

We need a lower bound that uniformly converge to 0. This is constructed using the same principle, except the special point $y_s \in M$ is chosen to be the point where the difference $\varphi_s - v_s$ achieves its infimum:

$$\varphi_s(y_s) - v_s(y_s) = \inf_{x \in M} \{\varphi_s(x) - v_s(x)\}.$$

The Laplacian of $\varphi_s - v_s$ now has to be non-negative at y_s, and we have identical chain of inequalities as in (5.12) in reverse order:

$$\begin{aligned} E_s(y_s) &\leq -s^2 h(y_s) e^{v_s(y_s)} \left[e^{\varphi_s(y_s) - v_s(y_s)} - 1 \right] \\ &\leq -s^2 h(y_s) e^{v_s(y_s)} \left[e^{\varphi_s(x) - v_s(x)} - 1 \right] \\ &= -s^2 h(y_s) e^{v_s(y_s)} e^{-v_s(x)} \left[e^{\varphi_s(x)} - e^{v_s(x)} \right] \end{aligned} \tag{5.14}$$

This leads to the desired lower bound

$$s^2 \left[e^{\varphi_s(x)} - e^{v_s(x)} \right] \geq E_s(y_s) \frac{e^{-v_s(y_s)} e^{v_s(x)}}{-h(y_s)}, \tag{5.15}$$

which decays to 0 uniformly as $s \to \infty$. The decaying upper bound (5.13) and lower bound (5.15) verify the claim (5.10).

Suppose that

$$\lim_{s \to \infty} \|\varphi_s - v_s\|_{W^{l,\infty}} = 0.$$

That is, for any multi-index J such that $|J| \leq l$, we have

$$\lim_{s \to \infty} \left\| \partial^J \varphi_s - \partial^J v_s \right\|_{L^\infty} = 0.$$

We wish to establish the convergence to the order $l+1$. The proof is substantially identical to the one for Claim (5.10), despite its involvement of rather tedious and lengthy bookkeeping of notations. Let I be a multi-index of length $l+1$. We apply ∂^I to (5.11):

$$\Delta \left(\varphi_s - v_s \right) = -s^2 h (e^{\varphi_s} - e^{v_s}) - E_s.$$

With caution to the commutation relation between covariant derivative and Laplacian stated in Lemma 2.3.11, one computes

$$
\begin{aligned}
& \Delta\left(\partial^I \varphi_s - \partial^I v_s\right) \\
= & \sum_{j\in\{I\}\cup M^l} \sum_{m^j(t)} \{ \left[a_{m^j(t)} \left(\partial^{I-j} h\right) \left(\partial^{m_i} v_s\right)^{t_i}\right] s^2 e^{v_s} - \left[a_{m_j(t)} \left(\partial^{I-j} h\right) \left(\partial^{m_i} \varphi_s\right)^{t_i}\right] s^2 e^{\varphi_s} \} \\
& -h\left[\left(\partial^I \varphi_s\right) s^2 e^{\varphi_s} - \left(\partial^I v_s\right) s^2 e^{v_s}\right] + \sum_{j\in\{I\}\cup M^l}^{l+1} Q^j(Rm)\left(\partial^j \varphi_s - \partial^j v_s\right) \\
& -\partial^I E_s.
\end{aligned}
\tag{5.16}
$$

Several notations above require explanations. These are algebraic expressions resulting from chain rules and product rules of differentiations, and the contributions of curvature tensors resulted form commuting ∂^I and Δ. First,

$$
M^l = \{r \in \mathbb{N}^n \mid |r| \leq l\},
$$

so that $j = I$ or some multi-index of length no greater than l. Each j in the index set generates a collection of pairs of the form

$$
m^j(t) := \{(m_1, \ldots, m_q), (t_1, \ldots, t_q) \mid m_i \in \mathbb{N}^n, t_i \in \mathbb{N}\}
$$

such that $|m_i| \leq l$ and

$$
m_1 t_1 + \ldots + m_q t_q = |j|.
$$

$a_{m(t)}$'s are then the appropriate constants in front of each function when differentiating the functions e^{v_s} and e^{φ_s} for $|j|$ times. For each j, $Q^j(Rm)$ is an algebraic combination of derivatives of the curvature tensors of (M,g) up to $|j|^{th}$ order, and is therefore smooth and uniformly bounded. We may combine the $Q^j(Rm)$'s into other terms in (5.16) and rewrite it into:

$$\begin{aligned} & \Delta\left(\partial^I\varphi_s - \partial^I v_s\right) \\ = \; & -s^2 h e^{\varphi_s}\left[1 - \frac{Q^I(Rm)}{s^2 e^{\varphi_s}}\right]\left(\partial^I\varphi_s - \partial^I v_s\right) \\ & + \sum_{j\in\{I\}\cup M^l}\left(A_{j,s} + B_{j,s}\right) \\ & + C_s - \partial^I E_s, \end{aligned} \tag{5.17}$$

where

$$A_{j,s} = \sum_{m^j(t)\neq((j),(1))} a_{m^j(t)}\left(\partial^{I-j}h\right)\left[\left(\partial^{m_i}v_s\right)^{t_i}\left(s^2e^{v_s} - s^2e^{\varphi_s}\right) + \left(\left(\partial^{m_i}v_s\right)^{t_i} - \left(\partial^{m_i}\varphi_s\right)^{t_i}\right)s^2e^{\varphi_s}\right], \tag{5.18}$$

$$\begin{aligned} B_{j,s} = \; & \left[a_{((j),(1))}\left(\partial^{I-j}h\right) - \frac{Q^j(Rm)}{s^2e^{v_s}}\right]s^2e^{v_s}\left(\partial^j v_s\right) \\ - \; & \left[a_{((j),(1))}\left(\partial^{I-j}h\right) - \frac{Q^j(Rm)}{s^2e^{\varphi_s}}\right]s^2e^{\varphi_s}\left(\partial^j \varphi_s\right), \end{aligned} \tag{5.19}$$

and

$$C_s = -h\left(\partial^I v_s\right)\left[s^2e^{\varphi_s} - Q^I(Rm)\right]\left(1 - e^{v_s-\varphi_s}\right). \tag{5.20}$$

One easily observes that for all j,

$$\lim_{s\to\infty}\left\|\frac{A_{j,s}}{s^2}\right\|_{L^\infty} = \lim_{s\to\infty}\left\|\frac{B_{j,s}}{s^2}\right\|_{L^\infty} = \lim_{s\to\infty}\left\|\frac{C_s}{s^2}\right\|_{L^\infty} = 0. \tag{5.21}$$

The decays of $\frac{A_{j,s}}{s^2}$ and $\frac{C_s}{s^2}$ follow easily from inductive hypothesis (all j are of lengths no greater than l), Lemma 2.2.7, Claim (5.10), and the facts that v_s are uniformly bounded in all Sobolev spaces and φ_s is uniformly bounded in L^∞. These facts also imply the decay of $\frac{B_{j,s}}{s^2}$. Indeed, by Claim (5.10), for each j there is a smooth function $\rho_j(s) \to 0$ smoothly in L^∞ so that

$$\frac{Q^j(Rm)}{s^2 e^{v_s}} = \frac{Q^j(Rm)}{s^2 e^{\varphi_s}} + \rho_j(s).$$

One can then rewrite

$$\begin{aligned} B_{j,s} &= \left[a_{((j),(1))}\left(\partial^{I-j}h\right) - \frac{Q^j(Rm)}{s^2 e^{\varphi_s}}\right]\left[\left(\partial^j \varphi_s\right)\left(s^2 e^{v_s} - s^2 e^{\varphi_s}\right) + \left(\partial^j v_s - \partial^j \varphi_s\right) s^2 e^{v_s}\right] \\ &\quad + \rho_j(s) s^2 e^{v_s}\left(\partial^j v_s\right), \end{aligned} \tag{5.22}$$

and the decay of $\frac{B_{j,s}}{s^2}$ in L^∞ follows.

We are in the position to re-apply the maximum principle as in the base case $|I| = 0$. Let $x_s \in M$ be the point so that

$$\partial^I \varphi_s(x_s) - \partial^I v_s(x_s) = \sup_{x\in M}\{\partial^I \varphi_s(x) - \partial^I v_s(x)\}.$$

Again, the Laplacian has to be non-positive at x_s, and we have, for all $x \in M$, that

$$
\begin{aligned}
0 &\geq \Delta\left(\partial^I \varphi_s - \partial^I v_s\right)(x_s) \\
&= -s^2 h(x_s) e^{\varphi_s(x_s)} \left[1 - \frac{Q^I(Rm)}{s^2 e^{\varphi_s}}\right](x_s)\left(\partial^I \varphi_s(x_s) - \partial^I v_s(x_s)\right) \\
&\quad + \sum_{j \in \{I\} \cup M^l} \left(A_{j,s}(x_s) + B_{j,s}(x_s)\right) \\
&\quad + C_s(x_s) - \partial^I E_s(x_s), \\
&\geq -s^2 h(x_s) e^{\varphi_s(x_s)} \left[1 - \frac{Q^I(Rm)}{s^2 e^{\varphi_s}}\right](x_s)\left(\partial^I \varphi_s(x) - \partial^I v_s(x)\right) \\
&\quad + \sum_{j \in \{I\} \cup M^l} \left(A_{j,s}(x_s) + B_{j,s}(x_s)\right) \\
&\quad + C_s(x_s) - \partial^I E_s(x_s),
\end{aligned}
\tag{5.23}
$$

The two expressions before and after the second $\geq$ are identical except that we replace x_s with x in the difference function $\partial^I \varphi_s - \partial^I v_s$ on the first line after the second $\geq$. For large enough s, we have

$$
1 - \frac{Q^I(Rm)}{s^2 e^{\varphi_s}} > 0
$$

on M and we may rearrange the (5.23) without reversing the direction of inequalities:

$$
\begin{aligned}
&\partial^I \varphi_s(x) - \partial^I v_s(x) \\
\leq\ & \frac{e^{-\varphi_s(x_s)}}{h(x_s)\left[1 - \frac{Q^I(Rm)}{s^2 e^{\varphi_s}}\right](x_s)} \left(\sum_{j \in \{I\} \cup M^l} \left[\frac{A_{j,s}(x_s)}{s^2} + \frac{B_{j,s}(x_s)}{s^2}\right] + \frac{C_s(x_s)}{s^2} - \frac{\partial^I E_s(x_s)}{s^2}\right)
\end{aligned}
\tag{5.24}
$$

By (5.21) and the fact that $E_s \to 0$ in all Sobolev spaces, the right hand side of this inequality decays to 0 as $s \to \infty$.

The lower bound for $\partial^I \varphi_s(x) - \partial^I v_s(x)$ is obtained similarly. For each s, there is a special point $y_s \in M$ such that

$$\partial^I \varphi_s(y_s) - \partial^I v_s(y_s) = \inf_{x \in M} \{\partial^I \varphi_s(x) - \partial^I v_s(x)\}.$$

The Laplacian of $\partial^I \varphi_s - \partial^I v_s$ has to be non-negative at y_s. Using identical arguments as the ones for upper bound (5.23) in reverse direction, we have, for all $x \in M$, that

$$\begin{aligned} & \partial^I \varphi_s(x) - \partial^I v_s(x) \\ \geq \quad & \frac{e^{-\varphi_s(y_s)}}{h(y_s)\left[1 - \frac{Q^I(Rm)}{s^2 e^{\varphi_s}}\right](y_s)} \left(\sum_{j \in \{I\} \cup M^l} \left[\frac{A_{j,s}(y_s)}{s^2} + \frac{B_{j,s}(y_s)}{s^2} \right] + \frac{C_s(y_s)}{s^2} - \frac{\partial^I E_s(y_s)}{s^2} \right) \end{aligned} \tag{5.25}$$

The right hand side again decays to 0 uniformly as $s \to \infty$ with the same arguments as in (5.24). Inequalities (5.24) and (5.25) establish the inductive step, and the lemma is therefore proved.

□

With Lemma 5.0.6 established, the Main Theorem follows trivially. Indeed, for all $l \in \mathbb{N}$, we have

$$\left\| \varphi_s - \log\left(\frac{c_2}{-h}\right) \right\|_{W^{l,\infty}} \leq \|\varphi_s - v_s\|_{W^{l,\infty}} + \left\| v_s - \log\left(\frac{c_2}{-h}\right) \right\|_{W^{l,\infty}} \to 0$$

as $s \to \infty$. Theorem 5.0.5 then follows easily from standard Sobolev compact embedding

$$W^{l,\infty} \hookrightarrow W^{l,p}$$

for any $l \in \mathbb{N}$ and $p \in [1, \infty]$.

□

Chapter 6

Baptista's Conjectures

We come back to Riemann surface $M = \Sigma$. The results collected so far prove two conjectures posed by Baptista [Ba]. The first one asserts that the natural L^2 metric on $\nu_{k,0}(s)$, when pulled back to $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ via Φ_s described in Lemma 4.1.1, evolves to a familiar one. We prove this result affirmatively, and use it to derive a general conjectural formula for the volume of the space $Hol_r(\Sigma, \mathbb{CP}^{k-1})$, which follows easily from the first conjecture. This formula has been verified by Speight for the case of degree 1 holomorphic maps on genus 0 Riemann surface (i.e. $\mathbb{S}^2$) in [Sp] using independent techniques and analysis.

6.1 The Evolution of L^2 Metrics on $\nu_{k,0}(s)$

We start with the definition of natural L^2 metric on $\mathcal{A}(H) \times \Omega^0(L) \times \ldots \times \Omega^0(L)$:

$$g_s((\dot{A}_s, \dot{\phi}_s), (\dot{A}_s, \dot{\phi}_s)) = \int_\Sigma \frac{1}{4s^2} \dot{A}_s \wedge \bar{*} \dot{A}_s + < \dot{\phi}_s, \dot{\phi}_s >_H vol_\Sigma \tag{6.1}$$

where $(\dot{A}_s, \dot{\phi}_s)$ denotes a tangent vector in $T_{(A,\phi)}(\mathcal{A}(H) \times \Omega^0(L)^k) \simeq \Omega^1(\Sigma) \oplus \Omega^0(L)^k$. By choosing tangent vectors orthogonal to $\mathcal{G}$-gauge transformations, (6.1) descends to a metric on the quotient space $\mathcal{A}(H) \times \Omega^0(L) \times \ldots \times \Omega^0(L)/\mathcal{G}$ and restricts to $\nu_{k,0}(s)$.

The L^2 metric for $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ is also well known, with Fubini-Study metric endowed on $\mathbb{CP}^{k-1}$. Given $f \in Hol_r(\Sigma, \mathbb{CP}^{k-1})$, the tangent space of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ at f can be identified with the space of sections of the pullback bundle of $T\mathbb{CP}^{k-1}$ via f:

$$T_f Hol_r(\Sigma, \mathbb{CP}^{k-1}) \simeq \Gamma(f^* T\mathbb{CP}^{k-1}).$$

Given $u, v \in T_f Hol_r(\Sigma, \mathbb{CP}^{k-1})$, they can be viewed as a pullbacked sections on Σ, which can be pushed forward by f to be tangent vectors on $\mathbb{CP}^{k-1}$, on which Fubini-Study metric ω_{FS} can be applied. We define

$$\langle u, v \rangle_{L^2} = \int_\Sigma \langle f_* u, f_* v \rangle_{\omega_{FS}} vol_\Sigma. \tag{6.2}$$

Here, the f_* denotes the pushforward of f.

Recall the correspondence

$$\Phi_s : Hol_r(\Sigma, \mathbb{CP}^{k-1}) \to \nu_{k,0}(s).$$

We are interested in pulling back g_s in (6.1) to $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ via Φ_s, denoted by g_s^*, and comparing it with $< \cdot, \cdot >_{L^2}$ in (6.2). It was conjectured by Baptista that, roughly, g_s approaches a constant multiple of $< \cdot, \cdot >_{L^2}$ as $s \to \infty$.

We carefully list the required data to proceed our analysis. Start with a holomorphic map $\tilde{\phi} : \Sigma \to \mathbb{CP}^{k-1}$. Equip $\mathbb{CP}^{k-1}$ with the Fubini-Study metric H_{FS}. There is a natural Hermitian metric on $\mathcal{O}(1)$ whose curvature form is a multiple of the Kähler form of Fubini-Study metric on $T\mathbb{CP}^{k-1}$. Explicitly, the metric is given locally at $[z_0 : \ldots : z_{k-1}] \in \mathbb{CP}^{k-1}$ by

$$\frac{1}{\sum_{i=1}^{k} |z_i|^2} |.|^2,$$

where $|.|$ is the Euclidean flat metric in the local trivialization. This metric carries the feature that its curvature form $F_{\mathcal{O}(1)}$ satisfies

$$\frac{\sqrt{-1}}{2\pi} \Lambda F_{\mathcal{O}(1)} = \omega_{FS},$$

the Kähler form of the Fubini-Study metric. Therefore,

$$\sqrt{-1}\Lambda F_{H_{\mathcal{O}(1)}} = \frac{1}{2\pi}(\omega_{FS}, \omega_{FS})_{\omega_{FS}} = \frac{1}{2\pi}.$$

On the degree one line bundle $\mathcal{O}(1)$, the metric above induces a curvature $F_{H_{\mathcal{O}(1)}}$ of constant trace. We will still denote this metric by H_{FS}. Recall the pullback construction of the line bundle L, sections ϕ, and background Hermitian metric as in Lemma 4.1.1:

$$\begin{array}{ccc} (L, H) & & (\mathcal{O}(1), H_{FS}) \\ \downarrow & & \uparrow {\scriptstyle z_1,\ldots,z_k} \\ \Sigma & \xrightarrow{\tilde{\phi}} & \mathbb{CP}^{k-1} \end{array}$$

where $L := \tilde{\phi}^*\mathcal{O}(1)$ and $H := \tilde{\phi}^* H_{FS}$. The global sections $z_1, \ldots, z_k$ on $\mathcal{O}(1)$ are pullbacked to L:

$$\phi := (\phi_i := \tilde{\phi}^* z_i)_i,$$

and $\tilde{\phi}$ also define a holomorphic structure $\bar{\partial}_L$ by pulling back the standard complex structure $\bar{\partial}_{\mathbb{CP}^{k-1}}$ on $\mathcal{O}(1)$. By the definition of H_{FS} on $\mathcal{O}(1)$, it is automatic that

$$\sum_{i=1}^{k} |\phi_i|_H^2 = 1.$$

We describe the variations of holomorphic maps and their corresponding pushforwards on $\nu_{k,0}(s)$. Given $\dot{\tilde{\phi}} \in T_{\tilde{\phi}} Hol_r(\Sigma, \mathbb{CP}^{k-1}) \simeq \Gamma(\phi^* T\mathbb{CP}^{k-1})$, we construct a smoothly varying curve $\tilde{\phi}(t)$ in $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ so that $\tilde{\phi}(0) = \tilde{\phi}$ and $\frac{\partial}{\partial t}|_{t=0}\tilde{\phi}(t) = \dot{\tilde{\phi}}$. $\tilde{\phi}(t)$ has the local coordinate expression

$$\tilde{\phi}(t) = \left[\tilde{\phi}_1(t), \ldots, \tilde{\phi}_k(t)\right]. \tag{6.3}$$

The corresponding family of sections in $\nu_{k,0}(s)$ are then defined by pulling back the global sections $z_1, \ldots, z_k$ via $\tilde{\phi}(t)$:

$$\phi_t = [\phi_{1,t}, \dots, \phi_{k,t}] \in \Omega^0(L) \times \dots \times \Omega^0(L), \tag{6.4}$$

where

$$\phi_{i,t} := \left(\tilde{\phi}(t)\right)^* (z_i).$$

The tangent vectors to ϕ_t are exactly the pullback of global sections $z_1, \dots, z_k$ via $\frac{\partial}{\partial t}\tilde{\phi}(t)$,:

$$\frac{\partial}{\partial t}\phi_t = \left(\frac{\partial}{\partial t}\phi_{1,t}, \dots, \frac{\partial}{\partial t}\phi_{k,t}\right),$$

where

$$\frac{\partial}{\partial t}\phi_{i,t} = \left(\frac{\partial}{\partial t}\tilde{\phi}_i(t)\right)^* z_i.$$

We denote

$$\dot{\phi} := \left(\frac{\partial}{\partial t}|_{t=0}\phi_{i,t}\right)_i.$$

All the pullback line bundles

$$L_t := \tilde{\phi}(t)^*\mathcal{O}(1)$$

are of the same degree and therefore isomorphic as complex line bundles. However, each of them is equipped with its own pullback holomorphic structure:

$$\bar{\partial}_{L_t} := \tilde{\phi}(t)^* \left(\bar{\partial}_{\mathbb{CP}^{k-1}}\right).$$

For convenience, we denote

$$\bar{\partial}_L := \bar{\partial}_{L_0}.$$

Each L_t is equipped with a background metric

$$H_t := \tilde{\phi}(t)^* H_{FS}$$

and denote $H := H_0$. With respect to H_t, we impose the orthogonality requirement in the definition of g_s on $\nu_{k,0}(s)$:

$$\left\langle \left(\frac{\partial}{\partial t} \phi_{i,t} \right), \phi_{i,t} \right\rangle_{H_t} = 0 \ \ \forall i, t. \tag{6.5}$$

To analyze g_s^*, we need to compute the pushforward of $\dot{\tilde{\phi}}$ under Φ_s, denoted by $\left(\dot{A}_s, \dot{\phi}_s \right)$ as in (6.1). For each t, our constructions above clearly imply

$$\bar{\partial}_{L_t} \phi_{i,t} = 0 \ \ \forall t, i.$$

By Theorem 4.1.2, or Theorem 5.0.4, we can then find a unique gauge $e^{2u_{s,t}} \in \mathcal{G}_{\mathbb{C}}$ so that

$$\left[\bar{\partial}_{L_t}, \phi_t, H_t e^{2u_{s,t}} \right] \in \mathcal{T}_{k,0}(s).$$

That is, the vortex equations are satisfied with the Hermitian structure $H_t e^{2u_{s,t}}$, and we denote

$$u_s := u_{s,0}.$$

This triplet above is further identified, via the identification

$$G_s : \mathcal{T}_{k,0}(s) \to \nu_{k,0}(s),$$

with

$$\left[D(e^{u_{s,t}*} \bar{\partial}_{L_t}), e^{u_{s,t}} \phi_t \right] \in \nu_{k,0}(s).$$

See Lemma 3.1.3 for the detailed descriptions. The map Φ_s is now explicitly written for each

t:

$$\Phi_s(\tilde{\phi}(t)) = \left[D(e^{u_{s,t}*}\bar{\partial}_{L_t}), e^{u_{s,t}}\phi_t\right].$$

Recall the gauge action on holomorphic structures:

$$e^{u_{s,t}*}\bar{\partial}_{L_t} = e^{u_{s,t}}\left(\bar{\partial}_L e^{-u_{s,t}}\right) = \bar{\partial}_{L_t} - \left(\frac{\partial u_{s,t}}{\partial \bar{z}}\right) d\bar{z},$$

we have

$$\Phi_s(\tilde{\phi}(t)) = \left[D\left(e^{u_{s,t}}(\bar{\partial}_L e^{-u_{s,t}})\right), e^{u_{s,t}}\phi_t\right], \tag{6.6}$$

where $D\left(e^{u_{s,t}}(\bar{\partial}_L e^{-u_{s,t}})\right)$ is the H_t-unitary connection with respect to the holomorphic structure

$$e^{u_{s,t}}\left(\bar{\partial}_L e^{-u_{s,t}}\right).$$

We can now identify $\dot{\phi}_s$. By our constructions and correspondences above, $\dot{\phi}_s$ is the pushforward of $\dot{\phi}$ under G_s. One notes that G_s multiplies each section ϕ_t by a gauge $e^{u_{s,t}}$ and is therefore linear on sections. Therefore, at $t = 0$, the pushforward of $\dot{\phi}$ is also given by the multiplication of the same function evaluated at 0:

$$\dot{\phi}_s = e^{u_s}\dot{\phi}. \tag{6.7}$$

$\dot{A}_s$ needs to be computed with caution. Let $\gamma \in \Omega^0(L)$ be the local holomorphic frame for L, with respect to the holomorphic structure $\bar{\partial}_L$. The background Hermitian metric is locally given by a smooth function H_t in this setting. Altering the holomorphic structure, we observe that the section $e^{u_{s,t}}\gamma$ is the local holomorphic frame with respect to the holomorphic structure $e^{u_{s,t}}(\bar{\partial}_L e^{-u_{s,t}})$. With respect to this frame, the same background Hermitian metric now has local coordinate description by the smooth function

$$H'_t = H_t e^{2u_{s,t}}.$$

We then compute the connection form $A_{s,t}$ of $D(e^{u_{s,t}}(\bar{\partial}_L e^{-u_{s,t}}))$ using the formula for H'_t-unitary connection forms:

$$\begin{aligned}
A_{s,t} &= (H'_t)^{-1}\partial(H'_t) \\
&= \frac{\left(\frac{\partial H_t}{\partial z} + 2H_t\frac{\partial u_{s,t}}{\partial z}\right)}{H_t}dz \\
&= \left[\frac{\partial}{\partial z}(\log H_t) + 2\frac{\partial u_{s,t}}{\partial z}\right]dz.
\end{aligned} \tag{6.8}$$

We differentiate $A_{s,t}$ with respect to t and evaluating it at $t = 0$ to obtain $\dot{A}_s$:

$$\dot{A}_s := \frac{\partial}{\partial t}|_{t=0}A_{s,t} = \frac{\partial}{\partial z}\left(\frac{\dot{H}}{H}\right) + 2\frac{\partial \dot{u_s}}{\partial z}dz,. \tag{6.9}$$

where

$$\dot{u_s} := \frac{\partial}{\partial t}|_{t=0}u_{s,t},$$

and

$$\dot{H} := \frac{\partial}{\partial t}|_{t=0}H_t.$$

With the pushforward tangent vector $(\dot{A}_s, \dot{\phi}_s)$ identified, we finally arrive at an explicit formula for g_s^*:

$$
\begin{aligned}
g_s^*\left(\dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right) &:= g_s\left(\Phi_{s,*}\left(\dot{\tilde{\phi}}\right), \Phi_{s,*}\left(\dot{\tilde{\phi}}\right)\right) \\
&= g_s\left((\dot{A}_s, \dot{\phi}_s), (\dot{A}_s, \dot{\phi}_s)\right) \\
&= \int_\Sigma \left(\frac{\left| \frac{\partial}{\partial z}\left(\frac{\dot{H}}{H}\right) + 2\frac{\partial \dot{u}_s}{\partial z} \right|^2}{4s^2} + \left\langle \dot{\phi}, \dot{\phi} \right\rangle_H e^{2u_s} \right) vol_\Sigma
\end{aligned} \tag{6.10}
$$

One should expect the first term in (6.10) to vanish as $s \to \infty$, and the second term to approach a multiple of square norm of $\dot{\phi}$. Namely, we expect (6.10) to approach the (multiple of) $< \cdot, \cdot >_{L^2}$ defined in (6.2). This is precisely the statement in the Baptista's Conjecture in [Ba].

Conjecture 6.1.1 (Baptista's Conjecture). *On $Hol_r(\Sigma, \mathbb{CP}^{k-1}) \backsimeq \nu_{k,0}(s)$, g_s^* defined in* (6.10) *converges in $H_{2,p}$ to a multiple of the ordinary L^2 metric on $Hol_r(\Sigma, \mathbb{CP}^{k-1})$.*

To be more mathematically precise, we state the following notion of convergence.

Definition 6.1.2 (Cheeger-Gromov Convergence). A family of n-dimensional Riemannian manifolds (M_s, g_s) is said to converge to a fixed Riemannian manifold (M, g) in $W^{l,p}$, in the sense of Cheeger-Gromov, if there is a locally finite covering chart $\{U_k, (x_i)\}$ on M and a sequence of diffeomorphisms $F_s : M \to M_s$, such that

$$
\left\| F_s^*(g_s)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) - g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \right\|_{W^{l,p}(U_k)} \to 0
$$

as $s \to \infty$.

We state Baptista's Conjecture in this level of mathematical rigor:

Proposition 6.1.3 (Precise Baptista's Conjecture). *Equipping $\mathbb{CP}^{k-1}$ with the Fubini-Study metric, the sequence of metrics g_s on $\nu_{k,0}(s)$ given by* (6.1) *Cheeger-Gromov converges* *smoothly to a multiple of the ordinary L^2 metric $< \cdot, \cdot >_{L^2}$ on $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ given by* (6.2)*. The family of diffeomorphisms are given by Φ_s.*

Proof. We first recall that for each t, the k-sections ϕ_t give rise to the function

$$h_t = -e^{2\psi_t} \sum_{i=1}^{k} |\phi_{i,t}|^2_{H_t} \tag{6.11}$$

as in the proof of Theorem 4.1.2, where

$$\Delta \psi_t = \sqrt{-1} \Lambda F_{H_t} - c_1.$$

However, since ϕ_t and H_t are pullbacked from the sections $z_1, \ldots, z_k$ on $\mathcal{O}(1)$ with constant H_{FS} norm of 1, it is clear that $\sum_{i=1}^k |\phi_{i,t}|^2_{H_t} = 1 \;\; \forall t$, and

$$h_t = -e^{2\psi_t}.$$

Throughout the proof, we use the following abbreviations for the initial value and variation of a family of functions f_t with parameter t:

$$f := f_0,$$

and

$$\dot{f} := \frac{\partial}{\partial t}|_{t=0} f_t.$$

For each t, recall the relation of $u_{s,t}$ and $\varphi_{s,t}$:

$$\varphi_{s,t} = 2(u_{s,t} - \psi_t),$$

it follows that $e^{2u_{s,t}} = -he^{\varphi_{s,t}}$ and

$$\frac{\partial \dot{u}_s}{\partial z} = \frac{1}{2}\left(\frac{\partial \dot{\varphi}_s}{\partial z} + 2\frac{\partial \dot{\psi}}{\partial z}\right).$$

The pullback metric g_s^* (6.10) can be rewritten as

$$g_s^*\left(\dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right) = \int_\Sigma \left(\frac{\left|\frac{\partial}{\partial z}\left(\frac{\dot{H}}{H}\right) + \frac{\partial \dot{\varphi}_s}{\partial z} + 2\frac{\partial \dot{\psi}}{\partial z}\right|^2}{4s^2} + \left\langle \dot{\phi}, \dot{\phi}\right\rangle_H \left(-he^{\varphi_s}\right)\right) vol_\Sigma \tag{6.12}$$

It is evident from our constructions that

$$\left\langle \dot{\phi}, \dot{\phi}\right\rangle_H = \left\langle \dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right\rangle_{H_{FS}}.$$

Moreover, viewing $\mathbb{CP}^{k-1}$ as $\mathbb{S}^{2k-1}/U(1)$, the Fubini-Study metric ω_{FS} is $\frac{1}{\pi}$ times the round metric of $\mathbb{S}^{2k-1} \hookrightarrow \mathbb{C}^k$, which is invariant under $U(1)$ action. We may rescale H_{FS} by $\frac{1}{\sqrt{\pi}}$ so that $< \cdot, \cdot >_{H_{FS}}$ is the same as the ordinary Fubini-Study metric.

We now allow $\tilde{\phi}$ to vary in any direction of the coordinates of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$. Fix a coordinate patch $U \subset Hol_r(\Sigma, \mathbb{CP}^{k-1})$ containing $\tilde{\phi}$, and view $\dot{\tilde{\phi}}$ as a parallel vector fields with respect the H_{FS}. Corresponding functions h, ψ, u_s, and φ_s are now smooth functions on $\Sigma \times Hol_r(\Sigma, \mathbb{CP}^{k-1})$. $g_s^*\left(\dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right)$ in (6.12) then defines a smooth function from U to $\mathbb{R}$. For a multi-index R, we may compute

$$\partial^R g_s^*,$$

where R is indexed with respect to the coordinates of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$. Denote z as the coordinate of Σ. Since all functions in the integrand of the integral (6.12) are smooth, one has

$$
\begin{aligned}
&\partial^R g_s^*\left(\dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right) \\
= \int_\Sigma \quad &\frac{\partial^R \left|\partial\left(\frac{\partial}{\partial z}(\log H) + \frac{\partial \varphi_s}{\partial z} - 2\frac{\partial \psi}{\partial z}\right)\right|^2}{4s^2} + \frac{1}{\sqrt{\pi}}\left[\partial^R \left\langle \dot{\tilde{\phi}}, \dot{\tilde{\phi}} \right\rangle_{H_{FS}}\right](-he^{\varphi_s}) \\
&+ \sum_{j \in \{R\} \cup M^R} A_j \left[\partial^j\left(-he^{\varphi_s}\right)\right] B^{R-j} \ \ vol_\Sigma.
\end{aligned} \tag{6.13}
$$

Here, ∂ is a first order derivative along some coordinate of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$. M^R is the set of all multi-indices with lengths less than $|R|$, as defined in the proof of Lemma 5.0.6. B^{R-j} are smooth functions defined by

$$
B^{R-j} = \partial^{R-j}\left\langle \dot{\tilde{\phi}}, \dot{\tilde{\phi}} \right\rangle_{H_{FS}},
$$

which are independent of s. The conclusion of the proposition follows from the following three conditions for every multi-index R, making the first and second term in (6.13) decay to 0 and second term approach to the desired quantity:

$$
\lim_{s\to\infty} \left\| \frac{\partial^R \left|\partial\left(\frac{\partial}{\partial z}(\log H) + \frac{\partial \varphi_s}{\partial z} - 2\frac{\partial \psi}{\partial z}\right)\right|^2}{4s^2} \right\|_{L^\infty(\Sigma)} = 0; \tag{6.14}
$$

$$
\lim_{s\to\infty} \left\|\partial^j\left(-he^{2\varphi_s}\right)\right\|_{L^\infty(\Sigma)} = 0 \ \ \forall j \text{ such that } 1 \le |j| \le |R|; \tag{6.15}
$$

and

$$
-he^{\varphi_s} \to c_2 \text{ in } L^\infty(\Sigma) \text{ as } s \to \infty, \tag{6.16}
$$

at every point of $U \subset Hol_r(\Sigma, \mathbb{CP}^{k-1})$. Note that the derivatives here are taken with respect to the coordinates of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$, but L^∞ are defined on functions of Σ. To show these three statements, we recall the definition of approximated solutions v_s and error E_s, now dependent on coordinates of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ as we vary $\tilde{\phi}$ arbitrarily:

$$v_s := \log\left(\frac{\Delta_\Sigma\left(-\log(-h)\right) - c(s)}{-s^2 h}\right) \tag{6.17}$$

with error

$$E_s = \Delta_\Sigma \log\left(\frac{\Delta_\Sigma\left(-\log(-h)\right) - c(s)}{s^2}\right) \tag{6.18}$$

so that

$$\Delta_\Sigma v_s + s^2 h e^{v_s} - c(s) = E_s.$$

Here, Δ_Σ denotes the Laplacian with respect to coordinates of Σ. It is straightforward to check that for all multi-indices R and $l \in \mathbb{N}$,

$$\left\|\partial^R v_s\right\|_{W^{l,\infty}(\Sigma)} \leq C_l, \tag{6.19}$$

$$\lim_{s\to\infty}\left\|\partial^R\left(-he^{v_s}\right)\right\|_{L^\infty(\Sigma)} = 0 \;\; \forall j \text{ such that } 1 \leq |j| \leq |R|, \tag{6.20}$$

$$-he^{v_s} \to c_2 \text{ in } L^\infty(\Sigma) \text{ as } s \to \infty, \tag{6.21}$$

and

$$\lim_{s\to\infty}\left\|\partial^R E_s\right\|_{W^{l,\infty}(\Sigma)} = 0. \tag{6.22}$$

at each point of $U \subset Hol_r(\Sigma, \mathbb{CP}^{k-1})$. Since H and ψ are independent of s, statements (6.14), (6.15), and (6.16) then follow from (6.19), (6.20), (6.21), and the lemma below:

Lemma 6.1.4. *For all multi-indices R, and $l \in \mathbb{N}$, we have*

$$\lim_{s\to\infty} \left\|\partial^R v_s - \partial^R \varphi_s\right\|_{W^{l,p}(\Sigma)} = 0,$$

at each point of the coordinate patch $U \subset Hol_r(\Sigma, \mathbb{CP}^{k-1})$. Here, ∂^R is the R^{th} derivative with respect to coordinates of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$.

Proof. The proof is identical to the proof of Lemma 5.0.6. One simply replace all $\partial^I \varphi_s$ and $\partial^I v_s$ there with $\partial^R \varphi_s$ and $\partial^R v_s$. The curvature terms $Q^j(Rm)$ are, in particular, independent of coordinates of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$.

We analyze the difference of equations satisfied by φ_s and v_s:

$$\Delta_\Sigma (\varphi_s - v_s) = -s^2 h e^{\varphi_s} + s^2 h e^{v_s} - E_s, \tag{6.23}$$

which is identical to (5.11) in Lemma 5.0.6. At each point of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$, statements (6.19) and (6.22) form all the properties of E_s and v_s required to proceed the proof of Lemma 5.0.6 and construct Claim (5.10) using maximum principles. Therefore, identical arguments apply toward $\partial^R v_s$ and $\partial^R \varphi_s$ to conclude the statement of the Lemma. □

We have proved, that on the coordinate patch $U \subset Hol_r(\Sigma, \mathbb{CP}^{k-1})$, the functions

$$\partial^R g_s^* \left(\dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right)$$

converges pointwise to the smooth function

$$\int_\Sigma \frac{c_2}{\sqrt{\pi}} \left[\partial^R \left\langle \dot{\tilde{\phi}}, \dot{\tilde{\phi}} \right\rangle_{H_{FS}}\right] = \frac{c_2}{\sqrt{\pi}} \partial^R \int_\Sigma \left\langle \dot{\tilde{\phi}}, \dot{\tilde{\phi}} \right\rangle_{H_{FS}},$$

for all multi-indices R, as $s \to \infty$. The limiting function is clearly a smooth function on U. Shrinking U if necessary, we may assume that U is compact with respect to the metric $< \cdot, \cdot >_{L^2}$ in (6.2). It then follows that the pointwise convergence of $\partial^R g_s^* \left(\dot{\tilde{\phi}}, \dot{\tilde{\phi}}\right)$ is actually uniform. This proves the smooth convergence of g_s^* to $< \cdot, \cdot >_{L^2}$ in the sense of Cheeger-Gromov.

$\square$

6.2 Volume of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$

The first conjecture on evolution of the L^2 metrics g_s on $\nu_{k,0}(s)$ easily proves another conjecture posed by Baptista in [Ba]. From the isometry we have just established:

$$\Phi_\infty : Hol_r(\Sigma, \mathbb{CP}^{k-1}) \to \nu_{k,0}(s),$$

we can directly compute the volume of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$. This space arises from theoretical physics as the moduli space of "charge d lumps" in the $\mathbb{CP}^{k-1}$ model on Σ. (See [Sp]).

As in [Ba], we denote the volume form of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ with respect to $< \cdot, \cdot >_{L^2}$ by $vol_{\mathcal{H}}$ and the volume form of $\nu_{k,0}(s)$ with respect to g_s by $vol_{\nu,s}$. Also, we denote the volume of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ and $\nu_{k,0}(s)$ with respect to the volume forms above by $\mathrm{Vol}_{\mathcal{H}}$ and $\mathrm{Vol}_{\nu,s}$, respectively. Baptista's Conjecture 6.1.3 implies that

$$\mathrm{Vol}_{\mathcal{H}} = \int_{Hol_r(\Sigma,\mathbb{CP}^{k-1})} vol_{\mathcal{H}} = \int_{\nu_{k,0}(s)} \lim_{s\to\infty} vol_{\nu,s} \tag{6.24}$$

Since $\nu_{k,0}(s)$ is an open dense subset of $\nu_k(s)$, they have the same volume and we may replace $\nu_{k,0}(s)$ by $\nu_k(s)$ and view $vol_{\mathcal{H}}$ and g_s as tensors on the entire $\nu_k(s)$. We have

$$\mathrm{Vol}_{\mathcal{H}} = \int_{Hol_r(\Sigma,\mathbb{CP}^{k-1})} vol_{\mathcal{H}} = \int_{\nu_k(s)} \lim_{s\to\infty} vol_{\nu,s} = \lim_{s\to\infty} \int_{\nu_k(s)} vol_{\nu,s}. \tag{6.25}$$

Let $[\omega_s] \in H^2(\nu_k(s), \mathbb{R})$ be the Kähler class of g_s. We pick a smooth representative from $[\omega_s]$ so that $vol_{\nu,s}$, a wedge power of $[\omega_s]$, is smooth and the third equality in (6.25) is justified. The third integral in (6.25) has been explicitly described and computed by Baptista in [Ba], and we briefly summarize here.

Let b be the genus of the Riemann surface Σ. For line bundle L of degree $r \geq 2b-2$, the moduli space $\nu_k(s)$ is isomorphic to the projective bundle

$$\nu_k(s) \simeq \mathbb{P}(V) \to J_\Sigma,$$

where J_Σ is the Jacobian torus of Σ. Each point in J_Σ identifies a holomorphic structure $\bar{\partial}$ of L, and the fiber over it is the projectivization of the vector space of holomorphic sections $H^0_{\bar{\partial}}(\Sigma, L)$ with respect to $\bar{\partial}$. For $r \geq 2b-2$, it is a classical fact that the dimension of $\mathbb{P}\left(H^0_{\bar{\partial}}(\Sigma, L^{\oplus k})\right)$ is uniform throughout J_Σ, and given by

$$q = b + k(r + 1 - b) - 1.$$

Perutz [P] and Baptista [Ba] have provided a cohomological formula of $[\omega_s]$ in terms of the Chern class of certain line bundle, and the standard integral symplectic form on J_Σ (also known as the Poincare dual of the theta divisor). Precisely, the formula is

$$[\omega_s] = \pi \left(\mathrm{Vol}_\Sigma - \frac{2\pi}{s^2} r \right) \eta + \frac{2\pi^2}{s^2} \theta. \tag{6.26}$$

Here,

$$\eta = c_1(\mathcal{L}),$$

the Chern class to the line bundle

$$\mathcal{B} \times_{U(1)} \mathbb{C} \to \nu_k(s),$$

where $\mathcal{B} \subset \mathcal{A}(H) \times \Omega^0(L) \times \ldots \Omega^0(L)$ is the subset of holomorphic pairs satisfying vortex equations (3.3) (without identifying the $\mathcal{G}$-gauge) and $U(1)$ acts on $\mathcal{B}$ by standard gauge actions defined in the preliminary Chapters. θ is the pullback of the normalized symplectic form on J_Σ via the projection

$$p : \nu_k(s) \to J_\Sigma.$$

In [Ba] and [B-D-W], it was shown that

$$p_*(\eta^{q-i}) = \frac{(k\theta)^{b-i}}{(b-i)!}.$$

Moreover, it is clear that

$$\int_{J_\Sigma} \theta^b = b!.$$

With these preliminaries, it is now straightforward to compute Vol_Σ ([Ba]).

$$\begin{aligned} \mathrm{Vol}_{\nu,s} &= \frac{1}{q}\int_{\nu_k(s)} [\omega_s]^r \\ &= \pi^q \sum_{i=0}^{b} \frac{g!k^{b-i}}{i!(q-i)!(b-i)!} \left(\frac{2\pi}{s^2}\right)^i \left(\mathrm{Vol}_\Sigma - \frac{2\pi}{s^2} r\right)^{q-i}. \end{aligned} \tag{6.27}$$

Letting $s \to \infty$, the only term that survives is when $i = 0$, and we have proved the second conjecture of Baptista:

Conjecture 6.2.1. *The volume of $Hol_r(\Sigma, \mathbb{CP}^{k-1})$ with respect to the ordinary L^2 metric $< \cdot, \cdot >_{L^2}$ is given by*

$$Vol_{\mathcal{H}} = \lim_{s\to\infty} Vol_\Sigma = (\pi\, Vol_\Sigma)^q \frac{k^b}{q!}, \tag{6.28}$$

where $q = b + k(r + 1 - b) - 1$.

For $g = 0$ ($\Sigma = \mathbb{S}^2$), and $r = 1$, this formula has been obtained previously by Speight in [Sp] using independent techniques that do not easily generalize to more spaces with higher genus or maps of higher degrees.

Chapter 7

Descriptions and Limiting Behaviors of the Entire Moduli Space

We have now completely described $\nu_{k,0}(s)$, its limiting descriptions, and confirmed the limiting identifying map $\Phi_\infty : (Hol_r(\Sigma, \mathbb{CP}^{k-1}), (6.2)) \to (\nu_{k,0}(s), (6.1))$ as an isometry. A natural advancement is to describe the entire moduli space $\nu_k(s)$ across singularities as $s \to \infty$. As we have expected, the presence of these singularities will be exhibited by some topological data. The first part of our results in this aspect is devoted to extending the vortices, obtained as smooth objects away from common zeros, across these singularities. This procedure can be classified as removing singularities and extending the vortices. The removal of singularities will be exhibited by loss of topological data, or the so called "bubbling". This will form the second part of the discussion, which is mainly a survey of established works. Both parts of the discussions are founded on Uhlenbeck's profound results for the compactness of connections with bounded curvatures, and we will begin by summarizing the relevant ones briefly. Most of the summaries are excerpted from [W].

7.1 Uhlenbeck's Results

The first result is the existence of Uhlenbeck gauge for connection forms whose curvatures are uniformly bounded in suitable Sobolev space. With this theorem, we are able to prove a classical compactness result for a sequence of connections with uniformly bounded curvatures in $W_2^{k,p}$. These results will ultimately lead us to construct the extension of vortices across the common zeros of k sections, where the Main Theorem 5.0.5 does not hold, as well as the observations of bubbling phenomenon of vortices, resulted from concentration of energies near common zeros, as $s \to \infty$.

Unless otherwise stated, we do not assume the connections to satisfy the vortex equations. Moreover, the boundary of the manifold M is not necessarily empty. The proofs of these theorems contain numerous profound and elegant techniques, and each of them provides us with powerful analytic tool to establish the results. Due to the limitation of the scope, we will only sketch the relevant parts of their proofs from [W] in this section.

We first introduce the notion of Uhlenbeck gauge, a Coulomb gauge with additional estimates on Sobolev norms. We will restrict the values of p, q in the range so that Sobolev estimates and embedding are valid:

If n is the dimension of the Riemannian manifold M, we let $1 < q \leq p < \infty$, $q \geq \frac{n}{2}$, $p > \frac{n}{2}$, and if $q < n$, assume $p \leq \frac{nq}{n-q}$.

Recall that for a connection form $A \in \Omega^1$, a gauge $u \in \mathcal{G}$ acts on A by:

$$u^* A = A + u^{-1} du.$$

Among the orbit of A by the action of $\mathcal{G}$, the following element is of particular importance. For our purposes, these connection forms allow us to perform regularity analysis from elliptic equations.

Definition 7.1.1. (Uhlenbeck Gauge)

A connection $A \in W_1^{1,p}(U)$ is in Uhlenbeck gauge if it satisfies

$$(i) d^* A = 0 \qquad (iii) \|A\|_{W_1^{1,q}} \leq C \|F_A\|_{L^q}$$

$$(ii) * A|_{\partial U} = 0 \qquad (iv) \|A\|_{W_1^{1,p}} \leq C \|F_A\|_{L^p}.$$

for some positive constant C.

The following theorem asserts that under appropriate conditions, one can always locally gauge transform a connection with small energy into Uhlenbeck gauge. For a Riemannian metric g and a given $q \in \mathbb{R}$, the L^q energy of a connection is given by

$$\mathcal{E}_q(A) = \int_U |F_A|_g^q dvol_g,$$

where F_A is the curvature of A.

Theorem 7.1.2. *(Existence of Uhlenbeck Gauge [W])*

Let $E \to (M, g)$ be a vector bundle over an n dimensional Riemannian manifold, p, q be in the correct Sobolev range described immediately above and G a compact Lie group. Then there exist positive constants C and ϵ so that the following holds:

Every point $x \in M$ has a neighborhood U such that for every connection $A \in W_1^{1,p}(U)$ with $\mathcal{E}_q(A) \leq \epsilon$, there exists a gauge $u \in \mathcal{G}^{2,p}$ so that u^*A is in Uhlenbeck gauge.

We first note that the theorem can be reduced considerably to two model cases on $\mathbb{R}^n$. One of them is the open unit ball in $\mathbb{R}^n$ with respect to the metric g $W^{1,\infty}$ close to the Euclidean metric. Since ∂M might not be empty, we must also consider an "egg", or the intersection of open unit ball and he half space $\mathbb{H}^n = \{(z_0, \ldots, z_{n-1}) \in \mathbb{R}^n | z_0 \geq 0\}$. We have,

Proposition 7.1.3. *(Uhlenbeck Gauge for the Model Cases [W]) Let $E \to (B, g)$ be a vector bundle over $B \subset \mathbb{R}^n$, where B is an open unit ball or an "egg". Let p, q be in the correct Sobolev range and G be a compact Lie group. Then there exist positive constants δ, C and ϵ so that the following holds:*

If $\|g - Id\|_{W_2^{1,\infty}} \leq \delta$ then for every connection $A \in W_1^{1,p}(B)$ with $\mathcal{E}_q(A) \leq \epsilon$, there exists a gauge $u \in \mathcal{G}^{2,p}$ so that u^*A is in Uhlenbeck gauge.

Theorem of Uhlenbeck gauge 7.1.2 follows rather easily from the model cases. The proof introduces important techniques on rescaling coordinates, which will be applied toward our later work.

Proof. of Theorem 7.1.2 assuming Proposition 7.1.3

All the conditions of Uhlenbeck gauge are coordinate independent and therefore it suffices to prove the theorem on some special local coordinates around x. For $x \in M$, there exists $\sigma \in (0, 1]$, and a diffeomorphism ψ so that $\psi : \sigma B \to M$ is a normal coordinate neighborhood around x such that $\|\psi_\sigma^* g - Id\|_{W_2^{1,\infty}}(\sigma B) \leq \delta$. This is possible from the smoothness and normality of g. Clearly, σB is an open ball if $x \in int(M)$ and an "egg" if $x \in \partial M$, both of radius σ.

The conditions of the model cases require an open unit ball B in flat metric. We consider the map $\psi_\sigma : B \to M$ defined by $\psi_\sigma(z) = \psi(\sigma z)$, and the pull back metric $\psi_\sigma^* g(z) = \sigma^2 \psi^* g(\sigma z)$ on B. The Riemannian manifold $(B, \psi_\sigma^* g)$ does not quite satisfy the condition of model cases, since the metric ψ_σ^* is not δ - close to Id anymore. However, the metric $g_0 = \sigma^{-2}\psi_\sigma^* g$ (thus $\psi_\sigma^* g = \sigma^2 g_0$) is $W_2^{1,\infty}$ close to Id and Theorem 7.1.3 applies to (B, g_0). The proof therefore boils down to scrutinizing the effect of the conformal scaling of the metrics $g_0 \to \sigma^2 g_0 = \psi_\sigma^* g$ on the validities of the conditions.

We first observe the energy. As we have computed (see Definition 2.3.4) that

$$
\begin{aligned}
\mathcal{E}_{\sigma^2 g_0}(A) &= \int_B \left(\sum_{i,j,k,l} \sigma^{-2} g_0^{ik} \sigma^{-2} g_0^{jl} (F_A)_{ij} (F_A)_{kl} \right)^{\frac{q}{2}} \sqrt{det(\sigma^2 g_0)} dvol_{Id} \\
&= \sigma^{n-2q} \mathcal{E}_{g_0}(A) \\
&\geq \mathcal{E}_{g_0}(A), \qquad (7.1)
\end{aligned}
$$

since $q \geq \frac{n}{2}$ and $\sigma \in (0, 1]$, we see that $\mathcal{E}_{\sigma^2 g_0}(A) \geq \mathcal{E}_{g_0}(A)$. Therefore, if $\mathcal{E}_{\sigma^2 g_0}$ is bounded by the constant ϵ in the model cases, $\mathcal{E}_{g_0}(A)$ is also bounded by ϵ. We are therefore able to find a gauge $u \in \mathcal{G}^{2,p}$ so that $u^* A$ is in Uhlenbeck gauge on (B, g_0). The first two Uhlenbeck gauge conditions (i) and (ii) are conformally invariant and therefore are still satisfied on $(B, \sigma^2 g_0 = \psi_\sigma^* g)$.

We need to make sure that $u^* A$ still satisfy conditions (iii) and (iv) of Uhlenbeck gauge, when the Riemannian metric on B is replaced with $\sigma^2 g_0$. The L^p norm of u^*A is

$$\begin{aligned} \|u^*A\|_{\sigma^2 g_0, L^p} &= \left[\int_B \left(\sum_{i,j} \sigma^{-2} g^{ij} (u^*A)_i (u^*A)_j \right)^{\frac{p}{2}} \sqrt{det(\sigma^2 g)} dvol_{Id} \right]^{\frac{1}{p}} \\ &= \sigma^{\frac{n-p}{p}} \|u^*A\|_{g_0, L^p} \leq \sigma^{\frac{n-2p}{p}} \|u^*A\|_{g_0, L^p}, \end{aligned}$$

since $\sigma \in (0, 1]$. For the covariant derivative, we note that the Christoffel symbols are conformally invariant, and so after a similar computation (with scale factor of σ^{-4} for two forms) , we have

$$\|\nabla^{\sigma^2 g_0} u^*A\|_{\sigma^2 g_0, L^p} = \sigma^{\frac{n-2p}{p}} \|\nabla^{g_0} u^*A\|_{g_0, L^p}.$$

The curvature F_A is a two form and therefore exhibits effect on conformal scale (See Lemma 2.3.5):

$$\|F_A\|_{\sigma^2 g_0, L^p} = \sigma^{\frac{n-2p}{p}} \|F_A\|_{g_0, L^p}.$$

Combining these inequalities, and with the granted result that $\|u^*A\|_{W_1^{1,p,g_0}} \leq C \|F_A\|_{L^p}$ (and for L^q as well), we have the same estimates with the same constants for the scaled metric $\sigma^2 g_0 = \psi_\sigma^* g$:

$$\|u^*A\|_{W_1^{1,p}, , \sigma^2 g_0} \leq \sigma^{\frac{n-2p}{p}} \|u^*A\|_{W_1^{1,p}, g_0} \leq C \sigma^{\frac{n-2p}{p}} \|F_A\|_{g_0, L^p} = C \|F_A\|_{\sigma^2 g_0, L^p}.$$

□

We now give a sketch of the proof of the model case. See Chapter 6 of [W] for details. The proof starts by establishing connectedness on the subset $\mathcal{A}_\epsilon$ of $W_1^{1,p}(B)$ with energy bounded by ϵ. One then proceeds to show that the subset $\mathcal{S}_\epsilon$ of $\mathcal{A}_\epsilon$ on which the conclusions of Theorem 7.1.3 hold true is non-empty, closed, open, and therefore the entire $\mathcal{A}_\epsilon$.

Proof. of Theorem 7.1.3-(Sketch)

Let

$$\mathcal{A}_\epsilon = \{A \in W^{1,p} | \mathcal{E}(A) \le \epsilon\},$$

and

$$\mathcal{S}_\epsilon = \{A \in \mathcal{A}_\epsilon \mid \text{Conclusions of Theorem 7.1.3 are True}\}.$$

We aim to show that $\mathcal{S}_\epsilon = \mathcal{A}_\epsilon$.

We may actually assume that the energy in the definition of $\mathcal{A}_\epsilon$ is defined with respect to the flat metric. Indeed, choose δ sufficiently small, requiring the metric g to be δ close to Id so that

$$\frac{1}{2}\mathcal{E}'(A) \le \mathcal{E}(A) \le 2\mathcal{E}'(A) \qquad \forall A \in W_1^{1,p}(B),$$

where $\mathcal{E}'(A)$ is the L^q energy of A with respect to Euclidean volume form $dvol_{Id}$. If $\mathcal{S}_\epsilon = \mathcal{A}_\epsilon$ is established with respect to the flat metric Id, by replacing ϵ with $\frac{\epsilon}{2}$, the inequality above gives us the same conclusion for $\mathcal{E}(A)$. Indeed, let $\mathcal{S}'_\epsilon$ and $\mathcal{A}'_\epsilon$ be the spaces defined above, but with respect to the flat metric. If we have $\mathcal{S}'_\epsilon = \mathcal{A}'_\epsilon$, and let $A \in \mathcal{A}_{\frac{\epsilon}{2}}$, we have $A \in \mathcal{A}'_\epsilon$, and therefore $A \in \mathcal{S}'_\epsilon$. Consequentially, $A \in \mathcal{S}_{\frac{\epsilon}{2}}$ and we have the desired result for a smaller ϵ, namely, $\mathcal{S}_{\frac{\epsilon}{2}} = \mathcal{A}_{\frac{\epsilon}{2}}$. It therefore suffices to prove the theorem with the L^q energy of A defined with respect to the flat metric Id:

$$\mathcal{E}_q(A) = \int_B |F_A|_{Id}^q dvol_{Id}.$$

- *Connectedness of* $\mathcal{A}_\epsilon$:

We aim to show that every connection $A \in \mathcal{A}_\epsilon$ can be connected by a path in $\mathcal{A}_\epsilon$ to the trivial connection $0 \in \mathcal{A}_\epsilon$, with respect to the $W_1^{1,p}$ topology.

The curve is given by $A_t(x) = tA(tx)$ so that $A_1 = A$ and $A_0 = 0$. The corresponding curve of curvature forms is then $F_{A_t}(x) = t^2 F_A$, making the energy $\mathcal{E}(A_t) = t^{2q-n}\mathcal{E}(A)$ along the curve. Since $q \geq \frac{n}{2}$, the energy stays bounded by ϵ, for all t and therefore the curve stays in $\mathcal{A}_\epsilon$. To show the continuity of the curve in $W_1^{1,p}$ topology, we list the required inequalities here. Since the metric is assumed to be flat, each of them can be obtained by ordinary calculus and some elementary manipulations. See the proof of Theorem 6.3 in [W] for detailed derivations.

The continuity at $t = 0$ follows from

$$\|A_t\|_{L^p}^p \leq Ct^{p-\frac{n}{2}} Vol(B)^{\frac{1}{2}} \|A\|_{1,p}^p,$$

where C is the constant in the Sobolev inequality embedding $L^{2p} \hookrightarrow W^{1,p}$. For the derivative, we have

$$\|\nabla A_t\|_{L^p}^p = t^{2p-n} \|\nabla A\|_{L^p}^p.$$

Clearly, since $p > \frac{n}{2}$, the right hand sides of both inequalities approaches 0 as $t \to 0$.

For continuity at $t_0 > 0$, we take a sequence of smooth connections $\{A^i\}$ converging to A in the $W_1^{1,p}$ norm. The smoothness of each A^i makes the $W_1^{2,p}$ norms of $A^i(x)$ uniformly bounded by a constant C_i on B. With these bounds, and some applications of triangle inequalities, we have

$$\|A_t - A_{t_0}\|_{L^p} \leq |t - t_0| {t_0}^{-\frac{n}{p}} \|A\|_{L^p} + \left(t^{-\frac{n}{p}} + {t_0}^{-\frac{n}{p}}\right) \|A - A^i\|_{L^p} + |t - t_0| Vol(B)^{\frac{1}{p}} C_i,$$

and

$$\|\nabla A_t - \nabla A_{t_0}\|_{L^p} \leq |t^2 - {t_0}^2| {t_0}^{-\frac{n}{p}} \|\nabla A\|_{L^p} + \left(t^{-\frac{n}{p}} + {t_0}^{-\frac{n}{p}}\right) \|\nabla A - \nabla A^i\|_{L^p} + |t - t_0| Vol(B)^{\frac{1}{p}} C_i.$$

One can readily observe that the right hand sides of both inequalities again approach 0 as $t \to t_0$ and $i \to \infty$. We therefore have established the continuity of the curve A_t, and the connectedness of $\mathcal{A}_\epsilon$.

- Closedness of $\mathcal{S}_\epsilon$

Pick a sequence of connections $\{A^i\} \subset \mathcal{S}_\epsilon$ converging to $A \in \mathcal{A}_\epsilon$ in $W_1^{1,p}$ norm. By the definition of $\mathcal{S}_\epsilon$, there exists a subsequence from $\{A^i\}$, still indexed by i, and a corresponding sequence of gauges $\{u^i\}$ so that $u^{i*}A^i$ is in Uhlenbeck gauge, with the same constant C in Theorem 7.1.3. We aim to find a gauge $u \in \mathcal{G}^{2,p}$ so that u^*A is in Uhlenbeck gauge with the same constant C.

A classical L^p curvature bound of connections A^i is given by (see (A.11) in [W]):

$$\|F_{A^i}\|_{L^p} \leq R\left(\left\|A^i\right\|_{W_1^{1,p}} + \left\|A^i\right\|^2_{W_1^{1,p}}\right) \leq R'.$$

The bound R' follows from the convergence of A^i in $W_1^{1,p}$. Let $\tilde{A}^i = u^{i*}A^i$. Since each $\tilde{A}^i$ is in Uhlenbeck gauge, their $W_1^{1,p}$ norms are uniformly bounded by the L^p norms of curvatures of A^i, which are uniformly bounded by R'. Therefore, Banach-Alaoglu theorem applies to yield a weakly convergent subsequence of $\{\tilde{A}^i\}$, with the weak limit $\tilde{A} \in W_1^{1,p}(B)$. Moreover, the compact embedding $W_1^{1,p} \hookrightarrow L^1_{2p}$ further provides a strongly convergent subsequence in L^{2p}, and we still denote the limit by $\tilde{A}$.

Next, we claim that the gauges $\{u_i\}$ converge (passing to subsequence if necessary) to $u \in \mathcal{G}^{2,p}$ in $\mathcal{C}^0$ topology and $u^*A = \tilde{A}$. The first part of the claim follows from Theorem 2.3.12 since there are uniform $W_1^{1,p}$ bounds on $\{A_i\}$ and $\{\tilde{A}_i\}$. Theorem 2.3.12 also provides uniform $W_1^{1,p}$ bound on $u_i{}^{-1}du_i = \tilde{A}_i - u_i{}^{-1}A_iu_i$ and $u_i{}^{-1}du_i \to u^{-1}du$ in L^{2p}, given that $p > \frac{n}{2}$. Since $\tilde{A}_i - u_i{}^{-1}A_iu_i \to \tilde{A} - u^{-1}Au$ in $W_1^{1,p}$, and thus in L^1_{2p} by the compact embedding $W_1^{1,p} \hookrightarrow L^{2p}$, the two convergences and the uniqueness of the limit imply the second statement of the claim.

It remains to verify that $\tilde{A}$ is in Uhlenbeck gauge. Conditions (i) and (ii) in Definition 7.1.1 (the Coulomb gauge) follow from integration by parts against an arbitrary smooth function ϕ on B. The two integrals are

$$\int_B \phi * d^* \tilde{A} = \int_B d\phi \wedge *(\tilde{A} - \tilde{A}_i)$$

and

$$\int_{\partial B} \phi * \tilde{A}|_{\partial B} = \int_B d\phi \wedge *(\tilde{A} - \tilde{A}_i) - \int_B \phi(d^*\tilde{A} - d^*\tilde{A}_i).$$

Both of them converge to 0 as $i \to \infty$ since $\tilde{A}_i \to \tilde{A}$ in $W_1^{1,p}$.

For the elliptic estimates (iii) and (iv), we notice the lower semicontinuity of $W_1^{1,p}$ norm and have

$$\left\|\tilde{A}\right\|_{W_1^{1,p}} \leq \liminf_{i\to\infty} \left\|\tilde{A}_i\right\|_{W_1^{1,p}} \leq \liminf_{i\to\infty} C\|F_{A_i}\|_{L^p}.$$

Since $\|F_A\|_{L^p}^p$ is continuous in $W_1^{1,p}$ (and $W_1^{1,q}$ as well), we have

$$\liminf_{i\to\infty} \|F_{A_i}\|_{L^p} = \|F_A\|_{L^p}$$

and (iii) of Definition 7.1.1 is established. The same procedure can be applied to the exponent q and therefore (iv) is established. We have verified the closedness of $\mathcal{S}_\epsilon$.

- Openness of $\mathcal{S}_\epsilon$

This is the main challenge of the theorem. We will therefore only lay out superficial outlines from [W]. We pick a connection $A_0 \in \mathcal{S}_\epsilon$ with energy smaller than ϵ and wish to find an open neighborhood (in $W_1^{1,p}$ topology) of A_0, on which every connection can be transformed into Uhlenbeck gauge. In fact, since the energy functional is gauge invariant, and $\mathcal{G}^{2,p}$ acts continuously on $W_1^{1,p}$, we may assume A_0 itself to be in Uhlenbeck gauge.

The neighborhood we aim to construct is the orbit of A_0 under actions of gauges given by the exponential map, whose existence is guaranteed by the implicit function theorem. The precise claim is the following:

Claim: There exists $\delta > 0, \epsilon > 0$, and $C > 0$ such that for every metric g on B with $\|g - Id\|_{W_2^{2,\infty}} \le \delta$, there exists a $W_1^{1,p}$- open ball with radius Δ, $B_\Delta(A_0)$, around A_0, on which every connection can be gauge transformed into Uhlenbeck gauge.

We roughly outline the proof of the claim here. One of the significance of δ is to keep the metric g close enough to Id so that $W_1^{1,p}$ norms on one forms with respect to those g's are equivalent to the Sobolev norm with respect to Id. δ might require further shrinking to ensure the estimates (iii) and (iv) in definition of Uhlenbeck gauge. Viewing connections as one forms with value in the Lie algebra $\mathfrak{g}$ with an Ad-invariant inner product, the implicit function theorem here asserts that there is an open ball B_λ in $\Omega^0(\mathfrak{g})$ with respect to $W^{2,p}(\mathfrak{g})$ norm of radius λ around $0 \in \mathfrak{g}$, so that for every $A \in B_\Delta(A_0)$, there is a unique solution $V \in B_\lambda$ satisfying

$$\begin{cases} d^*(exp(V) * A) = 0 & \text{on M} \\ *(exp(V) * A)|_{\partial B} = 0 & \text{on } \partial M. \end{cases}$$

The solutions are zeros to the mapping

$$D : W_1^{1,p}(B) \times \overline{W^{2,p}(\mathfrak{g})} \to \mathcal{Z})$$

defined by

$$D(A, V) = (d^*(exp(V) * A), *(exp(V) * A)|_{\partial B}) .$$

Here, $\overline{W^{2,p}(\mathfrak{g})}$ is the subspace of $W^{2,p}(\mathfrak{g})$ with zero mean value. The target $\mathcal{Z}$ is defined by

$$\mathcal{Z} = \{(f,\phi) \in L^p(\mathfrak{g}) \times W^{1,p}(\mathfrak{g})|_{\partial B} \mid \int_B f + \int_{\partial B} \phi = 0\}.$$

The proof of Theorem 6.3 in [W] clearly explains that the map D is well defined, and $\mathcal{Z}$ is a Banach space. To apply implicit function theorem, we need to ensure that at $(A_0, 0)$, the derivative of D with respect to the second variable is bijective. Direct computations along with the fact that A_0 is in Uhlenbeck gauge show that this linear operator at $(A_0, 0)$

$$\partial_2 D_{(A_0,0)} : \overline{W^{2,p}(\mathfrak{g})} \to \mathcal{Z}$$

is given by

$$\partial_2 D_{(A_0,0)}(\psi) = (\Delta\psi + *[d\psi \wedge *A_0], \frac{\partial \psi}{\partial \nu}),$$

where Δ is the Euclidean Laplacian and $\frac{\partial \psi}{\partial \nu}$ is the derivative of ψ along the normal direction to ∂B. To show the bijectivity, we decompose $\partial_2 D_{(A_0,0)}$ into a perturbation $T + S$, where

$$T(\psi) = \left(\Delta\psi, \frac{\partial \psi}{\partial \nu}\right), \qquad \text{and} \qquad S(\psi) = \left(*[d\psi \wedge *A_0], 0\right).$$

The surjectivity of T follows from Theorem 2.3.8 since the defining condition for the space $\mathcal{Z}$ is exactly (2.6), the sufficiency condition for solving the Neumann problem. Since the $\mathfrak{g}$-valued functions are assumed to be mean zero and solutions for Neumann problems are unique up to an additive constant, it follows that T is bijective. Shrinking δ if necessary, Theorem 2.3.9 ensures the upper bound for T^{-1}. If S can be controlled well enough so that

$$\|T^{-1}\|\|S\| < 1.$$

Lemma 2.2.6 them implies that $T + S$ is bijective. Direct computations show that $\|S\|$ is controlled by $\|A_0\|_{W_1^{1,p}}$, which is controlled by $\|F_{A_0}\|_{L^p} \leq \epsilon$ since it is in Uhlenbeck gauge. Shrinking ϵ if necessary, the bijectivity of $\partial_2 D_{(A_0,0)} = T + S$ is consequentially achieved.

We have therefore constructed a neighborhood of A_0, given by gauge orbits of exponential gauges, on which conditions (i) and (ii) of Uhlenbeck gauge are satisfied.

Next, we need to show that, shrinking the neighborhood $B_\lambda \subset \mathfrak{g}$ if necessary, the solutions obtained in the implicit function theorem above satisfy conditions (iii) and (iv) of Uhlenbeck gauge. To achieve these, one starts with constructing the following statement:

There exists constant δ and $\kappa > 0$ such that the followings are true. Fix a metric g δ-close to Id in $W_2^{1,\infty}$. For a connection A, with $\|A\|_{L^r} \leq \kappa$, satisfying conditions (i) and (ii) of Uhlenbeck gauge, A also satisfies conditions (iii) and (iv), the estimates of $W_1^{1,p}$ and $W_1^{1,q}$ norms by L^p and L^q norms of curvatures, respectively. Here, $r = \frac{nq}{n-q}$.

This statement follows from standard applications of Sobolev estimates of products, along with the L^r bound for A. Therefore, to complete the proof of openness, it remains to bound the connections obtained from implicit function theorem, $exp(V)^*A$, in L^r. To do so, we observe that

$$\|exp(V)^*A\|_{L^r} \leq \left\|Ad_{exp(V)}A\right\|_{L^r} + \|exp(-V)dexp(V)\|_{L^r}.$$

Since the inner product on $\mathfrak{g}$ is Ad-invariant, the first term is bounded by κ. The second term is bounded by the $W^{2,p}$ norm of V, which can be made arbitrarily small by shrinking λ. To this end, we have established the openness of $\mathcal{S}_\epsilon$.

Finally, it is clear that the trivial connection $0 \in \mathcal{S}_\epsilon$, and therefore $\mathcal{S}_\epsilon$ is a non-empty, open, and closed subset of the connected set $\mathcal{A}_\epsilon$. We conclude that $\mathcal{S}_\epsilon = \mathcal{A}_\epsilon$.

□

We have now learned that any connection with controlled energy can be gauge transformed into Uhlenbeck gauge, ensuring elliptic regularities for further analysis. Our next goal is to patch together the local results established above into a global one. The gauge transformations obtained locally does not necessarily form a global gauge transformation:

they might not be compatible with the transition functions of E, and therefore do not constitute a section of the bundle $Aut(E)$. However, one can show that under suitable conditions, we can adjust each gauge transformation so they can fit into a global one. If we apply this to a sequence of connections with uniform curvature bound in L^p, Theorem 7.1.2 provides a uniform $W_1^{1,p}$ bound for the sequence of connections, and the Banach-Alaoglu Theorem gives us a weak limit in $W_1^{1,p}$. We therefore have the weak compactness of the space of connections in $W_1^{1,p}$ with uniform curvature bounds.

Assume for now that M is compact.

Theorem 7.1.4. *(Weak Compactness Theorem [W])*

Given a sequence of connections $\{A^\nu\}$ in $W_1^{1,p}(E)$ so that $\|F_{A^\nu}\|_{L^p}$ are uniformly bounded, there exists a subsequence, still denoted by $\{A^\nu\}$, and a corresponding sequence of gauges $\{u^\nu\} \subset \mathcal{G}$ so that the sequence $\{u^{\nu *} A^\nu\}$ is weakly convergent in $W_1^{1,p}(E)$.

Proof. *(Sketch)*

The theorem is locally trivial. Let $C, \epsilon \in \mathbb{R}$ be the constants in Theorem 7.1.2, and U an open neighborhood in the trivializing open cover of M. Shrinking U if necessary, having the uniform L^p bound of the curvatures along with the fact that $q \leq p$, we obtain the required energy bounds to apply Theorem 7.1.2 on U. Since M is compact, we cover it with finitely many such open patches $M = \cup_{\alpha=1}^N U_\alpha$. For each $A^\nu|_{U_\alpha}$, the conditions for Theorem 7.1.2 are satisfied, and therefore we obtain a gauge $u_\alpha^\nu \in \mathcal{G}^{2,p}(U_\alpha)$ to turn $A_{U_\alpha}^\nu$ into a connection form in Uhlenbeck gauge. On each U_α, $\{u_\alpha^{\nu *} A^\nu|_{U_\alpha}\}$ are uniformly bounded in $W_1^{1,p}$ by the uniform L^p bound for their curvatures, and thus possesses a weakly convergent subsequence by the Banach-Alaoglu Theorem.

Major difficulties arise when we attempt to patch the gauge transformations into a global one. Recall that a gauge group, by definition, consists of global sections of the bundle $Aut(E)$. To be qualified for a global section, it has to be independent of local trivialization on the overlaps. That is to say, on $U_\alpha \cap U_\beta$, they must satisfy the equality

$$\phi_{\alpha\beta} u^\nu_\beta = u^\nu_\alpha \phi_{\alpha\beta},$$

or

$$\phi_{\alpha\beta} = (u^\nu_\alpha)^{-1} \phi_{\alpha\beta} u^\nu_\beta, \tag{7.2}$$

for all ν, where $\phi_{\alpha\beta}$ is the transition function of the vector bundle E on $U_\alpha \cap U_\beta$.

Let $u^\nu_{\alpha\beta} = (u^\nu_\alpha)^{-1} \phi_{\alpha\beta} u^\nu_\beta$, which satisfies the cocycle conditions for each α, β, and ν. They therefore form a collection of transition functions, supposedly for the vector bundle E. Such a requirement forces the two collections of transition functions, $\{\phi_{\alpha\beta}\}$ and $\{u^\nu_{\alpha\beta}\}$ to be in the same conjugacy class. In another words, it is required to replace u^ν_α by some $\tilde{u^\nu_\alpha}$ so that (7.2) holds true:

$$\phi_{\alpha\beta} = (\tilde{u^\nu_\alpha})^{-1} \phi_{\alpha\beta} (\tilde{u^\nu_\alpha}).$$

Theorem 7.1 and Lemma 7.2 in [W] provides us with such alternative gauges $\tilde{u^\nu_\alpha}$. Moreover, after picking a subsequence and shrinking each U_α to a smaller neighborhood V_α if necessary, it turns out that $\{\tilde{u}^{\nu *}_\alpha A^\nu_\alpha\}$ are uniformly bounded in $W^{1,p}_1(V_\alpha)$ for all α. Since there are finitely many α, they are uniformly bounded on the entire M. Banach-Alaoglu Theorem applies to $\{\tilde{u}^{\nu *}_1 A^\nu_1\}$, and there exists a weakly convergent subsequence $\{\tilde{u}^{\nu' *}_1 A^{\nu'}_1\}$ in $W^{1,p}_1(V_1)$. We may then follow the same procedures to take a subsequence of $\{\tilde{u}^{\nu' *}_1 A^{\nu'}_1\}$ weakly convergent on $W^{1,p}_1(V_2)$, and so on. Since there are finitely many α's, we have obtained the desired subsequence, weakly convergent on every $W^{1,p}_1(V_\alpha)$, and the proof is completed.

□

In fact, the compactness of the base manifold M can be somewhat relaxed. We recall the following topological definition:

Definition 7.1.5. *(Deformation Retract)*

A subset X of a smooth manifold M is called a deformation retract of M if there exists a continuous map $\Phi : [0,1] \times M \to M$ such that

$$\Phi(0,\cdot) = Id_M, \qquad \Phi(1,M) \subset X, \qquad \text{and } \Phi(t,\cdot)|_X = Id_X, \qquad \forall t \in [0,1].$$

The weak compactness theorem can be strengthened to a manifold M that is not necessarily compact but can be exhausted by increasing compact deformation retracts of M. Namely, the theorem stays true for $M = \cup_{k=1}^{\infty} M_k$, where $M_k \subset M_{k+1}$ and each M_k is a compact deformation retract of M.

Theorem 7.1.6. *(Strengthened Weak Compactness Theorem [W])*

Given M as described immediately above and a sequence of connections $\{A^\nu\}$ in $W^{1,p}_{1,loc}(E)$ such that for each k, there exists $B_k > 0$ and $\|F_{A^\nu}\|_{L^p(M_k)} \leq B_k, \forall \nu$.

Then, there exists a subsequence of $\{A^\nu\}$, and corresponding gauge transformations $\{u^\nu\} \subset \mathcal{G}^{2,p}_{loc}(E)$ so that $u^{\nu *}A^\nu|_{M_k}$ weakly converges on M_k for every k. (Note the independence of A^ν and u^ν of k).

The proof of this generalization is an easy consequence of the following proposition. It establishes a subsequence of $\{A^\nu\}$ that is uniformly, over both ν and k, bounded on all M_k. We will not present the proof here.

Proposition 7.1.7. *[W]*

Given the same M described above, and a sequence of connections $\{A^\nu\}$ in $W^{1,p}_{1,loc}(E)$, suppose that on each M_k, every subsequence $\{A^{\nu_i}\} \subset \{A^\nu\}$ possesses a further subsequence $\{A^{\nu_{k,i}}\}$ and corresponding gauges $\{u^{k,i}\}$ such that

$$\sup_{i\in\mathbb{N}} \left\| u^{k,i*} A^{\nu_{k,i}} \right\|_{W_1^{1,p}(M_k)} < \infty.$$

Then, there exists a subsequence of $\{A^{\nu_i}\} \subset \{A^\nu\}$*, and gauges* $\{u^i\}$*, so that* $\{u^{i*} A^{\nu_i}\}$ *is uniformly bounded on all deformation retracts* M_k*:*

$$\sup_{i\in\mathbb{N}} \left\| u^i * A^{\nu_i} \right\|_{W_1^{1,p}(M_k)} < \infty \qquad \forall k \in \mathbb{N}.$$

This is Proposition 7.6 in [W]. There, the result is stated for Sobolev $W_1^{1,p}$ norm for a general l. But $l = 1$ suffices to prove Theorem 7.1.6.

Proof. of Theorem 7.1.6 The theorem follows easily from Proposition 7.1.7 and Theorem 7.1.4. Pick a sequence $\{A^\nu\} \subset W^{1,p}_{1,loc}$ whose curvatures are uniformly (over ν) bounded in $L^p(M_k)$, for each k. That is, for each k, there exists $c_k > 0$ so that

$$\|F_{A^\nu}\|_{L^p(M_k)} \le C_k,$$

for all ν. By Theorem 7.1.4, on each M_k, there exists a subsequence $\{A^{\nu_{i,k}}\}_i \subset \{A^\nu_{M_k}\}$ and corresponding gauges $\{u^{i,k}\}_i$ so that the subsequence $u^{i,k*}A^{\nu_{i,k}}$ is weakly convergent in $W_1^{1,p}(M_k)$. By standard functional analysis fact, this subsequence is uniformly bounded in $W_1^{1,p}(M_k)$ and the condition in Proposition 7.1.7 is satisfied. We can therefore find a universal subsequence $\{A^{\nu_i}\} \subset \{A^\nu\}$ and corresponding gauges $\{u^i\}$ so that

$$\sup_{i\in\mathbb{N}} \left\| u^{i*} A^{\nu_i} \right\|_{W_1^{1,p}(M_k)} < \infty,$$

for all k. This subsequence is uniformly $W_1^{1,p}$-bounded on each M_k. It is then possible to pick a diagonal subsequence that is weakly $W_1^{1,p}$ convergent on all M_k. Precisely, let

$$\tilde{A}_0^i := u^{i*} A^{\nu_i}.$$

It is uniformly $W_1^{1,p}$-bounded on M_1, and therefore by Banach-Alaoglu it contains a weakly convergent subsequence $\{\tilde{A}_1^i\}_i$ on M_1. Inductively, $\{\tilde{A}_{k-1}^i\}_i$, weakly convergent on M_{k-1}, is uniformly, over i, $W_1^{1,p}$-bounded on M_j. It therefore contains a weakly convergent subsequence $\{\tilde{A}_k^i\}_i$ on M_k. The diagonal sequence $\{B_k\}$ is then obtained by

$$B_k = \tilde{A}_k^k,$$

which is weakly convergent on all M_k.

□

7.2 Extensions of Vortices with Bounded Energies

With all the classical results summarized, we now apply them to extend the asymptotic behaviors across the singularities. First, let us precisely acknowledge and understand the existence and nature of the possible singularities. We have restricted our discussion to the open subset $\nu_{k,0}(s)$ of $\nu_k(s)$ where sections do not vanish simultaneously. This leads to the non-vanishing of h, allowing us to take the logarithm to produce smooth functions u_∞. When h has zeros, the estimates in the Main Theorem 5.0.5 are no longer valid. One recalls, from the proof of 5.0.5, that when constructing super solution, we need to choose constants a and b so that for a positive constant c_2, we have $he^{av+b} + c_2 < 0$, where v is a solution to $\Delta v = \bar{h} - h$. These preparations allow the function $\varphi_+ = av + b$ to satisfy the condition of super solution:

$$\Delta\varphi_+ - c(s) + h_s e^{\varphi_+} = (a\bar{h} - c_1) + s^2(he^{av+b} + c_2) - ah \le 0$$

For h with zeros, this inequality can not be achieved at the zeros of h, where $he^{av+b} + c_2 > 0$. To bypass, one can perhaps pick functions v_s satisfying

$$\Delta v_s = s^2(\bar{h} - h)$$

and constants a,b such that $s^2 a\bar{h} < c_1 - s^2 c_2$ and $e^{av_s+b} - a > 0$. The functions $\varphi_{+,s} = av_s + b$ satisfy the defining property of super-solutions, but are nevertheless s-dependent. In fact, their L^∞ norms grow like s^2, and the corresponding functions u_s in the conclusion of the Main Theorem are not uniformly bounded anymore. Without its most vital condition, the convergence statement in the Main Theorem consequentially does not hold when h has zeros. Therefore, techniques from [K-W] are not sufficient to account all vortices.

We will then approach this problem from the original description of moduli space in Lemma 3.1.3, namely, $\nu_k(k,\tau)$. That is, we will vary connections and sections for a fixed Hermitian structure instead of looking for special metrics. The presence of these singularities motivates us to supply a more global but less explicit description of the adiabatic limits of vortices. Modeled on arguments and estimates in [L], [U], and [Sc], we aim to construct an extended limiting vortex that can be defined on the entire Riemann surface. Expectingly, the extended vortex will behave identically away from common zeros of ϕ_s's. However, we observe that the line bundle defined by the extended vortex is of lower degree than the original one. Analytically, we observe that the Yang-Mills-Higgs energy functional concentrates more and more around singularities, and curvatures finally evolve into point mass of zeros. Upon extending the vortices, these point masses were eliminated, and therefore the line bundle defined by the extended vortices possesses lower degree than the original one. In other words, we observe "bubbling" phenomena near the common zeros of ϕ_i's. We now describe these constructions explicitly, without any assumption on the existence or the location of the zeros of ϕ_s. We will write the connection D_s, its connection form A_s, or the connection defined by the 1 form A_s, D_{A_s}, interchangeably when no confusion arises. The central analytic tools used in this section will be Uhlenbeck's compactness results ([U],[W]) developed in the previous subsection and Hodge theory.

Let us first rewrite the Yang-Mills-Higgs energy functional in terms of its energy density. Recall that we may without loss of generality assume that $\tau = 1$.

$$YMH_s = \int_\Sigma e_s(D, \phi) vol_\Sigma$$

where the energy density e_s is

$$e_s(D, A) = \frac{1}{s^2}|F_D|^2 + \sum_{i=1}^k |D\phi_i|^2 + \frac{s^2}{4}\left[\left(\sum_{i=1}^k |\phi_i|_H^2 - 1\right)\right]^2, \tag{7.3}$$

and $|\ |_H$ are pointwise norms induced by the Hermitian metric and the background Kähler metric. Our first claim is that away from a singular set consisting of finite number of points, the Main Theorem still holds true. First, we need the following estimates on the energy density e_s. Unless stated otherwise, for $s \in [1, \infty]$, the pair (D_s, ϕ_s), or (A_s, ϕ_s), denotes a solution to the $s-$vortex equations (3.3) or (3.4), and $e_s = e(D_s, \phi_s)$. $B_r(z)$ denotes a geodesic ball of radius r centered around z, and when $z = 0$, we denote $B_r = B_r(0)$. The following lemma is quoted form [O]:

Lemma 7.2.1. *If $|\phi_s|_H^2 \geq \frac{1}{2}$ uniformly on a domain Ω, then there exists $C_1, C_2 > 0$ such that*

$$\Delta e_s \geq -C_1 e_s^2 - C_2|R|e_s$$

on Ω. Here, R is the scalar curvature of Σ.

Since curvature R and energy density e_s are uniformly bounded, adjusting the constant if necessary, the conclusion of this lemma may be strengthened to:

$$\Delta e_s \geq -Ce_s^2,$$

for some $C > 0$. Conditions for Lemma 2.3.6 are therefore satisfied. We have, on every ball B_R with radius R centered at z_0, the following inequality is true:

$$e_s(z_0) \leq C \int_{\mathbb{S}_R(z_0)} e_s \tag{7.4}$$

where $\mathbb{S}_R(z_0)$ is the sphere of radius R centered at z_0 and $C > 0$ is a constant inversely proportional to the volume of B. The estimate on e_s above leads to a local uniform bound of e_s by the total energy.

Theorem 7.2.2. *Assume that* $|\phi_s| \geq \frac{1}{2}$ *on* $B_R(z_0)$ *for all* s*. There exists a constant* $\epsilon_0 > 0$ *and* $C_0 > 0$ *such that if*

$$E_{R,s} = \int_{B_R(z_0)} e_s \leq \epsilon \leq \epsilon_0$$

then

$$sup_{B_{\frac{R}{2}(z_0)}} e_s \leq \frac{C_0}{R^2} E_{R,s}$$

Proof. We adapt the proof of Theorem 2.2 in [Sc]. Fix an s, and pick $R_1 = \frac{3}{4}R$, and $\sigma_0 \in (0, R_1)$ such that

$$(R_1 - \sigma_0)^2 \sup_{B_{\sigma_0}} e_s = \max_{\sigma \in (0,R_1]} (R_1 - \sigma)^2 \sup_{B_\sigma} e_s. \tag{7.5}$$

Such a σ_0 exists by continuity of the function

$$f(\sigma) = (R_1 - \sigma)^2 \sup_{B_\sigma} e_s$$

and compactness of the interval $[0, R_1]$. Clearly, there exists $z_0 \in \overline{B_{\sigma_0}}$ such that

$$e_0 := e_s(z_0) = \sup_{z \in B_{\sigma_0}} e_s.$$

Let $\rho_0 = \frac{1}{2}(R_1 - \sigma_0)$, one can readily show, from the constructions of the various radii above, that

$$\sup_{B_{\rho_0}(z_0)} e \leq 4e_0.$$

We also rescale the coordinates linearly around z_0 by a factor of $\lambda = \frac{1}{\sqrt{e_0}}$. More precisely, we have

$$\phi_\lambda(z) = \phi(z_0 + \lambda z),$$

and

$$A_\lambda(z) = \lambda A(z_0 + \lambda z).$$

With these, one can readily verify that

$$\sup_{B_{\frac{\rho_0}{\lambda}}} \tilde{e}_s \leq 4,$$

and

$$\tilde{e}_s(0) = 1,$$

where $\tilde{e}_s = e_s(A_\lambda, \phi_\lambda)$.

We claim that $\frac{\rho_0}{\lambda} = \rho_0\sqrt{e_0} < 1$. Suppose the contrary, that $\rho_0\sqrt{e_0} \geq 1$. The mean value estimate (7.4) applies to $e_s(A_\lambda, \phi_\lambda)$ on the ball B_1:

$$1 = \tilde{e}_s(0) \leq C \int_{B_1} \tilde{e}_s.$$

On the other hand, the rescaling of coordinates yields

$$\int_{B_1} \tilde{e}_s = \int_{B_{\frac{1}{\sqrt{e_0}}}} e_s \leq \int_{B_{\rho_0}} e_s \leq \int_{B_R} e_s \leq \epsilon.$$

The second "$\leq$" follows from the assumption on ρ_0 and e_0, which implies that $\frac{1}{\sqrt{e_0}} \leq \rho_0$. By choosing ϵ strictly smaller than $\frac{1}{C}$, the two inequalities above yield a contradiction, and we have proved the claim.

We now apply the mean value estimate (7.4) on $B_{\rho_0\sqrt{e_0}}$:

$$\begin{aligned} 1 = \tilde{e}_s(0) &\leq \frac{C}{(\rho_0\sqrt{e_0})^2} \int_{B_{\rho_0\sqrt{e_0}}} \tilde{e}_s \\ &= \frac{C}{(\rho_0\sqrt{e_0})^2} \int_{B_{\rho_0}(z_0)} e_s \leq \frac{C}{(\rho_0\sqrt{e_0})^2} E_R \end{aligned} \tag{7.6}$$

Recalling that $\rho_0 = \frac{1}{2}(R_1 - \sigma_0)$, equation (7.6) above implies that

$$\frac{1}{4}(R_1 - \sigma_0)^2 e_0 \leq C E_R.$$

Combining with equation (7.5) above, we have,

$$\max_{\sigma \in (0, R_1]} (R_1 - \sigma)^2 \sup_{B_\sigma} e_s \leq 4 C E_R.$$

By taking $\sigma = \frac{1}{2}R$ and $\epsilon_0 = \frac{1}{C}$, where C is the constant in the mean value inequality described above, we have proved the theorem.

□

Given a positive number ϵ, we call a ball "good" if the energies e_s over that ball are uniformly bounded by ϵ. We can now conclude the convergent behavior of vortices using Uhlenbeck's compactness result in [U]. One note that the constant $\frac{1}{2}$ in Theorem 7.2.2 is arbitrary and can be replaced by any positive constant. Following by a translation if necessary, we assume that z_0 is the origin in the theorem 7.2.2.

Proposition 7.2.3. *On a ball B_R where $e_s \leq \epsilon$, for all s, the corresponding family of connections A_s such that vortices (A_s, ϕ_s) solve the vortex equations* (3.3) *contains a convergent subsequence in $\mathcal{A}(H)$, in $W_1^{1,p}$, for all p in the correct Sobolev range.*

Proof. For each s, we linearly scale the coordinate by a factor of s, blowing up a good ball B_R into B_{sR}. The metric will be correspondingly scaled by $\frac{1}{s^2}$. Precisely, for each s, we form an isometry:

$$q_s : (B_{sR}, \frac{1}{s^2}\omega) \to (B_R, \omega).$$

Pulling back the connection and section by q_s, we have the pullback sections, connections, and curvatures:

$$\hat{\phi}_s(z) = \phi(\frac{z}{s}),$$

$$D_{\hat{A}_s}(z) = \frac{1}{s} D_s(\frac{z}{s}),$$

$$F_{\hat{A}_s}(z) = \frac{1}{s^2} F_s(\frac{z}{s}).$$

When integrating their square norms, the s-dependent coefficients of the terms above are absorbed into the volume forms of the new coordinate and we have:

$$\begin{aligned} & \int_{B_R} \frac{1}{s^2}|F_{A_s}|^2 + |D_{A_s}\phi_s|^2 + \frac{s^2}{4}\left(|\phi_s|^2 - 1\right)^2 vol_\Sigma \\ = & \int_{B_{sR}} |F_{\hat{A}_s}|^2 + |D_{\hat{A}_s}\phi_s|^2 + \left(|\hat{\phi}_s|^2 - 1\right)^2 vol_{s\Sigma} \end{aligned} \tag{7.7}$$

The adjusted vortex equations of $(D_{\hat{A}_s}, \hat{\phi}_s)$ are now without the parameter s:

$$\begin{cases} F^{(0,2)}_{\hat{A}_s} = 0 \\ D^{(0,1)}_{\hat{A}_s} = 0 \\ \sqrt{-1}\Lambda F_{\hat{A}_s} + \frac{1}{2}\left(|\hat{\phi}_s|^2_H - 1\right) = 0 \end{cases} \tag{7.8}$$

Apply Theorem 7.2.2 to $e(\hat{D}_s, \hat{\phi}_s)$, we have

$$\sup_{B_R} e_s = \sup_{B_{sR}} e(\hat{D}_s, \hat{\phi}_s) \leq \frac{C_0}{s^2} \int_{B_{sR}} e(\hat{D}_s, \hat{\phi}_s) \leq \frac{C_1}{s^2}.$$

Since the total energy is uniformly bounded above by the topological constraint, it then implies that

$$\frac{1}{s^2} \sup_{B_R} |F_{A_s}|^2 \leq \frac{C_1}{s^2},$$

which gives us the uniform L^∞ bound of F_{A_s} on B_R. In particular, curvature two forms are bounded in L^p, for all p. We may then apply the Weak Compactness Theorem 7.1.4 to obtain, from $\{A_s\}_s$, a subsequence, as well as a sequence of gauge transformations, so that after applying the gauge transformations and passing to the subsequence, the connections A_s converge weakly in $W^{1,p}_1$ to $A \in W^{1,p}_1$, for all p. Moreover, the connection one forms A_s are all in Uhlenbeck's gauge. That is, for all s, we have

$$\begin{cases} d^* A_s = 0 \\ *A_s|_{\partial B_R} = 0 \\ \|A_s\|_{W^{1,p}_1} \leq C \|F_{A_s}\|_{L^p} \end{cases} \tag{7.9}$$

In fact, since $W^{1,p}_1$ is compactly embedded in L^{2p}_1, for all p, and therefore the convergence $A_s \to A$ is a strong one in L^{2p}_1.

□

In fact, standard analysis of elliptic regularities guarantees the convergence above to be smooth.

Proposition 7.2.4. *On a ball B_R where $e_s \leq \epsilon$, for all s, the corresponding family of connections A_s can be gauge transformed to contain a subsequence convergent smoothly to a smooth connection.*

Proof. Take $p = 2$. We continue from the convergent sequence $\{A_s\}_s \subset L_1^4(B_R)$ constructed in the previous proposition. The proof is a standard elliptic regularity analysis (see, for example [DK]). It is in fact simpler since there is no quadratic term of connection involved in the curvature. The smoothness will follow from Sobolev estimates if we can show that for all k, there exists $C_k > 0$ such that

$$\|A_s\|_{W_1^{k,2}(B_R)} \leq C_k, \tag{7.10}$$

uniformly for all s. This shows that A_∞ is smooth and the k^{th} derivatives of A_s converge to that of A_∞ uniformly. Since $d^* A_s = 0$ and $*A_s|_{\partial B} = 0$, elliptic regularity implies that for all k, there is $B_k > 0$ such that

$$\|A_s\|_{W_1^{k+1,2}} \leq B_k \|dA_s\|_{W_1^{k,2}} = B_k \|F_{A_s}\|_{W_2^{k,2}},$$

uniformly for all s. It therefore suffices to bound the $W_2^{k,2}$ norms of the curvatures for each k. We proceed with induction on k. The case $k = 0$ is given. We assume $\|F_{A_s}\|_{W_2^{l,2}} < \infty$ for all positive integer $l < k$ and aim to bound $\nabla^k F_{A_s}$. In particular, we have

$$\|A_s\|_{W_1^{k-1,2}} \leq B_k \|F_{A_s}\|_{W_2^{k,2}}.$$

The $W_2^{k,2}$ bounds for F_{A_s} on B_R are the same as the bounds for the rescaled forms on B_{sR}. To bound $F_{\hat{A}_s}$, we recall the vortex equation for rescaled curvatures:

$$\sqrt{-1}\Lambda F_{\hat{A}_s} + \frac{1}{2}\left(|\hat{\phi}_s|_H^2 - 1\right) = 0.$$

The inductive hypothesis furthermore bounds the $W_1^{k,2}$ norm of $\hat{A}_s$. Apply ∇^k on each side of the equation above:

$$\sqrt{-1}\nabla^k(\Lambda F_{\hat{A}_s}) = -\frac{1}{2}\nabla^k(|\hat{\phi}_s|_H^2). \tag{7.11}$$

For $(1,1)$ forms $F_{\hat{A}_s}$, $\Lambda F_{\hat{A}_s} =< F_{\hat{A}_s}, \omega_s >_{\omega_s}$, where $\omega_s = \frac{1}{s^2}\omega$ is the rescaled volume form whose L^2 norm is controlled to any order. The proof therefore boils down to bounding the L^2 norm of $\nabla^k(|\hat{\phi}_s|_H^2)$. We note that from the H-compatibility of $D_{\hat{A}_s}$

$$\begin{aligned}
\nabla^k(|\hat{\phi}_s|_H^2) &= \nabla^{k-1}\nabla(|\hat{\phi}_s|_H^2) \\
&= \nabla^{k-1}\left[\left\langle \hat{\phi}_s, D_{\hat{A}_s}\hat{\phi}_s\right\rangle_H + \left\langle D_{\hat{A}_s}\hat{\phi}_s, \hat{\phi}_s\right\rangle_H\right] \\
&= \nabla^{k-1}\left[\left\langle d\hat{\phi}_s, \hat{\phi}_s\right\rangle_H + \left\langle \hat{\phi}_s, d\hat{\phi}_s\right\rangle_H \left\langle \hat{\phi}_s, \hat{A}_s\hat{\phi}_s\right\rangle_H + \left\langle \hat{A}_s\hat{\phi}_s, \hat{\phi}_s\right\rangle_H\right]
\end{aligned} \tag{7.12}$$

We recall the holomorphicity condition

$$D_{\hat{A}_s}\hat{\phi}_s = d\hat{\phi}_s + \hat{A}_s\hat{\phi}_s = 0.$$

This turns the right hand side of the third equation of (7.12) into

$$2\nabla^{k-1}\left\langle \hat{\phi}_s, \hat{A}_s\hat{\phi}_s\right\rangle_H + 2\nabla^{k-1}\left\langle \hat{A}_s\hat{\phi}_s, \hat{\phi}_s\right\rangle_H$$

These two terms are controlled by the $W_1^{k-1,2}$ norms of $\hat{A}_s$ and $W^{k-1,2}$ norms of $\hat{\phi}_s$. All of these are uniformly controlled from the inductive hypothesis above. We have therefore completed the inductive step and the proof.

□

Having constructed the smooth limiting connection on each good ball, we need to precisely piece them together to form a global connection away from singularities. To proceed, we first recognize the singular set where convergence of connections and curvatures is not available.

The "bad" balls are, naturally, those balls with energies greater than ϵ, regardless of how small their radii are. Clearly, YMH_s is unbounded in s, on any ball around common zeros of ϕ_s where $|\phi_s|_H^2 = 0$. That is, any ball containing common zero is bad. Fortunately, there can only be finitely many bad balls. Following similar reasoning in the Lemma 3.2 in [L], we can conclude that it is possible to cover $\Sigma - S$, where S is a finite set, with good balls. In another word, there can be only finitely many "bad" balls, the ones on which the energy is greater than ϵ.

Lemma 7.2.5. *Let Σ be a compact Riemann surface. Let (D_s, A_s) be a family of solutions to equation* (3.3)*, such that*

$$E_s = \int_\Sigma e_s$$

are uniformly bounded. Then, for all $\epsilon > 0$, there exists a subsequence of vortices, still denoted by (D_s, ϕ_s), a countable family of balls $\{B^\alpha\}_\alpha$, and a finite set of points $\{z_1, \ldots, z_N\}$ such that

$$\Sigma' = \Sigma - \{z_1, \ldots, z_N\} \subseteq \bigcup_\alpha \left(\frac{1}{2}B^\alpha\right)$$

and, for each α,

$$\limsup_s \int_{B^\alpha} e_s < \epsilon$$

Proof. The proof is an elementary covering argument. See Lemma 3.2 of [L].

□

We come back to the gluing of connections. For each α, the collection of "good" balls $\{B^\alpha\}$ gives rise to a collection of connections $\{D_s^\alpha\}$. The topological structure of the line bundle L provides transition functions on each overlap:

$$\gamma_{\alpha\beta}^s : B_\alpha \bigcap B_\beta \to U(1).$$

Each transition function is required to satisfy the following compatibility equation::

$$-d\gamma_{\alpha\beta}^s = A_\alpha^s \gamma_{\alpha\beta}^s - \gamma_{\alpha\beta}^s A_\beta^s, \tag{7.13}$$

where A_α^s is the connection form of D_s^α over B^α. Additionally, the cocycle conditions on $B^\alpha \bigcap B^\beta \bigcap B^\delta$ must be satisfied:

$$\gamma_{\alpha\beta}^s \gamma_{\beta\delta}^s \gamma_{\delta\alpha}^s = 1.$$

Since the connection forms converge uniformly and the transition functions take values in $U(1)$, equation (7.13) says that they are uniformly Lipschitz. Passing to a subsequence if necessary, we obtain a limiting connection form $\{A_\alpha\}$ and transition functions $\{\gamma_\alpha\}$ satisfying the compatibility equation. In other words, the connection forms $\{A_\alpha\}$ constitute a connection D on Σ'.

Having established the convergent behaviors away from the singular set, our next goal is to exhibit the concentration of energy near $z_1, \ldots, z_N$ as $s \to \infty$. These points prevent the smooth convergence of curvatures on the entire Riemann surface, and we wish to construct a natural smooth extension of the line bundle and vortices across these singularities.

For s large enough, so that $0 < \frac{1}{s} < \epsilon$, we define

$$\Sigma_{s,\rho} = \{z_i^s \in \Sigma : \int_{B_\rho(z_i^s)} e_s \geq \epsilon\}_{i=1}^{N_s}$$

From Lemma 7.2.5 above, we know that N_s's are uniformly bounded. By the compactness of Σ, passing to a subsequence if necessary, the singularities converge to points forming a singular set $S = \{z_1^\infty, \ldots, z_N^\infty\}$. Since Σ is compact, the countable cover in Lemma 7.2.5 has a finite subcover, and there is a constant $C > 0$ so that after passing into a subsequence, we have

$$\Sigma_{s,\rho} \subseteq \bigcup_{i=1}^{N_s} B_\rho(z_i^s) \subseteq \bigcup_{i=1}^{N} B_{C\rho}(z_i^\infty), \tag{7.14}$$

for all s.

Before exhibiting concentration of the energies near these points, we introduce the following quantity for a global meromorphic section ϕ. Near a zero or pole, say z_0, there exists an $r > 0$ so that

$$\phi = |\phi| e^{i\psi},$$

where $|\phi| > 0$ is smooth on $B_r(z_0) - \{z_0\}$. From the argument principle, we know that

$$\frac{1}{2\pi i}\int_{\partial B_r(z_0)} d\psi = \textit{number of zeros - number of poles of } \phi \textit{ inside } B_r(z_0)\textit{, counting multiplicities.}$$

We define

Definition 7.2.6. On the context immediately above, the local degree of a meromorphic section ϕ near a zero or pole z_0 is defined to be

$$\frac{1}{2\pi i}\int_{\partial B_r(z_0)} d\psi$$

Even though the quantity above is defined locally, it is a standard fact that the sum of local degrees of a global meromorphic section on all its zeros and poles is a topological quantity of the line bundle L. For a global holomorphic section, it is the degree of the line bundle.

The definition of local degree generalizes naturally to meromorphic k-sections. For $\phi = (\phi_1, \ldots, \phi_k)$ near one of its common zeros or poles z_0, we pick any single section with lowest order of vanishing (in absolute value), say ϕ_i, and define the local degree of ϕ by the local degree of ϕ_i.

Now we are ready to present the lemma on concentration of energies. Since poles or zeros of our k sections are invariant under gauge transformations, for $s < \infty$, the local degree of ϕ_s near a singularity z_i^∞ is independent of s. Let $\rho = \frac{1}{\sqrt{s}}$. We have,

Lemma 7.2.7. *In the context of* (7.14) *and Definition 7.2.6, we have, for each* $i \in \{1, \ldots, N\}$,

$$\liminf_{s\to\infty} \int_{B_{\frac{C}{\sqrt{s}}}(z_i^\infty)} e_s \geq 2\pi d_i,$$

where d_i *is the local degree of* k*-sections* $\phi_s = (\phi_1^s, \ldots, \phi_k^s)$ *near* z_0.

Proof. First, we recognize the fact that there exists a point $z_0 \notin \Sigma'$, such that

$$\inf_i |z_0 - z_i^s| \geq \frac{C}{\sqrt{s}}$$

for all s and i. This is an obvious consequence of lemma 7.2.5 and our choice of ρ. Theorem 7.2.2 thus implies the uniform bound on all compact set K contained in $B_{\frac{C}{2\sqrt{s}}}(z_0)$. That is, there exists a good ball, centered at z_0 and is far away from all the singularities:

$$\sup_K \left| \left(\sum_{i=1}^k |\phi_i^s|^2 - 1 \right) \right|^2 \leq \frac{C\epsilon}{s} \tag{7.15}$$

for all s.

Our strategy is to rescale the bad ball around z_i^∞ via a family of maps Ψ_s to "flatten" the ball and to evolve the Kähler metric ω of Σ into an Euclidean one, on which we may compute with the usual calculus. At the same time, the rescaling ensures the boundary of the bad ball to fall within good balls described in (7.15). Precisely, we define $\Psi_s : B_{\sqrt{s}} \to B_{\frac{C}{\sqrt{s}}}(z_i^\infty)$ by

$$\Psi_s(z) = \frac{Cz}{s} + z_i^\infty.$$

The sections and connection forms on $B_{\frac{C}{\sqrt{s}}}(z_i^\infty)$ are respectively pulled back to $B_{\sqrt{s}}$ via Ψ_s:

$$\hat{A}_s := \Psi_s^* A_s = \frac{1}{s} A_s(z_i^\infty + \frac{z}{s}),$$

$$\hat{\phi}_s := \Psi^* \phi_s = \phi_s(z_i^\infty + \frac{z}{s}).$$

It follows immediately that $\hat{\phi}_s$ approaches a constant section with value $\phi_\infty(z_i^\infty)$ as $s \to \infty$. The Kähler metric ω is smoothly rescaled and flattened as s increases:

$$\bar{\omega}_s := \Psi_s^* \omega = i e^{\rho_s} dz \wedge d\bar{z} \to i dz \wedge d\bar{z},$$

as $s \to \infty$. It is obvious that

$$\int_{B_{\frac{C}{\sqrt{s}}}} e_s = \int_{B_{\sqrt{s}}} e_1,$$

where $e_1 = e(\hat{A}_s, \hat{\phi}_s)$ and $(\hat{A}_s, \hat{\phi}_s)$ solves the vortex equation (7.8) without parameter . Since $\bar{\omega}_s$ converge smoothly to a flat metric as $s \to \infty$, we have

$$\begin{aligned} \int_{B_{\frac{C}{\sqrt{s}}}(z_i^\infty)} e_s \omega &= \int_{B_{\sqrt{s}}} e_1 \bar{\omega}_s \\ &\geq \int_{B_{\sqrt{s}}} e_1 i dz \wedge d\bar{z} + C_1(s), \end{aligned} \tag{7.16}$$

where $C_1(s)$ is an error term that approaches 0 as $s \to \infty$. Rewriting the energy functional

e_1, we have

$$
\begin{aligned}
& \int_{B_{\sqrt{s}}} e_1 idz \wedge d\bar{z} \\
= & \int_{B_{\sqrt{s}}} \left[\left(*F_{\hat{A}_s} + \frac{1}{2}(|\hat{\phi}_S|^2 - 1) \right)^2 - *F_{\hat{A}_s}(|\hat{\phi}_s|^2 - 1) + |D_{\hat{A}_s}\hat{\phi}_s|^2 \right] idz \wedge d\bar{z}
\end{aligned}
\tag{7.17}
$$

The first term in the integral above is 0 when $(\hat{A}_s, \hat{\phi}_s)$ satisfies (3.3) with $s = 1$. Moreover, $\hat{\phi}_s$ is $\hat{A}_s$-holomorphic when (3.3) is satisfied: $D^{(0,1)}_{\hat{A}_s}\hat{\phi}_s = 0$. Therefore, we have, using the fact that $D_{\hat{A}_s}$ are $H-$unitary connections, that

$$
\begin{aligned}
\int_{B_{\sqrt{s}}} |D_{\hat{A}_s}\hat{\phi}_s|^2 idz \wedge d\bar{z} &= \int_{B_{\sqrt{s}}} \left\langle D^{(1,0)}_{\hat{A}_s}\hat{\phi}_s, D^{(1,0)}_{\hat{A}_s}\hat{\phi}_s \right\rangle \\
&= \int_{B_{\sqrt{s}}} d\left\langle \hat{\phi}_s, D^{(1,0)}_{\hat{A}_s}\hat{\phi}_s \right\rangle + \int_{B_{\sqrt{s}}} \left\langle \hat{\phi}_s, D^{(0,1)}_{\hat{A}_s} D^{(1,0)}_{\hat{A}_s}\hat{\phi}_s \right\rangle \\
&= \int_{\mathbb{S}(\sqrt{s})} \left\langle \hat{\phi}_s, D^{(1,0)}_{\hat{A}_s}\hat{\phi}_s \right\rangle + \int_{B_{\sqrt{s}}} *F_{\hat{A}_s}|\hat{\phi}_s|^2 idz \wedge d\bar{z}
\end{aligned}
\tag{7.18}
$$

The third equality follows from Stoke's equation, where $\mathbb{S}(\sqrt{s})$ is the sphere of radius $\sqrt{s}$. For s large enough so the sphere is contained in a union of pull back of good balls described at the beginning of the proof via Ψ_s, we may write

$$
\hat{\phi}_s = \sum_{i=1}^{k} |\hat{\phi}^s_i|_H e^{i\psi_s},
$$

$$
\text{where } \sum_{i=1}^{k} |\hat{\phi}^s_i|^2_H \geq 1 - \frac{C}{\sqrt{s}}. \tag{7.19}
$$

With the connection form

$$\hat{A}_s = \hat{A}_{s,z} dz + \overline{\hat{A}_{s,z}} d\bar{z},$$

the covariant derivative of $\hat{\phi}_s$ can be computed so that

$$\begin{aligned} & \int_{S(\sqrt{s})} \left\langle \hat{\phi}_s, D^{(1,0)}_{\hat{A}_s} \hat{\phi}_s \right\rangle \\ = & \int_{S(\sqrt{s})} \left(\sum_{i=1}^{k} |\hat{\phi}_i^s|^2 - 1 \right) \left[d\psi_s - \hat{A}_{s,z} \right] + \int_{S(\sqrt{s})} \left[d\psi_s - \hat{A}_{s,z} \right] \\ & + \int_{S(\sqrt{s})} \left\langle \hat{\phi}_s, d \left(\sum_{i=1}^{k} |\hat{\phi}_i^s|_H^2 \right) e^{i\psi_s} \right\rangle \\ \geq & \frac{C\epsilon}{s} + \int_0^{2\pi} d\psi_s(\theta) - \int_{S(\sqrt{s})} \hat{A}_{s,z} - C_2(s), \end{aligned} \tag{7.20}$$

where $C_2(s) \to 0$ as $s \to \infty$. The inequality follows from (7.19), and the fact that $\hat{\phi}_s$ approaches a constant section as $s \to \infty$ (and thus $d\left(\sum_{i=1}^{k} |\hat{\phi}_i^s|_H^2\right) \to 0$). We also note that by the Stoke's Theorem,

$$\int_{S(\sqrt{s})} \hat{A}_{s,z} = \int_{B_{\sqrt{s}}} *F_{\hat{A}_{s,z}} i dz \wedge d\bar{z}.$$

Plugging these information and (7.20) into (7.17) and (7.18), we conclude that

$$\int_{B_{\sqrt{s}}} e_1 i dz \wedge d\bar{z} \geq 2\pi d_i + C_3(s),$$

where $C_3(s) \to 0$ as $s \to \infty$ and d_i is the local degree of the k-sections $\hat{\phi}_s$ near z_i^∞, independent of s and the rescaling. Letting $s \to \infty$, we have proved the lemma.

□

Lemma 7.2.7 directs us to extend the limiting data across singularities. Let

$$\Sigma'_s = \Sigma - \overline{\bigcup_{i=1}^{N_s} B_{\frac{C}{\sqrt{s}}}(z_i)}$$

and in the limit as $s \to \infty$,

$$\Sigma' = \Sigma - \{z_1^\infty, \ldots, z_N^\infty\}.$$

Pick a partition of unity $\{\eta_{i,s}\}_0^N$ subordinate to $\Sigma'_s \bigcup \left(\bigcup_{i=1}^{N_s} B_{\frac{C}{\sqrt{s}}}(z_i^s)\right)$. We see that $\eta_{i,s} \to \chi_{z_i^\infty}$ in L^p as $s \to \infty$. Here, $\chi_{z_i^\infty}$ is the characteristic function of the singleton $\{z_i^\infty\}$. When the vortex equations (3.3) are satisfied, the Yang-Mills-Higgs functional achieves its minimum, which is equal to a multiple of the degree of the line bundle, for all s (see [Br]). We then have, from Lemma 7.2.7, that

$$\begin{aligned} 2\pi r &= \int_\Sigma e_s \\ &= \lim_{s\to\infty} \left[\int_{\Sigma'_s} e_s + \sum_{i=1}^{N_s} \int_{B_{\frac{C}{\sqrt{s}}}(z_i^s)} \eta_{i,s} e_s\right] \\ &\geq \int_{\Sigma'} e_\infty + 2\pi \sum_{i=1}^{N} d_i \end{aligned} \tag{7.21}$$

Lemma 7.2.7 provides us with an upper bound of energies away from singularities, yielding sufficient tools to extend the vortices. We first extend the line bundle across singularities. Let $L' = L|_{\Sigma'}$. Each s defines a connection D_s for L' without altering its topological structure. Moreover, the inequality above indicates that the energies of vortices are bounded away from singularities. By the Main Theorem 5.0.5, the vortices converge in L^2 over any compact set $K \subset \Sigma'$. The boundedness of energies over s further motivates us to extend the limiting vortices (D, ϕ) and line bundle L' across the singularities.

The extensions of these data are quite dependent amongst each other. We first make sense of the extension of line bundle L' over Σ' to $\tilde{L}$ over Σ topologically. Start with $\Sigma = \Delta$, a complex disc, and $0 \in \Delta$ is the center of a bad ball. By the Main Theorem 5.0.5, the k sections are smooth and non-vanishing for all s, and the sum of their H norms approach a positive constant as $s \to \infty$, away from 0. We also denote $\Delta^* = \Delta - \{0\}$ to be the punctured disc. Since any complex line bundle over Δ^* is trivial due to its orientability, we let $\psi : \Delta^* \times \mathbb{C} \to L'$ be the global trivialization. The extension of L' will be made sense by embedding it into an Euclidean space and extending its trivialization ψ in the ambient space. The embedding is given by the k-sections, which gives rise to a map from Δ^* to $\mathbb{C}^k$:

$$\Psi(p) = (\phi_1(p), \ldots, \phi_k(p)).$$

One might notice the similarity of this map with the Φ^{-1} in Lemma 4.1.1. However, as the line bundle is trivial here, Ψ is well defined as it is, and we do not projectivise it. The map Ψ lifts to a map from L' to $\mathbb{C}^k \times \mathbb{C}$ naturally, by taking a point $q \in L'$, identified with $(p, \pi_2 \circ \psi^{-1}(q)) \in \Delta^* \times \mathbb{C}$, to $((\phi_1(p), \ldots, \phi_k(p)), \pi_2 \circ \psi^{-1}(q)) \in \mathbb{C}^k \times \mathbb{C}$. Here, π_2 is the projection to the second component. The lifted map is still denoted by Ψ. Following the discussions of Kodaira Embedding Theorem (Chapter 1 Section 4 in [G-H]), the map Ψ is indeed an embedding of L' into an Euclidean space $\mathbb{C}^k \times \mathbb{C}$, which depends crucially on the fact that the k sections do not vanish simultaneously on Δ^*. With this embedding available, we identify the trivialization ψ with $\Psi \circ \psi$, a map from $\Delta^* \times \mathbb{C}$ to $\mathbb{C}^k \times \mathbb{C}$ and seek an extension of this map.

One can observe that if the embedding Ψ extends across 0, the trivialization extends accordingly by continuity. The only possible difficulties to extending Ψ come from possible development of zeros or poles of the limiting k sections at 0. Poles can be ruled out due to the uniform (in s) upper bound of energies of vortices on the entire disc Δ. Common zeros are therefore the only possible hurdles, which prevent Ψ from being an embedding (see [G-H]). Before we deal with this case, we make a simple observation that if 0 is a singularity, the k sections must vanish simultaneously at 0. Otherwise, we may apply the main theorem with the non-vanishing k sections as the background sections to find a limiting Hermitian metrics that satisfy (4.3), which is smooth. Without any singularity, 0 can no longer be a center to a "bad" ball. Therefore, the singularities are precisely the common zeros of ϕ_i's. To overcome the presence of common zeros, we follow construction in [B-D-W] and factor out z^d from each section, where d is the least order of vanishing among all k sections. In another word, we consider the canonical defining section $\sigma \in H^0(\Delta, \mathcal{O}(d \cdot 0))$, and define a new k section $\tilde{\phi} \in H^0(\Delta, \mathcal{O}(L^{\oplus k}))$ so that

$$\phi = \tilde{\phi} \otimes \sigma,$$

and the trivialization ψ is now extended across 0 with ϕ replaced by $\tilde{\phi}$ in the definition of Ψ. This new section $\tilde{\phi}$ defines the same line bundle away from 0, as it is a multiple of ϕ by a non-vanishing function. The line bundle $\tilde{L}$ defined by $\tilde{\phi}$ is therefore an extension of L, that is, $\tilde{L}|_{\Delta^*} \cong L'$. Since $\tilde{\phi}$ now do not vanish simultaneously at 0 anymore, its local degree at the origin is 0. Comparing the local degree of ϕ and $\tilde{\phi}$ at 0, we conclude that

$$\tilde{L} \cong L \otimes \mathcal{O}(-d \cdot 0).$$

Pick a trivializing open cover of $L \to \Sigma$ so that each cover contains at most one "bad ball", whose centers are the common zeros of the k sections. We perform the techniques just developed to each cover, and construct an extended line bundle $\tilde{L}$ such that $\tilde{L}|_{\Sigma'} \cong L'$, and extended vortices as above. The degree of $\tilde{L}$ is $r - \sum d_i$, where d_i is the local degree of ϕ at z_i^∞. In another word, we have

$$\tilde{L} \cong L \otimes \mathcal{O}(-V),$$

where $V = \sum_{i=1}^N d_i z_i^\infty$ is the effective divisor of common zeros of the k sections, counted with multiplicities.

With the extended line bundle $\tilde{L}$ constructed topologically, we need to construct an extended complex structure on it. In another words, we need to extend the connections. One recalls that there is a limiting connection A_∞ smooth on Σ', with singularities precisely at $\{z_1^\infty, \ldots, z_N^\infty\}$. We need a reasonable smooth extension $\tilde{A}_\infty$ of A_∞ to the entire Σ, whose curvature has the topological invariance identical with that of $\tilde{L}$'s. Precisely, we look for $\tilde{A}_\infty$ so that

$$\int_\Sigma \sqrt{-1}\Lambda F_{\tilde{A}_\infty} = \int_{\Sigma'} \sqrt{-1}\Lambda F_{A_\infty}. \tag{7.22}$$

To do so, we recall that the sup norms of F_{A_s}'s are uniformly bounded away from singularities. Since values of functions at countably many points are irrelevant to Lebesgue integrals, they define a family of uniformly bounded Lebesgue measures on Σ. Namely, for $M \in \mathcal{M}(\Sigma)$, we define

$$\mu_s(M) = \int_M F_{A_s},$$

where $\mathcal{M}(\Sigma)$ is the sigma algebra of measurable subsets of Σ. Furthermore, by the dominated convergence theorem, the measures μ_s converge to a limiting measure μ_∞ pointwise. By the classical Lebesgue-Radon-Nikodym Theorem, it decomposes into an absolutely continuous part, and a singular part. The singular parts with respect to Lebesgue measure are precisely a combination Dirac delta measures. Since the curvatures converge smoothly away from singularities $\{z_1^\infty, \ldots, z_N^\infty\}$, the Dirac delta measures are supported on the singularities of F_{A_∞}, with certain multiplicity a_i at each z_i^∞. We have, for each $M \in \mathcal{M}(\Sigma)$, and as $s \to \infty$,

$$\int_M F_{A_s} \to \int_{M \bigcap \Sigma'} F_{A\infty} + \sum_{i=1}^N \int_{M \bigcap \{z_i^\infty\}} a_i \delta_{z_i^\infty}.$$

Topological constraints of the line bundle and Lemma 7.2.7 require that

$$\sum_{i}^{N} a_i = \sum_{i}^{N} d_i,$$

and

$$a_i \leq d_i.$$

implying that $a_i = d_i$ for each i. We anticipate the extended curvature to define a $(1,1)$ current identical to the first integral on the right side of the convergence immediately above. Let us first consider the cohomology class:

$$\Theta := [F_{A_\infty}] - \sum_{i=1}^{N} [d_i \delta_{z_i^\infty}] \in H^2(\Sigma').$$

Clearly, we see that

$$\frac{i}{2\pi}\Theta = c_1(L').$$

Recall the standard real Hodge orthogonal decomposition of a two form ψ on Σ':

$$\psi = \mathcal{H}(\psi) + dd^* G(\psi) + d^* dG(\psi),$$

where $\mathcal{H}$ is the orthogonal projection of a differential form onto the harmonic components, or the kernel of Laplacian $\Delta_d = d^*d + dd^*$. $G : \Omega^p \to \Omega^p$ is the Green operator on differential forms given by

$$(G\psi)(x) = \int_{\Sigma'} G(x,y)\psi(y)dy,$$

where $G(x,y)$ is the classical Green's function. The Poisson equation

$$\Delta_d \eta = \psi$$

has a weak solution precisely when $\mathcal{H}(\psi) = 0$. The solution is then

$$\eta = G\psi.$$

See, for example, [G-H] for a more complete descriptions of Hodge Theorem. Here, for the smooth limiting curvature two form F_{A_∞}, $d^*dGF_{A_\infty} = 0$ since F_{A_∞} is of the top dimension. The Hodge decomposition amounts to the existence a one form α so that

$$F_{A_\infty} - \mathcal{H}(F_{A_\infty}) = d\alpha \tag{7.23}$$

We note that Σ' can be exhausted by countably many compact deformation retract of Σ. Indeed, pick some $\epsilon_0 > 0$ so that the closure N coordinate balls centered at z_i^∞ with radius ϵ_0 are mutually disjoint. Pick a sequence $\epsilon_k \searrow 0$, and define

$$\Sigma_k := \Sigma - \bigcup_{i=1}^{N} \overline{B_{\epsilon_k}(z_i^\infty)}.$$

It is clear that each Σ_k is an increasing sequence of compact deformation retracts of Σ', and that

$$\Sigma' = \bigcup_{i=1}^{\infty} \Sigma_k.$$

Theorem 7.1.4 therefore applies to the connection form α, which can now be gauge transformed into Uhlenbeck gauge. Without changing the notation, it means precisely that

$$\begin{cases} d^*\alpha = 0 \\ *\alpha = 0 \text{ on } \partial\Sigma' = \{z_\infty^1, \ldots, z_\infty^N\} \\ \|\alpha\|_{W_1^{1,p}} \leq C \|F_{A_\infty} - \mathcal{H}(F_{A_\infty})\|_{L^p} \end{cases} \tag{7.24}$$

Next, we note that singularities on Σ are of codimension 2, and therefore

$$H^2(\Sigma', \mathbb{R}) \simeq H^2(\Sigma, \mathbb{R}).$$

Therefore, we may view the class Θ defined above as a class in $H^2(\Sigma, \mathbb{R})$ by excision. Pick a harmonic representative $F = \mathcal{H}(F) \in \Theta$ so that $dF = 0$ and $d^*F = 0$. Since F is closed, we may cover Σ with star-shaped open sets $\{U_j\}$ so that F is exact on each U_j. That is, there is $\tilde{\alpha}_j \in \Omega^1(U_j)$ such that

$$F = d\tilde{\alpha}_j.$$

Again, on each U_j, we have the uniform L^p bound of curvatures and may find a gauge g_j so that $g_j{}^*\tilde{\alpha}_j$ is in Uhlenbeck gauge. Denoting $g_j{}^*\tilde{\alpha}_j$ still by $\tilde{\alpha}_j$, since F is chosen to be harmonic, $d^*F = d^*d\alpha_j = 0$. We may then apply the elliptic regularity Theorem 2.3.13 with V, γ and ω all being 0 to conclude that $\tilde{\alpha}_j$ is smooth. Since Σ' is a countable union of compact deformation retracts of Σ, Theorem 7.5 in [W] applies to $\{\tilde{\alpha}_j\}$ so that after appropriate gauge transformations, these one forms satisfy (7.24) and define a global connection form on Σ. Moreover, Lemma 7.2 in [W] ensures that the gauges preserve the smoothness of $\{\tilde{\alpha}_j\}$, and we denote the global smooth connection one form by $\tilde{\alpha}$.

We now compare A_∞ and $\tilde{\alpha}$ on Σ'. From the derivations above, we have

$$\begin{cases} d(A_\infty - \tilde{\alpha}) = F_{A_\infty} - \mathcal{H}(F_{A_\infty}) \\ d^*(\tilde{\alpha} - \alpha) = 0 \end{cases}$$

These equations imply that

$$\Delta_d(A_\infty - \tilde{\alpha}) = d^*[F_{A_\infty} - \mathcal{H}(F_{A_\infty})].$$

But $d^*\mathcal{H}(F_{A_\infty}) = 0$ and $F_{A_\infty} = 0$ on Σ', we conclude that $A_\infty - \tilde{\alpha}$ is harmonic on Σ':

$$\Delta_d(A_\infty - \tilde{\alpha}) = 0.$$

Therefore, near each singularity z_i^∞, $A_\infty - \tilde{\alpha}$ is represented by the fundamental solution of harmonic form:

$$A_\infty - \tilde{\alpha} = b_i \log|z - z_i^\infty| dz + df,$$

for some smooth function f. However, since $d\tilde{\alpha}$ and dA_∞ both represent $c_1(L') \in H^2(\Sigma', \mathbb{R})$, it is required that on the punctured disc Δ^* centered at z_i^∞,

$$\int_{\Delta^*} dA_\infty = \int_{\Delta^*} d\tilde{\alpha},$$

which is only true if $b_i = 0$. This yields the global smooth one form $\tilde{\alpha}$ on Σ that agrees with A_∞ on Σ'. We have therefore constructed the extended smooth connection one form $\tilde{A_\infty} = \tilde{\alpha}$ on a different line bundle $\tilde{L}$ over Σ, where $L \simeq \tilde{L} \otimes \mathcal{O}(V)$.

We have the following conclusion on the asymptotic behaviors on the entire moduli space $\nu_k(s)$:

Away from the accumulation points of the common zeros of each $\phi_s = (\phi_1^s, \ldots, \phi_k^s)$ (singularities), the vortices (A_s, ϕ_s) solving s-vortex equations (3.3) converge smoothly to the vortices (A_∞, ϕ_∞) solving the ∞-vortex equations (3.4). Moreover, the limiting vortex can be smoothly extended across the singularities. However, the extended vortex is defined on $\tilde{L}$, with $L \simeq \tilde{L} \otimes$ (divisors defined by singularities).

7.3 Bubbling

This final subsection is a brief survey of established works. We have seen, from previous sections, that topological data is lost when extending vortices across singularities as a result of concentration of energies. At the adiabatic limit $s = \infty$, they are carried away in the forms "bubbles". Many descriptions of bubbling phenomenon are available, and are rapidly being updated, in contemporary literature. We have surveyed the results in [C-G-R-S], [G-S], [O], [Wo], [X], and [Z].

All of them concern the more general setting of symplectic vortex equations with target symplectic manifold (M, ω). More precisely, we consider a symplectic manifold (M, ω), and a Hamiltonian action of a compact Lie Group G on M with moment map $\mu : M \to \mathfrak{g}^*$. Let P be a principal G-bundle over a Riemann surface (Σ, ω_Σ), and J an ω-compatible complex structure on M. For a connection A of $\Omega^1(P)$, and section u (which is now a G-equivariant map from P to M), the symplectic vortex equations with parameter s is given by

$$\begin{cases} \bar{\partial}_{J,A}(u) = 0 \\ F_A + s^2(\mu \circ u)\omega_\Sigma = 0 \end{cases} \tag{7.25}$$

In [Z], analogous descriptions are given in this setting. In fact, our convergent conclusion in previous subsection is a special case of Proposition 18 in [Z]. There, similar rescaling as in Lemma 7.2.7 is performed near singularities (referred to as "marked points"). One recalls that upon zooming in near a singularity, we obtain a rescaled vortex equation without parameter (7.8), defined on a balls with increasing radii. At the limit $s = \infty$, we obtain a vortex equation on $\mathbb{C} \simeq \mathbb{R}^2$. This heuristic observation is precisely stated in [X] and [Z]. Furthermore, if the decay of the vortex on $\mathbb{C}$ is rapid enough near ∞, one may further extend the vortex to ∞, and define a vortex on $\mathbb{C} \bigcup \{\infty\} \simeq \mathbb{S}^2$.

In the language of [G-S], [Z], and [X], the limiting vortices described above are roughly classified as "raindrop" and spheres. The former corresponds to the limiting vortices that do not decay rapid enough, and therefore are defined on $\mathbb{C}$ as is. The latter model then corresponds to those vortices extended to $\mathbb{S}^2$. With certain properly defined tree relation and evaluation map, these data combine together to form a "stable map". This is the limiting object discussed in these papers. In [Z], it was shown that with certain mild conditions, if the singularities of the energy functionals do not all accumulate, there is a subsequence of the vortices convergent to a stable map in some suitable sense.

The rate of blow up of the energy density near a singularity z_i^∞ determines whether the bubble is a raindrop or a sphere. It turns out that the critical blow up rate is $\frac{1}{s^2}$. If

$$\lim_{s\to\infty} \frac{1}{s^2} e_s(z_i^\infty) < \infty,$$

the bubble is a raindrop. Otherwise, a sphere bubble is formed. In either case, each bubble carries some positive amount of energy, which is carried away as $s \to \infty$ (see [Z]). Our lower bound of energy given by local degree, computed in Lemma 7.2.7, is a special case of their results.

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