CURVES ON SURFACES

BY

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DISSERTATION

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Abstract

This dissertation is concerned with geometric and combinatoric problems of curves on surfaces. In Chapter 3, we show that certain families of iso-length spectral hyperbolic surfaces obtained via the Sunada construction are not generally simple iso-length spectral. In Chapter 4, We prove a strong form of finite rigidity for pants graphs of spheres. Specifically, for any $n \geq 5$ we construct a finite subgraph X_n of the pants graph $\mathcal{P}(S_{0,n})$ of the n-punctures sphere $S_{0,n}$ with the following property. Any simplicial embedding of X_n into any pants graph $\mathcal{P}(S_{0,m})$ of a punctured sphere is induced by an embedding $S_{0,n} \to S_{0,m}$.

To my family.

Acknowledgements

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Chapter 1

Introduction

Curves provide an important tool for studying the geometry and topology of surfaces. In this dissertation, we consider two very different problems in which curves can be used to provide information about surfaces.

The first involves isometric invariants of Riemannian metrics on a surface defined in terms of curves. We consider the *simple length spectrum*, the sequence of lengths of simple closed geodesics, and show that in a certain sense this provides more information about the surface than the length spectrum. In particular, we show that the simple length spectrum is able to distinguish many nonisometric pairs of hyperbolic surfaces which the length spectrum cannot.

The second problem we study is of a more geometric-topological nature. The set of all simple closed curves on a surface, up to isotopy, can be used to define combinatorial graphs and simplicial complexes on which the mapping class group acts. We consider a rigidity problem for finite subgraphs of one such graph called the *pants graph*.

In the next two sections we briefly describe our main results.

1.1 The Sunada construction and the simple length spectrum

Let S be a closed Riemannian suface. The **length spectrum** $L(S)$ of S is the set of all lengths of closed geodesics on S counted with multiplicities. Two surfaces S_1 and S_2 are said to be **iso-length spectral** if $L(S_1) = L(S_2)$.

In [Sun85], Sunada provided a method to construct iso-length spectral surfaces that are frequently not isometric (see also [Bus92, Ch.11-13]). This requires a notion from group theory.

Let G be a finite group. Two subgroups H and K of G are said to be **almost conjugate** if, for any $g \in G$,

$$|H \cap (g)| = |K \cap (g)|,$$

where (g) denotes the conjugacy class of g in G.

Theorem 1.1.1 (Sunada). Let S be a closed Riemannian surface, G a finite group, and H and K almost conjugate subgroups of G. If there is a surjective homomorphism from $\pi_1(S)$ onto G, then the finite covering spaces S_H and S_K of S corresponding to the subgroups H and K, respectively, are iso-length spectral.

When H and K are not conjugate in G, the surfaces S_H and S_K can often be shown to be nonisometric. For example, a generic hyperbolic metric on S will produce nonisometric S_H and S_K; see [Bus92, Ch.12.7].

The simple closed geodesics often carry more topological information. Accordingly, the **simple length spectrum** $L^s(S)$ of S is defined to be the set of all lengths of simple closed geodesics on S counted with multiplicities; see [MP08]. Two surfaces S_1 and S_2 are said to be **simple iso-length spectral** if $L^s(S_1) = L^s(S_2)$.

Question 1.1.2. Are there nonisometric simple iso-length spectral hyperbolic surfaces?

In [MP08], McShane and Parlier give example of pairs of 4-holed spheres with geodesic boundary which have the same *interior simple lengh spectrum* (one ignores the boundary lengths). They do in fact have different boundary lengths, and so they have different simple length spectra.

One can ask if Sunada's construction provides a positive resolution to Question 1.1.2.

Question 1.1.3. Does Sunada's construction, for a given homomorphism $\rho : \pi_1(S) \to G$, generically give simple iso-length spectral surfaces?

To answer Question 1.1.3, we choose one of the examples of almost conjugate subgroups Sunada provided in his paper [Sun85].

Example 1.1.4. $G = (\mathbb{Z}/8\mathbb{Z})^\times \ltimes \mathbb{Z}/8\mathbb{Z}$ with usual action of $(\mathbb{Z}/8\mathbb{Z})^\times$ on $\mathbb{Z}/8\mathbb{Z}$.

$H = \{(1,0),(3,0),(5,0),(7,0)\}$ and $K = \{(1,0),(3,4),(5,4),(7,0)\}$ are almost conjugate but not conjugate.

Our first theorem is the following.

Theorem 1.1.5. Let $S_{2,0}$ be a closed oriented surface of genus 2, G, H, and K the groups provided in the example above.

There is a surjective homomorphism $\rho : \pi_1(S_{2,0}) \to G$ such that, for almost every $[m] \in \mathcal{T}(S_{2,0})$, the corresponding iso-length spectral surfaces S_H and S_K are not simple iso-length spectral.

In fact, we prove a little bit more. We define the **length set** and the **simple length set** of a surface S to be the set of all lengths of closed geodesics on S without multiplicities and the set of all lengths of simple closed geodesics on S without multiplicities, respectively. Then from the proof of Theorem 1.1.5 we have the following theorem.

Theorem 1.1.6. The surfaces S_H and S_K in Theorem 1.1.5 have the same length set but they do not have the same simple length set.

This theorem shows that the construction of length equivalent surfaces in [LMNR07] does not necessarily give simple length equivalent surfaces.

A sketch of the proof of Theorem 1.1.5 is as follows. We begin by defining a surjective homomorphism $\rho : \pi_1(S_{2,0}) \to G$ and a closed curve α in $S_{2,0}$. By Sunada's construction, the covering spaces $\pi_H : S_H \to S_{2,0}$ and $\pi_K : S_K \to S_{2,0}$ corresponding to the subgroups H and K are iso-length spectral. We then show that, for almost every $[m] \in \mathcal{T}(S_{2,0})$, the induced metrics on S_H and S_K have the following property. In each of these two covering spaces S_H and S_K, there are exactly four closed geodesics having the same length as α, namely the two degree-one components of $\pi_H^{-1}(\alpha)$ (and $\pi_K^{-1}(\alpha)$) and their images under the lifts of the hyperelliptic involution $\tau : S_{2,0} \to S_{2,0}$. We also show that these four closed geodesics on S_H are nonsimple while the other four closed geodesics on S_K are simple. Therefore S_H and S_K are not simple iso-length spectral.

We remark on one subtlety of the proof. According to [Ran80], there are curves γ, γ' on $S_{2,0}$ such that for every hyperbolic metric m on $S_{2,0}$, $\text{length}_m(\gamma) = \text{length}_m(\gamma')$. Although these are nonsimple on $S_{2,0}$, they become simple in a finite sheeted cover, so must be accounted for in our proof.

1.2 Finite rigid subgraphs of the pants graphs of punctured spheres

We now suppose that S is an orientable surface of genus g with n punctures. To emphasize the topology of S, we write $S = S_{g,n}$ when convenient. The **complexity** of S is given by $\kappa(S_{g,n}) = 3g + n - 3$. There are two interesting combinatorial objects associated to the surface S using the simple closed curves: the curve complex and the pants graph. These are important in studying the mapping class group of S, or more precisely here, the **extended mapping class group** $\text{Mod}^{\pm}(S) = \pi_0(\text{Homeo}(S))$, where $\text{Homeo}(S)$ is the group of all homeomorphisms of S.

The **curve complex** $\mathcal{C}(S)$ of S has vertex set given by the set of isotopy classes of essential (i.e. nontrivial and nonperipheral) simple closed curves on S. A set of $k + 1$ vertices are the vertices of a k–simplex if and only if they can be realized pairwise disjointly on S, or equivalently, if they have have pairwise zero geometric intersection number (see Section 2.1 for more detailed definitions). The **pants graph** $\mathcal{P}(S)$ of S is a graph with vertex set given by pants decompositions (maximal collections of pairwise disjoint essential simple closed curves) up to isotopy. Two pants decompositions span an edge of $\mathcal{P}(S)$ if and only if they differ by an elementary move (see Section 2.3 for definitions). Every homeomorphism induces an automorphism of both $\mathcal{C}(S)$ and $\mathcal{P}(S)$, and homotopic homeomorphisms induce the same automorphism. Thus we obtain homomorphisms

$$\text{Mod}^{\pm}(S) \to \text{Aut}(\mathcal{C}(S)),$$

and

$$\text{Mod}^{\pm}(S) \to \text{Aut}(\mathcal{P}(S)).$$

The following is the main theorem of [Iva97, Kor99, Luo00].

Theorem 1.2.1 (Ivanov, Korkmaz, Luo). The map $\text{Mod}^{\pm}(S) \to \text{Aut}(\mathcal{C}(S))$ is surjective with finite kernel. Moreover, if $\kappa(S) \geq 3$, then this map is an isomorphism.

The difficult part of this theorem is proving that the map is surjective. Extracting out this part of the theorem, the authors prove that given any automorphism of $\mathcal{C}(S)$ there is some homeomorphism $S \to S$ which induces the given automorphism. This was generalized in [Irm04], [BM06] and [Sha07] to show that any locally injective simplicial map is induced by a homeomorphism of S (in particular, the locally injective map is in fact an automorphism).

This was recently generalized even further in [AL12] to the following.

Theorem 1.2.2 (**Aramayona–Leininger**). For any surface S, there exists a finite subcomplex $X \subset \mathcal{C}(S)$ with the following property. Any locally injective simplicial map $X \to \mathcal{C}(S)$ is the restriction of an automorphism of $\mathcal{C}(S)$, and in particular is induced by a homeomorphism of S.

For pants graphs, Margalit [Mar04] proved the analogue of Theorem 1.2.1.

Theorem 1.2.3 (**Margalit**). The map $\mathrm{Mod}^{\pm}(S) \to \mathrm{Aut}(\mathcal{P}(S))$ is surjective with finite kernel. Moreover, if $\kappa(S) \geq 3$, then this map is an isomorphism.

A natural question is whether or not Theorem 1.2.2 is true in this setting as well.

Question 1.2.4. Given a surface S, does there exist a finite subgraph $X \subset \mathcal{P}(S)$ such that every simplicial embedding $X \to \mathcal{P}(S)$ is the restriction of an automorphism of $\mathcal{P}(S)$, in particular, is induced by a homeomorphism of S?

For pants graphs, it turns out the Margalit's Theorem can be generalized well beyond what is possible for curve complexes. This is due to Aramayona [Ara10].

Theorem 1.2.5 (**Aramayona**). Let S and S' be compact orientable surfaces of negative Euler characteristic, such that the complexity of S is at least 2. Every injective simplicial map

$$\phi : \mathcal{P}(S) \to \mathcal{P}(S')$$

is induced by a π_1-injective embedding $f : S \to S'$.

(The precise meaning of *induced* will be clear from the following).

Theorems 1.2.2 and 1.2.5 suggest another question.

Question 1.2.6. Given a surface S, does there exist a finite subgraph $X \subset \mathcal{P}(S)$ such that, for any other surface S', any simplicial embedding $X \to \mathcal{P}(S')$ is induced by a π_1–injective embedding $f : S \to S'$?

Let $X \subset P(S_{g,n})$ be a subgraph, $\phi : X \to P(S_{g',m})$ be an injective simplicial map and $f : S_{g,n} \to S_{g',m}$ be a π_1-injective embedding. We say that f **induces** ϕ if there is a deficiency-$(3g+n-3)$ multicurve Q on $S_{g',m}$ with the following property (see Section 2.3 for definitions). The image $f(S_{g,n})$ is the unique non-pants component $(S_{g',m} - Q)_0 \subset S_{g',m} - Q$ and the simplicial map

$$f^Q : \mathcal{P}(S_{g,n}) \to \mathcal{P}(S_{g',m}),$$

defined by $f^Q(u) = f(u) \cup Q$ satisfies $f^Q(u) = \phi(u)$ for any vertex $u \in X$.

In the second part of this dissertation, we answer this second question in the positive, in the special case that S and S' are punctured spheres.

Theorem 1.2.7. For $n \geq 5$, there exists a finite subgraph $X_n \subset \mathcal{P}(S_{0,n})$ such that for any punctured sphere $S_{0,m}$ and any injective simplicial map

$$\phi : X_n \to \mathcal{P}(S_{0,m}),$$

there exists a π_1-injective embedding $f : S_{0,n} \to S_{0,m}$ unique up to isotopy that induces ϕ.

The sketch of the proof of Theorem 1.2.7 is as follow. We describe the finite subgraph $X_n \subset \mathcal{P}(S_{0,n})$ for every $n \geq 5$ and verify a number of properties to be used in the proof. The proof of the theorem is by induction on n. In section 4.2 we describe the proof for the case $n = 5$ which serves as the base case. This is stated as Lemma 4.2.1. The general cases is dealt with in Section 4.3. The beginning of the proof is the same for all $n \geq 6$. The case $n = 6$ requires a slightly different argument to finish, and we give this first. The final argument for the case $n \geq 7$ to complete is given at the end Section 4.3.

Chapter 2

Background

Here we provide some background and more detailed definitions of the objects discussed in this dissertation. References are provided throughout.

2.1 Curves and the geometric intersection number

Let $S = S_{g,n}$ be an orientable surface of genus g with n punctures. A closed curve on S is **essential** if it is homotopically nontrivial and nonperipheral. An essential simple closed curve is an embedded essential closed curve. We will consider essential (simple) closed curves up to homotopy and will not distinguish between curves and the homotopy classes they represent when it causes no confusion to do so.

Let α and β be closed curves on S. **The geometric intersection number** of α and β is defined by

$$i(\alpha,\beta) = \min_{\overline{\alpha},\overline{\beta}} |\left(\overline{\alpha} \times \overline{\beta}\right)^{-1}(\Delta)|,$$

where $\overline{\alpha}$ and $\overline{\beta}$ are in the homotopy classes $[\alpha]$ and $[\beta]$, respectively, $\overline{\alpha} \times \overline{\beta} : S^1 \times S^1 \to M \times M$, and $\Delta \subset M \times M$ is diagonal.

The intersection of any two curves mentioned in this dessertation refers to their geometric intersection number. Whenever we represent homotopy classes α and β by closed curves, we assume these intersect in $i(\alpha,\beta)$ number of points. Two curves α and β are **disjoint** if $i(\alpha,\beta) = 0$. Let A be a set of essential simple curves on S. We say that $\bigcup_{\alpha \in A} \alpha$ **fills** S if the complement is a disjoint union of disks or once-punture disks.

2.2 Teichmüller theory

Let $S = S_{g,0}$ be a closed oriented surface of genus $g \geq 2$. We denote the Teichmüller space of S by

$$\mathcal{T}(S) = \{[m] \mid m \text{ is a hyperbolic metric on } S\},$$

where $[m]$ represents the equivalence class via the equivalence relation $m \sim m'$ if there exists an isometry $f : (S,m) \to (S,m')$ such that $f \simeq id_S$, see e.g. [Bus92].

Given $[m] \in \mathcal{T}(S)$, the holonomy homomorphism

$$\rho_m : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$$

is well defined up to conjugation in $\mathrm{PSL}_2(\mathbb{R})$. This determines an embedding

$$\mathcal{T}(S) \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{conjugation} \tag{2.1}$$

by $[m] \mapsto [\rho_m]$.

Let α be an essential closed curve on S. The length function of α

$$\mathrm{length}_{(\cdot)}(\alpha) : \mathcal{T}(S) \to \mathbb{R}_+$$

is defined as the length of the m-geodesic homotopic to α. Using the holonomy homomorphism, one can compute

$$\mathrm{length}_{[m]}(\alpha) = 2\cosh^{-1}\left(\frac{|\mathrm{tr}(\rho_m(\alpha))|}{2}\right). \tag{2.2}$$

The embedding (2.1) makes $\mathcal{T}(S)$ into a real analytic surface. By (2.2), the length functions are analytic (see e.g. [Ker85] or [Abi80]). Since $\mathcal{T}(S)$ is connected, we then have the following theorem; see [MP08].

Theorem 2.2.1. Let $c \in \mathbb{R}$, α and β be closed curves on S. The function

$$f = c \cdot \mathrm{length}_{(\cdot)}(\beta) - \mathrm{length}_{(\cdot)}(\alpha) : \mathcal{T}(S) \to \mathbb{R}$$

is real analytic, in particular, $f \neq 0$ almost everywhere or $f = 0$ everywhere.

The next theorem provides a tool for dealing with the multiplicity phenomenon described in [Ran80].

Theorem 2.2.2. Let γ, γ' be closed curves on S and $k \in \mathbb{R}$.

If $\mathrm{length}_m(\gamma) = k \cdot \mathrm{length}_m(\gamma')$, for all $[m] \in \mathcal{T}(M)$, then $i(\gamma, \alpha) = k \cdot i(\gamma', \alpha)$, for all simple closed curves α on M.

Proof. For $k = 1$, a proof can be found in [Lei03] and we show how it can be adapted.

Given a simple closed curve α, as in [Lei03] one can construct a sequence $\{[S_N]\} \subset \mathcal{T}(M)$ by pinching α so that

$$\frac{1}{n} \cdot \mathrm{length}_{[S_N]}(\eta) \to i(\eta, \alpha),$$

for all closed curves η on M.

Now suppose $\mathrm{length}_{[m]}(\gamma) = k \cdot \mathrm{length}_{[m]}(\gamma')$ for all $[m] \in \mathcal{T}(M)$. Then

$$\frac{1}{n} \cdot \mathrm{length}_{[S_N]}(\gamma) \to i(\gamma, \alpha)$$

and

$$\frac{k}{n} \cdot \text{length}_{[S_N]}(\gamma') \to k \cdot i(\gamma', \alpha).$$

So $k \cdot i(\gamma', \alpha) = i(\gamma, \alpha)$. $\square$

The following theorem is Corollary 3.4 in [Lei03].

Theorem 2.2.3. Given γ and γ' closed curves on S, if

$$\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\gamma'),$$

for all $[m] \in \mathcal{T}(S)$, then $[\gamma] = \pm[\gamma']$ in $H_1(S)$.

2.3 Pants graphs

Let $S = S_{g,n}$ be an orientable surface of genus g with n punctures. We call a surface which is homeomorphic to $S_{0,3}$, **a pair of pants**. Let A be a set of pairwise disjoint essential simple closed curves on S. The **nontrivial component(s)** of the complement of the curves in A denoted $(S-A)_0$ is the union of the non-pants components of the complement.

A **multicurve** Q is a set of pairwise disjoint essential simple closed curves on S. Let Q_1 and Q_2 be multicurves. The intersection number of Q_1 and Q_2 is defined to be $i(Q_1, Q_2) = \sum i(\alpha_1, \alpha_2)$, where α_1 and α_2 are curves in Q_1 and Q_2, respectively. Observe that $Q_1 \cup Q_2$ is a multicurve if and only if $i(Q_1, Q_2) = 0$.

A **pants decomposition** P is a muticurve whose complement in S is a disjoint union of pairs of pants. Equivalently, a pants decomposition is a maximal set of pairwise disjoint essential simple closed curve on S, that is, a maximal multicurve. A pants decompositions always contains $3g+n-3$ curves and we call this number the **complexity** $\kappa(S)$ of S. The **deficiency** of a multicurve Q is the number $\kappa(S) - |Q|$. If Q is a deficiency-1 multicurve then $(S-Q)_0$ is homeomorphic to either $S_{0,4}$ or $S_{1,1}$.

Let P and P' be pants decompositions of S. P and P' differ by an **elementary move** if there are curves α, α' on S and a deficiency-1 multicurve Q such that $P = \{\alpha\} \cup Q, P' = \{\alpha'\} \cup Q$ and $i(\alpha, \alpha') = 2$ if $(S-Q)_0 \cong S_{0,4}$ or $i(\alpha, \alpha') = 1$ if $(S-Q)_0 \cong S_{1,1}$; see Figure 2.1 for examples of elementary moves.

The **pants graph** $\mathcal{P}(S_{g,n})$ of $S_{g,n}$ is a graph with the set of vertices

$$V_{\mathcal{P}(S_{g,n})} = \{P \mid P \text{ is a pants decomposition of } S_{g,n}\},$$

and the set of edges

$$E_{\mathcal{P}(S_{g,n})} = \{\{P, P'\} \in V_{\mathcal{P}(S_{g,n})} \times V_{\mathcal{P}(S_{g,n})} \mid P, P' \text{ differ by an elementary move}\}.$$

The pants graph $\mathcal{P}(S_{g,n})$ is connected, see[HLS00], which we view as a geodesic metric space by

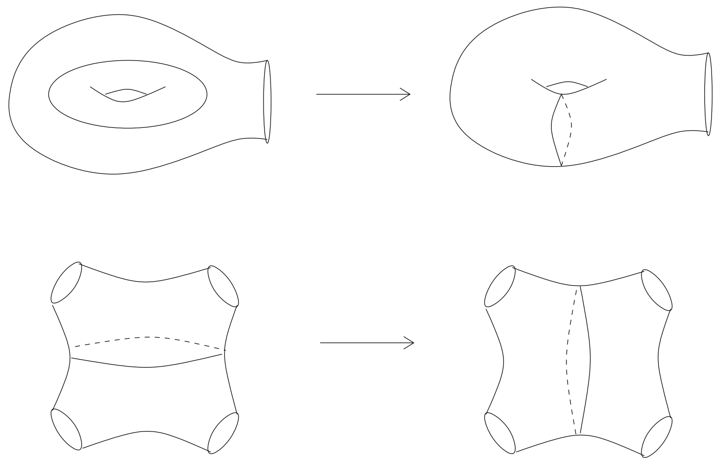

Figure 2.1: Two types of elementary moves.

requiring each edge to have length 1.

Given a multicurve Q, we let $\mathcal{P}_Q(S_{g,n})$ be the subgraph of $\mathcal{P}(S_{g,n})$ **induced** by the vertex set

$$V_{\mathcal{P}_Q(S_{g,n})} = \{P \in V_{\mathcal{P}(S_{g,n})} \mid Q \subset P\},$$

that is, the largest subgraph with $V_{\mathcal{P}_Q(S_{g,n})}$ as its vertex set. When the deficiency of Q is positive, it is easy to see that $\mathcal{P}_Q(S_{g,n}) \cong \mathcal{P}((S_{g,n} - Q)_0)$. Let Q_1 and Q_2 be multicurves. We observe that $\mathcal{P}_{Q_1}(S_{g,n}) \cap \mathcal{P}_{Q_2}(S_{g,n}) \neq \emptyset$ if and only if $Q_1 \cup Q_2$ is a multicurve if and only if $i(Q_1, Q_2) = 0$, and in this case, $\mathcal{P}_{Q_1}(S_{g,n}) \cap \mathcal{P}_{Q_2}(S_{g,n}) = \mathcal{P}_{Q_1 \cup Q_2}(S_{g,n})$.

A **Farey graph** $\mathbb{F}$ is a graph isomorphic to the standard Farey graph which has vertices

$$V = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, \frac{p}{q} \text{ is in lowest term} \right\} \cup \left\{ \frac{1}{0} = \infty \right\},$$

and edges

$$E_{\mathbb{F}} = \left\{ \left\{ \frac{p}{q}, \frac{s}{t} \right\} \in V \times V \mid |pt - qs| = 1 \right\}.$$

See Figure 2.2 for a picture of a part of the Farey graph.

The following Lemma combines the results from ([Min96], Section 3), and ([Mar04],Lemma 2).

Lemma 2.3.1. Let F be a subgraph of $\mathcal{P}(S_{g,n})$. Then F is isomorphic to a Farey graph if and only if there is a deficiency-1 multicurve Q such that $F = \mathcal{P}_Q(S_{g,n})$.

Note that as a consequence of Lemma 2.3.1, each edge e of $\mathcal{P}(S_{g,n})$ is contained in a unique Farey graph $\mathcal{P}_Q(S_{g,n})$ where $Q = P \cap P'$ is the deficiency-1 multicurve given by the intersection of its endpoints P and P'.

We also see from the Lemma that $\mathcal{P}(S_{0,4})$ and $\mathcal{P}(S_{1,1})$ are isomorphic to a Farey graph. Let α and β be two essential simple closed curves on $S_{0,4}$ such that $i(\alpha, \beta) = 2$. Then, as pants decompositions, α and β differ by an elementary move, i.e., they are two adjacent vertices in $\mathcal{P}(S_{0,4}) \cong \mathbb{F}$. Up to a homeomorphism on $S_{0,4}$, α and β are the curves on $S_{0,4}$ shown in Figure 2.2. Let $T_c^{\frac{1}{2}}$ be the half-twist around a curve c on $S_{0,4}$. Then we can see that

$$T_\beta^{\frac{1}{2}}(\alpha) = T_\alpha^{-\frac{1}{2}}(\beta),$$

and, together with α, β, these three vertices form a triangle in $\mathcal{P}(S_{0,4})$; see Figure 2.2.

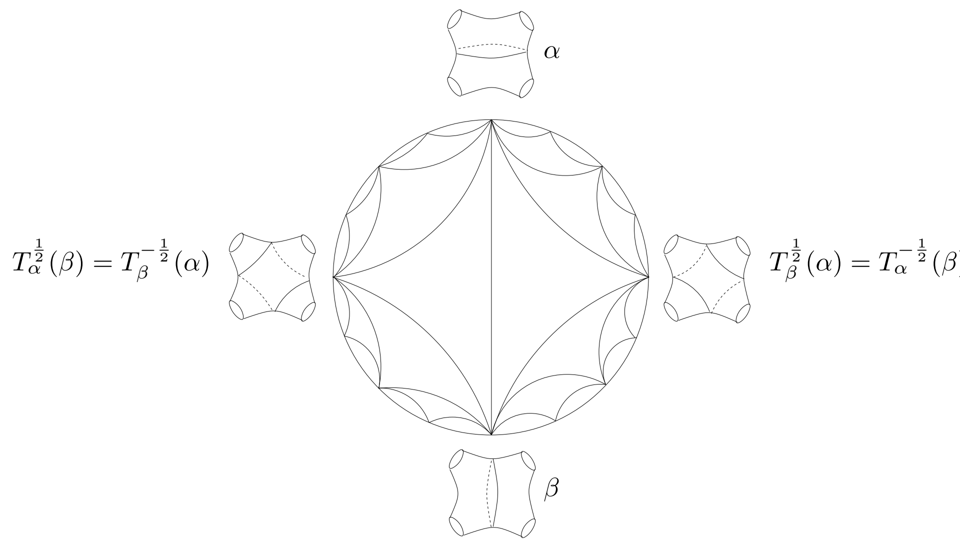

Figure 2.2: $\mathcal{P}(S_{0,4})$ and some curves representing its vertices.

By a **path** in $\mathcal{P}(S_{g,n})$, we always mean an edge path determined by a sequence of distinct adjacent vertices $v_1, ..., v_m$ of $\mathcal{P}(S_{g,n})$. A **cycle** in $\mathcal{P}(S_{g,n})$ is a subgraph homeomorphic to a circle. A cycle is called a **triangle**, **rectangle**, **pentagon** if it has $3, 4$ or 5 vertices, respectively.

A cycle $v_1, ..., v_m = v_1$ is called an **alternating cycle** if any two consecutive edges are in different Farey graphs. See Figure 2.3 for an example of an alternating pentagon in $\mathcal{P}(S_{0,5})$. The following is proved in [Mar04, Lemma 8].

Lemma 2.3.2. Let $X \subset \mathcal{P}(S_{0,5})$ be the pentagon shown in Figure 2.3 and let $\phi : X \to \mathcal{P}(S_{0,m})$ be an injective simplicial map. If $\phi(X)$ is an alternating pentagon, then there exists a deficiency-2 multicurve Q and a homeomorphism $f : S_{0,5} \to (S_{0,m} - Q)_0$ to a component of $S_{0,m} - Q$ such that $\phi|_X = f^Q|_X$.

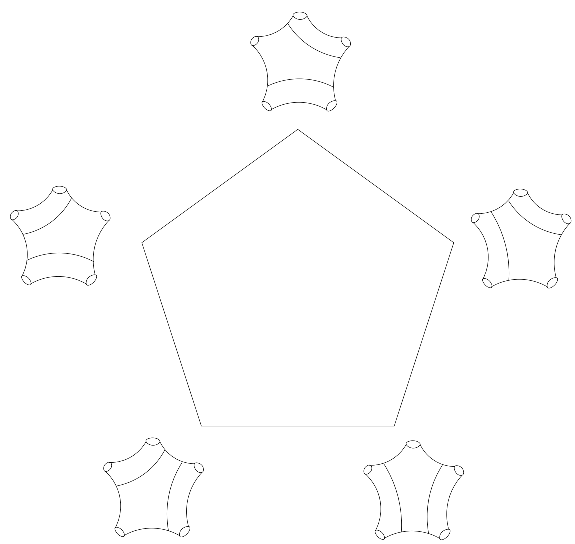

Figure 2.3: An alternating pentagon in $\mathcal{P}(S_{0,5})$.

Chapter 3

The Sunada construction and the simple length spectrum

Here we prove Theorem 1.1.5. Any curves mentioned in this chapter are essential and closed.

Let $S_{2,0}$ be a closed oriented surface of genus 2. We write the fundamental group of $S_{2,0}$ as $\pi_1(S_{2,0}) = \langle a, b, c, d | [a, b][c, d] = 1 \rangle$, see Figure 3.1. Let G, H and K be groups given in the example

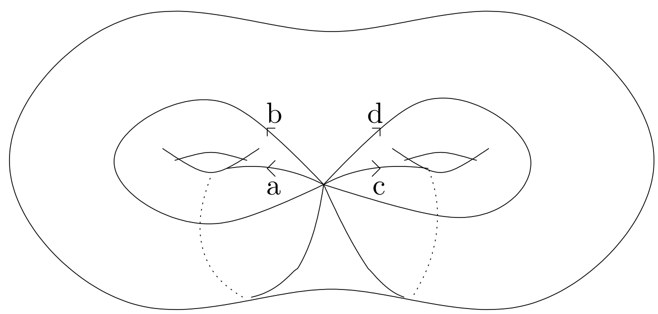

Figure 3.1: $S_{2,0}$ with the generators of $\pi_1(S_{2,0})$.

in Section 1.1. We define a surjective homomorphism $\rho : \pi_1(S_{2,0}) \to G$ by

$$\rho(a) = (3, 0), \quad \rho(b) = (5, 0), \quad \rho(c) = (1, 0), \quad \text{and} \quad \rho(d) = (1, 1).$$

Let $\pi : S \to S_{2,0}$, $\pi_H : S_H \to S_{2,0}$ and $\pi_K : S_K \to S_{2,0}$ be the covering spaces of $S_{2,0}$ corresponding to $\ker(\rho)$, $\rho^{-1}(H)$ and $\rho^{-1}(K)$, respectively.

To help visualize the covering space S, first we construct the covering space $\pi : S_N \to S_{2,0}$ corresponding to the subgroup $N = \mathbb{Z}/8\mathbb{Z}$ of G, as shown in Figure 3.2. Then we construct S from the surjective homomorphism $\sigma : \pi_1(S_N) \to N$, the restriction of ρ to $\pi_1(S_N) < \pi_1(S_{2,0})$, see Figure 3.3. Observe that the generator of $\mathbb{Z}/8\mathbb{Z} \cong N < G$ translates each piece in Figure 3.3 to the right, and sends the last piece to the first piece.

Lemma 3.0.3. Let $\alpha = abd[d, c^{-1}]d^{-1}$ be a closed curve on $S_{2,0}$. Then $\pi_H^{-1}(\alpha) = \beta_1^H \cup \cdots \cup \beta_5^H$, $\pi_K^{-1}(\alpha) = \beta_1^K \cup \cdots \cup \beta_5^K$ where $\pi_H|_{\beta_i^H}$, $\pi_K|_{\beta_i^K}$ are degree one, for $i = 1, 2$, and degree two, for $i = 3, 4, 5$. Furthurmore β_1^H, β_2^H are nonsimple and β_1^K, β_2^K are simple.

Proof. First we look at a component γ_1 of $\pi^{-1}(\alpha)$ in S, see Figure 3.5. Observe that the preimage of α is sixteen simple closed curves on S denoted $X = \{\gamma_1, \ldots, \gamma_{16}\}$. G acts on X and this action

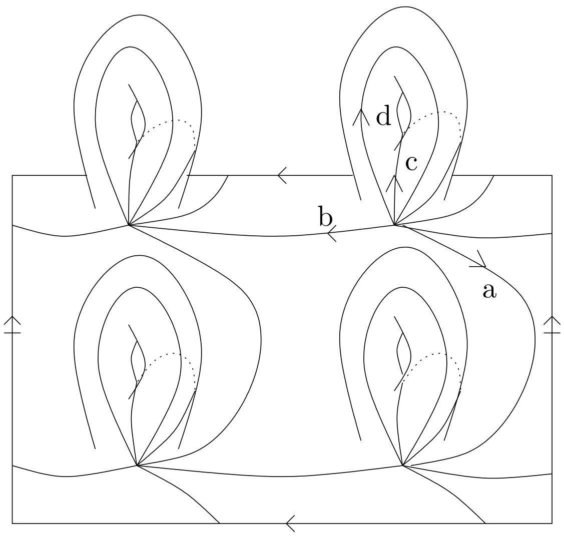

Figure 3.2: The covering space S_N.

is equivalent to the action of G on the cosets of $L = \mathrm{Stab}_G(\gamma_1) = \{(1,0),(7,0)\}$. More precisely, the bijection

$$G//L \to X$$

given by

$$gL \mapsto g \cdot \gamma_1$$

is equivariant with respect to the actions of G. We assume $\{\gamma_1, \ldots, \gamma_{16}\}$ are numbered so that

$$\begin{array}{rrrr} \gamma_1 \to L, & \gamma_2 \to (1,1)L, & \gamma_3 \to (1,2)L, & \gamma_4 \to (1,3)L, \\ \gamma_5 \to (1,4)L, & \gamma_6 \to (1,5)L, & \gamma_7 \to (1,6)L, & \gamma_8 \to (1,7)L, \\ \gamma_9 \to (3,0)L, & \gamma_{10} \to (3,3)L, & \gamma_{11} \to (3,6)L, & \gamma_{12} \to (3,1)L, \\ \gamma_{13} \to (3,4)L, & \gamma_{14} \to (3,7)L, & \gamma_{15} \to (3,2)L, & \gamma_{16} \to (3,5)L. \end{array}$$

We use the above representations to compute H and K orbits under the actions of H and K on X. Then the H orbits partition $\{\gamma_1, \ldots, \gamma_{16}\}$ as

$$\{\gamma_1, \gamma_9\}, \{\gamma_5, \gamma_{13}\}, \{\gamma_2, \gamma_8, \gamma_{10}, \gamma_{16}\}, \{\gamma_3, \gamma_7, \gamma_{11}, \gamma_{15}\}, \{\gamma_4, \gamma_6, \gamma_{12}, \gamma_{14}\}$$

and the K orbits partition $\{\gamma_1, \ldots, \gamma_{16}\}$ as

$$\{\gamma_1, \gamma_{13}\}, \{\gamma_5, \gamma_9\}, \{\gamma_2, \gamma_8, \gamma_{10}, \gamma_{14}\}, \{\gamma_3, \gamma_7, \gamma_{11}, \gamma_{15}\}, \{\gamma_4, \gamma_6, \gamma_{10}, \gamma_{16}\}.$$

All closed curves in each H orbit lie above exactly one closed curve on S_H and all closed curves in each K orbit lie above exactly one closed curve on S_K. So we can write $\pi_H^{-1}(\alpha) = \beta_1^H \cup \cdots \cup \beta_5^H$ and

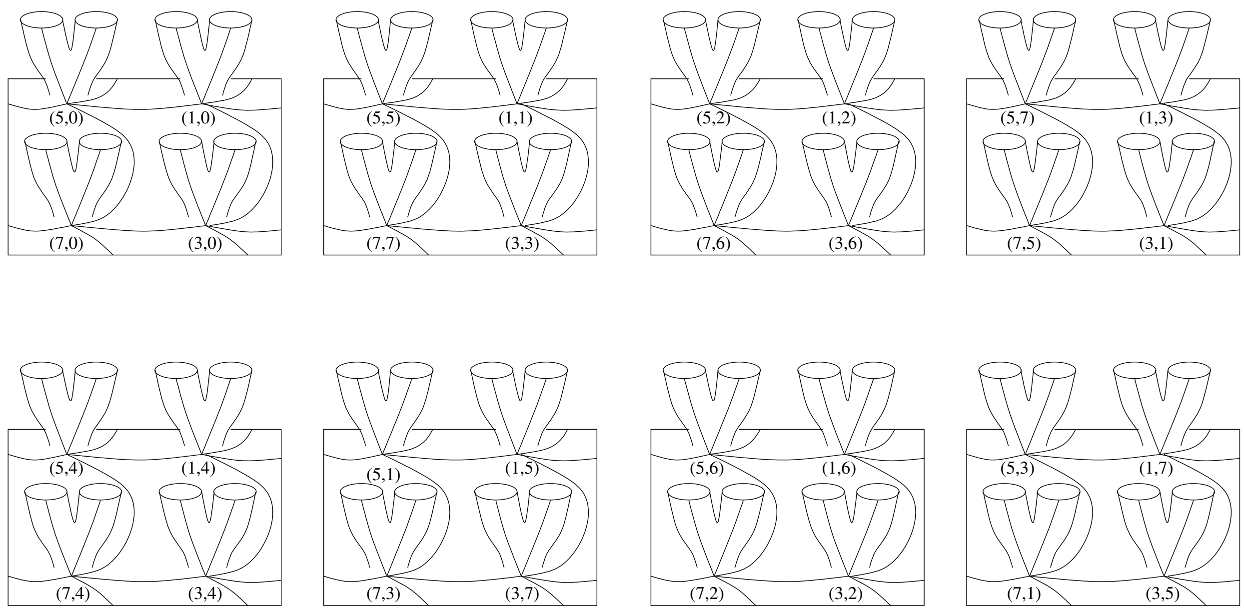

Figure 3.3: The covering space S. The sides of the eight squares are identified as in Figure 3.2. To determine the gluing on the 64 boundary components, the left boundary component from (x, y) is glued to the right boundary component from $(x, y+1)$, e.g. $(1, 0)$ and $(1, 1)$.

$\pi_K^{-1}(\alpha) = \beta_1^K \cup \cdots \cup \beta_5^K$. We may associate β_1^H, β_2^H, β_1^K and β_2^K with the orbits $\{\gamma_1, \gamma_9\}$, $\{\gamma_5, \gamma_{13}\}$, $\{\gamma_1, \gamma_{13}\}$ and $\{\gamma_5, \gamma_9\}$, respectively.

Next we observe that $\pi_H|_{\beta_i^H}$, $\pi_K|_{\beta_i^K}$ are degree one, for $i = 1, 2$, and degree two, for $i = 3, 4, 5$.

For the simplicity of β_1^H, β_2^H, β_1^K and β_2^K, we look at their associated orbits. We observe that γ_1 intersects $\gamma_9 = (3, 0) \cdot \gamma_1$ nontrivially by inspecting Figure 3.3 for the actions of G and Figure 3.5 for the picture of γ_1. Similarly we can show

$$\begin{aligned} \gamma_1 \cap \gamma_9 \ &\neq \emptyset, \quad \gamma_5 \cap \gamma_{13} \neq \emptyset, \\ \gamma_1 \cap \gamma_{13} &= \emptyset, \quad \gamma_5 \cap \gamma_9 \ = \emptyset. \end{aligned}$$

Since the H orbit $\{\gamma_1, \gamma_9\}$ corresponding to β_1^H contains intersecting curves, β_1^H is nonsimple. Similarly, β_2^H is also nonsimple. Since the K orbit $\{\gamma_1, \gamma_{13}\}$ corresponding to β_1^K contains pairwise disjoint curves, β_1^K is simple. Similarly, β_2^H is also simple. □

To prove Theorem 1.1.5, we will show that generically a hyperbolic metric on $S_{2,0}$ lifted to a hyperbolic metric on S_H has the property that there are exactly four closed curves on S_H having the same length as β_1^H(and β_2^H) and these four closed curves are nonsimple. In the previous Lemma, we found two such closed curves, namely β_1^H and β_2^H. Lemma 3.0.4 provides the other two closed curves and we will use Lemma 3.0.5 to show that there are exactly four such closed curves. Since S_K has a simple closed curve, β_1^K, of the same length in its lifted metric, S_H and S_K cannot be simple iso-length spectral.

Let $\tau : S_{2,0} \to S_{2,0}$ be the hyperelliptic involution. τ is isotopic to an isometry for any hyperbolic

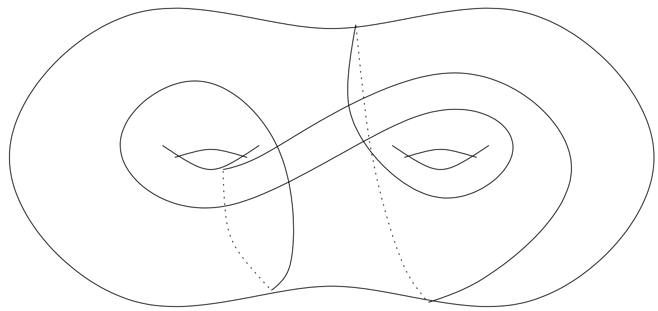

Figure 3.4: The closed curve α on $S_{2,0}$.

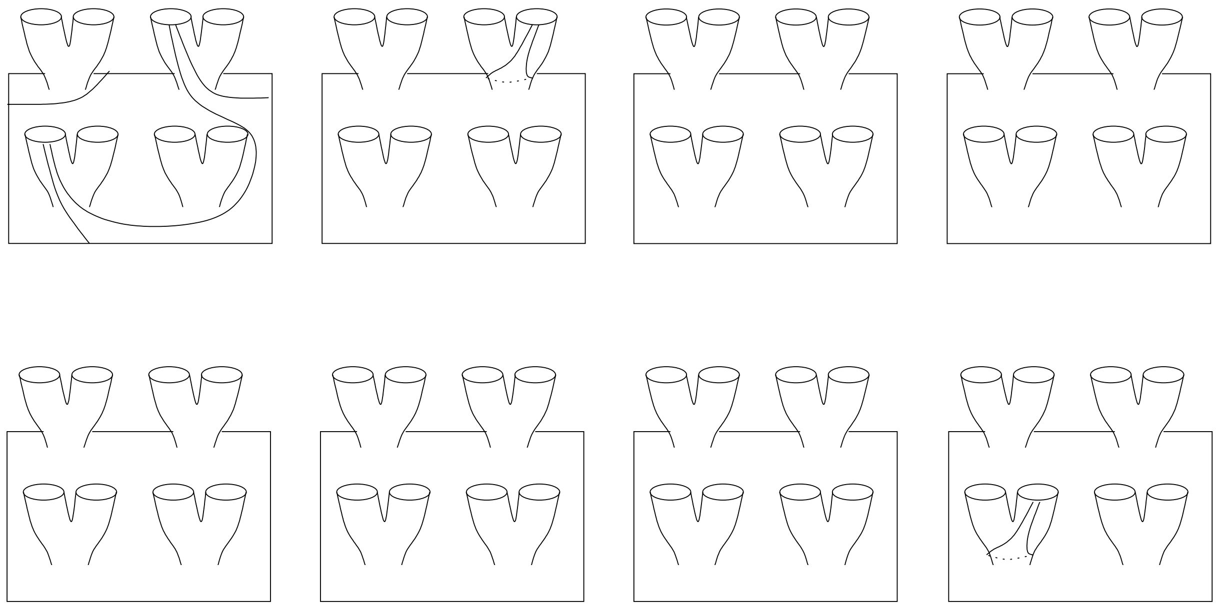

Figure 3.5: The covering space S and a component γ_1 of $\pi^{-1}(\alpha)$.

metric on $S_{2,0}$. So for any closed curve λ on $S_{2,0}$, $\text{length}_{S_{2,0}}(\lambda) = \text{length}_{S_{2,0}}(\tau(\lambda))$. For a specific basepoint, the induced map $\tau_* : \pi_1(S_{2,0}) \to \pi_1(S_{2,0})$ can be computed to be

$$\tau_*(a) = a^{-1}, \qquad \tau_*(b) = b^{-1},$$
$$\tau_*(c) = ac^{-1}dc^{-1}d^{-1}ca^{-1}, \quad \tau_*(d) = b^{-1}ad^{-1}ba^{-1}.$$

We have the following lemma.

Lemma 3.0.4. The hyperelliptic involution $\tau : S_{2,0} \to S_{2,0}$ lifts to $\tau_H : S_H \to S_H$ and $\tau_K : S_K \to S_K$. In particular, $\tau_H\left(\beta_i^H\right) \subset S_H$ is nonsimple and $\tau_K\left(\beta_i^K\right) \subset S_K$ is simple, for $i = 1, 2$.

Proof. Let $\psi : G \to G$ be the automorphism of G defined by $\psi(j,k) = (j,-k)$, for any element $(j,k) \in G$. Then we can compute $\psi \circ \rho = \rho \circ \tau_*$ and $H = \psi^{-1}(H)$. So $\rho^{-1}(H) = \rho^{-1}(\psi^{-1}(H)) =$

$\tau_*^{-1}(\rho^{-1}(H))$. Thus

$$\tau_*\left(\left(\pi_H\right)_*\left(\pi_1\left(M_H\right)\right)\right) = \tau_*\left(\rho^{-1}\left(H\right)\right) = \rho^{-1}\left(H\right) = \left(\pi_H\right)_*\left(\pi_1\left(M_H\right)\right).$$

Hence the lifting criterion implies that we may lift τ to τ_H. The existence of a lift τ_K to S_K is proven in the same way. ☐

Lemma 3.0.5. For almost every $[m] \in \mathcal{T}(S_{2,0})$, if γ is a closed curve, $k \in \mathbb{Q}$ and

$$k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$$

then $k = 1$ and $\gamma = \alpha$ or $\tau(\alpha)$.

This lemma will be used to show that the lifts of α to S_H and S_K have the appropriate multiplicity, and do not exhibit the phenomenon in [Ran80].

Proof. For any γ and any k, either $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$ is true for every $[m]$ or $k \cdot \text{length}_{[m]}(\gamma) \neq \text{length}_{[m]}(\alpha)$ for almost every $[m]$, by Theorem 2.2.1. So it suffices to show that if $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, for every $[m]$, then $k = 1$ and $\gamma = \alpha$ or $\tau(\alpha)$. Let y_1 be a simple

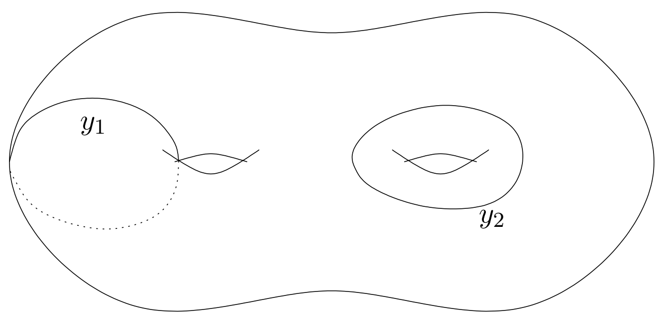

Figure 3.6: The simple closed curves y_1 and y_2 on the surface $S_{2,0}$.

closed curve as shown in Figure 3.6. The geometric intersection number of α and y_1 is $i(\alpha, y_1) = 1$. Since $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, by Theorem 2.2.2, $k \cdot i(\gamma, y_1) = i(\alpha, y_1) = 1$. Since the geometric intersection numbers are nonnegative intergers, $k = 1$. To prove that $\gamma = \alpha$ or $\tau(\alpha)$, we find some neccessary conditions for γ to have the same length as α, for every $[m] \in \mathcal{T}(S_{2,0})$

Let y_2 be the simple closed curve shown in Figure 3.6. Since $i(\gamma, y_2) = i(\alpha, y_2) = 0$ by Theorem 2.2.2, γ and α are contained in $S_{2,0} - y_2$.

We cut $S_{2,0}$ along the simple closed curve y_2 to get a torus with two holes and change the basis $\{a, b, d\}$ to the basis $\left\{a, b, x = da^{-1}\right\}$, see Figure 3.7. Then $\alpha = abxaba^{-1}b^{-1}x^{-1}$ and $\tau_*(\alpha) = a^{-1}b^{-1}b^{-1}x^{-1}ba^{-1}b^{-1}axb$. Consider the spine as shown in Figure 3.8. We homotope α and γ into the spine, as edge loops without backtracking. Then we can construct metrics on $S_{2,0}$ where the lengths of some of the edges are bounded while others tend to infinity and so that the lengths of α and γ in the spine are comparable (up to bounded additive error) to geodesic lengths. From this, we see that in order for γ to have the same length as α in $S_{2,0}$,

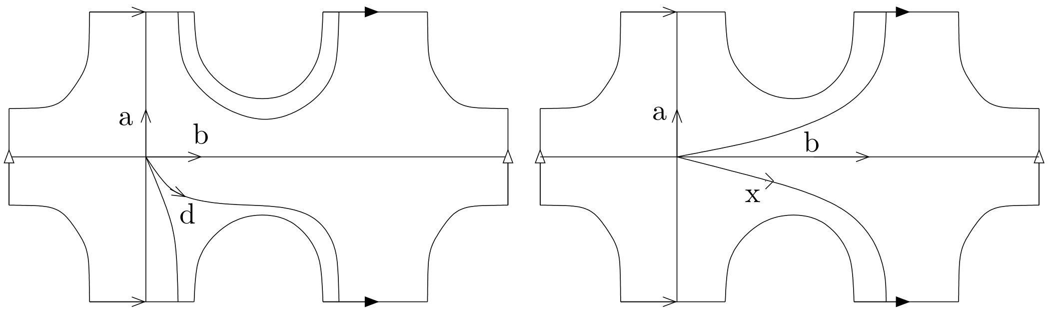

Figure 3.7: The torus with two holes, $S_{2,0} - y_2$.

$$\sharp \{a_1 \text{ edges of } \gamma\} = \sharp \{a_1 \text{ edges of } \alpha\} = 3,$$
$$\sharp \{x_1 \text{ edges of } \gamma\} = \sharp \{x_1 \text{ edges of } \alpha\} = 3,$$
$$\sharp \{b_1 \text{ edges of } \gamma\} + \sharp \{b_2 \text{ edges of } \gamma\} = \sharp \{b_1 \text{ edges of } \alpha\} + \sharp \{b_2 \text{ edges of } \alpha\} = 8.$$

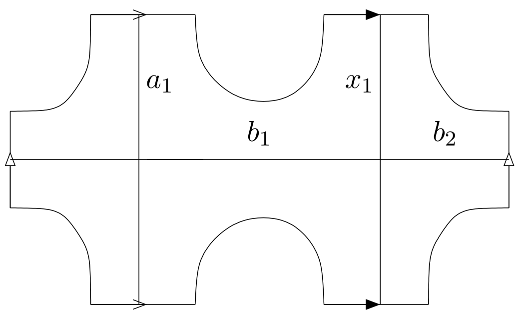

Figure 3.8: The torus with two holes, $S_{2,0} - x_2$ with spine.

Since $\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$ and $[\alpha] = [ab] \in H_1(S_{2,0})$, $[\gamma] = \pm [ab] \in H_1(S_{2,0})$, by Theorem 2.2.3. Thus from the observation of the edge counts above (replacing γ with γ^{-1} if necessary), we have the following conditions;

1. γ consists of exactly two a's, one a^{-1}, one x, and one x^{-1},
2. $\sharp \left\{b^{-1}\text{'s in } \gamma\right\} = \sharp \{b\text{'s in } \gamma\} - 1$, and
3. $\sharp \{b_1 \text{ edges of } \gamma\} + \sharp \{b_2 \text{ edges of } \gamma\} = 8$.

Next we find all closed curves on $S_{2,0}$ satisfying these three conditions. By the conditions above we know the exact number of a's, a^{-1}'s, x's, and x^{-1} that appear in γ. So we only need to determine the possible number of b's and b^{-1}. To do this, we note that while the number of a_1-edges and the number of x_1-edges can be computed directly by counting the number of $\left\{a, a^{-1}\right\}$ and $\left\{x, x^{-1}\right\}$, respectively, some combinations of x's and b's provide cancellations in the sum of b_1 and b_2-edge count. One example is that x alone contributes 2 to the sum of b_1 and b_2-edge count, b alone also contributes 2 to the sum of b_1 and b_2-edge count but xb contributes only 2 to the sum of b_1 and b_2-edge count.

Taking this type of cancellation into consideration, we can produce a list Ω of 4320 words in $\left\{a^{\pm 1}, b^{\pm 1}, x^{\pm 1}\right\}$ that contains all curves satisfying the three conditions, see Appendix A.

One can explicitly construct $[m] \in \mathcal{T}(S_{2,0})$, a hyperbolic metric on $S_{2,0}$ such that

$$\rho_m(a) = \begin{pmatrix} 5/3 & 3/4 \\ 3/4 & 5/4 \end{pmatrix},$$

$$\rho_m(b) = \begin{pmatrix} 4 & 0 \\ 0 & 1/4 \end{pmatrix},$$

$$\rho_m(x) = \begin{pmatrix} 5/3 & -16/3 \\ -1/3 & 5/3 \end{pmatrix}.$$

Then the trace of $\rho_m(\alpha)$ is

$$\operatorname{tr}(\rho_m(\alpha)) = 109505/2048.$$

By using Mathematica, we have that the elements in Ω having the same trace squared as α are α and $\tau(\alpha)^{-1}$, see Appendix B.

So, by equation (2.2), the only curves in Ω that have the same length in $S_{2,0}$ as α are α and $\tau(\alpha)$.

Thus if $\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, for every $[m] \in \mathcal{T}(S_{2,0})$, then $\gamma = \alpha$ or $\tau(\alpha)$. □

With the above three Lemmas, we are ready to prove Theorem 1.1.5.

Proof of Theorem 1.1.5. Let $\rho : \pi_1(S_{2,0}) \to G$ be the surjective homomorphism defined in this section.

Let $\alpha = abd[d, c^{-1}]d^{-1}$ be a closed geodesic on $S_{2,0}$.

By Lemma 3.0.3 and Lemma 3.0.4, for almost every $[m] \in \mathcal{T}(S_{2,0})$, there are four non-simple closed geodesics $\left\{\beta_1^H, \beta_2^H, \tau_H\left(\beta_1^H\right), \tau_H\left(\beta_2^H\right)\right\}$ on S_H having length $l = \text{length}_{[m]}(\beta_1^H) = \text{length}_{[m]}(\alpha)$ and there are four simple closed geodesics $\left\{\beta_1^K, \beta_2^K, \tau_K\left(\beta_1^K\right), \tau_K\left(\beta_2^K\right)\right\}$ on S_K having length l.

If γ^H is a closed geodesic on S_H having length

$$l = \text{length}_{[m]}(\beta_1^H) = \text{length}_{[m]}(\alpha),$$

then $\pi_H(\gamma^H)$ is a closed geodesic on $S_{2,0}$ having length

$$k \cdot l = k \cdot \text{length}_{[m]}(\beta_1^H) = k \cdot \text{length}_{[m]}(\gamma),$$

for some $k = 1, 1/2, 1/4, \text{or } 1/8$, since the degree of π_H and π_K is 8.

By Lemma 3.0.5, $k = 1$ and $\pi_H(\gamma^H) = \alpha$ or $\tau(\alpha)$. Thus γ^H is one of the four nonsimple closed curves above. Hence there are exactly four closed curves on S_H having length l and those four closed curves are nonsimple. Similarly, there are exactly four closed curves on S_K having length l and those four closed curves are simple.

Therefore S_H and S_K are not simple iso-length spectral. □

Proof of Theorem 1.1.6. As the proof of Theorem 1.1.5 shows, for almost every $[m] \in \mathcal{T}(S_{2,0})$, there is a simple closed geodesic on S_K with the same length as α on $S_{2,0}$, but no such simple geodesic on S_H. Therefore, S_H and S_K are not simple length equivalent. □

3.1 Final discussion

Theorem 1.1.5 should hold for any surjective homomorphism $\rho : \pi_1(S) \to G$ and for any closed surface S. Indeed, it can be shown that for G as in Theorem 1.1.5 and any ρ, there is a genus 2 or 3 subsurface $\Sigma \subset S$ so that the restriction $\rho|_{\pi_1(\Sigma)}$ is surjective. Then, one can list all such surjective homomorphisms and try to construct a curve α in Σ playing the role of α in the proof of Theorem 1.1.5. This does not seem to provide much new information, and even for the cases analyzed here, the resulting presentation is significantly more complicated. It would be interesting to find an approach that works for all homomorphisms simultaneously.

Another class of examples that would be interesting to analyze with respect to Question 1.1.3 are those given in [BT87] and [Bus86], as the construction is more geometric.

Chapter 4

Finite rigid subgraphs of the pants graphs of punctured spheres

In this chapter we prove Theorem 1.2.7. Because the proof is a somewhat involved, we outline the rest of the chapter briefly. To begin with, we describe the finite subgraph $X_n \subset \mathcal{P}(S_{0,n})$ for every $n \geq 5$ and verify a number of properties to be used in the proof. The proof of the theorem is by induction on n. In section 4.2 we describe the proof for the case $n = 5$ which serves as the base case. This is stated as Lemma 4.2.1. The general cases is dealt with in Section 4.3. The beginning of the proof is the same for all $n \geq 6$. The case $n = 6$ requires a slightly different argument to finish, and we give this first. The final argument for the case $n \geq 7$ to complete is given at the end of Section 4.3. We end the chapter with some thoughts about the general case.

Any curves mentioned in this chapter are essential simple closed curves.

4.1 The construction of X_n

In this section, we construct the finite subgraph X_n, for $n \geq 5$. To begin, consider $S_{0,n}$ as the double of a regular n-gon with vertices removed. Connect every non-adjacent pair of sides by a straight line segment and then double. The result is $S_{0,n}$ with a set of simple closed curves Γ_n. See Figure 4.1 for the case of $S_{0,8}$ and Figure 4.2 for the case of $S_{0,5}$. Label the sides of the n-gon

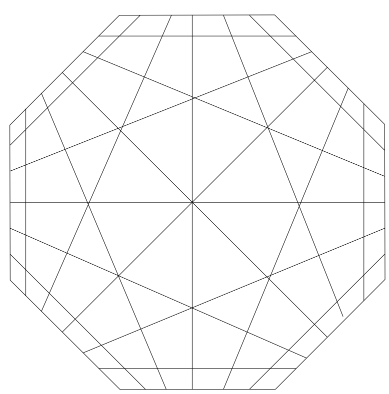

Figure 4.1: $S_{0,8}$ and the set of simple closed curves Γ_8.

cyclically as $1, ..., n$. In all that follows, we assume that any reference to these labels is taken modulo n (thus, if i is a label, so is $i+1$ and $i-1$). Given labels i and j, write $\alpha_{i,j}$ for the curve in

Γ_n obtained from the arc connecting the i^{th} side to the j^{th} side. For each i, we call $\alpha_{i,i+2}$ **a chain curve** of $S_{0,n}$. Compare [AL12, Section 3].

Let $W \subset \Gamma_n$ be a deficiency-m multicurve such that there is a nontrivial component $(S_{0,n} - W)_0$ of $(S_{0,n} - W)$ homeomorphic to a sphere with $m+3$ punctures. Let Γ^W_{m+3} be the subset of Γ_n whose every element is disjoint from every element of W.

Lemma 4.1.1. Let $W \subset \Gamma_n$ be a deficiency-m multicurve such that there is a nontrivial component $(S_{0,n} - W)_0$ of $(S_{0,n} - W)$ homeomorphic to a sphere with $m+3$ punctures.

Then there is a homeomorphism of pairs

$$h : (S_{0,m+3}, \Gamma_{m+3}) \to ((S_{0,n} - W)_0, \Gamma^W_{m+3}).$$

Proof. Consider $S_{0,n}$ as the double of a regular n-gon as described above. Then the curves in W are obtained from doubling pairwise disjoint arcs in the regular n-gon. There is a component Δ of the complement of those arcs that is doubled to produce $(S_{0,n} - W)_0$. Collapsing those arcs to points, Δ becomes an $(m+3)$-gon. Doubling, this defines a homeomorphism $h : S_{0,m+3} \to (S_{0,n} - W)_0$. The set Γ^W_{m+3} in $(S_{0,n} - W)_0$ is then the image under h of the set of curves obtained by doubling arcs connecting non-adjacent pair of sides of the $(m+3)$-gon, i.e., $h(\Gamma_n) = \Gamma^W_{m+3}$ as required. □

Let V_n be the set of vertices of $\mathcal{P}(S_{0,n})$ that correspond to pants decompositions consisting of curves from Γ_n.

Let Z_n be a subgraph of $\mathcal{P}(S_{0,n})$ induced by V_n, that is, Z_n is the largest subgraph with V_n as its vertex set.

Lemma 4.1.2. For $n \geq 5$, the subgraph Z_n is finite and connected.

Proof. The finiteness is obvious since the number of vertices is finite. We prove the connectedness by induction on n, the number of punctures. The base case when $n = 5$ is true since Z_5 is a pentagon.

For each $i = 1, ..., n$, let v_i be a vertex in Z_n corresponding to the pants decomposition $\{\alpha_{i,x} \in \Gamma_n | x \neq i-1, i, i+1\}$. It is not hard to see that two vertices v_i and v_{i+1} are connected in Z_n by a path of length $n-3$. Hence any two vertices v_i and v_j are connected in Z_n. Given any vertex v in Z_n, we will show that v is connected to one of the v_i's. Note that as a pants decomposition, v must contain a chain curve. Choose a chain curve $\alpha = \alpha_{s,s+2}$ in v. The nontrivial component of $S_{0,n} - \alpha$ is homeomorphic to $S_{0,n-1}$. By Lemma 4.1.1, $Z_n \cap P_\alpha(S_{0,n}) \cong Z_{n-1}$. By the induction hypothesis, the vertex v is connected to $v_s = \{\alpha_{s,x} \in \Gamma_n\}$ by a path in $Z_n \cap P_\alpha(S_{0,n})$. Therefore we conclude that Z_n is connected. □

Let

$$X_5 = Z_5 \cup \bigcup_{c \in \Gamma_5} T_c^{\pm\frac{1}{2}}(Z_5),$$

where $T_c^{\frac{1}{2}}$ is a simplicial map on $\mathcal{P}(S_{0,5})$ induced by the half-twist around the chain curve c; see figure 4.2.

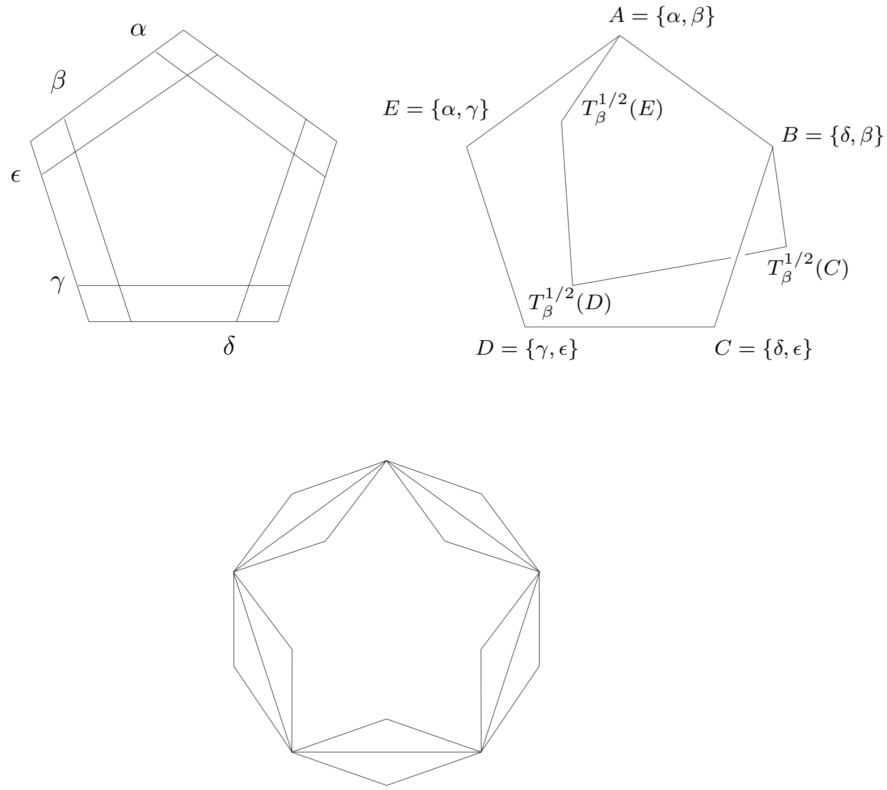

Figure 4.2: (top left)$S_{0,5}$ and simple closed curves in Γ_5, (top right)$Z_5 \cup T_\beta^{\frac{1}{2}}(Z_5)$ and (bottom)the thick pentagon $\widehat{Z_5}$.

We see that X_5 is a finite subgraph consisting of 11 alternating pentagons-one is Z_5 and the others 10 are the images of Z_5 under the twists. Each image of Z_5 shares an edge with Z_5, thus has two edges adjacent to Z_5 (in particular, X_5 is connected). These two edges from each of the 10 image pentagons form 10 triangles; each triangle has one of its edges in Z_5. In particular, each edge of Z_5 has two triangles attached to it. We call Z_5 **the core pentagon** of X_5 and call Z_5 together with these 10 triangles **the thick pentagon** $\widehat{Z_5}$ of X_5. See Figure 4.2. We also call any subgraph which is isomorphic to $\widehat{Z_5}$ a thick pentagon.

Lemma 4.1.3. If $Z \subset \mathcal{P}(S_{0,m})$ is an alternating pentagon then there exists a unique thick pentagon $\widehat{Z}$ containing Z and a unique subgraph $X \cong X_5$ containing $\widehat{Z}$.

Proof. Each edge of Z is contained in a unique Farey graph determined by the deficiency-1 multicurve corresponding to the intersection of its endpoints, (see [Mar04, Lemma 2] or [Ara10, Lemma 8]). Then the existence and uniqueness of $\widehat{Z}$ come from the fact that, in a Farey graph, each edge has exactly two triangles attached to it. For example, in Figure 4.2, the edge $\overline{AE} = \overline{\{\alpha, \beta\}\,\{\alpha, \gamma\}}$ has two triangles attaching to it and the remaining two vertices of the triangles are $T_\gamma^{\frac{1}{2}}(A) = T_\beta^{-\frac{1}{2}}(E)$ and $T_\beta^{\frac{1}{2}}(E) = T_\gamma^{-\frac{1}{2}}(A)$.

It remains to prove the existence and uniqueness of X. Since Z is an alternating pentagon, there exists a deficiency-2 multicurve Q such that $S_{0,m} - Q$ has exactly one nontrivial component $(S_{0,m} - Q)_0$ homeomorphic to $S_{0,5}$ and $Z \subset \mathcal{P}_Q(S_{0,m}) \cong \mathcal{P}((S_{0,m} - Q)_0) \cong \mathcal{P}(S_{0,5})$, [Ara10, Lemma 8]. Therefore we can write $Z = \{A = \{\alpha, \beta\} \cup Q, B = \{\delta, \beta\} \cup Q, ..., E = \{\alpha, \gamma\} \cup Q\}$; see figure 4.3. Write $\Gamma = \{\alpha, \beta, \delta, \epsilon, \gamma\}$. The subgraph

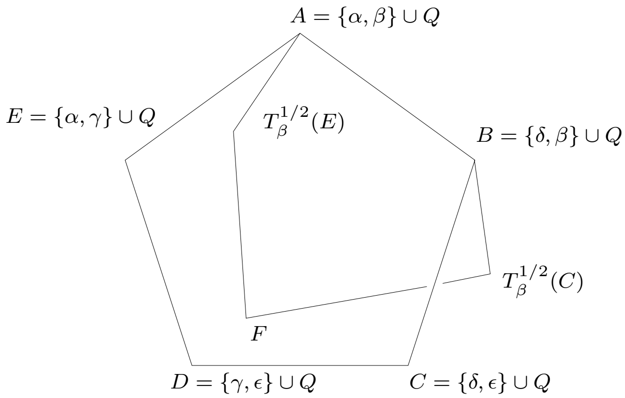

Figure 4.3: Z and two possible pictures of Y.

$$X = Z \cup \bigcup_{c \in \Gamma} T_c^{\pm\frac{1}{2}}(Z),$$

contains Z and $X \cong X_5$.

Next, let $X' \subset \mathcal{P}(S_{0,m})$ be a subgraph such that $Z \subset X'$ and $X' \cong X_5$. We will show that $X = X'$. Since $\widehat{Z}$ is the unique thick pentagon containing Z and $X' \cong X_5$, it follows that $\widehat{Z} \subset X'$. Let $Y \subset X'$ be a pentagon sharing only one of its edges with Z. Considering Figure 4.3, we assume without loss of generality that the shared edge is $\overline{AB}$ and $T_\beta^{\frac{1}{2}}(E)$ is a vertex of Y adjacent to $\overline{AB}$. Then another edge of Y adjacent to $\overline{AB}$ has either $T_\beta^{\frac{1}{2}}(C)$ or $T_\epsilon^{\frac{1}{2}}(B)$ as its endpoint. Let F be the remaining vertex of Y. We note that, as a pants decomposition, F must contain the multicurve Q. Then by direct calculations, if $T_\beta^{\frac{1}{2}}(C)$ is the endpoint, then the vertex $F = Q \cup \left\{ T_\beta^{\frac{1}{2}}(\gamma), T_\beta^{\frac{1}{2}}(\epsilon) \right\} = T_\beta^{\frac{1}{2}}(D)$ so $Y = T_\beta^{\frac{1}{2}}(Z)$, and if $T_\epsilon^{\frac{1}{2}}(B)$ is the endpoint, then there is no such Y since $i(T_\beta^{\frac{1}{2}}(\gamma), T_\epsilon^{\frac{1}{2}}(\beta)) = 4 > 0$. Hence any pentagon $Y \subset X'$ sharing only one edge with Z is $T_c^{\pm\frac{1}{2}}(Z)$ for some curve $c \in \Gamma$. We conclude that $X = X'$. This completes the proof. □

Let $\mathcal{G}$ be a subgraph of $\mathcal{P}(S_{0,n})$. We define **the thick graph** $\widehat{\mathcal{G}}$ for $\mathcal{G}$ to be the union of $\mathcal{G}$ with all triangles in $\mathcal{P}(S_{0,n})$ that share at least one of their edges with an edge in $\mathcal{G}$. If e is an edge, we call the thick graph $\hat{e}$ for e **the thick edge** which is the union of two triangles whose the common edge is e.

For a vertex $v \in Z_n$, let $\mathrm{st}(v)$ denote the closed star of v which is the union of all edges containing v. We define $\mathrm{st}_{Z_n}(v) = \mathrm{st}(v) \cap Z_n$, and let $\widehat{\mathrm{st}}_{Z_n}(v)$ be the thick graph for $\mathrm{st}_{Z_n}(v)$.

Lemma 4.1.4. Let v be a vertex of Z_n. Then $\mathrm{st}_{Z_n}(v)$ consists of $n-3$ edges. Moreover these $n-3$ edges are contained in $n-3$ distinct Farey graphs. Consequently, $\widehat{\mathrm{st}}_{Z_n}(v)$ consists of $n-3$ thick edges and these thick edges are contained in different Farey graphs.

Proof. Consider v as a pants decomposition on $S_{0,n}$. Then v contains $n-3$ simple closed curves. An edge which is adjacent to v corresponds to an elementary move. Forgetting a closed curve α in v gives a nontrivial component $(S_{0,n} - (v - \{\alpha\}))_0$ of $S_{0,n}$ homeomorphic to $S_{0,4}$. This nontrivial component contains two intersecting curves in Γ_n and one of these two curves is α. Hence there is only one elementary move which is able to change α to another curve in Γ_n. Forgetting two different curves in v gives two different nontrivial components homeomorphic to $S_{0,4}$. Therefore there are $n-3$ edges in Z_n adjacent to v and these edges are contained in $n-3$ different Farey graphs. The Lemma follows. □

Recall the description of $S_{0,5}$ as the double of a pentagon with vertices removed. Let $e : S_{0,5} \to S_{0,5}$ be the involution exchanging the two pentagons. We observe that the involution e induces a simplicial map on $\mathcal{P}(S_{0,5})$ whose the restriction to Z_5 is the identity and which restricts to a symmetry of X_5.

Lemma 4.1.5. Let $G = \mathrm{Sym}(X_5, Z_5)$ be the subgroup of the symmetry group $\mathrm{Sym}(X_5)$ of X_5 consisting of all elements that fix Z_5 pointwise. Then $G \cong \mathbb{Z}/2\mathbb{Z}$ generated by e.

Proof. Recall the thick pentagon $\widehat{Z_5}$ in Figure 4.2. There are 10 vertices of the triangles outside Z_5 and we number them as in Figure 4.4. Let V be the set of these 10 vertices. Each vertex in V

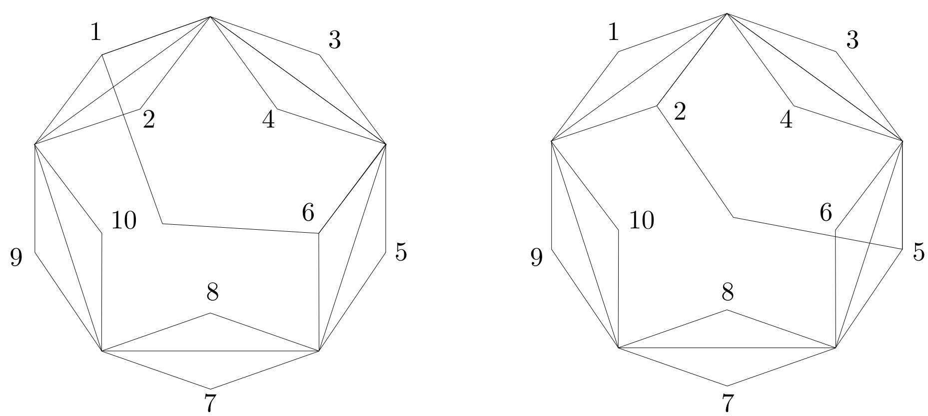

Figure 4.4: Thick pentagon $\widehat{Z_5}$ and the 10 labeled vertices.

is paired by two pentagons in X_5 to two vertices in V, for example, $\{1,6\}$ is a pair because 1 and 6 are in the same image pentagon shown in Figure 4.4. This forms 10 pairs of vertices, namely, $\{1,6\}, \{2,5\}, \{1,8\}, \{2,7\}, \{3,8\}, \{4,7\}, \{3,10\}, \{4,9\}, \{5,10\}, \{6,9\}$.

Let $\mathbb{S}_{\{x,y\}}$ be the symmetric group on two letters x and y. Note that $\mathbb{S}_{\{x,y\}} \cong \mathbb{Z}/2\mathbb{Z}$ generated by $\sigma_{\{x,y\}}$ which interchanges x and y.

Let $H = \mathbb{S}_{\{1,2\}} \times \mathbb{S}_{\{3,4\}} \times \mathbb{S}_{\{5,6\}} \times \mathbb{S}_{\{7,8\}} \times \mathbb{S}_{\{9,10\}} \cong (\mathbb{Z}/2\mathbb{Z})^5$. There is a natural injective homomorphism $\eta : G \to H$. Let g be a nonidentity element in G. Without loss of generality, we assume that $g(1) = 2$. By Lemma 4.1.3, g maps the pentagon in X_5 containing 1 and 6 to the pentagon in X_5 containing 2 and 5; see Figure 4.4. Thus $g(5) = 6$. By similar argument and the above observation of how vertices in V are paired to each others, we see that $\eta(g) = (\sigma_{\{1,2\}}, \sigma_{\{3,4\}}, \sigma_{\{5,6\}}, \sigma_{\{7,8\}}, \sigma_{\{9,10\}})$. Hence $\eta(G) \cong \mathbb{Z}/2\mathbb{Z}$ and so $G \cong \mathbb{Z}/2\mathbb{Z}$ generated by e. □

For $n > 5$, we construct X_n as follows. Let $W \subset \Gamma_n$ be a deficiency-2 multicurve such that the nontrivial component $(S_{0,n} - W)_0$ of $S_{0,n} - W$ is homeomorphic to $S_{0,5}$. By Lemma 4.1.1, we have a homeomorphism of pairs $h : (S_{0,5}, \Gamma_5) \to ((S_{0,n} - W)_0, \Gamma_5^W)$. Let

$$X_5^W = h^W(X_5) = \{h(u) \cup W \mid u \in X_5\},$$

where $h^W : P(S_{0,5}) \to P(S_{0,n})$ is the induced map of h as defined in Section 1.2 by $h^W(u) = h(u) \cup W$.

Finally we let

$$X_n = Z_n \cup \bigcup_W X_5^W,$$

where the union is taken over all deficiency-2 multicurves in Γ_n with a 5-puncture sphere component.

Lemma 4.1.6. For $n \geq 6, X_n$ has following properties.

1. $X_n \subset \mathcal{P}(S_{0,n})$ is connected.
2. For each chain curve α_i, $i = 1, ..., n$, let $X_{n-1}^i = X_n \cap \mathcal{P}_{\alpha_i}(S_{0,n})$. Then $X_{n-1}^i \cong X_{n-1}$. Moreover, this isomorphism is induced by $h : S_{0,n-1} \to (S_{0,n} - \alpha_i)_0$ as $h^{\alpha_i}(v) = h(v) \cup \alpha_i \in X_{n-1}^i$.
3. If $n \geq 7$ and α_i, α_j are disjoint chain curves then $X_{n-2}^{ij} = X_n \cap \mathcal{P}_{\{\alpha_i,\alpha_j\}}(S_{0,n}) \cong X_{n-2}$ with isomorphism $h^{\{\alpha_i,\alpha_j\}}$.

Proof. 1. Since the core pentagon of each X_5^W is in Z_n, X_5^W is connected to Z_n in X_n. By Lemma 4.1.2, Z_n is connected. Hence X_n is connected.

2. Fix a chain curve α_i. By Lemma 4.1.1, there is a homeomorphism of pairs $h : (S_{0,n-1}, \Gamma_{n-1}) \to ((S_{0,n} - \alpha_i)_0, \Gamma_{n-1}^{\alpha_i})$ and h induces an isomorphism from Z_{n-1} to $Z_n \cap \mathcal{P}_{\alpha_i}(S_{0,n})$. Moreover, for every deficiency-2 multicurve $W_0 \subset \Gamma_{n-1}$, $W = h(W_0) \cup \alpha_i$ is a deficiency-2 multicurve in Γ_n and h induces an isomorphism from $X_5^{W_0}$ to X_5^W. Therefore h induces a simplicial injection from X_{n-1} to $X_n \cap \mathcal{P}_{\alpha_i}(S_{0,n})$. We need to show that this is surjective.

Let W be a deficiency-2 multicurve in Γ_n with $(S_{0,n} - W)_0 \cong S_{0,5}$. If $i(\alpha_i, W) \neq 0$, then $X_5^W \cap \mathcal{P}_{\alpha_i}(S_{0,n}) = \emptyset$. For the rest of the proof we assume that $i(\alpha_i, W) = 0$.

If $\alpha_i \in W$, then $W_0' = W - \alpha_i$ is a deficiency-2 multicurve in $(S_{0,n} - \alpha_i)_0$. Setting $W_0 = h^{-1}(W_0')$, we have $W = h(W_0) \cup \alpha_i$. Thus X_5^W is the image of $X_5^{W_0}$ by h^{α_i}.

Suppose $\alpha_i \notin W$. Then α_i is contained in $(S_{0,n} - W)_0$, moreover $\alpha_i \in \Gamma_5^W$. Hence, by the definition of X_5^W, $X_5^W \cap \mathcal{P}_{\alpha_i}(S_{0,n}) \neq \emptyset$. Let u be a vertex in $X_5^W \cap \mathcal{P}_{\alpha_i}(S_{0,n})$. Consider u as a pants decomposition. We claim that there is a deficiency-2 multicurve W'' such that $\alpha_i \in W''$, $u \in X_5^{W''} \cap \mathcal{P}_{\alpha_i}(S_{0,n})$, and so from the case that $\alpha_i \in W''$, we see that u is in the image of h^{α_i}, as required. We prove the claim as follows. Since $u \in X_5^W \cap \mathcal{P}_{\alpha_i}(S_{0,n})$ and $\alpha_i \notin W$, $u = W \cup \{\alpha_i\} \cup \{x\}$ for some simple closed curve $x = y$ or $T_\beta^{\pm\frac{1}{2}}(y)$, with $y, \beta \in \Gamma_n^W$. Since $S_{0,n}$ has complexity at least 3 and α_i is a chain curve, there exists a close curve $\gamma \in W$ such that $W'' = (W - \{\gamma\}) \cup \{\alpha_i\}$ is a deficiency-2 multicurve with nontrivial component $(S_{0,n} - W'')_0 \cong S_{0,5}$. Then $u = W'' \cup \{\gamma\} \cup \{x\} \in X_5^{W''} \cap \mathcal{P}_{\alpha_i}(S_{0,n})$ as desired.

3. The statement is proved by applying 2 twice. □

4.2 The proof for $S_{0,5}$

We prove Theorem 1.2.7 for $n = 5$.

Lemma 4.2.1. Let $X_5 \subset \mathcal{P}(S_{0,5})$ be as above. Then for any punctured sphere $S_{0,m}$ and any injective simplicial map

$$\phi : X_5 \to \mathcal{P}(S_{0,m}),$$

there exists a π_1-injective embedding $f : S_{0,5} \to S_{0,m}$ unique up to isotopy that induces ϕ.

Proof. We show that ϕ maps the core pentagon Z_5 of the thick pentagon $\widehat{Z_5}$ in X_5 to an alternating pentagon in $\mathcal{P}(S_{0,m})$.

Let F be a Farey graph in $\mathcal{P}(S_{0,m})$. We claim that ϕ cannot map any three consecutive edges of Z_5 into F. To find a contradiction, we assume that ϕ maps three consecutive edges $\overline{ABCD}$ of Z_5 into F. Since ϕ is injective, ϕ maps all six triangles attaching to these edges to distinct triangles in F. Up to an automorphism of F, the image of the six triangles must be one of the two pictures in Figure 4.5. This implies that the distance of $\phi(A)$ and $\phi(D)$ in F is greater than the distance of $\phi(A)$ and $\phi(D)$ in $\mathcal{P}(S_{0,m})$, that is,

$$\mathrm{dist}_F(\phi(A), \phi(D)) = 3 > 2 = \mathrm{dist}_{\phi(X_5)}(\phi(A), \phi(D)) \geq \mathrm{dist}_{\mathcal{P}(S_{0,m})}(\phi(A), \phi(D)).$$

This is a contradiction to the fact, proven in [APS08], that F is isometrically embedded.

Next we claim that ϕ cannot map any two adjacent edges of Z_5 into F. To find a contradiction, we assume that ϕ maps $\overline{ABC}$ into F. The injectivity of ϕ implies that the two edges $\overline{\phi(A)\phi(B)}$ and $\overline{\phi(B)\phi(C)}$ are separated by at least three triangles in F. Let W be the deficiency-1 multicurve defining F, so that $\phi(A) = W \cup \{a\}$, $\phi(B) = W \cup \{b\}$. Then $\phi(C) = W \cup \left\{T_b^k(a)\right\}, k \geq 3/2$; see figure 4.6. By the previous claim the edges $\overline{\phi(A)\phi(E)}$ and $\overline{\phi(C)\phi(D)}$ are not in F. Then $\phi(E) = W' \cup \{a\}$, $\phi(D) = W'' \cup \left\{T_b^k(a)\right\}$; see figure 4.6. Since $i(a, T_b^k(a)) > 2$ when $k \geq 3/2$, $i(\phi(E), \phi(D)) > 2$. Hence there is no elementary move from $\phi(D)$ to $\phi(E)$; i.e. $\phi(D)$ and $\phi(E)$ are not connected by an edge. This is a contradiction, and so we conclude that $\phi(Z_5)$ is an alternating pentagon.

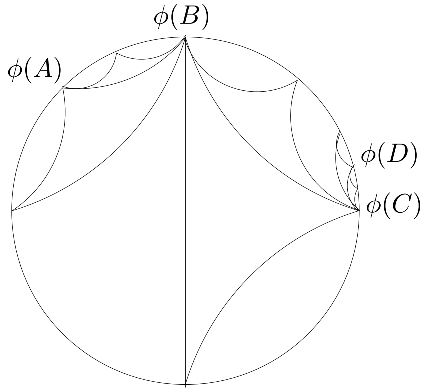

Figure 4.5: Possible images in F of three consecutive edges of $\hat{Z}_5$ and triangles of $\hat{Z}_5$ adjacent to them.

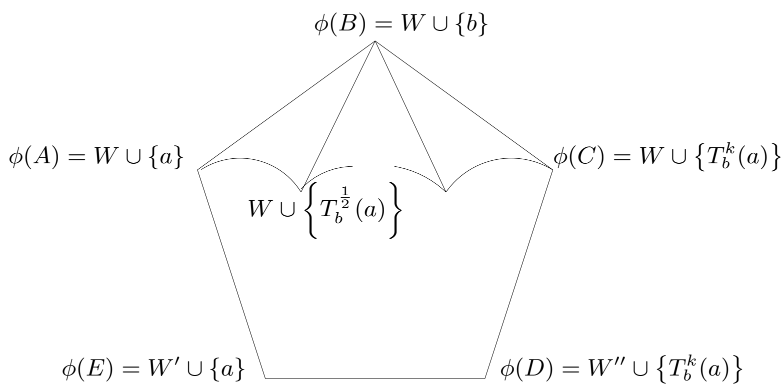

Figure 4.6: Image of two adjacent edges of Z_5 in F.

By Lemma 2.3.2, there exists a deficiency-2 multicurve Q and a homeomorphism $f : S_{0,5} \to (S_{0,m} - Q)_0$ to a component of $S_{0,m} - Q$ such that $\phi|_{Z_5} = f^Q|_{Z_5}$. Moreover, f is unique up to precomposing by e (and isotopy) since the pointwise stabilizer of Z_5 in $\mathrm{Mod}(S_{0,5})$ is generated by e. So $f^Q(X_5)$ is a subgraph of $\mathcal{P}(S_{0,m})$ containing $f^Q(Z_5) = \phi(Z_5)$. By Lemma 4.1.3, $f^Q(X_5) = \phi(X_5)$. Since $(f^Q)^{-1} \circ \phi$ is the identity map on Z_5, Lemma 4.1.5 implies that either $(f^Q)^{-1} \circ \phi = \mathrm{id}_{X_5}$ or $e \circ (f^Q)^{-1} \circ \phi = \mathrm{id}_{X_5}$. Hence $(f^Q \circ e)_{|X_5} = (f \circ e)^Q_{|X_5} = \phi$ or $f^Q{}_{|X_5} = \phi$. In either case ϕ is induced by an embedding. □

4.3 The general case

We now proceed to the proof of Theorem 1.2.7 in the general case.

Proof of Theorem 1.2.7. We prove the theorem by induction on n.

Lemma 4.2.1 proves the base case when $n = 5$.

Suppose the theorem is true for an $n \geq 5$. Consider an injective simplicial map $\phi : X_{n+1} \to \mathcal{P}(S_{0,m})$. For each chain curve α_i in $S_{0,n+1}, i = 1, ..., n+1$, recall $X_n^i = X_{n+1} \cap \mathcal{P}_{\alpha_i}(S_{0,n+1})$. By Lemma 4.1.6, $X_n^i \cong X_n$ and $X_n^i \subset \mathcal{P}_{\alpha_i}(S_{0,n+1}) \cong \mathcal{P}((S_{0,n+1} - \alpha_i)_0) \cong \mathcal{P}(S_{0,n})$. Given a vertex $u \in X_n^i$, consider u as a pants decomposition of $S_{0,n+1}$ so that $u_{\alpha_i} = u - \{\alpha_i\}$ is a pants decomposition of $(S_{0,n+1} - \alpha_i)_0$. Define an injective simplicial map

$$\phi_i : X_n^i \to \mathcal{P}(S_{0,m})$$

by $\phi_i(u) = \phi(u)$. By the induction hypothesis, there is an embedding $f_i : (S_{0,n+1} - \alpha_i)_0 \to S_{0,m}$ unique up to isotopy such that f_i induces ϕ_i, i.e., f_i has the following properties;

1. there is a deficiency-$(n-3)$ multicurve Q_i in $S_{0,m}$ such that $S_{0,m} - Q_i$ has only one nontrivial component $(S_{0,m} - Q_i)_0 \cong S_{0,n}$ and $f_i((S_{0,n+1} - \alpha_i)_0) = (S_{0,m} - Q_i)_0$,
2. the simplicial map $f_i^{Q_i} : \mathcal{P}_{\alpha_i}(S_{0,n+1}) \to \mathcal{P}(S_{0,m})$ defined by

$$f_i^{Q_i}(u) = f_i(u_{\alpha_i}) \cup Q_i$$

   satisfies $f_i^{Q_i}|_{X_n^i} = \phi_i$.

Under all the hypotheses above, we prove the following three lemmas.

Lemma 4.3.1. If α_i and α_j are disjoint chain curves then $i(Q_i, Q_j) = 0$.

Proof. Since $n+1 \geq 6$, $(S_{0,n+1} - \{\alpha_i, \alpha_j\})_0 \cong S_{0,n-1}$ with $n-1 \geq 4$. Since $\{\alpha_i, \alpha_j\} \subset \Gamma_{n+1}$, $Z_{n+1} \cap \mathcal{P}_{\{\alpha_i,\alpha_j\}}(S_{0,n+1}) \neq \emptyset$ and contains an edge e. Since e is an edge in

$$\mathcal{P}_{\{\alpha_i,\alpha_j\}}(S_{0,n+1}) = \mathcal{P}_{\{\alpha_i\}}(S_{0,n+1}) \cap \mathcal{P}_{\{\alpha_j\}}(S_{0,n+1}),$$

$\phi(e) = \phi_i(e) \subset \mathcal{P}_{Q_i}(S_{0,m})$ and $\phi(e) = \phi_j(e) \subset \mathcal{P}_{Q_j}(S_{0,m})$. Thus

$$\mathcal{P}_{Q_i}(S_{0,m}) \cap \mathcal{P}_{Q_j}(S_{0,m}) \neq \emptyset.$$

Hence $i(Q_i, Q_j) = 0$. □

Lemma 4.3.2. If $\alpha_i, \alpha_j, \alpha_k$ are pairwise disjoint chain curves and $u \in Z_{n+1} \cap \mathcal{P}_{\{\alpha_i,\alpha_j,\alpha_k\}}(S_{0,n+1})$ is any vertex, then the $n-2$ thick edges in $\widehat{\mathrm{st}}_{Z_{n+1}}(u)$ from Lemma 4.1.4 are mapped into $n-2$ distinct Farey graphs by ϕ.

Proof. Lemma 4.1.4 shows that the $n-2$ thick edges in $\widehat{\mathrm{st}}_{Z_{n+1}}(u)$ are contained in distinct Farey graphs. Then $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_i}(S_{0,n+1})$ contains $n-3$ thick edges and $f_i^{Q_i}$ maps the $n-3$ distinct Farey graphs containing these thick edges to distinct Farey graphs in $\mathcal{P}_{Q_i}(S_{0,m})$. The same is true if i is replaced by j or k.

Now for any two thick edges $\hat{e}_1$ and $\hat{e}_2$ in $\widehat{\mathrm{st}}_{Z_{n+1}}(u)$, $\hat{e}_1$ and $\hat{e}_2$ are both contained in at least one of $\mathcal{P}_{\alpha_i}(S_{0,n+1})$, $\mathcal{P}_{\alpha_j}(S_{0,n+1})$ or $\mathcal{P}_{\alpha_k}(S_{0,n+1})$. Assume that $\hat{e}_1$ and $\hat{e}_2$ are contained in $\mathcal{P}_{\alpha_i}(S_{0,n+1})$, then $f_i^{Q_i}(\hat{e}_1) = \phi(\hat{e}_1)$ and $f_i^{Q_i}(\hat{e}_2) = \phi(\hat{e}_2)$ are in different Farey graphs. □

Lemma 4.3.3. If $\alpha_i, \alpha_j, \alpha_k$ are pairwise disjoint chain curves then $Q_i \neq Q_j \neq Q_k \neq Q_i$.

Proof. Suppose $Q_i = Q_j$. Let u be a vertex in $Z_{n+1} \cap \mathcal{P}_{\{\alpha_i,\alpha_j,\alpha_k\}}(S_{0,n+1})$.

As in the previous proof, $f_i^{Q_i}$ and $f_j^{Q_j}$ map the $n-3$ thick edges of $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_i}(S_{0,n+1})$ and $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_j}(S_{0,n+1})$ into $n-3$ thick edges in $\mathcal{P}_{Q_i}(S_{0,m})$ and $\mathcal{P}_{Q_j}(S_{0,m}) = \mathcal{P}_{Q_i}(S_{0,m})$, respectively. There are $n-4$ thick edges in $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_i}(S_{0,n+1}) \cap \mathcal{P}_{\alpha_j}(S_{0,n+1})$ and $f_i^{Q_i}$ agrees with $f_j^{Q_j}$ on these thick edges because $f_i^{Q_i}(v) = \phi(v) = f_j^{Q_j}(v)$, for any vertex $v \in X_n^i \cap X_n^j$. Let $\widehat{e_i}$ be the thick edge in $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_i}(S_{0,n+1})$ but not in $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_j}(S_{0,n+1})$ and let $\widehat{e_j}$ be the thick edge in $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_j}(S_{0,n+1})$ but not in $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_i}(S_{0,n+1})$. Since Q_i is a deficiency-$(n-3)$ multicurve, there are $n-3$ Farey graphs in $\mathcal{P}_{Q_i}(S_{0,m}) = \mathcal{P}_{Q_j}(S_{0,m})$ that contain $f_i^{Q_i}(u) = f_j^{Q_j}(u)$. However, $f_i^{Q_i}(\widehat{e_i})$, $f_j^{Q_j}(\widehat{e_j})$ and the $n-4$ thick edges in $\widehat{\mathrm{st}}_{X_{n+1}}(u) \cap \mathcal{P}_{\alpha_i}(S_{0,n+1}) \cap \mathcal{P}_{\alpha_j}(S_{0,n+1})$ above all map into distinct Farey graphs by Lemma 5.2. Since this is $n-2 > n-3$, we obtain a contradiction. □

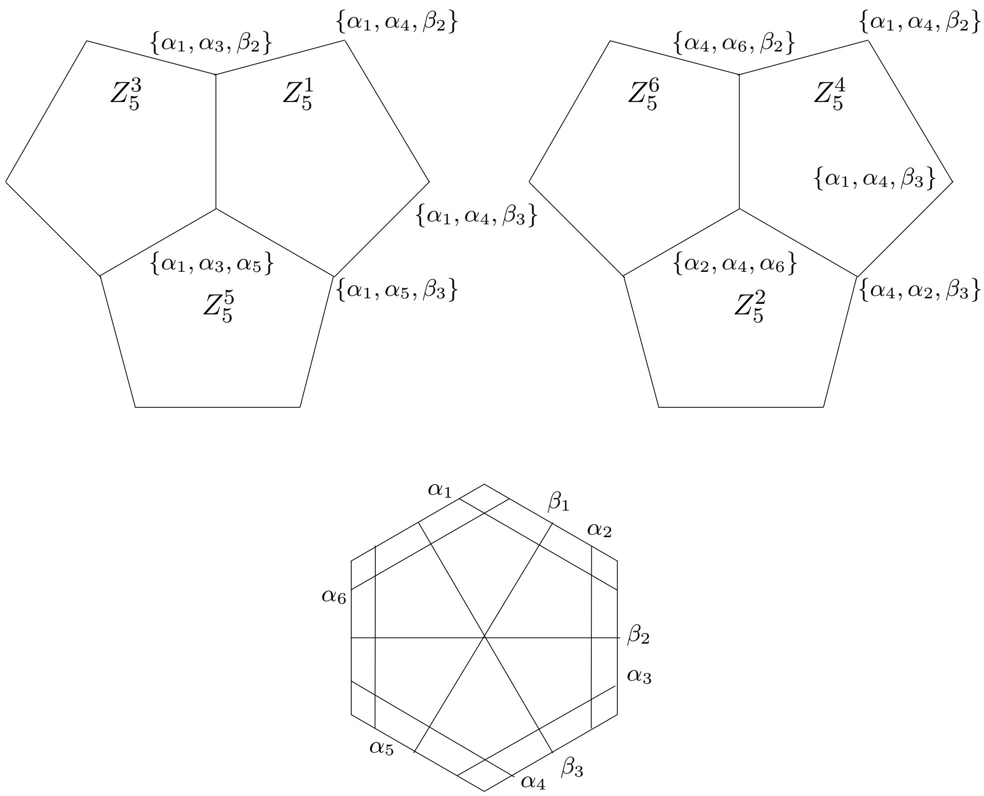

Figure 4.7: (upper) Z_6 as a union of $Z_5^1 \cup Z_5^3 \cup Z_5^5$ on the left and $Z_5^2 \cup Z_5^4 \cup Z_5^6$ on the right, the two subgraphs share three edges as labelled; (lower) $S_{0,6}$ with curves in Γ_6.

With the hypothesis as in Lemma 4.3.3, we must have

$$Q_i \cap Q_j = Q_i \cap Q_k = Q_j \cap Q_k = Q_i \cap Q_j \cap Q_k = Q,$$

Q has deficiency $n-2$ and there is a connected complexity $n-2$ component $(S_{0,m} - Q)_0$.

The proofs are different for $n+1 = 6$ and $n+1 \geq 7$. We first prove the theorem for $S_{0,n+1} = S_{0,6}$. We label the six chain curves by α_i for $i = 1, ..., 6$ and the three non-chain curves in Γ_6 by β_i for $i = 1, 2, 3$ as in Figure 4.7; also see the figure for the picture of Z_6 as a union of $Z_5^1 \cup Z_5^3 \cup Z_5^5$ and $Z_5^2 \cup Z_5^4 \cup Z_5^6$, where $Z_5^i = Z_6 \cap \mathcal{P}_{\alpha_i}(S_{0,6}) = X_5^i \cap Z_6$. Note that, in this case, Q_i is a deficiency-2 multicurve for all $i = 1, ..., 6$.

Let $i \neq j \in \{1, 3, 5\}$ or $i \neq j \in \{2, 4, 6\}$. Define $F_{i,j} : S_{0,6} \to S_{0,m}$ by

$$F_{i,j} = \begin{cases} f_i & \text{on } (S_{0,6} - \alpha_i)_0 \\ f_j & \text{on } (S_{0,6} - \alpha_j)_0 \end{cases}.$$

We claim that f_i and f_j agree on $(S_{0,6} - \{\alpha_i, \alpha_j\})_0$ so $F_{i,j}$ is well-defined. We prove the claim in the case when $(i, j) = (1, 5)$. For other cases the proof is similar.

X_5^1 and X_5^5 share one thick edge $\hat{e}$ having endpoints $\{\alpha_1, \alpha_5, \alpha_3\}$ and $\{\alpha_1, \alpha_5, \beta_3\}$. Moreover, $f_1^{Q_1}$ agrees with $f_5^{Q_5}$ on $\hat{e}$ as they are both equal to ϕ. Then we have that $f_1^{Q_1}$ and $f_5^{Q_5}$ agree on $\mathcal{P}_{\alpha_1}(S_{0,6}) \cap \mathcal{P}_{\alpha_5}(S_{0,6}) = \mathcal{P}_{\{\alpha_1,\alpha_5\}}(S_{0,6}) \cong \mathcal{P}(S_{0,4})$. Hence

$$f_1((S_{0,6} - \{\alpha_1, \alpha_5\})_0) = f_5((S_{0,6} - \{\alpha_1, \alpha_5\})_0),$$

and $f_5^{-1} \circ f_1$ is either the identity or one of the three hyperelliptic involutions on $(S_{0,6} - \{\alpha_1, \alpha_5\})_0$; we will show that the latter are not possible. Since $\alpha_1, \alpha_3, \alpha_5$ are pairwise disjoint chain curves, Lemma 4.3.1 and Lemma 4.3.3 shows that, for any $i \neq j \in \{1, 3, 5\}$, $i(Q_i, Q_j) = 0$ and $Q_i \neq Q_j$. Hence, for any $i \neq j \in \{1, 3, 5\}$, we have $\mathrm{def}(Q_i \cup Q_j) = 1$ and the symmetric difference $Q_i \triangle Q_j$ contains two simple closed curves. Let

$$q_1 \in Q_1 - (Q_3 \cup Q_5), \quad q_5 \in Q_5 - (Q_1 \cup Q_3), \quad q_3 \in Q_3 - (Q_1 \cup Q_5),$$

which are three distinct simple closed curves on $S_{0,m}$. We note that q_3 is a closed curve on $(S_{0,m} - (Q_1 \cup Q_5))_0 \cong S_{0,4}$ and q_3 cannot separate q_1 and q_5 since q_1 and q_5 are curves on $(S_{0,m} - Q_3)_0 \cong S_{0,5}$; see figure 4.8.

Since $f_1^{Q_1}$ and $f_5^{Q_5}$ agree on $\mathcal{P}_{\{\alpha_1,\alpha_5\}}(S_{0,6}) \cong \mathcal{P}(S_{0,4})$, f_1 and f_5 map any simple closed curves on $(S_{0,6} - \{\alpha_1, \alpha_5\})_0$ to simple closed curves on $(S_{0,m} - (Q_1 \cup Q_5))_0$. Thus $f_1(\alpha_5)$ is a curve in $(S_{0,m} - (Q_1))_0$ but not in $(S_{0,m} - (Q_1 \cup Q_5))_0$, and so $f_1(\alpha_5) = q_5$. Similarly $f_5(\alpha_1) = q_1$ and

$$f_1(\alpha_3) = q_3 = f_5(\alpha_3).$$

Therefore $f_1(\alpha_1) = q_1$ and $f_5^{-1} \circ f_1(\alpha_1) = \alpha_1$. We conclude that $f_5^{-1} \circ f_1$ is the identity on the

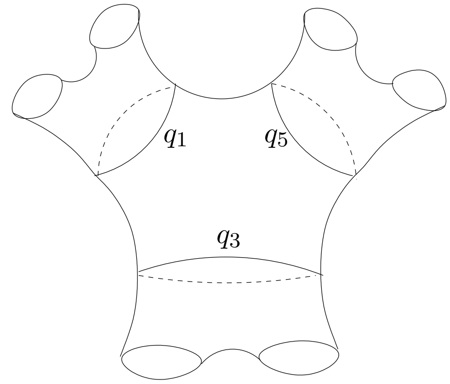

Figure 4.8: $f_1((S_{0,6}-\alpha_1)_0) \cup f_5((S_{0,6}-\alpha_5)_0) = (S_{0,m}-Q)_0$, together with the three curves q_1, q_3, q_5.

boundary of $(S_{0,6} - \{\alpha_1, \alpha_5\})_0$ hence cannot be one of the hyperelliptic involutions. So f_1 and f_5 agree on $(S_{0,6} - \{\alpha_1, \alpha_5\})_0$.

Observe that $F_{1,5} = F_{1,3} = F_{3,5}$ and $F_{2,4} = F_{2,6} = F_{4,6}$. For instance, to see that $F_{1,5} = F_{1,3}$, we note that f_3 agrees with f_5 on $(S_{0,6} - \{\alpha_3, \alpha_5\})_0$.

Define the common maps

$$f_{odd} = F_{1,5} = F_{1,3} = F_{3,5} \text{ and } f_{even} = F_{2,4} = F_{2,6} = F_{4,6}.$$

We have that f_{odd} induces the restriction map of ϕ on $X_5^1 \cup X_5^3 \cup X_5^5$ and f_{even} induces the restrict map of ϕ on $X_5^2 \cup X_5^4 \cup X_5^6$.

We show that $(S_{0,m} - Q_{odd})_0 = (S_{0,m} - Q_{even})_0$. Let $Q_{odd} = Q_1 \cap Q_3 \cap Q_5$ and $Q_{even} = Q_2 \cap Q_4 \cap Q_6$. As observed after Lemma 4.3.3, Q_{odd} and Q_{even} are deficiency-3 multicurves and

$$f_{odd}(S_{0,6}) = (S_{0,m} - Q_{odd})_0 \text{ and } f_{even}(S_{0,6}) = (S_{0,m} - Q_{even})_0.$$

Since $f_i^{Q_i}$ and $f_j^{Q_j}$ agree on the thick edge $\hat{e}_{i,j} \in X_5^i \cap X_5^j$ when $(i,j) \in \{(1,4),(3,6),(2,5)\}$,

$$f_{odd}^{Q_{odd}}(\hat{e}_{i,j}) = f_{even}^{Q_{even}}(\hat{e}_{i,j}) \in \mathcal{P}_{Q_{odd}}(S_{0,m}) \cap \mathcal{P}_{Q_{even}}(S_{0,m}) \neq \emptyset.$$

Hence $i(Q_{odd}, Q_{even}) = 0$. Moreover we have that $f_{odd}(\beta_k) = f_{even}(\beta_k)$, for each non-chain curve $\beta_k, k = 1,2,3$ since β_k's are interchanged in an elementary move that defines the edge $e_{i,j}$. The union $\bigcup_{k=1}^3 f_{odd}(\beta_k) = \bigcup_{k=1}^3 f_{even}(\beta_k)$ fills $(S_{0,m}-Q_{odd})_0$ and $(S_{0,m}-Q_{even})_0$. Since $(S_{0,m}-Q_{odd})_0$ is the unique subsurface filled by $\bigcup_{k=1}^3 f_{odd}(\beta_k)$, we conclude that $(S_{0,m}-Q_{odd})_0 = (S_{0,m}-Q_{even})_0$.

Next, we show that $f_{odd} = f_{even}$. Recall that we are considering $S_{0,6}$ as the double of a hexagon with vertices removed. Let r and e be homeomorphisms on $S_{0,6}$ induced by rotating the hexagon by π and exchanging the hexagons, respectively. Since $f_{odd}^{-1} \circ f_{even}$ preserves each non-chain curve $\beta_k, k = 1,2,3$, on $S_{0,6}$, $f_{odd}^{-1} \circ f_{even}$ is either the identity map, r, e, or $r \circ e$. We will show that only the first case is possible. The homeomorphism r induces a simplicial map on $P(S_{0,6})$ which exchanges the vertices $\{\alpha_1, \alpha_3, \alpha_5\}$ and $\{\alpha_2, \alpha_4, \alpha_6\}$. If $f_{odd}^{-1} \circ f_{even} = r$ then $\phi(\{\alpha_1, \alpha_3, \alpha_5\}) = \phi(\{\alpha_2, \alpha_4, \alpha_6\})$ which is a contradiction to the fact that ϕ is injective. Hence $f_{odd}^{-1} \circ f_{even} \neq r$. Since e and $r \circ e$ reverse orientation on each $\beta_i, i = 1, 2, 3$, they do not induce the identity map on the thick edge $\hat{e}_{i,j}, (i, j) \in \{(1, 4), (3, 6), (2, 5)\}$. So $f_{odd}^{-1} \circ f_{even} \neq e$ or $r \circ e$ because $f_{odd}^{-1} \circ f_{even}$ induces the identity map on those thick edges. Therefore $f_{odd}^{-1} \circ f_{even}$ is the identity map on $S_{0,6}$. We conclude that $f_{odd} = f_{even}$.

Let $f = f_{odd} = f_{even}$ and $Q = Q_{odd} = Q_{even}$. We show that $f : S_{0,6} \to S_{0,m}$ is the unique π_1–injective embedding with $\phi = f^Q$. The proof follows the same idea as in the proof of $f_{odd} = f_{even}$. So we give a brief explanation here. Suppose $h : S_{0,6} \to S_{0,m}$ is a π_1-injective embedding that induces $h^W = \phi$ for some deficiency-3 multicurve W on $S_{0,m}$. Since $f^Q = \phi = h^W$, $i(Q, W) = 0$ moreover $f(\beta_i) = h(\beta_i)$ for each non-chain curve $\beta_i, i = 1, 2, 3$. Since $(S_{0,m} - Q)_0$ and $(S_{0,m} - W)_0$ are unique subsurfaces filled by $\bigcup_{i=1}^3 f(\beta_i) = \bigcup_{i=1}^3 h(\beta_i)$, $(S_{0,m} - Q)_0 = (S_{0,m} - W)_0$ and hence $W = Q$. Since $h^{-1} \circ f$ preserves each non-chain curve on $S_{0,6}$ and induces the identity map on X_6, $h^{-1} \circ f$ is the identity map. Therefore $f = h$.

Next we prove the theorem for $S_{0,n+1}, n + 1 \geq 7$.
We define an embedding of $S_{0,n+1}$ to $S_{0,m}$ that induces ϕ. Let α_i and α_j be two disjoint chain curves on $S_{0,n+1}$. Define a homeomorphism $F_{ij} : S_{0,n+1} \to S_{0,m}$ by

$$F_{ij} = \begin{cases} f_i & \text{on } (S_{0,n+1} - \alpha_i)_0 \\ f_j & \text{on } (S_{0,n+1} - \alpha_j)_0 \end{cases} . \tag{4.1}$$

Note that, by Lemma 4.3.1 and 4.3.3, $i(Q_i, Q_j) = 0$ and $Q_i \neq Q_j$. We show that f_i agrees with f_j on $(S_{0,n+1} - \{\alpha_i, \alpha_j\})_0 \cong S_{0,n-1}$. Consider the restrictions of f_i and f_j as embeddings on $(S_{0,n+1} - \{\alpha_i, \alpha_j\})_0$. Let $X_{n-1}^{ij} = X_{n+1} \cap \mathcal{P}_{\{\alpha_i, \alpha_j\}}(S_{0,n+1}) \cong X_{n-1}$. Observe that

$$f_i^{Q_i}(v) = \phi(v) = f_j^{Q_j}(v),$$

for any vertex $v \in X_{n-1}^{ij}$. That is, both f_i and f_j induce the simplicial map $\phi_{ij} : X_{n-1}^{ij} \to \mathcal{P}(S_{0,m})$ defined by $\phi_{ij}(v) = \phi(v)$. Then the uniqueness statement in the induction hypothesis (which applies since $n - 1 \geq 5$) implies that f_i agrees with f_j on $(S_{0,n+1} - \{\alpha_i, \alpha_j\})_0$.

Lemma 4.3.4. For any four chain curves $\alpha_i, \alpha_j, \alpha_k$, and α_l such that $i(\alpha_i, \alpha_j) = 0 = i(\alpha_k, \alpha_l)$, we have $F_{ij} = F_{kl}$.

Proof. Consider a graph which has the set of vertices

$$V = \{\{i, j\} | \alpha_i, \alpha_j \text{ are disjoint chain curves on } S_{0,n+1}\},$$

and the set of edges

$$E = \{\{\{i, j\}, \{i, k\}\} | i(\alpha_j, \alpha_k) = 0\}.$$

Fix any two vertices $\{i, j\}$ and $\{k, l\}$. Since $n + 1 \geq 7$, it is not hard to see that $\{i, j\}$ and $\{k, l\}$

are connected by a path of length at most 4. Hence this graph is connected.

The Lemma now follows by proving that $F_{ij} = F_{ik}$ for $i(\alpha_i, \alpha_j) = 0 = i(\alpha_i, \alpha_k)$. We show that for any pair $(x, y), x \neq y \in \{i, j, k\}$, f_x agrees with f_y on $(S_{0,n+1} - \{\alpha_x, \alpha_y\})_0$. Since $(S_{0,n+1} - \{\alpha_x, \alpha_y\})_0$ has at least 5 punctures and $f_x^{Q_x}(v) = f_y^{Q_y}(v)$ for any vertex $v \in X_{n-1}^{xy}$, the uniqueness statement in the induction hypothesis implies that f_x agrees with f_y on $(S_{0,n+1} - \{\alpha_x, \alpha_y\})_0$. Hence $F_{ij} = F_{ik}$ as desired. □

Finally, we show that for any i, j with $i(\alpha_i, \alpha_j) = 0$, then $Q = Q_i \cap Q_j$ and $f = F_{ij} : S_{0,n+1} \to S_{0,m}$ is the unique π_1-injective embedding that induces $\phi = f^Q$. To show that f induces ϕ, given a vertex $v \in X_{n+1}$. $v \in X_{n-1}^{pk} = X_{n+1} \cap \mathcal{P}_{\{\alpha_p, \alpha_k\}}(S_{0,n+1})$ for some disjoint chain curves α_p, α_k. Then $Q = Q_i \cap Q_j = Q_p \cap Q_k$ and

$$f^Q(v) = F_{pk}^Q(v) = f_p^{Q_p}(v) = \phi(v),$$

The first equality comes from Lemma 4.3.4, the second equality comes from equation 4.1, and the third equality comes from the inductive hypothesis. Hence f induces ϕ.

For uniqueness, assume that there is a π_1-injective embedding $h : S_{0,n+1} \to S_{0,m}$ that induces $h^W = \phi$ for some deficiency-$n-2$ multicurve W on $S_{0,m}$. Let h_i and h_j be restrictions of h on $(S_{0,n+1} - \alpha_i)_0$ and $(S_{0,n+1} - \alpha_j)_0$, respectively. By assumption, h_i induces the same simplicial map on X_n^i as $f_i^{Q_i}$. Hence the uniqueness statement in the induction hypothesis implies that $f_i = h_i$ and $Q_i = W_i$. Similarly, we have $f_j = h_j$ and $Q_j = W_j$. Therefore $f = h$ and $Q = W$ as desired. □

4.4 Final discussion

One of the main difficulties in proving Theorem 1.2.7 is to construct the finite subgraphs X_n. To do this we can look for a candidate subgraph which, under additional hypothesis on the simplicial map, allows us to construct the embedding of the surface. For example, we have the condition on the simplicial map of Z_5 in Lemma 2.3.2. We then need to enlarge the candidate subgraph so that those extra conditions are encoded in the enlarged subgraph. But then another problem might arise which is that the induced map of the embedding that works on the original candidate subgraph might not control the added parts in the enlarged subgraph. For example, a simplicial embedding of the thick pentagon $\widehat{Z_5}$ ensures that the simplicial map restricted to Z_5 satisfies Lemma 2.3.2 and hence there is a candidate embedding of $S_{0,5}$. But the induced map may not agree with the simplicial map on $\widehat{Z_5} - Z_5$. Then we have to enlarge the subgraph further which might cause more problem.

It seems likely that Theorem 1.2.7 should be true for essentially any surface S. However it is unclear how to choose a subgraph $X \subset P(S)$.

Appendix A

We explain how to produce a list Ω of 4320 words in $\left\{a^{\pm 1}, b^{\pm 1}, x^{\pm 1}\right\}$ that contains all curves satisfying the following three conditions (from the proof of Lemma 3.0.5);

1. γ consists of exactly two a's, one a^{-1}, one x, and one x^{-1},

2. $\sharp\left\{b^{-1}\text{'s in } \gamma\right\} = \sharp\left\{b\text{'s in } \gamma\right\} - 1$, and

3. $\sharp\left\{b_1 \text{ edges of } \gamma\right\} + \sharp\left\{b_2 \text{ edges of } \gamma\right\} = 8$.

All combinations of x's and b's that provide at least one cancellation in the sum of b_1 and b_2-edge count are xb, $b^{-1}x$, $b^{-1}xb$, $x^{-1}b$, $b^{-1}x^{-1}$, and $b^{-1}x^{-1}b$. We use these combinations and condition 1 to make the table below and use conditions 2 and 3 to fill b or b^{-1} in the cells. For example, the case of row 1, column 2, we need to consider words constructed from $\left\{a, a, a^{-1}, x, x^{-1}b, b, b^{-1}\right\}$. Note that we leave a cell blank if it is impossible to fill in b or b^{-1}.

	x^{-1}	$x^{-1}b$	$b^{-1}x^{-1}$	$b^{-1}x^{-1}b$
x		b, b^{-1}	b, b	
xb	b, b^{-1}			b, b^{-1}
$b^{-1}x$	b, b			b, b
$b^{-1}xb$		b, b^{-1}	b, b	

We let Ω be the set of all permutation words we get from the table. Then Ω contains all closed curves on $S_{2,0}$ satisfying the three conditions.

Appendix B

We use Mathematica to compute the trace of $\rho_m(\omega)$ where ω is a word in Ω. The computation verifies that the elements in Ω having the same trace squared as α are α and $\tau(\alpha)^{-1}$. The following is the Mathematica code we used (the actual code can be found at `http://www.math.uiuc.edu/~rmaungc2/`). To simplify the computation, we assume that the elements of Ω all begin with a.

```
a = {{5/4, 3/4}, {3/4, 5/4}}
b = {{4, 0}, {0, 1/4}}
A = Inverse[a]
B = Inverse[b]
x = {{5/3, -16/3}, {-1/3, 5/3}}
X = Inverse[x]
L = {b, A, B, X, a, b, x}
Tr[a.L[[1]].L[[2]].L[[3]].L[[4]].L[[5]].L[[6]].L[[7]]]
109505/2048
J1 = {b, A, B.X, a, b, x}
K1 = Permutations[J1];
J2 = {b, A, X.b, a, B, x}
K2 = Permutations[J2];
J3 = {b, A, X, a, B, x.b}
K3 = Permutations[J3];
J4 = {b, A, B.X.b, a, B, x.b}
K4 = Permutations[J4];
J5 = {b, A, X, a, b, B.x}
K5 = Permutations[J5];
J6 = {b, A, B.X.b, a, b, B.x}
K6 = Permutations[J6];
J7 = {b, A, B.X, a, b, B.x.b}
K7 = Permutations[J7];
J8 = {b, A, X.b, a, B, B.x.b}
K8 = Permutations[J8];
For[i = 1, i < Length[K1] + 1, i++,
If[(Tr[a.K1[[i]][[1]].K1[[i]][[2]].K1[[i]][[3]].K1[[i]][[4]].K1[[i]][[5]].K1[[i]][[6]]])^2
```

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

1

113

For[$i = 1, i <$ Length[K2] + 1, i++,

If[(Tr[a.K2[[i]][[1]].K2[[i]][[2]].K2[[i]][[3]].K2[[i]][[4]].K2[[i]][[5]].K2[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

For[$i = 1, i <$ Length[K3] + 1, i++,

If[(Tr[a.K3[[i]][[1]].K3[[i]][[2]].K3[[i]][[3]].K3[[i]][[4]].K3[[i]][[5]].K3[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

For[$i = 1, i <$ Length[K4] + 1, i++,

If[(Tr[a.K4[[i]][[1]].K4[[i]][[2]].K4[[i]][[3]].K4[[i]][[4]].K4[[i]][[5]].K4[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

For[$i = 1, i <$ Length[K5] + 1, i++,

If[(Tr[a.K5[[i]][[1]].K5[[i]][[2]].K5[[i]][[3]].K5[[i]][[4]].K5[[i]][[5]].K5[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

For[$i = 1, i <$ Length[K6] + 1, i++,

If[(Tr[a.K6[[i]][[1]].K6[[i]][[2]].K6[[i]][[3]].K6[[i]][[4]].K6[[i]][[5]].K6[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

For[$i = 1, i <$ Length[K7] + 1, i++,

If[(Tr[a.K7[[i]][[1]].K7[[i]][[2]].K7[[i]][[3]].K7[[i]][[4]].K7[[i]][[5]].K7[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

206

315

For[$i = 1, i <$ Length[K8] + 1, i++,

If[(Tr[a.K8[[i]][[1]].K8[[i]][[2]].K8[[i]][[3]].K8[[i]][[4]].K8[[i]][[5]].K8[[i]][[6]]])^2

$== \left(\frac{109505}{2048}\right)$ ^2, Print[i]]]

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