COMBINATORICS OF A FAMILY OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH AN EYE TOWARDS TOPOLOGICAL TEMPERATURE

BY

CHRISTOPHER P. BONNELL

DISSERTATION

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Doctoral Committee:

Professor Richard Sowers, Chair
Associate Professor Robert Deville, Director of Research
Professor Jared Bronski
Assistant Professor Zoi Rapti

Abstract

Using the SISR asymptotic, a classification of a class of scale-separated stochastic differential equations is achieved using combinatorics and formal language theory. This is extended to a topological notion of temperature and qualitative results regarding the relatedness of scale separated stochastic dynamical systems by changes in temperature.

To my parents and grandparents,

for supporting my choices and

being very, very patient.

Acknowledgements

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Chapter 1

Intro and background: Multiscale SDE and Kramer's rate

1.1 Scale separated systems

Definition 1.1.1. *Let X_t^ϵ be a solution to the following Stochastic Differential Equation (SDE) [Øks03, KS91]:*

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \sqrt{\epsilon}\sigma(X_t^\epsilon)dW_t \tag{1.1}$$

where $X_0 = x_0$ and $X_t \in \mathbb{R}$

Definition 1.1.2. *Take a* **scale-separated stochastic system** *[PS08, DeV10] to be of the form*

$$\begin{aligned} dx = & \quad f(x,y)dt + \sqrt{\epsilon}h(x,y)dW_t^{(x)} \\ dy = & \quad \alpha g(x,y)dt + \sqrt{\alpha\epsilon}k(x,y)dW_t^{(y)} \end{aligned} \tag{1.2}$$

with $0 < \alpha, \epsilon << 1$, $f, g, h, k \in C^2$, and h is uniformly bounded below, $x \in \mathbb{R}^n, y \in \mathbb{R}$. The ODE given by setting $\epsilon = 0$ is called the **underlying deterministic system**

We make a few assumptions on the underlying deterministic system.

Assumption 1.1.1. *Dynamics*

1. *There is exactly one dimension of slow travel;*
2. *We assume that $k(x,y) = 0$, which is to say that the system moves deterministically in the slow direction of* (1.2)*;*
3. *The fast dynamics have no fixed points;*
4. *The stable manifolds of the dynamics are one-dimensional and are transverse to the dimensions of fast travel;*

5. *There is a connected, compact set $S \in \mathbb{R}^{n+1}$ such that every initial condition in $\mathbb{R}^{n+1}$ eventually enters S. That is, S is an attractor;*

6. *For every y with $(\mathbb{R}^n \times y) \cap S \neq \{\}$, $\exists x$ such that $(x, y) \in S$ and $f(x, y) = 0$.*

If we make the additional assumption that f and g of (1.2) can be derived from a single potential function V, we have the following system. In general, we will not have a potential function.

$$\begin{aligned} dx &= -V_x(x, y)dt + \sqrt{\epsilon} dW_t^{(x)} \\ dy &= -V_y(x, y)dt \end{aligned}$$

1.2 Kramer's rate

From DeVille [DeV10], and based on Kramers [Kra40] for a particle in a potential well, the inverse rate of travel to another potential well can be given by

$$\kappa \sim \nu(x) \exp\{(I(x))/\epsilon\}$$

Where ν is a prefactor based on curvature, I is a well depth,and ϵ is the noise strength. So, in the limit as $\epsilon \to 0$, the question of whether or not a jump occurs is one of whether or not $I > 0$ or vice-versa.

The Kramer's rate is essential to our analysis, as it provides a link between time and location that allows the system to be effectively tuned.

1.3 Quasipotential

Under very weak conditions, (1.2) is a vector field which can be assigned a quasipotential [WF12]. For the following, take D to be a basin of attraction to a unique attracting fixed point O.

Definition 1.3.1. *Let φ be a path with $\varphi_{T_1} = O$, and $\varphi_{T_2} = x_0 \in \partial D$. The* **action functional** *is* *defined to be*

$$S_{T_1T_2}(\varphi) = \begin{cases} \frac{1}{2}\int_{T_1}^{T_2} |\dot{\varphi}_t - b(\varphi_t)|^2\, dt & \text{if } \varphi \text{ is absolutely continuous} \\ \infty & \text{otherwise.} \end{cases} \tag{1.3}$$

Using this definition, we can define the quasipotential.

Definition 1.3.2. *The* **quasipotential** *of our dynamical system is defined to be*

$$V(O, x) = \inf\{S_{T_1T_2}(\varphi) : \varphi \in C_{T_1T_2}(\mathbb{R}^n), \varphi_{T_1} = O, \varphi_{T_2} = x, T_1 \leq T_2\}. \tag{1.4}$$

Note that this is an infimum over T_1, T_2, and φ.

Lemma 1.3.1 (Lemma 2.2 of [WF12]). *Suppose that the point O is a stable equilibrium position of (1.1), the domain D is attracted to O and $(b(x), n(x)) < 0 \forall x \in \partial D$. Then for any $\alpha > 0$ we have:*

1. *There exist positive constants a and T_0 such that for any function φ_t assuming its values in the set $D \cup \partial D \backslash \mathcal{E}_\alpha(O)$ for $t \in [0, T]$, we have the inequality $S_{OT}(\varphi) > a(T - T_0)$*

2. *There exist positive constants c and T_0 such that for all sufficiently small $\epsilon > 0$ and any $x \in D \cup \partial D \backslash \mathcal{E}_\alpha(O)$ we have the inequality*

$$P_x\{\zeta_\alpha > T\} \leq \exp\{-\epsilon^{-1} c(T - T_0)\}$$

where $\mathcal{E}_\alpha(a)$ is the α-neighborhood of the point a and $\zeta_\alpha = \inf\{t : X_t^\epsilon \notin D \backslash \mathcal{E}_\alpha(O)\}$.

Which is to say that the Kramers' rate holds, and we also have.

Theorem 1.3.2 (Theorem 2.1 of [WF12]). *Let O be a stable equilibrium position of (1.1) and suppose that the domain D is attracted to O and $(b(x), n(x)) < 0$ for $x \in \partial D$. Suppose furthermore that there exists a unique point $y_0 \in \partial D$ for which $V(O, y_0) = \min_{y \in \partial D} V(O, y)$. then*

$$\lim_{\epsilon \to 0} P_x\{\rho(X_{\tau^\epsilon}^\epsilon) < \delta\} = 1,$$

for every $\delta > 0$ and $x \in D$, where $\tau^\epsilon = \inf\{t : X_t^\epsilon \in \partial D\}$.

This, in English, says that if the quasipotential has a unique minimum on the boundary of the basin of attraction, then this will be the point of exit with extremely high probability.

Theorem 1.3.3 (Theorem 3.1 of [WF12]). *Suppose the vector field $b(x)$ for (1.1) has the decomposition $b(x) = -\nabla U(x) + l(x)$ where $U(x)$ is continuously differentiable on $D \cup \partial D$, $U(O) = 0$, and $\nabla U(x) \neq 0$ $\forall x \neq 0$ and $(l(x), \nabla U(x)) = 0$. Then, the quasipotential $V(O, x)$ with respect to O coincides with $2U(x)$ at all points $x \in D \cup \partial D$ for which $U(x) \leq U_0 = \min_{y \in \partial D} U(y)$. If U is twice continuously differentiable then the unique extremal of $s(\varphi)$ on the set of functions $\varphi_s, -\infty \leq s \leq T$ leading from O to x is given by*

$$\dot{\varphi}_s = \nabla U(\varphi_s) + l(\varphi_s).$$

In particular, if $l(x) = 0$ our system is gradient and thus quasipotential.

Of course, we chose quasipotential because there is a link between the quasipotential and the principal term of the asymptotics of the mean exit time of a trajectory starting at O in D. Namely, by theorem 4.1 of chapter 4 of [WF12].

Theorem 1.3.4 (Theorem 4.1 of [WF12]). *Let O be an asymptotically stable equilibrium position of (1.1) and assume that the domain $D \subseteq \mathbb{R}^n$ is attracted to O. Furthermore, assume that the boundary ∂D of D is a smooth manifold and $(b(x), n(x)) < 0$ for $x \in \partial D$, where $n(x)$ is the exterior normal of the boundary of D. Then for $x \in D$ we have*

$$\lim_{\epsilon \to 0} \epsilon \ln E_x \tau^\epsilon = \min_{y \in \partial D} V(O, y),$$

where τ^ϵ is the escape time with noise strength ϵ.

1.4 Markov Decomposition

Much in the same flavor as a Markov decomposition (see [Con78] for background on the idea), the scale-separated system, for fixed values of the slow variable, has a finite collection of attracting fixed points. These fixed points will, as the slow variable changes, form a collection of one-dimensional manifolds. (they are manifolds only because of genericity ruining the pitchfork bifurcation). A particle placed in the system will (in the limit as $\epsilon \to 0$, reside on these manifolds. Given that the system is assumed to be attracting at infinity, the system can be broken into a number of pieces, each corresponding to the basin of attraction for some manifold.

1.5 Specification to Fitzhugh-Nagumo (FN) system

A good prototype system defined above is the Fitzhugh-Nagumo system [Erm10], defined by:

$$\begin{aligned} \dot{x} = &\quad x - x^3/3 - y + F_{\text{ext}}, \\ \dot{y} = &\quad \alpha(x - A). \end{aligned}$$

It has one fixed point, and can be excited away from this point with either forcing or noise, whereon it goes on a large excursion before winding back up at the fixed point again. See Chapters 2 and 3 for more on this.

1.6 Motivation for SISR asymptotic

Starting with (1.2) and following [DeV10, DVEM05], we will define an asymptotic. If the noise is nonzero, a trajectory may leave a neighborhood of any slow manifold earlier than it could in the deterministic case. Consider starting at x_0 near a slow manifold $A(y)$. Take $\tau^\epsilon(y)$ to be the first passage time out of the basin of attraction of the fixed point. By 1.3.1 there is a function $I(y)$ such that

$$\lim_{\epsilon \to 0} \epsilon \log(\tau^\epsilon(y) = I(y)$$

If we consider the limit

$$\epsilon \log \alpha^{-1} \to \beta > 0$$

then a simple Kramers rate calculation reveals that the relative rate of jumping to the basin of attraction of a different slow manifold to the slow timescale, while $I(y) > \beta$, is 0. The same rate if $I(y) \geq \beta$ is ∞.

1.7 β

A way to think of β: β modifies the potential underlying the dynamics by filling up the potential wells to a height of β over their minima. The question that can be asked is the following: If I have two systems, A and B at whatever value of β, is there a third system C which appears as A for a certain β_A and B at some other β_B?

Chapter 2

Numerical and Asymptotic Study

2.1 SISR results

It's prudent to simulate those things proven asymptotically to ensure that the results are in agreement and that no higher-order correction actually has $O(1)$ impact. SISR has been studied in this manner for some time, but for completeness, I include a description of how such a validation would be achieved numerically.

2.1.1 Simulation design

The simulations themselves were designed from scratch in a variety of languages. First, in MATLAB to prove concept, then in C++ for speed and reliability. Eventually, for speed concerns, a CUDA implementation was devised but exhibited poor performance in the generation of random numbers. Most recently, a Java implementation was created with surprising performance results, but no better mathematical results than its C++ predecessors. They all follow the same basic design, which is

1. Implement the Euler-Murayama method (detailed below) on the appropriate differential equation
2. Collect data
3. Generate statistics of data

Typically these are done with as many threads as possible, and take between 10^5 and 10^9 excursions for each set of parameter values.

2.1.2 Simulation techniques

This can be simulated for numerical validation using the Euler-Maruyama method [AG11]

Algorithm. *For the SDE* $dX_t = a(X_t)dt + b(X_t)dW_t$, $X_0 = x_0$ *on the interval* $[0, T]$

1. *Take* $\tau_k = kT/N = k\delta$ *with* $N \in \mathbb{Z}$, $N \gg T$.

2. *Simulate the time evolution of an excursion using Euler's method plus noise:*

   - $X_{n+1} = X_n + a(X_n)\delta + b(X_n)\Delta W_n$
   - *where* $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$

In the simulation of deterministic equations, a higher-order method would be used for more regularity. In the stochastic case, variance reduction techniques would be more appropriate. [KP92] As we changed differential equations on a semi-regular basis, and the variance techniques would need to be re-derived each time, we decided that Euler-Murayama would be, though worse (alone) in each individual case, faster overall.

Chapter 3

Asymptotic study of the fixed-α asymptotic

As we wished to see what was true of more general systems than those in [DeV10], we were forced to make a decision. We could use a more general asymptotic to include higher-order effects, or we could use the same asymptotic on more general systems. Naturally, we did both, starting with the asymptotic. [BO99, Kee00, Hol09]

3.1 Definition and distinction from SISR

In fact, we not only studied a more general SISR asymptotic, but also a slightly different asymptotic which we called fixed-α

Definition 3.1.1. *The* **Fixed-α asymptotic** *is derived like SISR, but in the equation $\beta = \epsilon \log \alpha^{-1}$ we know that β corresponds to a height. Indeed it can locally be as big as the height of a potential well over it's minimum. If instead we take $\beta = \Delta h = \epsilon \log \alpha^{-1}$ we can arrive at a different asymptotic, namely $\alpha = \exp(\frac{-\Delta h}{\epsilon})$.*

This asymptotic fixes the scale separation, which would be immutable in many physical circumstances, and because of this it merits some investigation.

3.2 Asymptotics for SISR

The potential for FN has the form

$$V(x, y) \sim 1/4(1 - x^2)^2 - y\alpha(x + A).$$

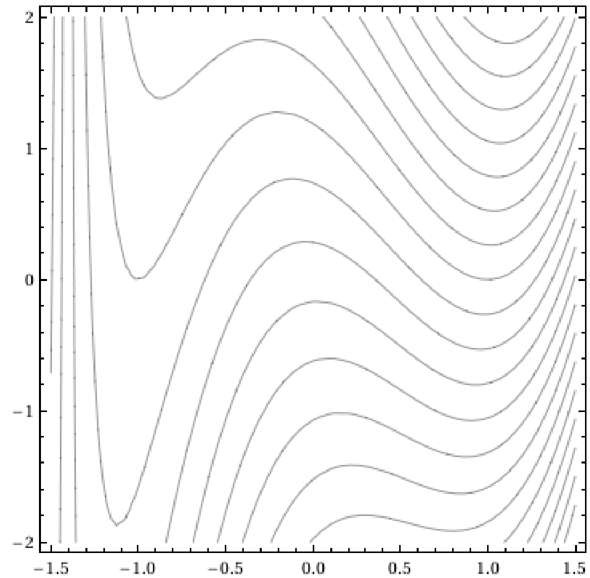

Figure 3.1: A contour map of the potential well for FN

This means that the Kramers' rate for FN depends on y, which in turn requires y to be a slow variable. From DeVille [DeV10] if we have a system of the form

$$\begin{aligned} dX_t^\epsilon = &\quad b(X_t^\epsilon)dt + \sqrt{\epsilon}\sigma(X_t^\epsilon)dW_t \\ \kappa(x) = &\quad \nu(x)e^{\rho(x)/\epsilon} \end{aligned}$$

we can derive an asymptotic approximation to the right-left jumping times.

First, we can define a hazard function $P = 1 - F$ where F is the CDF of the (right to left) jump event and find that

$$\frac{dP}{dt} = -P(t)\nu(y(t))e^{\rho(y(t))/\epsilon}$$

This can be transformed into terms of y alone,

$$\frac{dP}{dy} = -P(y)\frac{\nu(y)}{b(y)}e^{\rho(y)/\epsilon}$$

Of course, we can explicitly solve this, but the solution wil be ugly, and there will be very little to do with it.

Because of its relation to a CDF, we expect $F(-\infty) = 1$, and $F(\infty) = 0$ with a transition layer somewhere in between. If $\epsilon \ll 1$, the sign of ρ will determine whether $e^{\rho(y)/\epsilon}$ is small or large, which determines $\frac{dP}{dy}$. So, if we were to expand around any point, y_0 such that $\rho(y_0) = 0$ is the obvious one. We can exploit the relationship between the noise and timescale separation to get an asymptotic solution.

Fix a constant $\beta = \epsilon \log(\alpha)$ (SISR), and notice that $b = \alpha\tilde{b}$, and use this to obtain

$$\frac{dP}{dy} = -P(y)\frac{\nu(y)}{\tilde{b}(y)}e^{(\rho(y)-\beta)/\epsilon}.$$

Trying $y = y_0 + \epsilon(z + z_0)$ with $\rho(y_0) = \beta$, we find

$$\frac{dP}{dz} = -\epsilon P(z)\frac{\nu(y)}{\tilde{b}(y)}e^{\rho'(y_0)(z+z_0)}.$$

Now, let $z_0 = -\log(\epsilon)/\rho'(y_0)$ and we have

$$\frac{dP}{dz} = -P(z)\frac{\nu(y_0 + \epsilon z_0)}{\tilde{b}(y_0 + \epsilon z_0)}e^{\rho'(y_0)z}.$$

Now all the terms but the exponential are constant so we can solve explicitly. Let $A = \frac{\nu(y_0+\epsilon z_0)}{\tilde{b}(y_0+\epsilon z_0)}$ and $B = \rho'(y_0)$. Taken together, this gives

$$P(z) = \exp(\frac{A}{B}e^{Bz}).$$

This has the form of a Fisher-Tippet distribution, so we can calculate its moments.

$$m_1 = -\frac{\gamma + \log(A/B)}{B},$$

$$m_2 = \frac{\pi^2}{6B^2}.$$

These moments are in the z coordinate, in y we have

$$\langle Y^\epsilon \rangle = y_0 - \frac{\epsilon \log(\epsilon)}{\rho'(y_0)} - \epsilon\frac{\gamma + \log(\nu(y_0 + \epsilon z_0)/b(y_0 + \epsilon z_0)\rho'(y_0))}{\rho'(y_0)} + o(\epsilon)$$

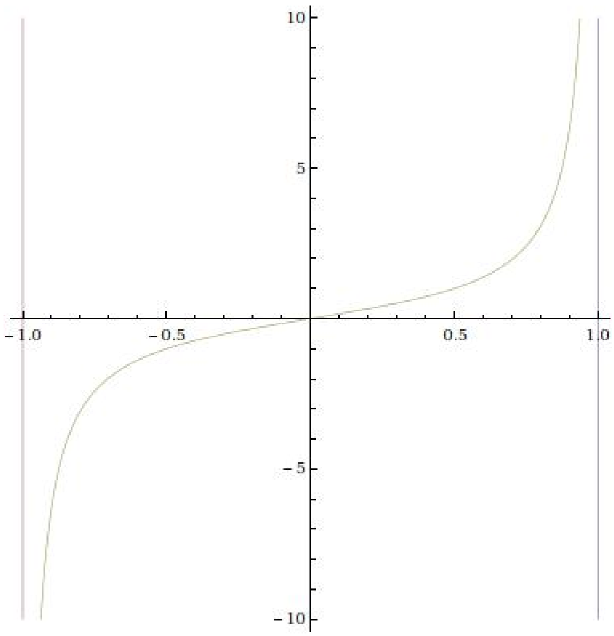

Figure 3.2: An infinitely long well

$$|(Y^\epsilon - \langle Y^\epsilon \rangle)^2| = \epsilon^2 \frac{\pi^2}{6\rho'(y_0)^2} + o(\epsilon^2)$$

3.3 Fixed-α

The SISR results are quite encouraging, and it's possible that we can recapture some of them with the fixed-α asymptotic, but a few initial notes are required. Namely, SISR fixes a height, but the new asymptotic does not. This being the case, it's easier to visualize things if we blow out the dynamics so that the $\epsilon = 0$ limit cycle spans the entirety of $-\infty$ to ∞. Using this model we can isolate the effects of perturbation on the location of the right to left jump more easily. For simplicity, we can model this with a quartic potential and set the separation equal to the height.

$$V(x,y) = \frac{12\rho(y)(x^4/4 - x^3\rho(y)/3 - x^2/2 + x\rho(y))}{6\rho(y)^2 - \rho(y)^4 - 8\rho(y) + 3}$$

There are 2 natural cases, being polynomial separation of manifolds $\rho(y) \sim y^{-p}$ and exponential separation $\rho(y) \sim e^{-y}$

3.3.1 Exponential Separation

Here we take $\rho(y) \sim e^{-y}$, and use $y = y_0 + \epsilon z$. Using this and following the ideas of the paper, we find

$$y_0 = \rho^{-1}(\epsilon \log(\alpha/\epsilon))$$

In this case, we can obtain a nice result [Gar09], namely

$$\lim_{\epsilon \to 0} m_1 = 1$$

$$\lim_{\epsilon \to 0} m_2 = 0$$

3.3.2 Polynomial Separation

In the case of a polynomial separation, m_1, the correction to the mean, dominates y_0, the actual mean. In this case, we have

$$\lim_{\epsilon \to 0} m_1/y_0 = \infty$$

3.3.3 Constant Separation

The polynomial separation wouldn't be a huge setback, except that using a constant separation, the Kramers' rate guarantees a jump in infinite time. As such, this asymptotic expansion (and the many others tried fruitlessly) can't distinguish polynomial and constant separations. A correction to this which fixes this technical problem was sought, but not found. In light of this, the direction of research went away from simple systems with harder asymptotics, and towards SISR on more general systems.

Chapter 4

Progression to more abstract systems

Although the fixed-α asymptotic gave some interesting results, the study of the case of more general systems using SISR proved more fruitful, and will dominate the remainder of the paper.

4.1 Assumptions on dynamics for this setup

Assumption 4.1.1. *First, we want that the vector field which gives the dynamics can be described by a quasi-potential.*

Assumption 4.1.2. Compactness, *meaning that the dynamics take place inside a compact set in* $\mathbb{R}^n$

Assumption 4.1.3. Genericity *in the sense that*

- *No two moves happen at the same y value*
- *Bifurcations in the dynamics are generic in the usual sense*
- *The β values at which we analyze the system are taken to be off of the set of measure zero where the system is not deterministic.*

Assumption 4.1.4. *The dynamics do not exhibit any fixed points (this is for convenience, and we will relax it eventually)*

Assumption 4.1.5. *Some point is attracting for each y value in the compact set considered (which may depend on β)*

Assumption 4.1.6. *The above attracting points correspond to basins of attraction which exist for some interval in the y direction with nonempty interior*

4.2 Definitions

Definition 4.2.1. *A* **rail** *is a tuple* $(L, D, M(\beta))$ *where*

- L *is a point in* $N-1$ *space which defines a line parallel to the slow direction.*
- $D \in \{+,-\}$ *is a direction of travel. An* **Upward Rail** *has* $D = +$*, and a* **Downward Rail** *has* $D = -$*.*
- $M(\beta)$ *is a function from* $\mathbb{R}_{>0} \to \mathcal{P}(\mathbb{R})$*.*

Note 4.2.1. *In a more practical light, the rails can be taken to be the stable manifolds at* $\beta = 0$*.*

Note 4.2.2. *If one wishes, as the dynamics take place in a compact set, and all transitions are assumed to be generic, the Stone-Weierstrauss [Rud91] theorem can be used to transform the vector field into a polynomial and the zero sets (hence the rails and stable manifolds) into algebraic varieties [Mum99].*

Definition 4.2.2. *A* **configuration** *is a collection of disjoint manifolds arranged on a collection of Rails which satisfies our assumptions. In general, this will be expressed by a jump sequence (as defined later).*

Definition 4.2.3. *Any generic transition derived from dynamics is called a* **Simple Legal Move** *when applied to a jump sequence. These will be enumerated in the subsequent chapters.*

Definition 4.2.4. *Any dynamically-derived sequence of simple legal moves is a* **Legal Move***.*

Definition 4.2.5. *A* **transition** *is a map between two Configurations given by a legal move. It may be used interchangeably with "move".*

Definition 4.2.6. *A* **jump** *is the location on a configuration where a particle moving under the dynamics of the system would switch between rails.*

4.2.1 Combinatorial and Categorical Formalisms

Take $S_{U,D,N}$ to be the collection of all configurations of lists of jumps corresponding to a manifold with U upward Rails, D downward Rails, and N jumps total.

There are transitions from this set of manifolds to a distinct, related one. These are generated by changing the intensity of the noise (β).

Definition 4.2.7. *For increasing β, we have*

1. *A manifold A at β_1* **fragments** *at some $\beta_2 > \beta_1$ if at β_2 there are manifolds B and C with $B \cap C = \{\}$ and $B \cup C \subset A$,*

2. *A manifold A at β* **shrinks** *if at some $\beta + \epsilon$ there is a manifold B with $B \subset A$,*

3. *A manifold A at β* **collapses** *if at some $\beta + \epsilon$ there is no manifold B with $B \subset A$.*

Definition 4.2.8. *For decreasing β, we have*

1. *A pair of manifolds A and B at β_1* **join** *if at some $\beta_2 < \beta_1$ there is a manifold C with $A \cup B \subset C$,*

2. *A manifold A at β* **grows** *if at some $\beta - \epsilon$ there is a manifold B with $A \subset B$,*

3. *A manifold A at β* **appears** *if at some $\beta - \epsilon$ there is no manifold B with $A \subset B$.*

4.2.2 Jump sequences

Consider any configuration. Start the jump sequence at $J = \{\}$, and proceed from $+\infty$ in the y direction, moving towards $-\infty$.

If a jump occurs, starting at manifold A, landing on B, then $A \to B$ should be appended to J. For example, if it is the first jump, we should have $J = \{A \to B\}$. If it is the second jump, $J = \{A \to B, C \to D\}$ etc.

The full jump sequence for a manifold will necessarily be finitely long, as the dynamics take place in a compact set. Given a jump sequence, one can reconstruct the dynamics, as seen below.

4.2.3 Building a manifold setup from a jump sequence.

If one is presented with a jump sequence J_k, it may be of interest to reconstruct an example of a manifold setup to which it corresponds.

Lemma 4.2.1. *If* $A_k \to B_k$ *is the first element in a jump string to involve* B_k *then* B_k *is downward moving.*

Proof. *If* B_k *were upward-moving instead, then this jump will take a particle to an upward-moving manifold. As this was the first element of the jump sequence to involve* B_k*, then the particle will move upwards forever, a violation of the compactness assumption.*

Chapter 5

An abstraction for the 1-1 case

5.1 Going from FN to rail setup

The FN system is a great starting point for a more general class of systems. FN has two stable manifolds, one on which a particle would move up in y, and one on which a particle would move down in y. We aim to capture this simple behavior and devise a way to encode it both usefully and descriptively.

5.2 Increasing vs. decreasing β, β-invertibility of all moves.

There are two ways to change the noise and thus the dynamics of a system. One is to add noise, and the other is to reduce noise. These are inverses, and thus it is prudent to consider only one, namely β increasing

Definition 5.2.1. *All moves are said to be β-invertible, meaning that if increasing β from a certain configuration by a certain amount causes a transition, reducing it by that same amount will return*

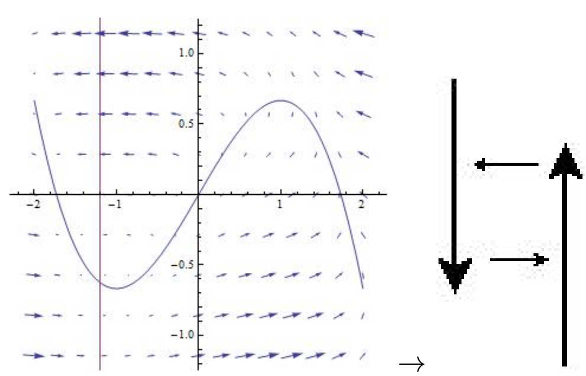

Figure 5.1: Going from FN to a more abstract system setup

the system to its initial configuration.

Definition 5.2.2. *We say that a move is invertible (with β taken to be increasing), if we can increase β to introduce the transition described by the move, and then increase β again to cause another move to return the system to its initial configuration.*

5.3 Basic legal moves in the 1-1 case

The case where there are only two manifolds U_1 and D_1, is called the $1-1$ case. There are only two jumps, namely $U_1 \to D_1$ and $D_1 \to U_1$. We denote them ">" and "<", respectively, for brevity.

If this is done, out of this sequence we can encode the dynamics, e.g. "$U \to D, D \to U, U \to D, U \to D, D \to U, D \to U, D \to U$" becomes the much more terse "><>><<<"

5.4 Counting and classifying

Lemma 5.4.1. 1. *Limit cycles correspond to substrings of the form "><".*

2. *pre-periodic orbits correspond to substrings of the form "> ... >< ... <"*

Proof. 1. *Following the dynamics, this sequence corresponds to moving up, jumping right, moving down, and then jumping left. Then, the sequence repeats, so it must be periodic. As the destinations of the jumps are fixed, at most 2 symbols can be observed in a limit cycle.*

2. *Observing the dynamics here, a particle falls from the left to the nearest "<" or, falls from the right to the nearest ">" and gets stuck in the limit cycle.*

Joining two adjacent manifolds into a larger manifold removes a single symbol, and splitting them is given by adding a symbol.

There is one restriction on the sequence. No sequence can start with "<" or end with ">" as this indicates either going up or down forever. As we are assuming compactness, these should be omitted.

Strings can be given an LCM, the longest common substring, and a GCD, the shortest common superstring.

These are well-studied by computer scientists. though each problem is technically NP-Hard for an arbitrary of strings, it is tractable for 2 inputs.

The LCS itself is non-unique, all strings of the same length can be embedded into an equal number of longer strings of any given constant length.

A table of these numbers follows, the horizontal represents the increasing length of the subsequence, the vertical the length of the target sequence.

X	1	2	3	4	5	6	7	8
1	1	-	-	-	-	-	-	-
2	3	1	-	-	-	-	-	-
3	7	4	1	-	-	-	-	-
4	15	11	5	1	-	-	-	-
5	31	26	16	6	1	-	-	-
6	63	57	42	22	7	1	-	-
7	127	120	99	64	29	8	1	-
8	255	247	219	163	93	37	9	1

Lemma 5.4.2. *This table follows the following "Pascal's triangle" rule: The leftmost entry is $2^n - 1$ where n is the length of the target sequence. Every other entry is the sum of the entry directly above it, and the one left-above it.*

Proof. *Suppose we have a string of k characters, and you wish to know how many strings of n characters it may be embedded into (here, of course, $k \leq n$). Recall that if $k = 1$, then there are $2^n - 1$ possibilities, as all but the string of identically not the character in question admit embedding. Recall also that if $k = n$ then it embeds only as itself, giving 1 embedding. Now, let's proceed by induction, supposing that we know how to embed strings of length up to $n - 1$ in sequences of length $n - 1$.*

Take a sequence of length k, embed it into one of length n. There are two distinct possibilities. One, the first character of it embeds into the target sequence, leaving an embedding of a string of length $k - 1$ into one of length $n - 1$, which we know how to do by induction. The other possibility is that the first character does not embed. In this case, we have a sequence of length k and we wish *to embed it into one of length $n-1$. Summing these two options gives us the number of possibilities for any $k \leq n$.*

5.5 Counting smaller sequences from which a larger sequence could emerge

In the above, the redundancy of sequences is accomplished by counting the number of ways, not the number of distinct sequences which could emerge. If we count instead the sequences, we realize that the number of sequences from which any given sequence could result is not constant accross sequences of a certain length, and in fact gives rise to the following pretty pictures (and their fourier transforms). Unfortunately, there seems to be no coherent information to be gotten this way. The frequency and spatial information differs radically between a number and its successors, and symmetries are few.

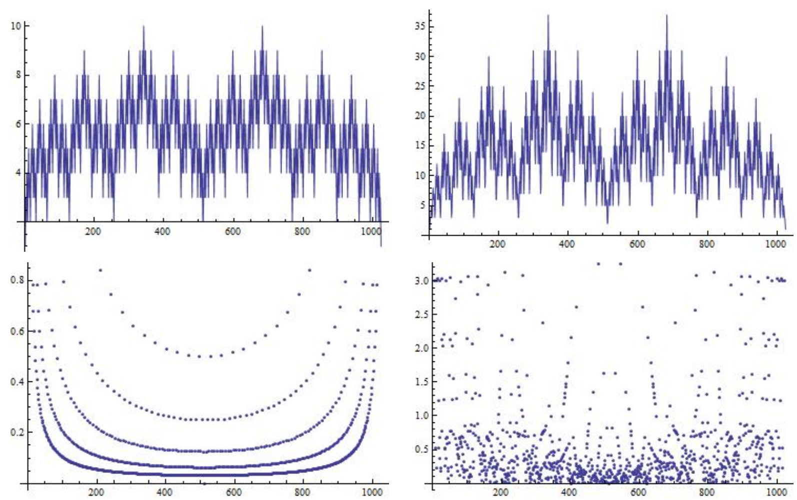

Figure 5.2: Top: Length 10 from length 9 embedding number, 10 from 8 embedding number. Bottom: Fourier transforms.

5.6 Mathematica Code

Returns whether or not a sequence can be embedded into another sequence using LongestCommonSequence, which measures the longest non-contiguous (but ordered) part of a sequence which can be found in both arguments

```
QSeqEmbed[one_, other_] :=
  With[{c = LongestCommonSequence[one, other]},
    (c == one) || (c == other)] ;
```

This generates a sequence of length at least PadToLen given by the binary representation of the number Number

```
Generate[Number_, PadToLen_] := Block[{a, i},
  a = IntegerString[Number, 2];
  For[i = 0, StringLength[a] < PadToLen, i++,
    a = StringJoin["0", a]];
  a]
```

Returns the number of sequences of length Len into which the sequence Seq can be embedded as above.

```
SeqEmbed[Seq_, Len_] := Module[{b, k}, b = 0;
  For[k = 0, k < 2^Len, k++,
    If[QSeqEmbed[Seq, Generate[k, Len]], b++, Null]];
   b]
```

Chapter 6

Total solubility of the 1-1 case

6.1 Classification: General Case

Theorem 6.1.1. *The* $1 - 1$ *case can be completely described not only up to valid configurations, but transitions between those configurations using a context sensitive grammar.*

Suppose we are given two dynamical systems under the assumptions descried above. Construct jump sequences from the manifolds, denote these A and B.

1. Using the method above, figure out which configuration has the most periodic orbits (WLOG, assume A). To state that A and B could be dynamics from the same system, we need to exhibit a system which is A at some β and B at some other.

2. Starting with B, use the method of removing pre-periodic jumps by increasing β until none remain.

3. Add periodic orbits by splitting the manifolds until the correct number of orbits (namely the number in A) exist.

4. Using the method for adding pre-periodic orbits by increasing β, add pre-periodic jumps until A is achieved.

6.1.1 Classification sidenote

Interestingly, if we forbid the transition where a manifold collapses, there is still a classification.

In this case, increasing β allows us to add elements. So, the problem boils down as follows. In this case, consider A WLOG to be the longer string. For A to follow from increasing β from B, the sequence B must be embeddable into A in the following sense: There needs to be a map

$f : [|B|] \to [|A|]$ with f strictly monotonic such that, if $A = (a_n)$ and $B = (b_n)$ then $b_n = a_{f(n)}$. This is equivalent to determining whether or not the longest common subsequence of A and B is B, i.e. whether or not $LCM(A, B) = B$.

6.2 Formal grammar

A context-free grammar [Cho56] which generates the “words” which comprise the one-one systems can be obtained, and further, all of the rules in this grammar correspond to β-increasing moves only. This grammar $G = (V, \Sigma, R, S)$is as follows.

$$
\begin{aligned}
S =& \quad \text{“}s\text{''}, \\
V =& \quad \{s, L, R\}, \\
\Sigma =& \quad \{<, >\}, \\
R =& \quad \{ \\
& \qquad s \to \quad RL, \\
& \qquad L \to \quad LL|RL| <, \\
& \qquad R \to \quad RR|RL| >, \\
& \quad \}
\end{aligned}
$$

There are things to prove, namely:

1. This generates only legal strings.

2. This generates all legal strings.

3. Each of the rules is actually a β-increasing legal move.

Lemma 6.2.1. *This grammar generates only legal strings.*

Proof. *The start rule corresponds to the string “$><$” which is legal, indeed we know that all strings in the 1-1 case are legal so long as they start with “$>$” and end with “$<$”. Each rule leaves the first character “$>$” and the last character as “$<$”, thus all strings generated are legal.*

Lemma 6.2.2. *Every legal string can be generated in this manner.*

Proof. *Take a legal string. We aim to follow the rules backwards to arrive at the start symbol. First,the string is comprised of “$<$” and “$>$”, so change this to “L” and “R”. Now, if there are any sub-strings comprised exclusively of Ls or Rs, follow the $L \to LL$ and $R \to RR$ rules backwards until only one of each remains. Now the sequence must alternate “RLRLR...RL” Starting on the left, apply the $R \to RL$ rule backwards then the $R \to RR$ rule. Alternate this process until only “RL” remains. Now note that the only start rule is $S \to RL$, so this is a string generated by the grammar.*

Lemma 6.2.3. *The moves correspond to β-increasing moves.*

Proof. *If we recall from the legal moves section, the only legal moves on a $1-1$ system are splittings of manifolds or collapses of manifolds. We can then fragment any manifold. This would introduce a new jump where formerly there was none, so we have $L \to LL|RL$, $R \to RR|RL$.*

Note 6.2.1. *Unfortunately, this isn't the entire grammar we could be interested in. This is the minimal non-contracting grammar which generates the strings. There are two contracting legal moves (which serve to trivialize the one-one case). The complete grammar is as follows, and is context sensitive:*

$$\begin{aligned}
S &= \text{“}s''\text{,} \\
V &= \{s, L, R\}, \\
\Sigma &= \{<, >\}, \\
R &= \{ \\
&\qquad s \to \quad RL, \\
&\qquad L \to \quad LL|RL|<, \\
&\qquad R \to \quad RR|RL|>, \\
&\quad Rx_1...x_nR \to, \quad Rx_1...x_n, \\
&\quad Lx_1...x_nL \to, \quad x_1...x_nL, \\
&\qquad \}
\end{aligned}$$

Lemma 6.2.4. *The above grammar still generates all legal sequences.*

Proof. *This is trivial, they were all generated before, now we could generate, if anything, more sequences.*

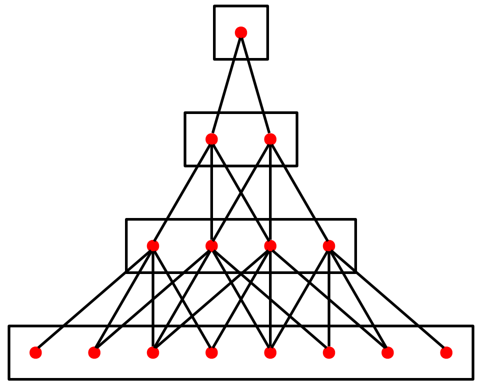

Figure 6.1: 3 iterations of the grammar, the nodes are boxed according to number of moves

Lemma 6.2.5. *The above grammar only generates legal sequences.*

Proof. *Every sequence that begins with R and ends with L is legal, the new rules do not enable removal of the starting R or ending L.*

Lemma 6.2.6. *The new moves are β-increasing and the set of moves encompasses all β increasing behaviors.*

Proof. *The moves correspond to a manifold shrinking out of existence, which is β increasing, and this set of rules encompasses all types of moves possible in the one-one case as listed in the legal moves section.*

A few pictures of the graph given by this grammar follow (see figures 6.2 and 6.2).

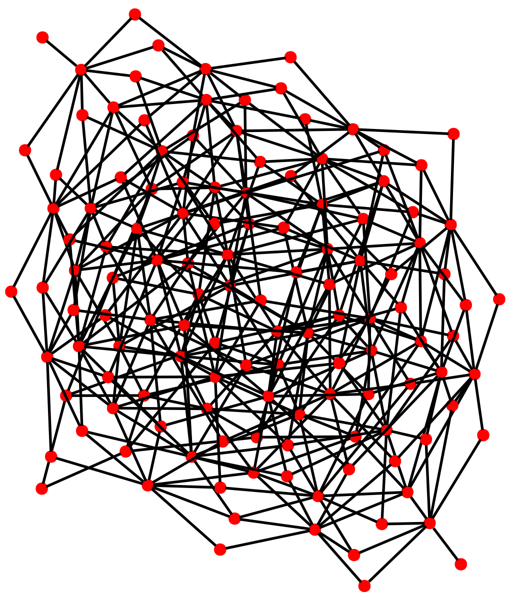

Figure 6.2: 6 iterations

Chapter 7

$1-2$, $2-2$, and $n-k$ examples with rules, behaviors and genericity results

7.1 Moves, new moves, and availability.

Recall the kinds of moves available in the $1-1$ system

1. A manifold may get shorter and potentially vanish

2. A manifold may break into two pieces

There's a lot of structure that we can exploit in the $1-1$ system to ensure that nothing too funny happens. Indeed, the interesting moves don't even start to show up until there are three total manifolds.

7.1.1 $2-1$

When there are 3 total manifolds, there is now interplay between not only the manifold sizes, but the relative well heights among all the varied destinations. It is possible for a manifold to get slightly shorter, and for the jump destination for the manifold to change. This move is called re-targeting and is illustrated in figure 7.1.1. Re-targeting is its own inverse.

Additionally, if manifolds are all getting shorter, one might get shorter faster than another one (indeed, it will in general) and this could cause the order of jumps to change. This move is reversible if the shortened manifolds both point in the same direction. (see figure 7.1.1)

If they do not point in the same direction, then the move is not necessarily invertible. This is because the manifolds have to shorten to accomplish this move, and now no longer overlap at all. No amount of shortening additional shortening can correct this in general.

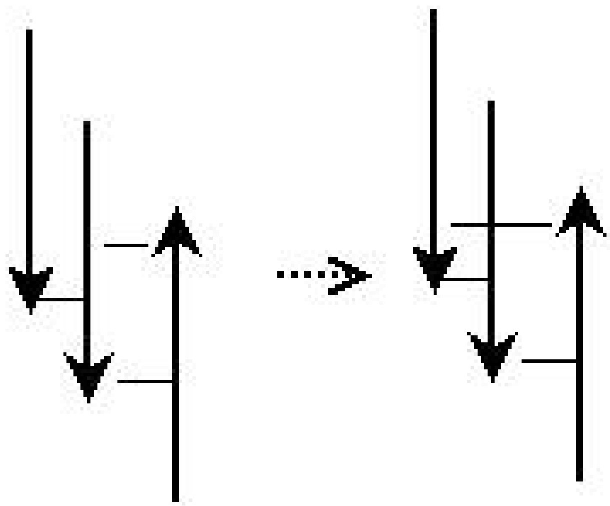

Figure 7.1: Re-targeting

7.1.2 $2-2$

At 4 total manifolds, there is a new move again. Now, there's enough space to have two jumps which are, due to the shrinking of the manifolds to which they are concerned, will be forced to pass through each other. (see figure 7.1.2)

7.2 The initial set $Start_{n,k}$

In the 1-1 case, the requirements that a system satisfy all assumptions requires that its jump sequences start with an up-down transition, and end with a down-up transition. There is a unique object which satisfies this, and it is in fact the initial object in the category $S_{1,1,*}$

If there are more than two manifolds, however, many configurations satisfy all the assumptions and involve all the rails.

Take $\mathrm{Start}_{j,k} \subset S_{j,k}$ with the restriction that if $a, b \in \mathrm{Start}_{j,k}$ then neither $a \to b$, nor $b \to a$.

Lemma 7.2.1. *if $a \in Start_{j,k}$ then, as a jump sequence, a has exactly $j+k$ elements. and must include exactly one manifold on each rail.*

Proof. *Suppose that this is not the case, and that the jump sequence J involves 2 manifolds on the*

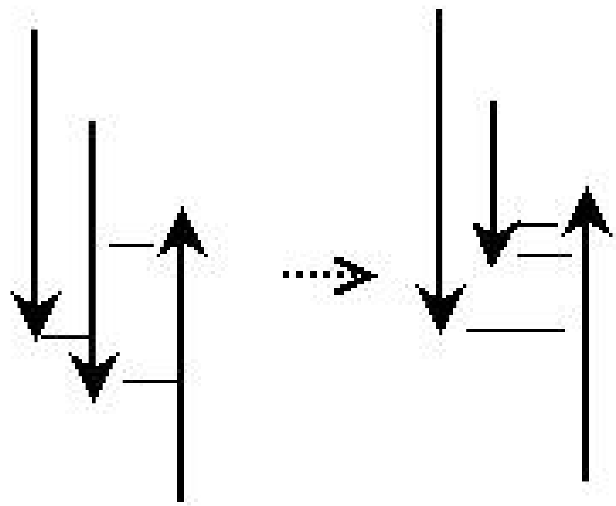

Figure 7.2: An Order Exchange which becomes possible at $2-1$

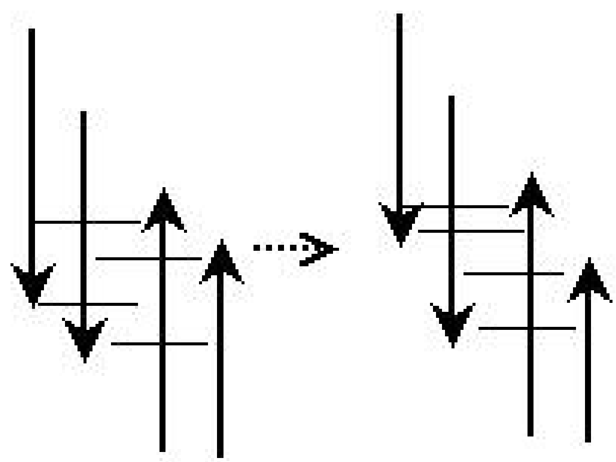

Figure 7.3: An Order Exchange which becomes possible at $2-2$

same rail $J = ...A^B...A^C...$ *Then, the jump sequence could be generated by a shorter jump sequence with only the latter of the two jumps included* $\tilde{J} = ...A^C....$ *There is a legal move* $\tilde{J} \to J$ *for* $j, k \geq 1$.

Therefore, no rail may appear more than once, and each must appear exactly once by the global assumptions on the system structure.

7.3 Representational Equivalence

Consider the following systems

$$(\textbf{??})J = U_1^{D_n}...U_n^{D_1}D_1^{U_n}...D_n^{U_1} \in S_{n,n,2n}J = U_1^{D_k}...U_{n-k-1}^{D_2}U_{n-k}^{U_{n-k-1}}...U_{n-1}^{U_n}U_n^{D_1}D_1^{U_{n-k}}...D_k^{U_1} \in S_{n,k,n+k} \tag{7.1}$$

This J is chosen because it represents a system of manifolds which is, in y, maximally overlapping, in a certain sense. However, it must be noted that it is representationally equivalent to several other jump sequences achieved by permuting the labels on the Us and Ds. To have any meaningful notion of uniqueness in $S_{n,k,*}$, we have to accept the following equivalence relation on jump sequences of configurations:

Definition 7.3.1. *We say that $A \sim B$ if $A = B$ or if there is some permutation σ of the numbers of the Us and Ds, or of the U and D symbols (if $n = k$) which causes $A = \sigma(B)$.*

Lemma 7.3.1. *$\sim$ is an equivalence relation.*

Proof. *$\sim$ is defined to be reflexive, and basic group theory provides that every permutation is invertible, so if $A \sim B$ then $B \sim A$ as if $A = \sigma(B)$ then $B = \sigma^{-1}A$ and transitive as multiplication of permutations is transitive.*

If we restrict ourselves to $S_{n,k,*}/\sim$, we have the following lemma.

Lemma 7.3.2. *$Start_{n,k}$ can be taken to be $\{J\}$ using the J from [?]*

Proof. *We know that any other element in $a \in Start_{n,k}$ has to have the same number of elements as J, and must have the same assortment of source symbols. Starting from the appropriate J, we can re-target and shorten to transpose symbols to show that $J \to a$.*

Lemma 7.3.3. *All representational equivalents of J can be generated from J.*

Proof. *These differ by order numbering of the up and down manifolds, and these can be interchanged by shrinking the manifolds in question and then order exchanging.*

7.4 Non-invertible moves

We have invertibility of moves in the $1-1$ case through what is actually a mild abuse of the assumptions. The move “ $><'' \rightarrow$ “ $>><''$, for example, along with the assumption that for each value of β, each value of y must have an attracting fixed point, tells us that the new $>$ symbol's manifold has its y values entirely in those of the $<$ symbol's manifold. Because of this, we can do “ $>><'' \rightarrow$ “ $><''$ and return to where we started. Naturally, we can still do this in the $n-k$ case, but only for these moves, there is one, namely the slip move, which involves manifolds getting shorter in a way that cannot be undone in some cases. Namely, if the slip pulls an upward going manifold past a downward going manifold, the move is not invertible.

7.5 Lattice description

Given that some moves are not invertible, if we have two configurations, a and b, there is no reason to expect that $a \rightarrow b$ or $b \rightarrow a$. But, if all the configurations can be taken to start from the same J, it makes sense to ask the following question: Given a and b in $S_{n,k,*}$ what is the longest jump sequence c which has $c \rightarrow a$ and $c \rightarrow b$, and what is the shortest jump sequence d which has $a \rightarrow d$ and $b \rightarrow d$.

Chapter 8

Categorical and formal language description of the $n - k$ case

Theorem 8.0.1. *Systems with n upward and k downward manifolds can be encoded, and their transitions can be completely described by a context sensitive grammar. This grammar accounts for all possible types of moves, and the situations in which they can be performed.*

8.1 Exhaustion of move types

Definition 8.1.1. *For the n, k system take $\{I_{a,b}(y)|1 \leq a \neq b \leq n + k\}$ as height functions corresponding to the jump height from wells a to b. If, for some a, $I_{a,b}(y) > \beta$ all b, then manifold a is stable at that y value for the temperature β.*

Definition 8.1.2. *Let $N(a, y, \beta) = |\{b|I_{a,b} \leq \beta\}|$ be the number of manifolds to which the manifold a is willing to jump at y for a certain β.*

$N(a, y, \beta) = |\{b|I_{a,b} \leq \beta\}|$, and by genericity can increase only in increments of 1 in y. In particular, this means that the manifold to which a particle on a jumps at the first y for which this function is > 0 is well defined.

Definition 8.1.3. *Let $S(a, \beta) = |\min_k \{I_k = [a_k, b_k]|I_j \cap I_k = 0 \forall i \neq j, I_{a,b}|_{(\cup_k I_k)^c} < \beta\}|$ the number of intervals in y on which the manifold a is stable at a particular β*

Lemma 8.1.1. *As a function of β, $S(a, \beta)$ can only generically increase or decrease in increments of 1.*

Proof. *If it increases or decreases by more than this, the function must have either two peaks with the same height, or two troughs with the same depth. Adding a bump function of height ϵ to one of the peaks or troughs is generic, and reduces the number.*

Lemma 8.1.2. *An increase by one of $S(a, \beta)$ indicates a manifold splitting.*

Lemma 8.1.3. *A decrease by one of $S(a, \beta)$ indicates a manifold collapse.*

As these are the only two possibilities for one manifold, this is all the single manifold behavior. For interaction behavior, the following may happen.

1. A manifold a may jump to b at a certain β, and then to c at a higher β.

2. A manifold a may be jumped two from two different manifolds, b, c. At some β, the $b \to a$ jump may be at a lower y than $c \to a$, and at some other β the order switches.

Definition 8.1.4. *Let $D(a, \beta) = b$ where b is the manifold to which a particle placed on a with temperature β will jump by following the dynamics.*
Taking y_a to be the y value where a begins, b is the manifold with $I_{a,b}(y) < \beta$ for the least $y > y_a$ with $N(a, y, \beta) > 0$

Defined this way, the first item corresponds to $D(a, \beta_1) = b$ (take this to happen at y_1), $D(a, \beta_2) = c$ (take this to happen at y_2) with $\beta_1 < \beta_2$. Winding this back through the definition, this corresponds to $\beta_1 = I_{a,b}(y_1) < I_{a,c}(y_1)$ and $\beta_2 = I_{a,c}(y_2) < I_{a,b}(y_2)$.

Lemma 8.1.4. *This is the only type of move which can be described using the same starting manifold and two separate ending manifolds.*

Proof. *If another were possible, it would have to involve $I_{a,b}(y)$. No change in dynamics happens unless, in changing beta, the relationship among these I's changes. If this happens without involving the above transition, we have that either they do not change sign relationship, or they become equal. The first indicates no change, and the second is non-generic. Thus, we have a contradiction.*

Note 8.1.1. *the only other thing which could involve 3 manifolds would be to have two jumps with the same destination, and change the order in which they occur.*

Note 8.1.2. *If 4 manifolds are involved, two jumps necessarily must happen, and the move can't be broken down into simpler steps. Thus, no re-targeting move could occur this way. The only remaining option is that 2 jumps occur, and their order is switched.*

Theorem 8.1.5. *In the case of greater than 4 manifolds involved, it is impossible to have a generic move which is not a composition of the above.*

Proof. *As it stands, such a move could not involve any two jumps changing order, nor any one jump changing destination, nor a manifold starting to exist or stopping. Said another way, the sources, targets, number, and order of jumps are all constant in such a move. If this were the case, the move has no effect on the level of jump sequences. Thus, any such move is the identity, a contradiction.*

8.2 The "big" grammar which gives a description of the configuration space and transitions for all n, k

Definition 8.2.1. *Define $M = \{manifolds\} = U \cup D$ where $U = \{upward\ manifolds\}$, $D = \{downward\ manifolds\}$, with $|M| = |U| + |D| = n + k$. Take $Start_{n,k}$ to be the minimal set of manifold configurations with n upward and k downward manifolds such that if $x, y \in Start_{n,k}$, then neither $x \to y$ nor $y \to x$*

Definition 8.2.2. *The grammar $G = (S, V, \Sigma, R)$ is as follows. $S =$ "s", $V = \{s, A^B | A \neq B \in M\}$ $\Sigma = \{i^j | 1 \leq i \neq j \leq n + k\}$ and R is detailed below.*

8.2.1 Start Rule

There is only one starting rule, namely $s \to a$ for $a \in Start_{n,k}$

8.2.2 Splitting Rules

Splitting rules are valid for any configuration with $i, j \geq 1$. In the following, consider $X, Y, \ldots$ to be sequences of symbols and $x, y, \ldots$ be single valid symbols. The rules are as follows:

$$a^b \to a^b a^b,$$

for any a, b.

$$Xa^b \to c^a X a^b,$$

$$d^b X \to b^d X d^c,$$

if $a \in D$ and $d \in U$.

8.2.3 Collapse Rules

Collapse rules are valid for any configuration with $i, j \geq 1$

$$a^b X a^x \to a^b X,$$

$$d^x X d^b \to X d^b,$$

if $a \in U$ and $d \in D$. If compactness of dynamics were not posited as an assumption, these rules would be dramatically easier, in that they would all be just

$$a^b \to \phi.$$

It is not immediately obvious, but the collapse is the inverse of split in the space of legal sequences. Though it may not appear as such, these rules are inverses.

Lemma 8.2.1. *The split and compactness rules are inverses.*

Proof. *The first of the split rules can obviously be undone with a collapse. The second rule is* $Xa^b \to c^a X a^b$*. For the original sequence to be legal, each manifold must be represented as a source, so one of the elements of* X *must have* c *as a source. That is, it can be written as* $X_1 c^x X_2 a^b$*, therefore, the result of the move is* $c^a X_1 c^x X_2 a^b$*, which can obviously be undone with a collapse. The third one is similar.*

To go the other way, if we collapse first, the first rule takes away an a^x*. If* x *is upward, there must be an* x *as a source to the right, and thus the second split rule restores it. The downward case is similar. The second collapse rule can be undone similarly. Thus if we have a configuration, we can do a split or collapse, then do the opposite to return to the initial configuration.*

8.2.4 Re-targeting rules

Re-targeting can begin with $n = 2, k = 1$ and, keeping with the $n \geq k$ formality, occurs for any with n, k larger than these.

$$c^x X a^b \to c^x X a^c$$

$$a^b X d^x \to a^d X d^x$$

for $c \in U$ and $d \in D$ Re-targeting is its own inverse provided it operates on a legal sequence.

8.2.5 $\geq 2, 1$ order exchange

Order exchange happens in two distinct cases, but in any case they involve the generic pulling-through of one jump through another. This cannot be broken down into simpler moves, and in some cases are not invertible!

In the simplest case, we have

$$a^c b^c \to b^c a^c$$

for $a, b \in U$ or $a, b \in D$, which is its own inverse. The other case is

$$a^c b^c \to b^c a^c$$

for $b \in D, a \in U$, which is not invertible.

8.2.6 General order exchange

In the larger setting, the order exchange is more complicated, but it is given as

$$a^c b^d \to b^d a^c$$

for $a, b \in U$ or $a, b \in D; c \neq b, a \neq d$ This type of move is again invertible, and self-invertible at that.

$$a^c b^d \to b^d a^c$$

for $a \in U, b \in D; c \neq b, d \neq a$ This latter move is again not invertible.

8.2.7 R

So, what is R?

$$R = \left\{ \begin{array}{ccc} s \to & x \in Start_{n,k}, & \\ a^b \to & a^b a^b, & \forall a, b, \\ Xa^b \to & c^a X a^b, & a \in D, \\ d^b X \to & b^d X d^c, & d \in U, \\ a^b X a^x \to & a^b X, & a \in U, \\ d^x X d^b \to & X d^b, & d \in D, \\ c^x X a^b \to & c^x X a^c, & c \in U, \\ a^b X d^x \to & a^d X d^x, & d \in D, \\ a^c b^d \to & b^d a^c, & a, b \in U \text{ or } a, b \in D, c \neq b, a \neq d, \\ a^c b^d \to & b^d a^c, & a \in U, b \in D; c \neq b, d \neq a. \end{array} \right\}$$

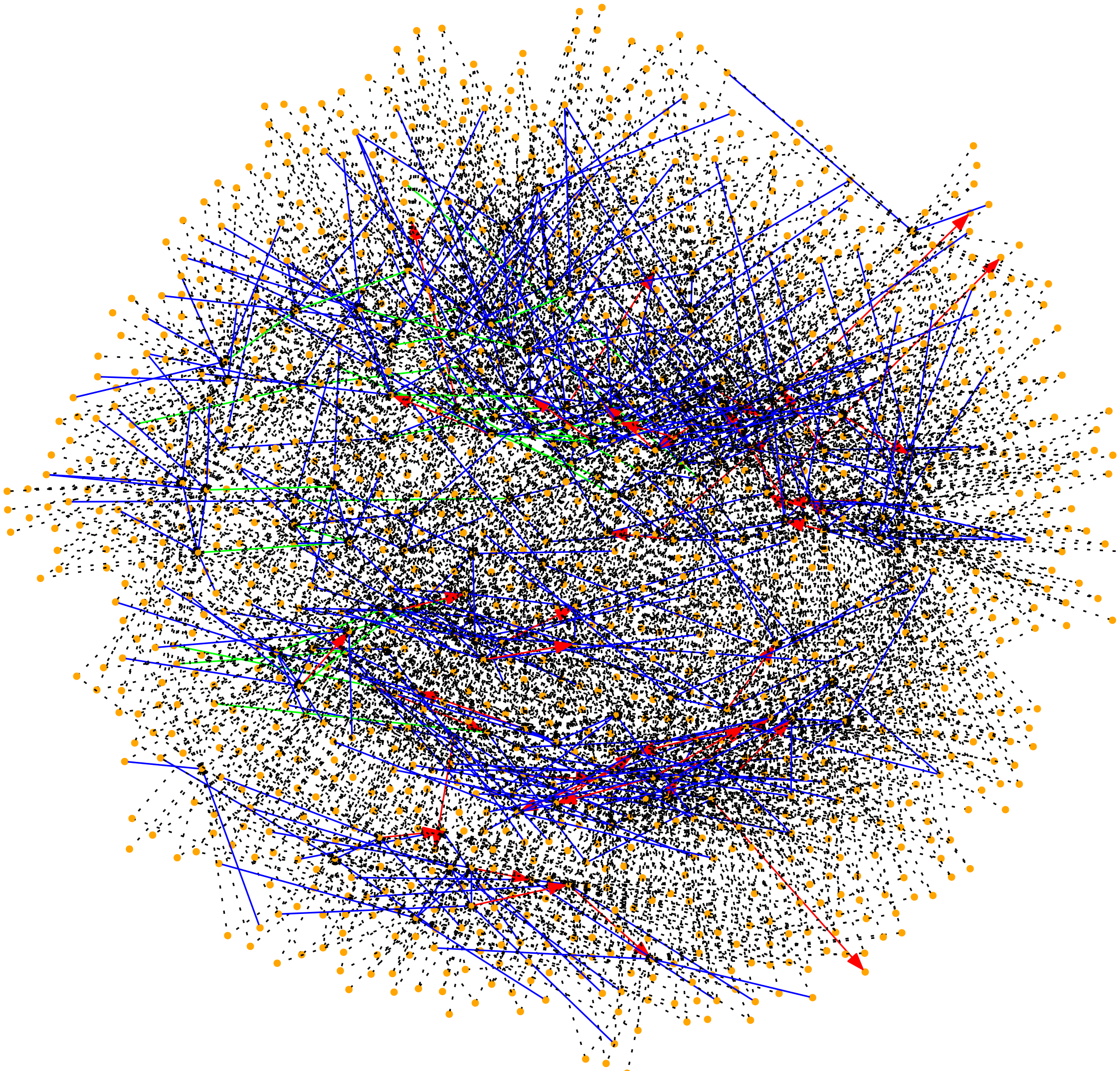

Figure 8.1: A graph of the $2-1$ grammar taken out 3 generations. The first one shows all moves, black are split/collapse, blue are re-targeting, green are invertible order exchanges, and the directed orange arrows are non-invertible order exchanges.

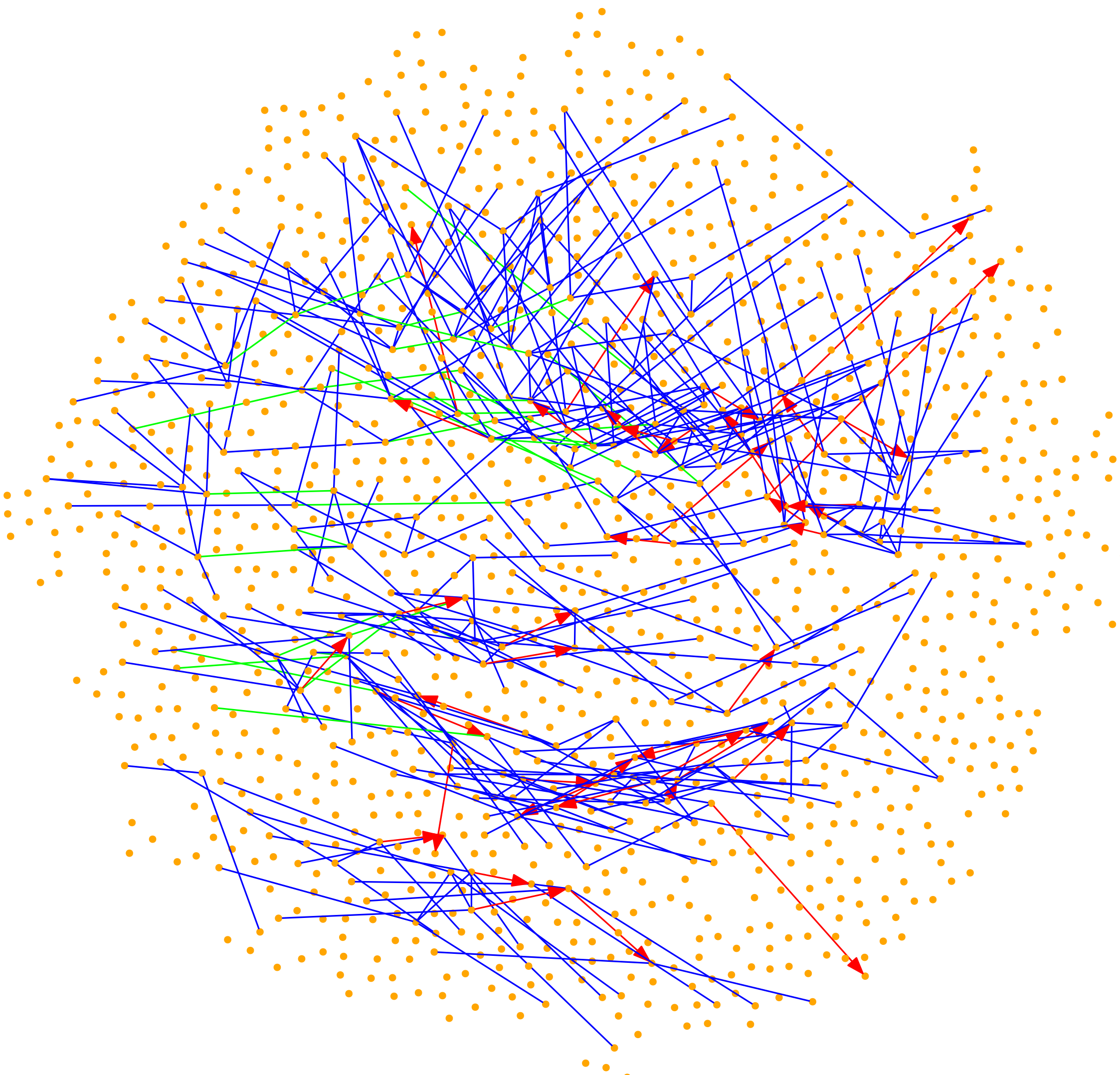

Figure 8.2: This is the same idea as before, with the black lines suppressed for legibility.

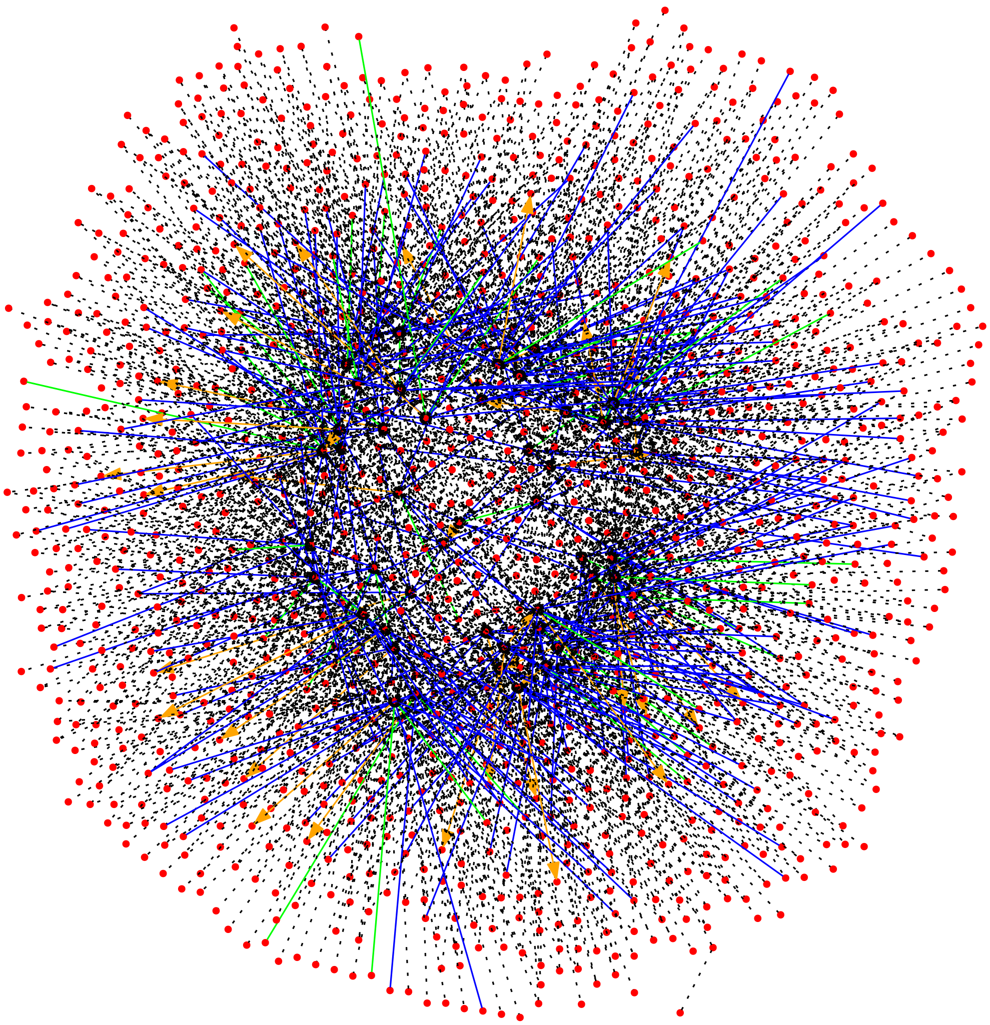

Figure 8.3: 2 generations of the $2-2$ grammar

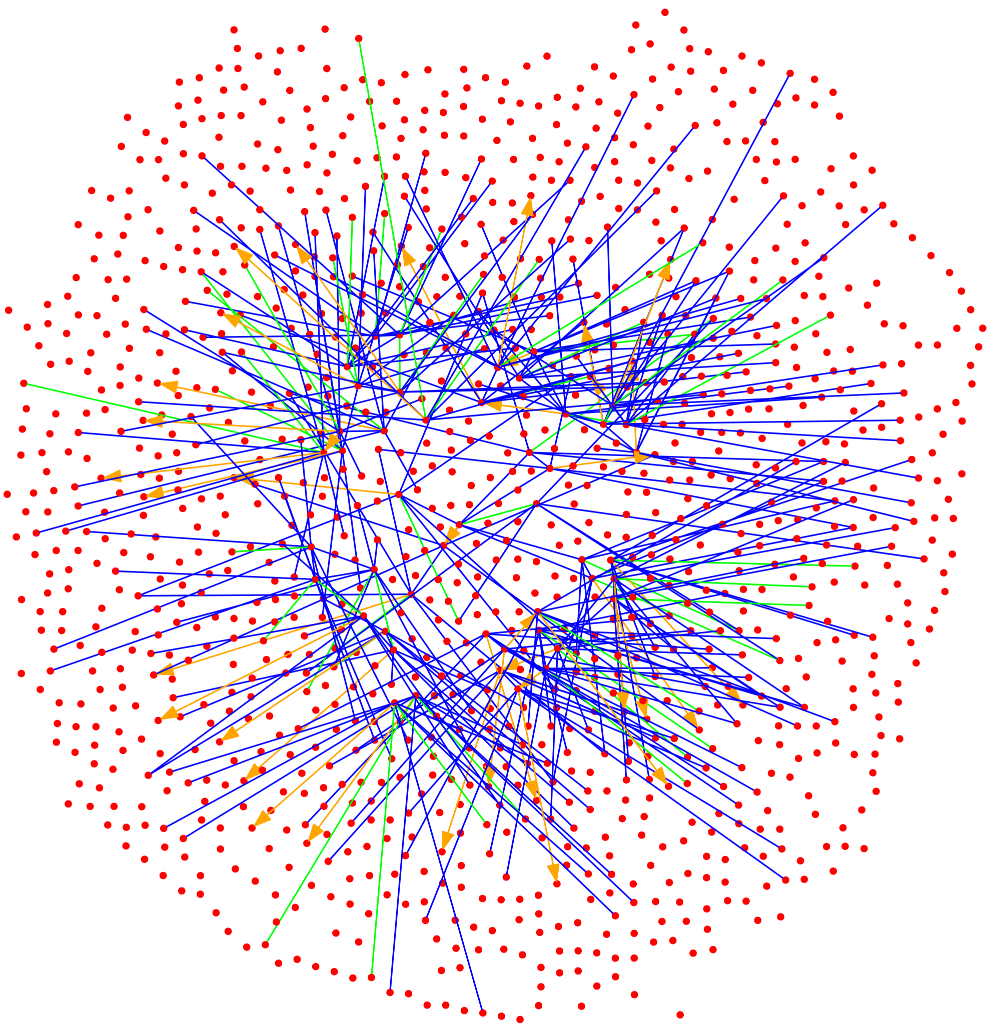

Figure 8.4: 2 generations of the $2-2$ grammar, with the black lines suppressed for legibility

Chapter 9

Transitivity

Theorem 9.0.2. *If two configurations are related by some path in the graph of legal moves, there is a configuration which traverses exactly this path.*

On the level of the potential, we can define a potential well with $-\exp\{-x^2/\sigma^2\}$. Say we want k rails in a system. Fix the kth rail $e_k = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^k$ where the 1 is in the kth place. Our wells will have the form $-\exp\{-(x-e_k)^T\Sigma^-1(x-e_k)\}$. This well is stable, and doesn't change with y at all, but we can do the following. The height may evolve in y, as may the Σ, giving $W_k(x, y) = -\exp\{-(x-e_k)^T\Sigma(y)^-1(x-e_k)\}f_k(y)$. This can be further simplified by taking $\Sigma_{ii} = \Sigma_{jj} \forall i, j$. Moreover, if we take exactly one pair $\Sigma_{ij} = \Sigma_{ji} > 0$, we can carefully control all the jumping behavior. In general, the diagonal elements should all be small. The smaller they are, the easier it is to see what's going on, but the more sensitive the system will be to minor noise fluctuations in simulation, and to β.

Definition 9.0.3. *A* **well** *will be taken to have the form*

$$R_j(x, y) = -f_j(y)\exp(-(x-e_j)^T(\sigma_j(y))^{-1}(x-e_j)).$$

Overall, our potential will have the form $V(x, y) = \Sigma_j R_j(x, y)$.

Lemma 9.0.3. *Let* $\beta > 0$*. A jump at* y *from* j *to* k *(assuming* $f_k(y) > 0$*) can be constructed by taking* $\Sigma_{jk} = \Sigma_{kj} = \epsilon_1$ *and* $\Sigma_{ii} = \epsilon_2, \forall i \neq k$*, and setting* $f_j(y)$ *appropriately.*

Proof. *Having* $\Sigma_{jk} \neq 0$ *predisposes the system to move in the direction indicated. All that remains is to calculate the amount of control that needs to be exerted to get a particle in the* j *well to the* k *well at* y*. We need then to calculate* $\inf_{T,\varphi,x}\{S_{0T}(\varphi)\ \varphi(0) = e_j, \varphi(T) = x, x \in \partial D\}$ *where*

$S_{0T}(\varphi) = \int_0^T |\dot{\varphi} - b(\varphi)|^2 dt$ and D is the basin of attraction for the fixed point of rail k. Normally this would involve a lot of variational calculus [GF00], but given that $S_{jk} \gg S_{ii} \forall i$ and also that the vector field is gradient, the path φ (so long as the jumps are in generic position) can be approximated be a straight line between e_k and e_j, and D should be the location of the maximum of the potential along that line. Along this line, we have

$$V(x,y) = \Sigma_j R_j(x,y) = -f_j(y)\exp\{-(x-e_j)^T(\sigma_j(y))^{-1}(x-e_j)\} - f_k(y)\exp\{-(x-e_k)^T(\sigma_k(y))^{-1}(x-e_k))\}.$$

We are considering the problem along the line $x(t) = e_j + t(e_k - e_j)$ So, we want

$$\begin{aligned}\tfrac{\partial}{\partial t}[-f_j(y)\exp\{-t^2(e_k-e_j)^T(\Sigma_j(y)^{-1})(e_k-e_j)\}\\ -f_j(y)\exp\{-(e_k-e_j)^T(\Sigma_k(y)^{-1})(e_k-e_j)\}\\ -t^2(e_k-e_j)^T(\Sigma_k(y)^{-1})(e_k-e_j))]\end{aligned}$$

Letting $(e_k - e_j)^T(\Sigma_j(y)^{-1})(e_k - e_j) = A$ and $(e_k - e_j)^T(\Sigma_k(y)^{-1})(e_k - e_j) = B$, we have

$$\frac{\partial}{\partial t}[-f_j(y)\exp(-t^2A) - f_j(y)\exp(-B - t^2B)] = 2tAf_j(y)\exp(-t^2A) + 2tBf_k(y)\exp(-B - t^2B).$$

Naturally the maximum on this line is attained by setting the equation above to 0. (Indeed it must be a maximum), so solve

$$\frac{tAf_j(y)}{2tBf_k(y)}\exp(B) = \exp(-t^2(B - A)).$$

We expect a $t = 0$ minimum, which we have, and the other critical point at somewhere $0 < t \leq 1$

$$t = \sqrt{\frac{B\ln(\frac{-f_jA}{f_kB}}{B - A}}$$

We can calculate both A and B, with our constraints above, giving $A = \dfrac{2}{\epsilon_1^2 - \epsilon_2^2}$, $B = \dfrac{2}{\epsilon_1}$

$$t = \sqrt{(\frac{1}{\epsilon_1} - 1)\ln(\frac{f_j\epsilon}{f_k(\epsilon_2^2 - \epsilon_1^2)})}$$

Now, for $t = 0$,we would have $f_j(y) = f_k(y)$ or $\epsilon = 1$, and for $t = 1$, using the simplifying *assumption that* $\epsilon_2^2 = \epsilon_1$*, we would have.*

$$\frac{f_j}{f_k} = \exp \frac{\epsilon_1}{(1-\epsilon_1)}$$

There is always space inside this range, so the construction is possible. If we let $D = (\frac{-f_j A}{f_k B})^{\frac{-B}{A-B}}$ *This gives us a*

$$V(x,y) = -f_j(y)D^A - f_k(y)D^B$$

which corresponds to a

$$\beta = f_j(y) - f_j(y)D^A - f_k(y)D^B \tag{9.1}$$

which will be a stable jump just so long as $f_k(y) > f_j(y)$.

Lemma 9.0.4. *Using a potential of the above form, it is possible to exhibit any configuration.*

Proof. *We have conditions for the stability of manifolds and for control of jumps. If we wish to have any particular configuration, all we have to do is iterate this procedure. As we want everything to be generic, we can easily take* f_i *and* Σ_i *all continuous in* y.

Lemma 9.0.5. *If* $A, B \in S_{n,k,*}$ *with* $B = S(A)$ *for* S *a legal move, then it is possible to exhibit a potential in the above form which exhibits both configurations,* A *and* B*, at different values of* β

Proof. *Choose a potential description for* A *and, aside from those symbols involved in the move, this system describes* B *as well. By genericity, it must describe the common features for some range of* β*, let's call this the admissible range* $[\beta_m, \beta_M]$ *The move must have one of the move types, and we proceed in cases.*

1. *The move is a split. In this case, we have one manifold breaking in to two pieces, which introduces a new* $M_1^{M_2}$ *type symbol. We need that the well for* M_1 *is entirely stable at* β_1*, and at some* $\gamma \in (\beta_m, \beta_M)$*, it becomes unstable at a point. This can be accomplished by controlling* f_{M_1} *and* f_{M_2} *as described in* (9.1)*. The idea is to cause the well to be stable but all points except some distinguished* y^* *when* $\beta = \gamma$*, at which point a jump is preferred by having* $(\Sigma_{M_1}(y^*))^{-1}_{M_1 M_2} = (\Sigma_{M_1}(y^*))^{-1}_{M_2 M_1} > 0$ *with all other non-diagonal elements* 0.

2. *The move is a collapse. This case is similar to the above, except that it removes a single* $M_1^{M_2}$ *type symbol, but it can be accomplished by having the well for this symbol so shallow that the entire manifold is unstable by some* $\gamma \in (\beta_m, \beta_M)$.

3. *The move is a retargeting, we have a symbol transition of* $M_1^{M_2} \to M_1^{M_3}$. *This gives that the well for this symbol has a jump preferred to* M^2 *at* β_m *For some* $\gamma \in (\beta_m, \beta_M)$, *we can find the last* y *which we will denote* y^* *at which the manifold is stable. To achieve the jump, we need that* $(\Sigma_{M_1}(y^*))^{-1}_{M_1 M_3} > (\Sigma_{M_1}(y^*))^{-1}_{M_1 M_2}$. *This causes the preferred jump to switch targets, as desired.*

4. *The move is an order exchange. This has the form* $M_1^{M_2} M_3^{M_4} \to M_3^{M_4} M_1^{M_2}$ *again can be accomplished by changing the heights alone. To achieve an order exchange at* $\gamma \in (\beta_m, \beta_M)$ *simply by having the respective well depths change at different rates so that one is preferred at* β_m *and the other at* γ.

Corollary 9.0.6. *If* $A, B \in S_{n,k,*}$ *with* $B = S_n \circ ... \circ S_1(A)$, *with the* S_i *legal moves, then it is possible to exhibit a potential in the above form which exhibits* $A, S_1(A), S_2(S_1(A)), ...B$ *at different values of* β

Proof. *This can be accomplished by iterating the above.*

Every path in the graph of legal moves can be represented as such a sequence, and thus our theorem is proven.

Chapter 10

Complexity results

With the grammars and categories defined as above, it is natural to ask whether or not different $S_{n,k}$ have different complexities, and in what sense this is the case, if so. Really, one would like to know if there's any point to increasing n and k beyond some certain threshold.

10.1 Move complexity and the set of legal moves

The first thing to notice when setting off in this matter is that the above grammatical results show that there are moves introduced until $n = k = 2$, at which point, for all $n, k \geq 2$ every type of move is represented. By genericity there are no more kinds of transitions which can occur. If we are to take our cue from this, it would seem plausible that the overall complexity could be bounded somehow.

10.2 Alphabetic complexity

However, if we add new manifolds, the alphabet with which we work continues to increase. This definitely means that there are more configurations possible, but are there more possible in any meaningful way? How would we tell?

Definition 10.2.1. *Let* $L(K) = \{S : S : K \to X, X, K \in S_{n,k,*}\}$. *Let* $D(K) = \{X : X = S(K), S \in L(K)\}$. *Take* $T(K) = \#\{unique\ elements\ in D(K)\}$.

Definition 10.2.2. *We will take a* **rail embedding** $F : S_{n,k,*} \to S_{n',k',*}$ *where either* $n' = n, k = k' + 1$ *or* $n = n' + 1, k = k'$ *to be any map such that, for any sequence* $X \in S_{n,k,*}$,

1. *The number of periodic orbits in X and $F(X)$ are the same.*

2. *If a symbol in the jump sequence for X is in the ith periodic orbit, it is in the ith periodic orbit for $F(X)$.*

3. *If a symbol in X is pre-periodic for the ith periodic orbit, it is still pre-periodic for the ith periodic orbit in $F(X)$.*

Definition 10.2.3. *$S_{n,k,*}$ and $S_{n',k',*}$ have* **The same complexity** *if $\exists F : S_{n,k,*} \to S_{n',k',*}$ a rail embedding such that $\forall X \in S_{n,k,*}$ $T(X) = T(F(X))$*

10.3 Description of a strictly n-k configuration for each n, k, and the sense in which it is irreducible

Definition 10.3.1. *In $S_{n,k,*}$ take $M = U_1^{D_k}...U_{n-k-1}^{D_2}U_{n-k}^{U_{n-k+1}}...U_{n-1}^{U_n}U_n^{D_1}D_k^{U_2}...D_1^{U_k}$*

You may recall the initial oscillator J, J and M are similar in construction, but while J has the maximum number of 2-cycles, M has one $n+k$-cycle.

Chapter 11

Topological Temperature

This classification allows us to make a few observations. First, if I have two configurations $A, B \in S_{n,k,*}$ for some n, k, it may be of interest to know whether or not $A \to B$ or $B \to A$ under legal moves, if so, how many, and in which direction. Additionally, the statement that neither $A \to B$ or $B \to A$ tells us that they can't both be exhibited by the same dynamical system without having to know any other particulars of the dynamics.

If, for example, $A \to B$ then we know that, if A and B were to both be exhibited by the same dynamical system, then B would necessarily occur at a higher β value than A and thus happens at a higher temperature. How much higher, or what the absolute temperature is, depend on details of the original system that are not encoded in our framework. What can be said, however, is that based only on the topology of the configurations, B must be at higher temperature. This is the basic notion of **Topological Temperature**.

In some cases, however, the analogy to regular temperature will not make sense. It is perfectly feasible to have $A \to B$ and $B \to A$, at which point A is hotter than B and B is hotter than A. Further, if neither of the systems comes from the other, they both do still exhibit their own configurations at certain β values, but comparing the temperatures is not meaningful, and it can't be said that either one is at higher temperature than the other.

Theorem 11.0.1. *If two configurations are given, C_1 and C_2 in $S_{n,k,*}$ one of the following must be true:*

1. *There is no path relating C_1 to C_2 in $S_{n,k,*}$*

2. *C_1 and C_2 are related by a sequence of legal moves and a minimum number of moves required is well-defined.*

Proof. *Using the lemmas below, we know that if* C_1 *and* C_2 *can be reduced to the same element of* $Start_{n,k}$ *in* N_1 *and* N_2 *invertible moves, respectively. Then their distance can be bounded by* $N = N_1 + N_2$*. Using this bound, take the grammar for* $S_{n,k,*}$ *replacing the start rule with* $s \to C_1$ *or* $s \to C_2$*, iterating the rules must reveal the other (if it is possible to do so) within* N *moves.*

If only one, C_1*, can be reduced to the designated element of the starting set with only* N_1 *invertible moves, but the other,* C_2*, can be generated from the same element in* N_2 *non-invertible moves, then only* $C_1 \to C_2$ *is possible and is bounded above by* $N_1 + N_2$

If neither can be reduced to any element of the start set, no bound is readily apparent, but the grammar will reveal through iteration whether or not they are indeed connected.

If each can be reduced to a separate element of the start set, then they are not connected by any path.

Lemma 11.0.2. *If* $A, B \in S_{n,k,*}$ *with* A *having* a *jumps and* B *having* b *jumps, then at least* $|a-b|$ *moves are required to transition from* A *to* B*.*

Proof. *The legal moves can at most add or subtract a single jump, so at least* $|a-b|$ *of them are required to go either* $A \to C$ *where* C *has* b *jumps or* $B \to C$ *whereC has* a *jumps.*

Lemma 11.0.3. *If sequences* $C_1, C_2 \in S_{n,k}$*, the graph distance from* C_1 *to* C_2*, is either undefined (because there is no path between* C_1, C_2*) is bounded above by* $2(|C_1| + |C_2|) + (|C_1| + |C_2|)^2$*.*

Proof. *To begin, add all the moves from* C_2 *to* C_1 *in whatever order possible (for* $|C_2|$ *additions. Then, rearrange (if possible) until some subsequence is the target,* C_2 *(this could take no more than* $(|C_1| + |C_2|)^2$ *moves given by the number of exchanges possible)) then at most* $|C_1| + |C_2|$ *retargets, then* $|C_1|$ *deletions.*

Of course, this begs the question answered by the following

Lemma 11.0.4. *There is no path in* $S_{2,2,*}$ *from* $C_1 = U_1^{D_1} D_1^{U_1} U_2^{D_2} D_2^{U_2}$ *to* $C_2 = U_1^{D_1} U_2^{D_2} D_1^{U_1} D_2^{U_2}$

Proof. *This is equivalent to inverting the move i described as "non-invertible". To invert this move, you need fo find some way to, through whatever means, get the* $U_2^{D_2}$ *term back on the left of* $D_1^{U_1}$*. This can not be done with order exchange, nothing can be collapsed, and the only things that* *can be added are further yet to the right of the existing $U_2^{D_2}$, which doesn't help. Retargeting can not change sources, thus there is no move sequence that takes $C_1 \to C_2$.*

These, if used iteratively, gives us.

Theorem 11.0.5. *If any configurations $C_1, ..., C_m$ are given, all residing in some $S_{n,k,*}$. One of the following is the case.*

1. *There is a dynamical system which exhibits all of the given configurations in that order and it can be constructed using the methods of the transitivity chapter.*
2. *The series of configurations cannot be derived, in that order, from any dynamical system.*

Chapter 12

Additional results

12.1 Relaxation of Compactness and lack of fixed points with impact on grammars

If the assumption of compactness is relaxed, our legality restrictions become much more lenient. Indeed, every sequence of moves is legal unless it contains a symbol of the form $x \to x$. Relaxing the assumption of fixed points can be handled in much the same way, but with the introduction of new symbols of the form X°, meaning that a particle on manifold X doesn't go anywhere. In this case, as a manifold shortens, it could go from a jump in the previous sense, to a fixed point behavior. This only requires the addition of a few extra rules in the formal language of the form

$$A^B \to A^\circ$$

$$D^x B^\circ \to D^x B^D \text{ for } D \text{ downward}$$

$$B^\circ U^x \to B^U U^x \text{ for } U \text{ upward}$$

.

12.2 Relaxation of consideration of only generic βs and the Markov description that can result

Relaxing this assumption leads to a lot of interesting new behavior. Using only the generic β values, the Kramers' rate description gives a clear target for every jump. At the singular β values, this need no longer be the case, and despite being a seemingly deterministic system, the Kramer's rate can ascribe a probability to jump to any of several target manifolds at the same instant. In this description, the system no longer merits analysis by a jump sequence, but instead each jump would have a corresponding Markov chain.

12.3 Noise in the slow direction

If we allow noise in the slow direction as well as the fast direction, instead of recovering a hybrid system from SISR, we more properly recover a stochastic hybrid system as detailed below.

Definition 12.3.1. *A stochastic hybrid system [HLS00] (or automaton) is a collection* $H = (Q, X, Inv, f, g, G, R)$ *where*

- Q *is a discrete variable taking countably many values in* $Q = \{q1, q2, ...\}$
- X *is a continuous variable taking values in* $X = \mathbb{R}^N$ *for some* $N \in \mathbb{N}$
- $Inv : Q \to 2^X$ *assigns to each* $q \in Q$ *an invariant open subset of* X
- $f, g : Q \times X \to TX$ *are vector fields*
- $G : E = Q \times Q \to 2^X$ *assigns to each* $e \in E$ *aguard* $G(e)$ *such that*
  - *For each* $e = (q, q') \in E$, $G(e)$ *is a measurable subset of* $\partial Inv(q)$ *(possibly empty)*
  - *For each* $q \in Q$*,the family* $\{G(e) : e = (q, q') \text{ for some } q \in Q\}$ *is a disjoint partition of* $\partial Inv(q)$*.*
- $R : E \times X \to P(X)$ *assigns to each* $e = (q, q') \in E$ *and* $x \in G(e)$ *a reset probability kernel on* X *concentrated on* $Inv(q')$*.Here* $P(X)$ *denote the family of all probability measures on* X*. Furthermore, for any measurable set* $A \subset Inv(q')$, $R(e, x)(A)$ *is a measurable function in* x*.*

The advantage of this framework is that it, on the one hand, subsumes the work done for systems of both 2 and more-than-2 manifolds. This is nice, and is of good validity so long as the timescale isn't rapidly-varying (being the case near the SISR limit) and provides a good counterpoint to the SISR analysis (being the other asymptotic regime).

Appendix A

Background and necessary results from various subjects

A.1 Hybrid Systems

The dynamics given by our scale separated systems, after having taken the SISR asymptotic and eliminated the noise, do not give a differential equation generically. That is to say, while a particle is on a manifold, simple dynamics govern its evolution, but at the end of a slow manifold, the particle moves infinitely fast to a different slow manifold. In reality, this is a simple case of a hybrid system [GST09], which is to say a combination of

1. A collection of differential equations on their respective domains

2. A collection of maps from the boundaries of these domains back into the same set.

For us, all the domains are lines, so the boundaries have 2 points. The dynamics are to move in a specified direction with unit rate. Our maps correspond to the jumps, and will be of primary interest.

A.2 Topology

It is important for the dynamics to be confined in a compact set. For our purposes (as we are working in $\mathbb{R}^n$) this is just a closed and bounded set [Mun00].

A.3 Category theory

A category [DF04] is a pair $C = (O, M)$ of objects O and morphisms M. Each morphism is of the form $A \to B$ for $A, B \in O$ and M is closed under composition. This is further subject to the restriction that $Id : A \to A$ must be in M for all A and the composition of maps needs to be associative.

A.4 Graph theory

We also use some standard results from graph theory. A Graph [NBW06] G is a collection $G = (V, E)$ of vertices V and directed edges E. The edges can be regarded as pairs (V_1, V_2) to indicate $V_1 \to V_2$ on the graph.

Given a graph $G = (V, E_1)$, its path completion $P = (V, E_2)$ is defined as follows: an edge exists in E_2 between V_1 and V_2 if a directed path in E_1 leads from V_1 to V_2. This includes loops from V_1 to itself. As it is stated, the path completion of a graph is a category.

A.5 Formal Grammars

A formal grammar [Cho56] is defined to be a collection $G = (N, \Sigma, P, S)$ being non-terminal symbols N, terminal symbols Σ, a start symbol (or symbols) $S \subset N$, and a set of rules P.

Each rule in P has the form $(\Sigma \cup N)^* N (\Sigma \cup N)^* \to (\Sigma \cup N)^*$. Here, $*$ is the Kleene star, giving the smallest collection of all strings of characters formed from the set on which it operates by concatenation, including the empty set.

In more mathematical terms, one can define a graph with vertices being strings in $(\Sigma \cup N)^*$ and edges being the maps in P.

Appendix B

Descriptions

There are four major descriptions of the situation as it is. Any space of configurations can be described by a Graph, a Category, a Markov Chain, and by a Context sensitive grammar. Each have their advantages.

B.1 Graph

As a graph, we have configurations corresponding to nodes, and directed edges corresponding to simple moves between configurations. The graph only illustrates the structure without defining it. If an actual particle were in a system which was changing temperature, it would follow this graph with dynamics on the graph itself. [DL10]

B.2 Category

As a category, we have configurations as objects, with morphisms existing between objects if a transition is possible. If one were to represent the category as a graph, it would have the structure of the path completion of the above graph. This is a disadvantage as the simple paths become hard to discern. This framework, however, gives a lot of very natural ideas about how to map between the graphs presented above.

B.3 Markov Chain

The Markov chain description [Nor98] relies on using the rules for the grammar as nonzero rates in a potentially infinite dimensional Markov matrix. This is interesting, but not immediately more useful than any of the other descriptions.

B.4 Context Sensitive Grammar

This format is perhaps the most accessible, although it carries the most baggage. The transient communicating class is represented by the choice of start symbol, and the moves are not just arrows, but manipulations of a string of symbols. This makes it very clear what types of moves are involved, and gives a constructive basis for the generation of sequences, as well as a very direct way to talk about whether or not two sequences have any relations.

Appendix C

Graphs and formal language validation

In the code below, for legibility, tabs have been replaced with the '.' character.

```
#!/usr/bin/env python2.7

import pydot

import re


#this little block controls which graphs to generate

#and calls the function to generate them.

for upr in [1,2,3]:

.for dnr in [1,2,3]:

..if upr>dnr:

...continue

..for gns in range(1,3):

...grammar.Grammar(upr,dnr,gns)


#This function runs the graph generators

def Grammar(uprails,downrails,generations):

.#the guts. Setup

.global n

.n=uprails

.global k

.k=downrails

.global gens
```

```
.gens=generations
.global seen
.seen = set([])
.global rules
.rules=[]
.global prerules
.prerules=[]
.new = []
.toonew=[initial()]
.for it in range(gens):
...#loop here.
...#figure out which of them are actually new.
...new=list(set(toonew)-seen);
...toonew=[]
...for node in new:
.....#add all the sequences we've seen already.
.....seen.add(node)
.....#figure out what new sequences we have to work with.
.....nodes=[]
.....toonew.extend(rulestep(node))
.for r in prerules:
...if not ((r[1],r[0],r[2]) in rules and r[2]!=4):
.....rules.append(r)
.for tn in toonew:
...seen.add(tn)
.makedot(set(rules))
.#makedot(rules)
.
```

```
#utility function to find new objects only.
def unseen(st):
..return [aa for aa in set(st) if aa not in set(seen)]

#for initial, i'm making the distinguished start sequence
#(maximal number of loops w/ minimal number of manifolds)
def initial():
..start=[];
..for i in range(n,0,-1):
....if i>=k and i<n:
......start.append('U%d>U%d,'%(i,i+1))..
....elif i>=k:
......start.append('U%d>D%d,'%(i,k))
....else:
......start.append('U%d>D%d,'%(i,k-i))
..for i in range(k):
....start.append('D%d>U%d,'%(k-i,k-i))
#returning as a string!
..return ''.join(start);
..
#determine if a string is legal or not. Every up manifold must go to another
#up manifold or a down manifold previous to itself in a sequence,
#every down must go to another down or an up subsequently.
def legal(st):
..#truth = ''
..truth =True
..for i in range(0,len(st)-6,6):
....if (st[i:i+2]==st[i+3:i+5]):
```

```
......truth = False
....elif  (st[i+3] == 'U' and st.find(st[i+3:i+5]+'>',0,i)==-1):
......truth = False
....elif  (st[i+3]=='D' and st.rfind(st[i+3:i+5]+'>',i)==-1):
......truth = False
..return truth


#Anything of the form Xa^b -> c^aXa^b for a=Di
#anything of the form d^bX -> d^bXc^d for d=Ui
def split(st):
..ret = []
..for i in range(0,len(st),6):
....drn = st[i]
....for j in range(0,len(st),6):
......if (drn == 'D' and j<i+1):
........for j in range(0,i+1,6):
..........for l in range(1,k+1):
............if l != int(st[i+1]):
..............out = st[:j]+ 'D%d>D'%l+st[i+1]+','+st[j:]
..............ret.append(out)
..............if not legal(out):
...............print('illegal split 1');
..........for l in range(1,n+1):
..............out = st[:j]+ 'U%d>D'%l+st[i+1]+','+st[j:]
..............ret.append(out)
..............if not legal(out):
...............print('illegal split 2');
..
```

```
......elif (drn=='U' and j>i+1):
........for j in range(i+6,len(st)+1,6):
..........for l in range(1,n+1):
............if l != int(st[i+1]):
..............out = st[:j]+ 'U%d>U'%l+st[i+1]+','+st[j:]
..............ret.append(out)
..............if not legal(out):
...............print('illegal split 3');
..........for l in range(1,k+1):
..............out = st[:j]+ 'D%d>U'%l+st[i+1]+','+st[j:]
..............ret.append(out)
..............if not legal(out):
...............print('illegal split 4');
..return ret

#so the rules can't implement representational equivalence, which means that
#the collapse rule can eventually descend to another of
#the equivalent members of the start class, but this isn't
#strictly an inverse of the splitting operation.
def collapse(st):
..ret=[]
..for i in range(0,len(st),6):
....drn = st[i]
....for j in range(0,len(st),6):
......if st[j:j+2]==st[i:i+2]:
........out = st[:i]+st[i+6:]
........if (drn=='U' and j<i):
..........ret.append(out)
```

```
.........if not legal(out):
...........print('illegal collapse! ' + st + ' -> ' +out+ ' '+ str(i) +' '+str(j))
.......elif (drn == 'D' and j>i):
.........ret.append(out)
.........if not legal(out):
...........print('illegal collapse! ' + st + ' -> ' +out+ ' '+ str(i) +' '+str(j))

..return ret


#type c^dXa^b -> c^dXa^c for c=Ui
#type c^dXa^b -> c^aXa^b for a=Di
def retarget(st):
..ret=[]
..for i in range(0,len(st),6):
....R=[]
....s1 = st[i:i+2]
....t1 = st[i+3:i+5]
....for j in range(0,len(st)-5,6):
......s2 = st[j:j+2]
......t2 = st[j+3:j+5]
......out = st[:i+3]+s2+','+st[i+6:]
......if j<i and st[j]=='U' and s1 != s2 and t1 != s2:
........ret.append(out)
........if not legal(out):
..........print(st[j]+'t ' + st)
..........print( '=  ' + out)
........if st == out:
..........print(st[i]+'RT same')
```

```
......elif j>i and st[j]=='D'and s1 != s2 and t1 != s2:
........ret.append(out)
........if not legal(out):
..........print(st[j]+'t ' + st)
..........print( '=  ' + out)
........if st == out:
..........print(st[i]+'RT same')
..return ret

#type a^db^c -> b^ca^d for a,b same direction c != a !=b , d != b (self-invertible)
def cisOE(st):
..ret=[]
..for i in range(0,len(st)-12,6):
....s1 = st[i:i+2]
....t1 = st[i+3:i+5]
....s2 = st[i+6:i+8]
....t2 = st[i+9:i+11]
....if (st[i] == st[i+6]) and not ((st[i+1] ==  st[i+7]) or (s1 == t2) or (s2 == t1)):
......out = (st[:i]+st[i+6:i+12]+st[i:i+6]+st[i+12:])
......if not legal(out):
........print('CE ' + st)
........print('=  ' + out)
......if st == out:
........print('COE same')
......ret.append(out)
..return ret

#type a^db^c -> b^ca^d for a,b different direction c != a !=b, d != b (not invertible)
```

```
#might need to be broken into cases!
def transOE(st):
..ret=[]
..for i in range(0,len(st)-7,6):
....s1 = st[i:i+2]
....t1 = st[i+3:i+5]
....s2 = st[i+6:i+8]
....t2 = st[i+9:i+11]
....if st[i] == 'U' and not ((st[i] ==  st[i+6]) or (s1 == t2) or (s2 == t1)):
......out = (st[:i]+st[i+6:i+12]+st[i:i+6]+st[i+12:])
......if(not legal(out)):
........print('TE ' +st)
........print('=  '+out)
......if st == out:
........print('TOE same')
......ret.append(out)
..return ret

#Take a list of strings and find all the possible destinations from that string.
def rulestep(i):
..ret=[]
..for j in split(i):
....ret.append(j)
....prerules.append((i,j,1))
..for j in collapse(i):
....ret.append(j)
....prerules.append((i,j,1))
..if n>1:
```

```
....for j in retarget(i):
......ret.append(j)
......prerules.append((i,j,2))
....for j in cisOE(i):
......ret.append(j)
......prerules.append((i,j,3))..
....for j in transOE(i):
......ret.append(j)
......prerules.append((i,j,4))
..return ret
..
#write it all out.
def makedot(edges):
..graph=pydot.Dot(graph_type = 'digraph', overlap='false')
..graph.set_node_defaults(label ='""', shape='point', color='red', nodesep='1')
..clusters= []
..for it in range(gens+1):
....clusters.append(pydot.Cluster('%d'%it));
..for x in seen:
....gen = (len(x)+1)/6 -n -k

....clusters[gen].add_node(pydot.Node(x))..
..for it in range(gens+1):
....graph.add_subgraph(clusters[it])
..for s in edges:
....edge=pydot.Edge(''.join(s[0]),''.join(s[1]))
....if s[2] == 1:
......edge.set_color("black")
```

```
......edge.set_arrowhead("false")
....elif s[2]==2:
......edge.set_color("blue")
......edge.set_arrowhead("false")
....elif s[2]==3:
......edge.set_color("green")
......edge.set_arrowhead("false")
....else:
......edge.set_color("orange")
....graph.add_edge(edge)
..filename= str(n)+str(k)+str(gens)
..graph.write(filename+'.dot')
..graph.write_pdf(filename+'.pdf', prog='neato')
```

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