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ANALYSES OF SKEW SLABS

A REPORT OF AN INVESTIGATION
CONDUCTED BY
THE ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS
IN COOPERATION WITH
THE PUBLIC ROADS ADMINISTRATION
FEDERAL WORKS AGENCY
AND
THE DIVISION OF HIGHWAYS
STATE OF ILLINOIS
BY
VERNON P. JENSEN

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ANALYSES OF SKEW SLABS

I. INTRODUCTION

1. Preliminary Remarks.—The modern highway, straightened to accommodate the greater speeds and more exacting safety requirements of present-day traffic, has required the design of increasing numbers of skew bridges. Despite this fact, relatively little is to be found in the literature regarding skew slabs. As a result of the lack of analytical data, the design of skew slabs is at present based largely on empirical rules formulated by judgment, and influenced by comparisons with rectangular slabs which can be analyzed. In this bulletin the behavior of skew slabs is determined by means of difference equations. The slabs studied have various angles of skew and various boundary conditions and are loaded uniformly or by wheel loads. Particular study is made of a simple-span slab-bridge with curbs.

Three publications on skew slabs are known to the author. In the first paper, Cecilia Vittoria Brigatti* attempts to apply Marcus' method of handling difference equations. Skew slabs uniformly loaded and either simply supported or fixed on all four edges are considered. Particular attention is given to the slab having all edges of the same length. The fundamental equations given by Brigatti are, however, believed to be in error. Certain checks which have been applied show a lack of agreement with results obtained by established equations. The error appears to be a function of the angle of skew, disappearing at zero skew.

In the second paper Adolf Anzelius† gives a solution of the differential equation for a skew slab loaded uniformly, simply supported on two opposite edges, and free on the other edges. The solution is in the form of a series for which coefficients must be determined, theoretically, from an infinite system of linear equations. Actually Anzelius neglects all but the first ten coefficients which he obtains from ten equations which are applicable to a particular slab of 45-degree skew. Only twisting moments are given, and these only qualitatively, so that checks by means of the relations of statics cannot be made. However, these qualitative results are in decided disagreement with the distribution of twisting moments found herein for a slab of

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identical shape. The equations given by Anzelius are too cumbersome to permit a direct check of his work.

The third paper, by Helmut Vogt,* is based on a doctoral dissertation† by the same author. Unfortunately, the publication omits all of the theoretical work, and gives merely a summary of the conclusions reached. These are based on analyses of slabs without curbs, having two opposite edges simply supported, and carrying uniformly distributed loads.

As stated previously, difference equations are applied to the problems studied in this bulletin. While they yield an approximate result, their use is particularly justifiable here because the complicated boundary conditions have so far prevented the expression of exact solutions for the skew slab. The magnitude of the inaccuracies which result from the substitution of a set of difference equations for a differential equation may be made to diminish by increasing the number of equations. Furthermore, where an exact solution is available, the degree of approximation may be determined by a direct comparison of the results obtained. Thus in the literature relating to the application of difference equations to slabs,‡ it has been shown that practical results are obtainable with reasonable labor provided that the bending moments do not vary too sharply. The important region of a simple slab-bridge where moments will vary sharply is under a wheel load in the central portion of the span. A method is indicated in this bulletin for correcting the moments given by the difference equations so as to account in a practical way for the assumed size of the loaded area under a truck wheel. For this purpose Westergaard’s approximate formulà is used in conjunction with certain results found herein.

The application of difference equations to slabs was developed extensively in Denmark by N. J. Nielsen§ and at about the same time in Germany by H. Marcus.** In this country an exposition of the method of Marcus was given in a paper by Joseph A. Wise†† who subsequently

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made further applications* to square slabs. D. L. Holl† has applied difference equations to slab problems in two papers published within the last five years. The majority of the slabs analyzed by difference equations have been square or rectangular, permitting the use of square or rectangular networks of points and relatively simple equations. Marcus has, however, indicated the form of some of the fundamental equations for radial networks or for skew networks of a restricted nature. Little use has been made of these networks and, as far as the author knows, there are no published records of slabs having free or elastically supported boundaries which have been analyzed by their means.

In order to develop a procedure for analyzing a skew slab-bridge with curbs, a study was made of various networks of points and the difference equations relating to these networks and relating to the various boundary conditions encountered. It was found that a square or rectangular network could be conveniently used for a number of angles of skew and proportions of sides, but that it was desirable also to develop equations applicable to a network of points formed by the intersections of lines parallel to the sides of the skew panel. Such equations were developed and applied first to a number of slabs under uniform load. Calculations were made later for a slab-bridge having a 26-ft. roadway, 45-degree skew, 19 ft.-6 in. span (normal to the abutments), ordinary curbs, and loading designated as H-20 by the American Association of State Highway Officials.

2. Acknowledgment.—The data contained in this bulletin form a part of the results of an investigation of concentrated loads on reinforced concrete bridge slabs being conducted in the Engineering Experiment Station in cooperation with the Public Roads Administration of the Federal Works Agency and the Illinois Division of Highways. The project is under the administrative direction of Dean M. L. ENGER, Director of the Engineering Experiment Station; Professor W. C. HUNTINGTON, Head of the Department of Civil Engineering; and Professor F. B. SEELY, Head of the Department of Theoretical and Applied Mechanics. F. E. RICHART, Research Professor of Engineering Materials, gives general supervision to the work of the investigation.

An Advisory Committee, having the following personnel, is in gen-

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eral charge of the plan of the program of the investigation: E. F. Kelley, Chief, Division of Tests, and L. A. Palmer, Associate Research Specialist, representing the Public Roads Administration; until his death, Mr. A. L. Gemeny was a representative of the Public Roads Administration; Ernst Liebermann, Chief Highway Engineer, and A. Benesch, Engineer of Grade Separations, representing the Illinois Division of Highways; F. E. Richart and N. M. Newmark, Research Assistant Professor of Civil Engineering, representing the University of Illinois.

As consultants to the Advisory Committee are W. M. Wilson, Research Professor of Structural Engineering and T. C. Shedd, Professor of Structural Engineering, both of the University of Illinois.

Acknowledgment is made of the valuable assistance given by members of the staff and the committee, and especially for the criticisms and suggestions which have been made by Professor Newmark. Mr. J. W. Allen, Special Research Graduate Assistant, and Mr. W. N. Findley, Instructor in Theoretical and Applied Mechanics, have made detailed computations, and Mr. Allen has contributed materially to the analytical results which are contained herein.

3. Notation.—The following notation is used in this bulletin:

- $w$ = deflection of slab, positive downward
- $x, y$ = horizontal rectangular coordinates
- $u, v$ = coordinate axes inclined to $x$ and $y$ as shown in Fig. 8
- $z$ = deflection of curb, positive downward
- $a, b, c, \ldots$ = letters designating points on the slab as shown in various figures. The letters $a$ and $b$ may also be used to represent the span or width of a slab as defined in the figures. For a rectangular network of points the subscripts $N, S, E, W$, and combinations are used to designate north, south, east, and west of an origin $O$.
- $w_a, w_b, \ldots w_1, w_2, \ldots$ = deflections of points $a, b, \ldots 1, 2, \ldots$ etc.
- $h$ = thickness of slab
- $E$ = modulus of elasticity of the material of the slab
- $E_1$ = modulus of elasticity of the material of the curb or edge beam
- $\mu$ = Poisson's ratio of the material of the slab
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\[ N = \frac{Eh^3}{12(1-\mu^2)}, \] measure of stiffness of the slab

\[ I_1 = \text{moment of inertia of the cross sectional area of the curb or edge beam} \]

\[ H = \frac{E_1I_1}{aN}, \] dimensionless quantity defining the relative stiffness of curb or edge beam to slab \((a = \text{span of slab})\)

\( \lambda, \lambda_x, \lambda_y, \ldots = \) distances between points or lines of the network as defined in various figures

\[ J = \frac{\lambda_y^3 E_1I_1}{\lambda_x^2 \lambda_z N}, \] dimensionless number proportional to the relative stiffness \(H\)

\( \alpha, \beta = \) portions of \(\lambda_x\) as defined in Fig. 8

\( A = \alpha/\lambda_x, \) an abbreviation

\( B = \beta/\lambda_x, \) an abbreviation

\[ C = \frac{\lambda_y^2}{\lambda_x^2} - AB, \] an abbreviation

\[ D = (1 - \mu) \frac{\lambda_y^2}{\lambda_x^2} = (1 - \mu) (C + AB), \] an abbreviation

\( P = \) concentrated load

\( p = \) distributed load per unit of area, positive when acting downward on the slab

\( q = \) intensity of load per unit of length, positive when acting downward on a curb or edge beam

\( V_x, V_y = \) vertical shear per unit of length, acting on sections normal to the \(x\) and \(y\) axes, respectively, positive on a rectangular element of a slab when acting upward on the side of the element having the smaller value of \(x\) or \(y\), respectively

\( M_x, M_y = \) bending moments per unit of length, acting on sections normal to the \(x\) and \(y\) axes, respectively, positive when producing compression at the top of the slab

\( M_{xy} = \) twisting moment per unit of length, acting on sections normal to the \(x\) and \(y\) axes, respectively, positive when producing compression at the top of the slab in the direction of the line \(x = y\)
\( M_{\text{curb}} = \) bending moment in the curb or edge beam, positive when it produces compression in the top

\( R_x, R_y = \) reactions per unit of length of slab, acting on sections normal to the \( x \) and \( y \) axes, respectively, positive in the same sense as the corresponding shears

\( R_n = \) reaction per unit of length of slab acting on an edge whose normal is taken as the direction of \( n \)

\( R_c = \) concentrated reaction at corner

\( N^2 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \) Laplace's operator in two variables

\( U = \nabla^2 w \)

\( \phi = \) angle of skew (see Figs. 7 and 8)

\( \delta = \) angle between \( v \) and \( y \) axes (see Fig. 8)

II. DERIVATION OF EQUATIONS

4. General Considerations.—For the purpose of the analysis, the slab is assumed to be of homogeneous, isotropic, and elastic material, and to be of constant thickness. The curb is replaced in the analysis by an edge beam of stiffness \( E_i I_i \) attached to the slab in such a way as to undergo the same deflection as the edge of the slab, and to transmit only vertical reactions to the slab at its edge.

The ordinary theory of thin plates requires that the deflected neutral surface of the slab shall satisfy the fundamental differential equation

\[ N \nabla^2 w = p \]  \hspace{1cm} (1)

in which \( p \) is the intensity of load, \( w \) is the deflection of the slab, Laplace's operator for two variables is

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \]  \hspace{1cm} (2)

and \( N \), a measure of the stiffness of the slab, is

\[ N = \frac{Eh^3}{12 (1 - \mu^2)}. \]  \hspace{1cm} (3)
ANALYSES OF SKEW SLABS

The equation for the deflection \( w \) must also satisfy the boundary conditions of the slab. When such a solution has been found it furnishes a complete answer to the question of the behavior of the slab, since bending moments, shears, and reactions may be obtained by differentiation according to the following formulas:

Moments:

\[
M_x = -N \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)
\]

\[
M_y = -N \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right)
\]

\[
M_{xy} = -N \left( 1 - \mu \right) \frac{\partial^2 w}{\partial x \partial y}
\]

Shears:

\[
V_x = -N \frac{\partial}{\partial x} (\nabla^2 w)
\]

\[
V_y = -N \frac{\partial}{\partial y} (\nabla^2 w)
\]

Reactions:

\[
R_x = V_x + \frac{\partial M_{xy}}{\partial y} = -N \left[ \frac{\partial^3 w}{\partial x^3} + (2 - \mu) \frac{\partial^3 w}{\partial x \partial y^2} \right]
\]

\[
R_y = V_y + \frac{\partial M_{xy}}{\partial x} = -N \left[ \frac{\partial^3 w}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x^2 \partial y} \right]
\]

Corner Reaction:

\[
R_c = \text{Sum of Twisting Moments on Intersecting Edges at Corner.}
\]

It is at once apparent that a knowledge of the exact deflection function permits the computation of the remaining quantities at any one of an infinite number of points of the slab. For the purpose of explaining the operations with difference equations the infinity of points on the slab is replaced by a finite number of regularly spaced points. At each point approximations to the load, shears, moments, etc. are expressible as linear functions of the deflections of neighboring points. This means that, without a knowledge of the exact deflection function, but knowing the deflections of a number of regularly and closely spaced points of the slab, one may determine bending
moments, etc. at these points with sufficient accuracy for engineering purposes. The linear relationships among the deflections are obtained by using finite differences in place of infinitesimally small differentials, and the equations are known as difference equations by analogy with the corresponding differential equations.

The advantage of using a conception of a finite number of points of the slab having deflections related by difference equations is that it reduces the problem from one of an infinite degree of indeterminateness to one of a finite degree. The method of solution then becomes somewhat analogous to the method of slope deflection for structures wherein a number of simultaneous equations are written and solved. In the slab the unknown quantities may be chosen as the deflections of the points of the network. In this case the known load at each point of the network is expressed by a difference equation analogous to (1). The resulting system of linear equations in the unknown deflections may then be solved. Bending moments, for example, may then be computed from the difference equations analogous to (4).

Some variation from the basic and simple procedure described in the preceding paragraph is possible. For rectangular slabs with simply supported boundaries it is convenient to use the method adopted by Marcus* of breaking Equation (1) into two parts which may be written as

\[ \nabla^2 U = \frac{p}{N} \]  \hspace{1cm} (8)

and

\[ \nabla^2 w = U. \]  \hspace{1cm} (9)

---

The difference analogs of these equations may then be solved in succession. The convenience of the method arises from the fact that these equations are simpler than (1) and, furthermore, the quantity $U$ must be zero on a simply supported straight boundary.

It has been noted that the finite number of points on the slab should form a regularly spaced network. In his excellent book, Marcus has indicated briefly how various networks, such as those shown in Fig. 1, may be used to express the difference equations corresponding to Equations (8) or (9). To these networks there is added in this bulletin a general skew network as shown in Fig. 2. The equations applicable to this network must reduce in special cases to those which apply to the networks shown in Fig. 1a, b, and c.

5. Difference Equations for Square and Rectangular Networks.—Let a plan of a slab be divided by equally spaced lines drawn parallel to rectangular axes $x$ and $y$, the interval between successive lines being $\lambda_x$ and $\lambda_y$ as shown in Fig. 3 for a region in the vicinity of any typical point $O$. Such a network may be employed conveniently, for example, when determining the maximum moment at the center of the uniformly loaded skew slab shown in Fig. 4.

A method of deriving a sufficient number of difference equations
for the determination of the bending moments in the slab shown in Fig. 4 is described in the following. Points in the neighborhood of a typical point \(O\) are lettered for identification as shown in Fig. 3 where a north, south, east, west system is indicated. Then the deflection of point \(O\) will be designated as \(w_0\), of point \(E\) as \(w_E\), etc. The identifying letters will also be used as subscripts for other quantities evaluated at those points.

Consider first the slope of the slab in the \(x\) direction at a point midway between points \(O\) and \(E\). In terms of finite differences this slope is

\[
\left( \frac{\partial w}{\partial x} \right)_E = \frac{(\text{slope})_E}{2} = \frac{w_E - w_0}{\lambda_x}.
\]  

Similarly, the corresponding slope at a point midway between \(O\) and \(W\) is

\[
\left( \frac{\partial w}{\partial x} \right)_W = \frac{(\text{slope})_W}{2} = \frac{w_0 - w_W}{\lambda_x}.
\]  

The curvature of the slab in the \(x\) direction at point \(O\) is the rate of change of slope, or

\[
\frac{(\text{slope})_E - (\text{slope})_W}{2} = \frac{(\text{curvature})_O}{\lambda_x}.
\]
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\[
\left( \frac{\partial^2 w}{\partial x^2} \right)_o = \frac{w_w - 2w_o + w_E}{\lambda_x^2}. \tag{12}
\]

Similarly, the curvature in the \( y \) direction at point \( O \) is

\[
\left( \frac{\partial^2 w}{\partial y^2} \right)_o = \frac{w_N - 2w_o + w_S}{\lambda_y^2}. \tag{13}
\]

Finally,

\[
(\nabla^2 w)_o = \left( \frac{\partial^2 w}{\partial x^2} \right)_o + \left( \frac{\partial^2 w}{\partial y^2} \right)_o = \frac{w_w - 2w_o + w_E}{\lambda_x^2} + \frac{w_N - 2w_o + w_S}{\lambda_y^2}, \tag{14}
\]

and, by change of variable,

\[
(\nabla^2 U)_o = \left( \frac{\partial^2 U}{\partial x^2} \right)_o + \left( \frac{\partial^2 U}{\partial y^2} \right)_o = \frac{U_w - 2U_o + U_E}{\lambda_x^2} + \frac{U_N - 2U_o + U_S}{\lambda_y^2}. \tag{15}
\]

The twists are most accurately represented by difference equations at the centers of the rectangles formed by the network. Thus from the previously given expressions for slope one finds the rate of change in the \( x \) direction of the slope in the \( y \) direction, evaluated at a point midway between points \( O \) and \( NE \), to be

\[
\left( \frac{\partial^2 w}{\partial x \partial y} \right)_{E,N} = \frac{w_{NE} - w_E - w_N + w_D}{\lambda_x \lambda_y}. \tag{16a}
\]

However, in order to determine the maximum resultant moment at a point such as \( O \), it is necessary to evaluate \( M_{xy} \) as well as \( M_x \) and \( M_y \) at that point. It is therefore necessary to evaluate the twist at the points of the network, giving for the larger enclosing rectangle,

\[
\left( \frac{\partial^2 w}{\partial x \partial y} \right)_o = \frac{w_{NE} - w_{SE} - w_{NW} + w_{SW}}{4\lambda_x \lambda_y}. \tag{16b}
\]

This is also the average of the four values of twist at the centers of the rectangles adjacent to point \( O \) as given by equations corresponding to (16a).
Equations corresponding to (8), (9), and (4) may now be written as follows:

\[
\frac{p_0}{N} = \frac{U_W - 2U_O + U_E}{\lambda_x^2} + \frac{U_N - 2U_O + U_S}{\lambda_y^2},
\]

\[
U_0 = \frac{w_W - 2w_O + w_E}{\lambda_x^2} + \frac{w_N - 2w_O + w_S}{\lambda_y^2},
\]

\[
(M_x)_0 = -N \left( \frac{w_W - 2w_O + w_E}{\lambda_x^2} + \mu \frac{w_N - 2w_O + w_S}{\lambda_y^2} \right)
\]

\[
(M_y)_0 = -N \left( \frac{w_N - 2w_O + w_S}{\lambda_y^2} + \mu \frac{w_W - 2w_O + w_E}{\lambda_x^2} \right)
\]

\[
(M_{xy})_0 = -N (1 - \mu) \frac{w_{NE} - w_{SE} - w_{NW} + w_{SW}}{4 \lambda_x \lambda_y}.
\]

The principal moments at any point \(O\) are then

\[
\frac{(M_{\text{max}})_0}{(M_{\text{min}})_0} = \frac{(M_x)_0 + (M_y)_0}{2} \pm \sqrt{\left( \frac{(M_x)_0 - (M_y)_0}{2} \right)^2 + (M_{xy})_0^2}.
\]

To illustrate the use of these equations, consider the slab shown in Fig. 4, and let \(\lambda_y = 2\lambda_x\). The boundary conditions on all edges are

\[
(w)_{\text{edge}} = 0; \quad (U)_{\text{edge}} = 0.
\]

The latter condition follows from the fact that the moments normal to the edge and the curvatures tangential to the edge must vanish at the edge. This condition permits (17) to be used directly to compute values of \(U\) at all points of the slab. These values of \(U\) in (18) permit the calculation of \(w\) at all points of the slab. These in turn are sufficient for the calculation of moments as indicated in (19) and (20).

Symmetry of the slab reduces the number of distinct interior points from 15 to 8 as shown in Fig. 4. Writing equations corresponding to (17) for each of these points in succession, and noting
that \( \lambda_x = b/6 \) and \( \lambda_y = b/3 \), yields the set of simultaneous linear equations:

\[
\begin{align*}
0 &= \frac{pb^2}{9N} + 10U_1 - 4U_2 \\
0 &= \frac{pb^2}{9N} - 4U_1 + 10U_2 - U_3 - 4U_4 \\
0 &= \frac{pb^2}{9N} - U_2 + 10U_3 - 4U_5 \\
0 &= \frac{pb^2}{9N} - 4U_2 + 10U_4 - U_5 - 4U_6 \\
0 &= \frac{pb^2}{9N} - 4U_3 - U_4 + 10U_5 - U_6 - 4U_7 \\
0 &= \frac{pb^2}{9N} - U_5 + 10U_6 - 4U_7 \\
0 &= \frac{pb^2}{9N} - 4U_4 - 4U_6 + 10U_7 - U_8 \\
0 &= \frac{pb^2}{9N} - 8U_5 - 2U_7 + 10U_8.
\end{align*}
\]

The solution\(^*\) of these equations is

\[
\begin{align*}
U_1 &= -0.29942 \frac{pb^2}{9N}, & U_5 &= -0.66191 \frac{pb^2}{9N}, \\
U_2 &= -0.49854 \frac{pb^2}{9N}, & U_6 &= -0.39387 \frac{pb^2}{9N}, \\
U_3 &= -0.41462 \frac{pb^2}{9N}, & U_7 &= -0.56920 \frac{pb^2}{9N}, \\
U_4 &= -0.59329 \frac{pb^2}{9N}, & U_8 &= -0.74337 \frac{pb^2}{9N}.
\end{align*}
\]

\(^*\)See Appendix A for notes on the solution of simultaneous equations.
Equations corresponding to (18) may now be written for each point of the network using the values of \( U \) which have just been obtained. The resulting equations are identical in form to (22), and are as follows:

\[
\begin{align*}
0 &= -0.29942 \frac{pb^i}{81N} + 10w_1 - 4w_2 \\
0 &= -0.49854 \frac{pb^i}{81N} - 4w_1 + 10w_2 - w_3 - 4w_4 \\
0 &= -0.41462 \frac{pb^i}{81N} - w_2 + 10w_3 - 4w_5 \\
0 &= -0.59329 \frac{pb^i}{81N} - 4w_2 + 10w_4 - w_5 \\
0 &= -0.66191 \frac{pb^i}{81N} + 4w_3 - w_4 + 10w_5 - w_6 - 4w_8 \\
0 &= -0.39387 \frac{pb^i}{81N} - w_6 + 10w_7 - 4w_7 \\
0 &= -0.56920 \frac{pb^i}{81N} - 4w_4 - 4w_5 + 10w_7 - w_8 \\
0 &= -0.74337 \frac{pb^i}{81N} - 8w_6 - 2w_7 + 10w_8
\end{align*}
\]

The deflections obtained from these equations are

\[
\begin{align*}
w_1 &= 0.13176 \frac{pb^i}{81N}, & w_5 &= 0.38549 \frac{pb^i}{81N}, \\
w_2 &= 0.25455 \frac{pb^i}{81N}, & w_6 &= 0.20293 \frac{pb^i}{81N}, \\
w_3 &= 0.22111 \frac{pb^i}{81N}, & w_7 &= 0.31249 \frac{pb^i}{81N}, \\
w_4 &= 0.32469 \frac{pb^i}{81N}, & w_8 &= 0.44523 \frac{pb^i}{81N}.
\end{align*}
\]
At point 8 the moments obtained from (19) and (25), when Poisson's ratio, \( \mu \), is 0.2, are
\[
\begin{align*}
(M_x)_s &= 0.0590pb^2 = 0.0787pa^2 \\
(M_y)_s &= 0.0401pb^2 = 0.0535pa^2 \\
(M_{xy})_s &= 0.0108pb^2 = 0.0144pa^2.
\end{align*}
\]
These give the principal moments
\[
\begin{align*}
(M_{\text{max}})_s &= 0.0852pa^2 \\
(M_{\text{min}})_s &= 0.0469pa^2
\end{align*}
\]
having a direction with the coordinate axes \( x \) and \( y \), respectively, defined by the equation
\[
(\theta)_s = \frac{1}{2} \tan^{-1} \frac{2(M_{xy})_s}{(M_x)_s - (M_y)_s} = \frac{1}{2} \tan^{-1} 1.144
\]
so that
\[
(\theta)_s = 24 \text{ deg.} \ 25 \text{ min.}
\]
Since the angle of skew is 26 deg. 35 min. it is seen that the maximum moment at the center of the slab is practically in the direction of the short span of the slab.

The intensity of maximum moment may be compared with that obtained for the superimposed rectangular slab \( ABCD \) shown in Fig. 5a. The network for the rectangular slab was made similar to that used for the skew slab, and is shown in Fig. 5b. The maximum
moment, at the center of the slab in the direction of the short span, is found, for Poisson’s ratio of 0.2, to be

$$(M_{\text{max}})_{\text{rect.}} = 0.0835pa^2$$

as compared with

$$(M_{\text{max}})_{\text{skew}} = 0.0852pa^2.$$  

It is seen that in this instance the rectangular slab furnishes a good approximation to the skew slab with respect to the maximum moment at the center.

A square network permits some simplification of the difference equations, since in this case

$$\lambda_x = \lambda_y = \lambda$$

in Equations (17), (18), and (19). The square network is applicable to certain slabs of 45 degree skew as well as to square and rectangular slabs. The skew slab shown in Fig. 6 is an illustration. A solution may be effected as before using Equations (17) and (18) and

$$\lambda_x = \lambda_y = \lambda = \frac{a}{4}$$

where $a$ is the short span of the slab. The moments found at the center of this slab by means of this network are, for Poisson’s ratio of 0.2,

$$M_x = 0.0389pa^2$$

$$M_y = 0.0898pa^2$$

$$M_{xy} = 0.0021pa^2,$$
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giving principal moments

\[ M_{\text{max}} = 0.0898pa^2 \]
\[ M_{\text{min}} = 0.0388pa^2 . \]

The principal moments are inclined in direction only about 2.5 degrees clockwise from the axes of y and x, respectively. The solution for this slab is discussed more in detail in Section 9.

The simplicity of the analysis of uniformly loaded simply supported skew slabs where square or rectangular networks may be used is evident from the foregoing. The solution of the simultaneous equations by the method given in Appendix A is not a lengthy matter requiring great accuracy with a calculating machine. As a matter of fact, when using the method careful slide-rule work is all that is required in solving the simultaneous equations in order to give final maximum moments of slide-rule accuracy.

The simplicity of the analysis is partly the result of the simple nature of the boundary conditions on the slab. When the edges are free to deflect or are supported by flexible beams or curbs, the conditions become more complicated and the resulting difference equations become more involved. The special considerations which must be given to a slab-bridge having any angle of skew and a general skew network are discussed in the next section. Simplifications are then made for special angles of skew and special networks.

It may be mentioned that the sets of simultaneous equations may be solved by a method of successive approximations, although the slow convergence of a straightforward procedure may be a serious handicap. Special devices and techniques have been developed for increasing the rate of convergence.* The development of a method of successive approximations is especially desirable, however, when the number of points in the network becomes very large.

6. General Skew Network.—Consider a slab-bridge having a parallelogram-shaped plan with any angle of skew, and with any given proportion of sides. Two edges are simply supported and two edges are stiffened by curbs. A network of points may be located on the slab as shown in Fig. 7 by noting the intersections of two sets of lines, each set being made up of equally spaced lines drawn parallel to one pair of sides of the slab. A third set of parallel lines, as shown in

*See, for example, George H. Shortley, Royal Weller, and Bernard Fried, “Numerical Solution of Laplace’s and Poisson’s Equations,” Bulletin No. 107, Eng. Exp. Sta., Ohio State University, Sept. 1940.
the figure, may then be drawn through the points of intersection. For
the purposes of this study the $x$-axis will be taken parallel to the
direction of flow of traffic, and consequently parallel to the curbs and
to one set of parallel lines. As shown in Fig. 7, the angle of skew will
then be defined as the angle between the $y$-axis and the simply sup-
ported edges of the slab.

The problem is to find a number of relationships between the
deflections of the slab at the points of the network, the number of
relationships to be equal to the number of points of the network.
For this purpose consider first a typical interior point $O$ and the
points in its immediate vicinity as shown in Fig. 8. The various
points in the neighborhood of $O$ have been lettered for identification.
These letters as subscripts on the deflection $w$ will indicate the de-
flexion of the slab at the corresponding points. Two inclined axes,
$u$ and $v$, are drawn in Fig. 8 in the directions of the inclined lines of
the network. The dimensions $\lambda_x$, $\lambda_u$, and $\lambda_v$ are defined as the dis-
tances between successive points of the network in the directions of
$x$, $u$, and $v$, respectively, whereas $\lambda_y$ is defined as the distance be-
tween successive lines drawn parallel to the $x$-axis as shown in Fig. 8.
The distance $\lambda_x$ is divided into two segments, $\alpha$ and $\beta$, as shown.
The $u$-axis and the $v$-axis make angles $\phi$ and $\delta$ respectively with
the $y$-axis.

The geometrical relationships just described define the relations

\[
\begin{align*}
\alpha + \beta &= \lambda_x \\
\sin \phi &= \beta/\lambda_u & \sin \delta &= \alpha/\lambda_x \\
\cos \phi &= \lambda_y/\lambda_u & \cos \delta &= \lambda_y/\lambda_x \\
\tan \phi &= \beta/\lambda_y & \tan \delta &= \alpha/\lambda_y
\end{align*}
\]

(26)
as reference to Fig. 8 will show. With the $u$ and $v$ directions established, one is now able to express derivatives of any function of $x$ and $y$, $f(x, y)$, in terms of corresponding derivatives with respect to $x$, $u$, and $v$.

The curvatures of the slab in the orthogonal directions $x$ and $y$ are expressed by the second derivatives of the deflected neutral surface $w$, namely, $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$, since the slopes of the deflected surface are small. In terms of second derivatives with respect to $x$, $u$, and $v$, the curvature in the $y$ direction may then be shown* to be given by the equation

$$\frac{\partial^2 w}{\partial y^2} = -\frac{\alpha \beta}{\lambda_y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\alpha \lambda_u^2}{\lambda_x \lambda_y^2} \frac{\partial^2 w}{\partial u^2} + \frac{\beta \lambda_v^2}{\lambda_x \lambda_y^2} \frac{\partial^2 w}{\partial v^2},$$

(27)

and consequently

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \left(1 - \frac{\alpha \beta}{\lambda_y^2}\right) \frac{\partial^2 w}{\partial x^2} + \frac{\alpha \lambda_u^2}{\lambda_x \lambda_y^2} \frac{\partial^2 w}{\partial u^2} + \frac{\beta \lambda_v^2}{\lambda_x \lambda_y^2} \frac{\partial^2 w}{\partial v^2}. \quad (28)$$

That is, the sum of the curvatures in two orthogonal directions is expressible in terms of the curvatures in three directions, no two of which are orthogonal. The differential operator in (28), when applied to $U$, gives

$$\nabla^2 U = \left(1 - \frac{\alpha \beta}{\lambda_y^2}\right) \frac{\partial^2 U}{\partial x^2} + \frac{\alpha \lambda_u^2}{\lambda_x \lambda_y^2} \frac{\partial^2 U}{\partial u^2} + \frac{\beta \lambda_v^2}{\lambda_x \lambda_y^2} \frac{\partial^2 U}{\partial v^2}. \quad (29)$$

*See Appendix B.
Equations (8) and (9), namely
\[ \nabla^2 U = \frac{p}{N}, \quad \nabla^2 w = U, \]
may now be expressed in terms of finite differences by the aid of (28), (29), and the previously given difference equation for second derivatives in the form of Equation (12). The result at any point \( O \) as defined by the network of Fig. 8 is

\[
\frac{p_0}{N} = (\nabla^2 U)_o = \left(1 - \frac{\alpha \beta}{\lambda^2_x}\right) \left(\frac{U_p - 2U_0 + U_r}{\lambda^2_x}\right) + \frac{\alpha \lambda^2_y}{\lambda^2_x \lambda^2_y} \left(\frac{U_p - 2U_0 + U_s}{\lambda^2_y}\right)
\]

\[ + \frac{\beta \lambda^2_y}{\lambda^2_x \lambda^2_y} \frac{U_q - 2U_0 + U_t}{\lambda^2_x}, \]

with a similar expression for \( \nabla^2 w \) at point \( O \). If, for abbreviation, one lets

\[ A = \frac{\alpha}{\lambda_x}, \quad B = \frac{\beta}{\lambda_x}, \quad C = \frac{\lambda_y^2}{\lambda_x^2} - AB \]

so that

\[ A + B = 1, \]

then the final forms of Equations (8) and (9) become

\[
\frac{p_0 \lambda^2_y}{N} = A(U_p + U_s) + B(U_q + U_t) - 2U_0 + C(U_w - 2U_0 + U_r) \quad (31)
\]

\[ U_0 \lambda^2_y = A(w_p + w_s) + B(w_q + w_t) - 2w_0 + C(w_w - 2w_0 + w_r). \quad (32) \]

When \( A = B = \frac{1}{2} \), Equations (31) and (32) reduce to the corresponding equations given by Marcus.*

Corresponding to Equation (32) for the value of \( U \) at point \( O \) the equations for \( U \) at points \( p, q, r, s, t, \) and \( w \) may be written by

a simple interchange of subscripts made according to the notation
given in Fig. 8. Thus

\[
\begin{align*}
U_p \lambda_y^2 &= A(w_a + w_o) + B(w_b + w_w) - 2w_p + C(w_n - 2w_p + w_q) \\
U_q \lambda_y^2 &= A(w_b + w_r) + B(w_c + w_o) - 2w_q + C(w_p - 2w_q + w_d) \\
U_r \lambda_y^2 &= A(w_q + w_f) + B(w_d + w_a) - 2w_r + C(w_o - 2w_r + w_e) \\
U_s \lambda_y^2 &= A(w_o + w_g) + B(w_r + w_h) - 2w_s + C(w_t - 2w_s + w_f) \\
U_t \lambda_y^2 &= A(w_w + w_h) + B(w_o + w_j) - 2w_t + C(w_k - 2w_t + w_s) \\
U_w \lambda_y^2 &= A(w_n + w_l) + B(w_p + w_e) - 2w_w + C(w_m - 2w_w + w_o).
\end{align*}
\]  

(33)

Equations (32) and (33), when substituted into (31), yield, after some simplification, the equation

\[
\frac{P_o \lambda_y^4}{N} = 2w_o(2 + A^2 + B^2 + 4C + 3C^2) \\
- 2(w_p + w_e)(2A + 2AC - BC) \\
- 2(w_e + w_r)(2B + 2BC - AC) \\
- 2(w_r + w_w)(2C + 2C^2 - AB) \\
+ (w_a + w_g)A^2 + (w_c + w_j)B^2 \\
+ (w_t + w_m)C^2 + 2(w_b + w_h)AB \\
+ 2(w_d + w_k)BC \\
+ 2(w_f + w_n)AC.
\]

(34)

This is the difference equation corresponding to the fourth order differential equation (1). It expresses a relationship between deflections of the slab at points of the skew network in the vicinity of a general point \( O \). If the points of the network of a given slab are numbered, an equation similar to (34) may be written for any interior point provided that it is surrounded by a sufficient number of points on all sides. If the point is on the boundary of the slab or
on a line adjacent to the boundary, there is required some modification of Equation (34) depending upon the type and orientation of the boundary.

Under certain special conditions shown in Fig. 9 the skew network becomes rectangular and Equation (34) reduces to a simpler form. In particular when \( \phi = 0 \) the constants take the values

\[
A = 1, \quad B = 0, \quad C = \frac{\lambda_y^2}{\lambda_x^2}.
\]

Equations (31) and (32) then reduce to (17) and (18), respectively, and (34) reduces to the required form

\[
\frac{p_0}{N} = 2w_0 \left[ 3 \left( \frac{1}{\lambda_x^4} + \frac{1}{\lambda_y^4} \right) + \frac{4}{\lambda_x^2 \lambda_y^2} \right] - 4 \left( \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} \right) \left( \frac{w_a + w_\delta}{\lambda_x^2} + \frac{w_\delta + w_\gamma}{\lambda_y^2} \right) + 2 \frac{w_a + w_\delta + w_\gamma + w_i}{\lambda_x^2 \lambda_y^2} + \frac{w_a + w_\delta}{\lambda_x^4} + \frac{w_a + w_\gamma}{\lambda_y^4}.
\]

Similarly, when \( \delta = 0 \) the network becomes rectangular, the constants become

\[
A = 0, \quad B = 1, \quad C = \frac{\lambda_y^2}{\lambda_x^2},
\]
and the equations reduce to their simpler forms. Finally, when \( \phi + \delta = \pi/2 \) the axes \( u \) and \( v \) are at right angles. Then

\[
A = \frac{\lambda_v^2}{\lambda_u^2 + \lambda_v^2}; \quad B = \frac{\lambda_u^2}{\lambda_u^2 + \lambda_v^2}; \quad C = 0;
\]

\[
\lambda_z^2 = \lambda_u^2 + \lambda_v^2; \quad \lambda_g^2 = \frac{\lambda_u^2 \lambda_v^2}{\lambda_u^2 + \lambda_v^2};
\]

and Equation (34) reduces to

\[
\frac{p_0}{N} = 2w_o \left[ 3 \left( \frac{1}{\lambda_u^4} + \frac{1}{\lambda_v^4} \right) + \frac{4}{\lambda_u^2 \lambda_v^2} \right] \\
- 4 \left( \frac{1}{\lambda_u^2} + \frac{1}{\lambda_v^2} \right) \left( \frac{w_p + w_x}{\lambda_u^2} + \frac{w_q + w_t}{\lambda_v^2} \right) \\
+ 2 \frac{w_r + w_w + w_b + w_h}{\lambda_u^2 \lambda_v^2} + \frac{w_n + w_q}{\lambda_u^4} + \frac{w_c + w_i}{\lambda_v^4}.
\]

Again, this is the required form for the rectangular network with axes in the \( u \) and \( v \) directions.

Consider now the effect of the boundaries of the slab on Equation (34) when point \( O \) is in the neighborhood of at least one boundary. Two types of boundary as shown in Fig. 7 will be considered. Because of the feasibility of treating a single load on a symmetrical structure as a combination of pairs of symmetrical and anti-symmetrical loads, it is sufficient to consider only one half of the total number of points on the bridge. Typical points will be taken from the upper half of the slab shown in Fig. 7. If the skew is opposite to that shown in Fig. 7 the slab may still be designed in the position shown provided that the direction of traffic is reversed in each lane. For an unsymmetrical structure an equation must be written for each point on the bridge. In such an instance the equations pertaining to the lower half of the bridge may be written at once by analogy with those given herein.

The boundary conditions at the simple supports require that there shall be no deflection and no moment normal to the edges of the slab on the lines of the supports. Expressed mathematically, this requires that

\[
(w)_{\text{edge}} = 0, \quad (\nabla^2 w)_{\text{edge}} = (U)_{\text{edge}} = 0.
\]  

(35)

Along the curbs three boundary conditions are stated: (1) there shall be no moment normal to the edge of the slab, (2) the curb shall be
considered as an edge beam which deflects equally with the slab, (3) the vertical reaction normal to the edge of the slab shall be transmitted to the curb as a vertical load. With the curb oriented as shown in Fig. 7, these conditions find expression as follows:

\[
(M_y)_{	ext{curb}} = -N \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right)_{	ext{curb}} = 0, \tag{36}
\]

\[
(w)_{	ext{curb}} = \text{Deflection of slab at curb} = z = \text{Deflection of curb}, \tag{37}
\]

\[
(R_y)_{	ext{curb}} = -N \left[ \frac{\partial}{\partial y} (\nabla^2 w) + (1 - \mu) \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} \right) \right]_{	ext{curb}}
= q - E_1 I_1 \frac{d^4 z}{dx^4}, \tag{38}
\]

where \(E_1 I_1\) is the "stiffness" of the curb, and \(q\) is an external downward load per unit of length of curb. Equation (36) may be written in the form

\[
(U)_{	ext{curb}} - (1 - \mu) \left( \frac{\partial^2 w}{\partial x^2} \right)_{	ext{curb}} = 0. \tag{39}
\]

If (37) is substituted into the right side of (38) and it is noted that \(\nabla^2 w = U\), then Equation (38) becomes

\[
\left[ \frac{E_1 I_1}{N} \frac{\partial^4 w}{\partial x^4} - \frac{\partial U}{\partial y} \right] - (1 - \mu) \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} \right)_{	ext{curb}} = \frac{q}{N}. \tag{40}
\]
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Let point 0 be chosen on the first interior line parallel to a simply supported edge as shown in Fig. 10. The first condition expressed by (35) requires that

\[ w_n = w_w = w_t = w_h = 0 \]  

(41)

and the second condition requires that

\[ U_w = U_t = 0. \]  

(42)

Equation (31) then becomes

\[ \frac{p_o \lambda^2}{N} = A(U_p + U_q) + BU_q - 2U_o + C(U_r - 2U_o). \]  

(43)

The values of \( U \) in (43) may then be expressed in terms of deflections of surrounding points by means of (32) and (33). The resulting equation, modified by (41), is

\[ \frac{p_o \lambda^2}{N} = w_o(4 + 2A^2 + B^2 + 8C + 5C^2) \]

\[ - (w_p + w_o)(4A + 4AC - BC) \]

\[ - 2w_q(2B + 2BC - AC) \]

\[ - 2w_r(2C + 2C^2 - AB) \]

\[ + (w_n + w_o)A^2 + w_iB^2 + w_oC^2 \]

\[ + 2w_AB + 2w_BC + 2w_AC. \]  

(44)

In an entirely analogous manner one finds, in the neighborhood of the opposite simply supported edge, the equation

\[ \frac{p_o \lambda^2}{N} = w_o(4 + 2A^2 + B^2 + 8C + 5C^2) \]

\[ - (w_p + w_o)(4A + 4AC - BC) \]

\[ - 2w_t(2B + 2BC - AC) \]

\[ - 2w_w(2C + 2C^2 - AB) \]

\[ + (w_n + w_o)A^2 + w_iB^2 + w_oC^2 \]

\[ + 2w_AB + 2w_BC + 2w_AC. \]  

(45)
Consider next an interior point $O$ near the curb as shown in Fig. 11. Boundary condition (39) may be applied at points $p$ and $q$ to express $U_p$ and $U_q$ in terms of deflections of points on the curb. Thus, from (39),

$$U_p \lambda_x^2 = (1 - \mu) \frac{\lambda_y^2}{\lambda_x^2} (w_n - 2w_p + w_q) = D (w_n - 2w_p + w_q) \tag{46}$$

where, for abbreviation,

$$D = (1 - \mu) \frac{\lambda_y^2}{\lambda_x^2}. \tag{47}$$

Similarly,

$$U_q \lambda_x^2 = D (w_p - 2w_q + w_d). \tag{48}$$

These values of $U$ may then be substituted into (31). The remaining values of $U$ to be substituted into (31) are given by (32) and (33). The substitutions give finally

$$\frac{p_0 \lambda_y^4}{N} = w_o (4 + A^2 + B^2 + 8C + 6C^2)
- w_p (2A + 2AC + 2AD - BC - BD)
- w_q (2B + 2BC + 2BD - AC - AD)
- (w_r + w_m) (4C + 4C^2 - AB)
- 2w_l (2B + 2BC - AC)
- 2w_s (2A + 2AC - BC)
+ w_n (AC + AD) + w_d (BC + BD)
+ (w_e + w_m) C^2 + 2w_f AC
+ w_s A^2 + 2w_n AB + w_l B^2 + 2w_s BC. \tag{49}$$
When point \( O \) is an interior point near a sharp corner of the slab, as shown in Fig. 12, the boundary conditions are

\[
\begin{align*}
W_n &= W_w = W_t = W_h = 0, \\
U_w &= U_t = 0, \\
U_p \lambda_y^2 &= D (-2w_p + w_q), \\
U_q \lambda_y^2 &= D (w_p - 2w_q + w_d).
\end{align*}
\]  

Equation (31), with \( U_0, U_r, \) and \( U_s \) given by (32) and (33), and modified by conditions (50) then gives the result

\[
\frac{p_0 \lambda_y^4}{N} = w_o (4 + A^2 + 8C + 5C^2) \\
- w_p (2A + 2AC + 2AD - BD) \\
- w_q (2B + 2BC + 2BD - AC - AD) \\
- w_r (4C + 4C^2 - AB) \\
- w_s (4A + 4AC - BC) \\
+ w_d (BC + BD) + w_e C^2 \\
+ 2w_f AC + w_e A^2.
\]  

In the neighborhood of the blunt corner shown in Fig. 13 the deflection equation is found from (31), using boundary condition (39) at point \( p \), namely,

\[
U_p \lambda_y^2 = D (w_n - 2w_p),
\]
and using conditions (35) at the simply supported edge,

\[ W_q = W_r = W_f = 0, \]
\[ U_q = U_r = 0. \]

The remaining values of \( U \) in Equation (31) may be expressed in terms of deflections of surrounding points by means of (32) and (33) with the final result

\[
\begin{align*}
\frac{p_0\lambda^4}{N} &= w_o \left( 4 + A^2 + B^2 + 8C + 5C^2 \right) \\
&\quad - w_o \left( 4C + 4C^2 - AB \right) \\
&\quad - w_p \left( 2A + 2AC + 2AD - BC \right) \\
&\quad - 2w_t \left( 2B + 2BC - AC \right) \\
&\quad - w_s \left( 4A + 4AC - BC \right) + w_o A^2 \\
&\quad + 2w_h AB + w_i B^2 + 2w_k BC \\
&\quad + w_m C^2 + w_n \left( AC + AD \right).
\end{align*}
\]

To determine the deflection equation which applies when point \( O \) is a general point on the curb as shown in Fig. 14 it is necessary to
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use condition (40) with the derivatives expressed by finite differences. From the geometrical relationships between slopes of a tangent plane in the directions of \( u, v, \) and \( y \) it can be shown that the first derivative of any function \( f \) with respect to \( y \), \( \frac{\partial f}{\partial y} \), evaluated at point \( O \), may be expressed in terms of finite differences in the neighborhood of point \( O \) by means of the equation

\[
\left( \frac{\partial f}{\partial y} \right)_0 = A \frac{f_p - f_s}{2\lambda_y} + B \frac{f_q - f_t}{2\lambda_y}.
\]  

(53)

Using this relationship in Equation (40), and expressing the remaining derivatives by finite differences in the usual way, one obtains the condition at a point \( O \) on the curb,

\[
\frac{q_0\lambda_y^3}{N} = J \left( w_m - 4w_r + 6w_o - 4w_t + w_e \right) - \frac{\lambda_y^2}{2} \left[ A(U_p - U_s) + B(U_q - U_d) \right]
- \frac{D}{2} \left[ A(w_n - 2w_p + w_q) - A(w_t - 2w_s + w_f) + B(w_p - 2w_q + w_d) - B(w_k - 2w_t + w_e) \right],
\]  

(54)

wherein

\[
J = \frac{\lambda_y^3}{\lambda_z^3} \frac{E_1I_1}{\lambda_zN}.
\]  

(55)

Equations (31) and (54) may now be combined so as to eliminate \( U_p \) and \( U_q \). The result is

\[
\frac{q_0\lambda_y^3}{N} + \frac{p_0\lambda_y^4}{2N} = J \left( w_m - 4w_r + 6w_o - 4w_t + w_e \right) + A\lambda_y^2U_s
+ B\lambda_y^2U_t + \frac{C}{2} \lambda_y^2(U_w + U_r) - (1 + C)\lambda_y^2U_0
+ \frac{D}{2} \left[ (A - 2B)w_t + (B - 2A)w_s + Aw_f + Bw_k \right]
- \frac{D}{2} \left[ (B - 2A)w_p + (A - 2B)w_q + Aw_n + Bw_d \right].
\]  

(56)

*See Appendix C.
In this equation $U_s$ and $U_t$ may be replaced by their expansions given in (33). The boundary condition $M_y = 0$ at points $w$, $O$, and $r$, gives further expansions similar to (46),

$$
\begin{align*}
U_w \lambda_y^2 &= D \left( w_m - 2w_e + w_o \right) \\
U_o \lambda_y^2 &= D \left( w_e - 2w_o + w_r \right) \\
U_r \lambda_y^2 &= D \left( w_o - 2w_r + w_r \right),
\end{align*}
$$

(57)

which may be substituted into (56). The deflections of external points, $w_m$, $w_p$, $w_q$, and $w_d$, which appear in the last bracketed term of (56) may be eliminated by using the three equations found by equating the right sides of (57) to the corresponding expansions of $U_w$, $U_o$, and $U_r$ given by (32) and (33). The final equation is

$$
\frac{q_o \lambda_y^3}{N} + \frac{p_o \lambda_y^4}{2N} = w_o \left( A^2 + B^2 + 4D + 6CD + 6J - 3D^2 \right) - (w_w + w_r) \left( 2D + 4CD + 4J - 2D^2 - AB \right) - w_s (2B + 2BC + 2BD - AC - AD) - w_e (2A + 2AC + 2AD - BC - BD) + (w_e + w_m) \left( CD + J - \frac{D^2}{2} \right) + w_r (AC + AD) + w_o A^2 + 2w_i AB + w_i B^2 + w_k (BC + BD).
$$

(58)

It remains to determine the equations which apply when point $O$ is on the curb near a corner. Certain limitations apparently exist relative to the applicability of particular boundary conditions at a
corner point. While these limitations are not well understood, there are fortunately certain checks which may be applied to the equations derived by a given choice of corner conditions. The method of checking will be discussed and applied later.

Consider the sharp corner shown in Fig. 15. As before, the network in the region external to the slab is shown dotted. The derivation proceeds exactly as for a general edge point up to and including the development of Equation (56). The values of $U$ in (56) may be expressed in terms of deflections from the equations

\[
\begin{align*}
U_w &= U_t = 0, \\
U_0 \lambda_y &= D (w_w - 2w_o + w_r), \\
U_r \lambda_y &= D (w_o - 2w_r + w_s), \\
U_s \lambda_y &= A(w_o + w_r) + B(w_r + w_h) - 2w_s + C(w_t - 2w_s + w_f),
\end{align*}
\]

and it may further be noted that

\[
w_w = w_t = w_h = 0.
\]

The first equation of (59) expresses the conditions that the moment sum is zero at points $w$ and $t$, and the second and third equations express the conditions that $M_y$ is zero at points $O$ and $r$. Two further conditions will be specified at the corner point, $w$, namely, the slope in the direction of the simply supported edge shall be zero, i.e.,

\[
\left( \frac{\partial w}{\partial u} \right)_w = 0,
\]

and the moment $M_y$ shall vanish, i.e.,

\[
U_w \lambda_y = D (w_m - 2w_w + w_o).
\]

The last equation, together with Equations (59) and (60), gives

\[
w_m = -w_o.
\]

Equation (61), expressed in terms of finite differences, gives

\[
\frac{w_n - w_t}{2\lambda_u} = 0, \quad w_n = 0,
\]

\[
\frac{w_n - w_t}{2\lambda_u} = 0, \quad w_n = 0,
\]
and, from the relation between slopes in the \(x, v,\) and \(u\) directions,* gives

\[ w_p - w_k = w_o - w_m. \]  

(64)

Equations (62), (63), and (64) are sufficient to eliminate \(w_k, w_o,\) and \(w_m\) from Equation (56). The remaining terms in \(w_p, w_q,\) and \(w_d\) may then be eliminated from (56) in the same manner as in the previous derivation for the general edge point, that is from the relations

\[ U_0 \lambda_y^2 = D(-2w_o + w_r) = A(w_p + w_s) + Bw_q - 2w_o + C(-2w_o + w_r) \]  

(65)

and

\[ U_r \lambda_y^2 = D(w_o - 2w_r + w_e) \]

\[ = A(w_q + w_f) + B(w_d + w_s) - 2w_r + C(w_o - 2w_r + w_e). \]  

(66)

The final result of all the substitutions in (56) is the equation

\[
\frac{q_0 \lambda_y^3}{N} + \frac{p_0 \lambda_y^4}{2N} = w_o \left( A^2 + 4D + 5CD + 5J - \frac{5}{2} D^2 - BD \right) \\
- w_r (2D + 4CD + 4J - 2D^2 - AB) \\
+ w_s \left( CD + J - \frac{D^2}{2} \right) \\
- w_s (2A + 2AC + 2AD - BD) \\
+ w_f (AC + AD) + w_a A^2. \]

(67)

As a check on the coefficients of all the deflections except \(w_o\) in (67), one may use the reciprocal relationship that the coefficient of any \(w_i\) in an equation expressing a unit load at point 2 is equal to the coefficient of \(w_o\) in an equation expressing a unit load at point 1.† Thus, the coefficient of \(w_r\) in (67) must be the same as the coefficient of \(w_w\) in (58), the coefficient of \(w_s\) in (67) must be the same as the coefficient of \(w_p\) in (51), etc. A further observation may be made in connection with the

*See Appendix C.
†This is a consequence of Maxwell's theorem.
sum of all of the coefficients, an observation which affords a check on
the coefficient of \( w_0 \). Note first that the sum of all the coefficients in
(44) may be obtained from the coefficients in (34) by discarding the
coefficients of points on the simply supported edge, that is of \( w_n, w_w, w_t, \) and \( w_h \), by changing the sign of the coefficients of points outside of
the slab boundaries, that is of \( w_m, w_k, \) and \( w_j \), and then summing. That
is, the coefficients of deflections on the simply supported edge vanish
and the coefficients of deflections exterior to the slab must be reflected
back, across the simply supported edge, with opposite sign. The same
adjustments permit the sum of the coefficients in (51) to be found
from the coefficients of (49). Finally, the same procedure applied to
the coefficients in (58) is found to give the sum of the coefficients
in (67).

The discussion just given constitutes reasonably good evidence that
suitable corner conditions have been used in the derivation of (67).
Further proof of this may be given for particular choices of angle of
skew and spacing of points in the network.

When point \( O \) is on the curb and is adjacent to the blunt corner of
the slab, the boundary conditions to be used in deriving the deflection
relationship are identical with those used in the derivation of (67).
One finds

\[
\frac{q_0 \lambda^2_y}{N} + \frac{p_0 \lambda^4_y}{2N} = w_0 \left( A^2 + B^2 + 4D + 5CD + 5J + BD - \frac{5}{2} D^2 \right)
\]

\[
- w_n (2D + 4CD + 4J - 2D^2 - AB)
\]

\[
- w_t (2A + 2AC + 2AD - BC)
\]

\[
- w_h (2B + 2BC + 2BD - AC - AD)
\]

\[
+ w_k A^2 + 2 w_k AB + w_j B^2
\]

\[
+ w_h (BC + BD) + w_m \left( CD + J - \frac{D^2}{2} \right).
\]

The same checks may be applied to the coefficients in (68) as were
applied to the coefficients in (67).

While some effort has been made to justify the conditions used in
deriving (67) and (68) and to verify the results, no specific limitations
on corner conditions have been stated, and no possible equations have
been mentioned other than those actually used. The most that can be said here is that the use of an equation such as (33) in the expansion of $U$ at the corner leads to a final result which violates the reciprocal relation between the coefficients. The idea of an imaginary extension of the slab beyond the actual boundaries is useful in deriving and using difference equations, but, in the neighborhood of a corner which is other than a right angle, the somewhat inadequate physical concept seems to require a break leading into the corner, perhaps as shown in Fig. 15. Thus, the expansion of $U$ at the corner is not permitted in terms of the deflections of points which continuously surround the corner. The nature of the singularity at an oblique or obtuse corner deserves further study.

A sufficient number of equations have now been derived to permit the determination of deflections at all points of the network for symmetrical or anti-symmetrical loads. The various types of equations are summarized in the following:

**Defining Network.**

![Diagram of Defining Network]

**Constants.**

\[
A = \frac{\alpha}{\lambda_z} ; \quad B = \frac{\beta}{\lambda_z} ; \quad A + B = 1 ;
\]

\[
C = \frac{\lambda_y^2}{\lambda_z^2} - \frac{\alpha\beta}{\lambda_z^2} = \frac{\lambda_y^2}{\lambda_z^2} - AB
\]

\[
D = (1 - \mu) \frac{\lambda_y^2}{\lambda_z^2} = (1 - \mu) (C + AB)
\]

\[
J = \frac{\lambda_y^3}{\lambda_z^3} \frac{E_1 I_1}{\lambda_z N}.
\]
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General Interior Point.

\[
\frac{p_0 \lambda^4}{N} = \frac{2w_o (2 + A^2 + B^2 + 4C + 3C^2)}{N} = 2w_o (2 + A^2 + B^2 + 4C + 3C^2) \\
- 2(w_p + w_s) (2A + 2AC - BC) \\
- 2(w_q + w_1) (2B + 2BC - AC) \\
- 2(w_r + w_w) (2C + 2C^2 - AB) \\
+ (w_a + w_g) A^2 + (w_c + w_j) B^2 \\
+ (w_c + w_m) C^2 + 2(w_b + w_k) AB \\
+ 2(w_d + w_k) BC \\
+ 2(w_f + w_n) AC.
\]

Interior Points Near Simply Supported Edges.

\[
\frac{p_0 \lambda^4}{N} = \frac{w_o (4 + 2A^2 + B^2 + 8C + 5C^2)}{N} = w_o (4 + 2A^2 + B^2 + 8C + 5C^2) \\
- (w_p + w_s) (4A + 4AC - BC) \\
- 2w_q (2B + 2BC - AC) \\
- 2w_r (2C + 2C^2 - AB) \\
+ (w_a + w_g) A^2 + w_c B^2 + w_c C^2 \\
+ 2w_h AB + 2w_d BC + 2w_f AC.
\]

\[
\frac{p_0 \lambda^4}{N} = \frac{w_o (4 + 2A^2 + B^2 + 8C + 5C^2)}{N} = w_o (4 + 2A^2 + B^2 + 8C + 5C^2) \\
- (w_p + w_s) (4A + 4AC - BC) \\
- 2w_q (2B + 2BC - AC) \\
- 2w_w (2C + 2C^2 - AB) \\
+ (w_a + w_g) A^2 + w_j B^2 + w_m C^2 \\
+ 2w_h AB + 2w_d BC + 2w_n AC.
\]
Interior Point Near Curb.—

\[
\frac{p_0 \lambda_v^4}{N} = w_0 (4 + A^2 + B^2 + 8C + 6C^2) \\
- w_p (2A + 2AC + 2AD - BC - BD) \\
- w_q (2B + 2BC + 2BD - AC - AD) \\
- (w_r + w_w)(4C + 4C^2 - AB) \\
- 2w_i (2B + 2BC - AC) \\
- 2w_j (2A + 2AC - BC) \\
+ w_n (AC + AD) + w_d (BC + BD) \\
+ (w_e + w_m)C^2 + 2w_f AC \\
+ w_o A^2 + 2w_h AB + w_i B^2 + 2w_k BC.
\] (49)

Interior Point Near Sharp Corner.—

\[
\frac{p_0 \lambda_v^4}{N} = w_0 (4 + A^2 + 8C + 5C^2) \\
- w_p (2A + 2AC + 2AD - BD) \\
- w_q (2B + 2BC + 2BD - AC - AD) \\
- w_r (4C + 4C^2 - AB) \\
- w_s (4A + 4AC - BC) \\
+ w_d (BC + BD) + w_c C^2 \\
+ 2w_f AC + w_g A^2.
\] (51)

Interior Point Near Blunt Corner.—

\[
\frac{p_0 \lambda_v^4}{N} = w_0 (4 + A^2 + B^2 + 8C + 5C^2) \\
- w_w (4C + 4C^2 - AB) \\
- w_p (2A + 2AC + 2AD - BC) \\
- 2w_i (2B + 2BC - AC) \\
- w_s (4A + 4AC - BC) + w_o A^2 \\
+ 2w_h AB + w_i B^2 + 2w_k BC \\
+ w_m C^2 + w_n (AC + AD).
\] (52)
ANALYSES OF SKEW SLABS

General Edge Point.—

\[
\frac{q_0 \lambda_y^3}{N} + \frac{p_0 \lambda_y^4}{2N} = w_o \left( A^2 + B^2 + 4D + 6CD + 6J - 3D^2 \right) - (w_e + w_v)(2D + 4CD + 4J - 2D^2 - AB) - w_s(2B + 2BC + 2BD - AC - AD) - w_t(2A + 2AC + 2AD - BC - BD) + (w_n + w_m) \left( CD + J - \frac{D^2}{2} \right) + w_f(AC + AD) + w_gA^2 + w_hAB + w_iB^2 + w_k(BC + BD).
\]

(58)

Edge Point Near Sharp Corner.—

\[
\frac{q_0 \lambda_y^3}{N} + \frac{p_0 \lambda_y^4}{2N} = w_o \left( A^2 + 4D + 5CD + 5J - \frac{5}{2} D^2 - BD \right) - w_r(2D + 4CD + 4J - 2D^2 - AB) + w_f \left( CD + J - \frac{D^2}{2} \right) - w_s(2A + 2AC + 2AD - BD) + w_f(AC + AD) + w_gA^2.
\]

(67)

Edge Point Near Blunt Corner.—

\[
\frac{q_0 \lambda_y^3}{N} + \frac{p_0 \lambda_y^4}{2N} = w_o \left( A^2 + B^2 + 4D + 5CD + 5J + BD - \frac{5}{2} D^2 \right) - w_u(2D + 4CD + 4J - 2D^2 - AB) - w_s(2A + 2AC + 2AD - BC) - w_t(2B + 2BC + 2BD - AC - AD) + w_gA^2 + 2w_hAB + w_iB^2 + w_k(BC + BD) + w_m \left( CD + J - \frac{D^2}{2} \right).
\]

(68)
After the deflections have been determined for a particular slab from a set of simultaneous linear equations it remains to determine the bending and twisting moments. As before, the derivatives required in Equations (4) must be expressed in terms of finite differences. For the points of the skew network, as defined in Fig. 8, the curvature in the $x$ direction at point $O$ is simply

$$\left( \frac{\partial^2 w}{\partial x^2} \right)_O = \frac{w_x - 2w_O + w_r}{\lambda_x^2}. \quad (69)$$

From Equation (27), the curvature in the $y$ direction is

$$\left( \frac{\partial^2 w}{\partial y^2} \right)_O = A \frac{w_p - 2w_O + w_s}{\lambda_y^2} + B \frac{w_q - 2w_O + w_t}{\lambda_y^2} - AB \frac{w_s - 2w_O + w_r}{\lambda_y^2}. \quad (70)$$

These, substituted in (4), are sufficient to determine the bending moments $M_x$ and $M_y$ at all interior points of the slab. Thus one finds

$$\begin{align*}
(M_x)_O &= -\frac{N}{\lambda_x^2} \left[ (D + \mu C) (w_x - 2w_O + w_r) \\
&\quad + \mu A (w_p - 2w_O + w_s) \\
&\quad + \mu B (w_q - 2w_O + w_t) \right] \\
(M_y)_O &= -\frac{N}{\lambda_y^2} \left[ (C - D) (w_x - 2w_O + w_r) \\
&\quad + A (w_p - 2w_O + w_s) \\
&\quad + B (w_q - 2w_O + w_t) \right].
\end{align*} \quad (71, 72)$$

The twisting moment,

$$M_{xy} = -N (1 - \mu) \frac{\partial^2 w}{\partial x \partial y},$$

is expressed in terms of finite differences at an interior point $O$ by means of the equation*

$$\begin{align*}
(M_{xy})_O &= -\frac{N(1 - \mu)}{2\lambda_x \lambda_y} \left[ w_q + w_t - w_p - w_s + (B - A) (w_s - 2w_O + w_r) \right]. \quad (73)
\end{align*}$$

*See Appendix B.
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When point $O$ is on the simply supported edge of the slab Equations (71), (72), and (73) become modified by the conditions

\[
\begin{align*}
\left. \begin{array}{l}
\dot{w}_p = w_0 = w_s = 0, \\
\dot{U}_o = 0, \\
\left( \frac{\partial \dot{w}}{\partial u} \right)_o = 0,
\end{array} \right\} \quad (74)
\end{align*}
\]

where the $u$ axis is in the direction of the supported edge. The second and third of these conditions find expression as follows:

\[
\begin{align*}
B (w_q + w_t) + C (w_w + w_r) &= 0, \\
w_q - w_t &= w_r - w_w,
\end{align*}
\]
as seen by reference to Equations (32) and (64). Therefore,

\[
\begin{align*}
w_w &= \frac{B - C}{B + C} w_r - \frac{2B}{B + C} w_q, \\
w_t &= \frac{C - B}{B + C} w_q - \frac{2C}{B + C} w_r.
\end{align*}
\]

The substitution of Equations (75) and the first of Equations (74) into (72) gives

\[
(M_y)_o = \frac{N (1 - \mu)}{\lambda_x \lambda_y} (w_r - w_q) \sin 2\phi, \quad (76)
\]

where $\phi$ is the angle of skew. Furthermore, from the relations between moments at a point and from the conditions

\[
(M_x + M_y)_o = (M_n + M_w)_o = 0,
\]
the remaining moments are found to be

\[
\begin{align*}
(M_x)_o &= -(M_y)_o, \\
(M_{xy})_o &= \frac{N(1 - \mu)}{\lambda_x \lambda_y} (w_r - w_q) \cos 2\phi, \\
(M_{max})_o &= -(M_{min})_o = (M_{nu})_o = \frac{N(1 - \mu)}{\lambda_x \lambda_y} (w_r - w_q).
\end{align*}
\]  
\ \ (77)

Along the curb \( M_y = 0 \), and consequently

\[\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} = 0.\]

Therefore, when point \( O \) is on the curb, the bending moment \( M_z \) in the slab adjacent to the curb is

\[
\begin{align*}
(M_z)_o &= -N(1 - \mu^2) \left( \frac{\partial^2 w}{\partial x^2} \right)_{curb} \\
&= -N(1 - \mu^2) \frac{w_w - 2w_o + w_r}{\lambda_x^2}.
\end{align*}
\]  
\ \ (78)

Furthermore, the bending moment in the curb is

\[ (M)_{curb} = -E_1 I_1 \frac{w_w - 2w_o + w_r}{\lambda_x^2}. \]  
\ \ (79)

The condition \( M_y = 0 \) is insufficient to yield an expression which will permit the twisting moment (73) to be expressed in terms of deflections of points on the slab. An approximation is therefore employed in which equations similar to (53) are used. Thus, where

\[
\begin{align*}
\left( \frac{\partial w}{\partial y} \right)_r &= A \frac{w_q - w_f}{2\lambda_y} + B \frac{w_d - w_s}{2\lambda_y}, \\
\left( \frac{\partial w}{\partial y} \right)_w &= A \frac{w_n - w_i}{2\lambda_y} + B \frac{w_p - w_k}{2\lambda_y}.
\end{align*}
\]  
\ \ (80)
one has

\[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)_o = \frac{1}{2\lambda_x} - \frac{\left( \frac{\partial w}{\partial y} \right)_r - \left( \frac{\partial w}{\partial y} \right)_o}{4\lambda_x \lambda_y} \]

From the conditions \( M_y = 0 \) at points \( w \) and \( r \), the quantities \( U_w \) and \( U_r \), given by (33) may also be expressed by (57), the equalities yielding the relations

\[
\begin{align*}
Aw_n + Bw_r &= 2w_o - Aw_t - Bw_k + (D - C) (w_n - 2w_o + w_0) \\
Aw_o + Bw_d &= 2w_r - Aw_f - Bw_s + (D - C) (w_o - 2w_r + w_e).
\end{align*}
\]

These, substituted into (81), give the twist at a general edge point. The twisting moment is then

\[ (M_{xy})_o = -N \left( 1 - \mu \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right)_o \]

Equation (82) becomes further modified by the corner conditions when either point \( w \) or point \( r \) is at a corner.

Summary of General Formulas for Moments:

Interior Point—

\[ (M_x)_o = -\frac{N}{\lambda_y^2} \left[ (D + \mu C) (w_w - 2w_o + w_r) \right. \]

\[ + \mu A (w_p - 2w_o + w_s) \]

\[ \left. + B (w_q - 2w_o + w_t) \right] \]

\[ (M_y)_o = -\frac{N}{\lambda_y^2} \left[ (C - D) (w_w - 2w_o + w_r) \right. \]

\[ + A (w_p - 2w_o + w_s) \]

\[ \left. + B (w_q - 2w_o + w_t) \right] \]
\[(M_{xy})_0 = -\frac{N (1 - \mu)}{2\lambda_x\lambda_y} \left[ (w_q + w_t - w_p - w_s) + (B - A) (w_w - 2w_o + w_r) \right]. \tag{73}\]

Points on Simply Supported Edge.—

\[
\begin{align*}
(M_x)_0 &= -(M_y)_0 = \frac{N(1 - \mu)}{\lambda_x\lambda_y} (w_q - w_r) \sin 2\phi \\
(M_{xy})_0 &= \frac{N(1 - \mu)}{\lambda_x\lambda_y} (w_r - w_q) \cos 2\phi \\
(M_{\text{max}})_0 &= -(M_{\text{min}})_0 = (M_{nt})_0 = \frac{N(1 - \mu)}{\lambda_x\lambda_y} (w_r - w_q) \\
(M_z)_0 &= -(M_y)_0 = \frac{N(1 - \mu)}{\lambda_x\lambda_y} (w_t - w_w) \sin 2\phi \\
(M_{xy})_0 &= \frac{N(1 - \mu)}{\lambda_x\lambda_y} (w_w - w_t) \cos 2\phi \\
(M_{\text{max}})_0 &= -(M_{\text{min}})_0 = (M_{nt})_0 = \frac{N(1 - \mu)}{\lambda_x\lambda_y} (w_w - w_t).
\end{align*}
\]

General Point on Curb.—

\[
\begin{align*}
(M_x)_0 &= -\frac{N(1 - \mu^2)}{\lambda_z^2} (w_w - 2w_o + w_r) \\
(M_y)_0 &= 0 \\
(M_{\text{curb}})_0 &= -\frac{E_I I_1}{\lambda_z^2} (w_w - 2w_o + w_r)
\end{align*}
\]

Point on Curb Near Sharp Corner.—

\[
\begin{align*}
(M_x)_0 &= \frac{N(1 - \mu^2)}{\lambda_z^2} (2w_o - w_r) \\
(M_y)_0 &= 0 \\
(M_{\text{curb}})_0 &= \frac{E_I I_1}{\lambda_z^2} (2w_o - w_r)
\end{align*}
\]

\[
(M_{xy})_0 = -\frac{N(1 - \mu)}{4\lambda_x\lambda_y} \left[ 2 (1 + C - D) (w_r - w_w) + 2A (w_t - w_r) \right. \\
&\left. + 2B (w_t - w_s) + (D - C) (w_o - w_m) \right]. \tag{82}\]

\[
(M_{xy})_0 = -\frac{N(1 - \mu)}{4\lambda_x\lambda_y} \left[ 2w_r - 2Aw_f - 2B (w_o + w_s) + (D - C) (w_o - 2w_r + w_e) \right]. \tag{83}\]
ANALYSES OF SKEW SLABS

Point on Curb Near Blunt Corner.—

\[
(M_x)_o = -\frac{N (1 - \mu^2)}{\lambda_e^2} (w'_w - 2w_o) \\
(M_y)_o = 0 \\
(M_{curb})_o = -\frac{E_1I_1}{\lambda_e^2} (w'_w - 2w_o) \\
(M_{xy})_o = \frac{N (1 - \mu)}{4\lambda_e \lambda_y} \left[ 2w'_w - 2A w_t + 2B (w_s - w_k) \\
+ (D - C) (w_m - 2w_w + w_o) \right].
\]

7. Special Case of 45-degree Skew Slab With Square Network.—
When a slab has a 45-degree skew it is convenient, at least for certain proportions of sides, to use a square network as shown for a typical slab in Fig. 22. In this case the constants take the values:

\[
\lambda_x = \lambda_y = \lambda \\
A = 0 \\
B = 1 \\
C = 1 \\
D = 1 - \mu \\
J = \frac{E_1I_1}{\lambda N}.
\]

The equations previously derived then become somewhat simplified, taking the following forms when expressed in terms of a north-south-east-west network as employed in Section 5.

General Interior Point.—

\[
\frac{p_0\lambda^4}{N} = 20w_o - 8(w_N + w_S + w_E + w_W) \\
+ 2(w_{NE} + w_{NW} + w_{SE} + w_{SW}) \\
+ w_{NN} + w_{SS} + w_{EE} + w_{WW}.
\]
Interior Point Near Simply Supported Edge.—

\[
\frac{p_0 \lambda^4}{N} = 18w_0 - 8(w_N + w_E) + 2w_{NE} \\
+ w_{NW} + w_{SE} + w_{NN} + w_{EE}.\] (86)

Interior Point Near Simply Supported Edge.—

\[
\frac{p_0 \lambda^4}{N} = 18w_0 - 8(w_W + w_S) + 2w_{SW} \\
+ w_{NW} + w_{SE} + w_{WW} + w_{SS}.\] (87)

Interior Point Near Curb.—

\[
\frac{p_0 \lambda^4}{N} = 19w_0 + (2-\mu)(w_{NW} + w_{NE}) \\
- 2(3-\mu)w_N - 8(w_W + w_E + w_S) \\
+ 2(w_{SE} + w_{SW}) + w_{WW} + w_{SS} + w_{EE}.\] (88)

Interior Point Near Sharp Corner.—

\[
\frac{p_0 \lambda^4}{N} = 17w_0 - 2(3-\mu)w_N + (1-\mu)w_{NW} \\
+ (2-\mu)w_{NE} - 8w_E + w_{SE} + w_{EE}.\] (89)

Interior Point Near Blunt Corner.—

\[
\frac{p_0 \lambda^4}{N} = 18w_0 + w_{NW} - 8(w_W + w_S) + 2w_{SW} \\
+ w_{SE} + w_{SS} + w_{WW}.\] (90)
ANALYSES OF SKEW SLABS

General Edge Point.—

\[ \frac{q_0\lambda^3}{N} + \frac{p\lambda^4}{2N} = \frac{1}{2} (11 - 6\mu - 5\mu^2 + 10J)w_o \]
\[ - 2(2 - \mu - \mu^2 + 2J)w_E \]
\[ + \frac{1}{2} (1 - \mu^2 + 2J)(w_{WW} + w_{EE}) \]
\[ - 2(3 - \mu)w_S + (2 - \mu)(w_{SW} + w_{SE}) \]
\[ + w_{SS}. \]  

Edge Point Near Sharp Corner.—

\[ \frac{q_0\lambda^3}{N} + \frac{p\lambda^4}{2N} = \frac{1}{2} (17 - 10\mu - 5\mu^2 + 10J)w_o \]
\[ - 2(2 - \mu - \mu^2 + 2J)w_E \]
\[ + \frac{1}{2} (1 - \mu^2 + 2J)w_{EE} \]
\[ + (1 - \mu)w_{SE}. \]  

Edge Point Near Blunt Corner.—

\[ \frac{q_0\lambda^3}{N} + \frac{p\lambda^4}{2N} = \frac{1}{2} (8 - 4\mu - 3\mu^2 + 6J)w_o \]
\[ - 2(2 - \mu - \mu^2 + 2J)(w_{W} + w_{E}) \]
\[ + \frac{1}{2} (1 - \mu^2 + 2J)(w_{WW} + w_{EE}) \]
\[ - 2(3 - \mu)w_S + (2 - \mu)(w_{SW} + w_{SE}) \]
\[ + w_{SS}. \]

Equations (85) to (93) permit one to write an equation for each point of the network for any loading. The solution of these equations gives the deflection of each point. Examples are given in Chapter III.

The equations for the moments at an interior point of the slab are
the same as expressed previously by Equations (19) with $\lambda_z = \lambda_y = \lambda$. These are modified at points near and on the edges of the slab in general by the same conditions as have been used in deriving the equations in Section 6. It will be recalled, however, that some doubt was expressed relative to the validity of considering the slab to be extended continuously around the corners of the slab. It would appear rather that the concept of the imaginary extension of the slab requires a broken surface outside of its actual area. Otherwise the overabundance of conditions at the corner results in inconsistent or impossible deflections. For example, in the blunt corner of the slab shown in Fig. 16 some of the conditions previously used at $O$ lead to the relation

$$w_W = -w_E$$

(see the corresponding Equation (62) at the sharp corner). However, the curvature normal to the simply supported edge at point $O_1$ must be zero, giving the condition

$$w_S = -w_E$$

Now each of these conditions relative to $w_E$ may be used in its place, but the two together may not be used to establish the equality of $w_W$ and $w_S$ as may be seen by considering the circumstances when the curb is infinitely stiff and therefore $w_W$ is zero. Obviously in this case $w_S$ is not also zero for every possible loading.

For the reasons just stated, no attempt is made to give the moments at the corners of the slab. These are believed to be of little actual significance because of the inadequacy of the ordinary theory in the neighborhood of the supports. The only effect of importance to be determined at the corners is the tendency of the corner to lift or press downward and it is believed that this tendency may be determined in other ways.
The equations for moments in the slab and curb at the various typical points are summarized as follows:

General Interior Point.—

\[
(M_x)_o = -\frac{N}{\lambda^2} \left[ w_w - 2w_o + w_E + \mu(w_N - 2w_o + w_S) \right]
\]

\[
(M_y)_o = -\frac{N}{\lambda^2} \left[ w_N - 2w_o + w_S + \mu(w_w - 2w_o + w_E) \right]
\]

\[
(M_w)_o = -\frac{N(1-\mu)}{4\lambda^2} \left( w_{NE} + w_{SW} - w_{NW} - w_{SE} \right).
\]

(94)

Interior Points Near Simply Supported Edges.—

Same as (94) except that

\[
w_w = w_S = 0,
\]

\[
w_{SW} = -w_o.
\]

Same as (94) except that

\[
w_N = w_E = 0,
\]

\[
w_{NE} = -w_o.
\]

General Point on Curb.—

\[
(M_x)_o = -\frac{N(1-\mu^2)}{\lambda^2} \left( w_w + 2w_o + w_E \right)
\]

\[
(M_y)_o = 0
\]

\[
(M_w)_o = \frac{N(1-\mu)}{4\lambda^2} \left[ 2(1+\mu) \left( w_w - w_E \right) + 2(w_{SE} - w_{SW}) + \mu(w_{EE} - w_{WW}) \right]
\]

\[
(M_{curb})_o = -\frac{E_1I_1}{\lambda^2} \left( w_w - 2w_o + w_E \right).
\]

(95)
Point on Curb Near Sharp Corner.—

Same as (95) except that

\[ w_W = w_S = 0 \]
\[ w_{WW} = w_{SW} = -w_0. \]

Point on Curb Near Blunt Corner.—

Same as (95) except that

\[ w_E = 0 \]
\[ w_{EE} = -w_0. \]

Points on Simply Supported Edges (except corner points).—

\[
(M_x)_o = -(M_y)_o = -\frac{N}{\lambda^2} (1 - \mu) (w_E - w_N)
\]
\[
(M_{xy})_o = 0.
\]

Numerical results obtained from the equations developed in this section are given in Chapter III.

8. Special Case of 30-degree Skew Slab With Network of Equilateral Triangles.—For slabs having a skew of 30 degrees it is often possible to use a regular network of equilateral triangles as shown in the headings of Tables 2 and 6. While this network requires certain proportions of width to span, the width of the actual structure may be altered slightly in the analysis without appreciable effect on the maximum moments.

For this network, the constants become

\[ A = B = C = \frac{1}{2} \]
\[ D = \frac{3}{4} (1 - \mu). \]
The equations developed in Section 6 then reduce to the following:

General Interior Point.—

\[
\frac{4p_0\lambda_y^4}{N} = 42w_o - 10(w_p + w_q + w_r + w_s + w_t + w_u) + 2(w_b + w_d + w_f + w_h + w_k + w_n) + w_a + w_c + w_d + w_e + w_g + w_h + w_i + w_j + w_m. \] (98)

Interior Points Near Simply Supported Edges.—

\[
\frac{4p_0\lambda_y^4}{N} = 40w_o - 11(w_p + w_a) - 10(w_q + w_r) + 2(w_b + w_d + w_f) + w_a + w_c + w_d + w_e + w_g. \] (99)

\[
\frac{4p_0\lambda_y^4}{N} = 40w_o - 11(w_p + w_a) - 10(w_u + w_t) + 2(w_b + w_h + w_k) + w_a + w_g + w_i + w_j + w_m. \] (100)

Interior Point Near Curb.—

\[
\frac{4p_0\lambda_y^4}{N} = 40w_o - 11(w_u + w_t) - \frac{13 - 3\mu}{2} (w_p + w_q) - 10(w_a + w_i) + \frac{5 - 3\mu}{2} (w_d + w_k) + 2(w_b + w_h + w_d) + w_c + w_d + w_e + w_i + w_j + w_m. \] (101)
Interior Point Near Sharp Corner.—

\[
\frac{4p_0\lambda^4}{N} = 38w_0 - \frac{15 - 3\mu}{2} w_p - \frac{13 - 3\mu}{2} w_q - 11(w_r + w_s) + \frac{5 - 3\mu}{2} w_d + 2w_f + w_e + w_g. \tag{102}
\]

Interior Point Near Blunt Corner.—

\[
\frac{4p_0\lambda^4}{N} = 39w_0 - (8 - 3\mu)w_p - 11(w_e + w_m) - 10w_r + 2(w_h + w_k) + w_e + w_f + w_m + \frac{5 - 3\mu}{2} w_h. \tag{103}
\]

General Edge Point.—

\[
\frac{2}{N} (2q_0\lambda^3 + p_0\lambda^4) = w_0 \left[ 2 + 21(1 - \mu) - \frac{27}{4} (1 - \mu)^2 + 24J \right] - (w_e + w_r) \left[ 12(1 - \mu) - \frac{9}{2} (1 - \mu)^2 + 16J - 1 \right] + (w_e + w_m) \left[ \frac{3}{2} (1 - \mu) + 4J - \frac{9}{8} (1 - \mu)^2 \right] - \frac{13 - 3\mu}{2} (w_r + w_s) + 2w_h + \frac{5 - 3\mu}{2} (w_f + w_k) + w_e + w_f. \tag{104}
\]
ANALYSES OF SKEW SLABS

Edge Point Near Sharp Corner.—

\[
\frac{2}{N} \left(2q_0\lambda_y^3 + p_0\lambda_y^4\right) = w_0 \left[ 18(1-\mu) - \frac{45}{8} (1-\mu)^2 + 20J + 1 \right]
\]

\[
-w_r \left[ 12(1-\mu) - \frac{9}{2} (1-\mu)^2 + 16J - 1 \right]
\]

\[
+ w_e \left[ \frac{3}{2} (1-\mu) - \frac{9}{8} (1-\mu)^2 + 4J \right]
\]

\[
-w_s \left[ \frac{6 + \frac{3}{2} (1-\mu) }{2} \right] + w_f \left[ 1 + \frac{3}{2} (1-\mu) \right] + w_g.
\]

(105)

Edge Point Near Blunt Corner.—

\[
\frac{2}{N} \left(2q_0\lambda_y^3 + p_0\lambda_y^4\right) = w_0 \left[ 2 + 21(1-\mu) - \frac{45}{8} (1-\mu)^2 + 20J \right]
\]

\[
-w_e \left[ 12(1-\mu) - \frac{9}{2} (1-\mu)^2 + 16J - 1 \right]
\]

\[
-w_s \left[ 5 + \frac{3}{2} (1-\mu) \right] - w_i \left[ 5 + \frac{3}{2} (1-\mu) \right]
\]

\[
+ w_e \left[ 1 + \frac{3}{2} (1-\mu) \right]
\]

\[
+ w_m \left[ \frac{3}{2} (1-\mu) - \frac{9}{8} (1-\mu)^2 + 4J \right]
\]

\[
+ 2w_h + w_g + w_f.
\]

(106)

Equations (98) to (106), inclusive, are sufficient to determine the system of simultaneous equations from which the deflections may
be obtained in a given slab of 30-degree skew. From the deflections the moments may be computed as follows:

General Interior Point. —

\[
(M_x) = \frac{N}{4\lambda_y^2} \left[ 6(1+\mu)w_o - (3-\mu)(w_t + w_r) \right] - 2\mu(w_p + w_q + w_s + w_t) \]

\[
(M_y) = \frac{N}{4\lambda_y^2} \left[ 6(1+\mu)w_o + (1-3\mu)(w_t + w_r) \right] - 2(w_p + w_q + w_s + w_t) \]

\[
(M_{xy}) = \frac{\sqrt{3} N(1-\mu)}{4\lambda_y^2} (w_p + w_s - w_q - w_t) \] (107)

Points on Simply Supported Edges (except corner points). —

\[
(M_x) = -(M_y) = \frac{3N}{4\lambda_y^2} (1 - \mu)(w_t - w_r) \] (108)

\[
(M_{xy}) = \frac{1}{\sqrt{3}} (M_y) \]

Point on Curb. —

\[
(M_x) = \frac{3N(1 - \mu^2)}{4\lambda_y^2} (2w_o - w_s - w_r) \]

\[
M_y = 0 \]

\[
(M_{xy}) = \frac{\sqrt{3} N (1-\mu)}{32\lambda_y^2} \left[ (1+3\mu)(w_s - w_m) + 2(5+3\mu)(w_o - w_r) + 16(w_s - w_t) \right] \] (110)

The equation for \(M_{xy}\) in (110) is obtained by using the condition \(M_y = 0\) at points \(w'\) and \(r'\) shown in the sketch, the deflections at
w' and r' being obtained by making the curve of deflection defined by the equation

\[ w = a_0 + a_1x + a_2x^2 + a_3x^3 \]  

(111)
pass through two points on each side of w' and r', respectively. When point O is adjacent to a corner the interpolation is modified so as to include three points instead of four.

III. Applications of Equations

9. Uniformly Loaded Slabs Simply Supported on Four Edges; Varying Skew.—Analysis was made of a group of uniformly loaded slabs in which the ratio of the long side to the short span was kept practically constant at a value of 2.0. These slabs were simply supported on all edges and varied in skew through angles of 0, 30, 45 and 60 degrees. An attempt was made to gain a rather complete picture of the distribution and direction of principal moments as well as the distribution of moments in the directions of the coordinate axes. The analysis of the slab of zero skew (rectangular slab) gives a basis for judging the effect of skew on these slabs. For all of the analyses Poisson's ratio was taken as 0.2.

Figure 17 shows a plan of the slabs analyzed, and shows values and directions of the principal moments obtained at the center of each slab. The curve at the bottom of the figure shows the variation of
maximum moment with angle of skew. It is to be noted that for this proportion of sides a considerable angle of skew has little effect on the coefficient of maximum moment, the coefficient for a 45-degree skew being 92 per cent of that for no skew. The values plotted on the curve for angles of skew, \( \phi \), of 75 degrees or more were obtained from the simple slab moment \( 1/8 \ \rho s^2 \) where \( s \) is the normal span given by the relation

\[ s = 2a \cos \phi. \]

For these large angles of skew the moment at the center of the slab is practically unaffected by the supports which are parallel to the \( x \) axis. That this is reasonable may be seen from the proportions of the slab sketched in Fig. 18 for a skew of 75 degrees.

Trajectories of principal moments were sketched as accurately as possible from the limited number of directions of principal moments computed at the points of the various networks. The trajectories are shown in Fig. 19 where the effect of increasing the angle of skew may be noted for angles up to 45 degrees. For the skew of 60 degrees the network was considered too coarse to yield a satisfactory picture of the trajectories. It may be observed that the direction of maximum moment at the center of these slabs remains practically in the direction of the short span for angles of skew up to 45 degrees.

Tables 1 to 4 and their accompanying sketches give more detailed information concerning this group of slabs. The networks are shown and deflections, moments, and directions of principal moments are given. The deflections for the rectangular slab, which was included for comparison, were available in Marcus' book, as noted in Table 1. For the slab of 30-degree skew the deflections were obtained from the solution of 9 equations of the type (98) to (103), inclusive, wherein
$\lambda_\nu = a/4$ and the intensity of load is constant and equal to $p$ for all points of the network. The moments for this slab were then computed from Equations (107) to (110), inclusive.

Similarly, for the slab of 45-degree skew the deflections were ob-
Table 1
DATA FOR UNIFORMLY LOADED SIMPLY SUPPORTED SLAB HAVING NO SKEW

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection*</th>
<th>$M_x$</th>
<th>$M_y$</th>
<th>$M_{xx}$</th>
<th>$M_{yy}$</th>
<th>$M_{max}$</th>
<th>$M_{min}$</th>
<th>$\theta$ deg.</th>
</tr>
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<tbody>
<tr>
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<td>0.05399</td>
<td>0.02324</td>
<td>0.03646</td>
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<td>2</td>
<td>0.07485</td>
<td>0.03866</td>
<td>0.04692</td>
<td>0</td>
<td>0.04822</td>
<td>0.02086</td>
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<tr>
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<td>0.02784</td>
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<td>0.05760</td>
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<td>-0.04310</td>
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<td></td>
</tr>
</tbody>
</table>


Reference is made to Appendix A for notes on the solution of simultaneous equations of the type encountered in the analysis of slabs by difference equations.

10. Uniformly Loaded Slabs Simply Supported on Two Edges; Varying Skew.—A second group of slabs, similar to the first except...
that the long edges were free to deflect, were analyzed for the effect of skew. Figure 20 shows a plan of the slabs and the maximum intensity of moment at the center and the distribution of moment along the edges of the slabs. The curves at the bottom of the figure show the variation of the maximum center moment and of maximum $M_x$ at the edge for various angles of skew. The maximum moment at the center of the slabs may be compared with the simple beam moment $0.125 pb^2$, where $b$ is the span measured normal to the supported edges. It will be seen that the rectangular slab of the given proportions has a center moment slightly under and a moment at the edge slightly over the simple beam moment. As the skew is increased the coefficient of maximum center moment also increases for a time, whereas the maximum moment in the direction of the edge decreases. However, when

### Table 2

**DATA FOR UNIFORMLY LOADED SIMPLY SUPPORTED SLAB HAVING A SKEW OF 30 DEGREES**

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection $16Nw/\rho a^4$</th>
<th>$M_x/\rho a^3$</th>
<th>$M_y/\rho a^3$</th>
<th>$M_{xx}/\rho a^2$</th>
<th>$M_{max}/\rho a^2$</th>
<th>$M_{min}/\rho a^2$</th>
<th>$\theta$ deg.</th>
</tr>
</thead>
<tbody>
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<td>0.02353</td>
<td>0.05445</td>
<td>0.01104</td>
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<tr>
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TABLE 3
DATA FOR UNIFORMLY LOADED SIMPLY SUPPORTED SLAB
HAVING A SKEW OF 45 DEGREES

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The angle of skew becomes very large the maximum moment at the center of the span must again become 0.125 $pb^2$ although it is conceivable that the maximum moment elsewhere may have a greater value.

Trajectories of principal moments are shown in Fig. 21. It is notable that the maximum moment in the central region of the slab is very nearly in the direction of the span normal to the abutments even for relatively large angles of skew. The departure from the normal direction is about 7 degrees for an angle of skew of 30 degrees, and is about 11 degrees for an angle of skew of 45 degrees, the inclination being from the normal toward the direction of the skew span. The indicated directions of principal moments in the central regions
ANALYSES OF SKEW SLABS

TABLE 4
DATA FOR UNIFORMLY LOADED SIMPLY SUPPORTED SLAB HAVING A SKEW OF 60 DEGREES

<table>
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<tr>
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<th>$M_0$ $\frac{pa^2}{o}$</th>
<th>$M_x$ $\frac{pa^2}{o}$</th>
<th>$M_{xx}$ $\frac{pa^2}{o}$</th>
<th>$M_{max}$ $\frac{pa^2}{o}$</th>
<th>$M_{min}$ $\frac{pa^2}{o}$</th>
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The deflections and moments are given for points 1 to 10. The deflections are in inches, and the moments are in foot-pounds. The angles of inclination of the moment trajectories with the free edges of the slabs may be found from the known stress-free condition of the edge, since this condition would make one set of trajectories necessarily intersect the edge at right angles. The reason for the discrepancy is that the analysis permits a shear and a twisting moment to exist on the edge, these to nullify one another as far as their resultant effects remote from the edge are concerned. However, near the edge, that is within a distance equal to one or two times the depth of the slab, their effects are noticeable and require correction.
Because of the foregoing considerations, the sketches of moment trajectories are interpreted as indicating the most favorable directions of reinforcement in the central regions of the slabs, but not in the vicinity of the edges. At a free edge a band of reinforcement should be placed parallel to the edge. The matter of reinforcement is discussed more fully in Section 15.

The details concerning the networks used in analyzing this group of slabs and numerical values of deflections and moments are given in Tables 5 to 8, inclusive, and their accompanying sketches. For any slab with free edges the number of simultaneous equations necessary for the calculation of deflections is increased over that used for an identical network on a simply supported slab because of the deflection of the edges. The general equations derived previously for slabs with curbs apply here when \( J \), the stiffness coefficient of the curb, and \( q \), the intensity of load on the curb, are made zero.

11. Simple Span Slab-Bridge With Curbs; 45-degree Skew.—Particular attention is given in this section to a slab-bridge having a 26-ft. roadway, a 45-degree skew, a span of 19 ft. 6 in. normal to the abutments, and a curb detail as shown in Fig. 22. Loads to be considered include the uniform dead load due to the weight of the slab,
Fig. 21. Trajectories of Principal Moments for Uniformly Loaded Skew Slabs Simply Supported on Two Edges Only
TABLE 5
DATA FOR UNIFORMLY LOADED SLAB WITH FREE EDGES HAVING NO SKEW

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<th>$M_y$ $\frac{\text{pa}^2}{\text{pa}^2}$</th>
<th>$M_{xy}$ $\frac{\text{pa}^2}{\text{pa}^2}$</th>
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</table>

The weight of the curb and handrail, and truck wheel loads approximated by placing distributed loads over small areas centered at various points of the network. Twelve simultaneous equations are required to determine the deflections at the points of the network shown in Fig. 22 for any system of symmetrical or anti-symmetrical loads.

The equations which lead to values of deflection are of the type (85) to (93), inclusive. It will be noted that Poisson's ratio, $\mu$, and the quantity $J$, proportional to the curb stiffness, must be given numerically before a solution can be effected. As before, Poisson's ratio was taken as 0.20. A value of $J$ was determined on the basis of an assumed depth of slab of 15 inches. Thus

$$
J = \frac{\lambda_y^3}{\lambda_x^3} \frac{E_1 I_1}{\lambda_y N} = \frac{E_1 I_1}{\lambda N} = \frac{5}{\sqrt{2}} \frac{12 (1 - \mu^2) I_1}{a h^3}
$$

$$
I_1 \approx \frac{17 (27)^3}{12} \text{ in.}^4
$$
This value of $J$ was used in the calculations in this section.

Consideration was given to the various loading conditions described in the headings of Tables 9 to 14, inclusive. Table 9 gives moment coefficients for dead load, including the weight of the curb and handrail. For the assumed depth of slab and a paving allowance of 25 lb. per sq. ft., $p = 212$ lb. per sq. ft. The provision for the weight of the curb and concrete handrail and spindles was taken as $q = 584$ lb. per ft., which permitted the use of a convenient relation $q = p\lambda/2$ in the equations.

Table 9 shows the greatest computed value of moment to be at points 8, near the blunt corner, where

$$M_{\text{max}} = 0.1260pa^2.$$ 

However, when combining the effects of live and dead loads, a more significant moment is near the center of the slab. At points 12

$$M_{\text{max}} = 0.1186pa^2,$$

the direction of this moment being practically at right angles to the abutments. If it is assumed that the maximum moment varies parabolically between points 11, 12 and the center of the slab, it is found that the maximum moment at the center of the slab is about 3 per cent higher than that at point 12. At points 9 the maximum moment is only slightly less than that at point 12 and is also practically normal.

Fig. 22
Table 6
DATA FOR UNIFORMLY LOADED SLAB WITH FREE EDGES HAVING A SKEW OF 30 DEGREES

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection $\frac{16Nw}{pa^2}$</th>
<th>$M_x$ $\frac{pa^2}{pa^2}$</th>
<th>$M_y$ $\frac{pa^2}{pa^2}$</th>
<th>$M_{xy}$ $\frac{pa^2}{pa^2}$</th>
<th>$M_{max}$ $\frac{pa^2}{pa^2}$</th>
<th>$M_{min}$ $\frac{pa^2}{pa^2}$</th>
<th>$\theta$ deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.10819</td>
<td>0.12670</td>
<td>0</td>
<td>0.04803</td>
<td>0.14285</td>
<td>-0.01615</td>
<td>18.5</td>
</tr>
<tr>
<td>2</td>
<td>1.09223</td>
<td>0.16308</td>
<td>0.04312</td>
<td>0.05056</td>
<td>0.19136</td>
<td>-0.01484</td>
<td>23.5</td>
</tr>
<tr>
<td>3</td>
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<td>0.29721</td>
<td>-0.02790</td>
<td>17</td>
</tr>
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<td>4</td>
<td>1.80309</td>
<td>0.25673</td>
<td>0</td>
<td>0.11184</td>
<td>0.33116</td>
<td>-0.00614</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>2.03391</td>
<td>0.33524</td>
<td>0</td>
<td>0.10786</td>
<td>0.36694</td>
<td>-0.0170</td>
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</tr>
<tr>
<td>6</td>
<td>1.89263</td>
<td>0.30585</td>
<td>0.04867</td>
<td>0.12046</td>
<td>0.36044</td>
<td>-0.00592</td>
<td>22.5</td>
</tr>
<tr>
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<td>1.33564</td>
<td>0.25887</td>
<td>0</td>
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<td>0.31258</td>
<td>-0.03371</td>
<td>22.5</td>
</tr>
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<td>1.12763</td>
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<td>0.05430</td>
<td>0.12192</td>
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<td>-0.01912</td>
<td>31</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0.00538</td>
<td>-0.00338</td>
<td>-0.00311</td>
<td>0.00821</td>
<td>-0.00621</td>
<td>-13</td>
</tr>
<tr>
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<td>0</td>
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<td>0.01105</td>
<td>0.00689</td>
<td>0.01380</td>
<td>-0.01380</td>
<td>-15</td>
</tr>
</tbody>
</table>

to the abutments. The distribution of moments in the curb is shown
in Fig. 23 where the greatest computed moment is given as

$$M_{curb} = 0.0421pa^2,$$

at point 4. While it is possible that a slightly higher value of moment
exists at a short distance to one side of point 4, the computed value
is evidently reasonably close to the maximum.

As a first approximation to the effect of truck rear wheel loads, the
areas surrounding points 11 and 12 were loaded by total loads $P$ by
letting

$$p_{11} = p_{12} = \frac{P}{\lambda x \lambda y} = \frac{P}{\lambda^2},$$

where $P$ is the magnitude of a single wheel load. The remainder of
the slab was left unloaded by making $q$ and $p$ zero at all other points
of the network. The coefficients of deflection obtained from the simul-
Table 7
DATA FOR UNIFORMLY LOADED SLAB WITH FREE EDGES HAVING A SKEW OF 45 DEGREES

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection (16N/m^2/\text{pa}^3)</th>
<th>(M_x/\text{pa}^2)</th>
<th>(M_y/\text{pa}^2)</th>
<th>(M_{xy}/\text{pa}^2)</th>
<th>(M_{max}/\text{pa}^2)</th>
<th>(M_{min}/\text{pa}^2)</th>
<th>(\theta) deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.45459</td>
<td>0.02224</td>
<td>0</td>
<td>0.00450</td>
<td>0.00285</td>
<td>-0.00062</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>0.87500</td>
<td>0.08843</td>
<td>0</td>
<td>0.02462</td>
<td>0.09482</td>
<td>-0.00839</td>
<td>14.5</td>
</tr>
<tr>
<td>3</td>
<td>0.44147</td>
<td>0.06086</td>
<td>0.01922</td>
<td>0.02230</td>
<td>0.07121</td>
<td>0.00882</td>
<td>24.5</td>
</tr>
<tr>
<td>4</td>
<td>1.20450</td>
<td>0.14388</td>
<td>0</td>
<td>0.03473</td>
<td>0.10338</td>
<td>-0.01800</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>0.82355</td>
<td>0.11623</td>
<td>0.04068</td>
<td>0.05691</td>
<td>0.14438</td>
<td>0.01254</td>
<td>28</td>
</tr>
<tr>
<td>6</td>
<td>0.42444</td>
<td>0.06766</td>
<td>0.03786</td>
<td>0.01045</td>
<td>0.09586</td>
<td>0.00966</td>
<td>35</td>
</tr>
<tr>
<td>7</td>
<td>1.37882</td>
<td>0.19871</td>
<td>0</td>
<td>0.00112</td>
<td>0.23416</td>
<td>-0.03345</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>1.09301</td>
<td>0.16584</td>
<td>0.03319</td>
<td>0.00475</td>
<td>0.21666</td>
<td>0.00205</td>
<td>30</td>
</tr>
<tr>
<td>9</td>
<td>0.78629</td>
<td>0.11830</td>
<td>0.07173</td>
<td>0.06144</td>
<td>0.18942</td>
<td>0.00074</td>
<td>38</td>
</tr>
<tr>
<td>10</td>
<td>0.42635</td>
<td>0.03859</td>
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<td>0.08116</td>
<td>0.14124</td>
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<td>-38</td>
</tr>
<tr>
<td>11</td>
<td>1.34615</td>
<td>0.22621</td>
<td>0</td>
<td>0.12824</td>
<td>0.25410</td>
<td>-0.04009</td>
<td>24</td>
</tr>
<tr>
<td>12</td>
<td>1.20113</td>
<td>0.19874</td>
<td>0.03573</td>
<td>0.12942</td>
<td>0.27499</td>
<td>-0.02902</td>
<td>30.5</td>
</tr>
<tr>
<td>13</td>
<td>1.03987</td>
<td>0.16908</td>
<td>0.07603</td>
<td>0.13273</td>
<td>0.23068</td>
<td>-0.01785</td>
<td>35</td>
</tr>
<tr>
<td>14</td>
<td>0.83479</td>
<td>0.13405</td>
<td>0.06311</td>
<td>0.14634</td>
<td>0.24916</td>
<td>-0.05200</td>
<td>35.5</td>
</tr>
<tr>
<td>15</td>
<td>0.59189</td>
<td>0.10169</td>
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<tr>
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<td>28.5</td>
</tr>
<tr>
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<td>0.19301</td>
<td>0.06352</td>
<td>0.15085</td>
<td>0.29146</td>
<td>-0.04003</td>
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<tr>
<td>18</td>
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<td>0.19287</td>
<td>0.07457</td>
<td>0.14634</td>
<td>0.29147</td>
<td>-0.02454</td>
<td>34</td>
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<tr>
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<td>0.00365</td>
<td>0</td>
<td>0.00365</td>
<td>0.00365</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
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<td>0.01362</td>
<td>-0.01362</td>
<td>0</td>
<td>0.01362</td>
<td>-0.01362</td>
<td>0</td>
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<tr>
<td>21</td>
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<td>-0.01050</td>
<td>0</td>
<td>0.01050</td>
<td>-0.01050</td>
<td>0</td>
</tr>
</tbody>
</table>

Simultaneous equations and the resulting moments are given in Table 10. The greatest moment is at point 12 where

\[ M_{max} = 0.541P. \]

The live load moments found for this loading are combined with the effects of front wheels and dead loads in Table 14.

Table 11 gives similar results for the wheel loads placed over points 8 and 9. The maximum moment in the slab is now at point 9, having a value

\[ M_{max} = 0.476P. \]
### Table 8
**Data for Uniformly Loaded Slab With Free Edges Having a Skew of 60 Degrees**

Load \( p \) per Unit Area. Poisson's Ratio \( \mu = 0.2 \)

\[
\begin{align*}
A &= 1 - \frac{\sqrt{3}}{2}, \quad B = \frac{\sqrt{3}}{2}, \quad C = 1 - \frac{\sqrt{3}}{2}, \quad D = \frac{1 - \mu}{4}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection ( \frac{16Nw}{pa^4} )</th>
<th>( M_x ) ( \frac{pa^2}{pa^3} )</th>
<th>( M_y ) ( \frac{pa^2}{pa^3} )</th>
<th>( M_{xy} ) ( \frac{pa^2}{pa^3} )</th>
<th>( M_{max} ) ( \frac{pa^2}{pa^3} )</th>
<th>( M_{min} ) ( \frac{pa^2}{pa^3} )</th>
<th>( \theta ) deg.</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>0</td>
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<td>0.03842</td>
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<td>18</td>
</tr>
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<tr>
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<td>0.01951</td>
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<td>0.08594</td>
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<tr>
<td>4</td>
<td>0.45406</td>
<td>0.08110</td>
<td>0</td>
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<td>0.05187</td>
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<td>0.08228</td>
<td>0.09026</td>
<td>0.14754</td>
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<td>0.04409</td>
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<td>0.16026</td>
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<td>53</td>
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<td>0.01070</td>
<td>0.05352</td>
<td>-0.03552</td>
<td>15</td>
</tr>
<tr>
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<td>0.03064</td>
<td>-0.03064</td>
<td>0.01763</td>
<td>0.05272</td>
<td>-0.03527</td>
<td>15</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0.00880</td>
<td>-0.00880</td>
<td>0.00313</td>
<td>0.01026</td>
<td>-0.01026</td>
<td>15</td>
</tr>
</tbody>
</table>

which is less than that found previously at point 12. However, the greatest moment in the curb has risen for the new position of loads as may be expected from the plan of the bridge.

A third symmetrical placing of loads is shown in Table 12. The coefficients of moment in the slab are definitely less than for previous
loadings, but the moment in the curb has increased again to a maximum value

\[ M_{\text{curb}} = 0.0933P_a \]

at point 4. If the loads \( P \) are taken as rear wheel loads for H-20 loading* with impact, and the dead load moment is included, the maximum moment in the curb becomes

\[
M_{\text{curb}} = 0.0421P a^3 + 0.0933P a \\
= 0.0421 (212) (19.5)^3 + 0.0933 (21 500) (19.5) \\
= 105 300 \text{ ft. lb.}
\]

Table 10
SLAB WITH CURBS ANALYZED FOR LOADS AT POINTS 11 AND 12

Total load on slab = 4P.

\[
\frac{J}{AN} = 1.5 \text{ for assumed curb.}
\]

Deflections and moment coefficients for loads \( P \) at points shown

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection ( \frac{Nw}{P\alpha^2} )</th>
<th>( M_x ) ( \frac{P}{P} )</th>
<th>( M_y ) ( \frac{P}{P} )</th>
<th>( M_{max} ) ( \frac{P}{P} )</th>
<th>( M_{min} ) ( \frac{P}{P} )</th>
<th>( \frac{M_{curb}}{P\alpha} )</th>
<th>( \phi ) deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.08249</td>
<td>0.00729</td>
<td>0</td>
<td>0.01342</td>
<td>0</td>
<td>0.00322</td>
<td>0.00322</td>
</tr>
<tr>
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<td>0.02179</td>
<td>0</td>
<td>0.03194</td>
<td>0</td>
<td>0.01405</td>
<td>0.01405</td>
</tr>
<tr>
<td>3</td>
<td>0.11777</td>
<td>-0.01699</td>
<td>0.06010</td>
<td>0.04070</td>
<td>0</td>
<td>0.03372</td>
<td>0.03372</td>
</tr>
<tr>
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<td>0</td>
<td>0.10138</td>
<td>0</td>
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<td>0.03872</td>
</tr>
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<td>0.05254</td>
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<tr>
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<td>0</td>
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<td>0.05254</td>
</tr>
<tr>
<td>9</td>
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<td>0.10582</td>
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<td>26.5</td>
</tr>
<tr>
<td>10</td>
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<td>0.21430</td>
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<td>0.28689</td>
<td>26.5</td>
<td></td>
</tr>
<tr>
<td>11</td>
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</tr>
<tr>
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<td>26.5</td>
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</tr>
<tr>
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<tr>
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<td>0</td>
<td>0.28689</td>
<td>26.5</td>
<td></td>
</tr>
</tbody>
</table>

Tables 13 and 14 give combined live and dead load moments for two cases of loading. First, with rear wheels alone on the slab at points 8 and 9, the maximum combined moment is at point 8, and has a magnitude of 20 130 ft. lb. per ft. Second, with rear wheels at points 11 and 12 and front wheels at points 5 and 6, the combined moment has a maximum of 21 820 ft. lb. per ft. at point 12. The direction of this maximum moment is only 6 degrees away from a line normal to the abutments. The trajectories of maximum moment have been sketched in Fig. 24 for this case of loading. It may be noted that the wheel and axle spacing used here differ somewhat from those required by the A.A.S.H.O. specifications.
12. Correction of Moments for Actual Diameter of Wheel Loads.— It is recognized that there is little basis for judging the actual concentration of load when an analysis is made by letting \( p = P/\lambda^2 \) at the loaded points and \( p = 0 \) at the remaining points of the network. To determine the effective concentration of a central load, five analyses were made by difference equations for comparison with the results obtained by other means. Because the other results were based on Poisson’s ratio of 0.15, this value was also used in these calculations with difference equations.

For the first case a square slab was analyzed using a square network of 36 equal squares. Two opposite edges were left free and two

---

### Table 11

**Slab With Curbs Analyzed for Loads at Points 8 and 9**

<table>
<thead>
<tr>
<th>Point</th>
<th>( \text{Deflection} ) ( Nw ) ( P )</th>
<th>( M_x ) ( P )</th>
<th>( M_y ) ( P )</th>
<th>( M_{xx} ) ( P )</th>
<th>( M_{yy} ) ( P )</th>
<th>( M_{max} ) ( P )</th>
<th>( M_{min} ) ( P )</th>
<th>( M_{curb} ) ( P_0 )</th>
<th>( \theta ) ( \text{deg.} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0.01916</td>
<td>0.03254</td>
<td>-0.01128</td>
<td>0.00939</td>
<td>30.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.31372</td>
<td>0.07261</td>
<td>0</td>
<td>0.07461</td>
<td>0.11920</td>
<td>-0.01667</td>
<td>0.03209</td>
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<td></td>
</tr>
<tr>
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<tr>
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</tr>
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<td>0</td>
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<td>0.11840</td>
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</tr>
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<td>+0.20063</td>
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</tr>
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<td>0.37131</td>
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<td>0.12357</td>
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<td>+0.22616</td>
<td>0.33943</td>
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<td>0.08568</td>
<td>0.27780</td>
<td>+0.05500</td>
<td>0.39260</td>
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<td>0.25625</td>
<td>0.14174</td>
<td>0.07618</td>
<td>0.21194</td>
<td>+0.05960</td>
<td>0.47120</td>
<td>47.5</td>
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</tr>
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<td>0.52437</td>
<td>0.18707</td>
<td>0.23117</td>
<td>0.15562</td>
<td>0.37129</td>
<td>+0.05695</td>
<td>0.49550</td>
<td>49</td>
<td></td>
</tr>
<tr>
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<td>0</td>
<td>-0.02067</td>
<td>0.02067</td>
<td>0</td>
<td>0.02667</td>
<td>-0.02667</td>
<td>0</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>14</td>
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<td>-0.07755</td>
<td>0.07755</td>
<td>0</td>
<td>0.07755</td>
<td>-0.07755</td>
<td>0</td>
<td>49</td>
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<td>0.02706</td>
<td>0</td>
<td>0.02706</td>
<td>-0.02706</td>
<td>0</td>
<td>49</td>
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<td>-0.00758</td>
<td>0</td>
<td>0.00758</td>
<td>-0.00758</td>
<td>0</td>
<td>49</td>
<td></td>
</tr>
</tbody>
</table>
ILLINOIS ENGINEERING EXPERIMENT STATION

**TABLE 12**

**SLAB WITH CURBS ANALYZED FOR LOADS AT POINTS 5 AND 6**

![Diagram of slab with curbs and points labeled 1 to 16.]

Deflections and moment coefficients for loads \( P \) at points shown

<table>
<thead>
<tr>
<th>Point</th>
<th>Deflection ( \frac{Nw}{P\times} )</th>
<th>( M_x ) ( \frac{P}{\times} )</th>
<th>( M_y ) ( \frac{P}{\times} )</th>
<th>( M_{xy} ) ( \frac{P}{\times} )</th>
<th>( M_{cor} ) ( \frac{P\times}{\alpha} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.21733</td>
<td>0.03745</td>
<td>0</td>
<td>0.02515</td>
<td>0.01655</td>
</tr>
<tr>
<td>2</td>
<td>0.20606</td>
<td>0.11923</td>
<td>0</td>
<td>0.08799</td>
<td>0.03269</td>
</tr>
<tr>
<td>3</td>
<td>0.25303</td>
<td>0.05904</td>
<td>0.11895</td>
<td>0.06099</td>
<td>0.03033</td>
</tr>
<tr>
<td>4</td>
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<td>0.21119</td>
<td>0</td>
<td>0.09039</td>
<td>0.09933</td>
</tr>
<tr>
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<td>0.47054</td>
<td>0.34833</td>
<td>0.24440</td>
<td>0.09099</td>
<td></td>
</tr>
<tr>
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<td>0.30927</td>
<td>0.20844</td>
<td>0.20397</td>
<td>0.07626</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.28230</td>
<td>0.11084</td>
<td>0</td>
<td>0.07401</td>
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</tr>
<tr>
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<td>0.14441</td>
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</tr>
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<td>10</td>
<td>0.17568</td>
<td>0.07099</td>
<td>0.02756</td>
<td>0.07866</td>
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<td>0.01376</td>
<td>0.09266</td>
<td>0.06124</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.27691</td>
<td>0.03776</td>
<td>0.09652</td>
<td>0.10770</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0.00767</td>
<td>0.00767</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>0.10067</td>
<td>0.10067</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>0.04451</td>
<td>0.04451</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>0.02904</td>
<td>0.02904</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Total load on slab = 4P.

\[ J = \frac{5.5}{\lambda N} \]

\[ = 1.5 \text{ for assumed curb and slab.} \]

were simply supported. Only the central point of the network was loaded. At that point the moment in the direction of the span, as found by difference equations, is

\[ M_x = 0.3660P. \]  

(113)

This moment may be considered to be made up of two parts, \( M_{ox} \) and \( M'_x \), where

\[ M_{ox} = \frac{P}{2.32 + \frac{8c}{a}} \]  

(114)
TABLE 13
SLAB WITH CURBS ANALYZED FOR DEAD LOAD AND LOADS AT POINTS 8 AND 9

<table>
<thead>
<tr>
<th>Point</th>
<th>$M_x$ ft. lb. per ft.</th>
<th>$M_y$ ft. lb. per ft.</th>
<th>$M_{xy}$ ft. lb. per ft.</th>
<th>$M_{max}$ ft. lb. per ft.</th>
<th>$M_{min}$ ft. lb. per ft.</th>
<th>$M_{curb}$ ft. lb.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4 170</td>
<td>0</td>
<td>1020</td>
<td>4 410</td>
<td>- 240</td>
<td>35 940</td>
</tr>
<tr>
<td>2</td>
<td>8 220</td>
<td>0</td>
<td>4180</td>
<td>9 970</td>
<td>- 1730</td>
<td>70 810</td>
</tr>
<tr>
<td>3</td>
<td>4 680</td>
<td>3 290</td>
<td>3240</td>
<td>7 300</td>
<td>670</td>
<td>94 220</td>
</tr>
<tr>
<td>4</td>
<td>10 950</td>
<td>0</td>
<td>7470</td>
<td>14 720</td>
<td>- 3700</td>
<td>84 380</td>
</tr>
<tr>
<td>5</td>
<td>9 120</td>
<td>6 800</td>
<td>6 500</td>
<td>14 530</td>
<td>1390</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5 660</td>
<td>6 770</td>
<td>3 820</td>
<td>9 940</td>
<td>2490</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>9 790</td>
<td>0</td>
<td>8 240</td>
<td>14 450</td>
<td>- 4600</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>14 400</td>
<td>10 200</td>
<td>7 540</td>
<td>20 130</td>
<td>4470</td>
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</tr>
<tr>
<td>9</td>
<td>13 880</td>
<td>12 160</td>
<td>6 660</td>
<td>19 740</td>
<td>6 310</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>7 260</td>
<td>6 220</td>
<td>4 830</td>
<td>11 060</td>
<td>2420</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>3 800</td>
<td>7 320</td>
<td>4 870</td>
<td>11 450</td>
<td>1640</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>9 570</td>
<td>10 360</td>
<td>7 340</td>
<td>17 420</td>
<td>2700</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>7 60</td>
<td>- 760</td>
<td>0</td>
<td>7 60</td>
<td>- 760</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>- 670</td>
<td>+ 670</td>
<td>0</td>
<td>6 70</td>
<td>- 670</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>- 100</td>
<td>+ 190</td>
<td>0</td>
<td>190</td>
<td>- 190</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>- 60</td>
<td>+ 60</td>
<td>0</td>
<td>60</td>
<td>- 60</td>
<td></td>
</tr>
</tbody>
</table>

is the moment produced by the load $P$ on a circular area of diameter $c$ at the center of a very long slab of span $a$, and $M_{x'}$ is a correction to be added to $M_{oz}$ in order to account for the actual boundary conditions on the cut ends of the slab. Equation (114) is Westergaard's approximate formula. The correction, $M_{x'}$, may be found from Bulletin 315, Tables 13 and 14 to be

$$M_{x'} = (0.108 - 0.049) P = 0.059P.$$


\[ M_x = 0.366P = \frac{P}{2.32 + \frac{8}{a}} + 0.059P, \]

Therefore, from (113)

from which it is found that

\[ \frac{c}{a} = 0.117 \]

or, since \( \lambda = a/6 \),

\[ c = 0.70\lambda. \]
ANALYSES OF SKEW SLABS

Dead Load: Uniform Load = 212 lb/ft²
Curb Load = 584 lb/ft²

H-20 Live Load Plus Impact: P₁ = 21,500 lb.
P₂ = 5,380 lb.

FIG. 24. TRAJECTORIES OF PRINCIPAL MOMENTS

Free Edge:
Mₓ = 0.366P  c = 0.70λ
(Ⅰ)-2 Edges Supported

Mₓ = 0.216P  c = 0.73λ
(Ⅱ)-4 Edges Supported

Mₓ = 0.254P  c = 0.74λ
(Ⅲ)-4 Edges Supported

Mₓ = 0.302P  c = 0.66λ
(Ⅳ)-4 Edges Supported

Mₓ = 0.226P  c = 0.70λ
(Ⅴ)-4 Edges Supported

FIG. 25. SQUARE SLABS INVESTIGATED FOR DISTRIBUTION OF LOAD
This means that a more accurate concept of the distribution of load in this case is one in which the load is uniformly distributed over a circular area having a diameter of about \( \frac{\sqrt{2}}{10} \) of the distance between adjacent points of the network.

The second, third, and fourth slabs to be similarly examined for distribution of load were all square slabs simply supported on 4 edges and loaded only at their central points, as shown in Fig. 25. The networks were all square of 16, 36, and 100 squares, respectively, so that \( \lambda \) took values of \( \frac{a}{4}, \frac{a}{6}, \) and \( \frac{a}{10} \). For the type of slab considered

\[
M_x = M_{ax} - 0.049P.
\]

The values of \( M_x \) were found to be 0.216\( P \), 0.254\( P \), and 0.302\( P \), respectively, giving values of \( c \) in Equation (114) of

\[
\begin{align*}
c &= 0.73\lambda \\
c &= 0.74\lambda \\
c &= 0.66\lambda.
\end{align*}
\]

Finally, a square slab simply supported on all edges and loaded at the center was analyzed using the 45-degree network of squares shown in Fig. 25. Here the results were

\[
\lambda = \frac{\sqrt{2}}{6} a; \quad M_x = 0.226P; \quad c = 0.70\lambda.
\]

Since the networks used in the analyses of skew slabs were all rather coarse, it was concluded that a reasonable assumption for the diameter of the loaded area would be

\[
c = 0.70\lambda.
\]

With this as a basis for judging the diameter of the loaded area when computing moments directly under a load, it is possible to make a correction to account for the difference between the moment due to a load on a circle of diameter 0.70\( \lambda \) and on a circle 1.25 ft. in diameter, the latter being prescribed in the A.A.S.H.O. specifications. This is accomplished by increasing the moment under the load by an amount

\[
M_{cor} = \frac{P}{2.32 + 8 \frac{1.25 \text{ ft.}}{a}} - \frac{P}{2.32 + 5.6 \frac{\lambda}{a}}. \quad (115)
\]
This simply means that the concentration of the load is increased by reducing the diameter of the loaded area from 0.70X to 1.25 ft. At any point the effects of loads remote from that point will be practically unchanged by the increased concentration of the loads. The slight difference in angle between the normal to the supports and the direction of the maximum moment does not alter the correction given by (115). In fact the correction (115) should be applied to the moment in every direction at the point of loading.

If the correction indicated by (115) is made to the moment of 21 820 ft. lb. per ft. which is indicated at point 12 in Table 14, it is found that

\[ M_{cor} = \frac{21500 \text{ lb.}}{2.32 + \frac{10}{19.5}} - \frac{21500 \text{ lb.}}{2.32 + \frac{5.6\sqrt{2}}{5}} \]

\[ = 7600 - 5510 = 2090 \text{ ft. lb. per ft.} \]

This correction is about 9.6 per cent of the previously computed moment, giving finally

\[ M_{max} = 23910 \text{ ft. lb. per ft.} \]

IV. INFLUENCE SURFACES

13. Derivation of Newmark’s Method of Determining Influence Surfaces by Difference Equations.—In determining the effect of truck wheel loads on a bridge slab it is convenient to use an influence surface for an effect at a particular point due to a unit load at any point on the slab. The ordinates which measure the effect, when plotted at the points of application of the unit load, form the influence surface. This surface may itself be regarded as the deflection of the slab to some scale due to an imposed system of fixed loads. N. M. Newmark has indicated how the system of fixed loads may be determined very simply by difference equations. His proof of the method* is essentially as follows:

Let

\[ Q_a = \text{Effect at point } a \]

\[ Q_a^{P_o=1} = \text{Effect at point } a \text{ due to } P = 1 \text{ at point } O. \]

For example,

\[ w_a P_0 = 1 = \text{Deflection at point } a \text{ due to } P = 1 \text{ at point } O. \]

\[ (M_x)_a P_0 = 1 = \text{Moment, } M_x, \text{ at point } a \text{ due to } P = 1 \text{ at point } O. \]

For any given loading the difference equation by which any effect is computed has the linear form

\[ Q_a = A w_a + B w_b + C w_c + \cdots + M w_m, \quad (116) \]

where \( A, B, C, \ldots, M \) are constants. In particular, when the deflections are due to a unit load at point \( O \),

\[ Q_a P_0 = 1 = A w_a P_0 = 1 + B w_b P_0 = 1 + \cdots + M w_m P_0 = 1. \quad (117) \]

An example of (116) is the equation for the bending moment in the curb at point \( a \),

\[ (M_{\text{curb}})_a = \frac{E_1 I_1}{\lambda^2} (2 w_a - w_W - w_E), \]

where points \( W \) and \( E \) are adjacent to and on either side of \( a \) and where \( \lambda \) is the distance between successive points.

Now, by Maxwell’s theorem of reciprocal deflections,

\[ w_a P_0 = 1 = w_0 P_{s=1}, \quad w_b P_0 = 1 = w_0 P_{s=1}, \quad (118) \]

e tc. Therefore, Equation (117) may be written as

\[ Q_a P_0 = 1 = A w_0 P_{s=1} + B w_0 P_{s=1} + C w_0 P_{s=1} + \cdots + M w_0 P_{s=1}. \quad (119) \]

Furthermore, since deflections are proportional to loads,

\[ A w_0 P_{s=1} = w_0 P_{s=A}, \quad B w_0 P_{s=1} = w_0 P_{s=B}, \quad (120) \]

e tc. Substitution into Equation (119) then gives

\[ Q_a P_0 = 1 = w_0 P_{s=A} + w_0 P_{s=B} + \cdots + w_0 P_{s=M} \]

\[ = w_0 \text{ due to loads } P_a = A, P_b = B, \ldots, P_m = M. \quad (121) \]
This equation states that any effect at a due to a unit load at \( O \) may be found as the deflection at \( O \) due to a system of loads \( A, B, \cdots, M \), applied at points \( a, b, \cdots, m \), respectively, where \( A, B, \cdots, M \) are the coefficients of \( w_a, w_b, \cdots, w_m \), respectively, in the linear Equation (116) for the desired effect at \( a \).

It may be observed that the system of loads \( A, B, \cdots, M \) for a given effect remains fixed when \( O \) is moved from point to point on the slab. Thus, the fixed system of loads produces a deflection of the slab in the form of the influence surface for the particular effect at a given point \( a \).

In the example mentioned previously the influence surface for the bending moment in the curb at point \( a \),

\[
(M_{\text{curb}})_a = \frac{E_1I_1}{\lambda^2} (2w_a - w_b - w_E),
\]

may be found as the deflection of the slab due to the loads

\[
P_a = \frac{2E_1I_1}{\lambda^2}, \quad P_E = P_W = -\frac{E_1I_1}{\lambda^2}.
\]

Newmark’s method may then be summarized as follows:

Where an effect \( Q_a \) at any point \( a \) in a slab is a linear relation of the deflections, as in the equation,

\[
Q_a = A_w + B_w + \cdots + M_w,
\]

an influence surface for \( Q_a \) may be obtained as the deflection of the slab due to a system of loads \( A, B, \cdots, M \), applied at the points \( a, b, \cdots, m \), respectively.

14. Influence Surfaces for Moments in Skew Slab-Bridges With Curbs.—The method derived in the preceding section was applied to the slab-bridge of 45-degree skew in determining the influence surfaces shown in Figs. 26 to 29, inclusive. Two stiffnesses of curb were investigated in order to make a better comparison of the resulting moments with those obtained for a right bridge of identical span. The ordinates to the influence surfaces are given in Table 15.

Consider first the influence surface for moments in the curb at point 4. Contour lines on the surface are shown in Fig. 26 for two
### Table 15
**Ordinates to Influence Surfaces for Moments**

![Contour lines on surfaces are shown in Figs. 20 to 29.](image)

<table>
<thead>
<tr>
<th>Point</th>
<th>Influence Ordinate</th>
<th>( M_{\text{curb}} ) Pt. 4</th>
<th>( M_{\text{curb}} ) Pt. 4</th>
<th>( M_1 ) Pt. 12</th>
<th>( M_2 ) Pt. 12</th>
<th>( M_3 ) Pt. 12</th>
<th>( M_4 ) Pt. 9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( J = 1.5 )</td>
<td>( J = 3.0 )</td>
<td>( J = 1.5 )</td>
<td>( J = 1.5 )</td>
<td>( J = 3.0 )</td>
<td>( J = 1.5 )</td>
</tr>
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</tr>
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<td></td>
</tr>
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<td>0.0456</td>
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</tr>
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</tr>
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<td>+0.0085</td>
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*Corrected for wheel load on 1.25 ft. dia. circle on slab of 19.5 ft. normal span.

The stiffnesses of curb. To obtain the ordinates to the surface as deflections of the slab, it is first noted that the equation for moment in the curb at point 4 is

\[
(M_{\text{curb}})_4 = \frac{E_1 I_1}{\lambda^2} (2w_4 - w_2 - w_7).
\]
The required loads on the curb are therefore

\[ P_4 = \frac{2EI_1}{\lambda^2}, \quad P_2 = -\frac{EI_1}{\lambda^2}, \quad \text{and} \quad P_7 = -\frac{EI_1}{\lambda^2}. \]

The deflections due to these loads will be the required influence ordinates. Since the loading is unsymmetrical it is convenient to break it into symmetrical and anti-symmetrical parts which may be added to give the required result. Expressed graphically, that is:

To obtain deflections due to the symmetrical loading requires nothing new in the application of Equations (85) to (93). It may be noted, however, that the loads are now at the edges of the slab, and their magnitudes are readily inserted into the equations by letting \( p=0 \) and \( q\lambda=\text{total load} \) in the proper equations. The deflections due to the anti-symmetrical loads are obtained from a second system of equations set up by noting that the points identified by the primed numbers in the heading of Table 15 have deflections opposite to those of the corresponding points having unprimed numbers. The sum of the two deflections at any point gives the required deflection or influence ordinate at that point.

From the contours of an influence surface it is possible to determine the positions of truck loadings which will produce maximum moment at a given point. After the influence surface has been determined, the spacing of the truck wheel loads becomes independent of the network. The maximum moment in the curb at point 4 was obtained for truck loads by giving the loads successive positions across the bridge, and plotting the sums of the influence ordinates after making allowance for the difference in magnitude of front and rear wheel loads. A truck in each lane may be handled separately. It
Fig. 27b. Influence Surface for Moment $M_1$ in the Slab at Point 12.
was found for $J = 1.5$, $P_1 = 21,500$ lb., and $P_2 = 5,380$ lb., that the greatest live load moment in the curb at point 4 is $74,900$ ft. lb. for the loads in the positions shown in Fig. 26a. By summing ordinates over the entire slab, the dead load effects found previously were checked. The total resulting moment due to live and dead load is

$$M_{\text{curb}} = 140,600 \text{ ft. lb.}$$

Comparison of this result with the previously found maximum moment of $105,300$ ft. lb., obtained with loads at interior net points only, shows that the previous placing of loads was unsatisfactory for the determination of the maximum live load effect in the curb.

Similar calculations were made to obtain the influence surface for moment in the curb at point 4 for $J = 3$, that is, for a curb twice as stiff as the previous one. Contours for the resulting surface are shown in Fig. 26b. The load positions indicated on the figure gave a maximum live load moment of $105,200$ ft. lb. The variation in moment in the curb with stiffness of curb is compared in Section 17 with a similar variation in the right bridge of identical span measured normal to the supports.

Figure 27 shows contours of influence surface for moments $M_1$ and $M_2$ in the slab at point 12 for $J = 1.5$ where the axes 1 and 2 are normal, and parallel respectively to the simple supports. These axes were chosen from the previous studies as being approximately in the directions of the principal moments at point 12 for design loading. To determine the loading which gives the influence surface as the deflection of the slab, it may be noted that

$$M_1 = \frac{M_x + M_y}{2} + M_{xy}$$

and

$$M_2 = \frac{M_x + M_y}{2} - M_{xy}$$

for the given orientation of axes. From Equations (94), with $\mu = 0.2$, $M_1$ and $M_2$ then become

$$M_1 = \frac{N}{h^2} \left[ 2.4 w_o - 0.6 (w_N + w_S + w_W + w_E) 
+ 0.2 (w_{NW} + w_{SE}) - 0.2 (w_{SW} + w_{NE}) \right]$$
M_2 = \frac{N}{\lambda^2} \left[ 2.4w_o - 0.6 (w_N + w_S + w_W + w_E) 
- 0.2 (w_{NW} + w_{SE}) + 0.2 (w_{SW} + w_{NE}) \right].

At point 12, therefore,

\( (M_1)_{12} = \frac{N}{\lambda^2} \left[ 1.8w_{12} + 0.2w_8 - 0.4w_9 - 0.8w_{10} - 0.6w_{11} \right] \)

\( (M_2)_{12} = \frac{N}{\lambda^2} \left[ 1.8w_{12} - 0.2w_8 - 0.8w_9 - 0.4w_{10} - 0.6w_{11} \right]. \)

For the determination of the influence surface for \( M_1 \) at point 12 the loads required are

\[ P_{12} = 1.8 \frac{N}{\lambda^2}; \quad P_8 = 0.2 \frac{N}{\lambda^2}; \quad P_9 = -0.4 \frac{N}{\lambda^2}; \]
\[ P_{10} = -0.8 \frac{N}{\lambda^2}; \quad \text{and} \quad P_{11} = -0.6 \frac{N}{\lambda^2}. \]

As before these were separated into symmetrical and anti-symmetrical sets of loadings, a solution being obtained for each set.

The final results and the positions of truck loadings to give maximum moments are shown in Fig. 27a. The influence surface similarly obtained for \( M_2 \) is shown in Fig. 27b. The influence ordinate at point 12 for \( M_1 \) at the same point is subject to the same error as discussed previously due to an effective distribution of load over a circle of diameter \( c = 0.70\lambda \). It may be corrected in the same manner by adding an increment

\[ M_{\text{cor}} = \frac{1}{2.32 + 8\frac{c}{a}} - \frac{1}{2.32 + 5.6\frac{\lambda}{a}}. \]

For the given dimensions and network

\[ M_{\text{cor}} = \frac{1}{2.32 + \frac{10}{19.5}} - \frac{1}{2.32 + \frac{5.6\sqrt{2}}{5}} = 0.353 - 0.256 = 0.097. \]
The corrected ordinate then becomes

\[(M_1)_{12}^{p_1=1} = 0.260 + 0.097 = 0.357.\]

The live load moment due to two trucks then becomes 14 470 ft. lb. per ft., which, together with the dead load moment of 9570 ft. lb. per ft. gives a design moment of 24 040 ft. lb. per ft. of slab. This is almost identical with the corrected value given on page 79 for loads on the points of the network.

The influence surface for \( M_2 \) at point 12 is represented in Fig. 27b. The ordinate at point 12, subject to the same correction noted above, becomes

\[(M_2)_{12}^{p_1=1} = 0.198 + 0.097 = 0.295.\]

The maximum live load moment due to two trucks in the position indicated in the figure, then becomes 9550 ft. lb. per ft. of slab. This, plus the dead load moment of 1580 ft. lb. per ft. gives 11 130 ft. lb. per ft. of slab, or approximately 46 per cent of the moment in the direction of the span.

To investigate the change in the moment at point 12 with stiffness of curb, the influence surface represented in Fig. 28 was found for \( J=3 \). The maximum live load moment due to two trucks is in this case 14 390 ft. lb. per ft. and the dead load moment is 8860 ft. lb. per ft., giving a total of 23 250 ft. lb. per ft. of slab. The reduction in moment in the slab due to doubling the stiffness of the curb is seen to be slight in this instance, the magnitude of the reduction being about 3.5 per cent.

Finally, to determine whether a greater maximum moment than found hitherto was possible at point 9, the influence surface for \( M_1 \) at point 9 was determined for \( J=1.5 \). The contours of this surface are shown in Fig. 29. The maximum live load moment was found to be 14 390 ft. lb. per ft. with the trucks in the position shown in the figure. With the dead load moment of 9470 ft. lb. per ft., the total moment becomes 23 860 ft. lb. per ft., less than 1 per cent below the maximum moment found at point 12.

V. DISCUSSION

15. Total Moments and Reactions From Statics.—For slabs simply supported on two opposite parallel supports, as illustrated in Section 10, the total bending moments on sections taken parallel to the sup-
Fig. 28. INFLUENCE SURFACE FOR MOMENT M, IN THE SLAB AT POINT 9 (Ordinary Curb)
ports may be determined from statics alone. Such moments were computed for various sections of the slabs analyzed in Tables 5, 6, and 7. The results are shown in Fig. 30 where comparisons are made with the corresponding moments obtained from the analyses by difference equations. In each instance the total distributed moment checks the requirement of statics within a small fraction of one per cent. The distributed moments normal to the cut sections were summed by Simpson's one-third rule after being computed in the usual way from the relation

\[ M_n = \frac{M_x + M_y}{2} + \frac{M_x - M_y}{2} \cos 2\theta + M_{xy} \sin 2\theta \]  

(122)

where the \( n \) axis makes an angle \( \theta \) with the \( x \) axis.

For the slab of 45-degree skew shown in Table 7 the position of the resultant reaction on one support was determined by taking moments about a line through points 16, 18, and 16. The distributed moments on the cut section were summed and then added to the moments from the distributed loads. To balance this moment the resultant of the reaction was found to have a position on the supported edge approximately at the quarter-point near to the blunt corner. The exact position is shown in Fig. 31. The same location of the resultant was obtained when moments were taken about a line through points 11-15.

The concentration of reaction near the blunt corner is consistent with the twisting effect of the skewed supports.

The agreement between the total distributed moments and the requirements of statics would seem to verify the analyses of these slabs, at least to the extent that consistent equations have been used.

Some calculations of distributed reactions have been made. These are incomplete due to the uncertainty of the actual state of moments at the corners of the slab. However, the indications are that there is no concentrated reaction at the sharp corner of the slab of 45-degree skew under uniform loading when the slab has the proportions given in Table 7. This belief results from a consideration of the equilibrium of a triangular portion of the slab obtained by passing a section through points 4, 5, 6, and 19 or through points 7, 5, and 20. On the other hand, there may be a concentration of reaction at the blunt corner. There may also be a singularity in the state of moments at the blunt corner. However, even if bending moments could be computed at the corner, their practical significance would be in considerable
FIG. 30. MOMENTS FROM STATICS COMPARED WITH TOTAL DISTRIBUTED MOMENTS
doubt because the assumed boundary conditions in the neighborhood of the corner could not be fulfilled in practice. Consequently, the lack of complete information at the corners, and the possibility of some inconsistencies at the blunt corner are not regarded as serious handicaps to an understanding of the general behavior of the slab.

16. Calculated Effectiveness of Reinforcement for Various Angles Between Main and Transverse Steel.—It is recalled that the direction of principal moment in the central region of a slab-bridge was found to be nearly normal to the supports for various loadings and for skews as great as 45 degrees. This fact makes it desirable to examine the practice which is frequently followed of placing the main bottom steel parallel to the curbs and the other layer of bottom steel parallel to the supports. When a layer of steel is placed at an angle with the direction of the moment which it is intended to resist, it is customary to restore its effectiveness in the direction of the moment by increasing its area according to the square of the secant of the angle of inclination. It will be shown here that this practice leaves the slab weak on sections which bisect the acute angle between the bands of steel.

Consider a slab having a layer of steel of percentage $p_1$ in the $x$ direction and a second layer of steel of percentage $p_2$ in a direction which makes an angle $\alpha$ with the $x$-axis as shown in Fig. 32a. To determine the effective percentage $p$ at any angle $\theta$ with the $x$-axis, it is noted first that the effectiveness of a single layer of steel varies as the square of the cosine of the angle between the direction of the steel and the direction in which the effectiveness is required. This results from the increased distance between the bars and the de-
creased components of the allowable tension in the bars as shown in Fig. 32b. Therefore, for the two layers of steel,

\[ p = p_1 \cos^2 \theta + p_2 \cos^2 (\theta + \alpha) \]

\[ = \frac{p_1 + p_2}{2} + \frac{p_1}{2} \cos 2\theta + \frac{p_2}{2} \cos 2(\theta + \alpha) \]

\[ = \frac{p_1 + p_2}{2} + \frac{p_1 + p_2 \cos 2\alpha}{2} \cos 2\theta - \frac{p_2 \sin 2\alpha}{2} \sin 2\theta. \]

Furthermore, the rate of change of \( p \) with respect to \( \theta \) is

\[ \frac{dp}{d\theta} = -(p_1 + p_2 \cos 2\alpha) \sin 2\theta - (p_2 \sin 2\alpha) \cos 2\theta \]

and therefore \( \frac{dp}{d\theta} = 0 \) when

\[ \tan 2\theta = -\frac{p_2 \sin 2\alpha}{p_1 + p_2 \cos 2\alpha}. \]

Therefore

\[ p_{\text{max}} = \frac{p_1 + p_2}{2} \pm \sqrt{\left( \frac{p_1 - p_2}{2} \right)^2 + p_1 p_2 \cos^2 \alpha}. \]

To determine the variation of the effective percentage \( p \) with \( \theta \) for a particular slab, consider the slab bridge with 45-degree skew which has been analyzed in previous sections. Suppose the bottom steel is placed parallel to the supports and curbs as shown in Fig. 33. The design moment in the direction of the span (normal to the sup-
ports) requires a percentage of steel \( p_0 \), but, since the steel is inclined at 45 degrees with the moment, its percentage is made

\[
p_1 = p_0 \sec^2 45^\circ = 2p_0.
\]

The moment in the direction of axis 2 was found to be 46 per cent of the design moment so that

\[
p_2 = 0.46p_0.
\]

With these percentages of steel, the maximum and minimum effective percentages are, from (125),

\[
\begin{align*}
p_{\text{max}} &= 2.26p_0 \\
p_{\text{min}} &= 0.20p_0.
\end{align*}
\]

This means that, in spite of the fact that the main steel has been doubled in area over the requirements in the direction of the normal span, there is some direction in which the effectiveness of the steel is less than half as great as the effectiveness provided in the direction of the minimum moment. Furthermore, in that direction of minimum effectiveness of steel the bending moment has a value between its maximum and minimum values. In this illustration the angle \( \theta \) for minimum effectiveness of steel is defined by the equation

\[
\tan 2\theta = -\frac{p_2 \sin 2\alpha}{p_1 + p_2 \cos 2\alpha} = -\frac{p_2}{p_1} = -0.23,
\]

so that

\[
\theta = 83^\circ 31'.
\]
The corresponding moment is

$$M = \frac{M_1 + M_2}{2} - \frac{M_1 - M_2}{2}\cos 2\phi$$

where $\phi = \theta - (\pi/4)$, so that, with $M_2 = 0.46M_1$,

$$M = 0.73M_1 - 0.27M_1(0.224) = 0.67M_1.$$  

Since the percentage $p_0$ exactly provides for the moment $M_1$, the minimum effective percentage $0.20p_0$ is all that is available to provide for a moment of $0.67M_1$.

To say, as a result of the preceding analysis, that stresses in the steel will be more than three times too high would be definitely misleading. For, if this were true, the moments in some directions would have to be many times those indicated by the analysis. In fact, it is very doubtful that the inclined steel, increased in percentage, is in a position to develop high stresses before the concrete has suffered considerable damage and a redistribution of moments takes place. Many anomalies could be mentioned in connection with steel inclined to the directions of principal moments. After all, the replacement of tension in the concrete by tension in the steel is an ideal which is at times incapable of accomplishment. What is pointed out here is that a serious deficiency in theoretical steel does exist in certain directions when steel bands do not cross at or near right angles, even though the percentage of inclined steel is increased by the square of the secant of the angle of inclination.

A graphical representation of the variation of effective percentage $p$ with the angle of inclination $\theta$ is shown in Fig. 34 for three arrangements of steel. The radial ordinate from $O$ to any curve indicates the effective percentage of steel in the direction of the ordinate for the arrangement of steel which applies to that curve. Curve $A$, for the steel at right angles, has the same form as that which represents the variation in moment when $M_2 = 0.5M_1$. This is practically the same ratio as has just been discussed. The interpretation of the diagram, on the basis of effective percentage of steel, is that a deficiency exists in any direction in which the radial ordinate for the actual arrangement of steel falls inside of curve $A$.

17. *Comparisons of Moments Calculated for Skew and Right Bridges.*—With the aid of influence surfaces, maximum moments in a
ANALYSES OF SKEW SLABS

Fig. 34. Variation in Effective Percentage for Various Arrangements of Steel.

skew slab-bridge were determined in Section 14 for two stiffnesses of curb. The bridge considered is shown in Fig. 22. The manner of variation of the live and dead load moments in the curb and slab is shown in Figs. 35 and 36 in comparison with corresponding variations of moment in a right bridge. The right bridge chosen for comparison was considered to have a 26-ft. roadway (27 ft.-6 in. c. to c. curbs) and a span of 19 ft.-6 in., the same as the normal span of the skew slab.
The stiffness of the curb relative to the slab in the skew bridge is defined here as

\[ H = \frac{E_1 I_1}{a N} \]

where \( a \) is the span normal to the supports. This gives a relation between \( H \) and \( J \). For the square network

\[ H = \frac{\lambda}{a} J. \]

Corresponding to the values \( J = 1.5 \) and \( J = 3 \) in the analyses by difference equations, one has, with \( \lambda/a = \sqrt{2/5} \), \( H = 0.424 \) and \( H = 0.848 \), respectively. The curves of moments for the right bridge were obtained from numerical values computed for \( H = 0.2, 0.5, \) and 1.0, using the tabular data given in Bulletin 315.

Figure 35 shows curves of moment in the curb of the right bridge for dead load only, live load only, and dead plus live load. The plotted points show the corresponding values of moment computed for the bridge having a 45-degree skew. In general the trend of variation of moments with stiffness of curb is similar for the skew and right bridge. The dead load moments in the curb of the skew bridge are from 13 to 26 per cent higher than the corresponding moments in the right bridge. The live load moments are from 2 to 6 per cent higher, and the total moments are from 6 to 14 per cent higher in the curb of the skew bridge. In each instance the smaller percentage applies to the lower value of curb stiffness for which analysis was made.

Figure 36 shows similar comparisons for maximum dead load, live load, and total moment in the slab. It is seen that the dead load moments are from 13 to 18 per cent higher, the live load moments from 7 to 3 per cent lower, and the total moments from 0 to 3 per cent higher in the skew bridge than in the right bridge. The first percentage given in each instance applies to the slab having the lower stiffness of curb.

It is apparent from the foregoing comparison that a satisfactory design of the slab of 45-degree skew would have resulted if the moment in the slab had been estimated from an analysis of a right bridge having the same span as the normal span of the skew bridge, and the same detail of curb. The estimated moment in the skew slab would have been practically identical with that found from an analysis of
the skew bridge. In the curb the moment estimated from the right bridge would have been about 6 per cent too low, a deficiency which is small enough to justify using the approximate method.

18. Summary and Discussion.—In this bulletin skew slabs are studied, particular attention being directed to the requirements for the analysis of the simple span slab-bridge with curbs. Difference equations are developed for a general system of skew coordinates.
permitting this type of bridge to be analyzed for any angle of skew and ratio of sides and for any stiffness of curb. It is hoped that a sufficient number of analyses may be made as indicated herein in order that a sounder theoretical basis can be established for simplified rules for the design of skew slab-bridges.

From the limited study that has been made thus far it seems probable that slab-bridges having ordinary curbs, two or more lanes of traffic, spans (normal to the abutments) of 20 ft. or less, and skews of 45 degrees or less may be designed on the basis of moments calculated from the coefficients given in Bulletin 315 for right bridges of identical span, width of roadway, and detail of curb. The direction of maximum moment at the center of such slabs is practically normal
to the supports for design loadings, the slight inclination being toward
the direction of the skew span.

In analyzing a particular bridge for highway truck loadings it was
found expedient to use influence surfaces for moments at critical
sections. For bridges limited as in the previous paragraph, the critical
section for moment in the slab is in the vicinity of the center of the
slab. The critical section for moment in the curb is shifted slightly
from the center of the curb toward the blunt corner of the slab as the
skew is increased. The influence surfaces for moment were obtained
by a method which is due to Newmark. The principle of the method
may be stated as follows:

Where an effect $Q_a$ at any point $a$ in a slab is a linear relation of the
deflections, as in the equation

$$Q_a = A w_a + B w_b + \cdots + M w_m,$$

an influence surface for $Q_a$ may be obtained as the deflection of the slab
due to a system of loads $A, B, \cdots, M$, applied at the points $a, b, \cdots, m$,
respectively.

Contours of influence surfaces for moments in curbs and slabs, ob-
tained by applying this principle, are shown in Section 14.

The problem of the proper arrangement of reinforcement is of
course related to the cost and practicability of placing many bars of
different lengths or at varying angles in the same slab. As far as the
directions of maximum moments are concerned it would be desirable,
for skews under 45 degrees and spans (normal to the abutments)
under 20 ft., to have the main steel at the center of the slab inclined
not more than 15 degrees from the line normal to the abutments, the
inclination, if any, being toward the direction of the skew span. Bands
of steel to take the moment in the curbs should be placed parallel with
the curbs within the foregoing limitations. The main steel in the slab
adjacent to the curbs may run into the curb in a direction parallel
with the main steel at the center of the slab, or it may run parallel
with the curb. In the latter case the direction of the main steel may
have to be changed gradually so as to meet the requirements given
previously for the central region of the slab. For the foregoing arrange-
ments of the main steel, the transverse steel may have a direction
parallel with the supports.

Tests of large reinforced concrete skew slabs with curbs are now
Table 16
SOLUTION OF SIMULTANEOUS EQUATIONS

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in progress. In these tests various arrangements of steel will be tried. The analytical studies are being extended and are being supplemented by tests of plaster models.

**APPENDIX A**

**Notes on Solution of Simultaneous Equations.**—The solution of a system of simultaneous linear equations, especially when the number of equations becomes greater than four or five, can be a laborious process often requiring in the separate steps of the solution an accuracy which is many times that desired in the final result. On the other hand, by a suitable arrangement of the work, the required labor can be materially lessened and the accuracy required in the individual steps can be made of the same order as that required in the final result.

The method of solution adopted by the author as best suited for the solution of the equations which arise from the application of difference equations to slabs has been called the “Doolittle Method” in Special Publication No. 28* of the United States Coast and Geodetic Survey where it is described. The method was developed by M. H. Doolittle as a modification of the Gaussian method of substitution, and was first presented in the “Coast and Geodetic Survey Report for 1878,” Appendix 8, pp. 115-118. Because the method does not seem to be familiar to most engineers, it is described here without proof.

Consider the system of equations (22) given in Section 5 for the values of $U$ and the second system (24) given for the deflections. It is to be noted that a form of symmetry exists among the coefficients.


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with respect to the main diagonal provided that the last equation in each set is divided by 2. This symmetry of coefficients is characteristic of all of the sets of simultaneous equations which are written for the solution of problems on slabs. Because of this symmetry a considerable shortening of labor is possible in the solution of the equations, although even without symmetry a systematic method of substitution is convenient and rapid.

Equations (22) are tabulated in the upper part of Table 16, the figures in the last column being sums of the coefficients in the rows. Since the figures to the left of the heavy line are repeated above the main diagonal terms, they need not actually be recorded. A figure in the summation column is then obtained by adding the coefficients in a column as far as the main diagonal term and then continuing the addition in a horizontal row. The solution of the equations is given in the remainder of the table.

The first step in the solution is to write in the first line the coefficients from the first equation. In the second line these are divided by the negative of the coefficient of $U_1$. This, in effect, gives a solution for $U_1$ in terms of the remaining variables and the constant. Each equation giving a particular $U$ in terms of the remaining ones is doubly underlined and checked by summing the coefficients horizontally for comparison with the coefficient in the summation column. The third line gives the coefficients of the second equation to the right of the heavy zigzag line. To these must be added the values given in the fourth line which represent $-4U_1$ obtained by multiplying the coefficient of $U_2$ in line 1 by the coefficients in line 2. The results of the addition are given in line 5. Line 6 gives $U_2$ in terms of the remaining variables and a constant. Again this line is checked before proceeding.

The process is continued in the manner described. It is important to note, however, that the multipliers of the coefficients which take the place of the dropped terms to the left of the main diagonal always appear immediately above the doubly underlined equations. These multipliers are indicated in bold-faced type in the table. Their appearance in these positions is a direct result of the symmetrical form of the coefficients, and their use in this manner is the essence of the abbreviated method of solution. The fact that checks may be made as the calculations proceed is a contributing factor to the value of the method. The further fact that the calculations may be made with slide-rule accuracy in the individual steps is demonstrated in the table. Line 36 in Table 16 indicates that

$$U_8 = -0.743 \frac{pb^2}{9N}.$$
It remains to substitute back into the doubly-underlined equations for the remaining values of \( U \). This may be done systematically as in lines 37 to 45 inclusive. The numbers in line 37 are taken from the column of constants in lines 2, 6, 10, 15, etc. The numbers in line 38 are products of \( U_s \) and the coefficients of \( U_s \) from the table. Thus

\[
U_7 = -0.393 \frac{pb^2}{9N} + 0.236 U_s = -0.569 \frac{pb^2}{9N}.
\]

In the tabulation the coefficient of \( U_7 \) is given in parentheses and is next used as a multiplier of coefficients of \( U_7 \) in the table to obtain the numbers in line 39. The value of \( U_6 \) is then obtainable. The substitution continues until all values of \( U \) are obtained. The actual process is shorter than its description.

Most of the calculations contained in Table 16 are now available and usable for the solution of the second set of equations for the deflections. All that is needed is to add one more column of constants to take the place of those headed \( pb^2/9N \). The coefficients in the new column of constants will be the numerical values of \( U \) just obtained except for the effect of dividing the last equation by 2. The solution leads to the values of deflection given in the text, the final accuracy being slide-rule accuracy if the individual calculations are made with a slide-rule.

**APPENDIX B**

*Derivation of \( V^2 w \) and \( \frac{\partial^2 w}{\partial x \partial y} \) in Terms of Curvatures in Three Directions.*—In this appendix, subscripts on \( w \) will indicate differentiation with respect to the variable noted, as, for example

\[
w_{xx} = \frac{\partial^2 w}{\partial x^2}, \quad w_{uu} = \frac{\partial^2 w}{\partial u^2}.
\]

etc. Then, with coordinate axes and angles defined in Fig. 8,

\[
w_{uu} = \frac{w_{xx} + w_{yy}}{2} - \frac{w_{xx} - w_{yy}}{2} \cos 2\phi - w_{xy} \sin 2\phi \quad (127)
\]

\[
w_{vv} = \frac{w_{xx} + w_{yy}}{2} - \frac{w_{xx} - w_{yy}}{2} \cos 2\delta + w_{xy} \sin 2\delta. \quad (128)
\]
Elimination of $w_{xy}$ from these equations gives

$$w_{yy} = \frac{\sin 2(\phi + \delta) - \sin 2\delta - \sin 2\phi}{\sin 2(\phi + \delta) + \sin 2\delta + \sin 2\phi} w_{xx}$$

$$+ \frac{2 \sin 2\delta}{\sin 2(\phi + \delta) + \sin 2\delta + \sin 2\phi} w_{uu}$$

$$+ \frac{2 \sin 2\phi}{\sin 2(\phi + \delta) + \sin 2\delta + \sin 2\phi} w_{vv}.$$

But, from the relations given in Fig. 8,

$$\sin 2(\phi + \delta) = 2 \frac{\lambda_x \lambda_y}{\lambda_u^2 \lambda_v^2} (\lambda_y^2 - \alpha \beta)$$

$$\sin 2\delta = 2 \frac{\alpha \lambda_y}{\lambda_v^2}$$

$$\sin 2\phi = 2 \frac{\beta \lambda_y}{\lambda_u^2}$$

and

$$\alpha + \beta = \lambda_x$$

$$\lambda_u^2 = \beta^2 + \lambda_y^2$$

$$\lambda_v^2 = \alpha^2 + \lambda_y^2.$$

Therefore

$$w_{yy} = -\frac{\alpha \beta}{\lambda_y^2} w_{xx} + \frac{\alpha \lambda_y^2}{\lambda_x \lambda_y^2} w_{uu} + \frac{\beta \lambda_x^2}{\lambda_x \lambda_y^2} w_{vv}$$

which is Equation (27) of the text. From this equation one obtains

$$\nabla^2 w = w_{xx} + w_{yy}$$

at once in terms of curvatures in the directions of $x$, $u$, and $v$.

The elimination of $w_{xy}$ from (127) and (128) yields, in a similar manner, the relation

$$w_{xy} = -\frac{\lambda_u^2}{2 \lambda_x \lambda_y} w_{uu} + \frac{\beta - \alpha}{2 \lambda_y} w_{xx} + \frac{\lambda_v^2}{2 \lambda_x \lambda_y} w_{vv}. \quad (129)$$
With the abbreviations
\[ A = \frac{\alpha}{\lambda_x}, \quad B = \frac{\beta}{\lambda_x} \]

and with the curvatures expressed in terms of finite differences according to the notation given in Fig. 8, Equation (129) becomes

\[ w_{x_0} = \frac{1}{2\lambda_x \lambda_y} \left[ w_q + w_t - w_q - w_s + (B-A) (w_x - 2w_0 + w_r) \right]. \] (130)

**APPENDIX C**

*Derivation of Relation Between Slopes in Three Directions.*—In discussing the slope of a function \( f \) in the direction of \( y \) in terms of slopes in the directions of \( u \) and \( v \), the relation

\[ \left( \frac{\partial f}{\partial y} \right)_o = A \frac{f_p - f_x}{2\lambda_y} + B \frac{f_q - f_t}{2\lambda_y} \] (53)

was given in the text. The relations between the axes and the positions of points are shown in Fig. 8. By passing a plane \( abcd \) tangent to \( f \) at point \( O \) as shown in Fig. 37, the following relations are found:

\[ \left( \frac{\partial f}{\partial y} \right)_o = \frac{f_1 - f_2}{2\lambda_y}, \]

\[ f_1 = f_q + \frac{\alpha}{\lambda_x} (f_p - f_q) = Af_p + Bf_q, \]

\[ f_2 = f_s + \frac{\beta}{\lambda_x} (f_t - f_s) = Bf_t + Af_s, \]

wherein

\[ A = \frac{\alpha}{\lambda_x}, \quad B = \frac{\beta}{\lambda_x}, \quad A + B = 1. \]

From these, Equation (53) follows immediately.

From Fig. 37 the slope in the direction of \( x \) appears either as the slope along the line \( pq \) or as the slope along the line \( ts \). The best
value from difference equations is the average of these two slopes, so that
\[
\left( \frac{\partial f}{\partial x} \right)_o = \frac{f_q - f_p}{2\lambda_x} + \frac{f_s - f_t}{2\lambda_x} = \frac{\lambda_v}{\lambda_x} \frac{f_q - f_t}{2\lambda_v} - \frac{\lambda_u}{\lambda_x} \frac{f_p - f_s}{2\lambda_u},
\]

or
\[
\frac{\partial f}{\partial x} = \frac{\lambda_v}{\lambda_x} \frac{\partial f}{\partial v} - \frac{\lambda_u}{\lambda_x} \frac{\partial f}{\partial u}.
\]

If this relationship is applied at point \( w \) in Fig. 15, with \( w_w = w_n = w_t = 0 \), Equation (64) results at once.
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