A STUDY OF CHATTER-INDUCED LOSS OF MECHANICAL CONTACT

BY

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THESIS

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This thesis concerns a recently discovered paradox in rigid body mechanics called *reverse chatter*, in which friction can cause the problem of two rigid bodies in sustained contact to admit multiple solutions. In particular, when the point of contact between the two bodies comes to rest, under certain conditions the rigid body model—coupled with Amontons-Coulomb friction and Stronge’s energetic *impact termination condition*—allows for sustained stick, as well as an infinite number of other trajectories, each involving an infinite number of impacts in finite time. A similar mechanism can occur between two bodies with a compliant contact model, though in this case there are only finitely many impacts. The purpose of the present work is to call attention to this reverse chatter phenomenon, and to explore it in greater depth. In particular, we investigate what becomes of reverse chatter under alternative impact termination conditions to that of Stronge, namely those of Newton and Poisson. We begin by establishing, for the first time to our knowledge, that Poisson’s impact termination condition is energetically consistent (i.e., it cannot generate energy during a frictional impact). We then show that reverse chatter is possible under Poisson’s impact termination condition, but not under Newton’s, thus establishing that, while the possibility of reverse chatter is sensitive to the impact termination condition used, it is not simply an artifact of Stronge’s hypothesis. Additionally, we consider what becomes of chatter in the presence of an external control scheme which attempts to keep two bodies in sustained contact. We find that chatter-like behavior is still possible, and can lead to a loss of contact followed by a sequence of impacts qualitatively similar to that observed when chalk hops on a chalkboard. It is argued that reverse chatter may be responsible for this and similar phenomena. Furthermore, the present results suggest that reverse chatter occurs under easily achievable laboratory conditions, setting the stage for reverse chatter to be studied experimentally in the future.
To my mother, whose unwavering love and countless sacrifices have made this work possible.
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LIST OF SYMBOLS

\(a_n\) Contact force independent acceleration in the direction of \(n\).
\(a_\tau\) Contact force independent acceleration in the direction of \(\tau\).
\(A\) Point of contact between \(\mathcal{B}_1\) and \(\mathcal{B}_2\).
\(\bar{b}_i\) \(2 \times 1\) eigenvector of \(\bar{B}\) corresponding to eigenvalue \(B_i\).
\(\bar{B}\) \(2 \times 2\) minor matrix of \(\bar{C}\) corresponding to components \(C_{22}, C_{23},\) and \(C_{33}\).
\(B_i\) Eigenvalue of \(\bar{B}\) corresponding to eigenvector \(\bar{b}_i\).
\(\mathcal{B}_i\) Rigid body \(i\).
\(c_{y,\theta}\) Commanded acceleration for \(y\) or \(\theta\).
\(\bar{C}\) \(3 \times 3\) or \(2 \times 2\) configuration matrix.
\(C_{ij}\) Element of \(\bar{C}\).
\(e\) Limiting ratio of the relative normal velocity just before subsequent impacts.
\(e_i\) Unit basis vector for Euclidian space.
\(\mathbb{E}^3\) Euclidian space.
\(f\) Negative of \(F_n\), or a continuous function.
\(f_s\) Sampling frequency.
\(f_c\) Filter cutoff frequency.
\(\mathbf{F}\) Force vector acting on \(\mathcal{B}_1\) as a result of contact with \(\mathcal{B}_2\).
\(\mathbf{F}_i\) Net external force vector acting on the center of mass of \(\mathcal{B}_i\).
\(\mathbf{F}_i^{\text{net}}\) Net force vector acting on \(\mathcal{B}_i\).
$F_n$ Scalar component of $F$ in the direction of $n$.

$F_n$ Vector component of $F$ in the direction of $n$.

$F_{sw}$ Minimum detectible force.

$F_\tau$ Scalar component of $F$ in the direction of $\tau$.

$F_\tau$ Vector component of $F$ which lies in the common tangent plane between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ at $A$.

$g$ Acceleration due to gravity on the surface of the earth.

$g$ Impact map.

$h$ Height, or a continuous function.

$\bar{h}$ $2 \times 1$ column matrix whose components are $C_{12}$ and $C_{13}$.

$\bar{I}$ $3 \times 1$ column matrix whose elements are the scalar components of $I$ in the $e_\tau$-basis.

$I$ Impulse vector due to $F$.

$I_i$ Scalar component of $I$ in the direction of $e_i$.

$I_n$ Scalar component of $I$ in the direction of $n$.

$I_n$ Vector component of $I$ in the direction of $n$.

$I_n^\dagger$ Value of $I_n$ for which $Z = 0$.

$I_\tau$ Scalar component of $I$ in the direction of $\tau$.

$\bar{I}_\tau$ $3 \times 1$ column matrix whose elements are the scalar components of $I_\tau$ in the $e_\tau$-basis.

$\bar{I}_\tau^*$ $2 \times 1$ column matrix whose elements are $I_2$ and $I_3$.

$I_\tau$ Vector component of $I$ which lies in the common tangent plane between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ at $A$.

$J_i$ Moment of inertia tensor of $\mathfrak{B}_i$, measured about its center of mass.

$k_i$ Radius of gyration of $\mathfrak{B}_i$ about the axis defined by $e_3$.

$k_{n,\tau}^{\pm,0}$ Rate constant (planar impacts only).

$k_{u,\theta}^{v,p,s,f,I,d}$ Gain parameter.

$\ell$ Length.
\( \mathbf{L}_i \)  
Angular momentum vector of \( \mathcal{B}_i \), measured about its center of mass.

\( m \)  
Effective mass of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \).

\( m_i \)  
Mass of \( \mathcal{B}_i \).

\( \bar{n} \)  
3 \times 1 \text{ column matrix whose elements are the scalar components of } \mathbf{n} \text{ in the } \mathbf{e}_i\text{-basis}.

\( \mathbf{n} \)  
Unit vector perpendicular to the common tangent plane between \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) at \( A \) and pointing toward \( \mathcal{B}_1 \).

\( \mathbb{N} \)  
The set of natural numbers: \{1, 2, 3,...\}.

\( O(q) \)  
Quantity of order \( q \).

\( \mathbf{P}_i \)  
Linear momentum vector of \( \mathcal{B}_i \).

\( \mathbf{r}_i \)  
Displacement vector from the center of mass of \( \mathcal{B}_i \) to \( A \).

\( r^n_i \)  
Scalar component of \( \mathbf{r}_i \) in the direction of \( \mathbf{n} \).

\( r^\tau_i \)  
Scalar component of \( \mathbf{r}_i \) in the direction of \( \mathbf{\tau} \).

\( R \)  
Torque.

\( \check{R}_k \)  
Unfiltered value of \( R \) at the \( k \)th sampling time.

\( s \)  
Riemann sum approximation to force integral.

\( S_x \)  
Force in the \( x \)-direction.

\( S_y \)  
Force in the \( y \)-direction.

\( \check{S}_{y,k} \)  
Unfiltered value of \( S_y \) at the \( k \)th sampling time.

\( t \)  
Time.

\( t_{sw} \)  
Time at a transition between position control and force control.

\( t^+ \)  
Time at which impact ends.

\( t^* \)  
Time at the transition between compression and restitution.

\( t^0 \)  
Time at which \( v_r \) vanishes.

\( t^- \)  
Time at which impact begins.

\( T \)  
Kinetic energy, or period of time.

\( \bar{T} \)  
2 \times 2 \text{ 90° counterclockwise rotation matrix}.
\( T_k \)  Duration of the \( k \)th free-flight period.

\( T_s \)  Sampling period.

\( u \)  Difference between \( y \) and its equilibrium value.

\( \bar{v} \)  \( 3 \times 1 \) column matrix whose elements are the scalar components of \( v \) in the \( \{e_i\} \)-basis.

\( v \)  Relative velocity vector of \( \mathcal{B}_1 \) with respect to \( \mathcal{B}_2 \) at \( A \).

\( v_i \)  Scalar component of \( v \) in the direction of \( e_i \).

\( v_n \)  Scalar component of \( v \) in the direction of \( n \).

\( v_n \)  Vector component of \( v \) in the direction of \( n \).

\( v_\tau \)  Scalar component of \( v \) in the direction of \( \tau \).

\( v_\tau \)  Vector component of \( v \) which lies in the common tangent plane between \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) at \( A \).

\( \bar{v}_\tau \)  \( 2 \times 1 \) column matrix whose elements are \( v_2 \) and \( v_3 \).

\( V_i \)  Linear velocity vector of the center of mass of \( \mathcal{B}_i \).

\( V_i^A \)  Linear velocity vector of the point on \( \mathcal{B}_i \) which coincides with \( A \).

\( W \)  Total work done on \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) during impact.

\( W_i \)  Work done on \( \mathcal{B}_i \) during impact.

\( W_n \)  Work done by \( F_n \) during impact.

\( x \)  Spatial coordinate.

\( y \)  Normal displacement between \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) at \( A \).

\( Z \)  Work integrand.

\( \alpha \)  Positive constant less than one.

\( \Gamma_i \)  Net external torque vector acting on \( \mathcal{B}_i \), measured about its center of mass.

\( \Gamma_{\text{net}}^i \)  Net torque vector acting on \( \mathcal{B}_i \), measured about its center of mass.

\( \delta_{ij} \)  Kronecker delta symbol.

\( \epsilon \)  Impact phase duration, a small quantity.

\( \epsilon_{ijk} \)  Levi-Civita symbol.
\( \varepsilon \) A small parameter.

\( \zeta_{y,\theta} \) Damping ratio for \( y \) or \( \theta \).

\( \eta_{N,P,S} \) Coefficient of restitution (dimensionless).

\( \theta \) Angular coordinate.

\( \mu \) Value of both \( \mu_s \) and \( \mu_k \).

\( \bar{\mu} \) Coefficient of friction in stick.

\( \mu_k \) Coefficient of kinetic friction.

\( \mu_s \) Coefficient of static friction.

\( \rho \) Limiting ratio of tangential and normal relative velocities before an impact.

\( \bar{\sigma} \) \( 3 \times 1 \) column matrix whose elements are the scalar components of \( \sigma \) in the \( e_i \)-basis.

\( \sigma \) Unit vector in the direction of slip.

\( \bar{\sigma}^* \) \( 2 \times 1 \) column matrix whose elements are \( \sigma_2 \) and \( \sigma_3 \).

\( \sigma_i \) Scalar component of \( \sigma \) in the direction of \( e_i \).

\( \tau \) Unit tangent vector (planar impacts only).

\( \phi \) Angle \( \bar{\sigma}^* \) makes with the direction defined by \( \bar{b}_1 \).

\( \omega \) Angular velocity.

\( \omega_c \) Filter cutoff angular frequency.

\( \omega_{y,\theta} \) Angular frequency for \( y \) or \( \theta \).

\( \Omega_i \) Angular velocity vector of \( \mathcal{B}_i \).
CHAPTER 1

INTRODUCTION

This thesis concerns a paradox in rigid body mechanics called reverse chatter, first discovered by Arne Nordmark, Harry Dankowicz, and Alan Champneys in 2011 (see [1]), in which friction can cause the loss of uniqueness of solutions to the problem of two bodies in sustained contact. We begin this chapter with a review of the relevant literature, and conclude with a summary of the remaining chapters, highlighting the original contributions and the arguments presented in this work.

1.1 Literature Review

The three main topics of relevance to the present work are (i) contact mechanics in general, (ii) paradoxes within contact mechanics, and (iii) contact transition control. We proceed to give an overview of the work that has already been done in these areas.

1.1.1 Contact Mechanics

Even before the advent of classical mechanics, a great deal of effort was made to understand mechanical contact, and collisional impact in particular. Stronge gives an abridged account of “the role of impact in the development of mechanics during the seventeenth and eighteenth centuries” in Appendix A of [2], which summarizes the contributions of Galileo Galilei, Marcus Marci, René Descartes, John Wallis, Christopher Wren, Christian Huyghens, Edmé Mariotte, Isaac Newton, and Leonhard Euler.

Of course, it was not until the publication of Newton’s *Philosophiae Naturalis Principia Mathematica* in 1686 that impact could be approached from a completely theoretical point of view. Indeed, one of the very first examples
given by Newton to demonstrate the consistency of his laws of motion with previous empirical observations was the collinear impact of spherical bodies supported by strings. In the Scholium of the chapter titled “Axioms, or Laws of Motion,” Newton describes his experiments with both elastic and inelastic spheres. Of his results with inelastic spheres, he writes:

[T]he experiments just described succeed equally as well with soft bodies as with hard bodies, evidently not depending at all on the condition of hardness. For if that rule is to be tested with bodies which are not perfectly hard, the reflection should simply be reduced by a certain proportion, according to the magnitude of the elastic force. According to the theory of Wren and Huygens, bodies which are absolutely hard return from one another with the same velocity with which they meet. But this will be confirmed more certainly of perfectly elastic bodies. In imperfectly elastic bodies, the velocity of the return should be reduced together with the elastic force; because that force (except when parts of the bodies are injured as a result of the impact, or undergo some sort of extension, as under a hammer) is certain and determined (as far as I can see), and makes bodies return from one another with a relative velocity which is in a given ratio to the relative velocity with which they collide. I tried this with balls of wool which were tightly bound and strongly compressed. First, by letting go of the hanging bodies and measuring their reflection, I found the magnitude of the elastic force; then, from this force, I determined the reflections that ought to occur in other cases of impacts, and experiments agreed. The balls always returned from one another with a relative velocity which was about 5/9 the velocity with which they collided. Balls of steel returned with almost the same velocity: others made of cork with a slightly smaller velocity: but in balls of glass the proportion was about 15/16. And in this way the third law, as far as it concerns collisions and reflections, has been proven by a theory which plainly agrees with experience ([3], trans. by John Sanders).

This ratio of the relative velocity of two bodies after an impact to that before the impact (typically taken to lie between 0 and 1) is now known as the kinematic coefficient of restitution or, to honor Newton, Newton’s coefficient of restitution (see, for example, [2] and [4–10]). As we will see, while New-
ton’s laws of motion govern the dynamics of a system during impact, it is the coefficient of restitution that determines the condition under which the impact ends, often called the *impact termination condition* for short. However, because Newton did not propose a model for the force of interaction between two bodies in contact, he could not apply his second law of motion during impact.

Near the turn of the eighteenth century, several years after the publication of Newton’s *Principia*, Guillaume Amontons published his laws of friction, which were later verified experimentally by Charles-Augustin de Coulomb (see, for example, Volume II, Chapter XXII of [11]). These laws, often treated as a single law (the so-called *Amontons-Coulomb law of friction*) provide a model for the force of interaction between two bodies in contact, and thus provide the final piece of the puzzle in classical, rigid body contact/impact mechanics, along with Newton’s laws of motion and Newton’s impact termination condition.

By the nineteenth century, however, it had become evident that while Newton’s coefficient of restitution agrees with experiments for the collinear impact configurations described in the *Principia*, a more adequate impact termination condition is needed for higher-dimensional impacts. In his *Traité de Mécanique*, published in 1811, Siméon Denis Poisson proposed another coefficient of restitution. Rather than considering only the state of the system before and after impact, Poisson distinguished between two phases of impact: (i) *compression*, during which the two objects impinge upon each other, and (ii) *restitution*, during which they retract from each other. His coefficient is based on the impulse (which he calls “quantity of motion,” following Newton’s terminology) imparted to each of the bodies during each of these two phases. In Chapter VII of Book IV, he writes:

*If the two bodies were not assumed to be perfectly elastic, [the quantity of motion impressed on each of these bodies in the normal direction] would be less in the second part of the impact than in the first; we should then assume for its value in the second part, a fraction f of its value [in the first]. This fraction f would depend on the degree of elasticity of the two bodies, and could only be determined by experiments made on bodies of the same kind of matter, in the simplest case, with respect to their original shape and motion* ([12], trans. by Maggie Sanders).
This ratio of the normal impulse imparted to each of the bodies during restitution to that imparted during compression (also typically taken to lie between 0 and 1) is now known as the kinetic coefficient of restitution, or, to honor Poisson, Poisson’s coefficient of restitution (see, for example, [2], [4–7], [9], [13], and [14]; note, however, that [5] mistakenly attributes the kinetic coefficient of restitution to Newton rather than to Poisson). To this day, the impact termination conditions of Newton and Poisson are the two most popular, the former being easier to implement, and the latter agreeing more closely with observations for complicated impact configurations.

In 1984, however, a professor at Stanford University named Thomas Kane discovered that Newton’s impact termination condition can, under certain circumstances, yield an increase in kinetic energy during frictional impacts, even when Newton’s coefficient lies between 0 and 1 (see [15]). This paradoxical behavior prompted the notion of energetic consistency, an impact termination condition being deemed energetically consistent if it cannot yield an increase in kinetic energy as long as the corresponding coefficient of restitution falls between 0 and 1. There was debate as to whether or not Poisson’s impact termination condition was energetically consistent, but a definitive proof eluded the mechanics community for many years. This inspired William Stronge, a professor at the University of Cambridge, to formulate his own energetic coefficient of restitution in 1990, the square of which is the ratio of the normal work done on each of the two impacting bodies during restitution to that done during compression (see [4]). Stronge’s impact termination condition is, by construction, energetically consistent; however, the debate over Poisson’s impact termination condition continued well into the nineties. In [4], Stronge shows that it dissipates energy in the special case of a planar impact with slip reversal during compression. Then, in 1992, a professor at Moscow Institute of Physics and Technology named Alexander Ivanov published a paper ([16]) in which he offered a proof that it was consistent in general. However, while the underlying idea of Ivanov’s proof is valid, one of the steps therein is erroneous. We will return to this flaw in Ivanov’s proof in a later chapter. Other impact termination conditions have been proposed and studied from the viewpoint of energetic consistency; see work by Chatterjee and Ruina (for example, [17]).

Once one has chosen an impact termination condition (be it Newton’s, Poisson’s, Stronge’s, or some other condition), it is possible—at least for so-
called planar impacts—to express the relative velocity of the two impacting bodies after the impact in terms of their relative velocity before the impact. This relation depends not only on the impact termination condition used; it also depends on the impact configuration itself. The set of all such relations for a particular impact termination condition is called an impact map. Nordmark et al. have constructed an explicit impact map for Stronge’s impact termination condition (see [18]); however, if explicit impact maps for Newton’s and Poisson’s conditions have been constructed, they do not appear to have been reported in the literature. In general, explicit impact maps cannot be formulated for three-dimensional impacts; nevertheless, great strides have been made in understanding them (see, for example, work done by Batlle in [6] and [19]).

1.1.2 Paradoxes in Contact Mechanics

While it might be said that Newton’s impact termination condition can lead to “paradoxical” behavior when it yields a gain in kinetic energy, this inconsistency is not an example of a true paradox. A paradox, in the most general sense of the word, is a problem which is not well posed mathematically. For example, any problem in classical physics for which no mathematical solution exists or for which there are multiple different solutions is a paradox. Several such problems have been identified in the classical theory of rigid body contact with friction. Here we give a few examples.

The Painlevé Paradoxes

In 1895, almost a century after Amontons published his laws of friction, a French mathematician by the name of Paul Painlevé posed a seemingly innocent problem: that of a rigid, slender rod sliding with friction on a rigid half-space. He, and others after him, showed that under certain conditions, the mathematical formulation of this problem (now recognized as a linear complementarity problem) has a unique solution; however, under certain other circumstances, it can have no solution, a finite number of distinct solutions, or infinitely many distinct solutions (see [20–22]; see also [23–28]). For example, in one case, there are two solutions: one corresponding to sustained sliding, and another corresponding to loss of contact (see Section 3.3.1
of [26]). These cases in which there is a loss of existence or uniqueness of solutions to the Painlevé problem are collectively referred to as the Painlevé paradoxes.

Several attempts to resolve the Painlevé paradoxes have been made. In 1905, the same year in which Painlevé published two of his papers on the subject, a French physicist and engineer named Léon Lecornu proposed a potential resolution in the form of “collision-less impacts”: discontinuous jumps in the normal velocity of the rod at the onset of the paradoxical behavior (see [29]; see also [27]). More recently, alternative resolutions have been pursued outside of the rigid body model. In the late nineties, for example, Prof. David Stewart at the University of Iowa used a time-stepping scheme to construct a numerical solution to the Painlevé problem, showed that this numerical solution converges, and argued that this resolves the existence problem in the case that the linear complementarity problem has no solution (see [24–26]). Later, in 2011, Profs. Arne Nordmark, Harry Dankowicz, and Alan Champneys analyzed the Painlevé problem with a compliant contact model, thus ensuring the existence of a solution, and considered the rigid body limit of zero compliance. They found that, when the linear complementarity problem has multiple solutions (e.g., both sustained sliding and loss of contact), the compliant model favors sustained contact, thus resolving the uniqueness problem in those cases (see [1]). The body of work mentioned here seems to have resolved the Painlevé paradoxes.

Reverse Chatter

Just as Nordmark et al. were resolving the last of the Painlevé paradoxes—indeed, in the very same paper ([1])—they discovered a similar but even more challenging paradox which they call reverse chatter. In this paradox, when the point of contact between two bodies in sustained sliding “sticks” (i.e., comes to rest momentarily), under certain conditions the rigid body model, coupled with Amontons-Coulomb friction and Stronge’s impact termination condition, admits multiple solutions. In particular, it allows for continued stick—which one would intuitively expect—as well as an infinite number of other trajectories, each of which involves an infinite number of impacts in finite time. However, unlike in the Painlevé case, Nordmark et al. showed that this new paradox cannot be resolved by considering a compliant contact
model in the rigid body limit. The issue of reverse chatter is therefore very much an open area of research, and in the present work we aim to explore the phenomenon in greater depth.

1.1.3 Contact Transition Control

During the second half of the twentieth century, even while impact mechanics was still being developed, engineers were already beginning to design ways to control impacting systems, especially robotic manipulators. In particular, there was much interest in keeping the end effector of a robotic arm in sustained contact with its environment. The two primary approaches to this task used today were developed in the early eighties. The first, hybrid position/force control, was developed by Prof. Marc Raibert and John Craig at the California Institute of Technology in 1981 (see [30]). As the name suggests, this approach regulates either the position of the end effector or the force it experiences, depending on whether or not it is in contact with its environment. The second approach, impedance control, was developed in 1985 by Prof. Neville Hogan at the Massachusetts Institute of Technology (see [31]). This approach regulates the mechanical impedance of the system during both contact and non-contact periods. As discussed by Tarn et al. in [32] and Goradia et al. in [33], there are advantages and disadvantages to each approach: while impedance control is fairly stable, it cannot regulate the contact force unless the exact nature of the environment is known and incorporated into the control scheme; contrariwise, while hybrid position/force control can regulate the contact force without detailed knowledge of the environment, it is prone to instabilities. The literature on this subject is vast; see, for example, [30–48]. In light of all this, a natural question to ask is what becomes of the phenomenon of reverse chatter in the presence of an external control scheme which is actively trying to keep two rigid bodies in sustained contact. We will address this question in the present work.

1.2 Thesis Summary

This thesis aims to call attention to the paradox of reverse chatter, and to explore the phenomenon in greater depth. Questions that will be addressed
include: Is Poisson’s impact termination condition truly energetically consistent? Is reverse chatter simply an artifact of Stronge’s impact termination condition, or can it occur for Poisson’s or Newton’s as well? Can chatter-like behavior occur in a compliant contact model in the presence external control? Is it possible to design an experiment in which one could observe reverse chatter?

The original contributions of this work are threefold. (i) We prove, for the first time (to our knowledge) since it was invented in 1811, that Poisson’s impact termination condition really is energetically consistent. (ii) We derive explicit impact maps for both Newton’s and Poisson’s impact termination conditions, and determine the conditions under which reverse chatter is possible for each impact map. We show that reverse chatter is possible under Poisson’s impact termination condition but not under Newton’s, establishing that, while the phenomenon is sensitive to the particular impact termination condition used, it is not peculiar to Stronge’s. (iii) We show that chatter-like behavior occurs in a compliant contact model, even in the presence of an external control scheme that is actively trying to keep the system in sustained contact. Indeed, reverse chatter causes a loss of contact, which then leads to a sequence of impacts not unlike those observed in chalk hopping on a chalkboard. It is argued that reverse chatter may be responsible for this and similar phenomena. Moreover, the particular scenario we investigate is easily achievable in a laboratory, providing a possible means by which to test reverse chatter experimentally.

The remainder of the thesis is organized as follows. Chapter 2 synthesizes the existing literature on rigid body impacts with friction, building up the theory from first principles and establishing a single, consistent set of notation, in which the contributions of this thesis will be expressed. Chapter 3 reviews the three impact termination conditions of Newton, Poisson, and Stronge. The flaw in Ivanov’s paper ([16]) is exposed and corrected, establishing for the first time to our knowledge a completely rigorous proof of the energetic consistency of Poisson’s condition. In Chapter 4, explicit impact maps for Newton’s and Poisson’s conditions are formulated alongside those developed for Stronge’s condition in [18]. Chapter 5 extends the material covered in Chapter 2 to sustained rigid body contact. In Chapter 6, reverse chatter is defined, and the results of Nordmark et al. in [1] for Stronge’s impact termination condition are reproduced and modified for Poisson’s im-
pact termination condition. Finally, Chapter 7 investigates what becomes of reverse chatter in the presence of the hybrid position/force control scheme developed by Tarn et al. in [32] and [34]. In Chapter 8, the conclusions of the present work are summarized, and some ideas are given for future work.
CHAPTER 2
RIGID BODY IMPACTS WITH FRICTION

We begin by formulating a general theory for impacts between two rigid bodies in the presence of friction, guided by Keller [13], Stronge [2], Ivanov [16], Nordmark et al. [18], and Batlle [19]. Consider two arbitrary rigid bodies, $B_1$ and $B_2$, as shown in Figure 2.1. The first, $B_1$, has mass $m_1$ and moment of inertia $J_1$ about its center of mass. It moves with velocity $V_1$ at its center of mass, and rotates with angular velocity $\Omega_1$. Furthermore, it is acted upon by a net force $F_1$ and a net torque $\Gamma_1$ about its center of mass. Similarly, the second body, $B_2$, has mass $m_2$ and moment of inertia $J_2$ about its center of mass. It moves with velocity $V_2$ at its center of mass, and rotates with angular velocity $\Omega_2$. Furthermore, it is acted upon by a net force $F_2$ and a net torque $\Gamma_2$ about its center of mass. We assume that we are working in an inertial reference frame.

Now suppose that $B_1$ and $B_2$ collide at a common point $A$. Let $r_1$ and $r_2$ be the displacement vectors which point from the center of mass of $B_1$ to $A$ and from that of $B_2$ to $A$, respectively. Suppose further that at least one of $B_1$ and $B_2$ is topologically smooth at $A$, so that there exists a common tangent plane to both bodies at that point. Now let $F$ be the force experienced by $B_1$ as a result of contact. We assume that this force is “friction-like,” though we will defer an explanation of this term for the moment. By Newton’s third law of motion, the force experienced by $B_2$ as a result of contact is $-F$.

Since the bodies are rigid (i.e., infinitely stiff), we assume that the impact must be instantaneous. That is, there is an instantaneous change in the linear and angular velocities of the two bodies as a result of the impact. Our goal is to determine a rule which gives the outgoing velocities in terms of the incoming velocities (we refer to such a rule as an impact map). At first this task might seem impossible, since the laws of mechanics come in the form of differential equations that must be integrated over some nonzero period of time in order to yield changes in the various kinematic quantities.
involved. To get around this, we will use a clever trick: initially we will take the duration of the impact phase to be small but nonzero, calculate the resulting change in the velocities, and then neglect the time that passed during the impact. Mathematically, we let the impact begin at time \( t^- \) and end at time \( t^+ := t^- + \epsilon \), and then consider the limit as \( \epsilon \to 0 \).

2.0.1 The Equations of Motion

The linear and angular momenta of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are given by

\[
\begin{align*}
P_1 &= m_1 V_1, & L_1 &= J_1 \cdot \Omega_1, \\
P_2 &= m_2 V_2, & L_2 &= J_2 \cdot \Omega_2,
\end{align*}
\]

(2.1)

respectively, and during the impact phase, the net forces and torques acting on them are

\[
\begin{align*}
F_1^\text{net} &:= F_1 + F, & \Gamma_1^\text{net} &:= \Gamma_1 + r_1 \times F, \\
F_2^\text{net} &:= F_2 - F, & \Gamma_2^\text{net} &:= \Gamma_2 - r_2 \times F,
\end{align*}
\]

(2.2)

respectively. The Newton-Euler equations of motion are obtained by setting \( \frac{dP_i}{dt} = F_i^\text{net} \) and \( \frac{dL_i}{dt} = \Gamma_i^\text{net} \) (for \( i = 1, 2 \)), yielding

\[
\begin{align*}
\frac{d}{dt} (m_1 V_1) &= F_1 + F, & \frac{d}{dt} (J_1 \cdot \Omega_1) &= \Gamma_1 + r_1 \times F, \\
\frac{d}{dt} (m_2 V_2) &= F_2 - F, & \frac{d}{dt} (J_2 \cdot \Omega_2) &= \Gamma_2 - r_2 \times F.
\end{align*}
\]

(2.3)
These equations tell us what happens during the impact phase; they do not tell us when the impact phase ends. For this reason, an impact termination condition must be specified a priori. For rigid body impacts, there are several such conditions in use, and we will return to them in Chapter 3.

Now in order to solve (2.3), we might think that we would have to specify the form of $F$. However, as we will soon see, by replacing time with another independent variable, we will only need to impose constraints on the components of $F$.

2.0.2 The Refined Equations of Motion

It turns out that our assumption of instantaneous impact allows us to simplify, or refine, the equations of motion. To do so, we integrate (2.3) from $t = t^-$ to $t \in (t^-, t^+]$, taking the limit as $\epsilon \to 0$. We assume that $J_1$, $J_2$, $F_1$, $F_2$, $\Gamma_1$, $\Gamma_2$, $r_1$, and $r_2$ are all independent of the impact phase duration $\epsilon$, so that they are constant during the impact phase in the limit as $\epsilon \to 0$. However, the same cannot be true of $F$, as it is responsible for the change in the velocities during the impact phase; in other words, it must yield some nonzero impulse during the instantaneous impact phase. Let $I(t)$ be the impulse which has been imparted by $F$ to $B_1$ at any time $t$, so that $I(t^-) := F$ and $I(t^-) := 0$.

Henceforth, for convenience, we will use the following shorthand notation. Any quantity $Q$ at time $t^o$, where $o$ is any symbol, will be denoted $Q^o$. For example, $I^- := I(t^-)$, and so on. In addition, the change in $Q$ between time $t^-$ and any time $t \in (t^-, t^+]$ will be denoted $\Delta Q$. That is, $\Delta Q := Q - Q^-$. Note that, since $Q^-$ is a constant, $dQ = d(\Delta Q)$.

Taking all of this into account, after integrating (2.3), we obtain

$$
m_1 \Delta V_1 = I, \quad J_1 \cdot \Delta \Omega_1 = r_1 \times I,
$$

$$
m_2 \Delta V_2 = -I, \quad J_2 \cdot \Delta \Omega_2 = -r_2 \times I.
$$

(2.4)

This is the form of the equations of motion used by Ivanov in [16]. Differentiating again, and recalling that $dQ = d(\Delta Q)$ for any quantity $Q$, we obtain
the refined equations of motion:

\begin{align}
    m_1 dV_1 &= dI, \\
    J_1 \cdot d\Omega_1 &= r_1 \times dI, \\
    m_2 dV_2 &= -dI, \\
    J_2 \cdot d\Omega_2 &= -r_2 \times dI.
\end{align}

(2.5)

This is the form of the equations of motion used by Stronge in [2] and Nordmark et al. in [18]. Note that, in effect, the assumption of instantaneous impact allows us to neglect all external forces and torques, and to treat \( J_1, J_2, r_1, \) and \( r_2 \) as constant. Note also that the total linear momentum of the system is conserved throughout the impact phase, since \( m_1 dV_1 + m_2 dV_2 = dI - dI \equiv 0 \). The total angular momentum of the system, however, is not conserved, since \( F \) is not a central force.

2.0.3 The Relative Velocity

We may simplify our analysis even further by considering only the relative velocity of \( \mathcal{B}_1 \) with respect to \( \mathcal{B}_2 \) at the point \( A \), for once the total impulse has been determined, the changes in the individual velocities can be extracted using (2.4) evaluated at time \( t = t^+ \). The velocity of the point on \( \mathcal{B}_1 \) which coincides with \( A \) is given by \( V_1^A := V_1 + \Omega_1 \times r_1 \). Likewise, the velocity of the point on \( \mathcal{B}_2 \) which coincides with \( A \) is given by \( V_2^A := V_2 + \Omega_2 \times r_2 \). We define the relative velocity of \( \mathcal{B}_1 \) with respect to \( \mathcal{B}_2 \) at \( A \) as

\[
    v := V_1^A - V_2^A = V_1 + \Omega_1 \times r_1 - V_2 - \Omega_2 \times r_2.
\]

(2.6)

(we assume that, in order for an impact to occur, \( v^- \neq 0 \)). Taking finite and infinitesimal differences, we have the following:

\[
    \Delta v = \Delta V_1 + \Delta \Omega_1 \times r_1 - \Delta V_2 - \Delta \Omega_2 \times r_2, 
\]

(2.7)

\[
    dv = dV_1 + d\Omega_1 \times r_1 - dV_2 - d\Omega_2 \times r_2.
\]

(2.8)

From here we can obtain either \( \Delta v \) or \( dv \) in terms of the impulse \( I \) by using (2.4) or (2.5), respectively.
2.0.4 The Unit Normal Vector, Compression versus Restitution, & Various Impact Configurations

Until now, we have not assumed any particular basis for Euclidian space. However, we anticipate putting “friction-like” restrictions on the contact force $F$, in accordance with the Amontons-Coulomb law of friction. Therefore, let the unit normal vector $n$ be the constant unit vector which is perpendicular to the common tangent plane of the two bodies at $A$ and points in the direction of $B$. If $n$ happens to be parallel to both $r_1$ and $r_2$ (i.e., $r_1 \times n = r_2 \times n = 0$), the impact configuration is called central; otherwise, it is called eccentric.

Now the component of $v$ in the direction of $n$ is given by $v_n := v \cdot n$, and we define the normal velocity vector as $v_n := v_n n$. When $v_n < 0$, we say that the system undergoes compression, and when $v_n > 0$, we say that the system undergoes restitution. Clearly $v_n$ cannot be positive at the onset of impact, so it follows that $v_n^- \leq 0$. A phenomenon called grazing incidence (or simply grazing) occurs in the special case that $v_n^- = 0$ (we will return to this later). If, at the onset of impact, the velocity is entirely in the normal direction (i.e., $v_n^- = v^-$), the impact configuration is called direct; otherwise, it is called oblique.

Likewise, the components of $F$ and $I$ in the direction of $n$ are given by $F_n := F \cdot n$ and $I_n := I \cdot n$, and we define the normal force and impulse vectors as $F_n := F_n n$ and $I_n := I_n n$, respectively. We assume that $F$ opposes overlapping of the two bodies, so that $F_n = I_n'(t) > 0$. Since $I_n^- = 0$, it follows that $I_n(t) \geq 0$. That is, $I_n$ is non-negative and increases monotonically with time. As such, it can be taken as the independent variable instead of the time $t$.

2.0.5 The Common Tangent Plane & Stick versus Slip

The remaining vector components of $v$, $F$, and $I$ lie in the common tangent plane. In particular, the tangential velocity, force, and impulse vectors are defined as $v_\tau := v - v_n$, $F_\tau := F - F_n$, and $I_\tau := I - I_n$, respectively. When $v_\tau = 0$, we say that the system is in relative stick, or simply stick (for sustained stick, we require $v_\tau \equiv 0$), and when $v_\tau \neq 0$, we say that the system is in relative slip, or simply slip.
2.0.6 The Amontons-Coulomb Law of Friction

We may now put explicit, “friction-like” restrictions on the contact force \( F = \frac{dI}{dt} \) using the Amontons-Coulomb law of friction. During sustained stick, the Amontons-Coulomb law states that \( F_\tau \) is consistent with \( v_\tau \equiv 0 \), provided the magnitudes of \( F_n \) and \( F_\tau \) are related such that

\[
||F_\tau|| < \mu_s ||F_n|| \Leftrightarrow ||dI_\tau|| < \mu_s ||dI_n|| \quad \text{during stick,} \tag{2.9}
\]

where \( \mu_s \) is a non-negative constant called the coefficient of static friction, which is a physical property of the two bodies \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \).

During relative slip, on the other hand, the magnitudes of the normal and tangential components of \( F \) are proportional, so that

\[
||F_\tau|| = \mu_k ||F_n|| \Leftrightarrow ||dI_\tau|| = \mu_k ||dI_n|| \quad \text{during slip,} \tag{2.10}
\]

where \( \mu_k \) is a non-negative constant called the coefficient of kinetic friction, which is also a physical property of the two bodies \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \). Furthermore, the tangential vector components of \( F \) and \( v \) point in opposite directions, so that

\[
\frac{F_\tau}{||F_\tau||} = \frac{dI_\tau}{||dI_\tau||} = -\frac{v_\tau}{||v_\tau||} =: -\sigma \quad \text{during slip,} \tag{2.11}
\]

where we have defined the unit vector in the direction of slip \( \sigma := v_\tau / ||v_\tau|| \). Note that we may divide by \( ||F_\tau|| \)—and hence \( ||dI_\tau|| \)—and \( ||v_\tau|| \) because these quantities are non-zero during slip. In what follows we will assume that the coefficients of static and kinetic friction are equal (i.e., \( \mu_s = \mu_k = \mu \), where \( \mu \) is simply referred to as the coefficient of friction). This makes the contact force continuous across transitions between stick and slip.

Before we consider rigid body impacts with friction in complete generality, we will first consider two special cases, namely, one-dimensional (or “collinear”) impacts, and two-dimensional (or “planar”) impacts.

2.1 Collinear (One-Dimensional) Impacts

Consider first impact configurations which are both central (i.e., \( \mathbf{r}_1 \times \mathbf{n} = \mathbf{r}_2 \times \mathbf{n} = 0 \)) and direct (i.e., \( v_\tau^- = 0 \)) (note that, in order for there to be
an impact at all, it must be the case that $v_n^- < 0$). It follows that there is no contact force in the tangential direction during the entire impact phase (i.e., $I_\tau \equiv 0$, so that $I \equiv I_n$), and, as we will soon prove, all relative motion occurs along the line defined by $n$. For this reason, such impacts are called one-dimensional or collinear.

Using (2.5), we see that $d\Omega_1 \equiv d\Omega_2 \equiv 0$. The only part of (2.8) that survives is

$$dv = dV_1 - dV_2 = \frac{1}{m} dI,$$

(2.12)

where $1/m := 1/m_1 + 1/m_2$. Furthermore, since $I \equiv I_n$, we have that $dV_\tau \equiv 0$, and $v_\tau \equiv 0$. We are left simply with

$$\frac{dv_n}{dI_n} = \frac{1}{m},$$

(2.13)

or, integrating from $I_n = I_n^-$ to $I_n \in (I_n^-, I_n^+)$,

$$v_n(I_n) = v_n^- + \frac{1}{m} I_n \quad \text{for} \quad 0 \leq I_n \leq I_n^+.$$  

(2.14)

That is, $v_n$ increases linearly with $I_n$ from its initial (negative) value of $v_n^-$. If we take $I_n$ as our independent variable, we simply need to specify the impact termination condition in order to determine the outgoing relative normal velocity $v_n^+$ in terms of the incoming relative normal velocity $v_n^-$. We will return to this task in Chapter 4.

### 2.2 Planar (Two-Dimensional) Impacts

Consider next impact configurations in which $r_1$, $r_2$, and $v$ initially lie in a principal plane of inertia common to both bodies. It follows that $v$, and hence $F$, are confined to this principal plane throughout the impact phase. For this reason, such impacts are called two-dimensional or planar.

#### 2.2.1 The Unit Tangent Vector, Positive versus Negative Slip, & Friction in Two Dimensions

We may define the unit tangent vector $\tau$ to be either of the two unit vectors which are perpendicular to $n$ and lie in the common principal plane of inertia.
Without loss of generality, let \( \tau := -v^-/||v^-|| \) if \( v^- \neq 0 \). Otherwise, simply select \( \tau \) at random.

Now the component of \( v \) in the direction of \( \tau \) is given by \( v_\tau := v \cdot \tau \), and it follows that \( v \tau = v_\tau \tau \). Recall that when \( v \tau \neq 0 \), we say that the system is in slip. More specifically, when \( v \tau > 0 \), we say that the system is in positive slip, and when \( v \tau < 0 \), we say it is in negative slip. Furthermore, by construction, if the system starts in slip, it starts in negative slip (i.e., \( v^- \leq 0 \)). Combining this with the fact that \( v^- \leq 0 \), we see that the initial velocity components are confined to the third quadrant of the \( v_\tau v_n \)-plane (henceforth referred to as velocity space), including the negative \( v_\tau \)- and \( v_n \)-axes (see 2.3).

Likewise, the component of \( I \) in the direction of \( \tau \) is given by \( I_\tau := I \cdot \tau \), and it follows that \( I_\tau = I_\tau \tau \). During relative stick, (2.9) states that \( |dI_\tau| < \mu |dI_n| \), where we have replaced \( \mu_s \) with \( \mu \). Equivalently,

\[
\frac{|dI_\tau|}{|dI_n|} < \mu \quad \text{during stick.} \quad (2.15)
\]

During relative slip, (2.10) states that \( |dI_\tau| = \mu |dI_n| \), where we have replaced \( \mu_k \) with \( \mu \). Given that \( dI_n > 0 \), we have that \( dI_\tau = \pm \mu dI_n \), where the sign depends on the direction of slip. In particular, according to (2.11),

\[
\frac{dI_\tau}{dI_n} = \begin{cases} 
-\mu & \text{during positive slip,} \\
+\mu & \text{during negative slip.} 
\end{cases} \quad (2.16)
\]

### 2.2.2 The Equations of Motion in Two Dimensions

Given the special relationship between the normal and tangential components of \( I \), it is only natural to use \( e_1 := n \), \( e_2 := \tau \), and \( e_3 := n \times \tau \) as our basis for Euclidian space. Let \( k_1 \) and \( k_2 \) be, respectively, the radii of gyration of \( B_1 \) and \( B_2 \) about the axis defined by \( e_3 \). In addition, let \( r_i = r_i^n n + r_i^\tau \tau \) for \( i = 1, 2 \). In this way, (2.5) becomes

\[
\begin{align*}
m_1 dV_1 &= dI_n n + dI_\tau \tau, \\
m_1 k_1^2 d\Omega_1 &= (r_1^n dI_\tau - r_1^\tau dI_n) e_3, \\
m_2 dV_2 &= -(dI_n n + dI_\tau \tau), \\
m_2 k_2^2 d\Omega_2 &= (r_2^n dI_\tau - r_2^\tau dI_n) e_3.
\end{align*} \quad (2.17)
\]
From the righthand column of (2.17) we have that
\[
\begin{align*}
\frac{d\Omega_1 \times r_1}{r_1} &= \left( \frac{1}{m_1 k_1^2} \right) (r_1^n dI_n - r_1^\tau dI_\tau) (r_1^n \tau - r_1^\tau n), \\
\frac{d\Omega_2 \times r_2}{r_2} &= \left( \frac{1}{m_2 k_2^2} \right) (r_2^n dI_n - r_2^\tau dI_\tau) (r_2^n \tau - r_2^\tau n).
\end{align*}
\]  
(2.18) (2.19)

Using the lefthand column of (2.17), as well as (2.18), in (2.8), we find that
\[
\begin{align*}
\frac{dv}{dI_n} &= \frac{1}{m} (dI_n + dI_\tau) \tau + \left( \frac{1}{m k_1^2} \right) (r_1^n dI_n - r_1^\tau dI_\tau) (r_1^n \tau - r_1^\tau n) \\
&\quad - \left( \frac{1}{m k_2^2} \right) (r_2^n dI_n - r_2^\tau dI_\tau) (r_2^n \tau - r_2^\tau n),
\end{align*}
\]
(2.20)
or, in matrix component form,
\[
\begin{bmatrix}
\frac{dv_n}{dI_n} \\
\frac{dv_\tau}{dI_n}
\end{bmatrix} =
\begin{bmatrix}
C_{nn} & C_{n\tau} \\
C_{n\tau} & C_{\tau\tau}
\end{bmatrix}
\begin{bmatrix}
dI_n \\
dI_\tau
\end{bmatrix},
\]
(2.21)

where
\[
\begin{align*}
C_{nn} &:= \frac{1}{m} + \frac{(r_1^\tau)^2}{m_1 k_1^2} + \frac{(r_2^\tau)^2}{m_2 k_2^2}, \\
C_{n\tau} &:= - \left( \frac{r_1^n r_1^\tau}{m_1 k_1^2} + \frac{r_2^n r_2^\tau}{m_2 k_2^2} \right), \\
C_{\tau\tau} &:= \frac{1}{m} + \frac{(r_1^n)^2}{m_1 k_1^2} + \frac{(r_2^n)^2}{m_2 k_2^2}.
\end{align*}
\]  
(2.22)

Note that both $C_{nn}$ and $C_{\tau\tau}$ are strictly positive, but that $C_{n\tau}$ can be positive, negative, or zero. Furthermore, straightforward computation reveals that
\[
C_{nn} C_{\tau\tau} - C_{n\tau}^2 = \frac{1}{m^2} + \frac{1}{m} \left( \frac{r_1^2}{m_1 k_1^2} + \frac{r_2^2}{m_2 k_2^2} \right) + \frac{(r_1^\tau r_2^n - r_2^\tau r_1^n)^2}{m_1 k_1^2 m_2 k_2^2} > 0,
\]
(2.23)
where $r_i^2 := (r_i^n)^2 + (r_i^\tau)^2$ for $i = 1, 2$. That is, the determinant of the $2 \times 2$ matrix in (2.21) is positive. Carrying out the matrix multiplication in (2.21), and dividing by $dI_n$, we obtain
\[
\begin{align*}
\frac{dv_n}{dI_n} &= C_{nn} + \frac{dI_\tau}{dI_n} C_{n\tau}, \\
\frac{dv_\tau}{dI_n} &= C_{n\tau} + \frac{dI_\tau}{dI_n} C_{\tau\tau}.
\end{align*}
\]  
(2.24)
Now according to (2.16), the value of $\frac{d}{dI_n} = -\mu$ in positive slip and $+\mu$ in negative slip. During sustained stick, however, it is determined by enforcing $v_\tau \equiv 0$. Setting $\frac{dv_\tau}{dI_n} = 0$ in (2.24) yields

$$\frac{dI_\tau}{dI_n} = -\frac{C_{nt}}{C_{\tau\tau}} =: \bar{\mu} \quad \text{during sustained stick} \quad (2.25)$$

(we are allowed to divide by $C_{\tau\tau}$ because it is strictly positive; whether sustained stick actually happens depends on the value of $\mu$, as discussed in Section 2.2.3). Thus, in each of the three cases, the rate of change of each of $v_n$ and $v_\tau$ is constant. We may therefore define the following rate constants

$$k^+_n := C_{nn} - \mu C_{n\tau}, \quad k^+_\tau := C_{n\tau} - \mu C_{\tau\tau},$$

$$k^-_n := C_{nn} + \mu C_{n\tau}, \quad k^-_\tau := C_{n\tau} + \mu C_{\tau\tau},$$

$$k^0_n := \frac{(C_{nn}C_{\tau\tau} - C_{n\tau}^2)}{C_{\tau\tau}}, \quad k^0_\tau := 0, \quad \text{(2.26)}$$

so that

$$\frac{dv_i}{dI_n} = \begin{cases} 
  k^+_i & \text{during positive slip,} \\
  k^-_i & \text{during negative slip,} \\
  k^0_i & \text{during sustained stick,}
\end{cases} \quad \text{for } i = n, \tau. \quad (2.27)$$

This means that $v_n$ and $v_\tau$ are piecewise linear in $I_n$, and hence that the ordered pair $(v_\tau, v_n)$ lives on piecewise straight lines in velocity space (with slopes given by $k^0_n/k^0_\tau$, where $o = +, -, 0$), as shown in Figure 2.3.

Note that, from (2.23) and the fact that $C_{\tau\tau} > 0$, it follows that $k^0_n$ is strictly positive (it cannot be negative or zero). However, each of the rate constants during slip (i.e., $k^+_n, k^-_n, k^+_\tau, k^-_\tau$) can be positive, negative, or zero, depending on the impact configuration. Ostensibly, this makes for $3^4 = 81$ sign combinations. However, only 9 of these combinations are admitted by (2.23) and (2.26). In particular, the curves $k^+_n = 0$, $k^-_n = 0$, $k^+_\tau = 0$, and $k^-_\tau = 0$ separate the strip defined by $|C_{n\tau}| < \sqrt{C_{nn}C_{\tau\tau}}$ and $\mu \geq 0$ into five open sets, corresponding to distinct sign combinations, as shown in Figure 2.2 (cf. Figure 2 of [18]). It follows that a velocity space trajectory cannot move horizontally to the left, vertically downward, or downward and to the left.
2.2.3 Initial Conditions & the Evolution of Planar Impact

As noted in Section 2.2.1, the initial velocity components lie in the third quadrant of velocity space (i.e., the $v_\tau v_n$-plane). There are clearly three distinct possibilities: either (i) $(v_\tau, v_n)$ starts within the third quadrant (i.e., $v_\tau, v_n < 0$), (ii) $(v_\tau, v_n)$ starts on the negative $v_\tau$-axis (i.e., $v_n = 0$), or (iii) $(v_\tau, v_n)$ starts on the negative $v_n$-axis (i.e., $v_\tau = 0$).

Initial Slip without Grazing: $v_\tau^-, v_n^- < 0$

Suppose that initially there is both normal and tangential relative velocity (i.e., $v_\tau^- < 0$ and $v_n^- < 0$). This corresponds to initial (negative) slip without grazing. Clearly, either $v_\tau$ will remain negative throughout the impact phase, or it will vanish at some point in time. If this happens, there may be a transition, either to positive slip or sustained stick. The system cannot remain in negative slip because the rate of change of $v_\tau$ in negative slip (i.e., $k_\tau^-$) is constant according to (2.27), and if $v_\tau$ is passing through zero, it must be increasing (i.e., $k_\tau^+ > 0$); in order to remain in negative slip, it would have to start decreasing again (a contradiction).

Sticking: A Transition to Sustained Stick According to (2.15) and (2.25), stick persists as long as $|dI_\tau/dI_n| < \mu$ and $dI_\tau/dI_n = \bar{\mu}$. Therefore, if $|\bar{\mu}| < \mu$, a transition to sustained stick (or sticking) will occur when $v_\tau$ vanishes. Note that, once the system is in sustained stick, it will remain so, as $dv_\tau/dI_n = 0$ in sustained stick.
Slip Reversal: A Transition to Positive Slip  
If, on the other hand, $|\tilde{\mu}| \geq \mu$, slip reversal (a transition to positive slip) can occur. A period of positive slip requires that $k_\tau^+ > 0$, or equivalently, that $\tilde{\mu} < -\mu$. Hence, there will be a transition to positive slip when $v_\tau$ vanishes if $\tilde{\mu} < -\mu$. Again, once the system is in positive slip, it will remain so, since $v_\tau$ will continue to increase until the impact terminates.

What about when $\tilde{\mu} = -\mu$ or $\tilde{\mu} \geq \mu$? The former is equivalent to $k_\tau^+ = 0 =: k_\tau^0$ and $k_n^+ = k_n^0$. That is, when $\tilde{\mu} = -\mu$, sticking and slip reversal result in the same behavior: the velocity space trajectory is “continuous” across $\tilde{\mu} = -\mu$. This case can therefore be considered to result in either sticking or slip reversal. Here, we will take $\tilde{\mu} = -\mu$ to result in sticking. In regard to $\tilde{\mu} \geq \mu$, it turns out that this cannot happen: $\tilde{\mu} \geq \mu$ is equivalent to $k_\tau^- \leq 0$. But in order for $v_\tau$ to vanish in the first place, $k_\tau^-$ must be positive. Hence, it cannot happen that $\tilde{\mu} \geq \mu$ when $v_\tau$ becomes zero.

We may summarize the preceding discussion with the following rule: If $v_\tau$ vanishes at some point during the impact phase, slip reversal will occur if (and only if) $\tilde{\mu} < -\mu$, and sticking will occur if (and only if) $-\mu \leq \tilde{\mu} < \mu$. In short, we have

$$\text{Slip Reversal} \iff \tilde{\mu} < -\mu,$$

$$\text{Sticking} \iff -\mu \leq \tilde{\mu} < \mu.$$  

Dynamical Jamming  
A phenomenon called dynamical jamming, or simply jam, occurs when $v_n$ initially decreases during compression (i.e., $k_n^-$ is negative). In that case, $k_\tau^-$ must be positive in order to satisfy (2.23). That $k_n^-$ is negative and $k_\tau^-$ is positive implies that $|\tilde{\mu}| < \mu$. That is, dynamical jamming always results in sticking during the compression phase.

Grazing Incidence: $v_n^- = 0$

Suppose that initially there is relative tangential velocity, but no relative normal velocity (i.e., $v_\tau^- < 0$ and $v_n^- = 0$). This corresponds to grazing incidence with negative slip. In this case, there may not even be an impact in the first place. If $v_n$ increases or remains the same during negative slip
(i.e., $k_n^- \geq 0$), the two bodies separate immediately after grazing, and there is no impact. The only way for an impact to happen is for $k_n^-$ to be negative, so that jam occurs. That is, grazing incidence results in an impact if and only if jam occurs.

Initial Stick: $v_r^- = 0$

Suppose that initially there is relative normal velocity, but no relative tangential velocity (i.e, $v_r^- = 0$ and $v_n^- < 0$). This corresponds to initial stick. Recall that this is the only scenario for which we did not specify the tangential unit vector $\tau$. In this case, one of three things will happen: either the system will remain in stick, there will be an immediate transition to positive slip, or there will be an immediate transition to negative slip. To decide exactly what happens, the same reasoning as that used to obtain (2.28) applies.

This time, however, it is possible that $\tilde{\mu} \geq \mu$.

A period of negative slip requires that $k_r^- < 0$, or equivalently, that $\tilde{\mu} > \mu$. Hence, there will be a transition to negative slip if $\tilde{\mu} > \mu$, and negative slip will persist until the impact terminates. In the case that $\tilde{\mu} = \mu$, we find that this is equivalent to $k_r^- = 0 =: k_r^0$ and $k_n^- = k_n^0$. That is, when $\tilde{\mu} = \mu$, sustained stick and a transition to negative slip result in the same behavior: the velocity space trajectory is “continuous” across $\tilde{\mu} = \mu$. This case can therefore be considered to result in either sticking or a transition to negative slip. Here, for the sake of consistency with (2.28), we will take $\tilde{\mu} = \mu$ to result in sticking.

We may summarize the preceding discussion with the following rule: If $v_r^- = 0$, a transition to positive slip will occur if (and only if) $\tilde{\mu} < -\mu$, sustained stick will occur if (and only if) $|\tilde{\mu}| \leq \mu$, and a transition to negative slip will occur if (and only if) $\tilde{\mu} > \mu$. In short, we have

\[
\begin{align*}
\text{Positive Slip} & \quad \text{iff} \quad \tilde{\mu} < -\mu, \\
\text{Sustained Stick} & \quad \text{iff} \quad |\tilde{\mu}| \leq \mu, \\
\text{Negative Slip} & \quad \text{iff} \quad \tilde{\mu} > \mu.
\end{align*}
\] (2.29)
2.2.4 Explicit Planar Impact Processes

Following Nordmark et al. in [18], we may now explicitly identify all possible impact processes (i.e., all conceivable combinations of positive slip, negative slip, sustained stick, compression, and restitution) for planar impacts. There are a total of ten distinct possibilities:

Either the system starts in stick, in which case it either remains in sustained stick, immediately transitions to positive slip, or immediately transitions to negative slip; or else the system starts in negative slip, in which case either it remains in negative slip, sticking occurs, or slip reversal occurs. Both sticking and slip reversal can occur during compression, restitution, or at the transition between compression and restitution.

We will now investigate under what circumstances these events may occur.

Case #1: Sustained Initial Stick

According to (2.29), initial stick \( (v^{-} = 0) \) will persist if and only if \( |\tilde{\mu}| \leq \mu \). Under these circumstances, \( v_{\tau} \) will remain zero throughout the impact phase, and \( v_{n} \) will increase at a constant rate of \( k_{n}^{0} \), until the impact terminates. This corresponds to a vertical trajectory in velocity space; see Figure 2.3(a).

Case #2: Initial Stick to Positive Slip

According to (2.29), initial stick \( (v^{-} = 0) \) will immediately transition to positive slip if and only if \( \tilde{\mu} < -\mu \). Under these circumstances, \( v_{\tau} \) will increase at a constant rate of \( k_{\tau}^{+} \), and \( v_{n} \) will increase at a constant rate of \( k_{n}^{+} \) \( (k_{n}^{+} \) must be positive according to Figure 2.2), until the impact terminates. This corresponds to a trajectory which moves upward and to the right in velocity space; see Figure 2.3(b).
Figure 2.3: Representative velocity space trajectories. Cases #1-10 are shown in (a)-(j), respectively.
Case #3: Initial Stick to Negative Slip

According to (2.29), initial stick \( (v_\tau^- = 0) \) will immediately transition to negative slip if and only if \( \tilde{\mu} > \mu \). Under these circumstances, \( v_\tau \) will decrease at a constant rate of \( k_\tau^- \), and \( v_n \) will increase at a constant rate of \( k_n^+ \) (\( k_n^+ \) must be positive according to Figure 2.2), until the impact terminates. This corresponds to a trajectory which moves upward and to the left in velocity space; see Figure 2.3(c).

Case #4: Sustained Initial Slip

Initial slip \( (v_\tau^- < 0) \) will persist if (and only if) no transitions occur. This will be the case if \( v_\tau \) does not increase (i.e., \( k_\tau^- \leq 0 \), or equivalently, \( \tilde{\mu} \geq \mu \)), or if the impact terminates at or before the moment \( v_\tau \) would vanish; see Figure 2.3(d).

When can \( v_\tau \) vanish? Henceforth in this section we take \( v_\tau^- < 0 \), and we pause here to consider under what circumstances \( v_\tau \) can vanish during the impact phase. Clearly, \( v_\tau \) can only vanish if \( k_\tau^- \) is positive, or equivalently, if

\[
\tilde{\mu} < \mu. \tag{2.30}
\]

Moreover, recall that the slope of velocity space trajectories during negative slip is fixed by the impact configuration, and given by

\[
k_n^- / k_\tau^- = \frac{C_{nn} + \mu C_{n\tau}}{C_{n\tau} + \mu C_{\tau\tau}}. \tag{2.31}
\]

If \( k_n^- \) is negative, the slope is negative, and jam occurs no matter where the system starts in velocity space. Otherwise, the initial conditions \( v_n^- \) and \( v_\tau^- \) determine whether \( v_\tau \) can vanish during compression, during restitution, or at the transition between compression and restitution. If \( k_n^- = 0 \), the slope is zero; \( v_\tau \) can vanish during compression only if \( v_n^- < 0 \), and \( v_\tau \) can vanish at the transition between compression and restitution only if \( v_n^- = 0 \). If \( k_n^- \) is positive, the slope is positive, and we consider the line defined by \( v_n = (k_n^- / k_\tau^-) v_\tau \): \( v_\tau \) can only vanish during compression if \( (v_\tau^-, v_n^-) \) lies below this line; \( v_\tau \) can only vanish during the transition between compression and restitution if \( (v_\tau^-, v_n^-) \) lies on this line; and \( v_\tau \) can only vanish during
restitution if \((v^-_n, v^-_\tau)\) lies above this line. In short,

\[
\begin{align*}
k^-_n > 0, v^-_\tau \to 0 \text{ in compression} & \Rightarrow v^-_n/v^-_\tau > k^-_n/k^-_\tau, \\
k^-_n > 0, v^-_n \to 0 \text{ in restitution} & \Rightarrow v^-_n/v^-_\tau < k^-_n/k^-_\tau, \\
k^-_n > 0, v^-_n \to 0 \text{ in-between} & \Rightarrow v^-_n/v^-_\tau = k^-_n/k^-_\tau.
\end{align*}
\] (2.32)

Case #5: Sticking in Restitution

According to (2.28) and (2.32), sticking can only occur during restitution if

\(-\mu \leq \bar{\mu} < \mu\) (this is consistent with (2.30)), \(k^-_n > 0\), and \(v^-_n/v^-_\tau < k^-_n/k^-_\tau\); see Figure 2.3(e).

Case #6: Slip Reversal in Restitution

According to (2.28) and (2.32), slip reversal can only occur during restitution if \(\bar{\mu} < -\mu\) (this is consistent with (2.30)), \(k^-_n > 0\), and \(v^-_n/v^-_\tau < k^-_n/k^-_\tau\); see Figure 2.3(f).

Case #7: Sticking in Compression

According to (2.28) and (2.32), sticking can only occur during compression if

\(-\mu \leq \bar{\mu} < \mu\) (this is consistent with (2.30)) and either (i) \(k^-_n < 0\) (dynamical jamming), (ii) \(k^-_n = 0\) and \(v^-_n < 0\), or (iii) \(k^-_n > 0\) and \(v^-_n/v^-_\tau > k^-_n/k^-_\tau\); see Figure 2.3(g).

Case #8: Slip Reversal in Compression

According to (2.28) and (2.32), slip reversal can only occur during compression if \(\bar{\mu} < -\mu\) (this is consistent with (2.30)), and either (i) \(k^-_n = 0\) and \(v^-_n < 0\), or (ii) \(k^-_n > 0\) and \(v^-_n/v^-_\tau > k^-_n/k^-_\tau\); see Figure 2.3(h).

Case #9: Sticking between Compression & Restitution

According to (2.28) and (2.32), sticking can only occur at the transition between compression and restitution if \(-\mu \leq \bar{\mu} < \mu\) (this is consistent with (2.30)), \(k^-_n > 0\), and \(v^-_n/v^-_\tau = k^-_n/k^-_\tau\); see Figure 2.3(i).
Case #10: Slip Reversal between Compression & Restitution

According to (2.28) and (2.32), slip reversal can only occur at the transition between compression and restitution if \( \tilde{\mu} < -\mu \) (this is consistent with (2.30)), \( k_n^- > 0 \), and \( v_n^- / v_r^- = k_n^- / k_r^- \); see Figure 2.3(j).

As mentioned, representative velocity space trajectories are shown in Figure 2.3 (cf. Figure 3 of [18]). Note that there is some ambiguity as to whether or not Cases #4-10 actually occur, because we still do not know when the impact terminates. In fact, to construct explicit impact maps, we must choose an impact termination condition. We will return to this task in Chapter 4.

2.3 General (Three-Dimensional) Impacts

We now consider rigid body impacts with friction in three dimensions. Let \( \mathbf{e}_2 \) be an arbitrary unit vector in the common tangent plane. We choose as our basis for Euclidian space \( \mathbf{e}_1 := \mathbf{n}, \mathbf{e}_2, \) and \( \mathbf{e}_3 := \mathbf{n} \times \mathbf{e}_2 \).

With this basis defined, any vector \( \mathbf{a} \) (e.g., \( \mathbf{v} \) or \( \mathbf{I} \)) can be written in component form as

\[
\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3, \tag{2.33}
\]

where \( a_i := \mathbf{a} \cdot \mathbf{e}_i \) is called the component of \( \mathbf{a} \) in the direction of \( \mathbf{e}_i \). Note that, in terms of the notation we have been using so far, \( a_n = a_1, a_n = a_1 \mathbf{e}_1, \) and \( \mathbf{a}_r = a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 \).

2.3.1 Friction in Three Dimensions

During relative stick, (2.9) states that \( ||d\mathbf{I}_r|| = \sqrt{(dI_2)^2 + (dI_3)^2} < \mu \ dI_n \), where we have replaced \( \mu_s \) with \( \mu \). Dividing by \( dI_n \), we have

\[
\frac{||d\mathbf{I}_r||}{dI_n} = \sqrt{\left(\frac{dI_2}{dI_n}\right)^2 + \left(\frac{dI_3}{dI_n}\right)^2} < \mu \quad \text{during stick.} \tag{2.34}
\]
During relative slip, (2.10) states that \( \|dI_T\| = \sqrt{(dI_2)^2 + (dI_3)^2} = \mu \: dI_n \), where we have replaced \( \mu_k \) with \( \mu \). Dividing by \( dI_n \), we have

\[
\frac{\|dI_T\|}{dI_n} = \sqrt{\left(\frac{dI_2}{dI_n}\right)^2 + \left(\frac{dI_3}{dI_n}\right)^2} = \mu \quad \text{during slip.} \tag{2.35}
\]

Using (2.35) in (2.11), we obtain the following during relative slip:

\[
\frac{dI_T}{dI_n} = -\mu \sigma, \tag{2.36}
\]
or, in component form,

\[
\frac{dI_2}{dI_n} = -\mu \frac{v_2}{\sqrt{v_2^2 + v_3^2}}, \tag{2.37}
\]
\[
\frac{dI_3}{dI_n} = -\mu \frac{v_3}{\sqrt{v_2^2 + v_3^2}}. \tag{2.38}
\]

The Sliding Velocity

We pause here to note that, rather than expressing \( \mathbf{v}_r \) in terms of its basis components, we may do so using its magnitude and direction. The magnitude of \( \mathbf{v}_r \) is given by

\[
|v_r| := \|\mathbf{v}_r\| = \sqrt{v_2^2 + v_3^2}, \tag{2.39}
\]
and we have already defined the slip direction \( \sigma := \mathbf{v}_r / |\mathbf{v}_r| \). In terms of these quantities,

\[
\mathbf{v}_r = |v_r| \sigma = v_r (\cos \theta \: \mathbf{e}_2 + \sin \theta \: \mathbf{e}_3), \tag{2.40}
\]
where \( \theta \) is the angle \( \mathbf{v}_r \) makes with the positive \( \mathbf{e}_2 \)-direction, i.e.,

\[
\tan \theta = v_3 / v_2. \tag{2.41}
\]

In general, \( \sigma \) can vary, both continuously and discontinuously, during the impact phase.
2.3.2 The Equations of Motion in Three Dimensions

Substituting (2.5) into (2.8), we obtain

\[
dv = \frac{1}{m} dI + [J_1^{-1} \cdot (r_1 \times dI)] \times r_1 + [J_2^{-1} \cdot (r_2 \times dI)] \times r_2.
\] (2.42)

Note that, since moment of inertia tensors are positive-definite, their inverses are well-defined. Using indicial notation\(^1\) in terms of the basis we have chosen, (2.42) becomes

\[
dv_i = C_{ij} \, dI_j,
\] (2.43)

where

\[
C_{ij} := \frac{1}{m} \delta_{ij} + \epsilon_{ikl} \epsilon_{jmn} \left[ (J_1^{-1})_{km} (r_1)_l (r_1)_n + (J_2^{-1})_{km} (r_2)_l (r_2)_n \right],
\] (2.44)

and summation over repeated indices from 1 to 3 is implied (this is equivalent to Equation (4.7) of [2]). Note that \(C_{ji} = C_{ij}\), so that there are only six distinct values contained in (2.44). They are given below:

\[
C_{11} := \frac{1}{m} + (J_1^{-1})_{33} (r_1)_3^2 - 2(J_1^{-1})_{23} (r_1)_2 (r_1)_3 + (J_1^{-1})_{22} (r_1)_2^2
\]
\[+ (J_2^{-1})_{33} (r_2)_3^2 - 2(J_2^{-1})_{23} (r_2)_2 (r_2)_3 + (J_2^{-1})_{22} (r_2)_2^2,
\] (2.45)

\[
C_{12} := - (J_1^{-1})_{33} (r_1)_1 (r_1)_2 + (J_1^{-1})_{23} (r_1)_1 (r_1)_3 + (J_1^{-1})_{13} (r_1)_2 (r_1)_3 - (J_1^{-1})_{12} (r_1)_2 (r_1)_3
\]
\[+ (J_2^{-1})_{33} (r_2)_1 (r_2)_2 + (J_2^{-1})_{23} (r_2)_1 (r_2)_3 + (J_2^{-1})_{13} (r_2)_2 (r_2)_3 - (J_2^{-1})_{12} (r_2)_2 (r_2)_3,
\] (2.46)

\[
C_{13} := (J_1^{-1})_{23} (r_1)_1 (r_1)_2 - (J_1^{-1})_{13} (r_1)_2^2 - (J_1^{-1})_{22} (r_1)_1 (r_1)_3 + (J_1^{-1})_{12} (r_1)_2 (r_1)_3
\]
\[+ (J_2^{-1})_{23} (r_2)_1 (r_2)_2 - (J_2^{-1})_{13} (r_2)_2^2 - (J_2^{-1})_{22} (r_2)_1 (r_2)_3 + (J_2^{-1})_{12} (r_2)_2 (r_2)_3,
\] (2.47)

\[
C_{22} := \frac{1}{m} + (J_1^{-1})_{33} (r_1)_1^2 - 2(J_1^{-1})_{13} (r_1)_1 (r_1)_3 + (J_1^{-1})_{11} (r_1)_3^2
\]
\[+ (J_2^{-1})_{33} (r_2)_1^2 - 2(J_2^{-1})_{13} (r_2)_1 (r_2)_3 + (J_2^{-1})_{11} (r_2)_3^2,
\] (2.48)

\(^1\)For a review of indicial notation, see Appendix A.
\[ C_{23} := -(J^{-1})_{23}(r_1)_1^2 + (J^{-1})_{13}(r_1)_1(r_1)_2 + (J^{-1})_{12}(r_1)_1(r_1)_3 - (J^{-1})_{11}(r_1)_2(r_1)_3 \\
- (J^{-1})_{23}(r_2)_1^2 + (J^{-1})_{13}(r_2)_1(r_2)_2 + (J^{-1})_{12}(r_2)_1(r_2)_3 - (J^{-1})_{11}(r_2)_2(r_2)_3, \]
\[ (2.49) \]

\[ C_{33} := \frac{1}{m} + (J^{-1})_{22}(r_1)_1^2 - 2(J^{-1})_{12}(r_1)_1(r_1)_2 + (J^{-1})_{11}(r_1)_2^2 \\
+ (J^{-1})_{22}(r_2)_1^2 - 2(J^{-1})_{12}(r_2)_1(r_2)_2 + (J^{-1})_{11}(r_2)_2^2. \]
\[ (2.50) \]

In matrix form, (2.43) becomes
\[
\begin{bmatrix}
dv_n \\
dv_2 \\
dv_3
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{12} & C_{22} & C_{23} \\
C_{13} & C_{23} & C_{33}
\end{bmatrix}
\begin{bmatrix}
dI_n \\
dI_2 \\
dI_3
\end{bmatrix}.
\]
\[ (2.51) \]

It can be shown, from the fact that both \( J^{-1}_1 \) and \( J^{-1}_2 \) are positive-definite, that all principal minors of the 3 \times 3 matrix in (2.51) (henceforth referred to as \( \bar{C} \)) are positive. By Sylvester’s criterion, we may conclude that \( \bar{C} \) is positive-definite. Letting \( \bar{v} := [v_n, v_2, v_3]^T \) and \( \bar{I} := [I_n, I_2, I_3]^T \), we may rewrite (2.51) even more succinctly as
\[ d\bar{v} = \bar{C}d\bar{I}. \]
\[ (2.52) \]

Now, carrying out the matrix multiplication in (2.51), and dividing by \( dI_n \), we obtain
\[
\frac{dv_n}{dI_n} = C_{11} + C_{12} \frac{dI_2}{dI_n} + C_{13} \frac{dI_3}{dI_n}, \\
\frac{dv_2}{dI_n} = C_{12} + C_{22} \frac{dI_2}{dI_n} + C_{23} \frac{dI_3}{dI_n}, \\
\frac{dv_3}{dI_n} = C_{13} + C_{23} \frac{dI_2}{dI_n} + C_{33} \frac{dI_3}{dI_n},
\]
\[ (2.53) \]

or, letting \( \bar{I}_r := [0, I_2, I_3]^T \) and noting that in matrix form \( \bar{n} = [1, 0, 0]^T \),
\[
\frac{d\bar{v}}{dI_n} = \bar{C}(\bar{n} + \frac{d\bar{I}_r}{dI_n}).
\]
\[ (2.54) \]
Sustained Stick

During sustained stick, \( \mathbf{v}_r \equiv \mathbf{0} \). Setting \( dv_2 = dv_3 = 0 \) in (2.51) yields

\[
\frac{d}{dI_n} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} = - \begin{bmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} C_{12} \\ C_{13} \end{bmatrix} = \frac{1}{C_{22}C_{33} - C_{23}^2} \begin{bmatrix} C_{13}C_{23} - C_{12}C_{33} \\ C_{12}C_{23} - C_{13}C_{22} \end{bmatrix}.
\] (2.55)

(We may invert the minor matrix because \( \bar{C} \) is positive-definite.) That is, during sustained stick, \( dI_2/dI_n \) and \( dI_3/dI_n \) are constant, and from (2.53), \( dv_n/dI_n \) is constant as well. In particular, from (2.53),

\[
\frac{dv_n}{dI_n} = C_{11} - \mu \left( \frac{C_{12}v_2 + C_{13}v_3}{\sqrt{v_2^2 + v_3^2}} \right),
\]

\[
\frac{dv_2}{dI_n} = C_{12} - \mu \left( \frac{C_{22}v_2 + C_{23}v_3}{\sqrt{v_2^2 + v_3^2}} \right),
\]

\[
\frac{dv_3}{dI_n} = C_{13} - \mu \left( \frac{C_{23}v_2 + C_{33}v_3}{\sqrt{v_2^2 + v_3^2}} \right).
\] (2.58)

Since \( \bar{C} \) is positive definite, we have that \( k_n^0 \) is positive. That is, during sustained stick, \( v_n \) increases at a constant rate of \( k_n^0 \), as defined above. Moreover, we find that

\[
\left| \frac{d\mathbf{I}_r}{dI_n} \right| = \sqrt{\frac{(C_{13}C_{23} - C_{12}C_{33})^2 + (C_{12}C_{23} - C_{13}C_{22})^2}{C_{22}C_{33} - C_{23}^2}} =: \tilde{\mu}.
\] (2.57)

Note that \( \tilde{\mu} \) as given above is strictly positive.

Relative Slip

During relative slip, substitution of (2.37) into (2.53) yields

\[
\frac{dv_n}{dI_n} = C_{11} - \mu \left( \frac{C_{12}v_2 + C_{13}v_3}{\sqrt{v_2^2 + v_3^2}} \right),
\]

\[
\frac{dv_2}{dI_n} = C_{12} - \mu \left( \frac{C_{22}v_2 + C_{23}v_3}{\sqrt{v_2^2 + v_3^2}} \right),
\]

\[
\frac{dv_3}{dI_n} = C_{13} - \mu \left( \frac{C_{23}v_2 + C_{33}v_3}{\sqrt{v_2^2 + v_3^2}} \right).
\] (2.58)

Alternatively, letting \( \bar{\sigma} := [0, \sigma_2, \sigma_3]^T \), substitution of (2.36) into (2.54) yields

\[
\frac{d\bar{\sigma}}{dI_n} = \bar{C}(\bar{n} - \mu \bar{\sigma}).
\] (2.59)
The Sliding Velocity Flow

Following Batlle in [19], let us now focus on the dynamics within the tangent plane. Letting \( \bar{v}^* := [v_2, v_3]^T \), \( \bar{\sigma}^* := [\sigma_2, \sigma_3]^T \), \( \bar{h} := [C_{12}, C_{13}]^T \), and

\[
\bar{B} := \begin{bmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix},
\]

so that

\[
\bar{C} = \begin{bmatrix} C_{11} & \bar{h}^T \\ \bar{h} & \bar{B} \end{bmatrix},
\]

expansion of (2.59) yields

\[
\frac{d\bar{v}^*}{dI_n} = \bar{h} - \mu \bar{B} \bar{\sigma}^*.
\]

Associated with this differential equation in \( v_2 \) and \( v_3 \) is a vector field, the isoclines of which are the curves along which \( d\bar{v}_\tau / dI_n \) is constant. The isoclines are therefore given by solving \( d\bar{v}_\tau / dI_n = \bar{c} \), where \( \bar{c} \) is a constant column matrix, and straightforward manipulation shows that they are defined by

\[
\bar{\sigma}^* = \bar{B}^{-1}(\bar{h} - \bar{c})/\mu,
\]

where, in order to enforce the fact that \( \sigma \) is a unit vector, we must have \( \mu = ||\bar{B}^{-1}(\bar{h} - \bar{c})|| \). Now (2.63) tells us that, along the isoclines, \( \bar{\sigma}^* \) is constant; that is, the isoclines are straight half-lines which emanate from the origin of the \( v_2v_3 \)-plane. For example, when \( \mu = \bar{\mu} := ||\bar{B}^{-1}\bar{h}|| \), there exists an isocline (called the critical isocline),

\[
\bar{\sigma}_{c}^* := \frac{\bar{B}^{-1}\bar{h}}{\bar{\mu}},
\]

along which \( d\bar{v}_\tau / dI_n = 0 \).

Particularly important among the isoclines are those along which \( d\bar{v}_\tau / dI_n \) is parallel to the direction of the isocline (i.e., those along which \( d\theta / dI_n = 0 \), which we will call asymptotes), for these give the slip directions immediately before and after transitions in the slip direction. To see this, we differentiate (2.41) with respect to \( I_n \), and express the result in terms of \( |v_\tau| \) and \( \theta \) using
(2.58) and the relations $v_2 = |v_r| \cos \theta$, $v_3 = |v_r| \sin \theta$. In this way, we obtain

$$|v_r| \frac{d\theta}{dI_n} = C_{13} \cos \theta - C_{12} \sin \theta + \mu[C_{23}(\sin^2 \theta - \cos^2 \theta) + (C_{22} - C_{33}) \cos \theta \sin \theta].$$

(2.65)

This means that, for well-defined values of $\theta$ (i.e., away from $|v_r| = 0$), as $|v_r|$ approaches 0, $d\theta/dI_n$ grows without bound (unless, of course, the right hand side is zero, in which case $d\theta/dI_n = 0$). Thus, $\theta$ approaches the nearest value for which $d\theta/dI_n = 0$. But this is precisely how we defined the asymptotes! We conclude that, immediately before the sliding velocity vanishes, $\sigma$ is parallel to an asymptote. Likewise, if friction is not strong enough to prevent slipping in a new direction (i.e., $\mu < \tilde{\mu}$), the new direction must be an asymptote as well: for if sliding resumed in a direction which was not an asymptote, it would very quickly approach one anyway, since $d\theta/dI_n$ is infinite for vanishingly small values of $|v_r|$. We conclude that $\sigma$ is parallel to an asymptote immediately after a transition in the slip direction, and subsequently remains constant for the rest of the impact phase.

We now turn our attention to solving for the directions $\tilde{\sigma}^*$ of the asymptotes. By definition, the asymptotes satisfy

$$(\tilde{T}\tilde{\sigma}^*)^T(\tilde{h} - \mu\tilde{B}\tilde{\sigma}^*) = 0,$$

(2.66)

where

$$\tilde{T} := \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

(2.67)

is the $90^\circ$ counterclockwise rotation matrix. Before we solve for the asymptotes, however, we note that because $\tilde{B}$ is positive definite, it has two positive eigenvalues $B_1$ and $B_2$ (with $B_1 > B_2$), and two corresponding orthonormal eigenvectors $\tilde{b}_1$ and $\tilde{b}_2$. These satisfy the following relations:

$$\tilde{B}\tilde{b}_i = B_i\tilde{b}_i \Leftrightarrow \tilde{B}^{-1}\tilde{b}_i = (1/B_i)\tilde{b}_i \quad \text{for } i = 1, 2.$$  

(2.68)

Since the eigenvectors are orthogonal, we may use them as a basis for our vector space. In particular, we may write

$$\tilde{h} = h_1\tilde{b}_1 + h_2\tilde{b}_2,$$

(2.69)
where \( h_1 := \bar{h}^T \bar{b}_1 \) and \( h_2 := \bar{h}^T \bar{b}_2 \), and

\[
\bar{\sigma}^* = \cos \phi \bar{b}_1 + \sin \phi \bar{b}_2, \tag{2.70}
\]

where \( \phi \) is the angle \( \bar{\sigma}^* \) makes with the direction defined by \( \bar{b}_1 \). We take the direction of \( \bar{b}_1 \) to be that for which \( h_1 \geq 0 \), and \( \bar{b}_2 \) to be given by

\[
\bar{b}_2 = \bar{T} \bar{b}_1, \tag{2.71}
\]

In this way, since \( \bar{T}^{-1} \equiv \bar{T}^T \equiv -\bar{T} \),

\[
\bar{T}^T \bar{b}_2 = \bar{b}_1, \quad \bar{T}^T \bar{b}_1 = -\bar{b}_2. \tag{2.72}
\]

Finally, we find that the direction of the critical isocline is given by

\[
\bar{\sigma}^*_c := \frac{\bar{B}^{-1} \bar{h}}{\bar{\mu}} = \frac{\bar{B}^{-1} (h_1 \bar{b}_1 + h_2 \bar{b}_2)}{\bar{\mu}} = (h_1 / B_1) \bar{b}_1 + (h_2 / B_2) \bar{b}_2, \tag{2.73}
\]

which lies in the same quadrant as \( \bar{h} \).

Now, solving (2.66) for \( \mu \) yields

\[
\mu = \frac{(T \bar{\sigma}^*)^T \bar{h}}{(T \bar{\sigma}^*)^T \bar{B} \bar{\sigma}^*} = \frac{h_1 / \cos \phi - h_2 / \sin \phi}{B_1 - B_2}, \tag{2.74}
\]

and it is straightforward to show that

\[
\left( \mu \cos \phi - \frac{h_1}{B_1 - B_2} \right) \left( \mu \sin \phi + \frac{h_2}{B_1 - B_2} \right) = \frac{-h_1 h_2}{(B_1 - B_2)^2}. \tag{2.75}
\]

The only variable in (2.75) is \( \phi \). If we define two new variables

\[
x := \mu \cos \phi, \quad y := \mu \sin \phi, \tag{2.76}
\]

then solving (2.75) is equivalent to solving the following two equations simultaneously:

\[
\left( x - \frac{h_1}{B_1 - B_2} \right) \left( y + \frac{h_2}{B_1 - B_2} \right) = \frac{-h_1 h_2}{(B_1 - B_2)^2}; \tag{2.77}
\]

\[
x^2 + y^2 = \mu^2. \tag{2.78}
\]
Figure 2.4: Tangential velocity space, with representations of the three conic sections defined in the text. Intersections between the circle and the hyperbola correspond to asymptotic isoclines, or “asymptotes”. The slip direction $\mathbf{\sigma}$ is tangent to an asymptote immediately before and after jumps in the slip direction caused by the vanishing of the sliding velocity.

Intersections between the circle and the ellipse correspond to orthogonal isoclines, which separate velocity space into regions where $d|v_\tau|/dI_n$ is positive and regions where $d|v_\tau|/dI_n$ is negative. The fact that the ellipse does not intersect the second branch of the hyperbola implies that the outgoing slip direction, when it exists, is unique.

Now (2.77) defines a pair of hyperbola whose asymptotes are the lines defined by $\bar{b}_1$ and $\bar{b}_2$, and (2.78) defines a circle of radius $\mu$. The directions of the asymptotic isoclines are therefore given by the intersections between these hyperbola and this circle (see Figure 2.4; cf. Figure 5 of [19]).

We note in passing that

$$
\frac{d|v_\tau|}{dI_n} = \frac{d}{dI_n} \sqrt{v_\tau \cdot v_\tau} = \frac{1}{|v_\tau|} v_\tau \cdot \frac{d v_\tau}{dI_n} = \mathbf{\sigma} \cdot \frac{d v_\tau}{dI_n} = (\bar{\sigma}^*_c)^T \frac{d \bar{v}^*_\tau}{dI_n}. \tag{2.79}
$$

For $\mu = 0$, (2.62) reduces to $d\bar{v}^*_\tau/dI_n = \bar{h}$, so that there are two asymptotes: one in the direction of $\bar{h}$ (for which, according to (2.79), $d|v_\tau|/dI_n > 0$) and one opposite (for which $d|v_\tau|/dI_n < 0$). As $\mu$ increases toward infinity, the former approaches $\text{sgn}(h_2) \bar{b}_2$, passing through $\bar{\sigma}^*_c$ at $\mu = \bar{\mu}$ (at which point $d|v_\tau|/dI_n$ becomes negative), while the latter approaches $-\bar{b}_1$. At a
certain value of \( \mu \), the circle will intersect the other branch of the hyperbola, a bifurcation which results in two new asymptotes which, as \( \mu \) increases toward infinity, approach \( \bar{b}_1 \) and \(-\text{sgn}(h_2)\bar{b}_2\), respectively.

Uniqueness of the Outgoing Asymptote

We proceed to show that \( d|v_\tau|/dI_n < 0 \) for these two bifurcation asymptotes, and hence that there is at most one asymptote for which \( d|v_\tau|/dI_n > 0 \). (Because the vector field points away from the origin along this asymptote, we will call it the “outgoing” asymptote.) To do so, we identify those isoclines along which \( dv_\tau/dI_n \) is perpendicular to the direction of the isocline, for these isoclines by their very nature separate the \( v_1v_2 \)-plane into regions of positive and negative values for \( d|v_\tau|/dI_n \). These “orthogonal” isoclines satisfy

\[
(\bar{\sigma}^*)^T(\bar{\sigma} - \mu \bar{B} \bar{\sigma}^*) = 0, \tag{2.80}
\]

and solving (2.80) for \( \mu \) yields

\[
\mu = \frac{(\bar{\sigma}^*)^T \bar{h}}{(\bar{\sigma}^*)^T \bar{B} \bar{\sigma}^*} = \frac{h_1 \cos \phi + h_2 \sin \phi}{B_1 \cos^2 \phi + B_2 \sin^2 \phi}. \tag{2.81}
\]

It is straightforward to show that

\[
\left(\frac{\mu \cos \phi - \frac{h_1}{2B_1}}{B_2}\right)^2 + \left(\frac{\mu \sin \phi - \frac{h_2}{2B_2}}{B_1}\right)^2 = \frac{h_2^2 B_1 + h_1^2 B_2}{4B_1^2 B_2^2}. \tag{2.82}
\]

Again letting

\[
x := \mu \cos \phi, \quad y := \mu \sin \phi, \tag{2.83}
\]

we see that solving (2.82) is equivalent to solving the following two equations simultaneously:

\[
\left(\frac{x - \frac{h_1}{2B_1}}{B_2}\right)^2 + \left(\frac{y - \frac{h_2}{2B_2}}{B_1}\right)^2 = \frac{h_2^2 B_1 + h_1^2 B_2}{4B_1^2 B_2^2}, \tag{2.84}
\]

\[
x^2 + y^2 = \mu^2. \tag{2.85}
\]

Now (2.84) defines an ellipse whose major and minor axes are the lines defined by \( \bar{b}_1 \) and \( \bar{b}_2 \), and (2.85) defines a circle of radius \( \mu \). The directions of the
orthogonal isoclines are therefore given by the intersections between this ellipse and this circle (see Figure 2.4).

For $\mu = 0$, recall that (2.62) reduces to $d\bar{v}_r^*/dI_n = \bar{h}$, so that there are two orthogonal isoclines, both perpendicular to the direction defined by $\bar{h}$. As we noted earlier, these two orthogonal isoclines separate the plane into two regions: one in which $d|v_r|/dI_n > 0$, and one in which $d|v_r|/dI_n < 0$. As $\mu$ increases, there is a bifurcation which results in the appearance of a second region in which $d|v_r|/dI_n > 0$; however, it turns out that the two bifurcation asymptotes do not lie in this region. To show this, we find the intersections between the ellipse defined by (2.84) and the hyperbola defined by (2.77). Straightforward substitution reveals that both (2.84) and (2.77) are satisfied by the origin and the point defined by $\tilde{\mu}\tilde{\sigma}^c$, as given by the numerator in (2.73), both of which lie on one branch of the hyperbola. Solving (2.77) for $y$, substituting the result into (2.84), and simplifying, we find that the $x$-coordinates of the other two intersections between the hyperbola and the ellipse are given by the solution to the following quadratic equation:

$$h_2^2 + [h_1 - (B_1 - B_2)x]^2 = 0. \quad (2.86)$$

Namely, the $x$-coordinates of the intersections are

$$\frac{h_1 \pm ih_2}{B_1 - B_2}. \quad (2.87)$$

The only way these can be real is if $h_2 = 0$. However, examination of (2.77) shows that $y$ is indeterminate when $h_2 = 0$ and $x$ is given by the above. We conclude that the ellipse does not intersect the other branch of the hyperbola, and it follows that the two bifurcation asymptotes do not lie in the second region in which $d|v_r|/dI_n > 0$. Hence, there is at most one asymptote for which $d|v_r|/dI_n > 0$.

### 2.3.3 Initial Conditions & the Evolution of 3D Impact

We may now generalize the results of Section 2.2.3 to three dimensions. There are still three distinct possibilities for the initial values of $v_n$ and $|v_r|$, corresponding to (i) initial slip without grazing, (ii) grazing incidence, and (iii) initial stick.
Initial Slip without Grazing: $v_n^-, |v_\tau^-| \neq 0$

Suppose that initially there is both normal and tangential relative velocity (i.e., $v_n^-, |v_\tau^-| \neq 0$). This corresponds to initial slip without grazing. If $|v_\tau|$ vanishes during the impact phase, either sticking will occur, or slip will resume in a different direction.

**Sticking** In direct parallel to the case of planar impacts, we take sticking to occur when $|v_\tau|$ vanishes if (and only if) $\bar{\mu} \leq \mu$. This time, $\bar{\mu}$ is given by (2.57), and we drop the absolute value because in this case $\bar{\mu}$ is strictly positive. Note that, immediately before the magnitude of $v_\tau$ vanishes, it will be pointing in the direction of one of the three asymptotes for which $d|v_\tau|/dI_n < 0$.

**Slip Resumption** Likewise, slip resumes in a different direction when $|v_\tau|$ vanishes if (and only if) $\bar{\mu} > \mu$. Again, note that immediately before the magnitude of $v_\tau$ vanishes, it will be pointing in the direction of one of the three asymptotes for which $d|v_\tau|/dI_n < 0$, and that that the new sliding direction must be given by the (unique) asymptote for which $d|v_\tau|/dI_n > 0$.

Grazing Incidence: $v_n^- = 0$

Suppose that initially there is relative tangential velocity, but no relative normal velocity (i.e., $v_n^- = 0$ and $|v_\tau^-| \neq 0$). This corresponds to grazing incidence. Just as in the case of planar impacts, there will be an impact in this case if and only if $dv_n/dI_n$ is negative upon contact (this is the analog of dynamical jamming in three dimensions).

Initial Stick: $|v_\tau^-| = 0$

Suppose that initially there is relative normal velocity, but no relative tangential velocity (i.e., $v_n^- \neq 0$ and $|v_\tau^-| = 0$). This corresponds to initial stick. In this case, one of two things will occur: either the system will remain in stick, or slip will begin in a certain other direction. To decide exactly what happens, the same reasoning as in the case of initial slip without grazing applies: stick will persist if (and only if) $\bar{\mu} \leq \mu$, and slip will begin in the direction of the unique asymptote for which $d|v_\tau|/dI_n > 0$ if (and only if) $\bar{\mu} > \mu$.  
Now that we have formulated a theory for what happens during an impact between two rigid bodies in the presence of friction, we are ready to consider when the impact phase ends, i.e., various impact termination conditions.

3.1 The Hypotheses of Newton, Poisson, & Stronge

The three most common impact termination conditions are those attributed to Newton, Poisson, and Stronge. In each case, the impact phase is assumed to consist of exactly one phase of compression followed immediately by exactly one phase of restitution (we will denote by $t^*$ the time at which compression ends and restitution begins). The impact phase is then taken to end when the value of a certain quantity reaches a fraction of its value at some other point during the impact phase. This fraction is in turn a function of the corresponding coefficient of restitution $\eta$, which is a non-negative constant which we take to be less than or equal to 1 (i.e., $\eta \in [0, 1]$).

3.1.1 Newton’s Hypothesis

Newton’s coefficient of restitution $\eta_N$ is defined as the ratio between the magnitude of the normal component of the relative velocity at the end of the impact phase and that at the onset of impact:

$$\eta_N := -\frac{v_n^+}{v_n^-}. \quad (3.1)$$

According to Newton’s hypothesis, the value of $\eta_N$ for any two bodies can be determined by experiment and specified a priori, so that, for any impact configuration, the impact phase ends as soon as $v_n = -\eta_N v_n^-$ (see [3]; see also [2] and [4–10]).
3.1.2 Poisson’s Hypothesis

Poisson’s coefficient of restitution $\eta_P$ is defined as the ratio between the normal impulse imparted to $B_1$ during restitution and that imparted during compression:

$$\eta_P := \frac{I_n^+ - I_n^*}{I_n^*} = \frac{I_n^+}{I_n^*} - 1. \quad (3.2)$$

According to Poisson’s hypothesis, the value of $\eta_P$ for any two bodies can be determined by experiment and specified a priori, so that, for any impact configuration, the impact phase ends as soon as $I_n = (1 + \eta_P)I_n^*$ (see [12]; see also [2], [4–7], [9], [13], and [14]).

3.1.3 Stronge’s Hypothesis

Just as Newton’s coefficient of restitution involves the normal component of the relative velocity and Poisson’s coefficient of restitution involves the normal component of impulse, Stronge’s coefficient of restitution $\eta_S$ involves the work done on the system by the normal component of the contact force. Before we can write down the definition of $\eta_S$, then, we must derive an expression for the work done on the system.

By the same reasoning that we used to derive the refined equations of motion, the external forces and torques acting on $B_1$ and $B_2$ do no work on the system during any period of time within the impact phase; the only work is done by the contact force $F$. Denote by $W_1$ the work done on $B_1$. It is given by

$$W_1 = \int F \cdot V_1 dt + \int r_1 \times F \cdot d\Theta_1, \quad (3.3)$$

where $d\Theta_1 := \Omega_1 dt$, and the integrals are taken over any one period of time within the impact phase. Similarly, denote by $W_2$ the work done on $B_2$. It is given by

$$W_2 = -\int F \cdot V_2 dt - \int r_2 \times F \cdot d\Theta_2, \quad (3.4)$$

where $d\Theta_2 := \Omega_2 dt$. The total work $W$ done on the system is simply the sum of $W_1$ and $W_2$. Adding (3.3) to (3.4), and noting that $Fd\Omega = dI$ and $a \cdot b \times c = c \cdot a \times b$ for any three vectors $a$, $b$, and $c$, we find that

$$W = \int (V_1 + \Omega_1 \times r_1 - V_2 - \Omega_2 \times r_2) \cdot dI. \quad (3.5)$$
But the term in parentheses is simply $v$, according to (2.6). Thus,

$$ W = \int v \cdot dI = \int v \cdot \frac{dI}{dI_n} dI_n. $$

(3.6)

The work done by the normal component of the contact force is given by

$$ W_n := \int v \cdot \frac{dI_n}{dI_n} dI_n = \int v \cdot n \ dI_n = \int v_n \ dI_n. $$

(3.7)

The square of Stronge’s coefficient of restitution is defined as the ratio between the magnitude of work done by the normal component of the contact force during restitution and that done during compression:

$$ \eta_S^2 := -\int_{I_n^+}^{I_n^-} v_n \ dI_n \bigg/ \int_0^{I_n^+} v_n \ dI_n. $$

(3.8)

According to Stronge’s hypothesis, the value of $\eta_S$ for any two bodies can be determined by experiment and specified \textit{a priori}, so that, for any impact configuration, the impact phase ends as soon as

$$ \int_{I_n^+}^{I_n^-} v_n \ dI_n' = -\eta_S^2 \int_0^{I_n^+} v_n \ dI_n $$

(3.9)

(see [4] and [2]; see also [7]).

3.1.4 Equivalence of the Hypotheses for Balanced Collisions

In general, the impact termination conditions of Newton, Poisson, and Stronge are not equivalent. However, during so-called \textit{balanced collisions} (in which $h^T dI_r = 0$; cf. [6]), $v_n$ is linear in $I_n$ according to (2.51), so that the lines $v_n = v_n^- + (1/m)I_n$, $v_n = 0$, $I_n = 0$, and $I_n = I_n^+$ form two similar, right triangles in the $I_n v_n$-plane, as shown in Figure 3.1. By definition, $\eta_N$ is the ratio of the two vertical leg lengths, $\eta_P$ is the ratio of the two horizontal leg lengths, and $\eta_S^2$ is the ratio of the two areas. Because the triangles are similar, it follows that

$$ \eta_N = \eta_P = \eta_S \quad \text{for balanced collisions.} $$

(3.10)
Figure 3.1: Plot of $v_n$ versus $I_n$ for balanced collisions. The lines $v_n = v_n^- + (1/m)I_n$, $v_n = 0$, $I_n = 0$, and $I_n = I_n^+$ form two similar, right triangles in the plane. By definition, $\eta_N$ is the ratio of the two vertical leg lengths, $\eta_P$ is the ratio of the two horizontal leg lengths, and $\eta_s^2$ is the ratio of the two areas. Because the triangles are similar, it follows that $\eta_N = \eta_P = \eta_s$ for balanced collisions.

3.2 Energetic Consistency of an Impact Termination Condition

Friction is a non-conservative force: it dissipates energy. Therefore, the collisions we have been considering, in which the only force that is non-negligible is the friction-like contact force $F$, should not result in an increase in the total energy of the system (we allow the total energy to remain constant in cases where $\mu = 0$ or there is no relative tangential motion). Because we are concerned only with the mechanics of rigid body impacts, we need only consider mechanical energy, and since we take impact to be instantaneous (so that there is no change in the configuration of the system during the impact phase), the only form of mechanical energy that changes is the kinetic energy $T$. By the work-energy theorem, the change in kinetic energy of the system during the impact phase is precisely equal to the total work done on the system during the impact phase, as given by (3.6):

$$\Delta T = \int_0^{I_n^+} v \cdot \frac{dI}{dI_n} dI_n. \quad (3.11)$$

We say that an impact termination condition is energetically consistent if it cannot yield an increase in the kinetic energy of the impacting system during
the impact phase. That is, an impact termination condition is energetically consistent if (and only if), for all impacts,

$$\Delta T \leq 0. \quad (3.12)$$

We proceed to consider under what circumstances the three impact termination conditions (3.1), (3.2), and (3.8) are energetically consistent.

### 3.2.1 Consistency of Stronge’s Hypothesis

It turns out that Stronge’s impact termination condition is energetically consistent, since the fact that $\eta_S \in [0, 1]$ guarantees that $\Delta T \leq 0$. To see this, we begin by decomposing $dI/dI_n$ in (3.11) into its normal and tangential components:

$$\Delta T = \int_0^{I_n^+} v_n \, dI_n + \int_0^{I_n^+} v \cdot \frac{dI}{dI_n} \, dI_n, \quad (3.13)$$

where the first term follows from (3.7). During relative stick, the second term is zero, and during relative slip, it can be simplified using (2.11) and (2.35), so that, in either case,

$$\Delta T = \int_0^{I_n^+} (v_n - \mu |v_\tau|) \, dI_n, \quad (3.14)$$

where $|v_\tau|$ is given by (2.39). Next, we break the first integral in (3.13) into integrals over compression and restitution:

$$\Delta T = \int_0^{I_n^*} v_n \, dI_n + \int_{I_n^*}^{I_n^+} v_n \, dI_n - \mu \int_0^{I_n^+} |v_\tau| \, dI_n. \quad (3.15)$$

According to Stronge’s impact termination condition, the first two terms in (3.15) are related by (3.8), so that

$$\Delta T = (1 - \eta_S^2) \int_0^{I_n^*} v_n \, dI_n - \mu \int_0^{I_n^+} |v_\tau| \, dI_n. \quad (3.16)$$

Now we want $\Delta T$ to be less than or equal to zero. The first integral is negative because $v_n$ is non-positive during compression, and the second integral is non-negative because $|v_\tau|$ is a non-negative quantity. Thus, as long as $1 - \eta_S^2 \geq 0$
(or equivalently, as long as \(\eta_s \in [0, 1]\)), \(\Delta T \leq 0\). This means that Stronge’s impact termination condition is indeed energetically consistent.

3.2.2 Consistency of Newton’s and Poisson’s Hypotheses

We proceed to investigate whether Newton’s and Poisson’s hypotheses are energetically consistent.

An Obvious Theorem

We see from (3.14) that the change in the kinetic energy is the integral of a certain function

\[
Z(I_n) := v_n - \mu |v_r|
\]

over the interval \(I_n \in [0, I_n^+]\). How are we to show that the definite integral of a function is negative? Following Ivanov in [16], we note the following sufficient condition:

**Theorem 1.** Suppose that the function \(f(x)\) satisfies the following conditions: (i) \(f\) is continuous on the closed interval \(x \in [a, b]\), (ii) \(f(a) < 0\), (iii) \(f'\) is non-negative and non-increasing on the open interval \(x \in (a, b)\), and (iv) \(f(b) > 0\). Suppose further that (v) \(b - c \leq c - a\), where \(c\) is the value of \(x\) on this interval at which \(f(x)\) becomes zero. Then

\[
\int_a^b f(x)dx \leq 0.
\]

This result is obvious from inspection of a sufficiently accurate sketch (see, for example, Figure 3.2). Essentially, condition (iii) says that \(f\) is concave for \(x \in [a, b]\), so that its “mirror image” (the lower boundary of the purple region in Figure 3.2) is convex. This means that the mirror image is greater than the original function for \(x \in (2c - b, c)\), which, coupled with conditions (i), (ii), (iv), and (v), guarantees that the magnitude of the (negative) area under \(f(x)\) for \(x \in [a, c]\) is greater than that of the (positive) area under \(f(x)\) for \(x \in [c, b]\). We will use this result with \(f(x) = Z(I_n), a = 0,\) and \(b = I_n^+\) to prove the consistency of Poisson’s hypothesis, which we will see guarantees condition (v). Newton’s hypothesis, however, does not guarantee condition (v), and is not in general energetically consistent.
Figure 3.2: Representative sketch illustrating the validity of Theorem 1. The purple region is a rotated copy of the red region, overlayed on top of the blue region. Conditions (i)-(v) guarantee that the area over $f(x)$ for $x \in [a, c]$ (the union of the blue and purple regions) is greater than that under $f(x)$ for $x \in [c, b]$ (the red region). In other words, not all of the blue region can be covered by the red region. This theorem is invoked to prove the energetic consistency of Poisson’s impact termination condition.

The Work Integrand for Rigid Body Impacts

We begin by showing that $Z(I_n) := v_n - \mu |v_\tau|$ satisfies conditions (i)-(iv) of Theorem 1. Clearly, $Z$ is continuous during the impact phase, since both $v_n$ and $|v_\tau|$ are continuous. Thus, condition (i) is satisfied. Now

$$Z(0) = v_n^+ - \mu |v_\tau|^+ < 0,$$  (3.19)

since $v_n^- \leq 0$ and $|v_\tau|^+ \geq 0$, but $v_n^-$ and $|v_\tau|^-\!$ cannot both be zero simultaneously. Thus, condition (ii) is also satisfied.

Condition (iii) is more challenging. We have that

$$Z'(I_n) = \frac{dv_n}{dI_n} - \mu \frac{d|v_\tau|}{dI_n}.$$  (3.20)

During periods of sustained stick, $d|v_\tau|/dI_n \equiv 0$, and, using (2.56),

$$Z'(I_n) = \frac{dv_n}{dI_n} = \frac{1}{C_{11}^{-1}} > 0.$$  (3.21)

That is, during periods of sustained stick, $Z'$ is constant and positive, so
condition (iii) is satisfied during sustained stick. During sustained slip,

\[
\frac{dv_n}{dI_n} = n \cdot \frac{dv}{dI_n} = \bar{n}^T \frac{d\bar{v}}{dI_n},
\]

(3.22)

and, recalling (2.79),

\[
\frac{d|v_\tau|}{dI_n} = (\bar{\sigma}^*)^T \frac{d\bar{v}_\tau^*}{dI_n} = \bar{\sigma}^T \frac{d\bar{v}}{dI_n}.
\]

(3.23)

Thus, using (2.59),

\[
Z'(I_n) = (\bar{n} - \mu \bar{\sigma})^T \bar{C}(\bar{n} - \mu \bar{\sigma}),
\]

(3.24)

which is positive because \(\bar{C}\) is positive definite. Furthermore, differentiating this expression, we find that

\[
Z''(I_n) = -\mu \frac{d\bar{\sigma}}{dI_n}^T \bar{C}(\bar{n} - \mu \bar{\sigma}) \frac{d\bar{\sigma}}{dI_n} = -2\mu \frac{d\bar{\sigma}}{dI_n}^T \bar{C}(\bar{n} - \mu \bar{\sigma}),
\]

(3.25)

where the last equality follows from the fact that \(\bar{C}\) is symmetric. Now

\[
\frac{d\sigma}{dI_n} = \frac{1}{|v_\tau|^2} \left( |v_\tau| \frac{dv}{dI_n} - v_\tau \frac{d|v_\tau|}{dI_n} \right) = \frac{1}{|v_\tau|} \left[ \frac{dv}{dI_n} - \left( \frac{dv}{dI_n} \cdot n \right) n - \left( \frac{dv}{dI_n} \cdot \sigma \right) \sigma \right].
\]

(3.26)

In matrix notation, letting \(\bar{q} := \bar{C}(\bar{n} - \mu \bar{\sigma})\), and using (2.59) again, we have

\[
Z''(I_n) = -\frac{2\mu}{|v_\tau|} \left[ \bar{q}^T \bar{q} - (\bar{q}^T \bar{n})^2 - (\bar{q}^T \bar{\sigma})^2 \right],
\]

(3.27)

which is non-positive according to Bessel’s inequality. Hence, we have shown that condition (iii) is satisfied during periods of sustained stick and sustained slip. To show that this condition is always satisfied, we must show that it holds during transitions from sustained slip to sustained stick and during discontinuous changes in the slip direction, as this will cause a jump in \(Z'\).

**Sticking** When a transition from sustained slip to sustained stick occurs, it turns out that the sliding direction immediately beforehand doesn’t matter.
Immediately after the transition, \( Z' \) will be given by (2.56):

\[
Z'_{\text{after}} = C_{11} - \frac{C_{13}^2 C_{22} - 2C_{12} C_{13} C_{23} + C_{12}^2 C_{33}}{C_{22} C_{33} - C_{23}^2}.
\] (3.28)

Immediately before the transition, \( \sigma = \sigma_1 \), and \( Z' \) is given by (3.24):

\[
Z'_{\text{before}} = (\bar{n} - \mu \bar{\sigma})^T \bar{C}(\bar{n} - \mu \bar{\sigma}) = \bar{n}^T \bar{C} \bar{n} - 2\mu \bar{\sigma}_1^T \bar{C} \bar{n} + \mu^2 \bar{\sigma}_1^T \bar{C} \bar{\sigma}_1
\]

\[
\geq \bar{n}^T \bar{C} \bar{n} - \frac{(\bar{\sigma}_1^T \bar{C} \bar{n})^2}{\bar{\sigma}_1^T \bar{C} \bar{\sigma}_1}.
\] (3.29)

Recall that until this point \( e_2 \) has been arbitrary. Now, however, we will take \( e_2 = \sigma_1 \). In this way, we have that

\[
Z'_{\text{before}} \geq C_{11} - \frac{C_{12}^2}{C_{22}},
\] (3.30)

and the jump in \( Z' \) can be estimated as follows:

\[
\Delta Z' := Z'_{\text{after}} - Z'_{\text{before}}
\]

\[
\leq C_{11} - \frac{C_{13}^2 C_{22} - 2C_{12} C_{13} C_{23} + C_{12}^2 C_{33}}{C_{22} C_{33} - C_{23}^2} - \left( C_{11} - \frac{C_{12}^2}{C_{22}} \right)
\]

\[
= \frac{C_{12}^2 (C_{22} C_{33} - C_{23}^2) - C_{22} (C_{13}^2 C_{22} - 2C_{12} C_{13} C_{23} + C_{12}^2 C_{33})}{C_{22} (C_{22} C_{33} - C_{23}^2)}
\]

\[
= -\frac{(C_{12} C_{23} - C_{13} C_{22})^2}{C_{22} (C_{22} C_{33} - C_{23}^2)},
\] (3.31)

which, since \( \bar{C} \) is positive definite, is negative. Thus, we have shown that condition (iii) is satisfied during a transition from slip to sustained stick.

**Slip Resumption**  It is this step in Ivanov’s proof that is erroneous. Close inspection of his work reveals a typo in Equation (3.7) of [16], which enables him to factor \((\cos \xi - 1)\) out of his expression for \( \Delta \Phi \) (what we have been calling \( \Delta Z' \)). His result implies that \( \Delta Z' \leq 0 \) for arbitrary changes in the slip direction \( \xi \) (what we have been calling \( \phi \))—not just changes between asymptotes. This is simply not true, as can be shown via direct calculation (see, for example, Figure 3.3). To show that the change in \( Z' \) is negative
Figure 3.3: Plot of $\Delta Z'$ for arbitrary incoming and outgoing slip directions $\phi_1$ and $\phi_2$. The parameters used for this example are as follows: $\bar{h} = [1/2, (1/2)^*(5/2)]^T$, $\bar{B} = [11/2, 0; 0, 1/2]$, and $\mu$ is the mean of the critical friction coefficient $\tilde{\mu}$ and the value of $\mu$ at which the bifurcation in the number of asymptotes occurs.

during an instantaneous change in the slip direction (i.e., when the tangential velocity vanishes but $\tilde{\mu} > \mu$), we must make use of the fact that the slip direction is tangent to an asymptote immediately before and after such transitions. We begin by making the following observations. First, the slip direction immediately prior to the transition, $\sigma_1$, will point along one of the three asymptotes for which $d|v_r|/dn < 0$. Invoking (2.66), (2.79), and (2.62), and letting $\bar{p}_1 := \bar{h} - \mu \bar{B} \bar{\sigma}_1^*$, we find that $\tilde{\sigma}_1^*$ satisfies the following:

$$ (\bar{T} \tilde{\sigma}_1^*)^T \bar{p}_1 = 0, \quad (3.32) $$

$$ \bar{p}_1^T \tilde{\sigma}_1^* < 0. \quad (3.33) $$

From (3.32), we see that $\tilde{\sigma}_1^*$ and $\bar{p}_1$ are parallel, so that we may write

$$ \bar{p}_1 = (\bar{p}_1^T \tilde{\sigma}_1^*) \tilde{\sigma}_1^*. \quad (3.34) $$

Similarly, the slip direction after the transition, $\sigma_2$, will point along the unique asymptote for which $d|v_r|/dn > 0$. Thus, letting $\bar{p}_2 := \bar{h} - \mu \bar{B} \bar{\sigma}_2^*$, we
find that \( \bar{\sigma}_2^* \) satisfies the following:

\[
(\bar{T}\bar{\sigma}_2^*)^T \bar{p}_2 = 0, \tag{3.35}
\]
\[
\bar{p}_2^T \bar{\sigma}_2^* > 0. \tag{3.36}
\]

Just as before, we see that \( \bar{\sigma}_2^* \) and \( \bar{p}_2 \) are parallel, so that

\[
\bar{p}_2 = (\bar{p}_2^T \bar{\sigma}_2^*) \bar{\sigma}_2^*. \tag{3.37}
\]

Now we are ready to calculate the jump in \( Z' \). Using (3.24),

\[
\Delta Z' = (\bar{n} - \mu \bar{\sigma}_2)^T \bar{C}(\bar{n} - \mu \bar{\sigma}_2) - (\bar{n} - \mu \bar{\sigma}_1)^T \bar{C}(\bar{n} - \mu \bar{\sigma}_1). \tag{3.38}
\]

Simplifying using (2.61), and recalling that \( \bar{n} = [1, 0, 0]^T \) and \( \bar{\sigma}_i = [0, (\bar{\sigma}_i^*)^T]^T \) for \( i = 1 \) and 2, we find that

\[
\Delta Z' = -2\mu \bar{h}^T \bar{\sigma}_2^* + \mu^2 (\bar{\sigma}_2^*)^T \bar{B} \bar{\sigma}_2^* + 2\mu \bar{h}^T \bar{\sigma}_1^* - \mu^2 (\bar{\sigma}_1^*)^T \bar{B} \bar{\sigma}_1^*. \tag{3.39}
\]

Now this can be rewritten in terms of \( \bar{p}_1 \) and \( \bar{p}_2 \) as follows:

\[
\Delta Z' = \mu (\bar{p}_2^T \bar{\sigma}_1^* - \bar{p}_1^T \bar{\sigma}_2^* + \bar{p}_1^T \bar{\sigma}_1^* - \bar{p}_2^T \bar{\sigma}_2^*). \tag{3.40}
\]

Substituting (3.34) and (3.37) and factoring, this becomes

\[
\Delta Z' = \mu [1 - (\bar{\sigma}_1^*)^T (\bar{\sigma}_2^*)] [\bar{p}_1^T \bar{\sigma}_1^* - \bar{p}_2^T \bar{\sigma}_2^*]. \tag{3.41}
\]

Now the first bracketed term is positive since \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) are unit vectors, and the second bracketed term is negative by (3.33) and (3.36). Hence, \( \Delta Z' \) is negative, and condition (iii) is always satisfied.

As for condition (iv), we note that it is possible that \( Z(I_n^+) > 0 \) (for example, if \( |v_\tau|^+ = 0 \), since \( v_\tau^+ \geq 0 \)), and that, if \( Z(I_n^+) \leq 0 \), by condition (iii), (3.18) is trivial. We may therefore suppose that condition (iv) is satisfied in order to proceed.

To summarize, we have shown that conditions (i)-(iv) are satisfied by the work integrand for rigid body impacts, regardless of the choice of termination

\footnote{Credit for this insight is due to Arne Nordmark.}
Consistency of Poisson’s Hypothesis

It turns out that, just as in the case of Stronge’s hypothesis, the fact that $\eta_P \in [0, 1]$ guarantees that $\Delta T \leq 0$. We shall now show that $\eta_P \in [0, 1]$ ensures that condition (v) of Theorem 1 is satisfied (recall that in this context $a = 0$ and $b = I_n^+)$. Let $I_n^+$ denote the value of $I_n$ at which $Z$ becomes zero (the analog of $c$ in Theorem 1). We begin by noting that, at the transition between compression and restitution,

$$Z(I_n^+) = v_n^+ - \mu |v_r| = -\mu |v_r| \leq 0, \quad (3.42)$$

since by definition, $v_n^+ = 0$, and $|v_r| \geq 0$. From this it follows that

$$I_n^+ \geq I_n^*, \quad (3.43)$$

since $Z$ is non-decreasing and starts negative. Taking the reciprocal of both sides, multiplying both sides by $I_n^+$ (which is positive), and subtracting 1 from both sides, we find that

$$\frac{I_n^+ - I_n^+}{I_n^+} \leq \frac{I_n^+ - I_n^*}{I_n^*} =: \eta_P \quad (3.44)$$

Hence, as long as $\eta_P \in [0, 1]$,

$$I_n^+ - I_n^+ \leq I_n^+ \quad (3.45)$$

and condition (v) is satisfied. This means that Poisson’s hypothesis is energetically consistent.

Inconsistency of Newton’s Hypothesis

Because the fact that $\eta_N \in [0, 1]$ does not ensure that condition (v) of Theorem 1 is satisfied, the inconsistency of Newton’s hypothesis cannot be ruled out. We defer a numerical example of this for Section 4.2.1.
CHAPTER 4

EXPLICIT RIGID BODY IMPACT MAPS

In Chapter 2, we found that the equations of motion for collinear and planar impacts can be solved analytically, and in Section 2.2.4, we listed all conceivable planar impact processes. However, at the time we did not know which process would actually occur because we did not know when the impact phase was going to end. In Chapter 3, we discussed the three most commonly used impact termination conditions. Therefore, for each termination condition, we can now determine under what conditions each process occurs and formulate an explicit impact map $g : (v_n^-, v_r^-) \mapsto (v_n^+, v_r^+)$ relating the outgoing relative velocities to the incoming relative velocities. Unfortunately, the same cannot be done for three-dimensional impacts, as the equations of motion in that case cannot be solved analytically.

4.1 Collinear Impacts

We saw in Section 3.1.4 that in the case of collinear impacts, the hypotheses of Newton, Poisson, and Stronge are equivalent. Hence, there is only one impact map for collinear impacts, $g_c$, which is given by

\[
(v_n^+, v_r^+) = g_c(v_n^-, v_r^-) = (-\eta v_n^-, 0),
\]

where $\eta := \eta_N = \eta_P = \eta_S$ is the unique coefficient of restitution.

4.2 Planar Impacts

We proceed to formulate explicit impact maps for planar impacts, that is, Cases #1-10 of Section 2.2.4, using the hypotheses of Newton, Poisson, and Stronge. In each of the ten cases, the impact map is determined by evaluating
\(v_n(I_n)\) and \(v_\tau(I_n)\) at \(I_n = I_n^+\). The expression for \(I_n^+\) in turn depends on the particular hypothesis.

The Integrated Equations of Motion

Recall that, for Cases #1-4, the velocity space trajectory consists of only one segment, so that, integrating (2.27), we have

\[
v_n(I_n) = v_n^- + k_n^1 I_n \quad \text{for} \quad 0 \leq I_n \leq I_n^+; \tag{4.2}
\]

\[
v_\tau(I_n) = v_\tau^- + k_\tau^1 I_n \quad \text{for} \quad 0 \leq I_n \leq I_n^+; \tag{4.3}
\]

where \(k_n^1\) and \(k_\tau^1\) are the corresponding normal and tangential rate constants, which depend on the Case (see Table 4.1).

For Cases #5-10, the velocity space trajectory consists of two segments, and the transition between the two occurs at the moment when \(v_\tau\) vanishes (we will refer to this moment in time as \(t^0\)). In accordance with our shorthand notation, let \(I_n^0\) be the value of \(I_n\) at time \(t = t^0\). In this way, integrating (2.27), we have

\[
v_n(I_n) = \begin{cases} 
v_n^1 := v_n^- + k_n^1 I_n, & 0 \leq I_n \leq I_n^0, \\
v_n^11 := v_n^- + k_n^1 I_n^0 + k_n^11 (I_n - I_n^0), & I_n^0 < I_n \leq I_n^+; 
\end{cases} \tag{4.4}
\]

\[
v_\tau(I_n) = \begin{cases} 
v_\tau^1 := v_\tau^- + k_\tau^1 I_n, & 0 \leq I_n \leq I_n^0, \\
v_\tau^11 := v_\tau^- + k_\tau^1 I_n^0 + k_\tau^11 (I_n - I_n^0), & I_n^0 < I_n \leq I_n^+; 
\end{cases} \tag{4.5}
\]

where \(k_n^1, k_n^11, k_\tau^1,\) and \(k_\tau^11\) are the corresponding normal and tangential rate constants for the two velocity space segments (again, see Table 4.1).

Now, by definition, \(I_n^0\) is the root of \(v_\tau^1(I_n) = 0\):

\[
I_n^0 := \frac{v_\tau^-}{k_\tau^1}, \tag{4.6}
\]

where we may divide by \(k_\tau^1\) because it is non-zero for Cases #5-10. Solving (4.4) and (4.5) for \(I_n\) and setting the corresponding expressions equal to each other, we obtain a first integral of the equations of motion. Focusing on the first segment, we find that the quantity \(k_\tau^1 v_n - k_n^1 v_\tau\) is constant. That is,

\[
k_\tau^1 v_n - k_n^1 v_\tau = k_\tau^1 v_n^- - k_n^1 v_\tau^- \quad \text{for} \quad 0 \leq I_n \leq I_n^0 \tag{4.7}
\]
Table 4.1: Rate constants for planar impacts.

<table>
<thead>
<tr>
<th>Case</th>
<th>$k^I_n$</th>
<th>$k^I_n$</th>
<th>$k^I_\tau$</th>
<th>$k^{II}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>$k^0_n$</td>
<td>-</td>
<td>$k^+_\tau$</td>
<td>-</td>
</tr>
<tr>
<td>#2</td>
<td>$k^+_n$</td>
<td>-</td>
<td>$k^-\tau$</td>
<td>-</td>
</tr>
<tr>
<td>#3</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>#4</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^+_\tau$</td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
<td>$k^+_\tau$</td>
</tr>
<tr>
<td>#6</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
</tr>
<tr>
<td>#7</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^+_\tau$</td>
<td></td>
</tr>
<tr>
<td>#8</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
</tr>
<tr>
<td>#9</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
</tr>
<tr>
<td>#10</td>
<td>$k^-n_0$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
<td>$k^-\tau$</td>
</tr>
</tbody>
</table>

(cf. Equation (15) of [18]). This is also the case for the duration of the impact phase in Cases #1-4. In particular, for Cases #1-4,

$$v^+_\tau = v^-_\tau + \frac{k^I_n}{k^{II}_n} (-v^-_n + v^+_n). \quad (4.8)$$

Focusing on the second segment, we find that

$$k^{II}_n v^+_\tau = k^{II}_\tau (-k^I_n I^0_n - v^-_n + v^-_n) \quad \text{for} \quad I^0_n < I_n < I^*_n. \quad (4.9)$$

In particular, for Cases #5-10,

$$v^+_\tau = \frac{k^I_n}{k^{II}_n} \left( \frac{k^I_n}{k^{II}_\tau} v^-_\tau - v^-_n + v^+_n \right). \quad (4.10)$$

(Note that (4.10) reduces to (4.8) in the limit as $k^{II}_n \to k^I_n$ and $k^{II}_\tau \to k^I_\tau$, as we would expect.) This means that, when stating an impact map, it suffices simply to specify the expression for $v^+_n$, since $v^+_\tau$ can be recovered from (4.8) or (4.10).

The Transition between Compression & Restitution

The transition between compression and restitution occurs when $v_n$ vanishes. This means that $I^*_n$ is the root of $v_n(I_n) = 0$, and will therefore depend on the Case. For Cases #1-4, #5, and #6 (in the latter two, $0 < I^*_n < I^0_n$),
\( I^*_n \) is the root of \( v^1_n(I_n) = 0 \). For Cases #7 and #8, \( I^0_n < I^*_n < I^+_n \), and \( I^*_n \) is the root of \( v^{II}_n(I_n) = 0 \). For Cases #9 and #10 we may use either root, since (4.4) is continuous across \( I_n = I^0_n \), and in these Cases, \( I^*_n = I^0_n \). Putting everything together, then, we may write

\[
I^*_n := \begin{cases} 
-v^-_n/k^I_n & \text{for Cases #1-6 (and #9,10)}, \\
I^0_n - (v^-_n + k^I_n I^0_n)/k^{II}_n & \text{for Cases #7-10},
\end{cases}
\]

(4.11)

where \( I^0_n \) is given by (4.6), and \( k^I_n \) and \( k^{II}_n \) are non-zero for the Cases in which we have divided by them.

The Three Impact Maps for Planar Impacts

Based on the preceding discussion, we may identify three distinct “components” of a planar impact map: \( g_I \) for Cases #1-4, in which the velocity space trajectory consists of only one segment and \( I^*_n \) is given by the first expression in (4.11); \( g^{II}_n \) for Cases #5 and #6, in which the velocity space trajectory consists of two segments and \( I^*_n \) is given by the first expression in (4.11); and \( g^{III}_n \) for Cases #7 and #8, in which the velocity space trajectory consists of two segments and \( I^*_n \) is given by the second expression in (4.11). (For Cases #9 and #10, either \( g^{II}_n \) or \( g^{III}_n \) may be used.) To go any further, we must specify an impact termination condition.

### 4.2.1 Planar Impacts with Newton’s Hypothesis

With Newton’s hypothesis, the impact map is trivial: \( v^+_n = -\eta_N v^-_n \), with \( v^+_n \) given by (4.8) for Cases #1-4, and (4.10) for Cases #5-10. We can thus determine the value of \( I^+_n \) by setting \( v_n(I^+_n) = -\eta_N v^-_n \) in (4.2) and (4.4), yielding

\[
I^+_n := \begin{cases} 
-(1 + \eta_N) v^-_n/k^I_n & \text{for Cases #1-4}, \\
I^0_n - [(1 + \eta_N) v^-_n + k^I_n I^0_n]/k^{II}_n & \text{for Cases #5-10}.
\end{cases}
\]

(4.12)

We may now determine when Cases #1-4 occur for Newton’s hypothesis. Recall that Cases #4-10 only occur if \( v^-_n \neq 0 \). Under this condition, Case #4 occurs if either \( \mu \geq \mu \) or the impact phase terminates at or before \( t^0 \); otherwise, one of Cases #5-10 occurs. Now the impact phase will terminate
at or before \(t^0\) if (and only if) \(I_n^+ \leq I_n^0\). For Case #4, using (4.12) and (4.6), we see that \(I_n^+ \leq I_n^0\) is equivalent to

\[
-(1 + \eta_N) \frac{v_n^-}{k_n^-} \leq -\frac{v_n^-}{k_r^-}.
\] (4.13)

Hence, if \(v_r^- \neq 0\), Case #4 will occur if either \(\bar{\mu} \geq \mu\) or (4.13) holds; otherwise, one of Cases #5-10 will occur. The entire decision process for deciding which Case occurs is summarized in the flowchart shown in Figure 4.1 (cf. Figure 5 of [18] and Figure 4 of [1]).

**Example: Energetic Inconsistency of Newton’s Hypothesis**  As an aside, recall that Newton’s hypothesis is not generally energetically consistent. We are now at a point where we can show this with a numerical example.

Consider a planar impact with \(C_{nn} = C_{\tau\tau} = 4, C_{n\tau} = 3, \mu = 1, \eta_N = 0.8, v_n^- = -1, \) and \(v_r^- = -0.6\). This can be interpreted as an impact between a rigid rod and a half-space (cf. [16]). From (2.26), we find that

\[
\begin{align*}
k_n^+ &:= C_{nn} - \mu C_{n\tau} = 1, & k_r^+ &:= C_{n\tau} - \mu C_{\tau\tau} = -1, \\
k_n^- &:= C_{nn} + \mu C_{n\tau} = 7, & k_r^- &:= C_{n\tau} + \mu C_{\tau\tau} = 7, \\
k_n^0 &:= (C_{nn} C_{\tau\tau} - C_{n\tau}^2)/C_{\tau\tau} = 1.75, & k_r^0 &:= 0,
\end{align*}
\] (4.14)

and from (2.25),

\[
\bar{\mu} := -\frac{C_{n\tau}}{C_{\tau\tau}} = -0.75.
\]

Following the flowchart in Figure 4.1, we conclude that this example falls into Case #7, so that

\[
\begin{align*}
k_n^1 &= k_n^- = 7, & k_r^1 &= k_r^- = 7, \\
k_n^\| &= k_n^0 = 7/4, & k_r^\| &= k_r^0 = 0.
\end{align*}
\] (4.15)

It follows from (4.6) that

\[
I_n^0 := -\frac{v_r^-}{k_r^1} = \frac{3}{35},
\]
Figure 4.1: Flowchart for deciding which of Cases #1-10 occurs.
and from (4.12) that $I^+_n = 0.7714...$. From (4.4) and (4.5), we have

$$v_n(I_n) = \begin{cases} 
-1 + 7I_n, & 0 \leq I_n \leq I^0_n \\
-0.55 + 1.75I_n, & I^0_n < I_n \leq I^+_n 
\end{cases}, \quad (4.16)$$

$$v_\tau(I_n) = \begin{cases} 
-0.6 + 7I_n, & 0 \leq I_n \leq I^0_n \\
0, & I^0_n < I_n \leq I^+_n 
\end{cases}. \quad (4.17)$$

Finally, from (3.14), we find the change in kinetic energy to be

$$\Delta T = \int_{I^+_n}^{I^0_n} (v_n - \mu |v_\tau|) dI_n = \int_{I^+_n}^{I^0_n} (-1 + 7I_n - \mu | - 0.6 + 7I_n|) dI_n + \int_{I^+_n}^{I^0_n} (-0.55 + 1.75I_n) dI_n = 0.0514..., \quad (4.18)$$

which is positive, in violation of (3.12). □

4.2.2 Planar Impacts with Stronge’s Hypothesis

According to Stronge’s impact termination condition, (3.8), $I^+_n$ is one of the two solutions to the following quadratic equation:

$$\int_{I^+_n}^{I^0_n} v_n dI_n = -\eta_S^2 \int_{I^0_n}^{I^+_n} v_n dI_n. \quad (4.19)$$

The correct choice is the one which yields a positive value for $v^+_n$. In particular, for Cases #1-4, we find that

$$I^+_n = -(1 + \eta_S) \frac{v^-_n}{k^+_n}. \quad (4.20)$$

We may now determine when Cases #4-10 occur. Recall that Cases #4-10 only occur if $v^-_\tau \neq 0$. Under this condition, Case #4 occurs if either $\bar{\mu} \geq \mu$ or the impact phase terminates at or before $t^0$; otherwise, one of Cases #5-10 occurs. Now the impact phase will terminate at or before $t^0$ if (and only if) $I^+_n \leq I^0_n$. For Case #4, using (4.20) and (4.6), we see that $I^+_n \leq I^0_n$ is equivalent to

$$-(1 + \eta_S) \frac{v^-_n}{k^+_n} \leq -\frac{v^-_\tau}{k^-_\tau}. \quad (4.21)$$

Hence, if $v^-_\tau \neq 0$, Case #4 will occur if either $\bar{\mu} \geq \mu$ or (4.21) holds; otherwise, one of Cases #5-10 will occur. Since (4.21) has the same form as (4.13),
we may still use the flowchart in Figure 4.1 when using Stronge’s impact
termination condition.

Following Nordmark et al. in [18] and [1], we proceed to derive the explicit
form of $g_I$, $g_{II}$, and $g_{III}$ for Stronge’s condition.

Cases #1-4 ($g_I$) for Stronge’s Hypothesis

Evaluating (4.2) and (4.3) at (4.20), and using the first expression for $I_n^*$ in
(4.11), we find that $g_I$ is given by

$$v_n^+ = -\eta Sv_n^-,$$

with $v^+ \tau$ given by (4.8).

Cases #5,6 and #9,10 ($g_{II}$) for Stronge’s Hypothesis

Solving (4.19) for Cases #5 and #6 with the first expression for $I_n^*$ in (4.11),
and evaluating (4.4) and (4.5) at the result, we find that $g_{II}$ is given by

$$v_n^+ = \sqrt{(1 - \frac{k_{II}^n}{k_{II}^n}) (\frac{k_{II}^n}{k_{II}^\tau} v^- - v_n^-)^2 + \eta^2 S \frac{k_{II}^n}{k_{II}^n} (v_n^-)^2},$$

with $v^+ \tau$ given by (4.10).

Cases #7-10 ($g_{III}$) for Stronge’s Hypothesis

Solving (4.19) for Cases #7-10 with the second expression for $I_n^*$ in (4.11),
and evaluating (4.4) and (4.5) at the result, we find that $g_{III}$ is given by

$$v_n^+ = \eta S \sqrt{\left[v_n^- + \left(\frac{k_{II}^n - k_{II}^\tau}{k_{II}^\tau}\right) v^- \right] v_n^-} - k_{II}^n \left[\frac{k_{II}^n - k_{II}^\tau}{(k_{II}^\tau)^2}\right] (v^-)^2,$$

or, provided $k_{II}^n \neq 0$,

$$v_n^+ = \eta S \sqrt{1 - \frac{k_{II}^n}{k_{II}^\tau}} \left(\frac{k_{II}^n}{k_{II}^\tau} v_n^- - v_n^-\right)^2 + \frac{k_{II}^n}{k_{II}^\tau} (v_n^-)^2.$$
Either way, $v_\tau^+$ is given by (4.10).

This set of impact maps is consistent with those derived in [18] and reproduced in [1]. Notice that both $g_{II}$ and $g_{III}$ reduce to $g_I$ in the limit as $k_{II} \to k_I$ and $k_{III} \to k_I$, as we would expect.

4.2.3 Planar Impacts with Poisson’s Hypothesis

According to Poisson’s impact termination condition, (3.2),

$$I_n^+ = (1 + \eta_P)I_n^*.$$ (4.26)

With $I_n^+$ specified, we may now determine when Cases #4-10 occur. Recall that Cases #4-10 only occur if $v^- \neq 0$. Under this condition, Case #4 occurs if either $\bar{\mu} \geq \mu$ or the impact phase terminates at or before $t^0$; otherwise, one of Cases #5-10 occurs. Now the impact phase will terminate at or before $t^0$ if (and only if) $I_n^+ \leq I_0^n$. Using (4.11) and (4.6), we see that, for Case #4, this is equivalent to

$$-(1 + \eta_P)\frac{v^-}{k^-} \leq -\frac{v^-}{k^-}.$$ (4.27)

Hence, if $v^- \neq 0$, Case #4 will occur if either $\bar{\mu} \geq \mu$ or (4.27) holds; otherwise, one of Cases #5-10 will occur. Since (4.27) has the same form as both (4.13) and (4.21), we may still use the flowchart in Figure 4.1 when using Stronge’s impact termination condition.

Now, using the same procedure as before, we derive the explicit form of $g_I$, $g_{II}$, and $g_{III}$ for Poisson’s condition.

Cases #1-4 ($g_I$) for Poisson’s Hypothesis

Evaluating (4.2) and (4.3) at (4.26), and using the first expression for $I_n^*$ in (4.11), we find that $g_I$ is given by

$$v_n^+ = -\eta_P v_n^-.$$ (4.28)

with $v_\tau^+$ given by (4.8).
Cases #5,6 and #9,10 ($g_{II}$) for Poisson’s Hypothesis

Evaluating (4.4) and (4.5) at (4.26), and using the first expression for $I_n^*$ in (4.11), we find that $g_{II}$ is given by

$$v_n^+ = -\frac{k_{II}^n}{k_I^n} \left[ \eta_P + \left( \frac{k_{II}^n - k_I^n}{k_{II}^n} \right) \right] v_n^- + \left( \frac{k_{II}^n - k_I^n}{k_I^r} \right) v_r^-, \quad (4.29)$$

with $v_r^+$ given by (4.10).

Cases #7-10 ($g_{III}$) for Poisson’s Hypothesis

Evaluating (4.4) and (4.5) at (4.26), and using the second expression for $I_n^*$ in (4.11), we find that $g_{III}$ is given by

$$v_n^+ = -\eta_P v_n^- - \eta_P \left( \frac{k_{II}^n - k_I^n}{k_I^r} \right) v_r^-, \quad (4.30)$$

with $v_r^+$ given by (4.10).

Again, just as in the case of Stronge’s hypothesis, we see that both $g_{II}$ and $g_{III}$ reduce to $g_I$ in the limit as $k_{II}^n \to k_I^n$ and $k_{II}^r \to k_I^r$, as expected.
CHAPTER 5

GENERAL RIGID BODY MOTION

In addition to the impulsive contacts (or impacts) we have considered so far, there are clearly instances in which two rigid bodies remain in contact for a finite (i.e., non-zero) period of time. We refer to these as periods of sustained contact. There are also periods during which the two bodies are not in contact; we refer to these as periods of free flight. Here we will extend the theory of rigid body impacts with friction we have already developed to include periods of sustained contact and free flight. In anticipation of our discussion of dynamical chatter in Chapter 6, we will only expound upon planar motion here.

5.1 The Full Equations of Motion

Consider again the two arbitrary rigid bodies, \( B_1 \) and \( B_2 \), we met in Chapter 2. The general equations of motion for these bodies during a period of contact are given by (2.3):

\[
\begin{align*}
\frac{d}{dt} (m_1 V_1) & = F_1 + \mathbf{F}, \\
\frac{d}{dt} (J_1 \cdot \Omega_1) & = \Gamma_1 + \mathbf{r}_1 \times \mathbf{F}, \\
\frac{d}{dt} (m_2 V_2) & = F_2 - \mathbf{F}, \\
\frac{d}{dt} (J_2 \cdot \Omega_2) & = \Gamma_2 - \mathbf{r}_2 \times \mathbf{F}.
\end{align*}
\] (5.1)

Here we do not assume that the contact phase is instantaneous, so that we may not neglect the external forces (\( F_1 \) and \( F_2 \)) and torques (\( \Gamma_1 \) and \( \Gamma_2 \)), nor may we treat \( J_1, J_2, r_1, \) and \( r_2 \) as constant. There is therefore no way to “refine” the equations as we did in Section 2.0.2; we must deal with them in their entirety.
5.2 The Relative Velocity

Recall that the relative velocity \( \mathbf{v} \) of \( \mathcal{B}_1 \) with respect to \( \mathcal{B}_2 \) at their contact point \( A \) is given by (2.6):

\[
\mathbf{v} = \mathbf{V}_1 + \mathbf{r}_1 - \mathbf{V}_2 - \mathbf{r}_2. \tag{5.2}
\]

Taking the derivative of \( \mathbf{v} \) with respect to time \( t \), we obtain the following:

\[
\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{V}_1}{dt} + \frac{d\mathbf{r}_1}{dt} - \frac{d\mathbf{V}_2}{dt} - \frac{d\mathbf{r}_2}{dt}. \tag{5.3}
\]

During sustained contact, then, the relationship between \( \mathbf{v} \) and \( \mathbf{F} \) is not linear, as it was in the case of rigid body impact (cf. (2.51)). Fortunately, however, for certain contact configurations the linearity can be preserved, as we will soon see.

5.3 The Equations of Motion for Planar Motion

Consider contact configurations in which \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) lie in a principal plane of inertia common to both bodies, and that \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are perpendicular to this plane. It follows that, if \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{v} \) initially lie in this plane, \( \mathbf{v} \), and hence \( \mathbf{F} \), are confined to this plane throughout the contact phase. For this reason, such contact configurations are called planar.

We may therefore greatly simplify the equations of motion (5.1). We choose as basis vectors \( \mathbf{n}, \mathbf{r} \) (as defined in Section 2.2), and \( \mathbf{e}_3 := \mathbf{n} \times \mathbf{r} \). In addition, let \( \mathbf{F}_i = F_i^n \mathbf{n} + F_i^\tau \mathbf{r} \) and \( \Gamma_i = \Gamma_i \mathbf{e}_3 \) for \( i = 1, 2 \). In this way, (5.1) becomes

\[
\begin{align*}
\frac{d}{dt} \mathbf{V}_1 &= (F_1^n + F_n) \mathbf{n} + (F_1^\tau + F_\tau) \mathbf{r}, \\
\frac{1}{k_1} \frac{d}{dt} \mathbf{r}_1 &= (\Gamma_1 + r_1^n F_\tau - r_1^\tau F_n) \mathbf{e}_3, \\
\frac{d}{dt} \mathbf{V}_2 &= (F_2^n - F_n) \mathbf{n} + (F_2^\tau - F_\tau) \mathbf{r}, \\
\frac{1}{k_2} \frac{d}{dt} \mathbf{r}_2 &= (\Gamma_2 + r_2^n F_\tau - r_2^\tau F_n) \mathbf{e}_3. \tag{5.4}
\end{align*}
\]

It follows that the expression for \( \frac{d\mathbf{v}}{dt} \) during sustained planar contact, (5.3), contains all of the terms it did for planar impacts, (2.21), as well as additional terms which depend only on the configuration of the system and the external forces and torques, but not on \( \mathbf{F} \). In particular, during sustained
planar contact,

\[
\frac{d}{dt} \begin{bmatrix} v_n \\ v_\tau \end{bmatrix} = \begin{bmatrix} C_{nn} & C_{n\tau} \\ C_{n\tau} & C_{\tau\tau} \end{bmatrix} \begin{bmatrix} F_n \\ F_\tau \end{bmatrix} + \begin{bmatrix} a_n \\ a_\tau \end{bmatrix},
\]

(5.5)

where \( C_{nn}, C_{n\tau}, \) and \( C_{\tau\tau} \) are given by (2.22) as before (though now they vary in time), and

\[
a_n := \frac{F^n_1}{m_1} - \frac{\Gamma_1 r^n_1}{m_1 k_1^2} - \Omega_1 \frac{dr^n_1}{dt} - \frac{F^n_2}{m_2} + \frac{\Gamma_2 r^n_2}{m_2 k_2^2} + \Omega_2 \frac{dr^n_2}{dt},
\]

(5.6)

\[
a_\tau := \frac{F^n_1}{m_1} + \frac{\Gamma_1 r^n_1}{m_1 k_1^2} + \Omega_1 \frac{dr^n_1}{dt} - \frac{F^n_2}{m_2} - \frac{\Gamma_2 r^n_2}{m_2 k_2^2} - \Omega_2 \frac{dr^n_2}{dt}.
\]

(5.7)

Note that, during sustained contact, \( v_n \equiv 0 \) (so that \( dv_n/dt \equiv 0 \)) and \( F_n > 0 \).

5.4 Modes of Sustained Planar Contact

Just as in the case of rigid body impacts, there are two primary modes of sustained contact, namely slip and stick. Slip can in turn be either positive or negative, in accordance with the sign of \( v_\tau \).

5.4.1 Sustained Slip

When \( v_\tau \neq 0 \), we say that the system undergoes sustained slip. According to the Amontons-Coulomb law of friction, and in particular (2.10),

\[
|F_\tau| = \mu F_n \quad \text{during slip},
\]

(5.8)

where we have set \( \mu_k = \mu \).

Sustained Positive Slip

When \( v_\tau > 0 \), we say that the system is in a state of positive slip. In this case, (2.11) implies that \( F_\tau < 0 \), and from (5.8), we have

\[
F_\tau = -\mu F_n.
\]

(5.9)
Substituting (5.9) into (5.5) with \(dv_n/dt = 0\), we find that

\[
F_n = -\frac{a_n}{C_{nn} - \mu C_{n\tau}} = -\frac{a_n}{k_n^+} \quad \text{during positive slip.} \tag{5.10}
\]

From (5.9), then, we have

\[
F_{\tau} = -\mu F_n = \frac{\mu a_n}{k_n^+} \quad \text{during positive slip} \tag{5.11}
\]

(cf. Equation (2.22) of [1]).

Sustained Negative Slip

Likewise, when \(v_\tau < 0\), we say that the system is in a state of negative slip. In this case, (2.11) implies that \(F_\tau > 0\), and from (5.8), we have

\[
F_\tau = +\mu F_n. \tag{5.12}
\]

Substituting (5.12) into (5.5) with \(dv_n/dt = 0\), we find that

\[
F_n = -\frac{a_n}{C_{nn} + \mu C_{n\tau}} = -\frac{a_n}{k_n^-} \quad \text{during negative slip.} \tag{5.13}
\]

From (5.12), then, we have

\[
F_{\tau} = +\mu F_n = -\frac{\mu a_n}{k_n^-} \quad \text{during negative slip} \tag{5.14}
\]

(cf. Equation (2.23) of [1]).

5.4.2 Sustained Stick

When \(v_\tau \equiv 0\) (and hence \(dv_\tau/dt \equiv 0\), we say that the system is in a state of sustained stick. According to the Amontons-Coulomb law of friction, and in particular (2.9),

\[
|F_\tau| < \mu F_n \quad \text{during stick,} \tag{5.15}
\]

where we have set \(\mu_s = \mu\). Setting \(dv_\tau/dt = 0\) in (5.5), we find that

\[
F_\tau = -(a_\tau + C_{n\tau} F_n)/C_{\tau\tau} \tag{5.16}
\]

during sustained stick.
5.5 Transitions between Planar Contact Modes

Clearly, a system of two rigid bodies in contact will either start in slip or stick. We would like to predict how the state of the system evolves in time. Unless there is spontaneous lift-off (e.g., when \( a_n > 0 \)), it is tempting to apply the same conditions developed in Section 2.2.3, with \( \bar{\mu} \) replaced by

\[
\mu^* := \frac{F_\tau}{F_n} \bigg|_{\text{stick}} = \frac{C_{n\tau} a_n - C_{n\tau} a_\tau}{C_{n\tau} a_\tau - C_{\tau\tau} a_n}.
\]  

(5.17)

That is, we might be tempted to assume the following:

- If the system starts in stick, it will remain in stick as long as \( |\mu^*| \leq \mu \). A transition to positive (negative) slip will occur precisely when \( \mu^* < -\mu \) (\( \mu^* > \mu \)).

- If the system starts in negative slip, it will remain in negative slip until \( v_\tau \) vanishes, at which point there will be a transition to positive (negative) slip if \( \mu^* < -\mu \) (\( \mu^* > \mu \)) or sustained stick if \( |\mu^*| \leq \mu \).

However, the situation is not so simple. It turns out that, when a system stops slipping, there is an additional possibility allowed by our rigid body model: a phenomenon called reverse chatter. We shall now determine under what conditions reverse chatter can occur.
CHAPTER 6
DYNAMICAL CHATTER

6.1 The Example of the Bouncing Ball

Consider the familiar game of throwing a ball vertically against the ground, and watching it bounce up and down repeatedly until it eventually comes to rest. As a reasonable approximation, we may take the surface of the earth to be perfectly flat, and its gravitational field $g$ uniform. Here we model the ball as a rigid sphere, and the earth as a rigid half-space, so that all collisions between the ball and the ground are collinear, and their relative velocity is simply that of the ball.

Let $y(t)$ be the vertical displacement between the ball and the ground. During periods of free flight, the motion of the ball is governed by the equation $\ddot{y}(t) = -g$ (where we have switched from Leibniz’s to Newton’s notation for time-derivatives), which has the general solution

$$y(t_0 + t) = y(t_0) + \dot{y}(t_0)t - \frac{1}{2}gt^2. \tag{6.1}$$

When impact occurs, we apply (4.1), so that the velocity jumps instantaneously from $\dot{y}$ to $-\eta \dot{y}$, where $\eta \in [0, 1]$ is the coefficient of restitution, and another period of free flight ensues.

If the ball is thrown from an initial height of $h$ with vertical velocity $v_0$, it will hit the ground after a period of time $T_0$ given by

$$T_0 = \frac{v_0 + \sqrt{v_0^2 + 2gh}}{g}, \tag{6.2}$$

which is obtained by setting (6.1) equal to zero with $y(t_0) = h$ and $\dot{y}(t_0) = v_0$, 

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and solving for $t$. At that moment, the velocity is

$$\dot{y}(t_0 + T_0) = v_0 - gT_0 = -\sqrt{v_0^2 + 2gh}. \tag{6.3}$$

The velocity after the first impact is then simply

$$v_1 = -\eta \dot{y}(t_0 + T_0) = \eta \sqrt{v_0^2 + 2gh}. \tag{6.4}$$

At this point, a new period of free flight begins, whose duration is

$$T_1 = \frac{2v_1}{g} = \frac{2\eta \sqrt{v_0^2 + 2gh}}{g}, \tag{6.5}$$

which is obtained by setting (6.1) equal to zero with $y(t_0) = 0$ and $\dot{y}(t_0) = v_1 = \eta \sqrt{v_0^2 + 2gh}$, and solving for $t$. Indeed, the velocity of the ball after the $k$th impact is

$$v_k = \eta^k \sqrt{v_0^2 + 2gh}, \tag{6.6}$$

and the duration of the subsequent free flight phase

$$T_k = \frac{2v_k}{g} = \frac{2\eta^k \sqrt{v_0^2 + 2gh}}{g}, \tag{6.7}$$

a claim which can easily be proven by induction on $k$. To wit, suppose that (6.6) holds for some positive integer $k$. The velocity of the ball after the following free flight phase is simply $-v_k$, and that after the $(k + 1)$th impact given by

$$v_{k+1} = (-\eta)(-v_k) = \eta v_k = \eta^{k+1} \sqrt{v_0^2 + 2gh}. \tag{6.8}$$

We have thus shown that (6.6) holds for $k = 1$, and that if it holds for some positive integer $k$, it holds for $k + 1$ as well. By the principle of mathematical induction, (6.6) holds for all positive integers $k$.

The total amount of time $T$ it takes for the ball to come to rest is therefore given by

$$T - T_0 = \sum_{k=1}^{\infty} T_k = \frac{2\sqrt{v_0^2 + 2gh}}{g} \left( \sum_{k=0}^{\infty} \eta^k - 1 \right). \tag{6.9}$$

Now the sum in (6.9) converges to $1/(1 - \eta)$ provided $|\eta| < 1$, and we have already imposed the condition that $\eta \in [0, 1]$. If $\eta = 1$, the sum does not converge: $T$ is infinite, and the ball continues to bounce forever. This is what
we would expect, since \( \eta = 1 \) represents impacts which are perfectly elastic, and which therefore do not dissipate energy. If, however, \( \eta \in [0, 1) \), we have that
\[
T - T_0 = \frac{2\eta}{1 - \eta} \frac{\sqrt{v_0^2 + 2gh}}{g}.
\]
Thus, the ball comes to rest after an infinite number of impacts in finite time. Furthermore, if we fix \( T \), (6.10) constitutes a single equation for both \( h \) and \( v_0 \), so that we may solve for one of these quantities in terms of the other. In particular,
\[
h = \frac{1}{2g} \left[ \left( \frac{1 - \eta}{1 + \eta} \right)^2 (gT - v_0)^2 - v_0^2 \right].
\]
That is, given any value of \( v_0 \), (6.11) gives the value of \( h \) for which the ball comes to rest after time \( T \). Hence, there are an infinite number of initial conditions \((h, v_0)\) which make the ball come to rest after a given time \( T \).

6.2 Dynamical Chatter Defined

In the previous example, the ball-earth system exhibits a phenomenon called dynamical chatter (or simply chatter). In general, we say that any system of two rigid bodies exhibits chatter if said bodies undergo a sequence of infinitely many impacts in finite time. More precisely, chatter occurs if the infinite sequence of times \( \{t_k\} \) at which impacts occur has a finite limit point \( t_\infty \). If the limit point \( t_\infty \) is reached as \( t \) increases (i.e., \( t_\infty > t_k \ \forall \ k \)), as it does in the example of the bouncing ball, the system is said to exhibit forward chatter. If, on the other hand, \( t_\infty \) is reached as \( t \) decreases (i.e., \( t_\infty < t_k \ \forall \ k \)), the system is said to exhibit reverse chatter. In Section 6.3, we will examine how the Amontons-Coulomb law of friction for rigid bodies can induce reverse chatter.

6.3 Reverse Chatter Induced by Friction

In Section 6.1, the ball had only a single degree of freedom, and therefore came to rest after time \( T \). However, in general it could have been thrown
with some initial horizontal (tangential) velocity component, in which case it might have continued to move in the tangential direction (e.g., by sliding on the ground) after it had stopped bouncing. That is, our rigid body impact model allows forward chatter to precede a period of sustained contact.

Might this model also allow reverse chatter to follow a period of sustained contact? Could, for example, a rigid body in sustained contact with a flat surface begin to bounce with ever increasing amplitude under the assumptions of our rigid body impact model? Surprisingly, the answer is yes! We will proceed to derive the conditions under which this can happen.

6.3.1 Conditions for Reverse Chatter

Suppose we are given a system of two rigid bodies in sustained contact, so that the governing equations of motion are given by (5.5), which we reproduce here:

\[
\begin{bmatrix}
\frac{dv_n}{dt} \\
\frac{dv_\tau}{dt}
\end{bmatrix} =
\begin{bmatrix}
C_{nn} & C_{n\tau} \\
C_{n\tau} & C_{\tau\tau}
\end{bmatrix}
\begin{bmatrix}
F_n \\
F_\tau
\end{bmatrix} +
\begin{bmatrix}
a_n \\
a_\tau
\end{bmatrix}.
\] (6.12)

We are interested in the possible occurrence of an “impact” even though the normal velocity \(v_n\) is zero. To see whether this is possible, we will give our rigid bodies a small negative relative normal velocity \(v_n^-\) (so that \(|v_n^-| \ll 1\)) at some time \(t^-\) (at which time \(v_\tau\) is equal to some value \(v_\tau^-\)), analyze the dynamics of the system, and then consider the limit as \(v_n^- \to 0\).

Immediately after the impact, at time \(t^+ = t^-\), the bodies have relative velocities \((v_n^+, v_\tau^+)\) given by

\[
(v_n^+, v_\tau^+) = g(v_n^-, v_\tau^-),
\] (6.13)

where \(g\) is either \(g_I\), \(g_{II}\), or \(g_{III}\), depending on the Case, and \(v_n^+\) is positive.

At this point a period of free flight ensues, during which \(F_n \equiv F_\tau \equiv 0\), and

\[
\begin{bmatrix}
\frac{dv_n}{dt} \\
\frac{dv_\tau}{dt}
\end{bmatrix} =
\begin{bmatrix}
a_n(t) \\
a_\tau(t)
\end{bmatrix},
\] (6.14)

where we have reminded ourselves that \(a_n\) and \(a_\tau\) are functions of configuration, and hence time. We now expand \(a_n(t)\) and \(a_\tau(t)\) about \(t = t^+\):

\[
a_n(t) = a_n(t^+) + O(t - t^+) = a_n^+ + O(t - t^+),
\] (6.15)
\[ a_r(t) = a_r(t^+) + O(t - t^+) = a_r^+ + O(t - t^), \]  
\hspace{1cm} (6.16)

where we have made the definitions

\[ a_r^+ := a_r(t^+), \quad a_r^+ := a_r(t^+). \]  
\hspace{1cm} (6.17)

Substituting (6.15) and (6.16) into (6.14) and integrating, it follows that, during the free flight phase,

\[ v_n(t) = v_n^+ + a_n^+(t - t^+) + O(t - t^+)^2, \]  
\hspace{1cm} (6.18)

\[ v_r(t) = v_r^+ + a_r^+(t - t^+) + O(t - t^+)^2, \]  
\hspace{1cm} (6.19)

and

\[ y(t) = v_n^+(t - t^+) + \frac{1}{2} a_n^+(t - t^+)^2 + O(t - t^+)^3, \]  
\hspace{1cm} (6.20)

where \( y \) is the normal displacement of the two bodies. In particular, if \( a_r^+ < 0 \), there is a time \( t' \) at which the rigid bodies come into contact again, so that

\[ y(t') = v_n^+(t' - t^+) + \frac{1}{2} a_n^+(t' - t^+)^2 + O(t' - t^+)^3 = 0. \]  
\hspace{1cm} (6.21)

Solving for the nonzero root, \( t' - t^+ \), we find that

\[ t' - t^+ = -\frac{2v_n^+}{a_n^+} + O(t' - t^+)^2 = O(v_n^+). \]  
\hspace{1cm} (6.22)

Substituting this result back into (6.18) and (6.19), we obtain

\[ v'_n := v_n(t') = v_n^+ - 2v_n^+ + O(v_n^+)^2 = -v_n^+ + O(v_n^+)^2, \]  
\hspace{1cm} (6.23)

\[ v'_r := v_r(t') = v_r^+ - \frac{2a_r^+}{a_n^+} v_n^+ + O(v_n^+)^2. \]  
\hspace{1cm} (6.24)

At this point another impact will occur, and then another, and so on.

Now consider the ratio \( e \) of the normal velocity just before the second impact, \( v'_n \), to that just before the first impact, \( v_n^- \), in the limit as the latter approaches zero:

\[ e := \lim_{v_n^- \to 0} \frac{v'_n}{v_n^-} = \lim_{v_n^- \to 0} \frac{-v_n^+ + O(v_n^+)^2}{v_n^-} \]  
\hspace{1cm} (6.25)

(note that \( e \) is positive by definition). For example, in Cases #1-4, \( \mathbf{g} = \mathbf{g}_1 \).
so that \( v_n^+ = -\eta v_n^- \) (where \( \eta \) is the appropriate coefficient), and

\[
e = \eta + \lim_{v_n^- \to 0} O (v_n^-) = \eta.
\] (6.26)

Now a necessary condition for reverse chatter is that \( e \) be greater than 1 for this and all subsequent iterations of impact followed by free flight. Since \( \eta \in [0, 1] \), it follows that reverse chatter cannot occur for Cases #1-4. The same conclusion holds in all Cases with Newton’s hypothesis.

In Cases #5-10, on the other hand, \( v_n^+ = O(|v_n^-|) \), and in order to guarantee that \( v_n^- \ll 1 \), we take \( |v_\tau^-| \ll 1 \). Neglecting \( O(v_n^+)^2 \) terms, then, we have

\[
e = - \lim_{v_n^- \to 0} \frac{v_n^+}{v_n^-}.
\] (6.27)

In these Cases, therefore, \( e \) depends on the following quantity:

\[
\rho^- := \lim_{v_n^- \to 0} \frac{v_\tau^-}{v_n^-},
\] (6.28)

that is, the initial ratio of the tangential and normal velocities (note that \( \rho^- \) is also positive by definition). Thus, in principle we can find conditions on \( \rho^- \) under which \( e > 1 \). To extend these conditions to all iterations, we will consider the same ratio just before the following impact:

\[
\rho' := \lim_{v_n^- \to 0} \frac{v_\tau'}{v_n'},
\] (6.29)

Notice that in Cases #5, #7, and #9 (transitions from slip to sustained stick), \( v_n^+ = 0 \), and the above reduces to

\[
\rho' = \frac{2a_n^+}{a_n^+}.
\] (6.30)

Regardless of the value of \( \rho^- \). That is, in these cases, the ratio of \( v_\tau \) to \( v_n \) is \( 2a_n^+ / a_n^+ \) after every impact subsequent to the first. Thus, we may guarantee that reverse chatter is possible by finding the conditions on \( \rho^- \) under which \( e > 1 \) and then imposing those conditions on \( 2a_n^+ / a_n^+ \).

Thus far, we have established that in order for reverse chatter to be possible, the system should start in (negative) slip, and subsequently stick. We therefore have that \( k_n^I = k_n^-, k_n^{II} = k_n^0 \), and, since \( \bar{\mu} < \mu \), \( k_n^- > 0 \). Here we
will not consider the possibility of jamming and assume that \( k_n^- > 0 \) (as it turns out, the same results are obtained when \( k_n^- < 0 \); see [1]). To go any further, however, we must choose an impact termination condition.

Reverse Chatter with Stronge’s Hypothesis

Following Nordmark et al. in [1], we note that, for Cases #5 and #9, \( g = g_{II} \), \( \rho^- \geq k^-_r/k_n^- \), and we find that

\[
e^2 = \left(1 - \frac{k_n^0}{k_n^-}\right) \left(\frac{k_n^-}{k_r^-} \rho^--1\right)^2 + \eta_S^2 \frac{k_n^0}{k_n^-}.
\] (6.31)

Likewise, for Case #7, \( g = g_{III} \), \( \rho^- < k^-_r/k_n^- \), and we find that

\[
e^2 = \eta_S^2 \left[ \left(1 - \frac{k_n^0}{k_n^-}\right) \left(\frac{k_n^-}{k_r^-} \rho^-1\right)^2 + \frac{k_n^0}{k_n^-} \right].
\] (6.32)

Now both (6.31) and (6.32) involve the following quantity:

\[
1 - \frac{k_n^0}{k_n^-} = \frac{k_n^- - k_n^0}{k_n^-} = C_{nr} \left(\frac{k_r^-}{k_n^- C_{rr}}\right),
\] (6.33)

the sign of which is the same as that of \( C_{nr} \). Let us assume for the present that \( C_{nr} < 0 \) (we will return to the case in which \( C_{nr} > 0 \) later; note that when \( C_{nr} = 0 \), both (6.31) and (6.32) reduce to \( e^2 = \eta_S^2 \), and reverse chatter is not possible). Setting (6.31) greater than 1, we obtain

\[
\frac{k_r^-}{k_n^-} \left(1 - \sqrt{\frac{k_n^- - \eta_S^2 k_n^0}{k_n^- - k_n^0}}\right) < \rho^- < \frac{k_r^-}{k_n^-} \left(1 + \sqrt{\frac{k_n^- - \eta_S^2 k_n^0}{k_n^- - k_n^0}}\right).
\] (6.34)

But the left hand inequality is already taken care of by the fact that, in Cases #5 and #9, \( \rho^- \geq k^-_r/k_n^- \), so that only the right hand inequality is non-trivial. Likewise, setting (6.32) greater than 1, we obtain

\[
\frac{k_r^-}{k_n^-} \left(1 - \frac{1}{\eta_S} \sqrt{\frac{k_n^- - \eta_S^2 k_n^0}{k_n^- - k_n^0}}\right) < \rho^- < \frac{k_r^-}{k_n^-} \left(1 + \frac{1}{\eta_S} \sqrt{\frac{k_n^- - \eta_S^2 k_n^0}{k_n^- - k_n^0}}\right).
\] (6.35)

But the right hand inequality is already taken care of by the fact that, in Case #7, \( \rho^- < k^-_r/k_n^- \). Hence, only the left hand inequality is non-trivial.
Notice that in both cases, the quantity
\[ k_n^- - \eta_S^2 k_n^0 \] (6.36)
governs the sign of the radicand. In order for the radicand to be non-negative, and thus to yield a real-valued result, the above quantity must be non-positive. This results in the additional requirement that
\[ \eta_S^2 \geq k_n^- / k_n^0. \] (6.37)

Putting everything together, we see that reverse chatter can occur at a transition from slip to sustained stick when \( C_n \tau < 0 \) if (i) \( a_n^+ < 0 \), (ii) \( \eta_S^2 \geq k_n^- / k_n^0 \), and
\[ (iii) \quad \frac{k_n^-}{k_n^0} \left( 1 - \frac{1}{\eta_S} \sqrt{\frac{k_n^- - \eta_S^2 k_n^0}{k_n^- - k_n^0}} \right) < \frac{2a_n^+}{a_n^+ k_n^+} < \frac{k_n^-}{k_n^0} \left( 1 + \sqrt{\frac{k_n^- - \eta_S^2 k_n^0}{k_n^- - k_n^0}} \right), \] (6.38)
which we can rewrite slightly more succinctly as (iii) \( -1 < \gamma_S^- < \eta_S \), where
\[ \gamma_S^\pm := \eta_S \left( \frac{2a_n^+ k_n^\pm}{a_n^+ k_n^\pm} - 1 \right) \sqrt{\frac{k_n^\pm - k_n^0}{k_n^\pm - \eta_S^2 k_n^0}}. \] (6.39)

To obtain the equivalent conditions when \( C_n \tau > 0 \), we will employ a clever trick. Notice that we can switch the sign of \( C_n \tau \) by reversing the direction of the tangential unit vector \( \tau \). Furthermore,
\[ k_n^- \bigg|_{C_n \tau < 0} = k_n^+ \bigg|_{C_n \tau > 0}, \] (6.40)
\[ k_n^- \bigg|_{C_n \tau < 0} = -k_n^+ \bigg|_{C_n \tau > 0}. \] (6.41)
Thus, we obtain the conditions for \( C_n \tau > 0 \) by applying the following transformation: \( a_\tau^+ \rightarrow -a_\tau^+ \), \( k_\tau^- \rightarrow k_\tau^+ \), and \( k_\tau^- \rightarrow -k_\tau^+ \). In particular, the only changes are that (ii) becomes \( \eta_S^2 \geq k_n^+ / k_n^0 \), and (iii) becomes \( -1 < \gamma_S^\pm < \eta_S \) (this is equivalent to Equations (4.22) and (4.23) of [1]). In general, then, we may write (ii) as \( \eta_S^2 \geq k_n^\pm / k_n^0 \) and (iii) as \( -1 < \gamma_S^\pm < \eta_S \), where “+” is chosen when \( C_n \tau > 0 \) and “−” when \( C_n \tau < 0 \).
Reverse Chatter with Poisson’s Hypothesis

Now we will follow the same procedure as before, but with Poisson’s impact termination condition. For Cases #5 and #9, \( g = g_{II} \), \( \rho^- \geq k^-_\tau / k^-_n \), and we find that

\[
e = \frac{k^0_n}{k^-_n} \left[ \eta_P + \left( \frac{k^0_n - k^-_n}{k^0_n} \right) \right] - \left( \frac{k^0_n - k^-_n}{k^-_\tau} \right) \rho^-.
\]

Likewise, for Case #7, \( g = g_{III} \), \( \rho^- < k^-_\tau / k^-_n \), and we find that

\[
e = \eta_P + \eta_P \left( \frac{k^0_n - k^-_n}{k^-_\tau} \right) \rho^-.
\]

Now both (6.42) and (6.43) involve the following quantity:

\[
k^0_n - k^-_n = -C_{n\tau} \left( \frac{k^-_\tau}{C_{\tau\tau}} \right),
\]

the sign of which is the opposite of that of \( C_{n\tau} \). Again we assume, for the present, that \( C_{n\tau} < 0 \). Setting (6.42) greater than 1, we obtain

\[
\rho^- < \frac{k^-_\tau}{k^-_n} \left( 1 + \frac{k^-_n - \eta_P k^0_n}{k^-_n - k^0_n} \right).
\]

Likewise, setting (6.43) greater than 1, we obtain

\[
\rho^- > \frac{k^-_\tau}{k^-_n} \left( 1 - \frac{1}{\eta_P} \frac{k^-_n - \eta_P k^0_n}{k^-_n - k^0_n} \right).
\]

In order to guarantee that the right hand sides of (6.45) and (6.46) have the appropriate relation to \( k^-_\tau / k^-_n \) (i.e., greater than or equal to and less than or equal to \( k^-_\tau / k^-_n \), respectively), we also require that \( k^-_n - \eta_P k^0_n \geq 0 \), or equivalently,

\[
\eta_P \geq \frac{k^-_n}{k^0_n}.
\]

Putting everything together, we see that reverse chatter can occur at a transition from slip to sustained stick when \( C_{n\tau} < 0 \) if (i) \( a^+_n < 0 \), (ii) \( \eta_P \geq k^-_n / k^0_n \), and

\[
(iii) \frac{k^-_\tau}{k^-_n} \left( 1 - \frac{1}{\eta_P} \frac{k^-_n - \eta_P k^0_n}{k^-_n - k^0_n} \right) < 2a^+_\tau < \frac{k^-_\tau}{k^-_n} \left( 1 + \frac{k^-_n - \eta_P k^0_n}{k^-_n - k^0_n} \right),
\]
which we can rewrite slightly more succinctly as (iii) $-1 < \gamma_- < \eta_P$, where

$$\gamma_- := \eta_P \left( \frac{2 a_+^+ k_+^+}{a_n^+ k_\tau^+} - 1 \right) \left( \frac{k_n^+ - k_n^0}{k_n^+ - \eta_P k_n^0} \right). \quad (6.49)$$

We obtain the equivalent conditions for $C_{n\tau} > 0$ by applying the transformation $a_+^+ \rightarrow -a_+^-$, $k_n^- \rightarrow k_n^+$, and $k_\tau^- \rightarrow -k_\tau^+$. The only changes are that (ii) becomes $\eta_P \geq k_n^+ / k_n^0$, and (iii) becomes $-1 < \gamma_+^P < \eta_P$. In general, then, we may write (ii) as $\eta_P \geq k_n^+ / k_n^0$ and (iii) as $-1 < \gamma_+^P < \eta_P$, where “+” is chosen when $C_{n\tau} > 0$ and “−” when $C_{n\tau} < 0$.

### 6.3.2 Reverse Chatter for $\eta_S = \eta_P = 1$

Henceforth, without loss of generality, we will restrict our attention to cases in which $C_{n\tau} > 0$. When $\eta_S = \eta_P = 1$, we find that the conditions for reverse chatter with Stronge’s impact termination condition and Poisson’s impact termination condition coincide. In particular, the conditions reduce to (i) $a_n^+ < 0$, (ii) $k_n^0 \geq k_n^+$ (which is equivalent to $k_\tau^+ \leq 0$), and

$$\text{(iii) } \frac{k_+^+}{k_n^+} < \frac{a_+^+}{a_n^+} < 0. \quad (6.50)$$

We can simplify this further by noting that, in order to satisfy the first inequality, $k_+^+$ must be strictly negative, and $k_n^+$ must be strictly positive. Thus, the above can be rewritten as

$$\text{(iii) } a_+^+ k_n^+ - a_n^+ k_\tau^+ < 0 < a_+^+, \quad (6.51)$$

in agreement with Equation (4.24) of [1].

### 6.4 Summary

To summarize, we have seen that for planar impacts in which sticking occurs, the ratio $e$ can be greater than one, even in the limit as both $v_n^-$ and $v_\tau^-$ approach zero. How might we achieve this situation in practice? Clearly, one possibility is sustained rigid body sliding where the tangential velocity vanishes, for at that instant both $v_n$ and $v_\tau$ are zero. We conclude that,
during planar rigid body sliding, when sliding halts and we would ordinarily assume that sticking occurs, under the given conditions both sticking and reverse chatter are possible. Such conditions exist for both Stronge’s impact termination condition and Poisson’s (and these are identical in the case of perfectly elastic impacts), but not for Newton’s. It can be shown (see [1] for the details) that halted sliding is the only instance of sustained rigid body contact in which reverse chatter is a possibility.

Within the rigid body paradigm, there is no way to decide what happens, and we are faced with a paradox. Worse still, even if we could somehow surmise that chatter occurs instead of sustained stick, what would the normal displacement/velocity be after a given time $T$? There are infinitely many possibilities, according to the argument at the end of Section 6.1! Furthermore, attempts to pinpoint the particular solution by considering a compliant contact model in the rigid body limit of zero compliance have been unsuccessful: in the rigid body limit, all of the infinitely many solutions collapse together (for details, see [1]). In short, under the rigid body model and the aforementioned conditions, it would seem that there is no unique solution to the governing initial value problem.
One may be tempted to think reverse chatter it is simply an artifact of the rigid body model, and is therefore not actually physically realizable. However, we will now demonstrate that chatter-like behavior occurs for a compliant model wherein the bodies are repelled by a strong conservative force when in contact, indicating that chatter is not peculiar to the rigid body paradigm. We will then investigate numerically what becomes of chatter-like behavior in the presence of external control, an easily achievable scenario in which one could test reverse chatter experimentally.

7.1 Chatter-Like Behavior in a Compliant Model

Following Nordmark et al. in [1], consider a compliant contact model—with equations of motion given by (5.5)—in which \(y\), the normal displacement of the two bodies, is allowed to be negative, and the contact force is modeled as follows. The normal component, \(F_n\), is given by the following equation:

\[
F_n = -h(y/\varepsilon^2),
\]

(7.1)

where \(h\) is a continuous function satisfying \(h(x) = 0\) for \(x \geq 0\), \(h'(x) > 0\) for \(x < 0\), and \(h(x) \to -\infty\) as \(x \to -\infty\), and \(\varepsilon\) is a small parameter which governs the stiffness. Since this force is conservative, and contact is assumed to terminate when \(y = 0\), the work done by this force is the same during both compression and restitution. This compliant model is therefore analogous to the rigid body model when \(\eta_S = 1\).\(^1\) The tangential component, \(F_t\), obeys

\(^1\)Note, however, that this model is not analogous to the rigid body model when \(\eta_P = 1\).
Coulomb’s law of friction:

\[
F_r = \begin{cases} 
0, & y \geq 0, \\
-\mu F_n, & \text{in positive slip}, \\
+\mu F_n, & \text{in negative slip}, \\
-(a_r + C_{nt} F_n)/C_{rt}, & \text{in sustained stick}.
\end{cases} 
\]  

(7.2)

It can be shown (see Sections 5.1-3 of [1]) that for such a model, there are extra conditions for chatter-like behavior in addition to those established in Chapter 6. A system is said to exhibit chatter-like behavior (or simply chatter) if, when the tangential velocity vanishes during a period of sustained slip, the system undergoes a finite number of impacts, wherein the normal velocity just after subsequent impacts increases linearly (see Figures 7.2-7.5). It turns out that this motion is due to an instability in the normal dynamics about its equilibrium state in stick, as shown in Figure 7.5 (see [1] for more details). The extra conditions for chatter-like behavior are

\[
k^+_n > 0 \quad \text{and} \quad k^+_n/k^0_n < 1/2
\]  

(7.3)

when coming from positive slip (if \(h(x) = x\) for \(x < 0\)), and

\[
a^+_n k^-_n - a^+_n k^+_n > 0
\]  

(7.4)

when coming from negative slip, regardless of the form of \(h\).

### 7.1.1 A Concrete Example

As an example, let \(\mathcal{B}_1\) be a rigid rod of mass \(m_1 = 1\) and length \(\ell = 2\) (so that \(k_1 = \ell/\sqrt{12} = 1/\sqrt{3}\)), let \(\mathcal{B}_2\) be a stationary half-space (so that \(m = m_1 = 1\)), and let the coefficient of friction between them be \(\mu = 0.9\). Suppose further that \(\mathcal{B}_1\) and \(\mathcal{B}_2\) undergo planar motion. Let their contact point \(A\) (at the end of the rod) have tangential and normal coordinates \((x, y)\), and let \(\theta\) be the angle the rod makes with the horizontal, so that \(r^+_1 = \cos \theta\) and \(r^+_1 = -\sin \theta\) (see Figure 7.1). From (2.22), we have that

\(^2\)This example is essentially identical to the one given in [1].
Figure 7.1: Example system which exhibits chatter-like behavior, along with a graphical illustration of the notation used for various quantities.

\[ C_{nn} = 1 + 3 \cos^2 \theta, \]
\[ C_{nt} = 3 \cos \theta \sin \theta, \]
\[ C_{\tau\tau} = 1 + 3 \sin^2 \theta. \]  

(7.5)

Furthermore, let \( \mathcal{B}_1 \) be acted upon by forces \( F_1^\tau = S_x, \) \( F_1^n = S_y, \) and torque \( \Gamma_1 = R. \) It follows from (5.6) that

\[ a_n = S_y - 3R \cos \theta + \omega^2 \sin \theta, \]
\[ a_\tau = S_x - 3R \sin \theta - \omega^2 \cos \theta, \]  

(7.6)

where \( \omega := \frac{d\theta}{dt}. \) The full equations of motion are then given by

\[ \frac{dx}{dt} = v_\tau, \quad \frac{dv_\tau}{dt} = a_\tau + C_{\tau\tau}F_\tau + C_{nt}F_n, \]
\[ \frac{dy}{dt} = v_n, \quad \frac{dv_n}{dt} = a_n + C_{nt}F_\tau + C_{nn}F_n, \]
\[ \frac{d\theta}{dt} = \omega, \quad \frac{d\omega}{dt} = 3R - 3 \sin \theta F_\tau - 3 \cos \theta F_n, \]  

(7.7)

where the first two lines of (7.7) follow from (5.5), and the last line of (7.7) follows from the second line of (5.4). We model the normal component of the contact force with the function \( h(x) = x \) for \( x < 0, \) so that (cf. (7.1))

\[ F_n = \begin{cases} 
0, & y \geq 0, \\
-y/\varepsilon^2, & y < 0,
\end{cases} \]  

(7.8)
where $\varepsilon = 10^{-5}$. As always, $F_\tau$ is given by (7.2).

For now, we restrict attention to the case when $R \equiv 0$, $S_x \equiv 0.5$ and $S_y \equiv -1$ (this can be interpreted as a gravity-like force acting on the rod, with the half-space inclined at an angle of $\arctan 0.5$ with respect to any gravitational equipotential surface). We choose initial conditions $x(0) = 0$, $y(0) = y_0$, $\theta(0) = 1.18$, $v_\tau(0) = v_0$, $v_n(0) = 0$, and $\omega(0) = 0$, where $y_0$ is chosen so as to make $dv_n/dt = 0$ initially. With $v_0 = 0.1$, substitution of (7.5) and (7.6) into (7.7) yields, for positive slip,

$$y_0 = \varepsilon^2 \left( \frac{S_y - 3R \cos \theta + \omega^2 \sin \theta}{1 + 3 \cos^2 \theta - 3\mu \sin \theta \cos \theta} \right), \quad (7.9)$$

which, for the parameters we have chosen, comes out to approximately $-2.06 \times 10^{-10}$. It can be shown that when the initial sliding halts, the various parameters and state variables satisfy the conditions for chatter-like behavior, and this is confirmed via simulation. See Figures 7.2-7.5 (the code used to generate these figures can be found in Listing B.1 of Appendix B); cf. Figures 11-13 of [1]. Furthermore, when $v_0$ is changed to $-0.1$ and $y_0$ to its equilibrium value in negative slip (which is simply (7.9) with $-\mu$ in the place of $\mu$), the conditions for chatter-like behavior are not met, and simulation (the results of which are not included here) confirms that chatter does not occur; instead, the system simply transitions to sustained stick when the sliding velocity vanishes.
Figure 7.2: Plots of the normal displacement $y$ of the contact point $A$ in Figure 7.1 versus (a) time $t$ and (b) the tangential distance $x$, in the absence of external control. Such motion is characteristic of chatter-like behavior.
Figure 7.3: Plots of (a) the angle of orientation $\theta$ of the example system in Figure 7.1 and (b) the tangential distance $x$ versus time $t$, in the absence of external control. Note that $\theta$ increases monotonically; in fact, it eventually reaches $\pi$ (not shown), meaning that the rod topples over.
Figure 7.4: Plots of (a) the tangential velocity $v_\tau$ and (b) the normal velocity $v_n$ of the contact point $A$ in Figure 7.1 versus time $t$, in the absence of external control. Such motion is characteristic of chatter-like behavior. Note the linear increase in $v_n$ after subsequent impacts.
Figure 7.5: Plot of the normal velocity $v_n$ versus the normal distance $y$ of the contact point $A$ in Figure 7.1, in the absence of external control. The dot in the middle corresponds to the equilibrium state in stick, and the dashed vertical line shows the value of $y$ below which stick is stable. If the phase plane trajectory crosses this line, it can be shown, to first order, that when it comes around, it will cross the line again at an even larger value of $v_n$, an instability resulting in ever increasing values for $v_n$ immediately after successive impacts (see [1] for more details). Such motion is characteristic of chatter-like behavior.
7.2 Chatter-Like Behavior in the Presence of Control

The previous analysis and example show that chatter-like behavior can occur under constant external forces and torques, such as gravity. However, such a scenario would be somewhat difficult to achieve experimentally. Indeed, during an experiment, one would presumably control the external forces and torques on the rod, so as not to allow the system to chatter out indefinitely. A natural question to ask, then, is whether chatter can occur under external control. In particular, what becomes of chatter when the control scheme is actively trying to keep the rod in contact with the plane? Given that varying forces are essentially constant over small time scales, it seems reasonable that chatter should start to occur, but then get drowned out by the control dynamics. In fact, this is precisely what happens. There is a twist, however, as we will see.

7.2.1 Hybrid Position/Force Control

Let us simulate the system from Section 7.1.1, except instead of keeping all the external loads constant, we will vary $S_y$ and $R$ according to the hybrid position/force control scheme developed by Tarn et al. in [32] and [34] (we leave $S_x$ constant).

We distinguish between two distinct phases of this scheme: position control and force control. During position control, the loads attempt to make the differences between the generalized coordinates and their “desired” values behave like damped harmonic oscillators, so that the generalized coordinates approach their desired values in the limit as $t \to \infty$. In our example, the desired values of $y$ and $\theta$ are simply their initial values, though they could in general be functions of time. During force control, the loads attempt to make the differences between the generalized constraint impulses and their “desired” values behave like damped harmonic oscillators, so that the generalized constraint forces approach their desired values in the limit as $t \to \infty$.

In our example, the only constraint force is $F_n$, and the desired value of $f := -F_n$ (the reason for the minus sign will become clear shortly) is that which corresponds to the desired value of $y$. Denoting the desired values of $y$ and $\theta$ by $y_d$ and $\theta_d$, respectively, we see from (7.8) that the desired value
of $f$ is given by
\[ f_d := y_d / \varepsilon^2. \tag{7.10} \]

In an experiment, of course, the force-displacement relationship between the two bodies would be unknown, and the values of $y_d$ and $f_d$ would have to be guessed and measured, respectively. A transition between position control and force control (or vice versa) is triggered when the measured value of the constraint force crosses its “minimum detectible value” $F_{sw}$, a parameter of the particular force sensor used in the experiment: when $F_n < F_{sw}$ we use position control, and when $F_n > F_{sw}$ we use force control.

Perfect Control

Returning to (7.7), we see that if we make
\[ S_y = c_y + 3R \cos \theta - \omega^2 \sin \theta - C_{n\tau} F_{\tau} - C_{mm} F_n, \tag{7.11} \]
then the equation for $y$ reduces to $\ddot{y} = c_y$. Similarly, if we make
\[ R = \frac{1}{3} c \theta + \sin \theta F_{\tau} + \cos \theta F_n, \tag{7.12} \]
then the equation for $\theta$ reduces to $\ddot{\theta} = c_\theta$. We shall refer to $c_y$ and $c_\theta$ as the “commanded accelerations” for $y$ and $\theta$, respectively.

**Position Control** During position control, set
\[ c_y = \ddot{y}_d + k_v^y (\dot{y}_d - v_n) + k_p^y (y_d - y), \tag{7.13} \]
where $k_v^y$ and $k_p^y$ are positive constants called gain parameters.\(^3\) In this way, the equation for $y$ becomes
\[ (\ddot{y}_d - \ddot{y}) + k_v^y (\dot{y}_d - \dot{y}) + k_p^y (y_d - y) = 0, \tag{7.14} \]
and the difference $y_d - y$ behaves like a damped harmonic oscillator with damping ratio
\[ \zeta_y = \frac{k_v^y}{2 \sqrt{k_p^y}} \tag{7.15} \]

\(^3\)The subscripts are $v$ for “velocity” and $p$ for “position.”
and, if $\zeta_y < 1$, angular frequency

$$\omega_y = \sqrt{k_p^y (1 - \zeta_y^2)}. \quad (7.16)$$

Then, in the limit as $t \to \infty$, $y$ converges to $y_d$.

In exactly the same manner, set

$$c_\theta = \ddot{\theta}_d + k_v^\theta (\dot{\theta}_d - \omega) + k_p^\theta (\theta_d - \theta), \quad (7.17)$$

where $k_v^\theta$ and $k_p^\theta$ are again positive gain parameters. In this way, the equation for $\theta$ becomes

$$(\ddot{\theta}_d - \ddot{\theta}) + k_v^\theta (\dot{\theta}_d - \dot{\theta}) + k_p^\theta (\theta_d - \theta) = 0, \quad (7.18)$$

and the difference $\theta_d - \theta$ behaves like a damped harmonic oscillator with damping ratio

$$\zeta_\theta = \frac{k_v^\theta}{2 \sqrt{k_p^\theta}} \quad (7.19)$$

and, if $\zeta_\theta < 1$, angular frequency

$$\omega_\theta = \sqrt{k_p^\theta (1 - \zeta_\theta^2)}. \quad (7.20)$$

Then, in the limit as $t \to \infty$, $\theta$ converges to $\theta_d$.

**Force Control** During force control, leave (7.17) the way it is, but change (7.13) to

$$c_y = \alpha \ddot{y} + k_s (\dot{f}_d - \dot{f}) + k_f(f_d - f) + k_I \int_{t_{sw}}^{t} (f_d - f) \, dt', \quad (7.21)$$

where $\alpha$ is a positive constant less than (but close to) one, $k_s$, $k_f$, and $k_I$ are positive gain parameters,\(^4\) $f := -F_n$ per the previous discussion, and $t_{sw}$ is the time at which the current phase of force control started. Note that we have tacitly used the value of $\ddot{y}$ to determine the value of $\ddot{y}$, an obvious fallacy. However, let us assume for now that we can do this, see where it leads, and then address the issue of causality later when we discuss the inclusion of sampling and filtering. In any case, if we could use the value of $\ddot{y}$ in the

\(^4\)The subscripts are $s$ for “surge,” $f$ for “force,” and $I$ for “impulse.”
equation for $y$, the equation would become

$$(1 - \alpha)\ddot{y} = k_s(\dot{f}_d - \dot{f}) + k_f(f_d - f) + k_I \int_{t_{sw}}^{t} (f_d - f)dt'. \quad (7.22)$$

Then, if $(1 - \alpha)$ were sufficiently small and $\ddot{y}$ remained finite, the difference between the desired and actual values of the impulse due to $f$ would behave almost like a damped harmonic oscillator. We might hope that by incorporating some form of positive acceleration feedback, the actual impulse might converge to its desired value as $t \to \infty$. In fact, this is precisely what occurs. See [32] and [34] for comparisons with experiment; we will see simulation results shortly.

Recall that until now we have not presumed to know the force-displacement relationship between the two bodies. However, if we approximate the relationship with a linear function, $\dot{f}$ is proportional to $\dot{y}$,\footnote{By defining $f$ as $-F_n$, we make the constant of proportionality between $f$ and $y$ positive. If we had defined $f$ as $F_n$, the proportionality constant would have been negative.} and (7.21) becomes

$$c_y = \alpha \ddot{y} + k_d(\dot{y}_d - \dot{y}) + k_f(f_d - f) + k_I \int_{t_{sw}}^{t} (f_d - f)dt', \quad (7.23)$$

where $k_d$ is a different gain parameter from $k_s$. In other words, under the given approximation, we can measure $\dot{y}$ instead of $\dot{f}$, and since we were already measuring $\dot{y}$ for the position control phase, we save ourselves some work. In our example, this approximation is exact (cf. (7.8)), and we have that $k_s = \epsilon^2 k_d$. In practice, however, we would not know the precise relationship between $k_s$ and $k_d$.

Inclusion of Sampling and Filtering

As noted earlier, we cannot use the current value of $\ddot{y}$ in the equation for $y$ without violating causality. However, during an actual experiment, we would presumably measure (or sample) the various state variables at discrete times, use these values in our forcing terms (7.11) and (7.12), and then filter the results before plugging them into (7.7). We can avoid violating causality, therefore, by using a recent measurement of $\ddot{y}$ instead of the current value.

To this end, let the sampling times be denoted \{tk\} for $k \in \mathbb{N}$ such that $t_1 = 0$. For simplicity, we will keep the time increment between samples constant,
so that $t_{k+1} - t_k = T_s \forall k$, where $T_s$ is the sampling period. A subscript on a variable will indicate that it has been sampled at the corresponding time (i.e., $y_k := y(t_k)$ and so on). Then, during the $k$th sampling period, the unfiltered external torque and vertical force become

$$\tilde{R}_k = \frac{1}{3} c_{\theta,k} + \sin \theta_k F_{\tau,k} + \cos \theta_k F_{n,k},$$

(7.24)

$$\tilde{S}_{y,k} = c_{y,k} + 3 \tilde{R}_k \cos \theta_k - \omega_k^2 \sin \theta_k - C_{n\tau,k} F_{\tau,k} - C_{nn,k} F_{n,k},$$

(7.25)

respectively, where, from (7.17), (7.13), and (7.23),

$$c_{\theta,k} = \ddot{\theta}_d + k_{\theta} \left( \dot{\theta}_d - \omega_k \right) + k_{p}^\theta (\theta_d - \theta_k);$$

(7.26)

$$c_{y,k} = \begin{cases} \ddot{y}_d + k_{y}^\theta (\dot{y}_d - v_{n,k}) + k_{p}^y (y_d - y_k), & \text{position control,} \\ \alpha c_{y,k-1} + \bar{k}_{d}(y_d - v_{n,k}) + k_{f} (f_d - f_k) + k_{I} s_k, & \text{force control} \end{cases},$$

(7.27)

the integral term being approximated by the Riemann sum

$$s_k = s_{k-1} + (f_d - f_k) T_s;$$

(7.28)

$$f_k = -F_{n,k};$$

from (7.5),

$$C_{nn,k} = 1 + 3 \cos^2 \theta_k,$$

(7.29)

$$C_{n\tau,k} = 3 \cos \theta_k \sin \theta_k;$$

and $F_{n,k}$ and $F_{\tau,k}$ are the measured values of $F_n$ and $F_\tau$, respectively. For the purposes of simulation, we will assume that our force sensor is completely accurate, so that

$$F_{n,k} = \begin{cases} 0, & y_k \geq 0, \\ -y_k/\varepsilon^2, & y_k < 0, \end{cases}$$

(7.30)

and

$$F_{\tau,k} = \begin{cases} 0, & y_k \geq 0, \\ -\mu F_{n,k}, & y_k \text{ measured in positive slip}, \\ +\mu F_{n,k}, & y_k \text{ measured in negative slip}, \\ -(a_{\tau,k} + C_{n\tau,k} F_{n,k})/C_{\tau\tau,k}, & y_k \text{ measured in sustained stick}, \end{cases}$$

(7.31)
where
\[
a_{x,k} = S_x - 3R_k \sin \theta_k - \omega_k^2 \cos \theta_k.
\] (7.32)

The values of \(R\) and \(S_y\) that are plugged into (7.7) are obtained by passing \(\tilde{R}\) and \(\tilde{S}_y\) through a filter. The filter can be of either the analog or the digital type. Since, in reality, the applied forces and torques are continuous in time, we opt for an analog filter here. Besides that, we already know that chatter can occur with constant loads; what we are really interested in here is whether it can occur with continuously varying loads. Furthermore, digital filters are prone to instabilities for this example, as we will see shortly. For simplicity, then, we choose a first-order, low-pass analog filter with cutoff frequency \(\omega_c\), so that
\[
\frac{dS_y}{dt} + \omega_c S_y = \omega_c \tilde{S}_{y,k},\]
(7.33)
\[
\frac{dR}{dt} + \omega_c R = \omega_c \tilde{R}_k.
\] (7.34)

This completes the incorporation of sampling and filtering. The equations of motion (7.7) are simply augmented by the above two equations, with \(\tilde{R}_k\) and \(\tilde{S}_{y,k}\) given by (7.24), (7.25), and the equations that follow.

**Example: Purely Vertical Motion** To validate our control scheme, we will consider first the special case in which the rod in Figure 7.1 is vertical (i.e., \(\theta \equiv \pi/2\)) and is forced into contact with the half-space without any horizontal velocity, with both \(S_x\) and \(R\) equal to zero. It follows that all impacts are collinear, with motion only in the \(y\)-direction. We will consider both the analog filter represented by (7.33) and a corresponding digital filter in order to demonstrate the advantages of the former over the latter. To convert an analog filter to a digital filter, we simply approximate all of the derivatives involved with finite differences. For example, if we use a backward difference approximation to \(dS_y/dt\) in (7.33), we get
\[
\frac{S_{y,k} - S_{y,k-1}}{t_k - t_{k-1}} + \omega_c S_{y,k} = \omega_c \tilde{S}_{y,k},
\] (7.35)
from which we obtain
\[
S_{y,k} = \beta S_{y,k-1} + (1 - \beta) \tilde{S}_{y,k},
\] (7.36)
where \( \beta := 1/(1 + \omega c T_s) \). That is, the external force is constant during each sampling period.

The equations of motion (7.7) during the \( k \)th sampling period reduce to

\[
\ddot{y} = S_y(t) + F_n,
\]

where \( S_y(t) \) is given either by \( S_{y,k} \) for a digital filter, or by the solution to (7.33) for an analog filter. In particular, solving (7.33) for the general solution, we find that \( S_y(t) \) takes the following form:

\[
S_y(t) = \begin{cases} 
S_{y,k}, & \text{digital filter}, \\
\tilde{S}_{y,k} + \delta e^{-\omega_c t}, & \text{analog filter},
\end{cases}
\]

(7.37)

where \( \delta \) is a constant. Let \( u := y - \varepsilon^2 \tilde{S}_{y,k} \). Then, during contact, we have

\[
\ddot{u} + u/\varepsilon^2 = \begin{cases} 
S_{y,k} - \tilde{S}_{y,k}, & \text{digital filter}, \\
\delta e^{-\omega_c t}, & \text{analog filter},
\end{cases}
\]

(7.38)

The general solution for \( u \), then, is of the following form:

\[
u(t) = \begin{cases} 
A + B \cos (t/\varepsilon + C), & \text{digital filter}, \\
D e^{-\omega_c t} + E \cos (t/\varepsilon + F), & \text{analog filter},
\end{cases}
\]

(7.39)

where \( A, B, C, D, E, \) and \( F \) are constants. It is clear from the above expressions that, with a digital filter, \( u \) (and hence \( y \)) simply oscillates about a fixed value during each sampling period. This can cause problems if the sampling period is close to an integer multiple of the period of oscillation, as shown in Figure 7.6(a), where the system is sampled near a peak each time, driving \( y \) up and away from its desired value. Because of this, the system is unable to transition to force control by the end of the simulation. On the other hand, with an analog filter, \( y \) oscillates about an exponentially decaying function during each sampling period, as shown in Figure 7.6(b). In this case, it is found that, even with the same sampling frequency, a transition to sustained force control occurs relatively quickly, and \( y \) subsequently approaches its desired value. □
Figure 7.6: Plots of the normal distance $y$ of the contact point $A$ in Figure 7.1 versus time $t$ for purely vertical motion in the presence of external control with (a) a digital filter and (b) an analog filter. The only nonzero initial condition is $y(0) = 1 \times 10^{-5}$, and the various control parameters used are the same as those used to generate Figures 7.7-7.10.

Globally, the plots seem almost identical. However, closer inspection reveals that the system is unable to transition to force control with the digital filter, even by time $t = 5$, whereas it transitions to sustained force control at $t = 2.146$ with the analog filter. The reason for this is explained in the text.
Simulation Results

Note that, in the absence of external control, the oscillations in Figure 7.6 would not be damped, and sustained contact would never be achieved. We therefore have confidence that our control scheme is working properly, and we proceed to simulate the system in Figure 7.1, with the same parameters and initial conditions as those used to generate Figures 7.2-7.5, but this time in the presence of the control scheme we have developed. In light of the preceding discussion, we will use an analog filter instead of a digital one. Since the applied normal force and torque are no longer constant, their initial conditions will be their values from the previous example, namely, \( S_y(0) = \tilde{S}_{y,1} = -1 \) and \( R(0) = \tilde{R}_1 = 0 \). The sampling and filter cutoff frequencies are chosen to coincide with those used by Tarn et al. in their experiments, so that \( f_s = 1/T_s = 10^3 \) and \( f_c = \omega_c/2\pi = 500 \). The rest of the control parameters are as follows: \( F_{sw} = 0.01 \), \( y_d = y_0 \), \( \theta_d = 1.18 \), \( k^y_p = 10^3 \), \( \zeta_y = 0.1 \), \( k^\theta_p = 10 \), \( \zeta_\theta = 0.5 \), \( \alpha = 0.85 \), \( k_d = 15 \), \( k_f = 0.06 \), \( k_I = 0.8 \) (where the last four values are taken from Tarn et al. as well). Note that, because \( y_d \) and \( \theta_d \) are constant, their time derivatives are zero. The results of the simulation are shown in Figures 7.7-7.10 (the code used to generate these figures can be found in Listing B.2 of Appendix B).
Figure 7.7: (a) Plot of the normal displacement $y$ of the contact point $A$ in Figure 7.1 versus the tangential distance $x$, in the presence of external control. (b) Closeup view of the same near the first instant the tangential velocity vanishes, showing the onset of chatter-like behavior (cf. Figure 7.2(b)).
Figure 7.8: Plot of the angle of orientation $\theta$ of the example system in Figure 7.1 versus time $t$, in the presence of external control. Notice that, in the presence of control, the angle oscillates about its desired value instead of increasing monotonically, as it does in Figure 7.3(a).
Figure 7.9: Plots of (a) the tangential velocity $v_\tau$ and (b) the normal velocity $v_n$ of the contact point $A$ in Figure 7.1 versus time $t$, in the presence of external control, near the first instant the tangential velocity vanishes, showing the onset of chatter-like behavior (cf. Figure 7.4). The almost vertical jumps correspond to impacts.
Figure 7.10: (a) Plot of the normal velocity $v_n$ versus the normal distance $y$ of the contact point $A$ in Figure 7.1, in the presence of external control, near the first instant the tangential velocity vanishes, showing the onset of chatter-like behavior (cf. Figure 7.5). (b) Plot of the applied normal force $S_y$ (solid curve) and unfiltered force $\tilde{S}_{y,k}$ (dotted curve) versus time $t$ near the aforementioned instant. $S_y$ starts near its equilibrium value at $-2$, and then changes due to the onset of chatter-like behavior. At $t = 0.073$, there is a transition from force control to position control, and the subsequent delay in $S_y$ is what causes the relatively large free-flight oscillation.
At first glance, Figure 7.7(a) does not look like chatter. In fact, the large oscillations which are apparent in Figure 7.7(a) are not chatter, but merely the motion prescribed by the control scheme. Closer inspection reveals that chatter is responsible for the loss of contact which leads to these relatively larger oscillations, as shown in Figure 7.7(b). There can be no doubt that this initial motion is chatter, as the corresponding plots of the relative velocities and phase plane confirm (see Figures 7.9 and 7.10(a); cf. Figures 7.2(b), 7.4, and 7.5). In fact, chatter occurs not once, but twice: after the first set of oscillations dies out, the rod comes back into sustained contact with the plane, and because the orientation of the rod is being controlled as well, so that it does not vary appreciably (see Figure 7.8), the state of the system is similar to what it was in the beginning, and when the sliding velocity vanishes a second time, chatter occurs again just short of $x = 1.2$.

The reader may be wondering why the first large oscillation in Figure 7.7(a) is so much larger than the oscillations in Figure 7.7(b). The reason is that, shortly before this large oscillation (at $t = 0.073$), the system is sampled in free-flight. This prompts the controller to switch from force control to position control, and the desired external force becomes appreciably less. However, due to the finite bandwidth of the filter, the applied force does not change instantaneously; see Figure 7.10(b). The rod is therefore pushed back into contact with the plane with too much force. The end result is that the rod is sent back into free flight with a much larger velocity than before.

We pause here to note the striking similarity between Figure 7.7(a) and the chalk pattern shown in Figure 7.11. In both cases, there is an initial period of contact, during which there are small-amplitude oscillations with
impacts, i.e., those shown in Figure 7.7(b), and the solid line on the left side of Figure 7.11, which contains some impacts as well. There is then a period of relatively larger oscillations with distinct impacts, i.e., the first set of oscillations visible in Figure 7.7(a) just short of $x \approx 0.3$, and the dotted line in the middle of Figure 7.11. Finally, the oscillations become smaller again and die out, corresponding to the relatively flat line in Figure 7.7(a) between $x \approx 0.3$ and just short of $x = 1.2$ and the solid line on the right side of Figure 7.11 (the second sequence of large-amplitude oscillations is not present in Figure 7.11). Previous attempts to explain such “hopping” behavior in chalk (for example, [27]) have attributed it to jamming behavior resulting from the Painlevé paradox. However, Nordmark et al. showed in [1] that there is no ambiguity due to the Painlevé paradox in this case (recall the discussion in Section 1.1.2), and in fact, jam does not occur here. In the end, only experiment can decide whether chatter is responsible for the hopping behavior shown in Figure 7.11, but it should be evident that chatter is just as likely a suspect, if not a more likely one, than the Painlevé paradox.

One might object that the hybrid control scheme outlined here, which has been employed in robotic manipulators, is not an accurate representation of the algorithm used by the human brain. In particular, one might argue that the transitions between position and force control require more sensitive and rapid measurements of the system than the autonomic nervous system is capable of. The pursuit of this debate would require further investigation.

7.3 Summary

In this chapter we have seen that chatter-like behavior can occur in a compliant contact model, even in the presence of external control which is actively trying to keep two bodies in sustained contact. Indeed, the finite bandwidth of the controller can allow chatter-like oscillations to build up, resulting in a transition from force control to position control. This then leads to a sequence of impacts not unlike those observed in chalk hopping on a blackboard. It is argued that chatter may be responsible for this and similar phenomena.
CHAPTER 8

CONCLUSION

This thesis has sought to call attention to a phenomenon called reverse chatter, first discovered in 2011 by Arne Nordmark, Harry Dankowicz, and Alan Champneys (see [1]), in which friction causes the loss of uniqueness of solutions to the problem of two rigid bodies in sustained contact. A similar mechanism causes a loss of contact between two bodies in a compliant contact model. Until now, it had only been shown that reverse chatter was possible in the rigid body model with Stronge’s impact termination condition. In Chapter 3, building upon work done by Alexander Ivanov in 1992 (see [16]), we established, for the first time to our knowledge, a completely rigorous proof that Poisson’s impact termination condition is energetically consistent. In Chapter 4, we formulated explicit impact maps using the impact termination conditions of Newton and Poisson, and in Chapter 6, we investigated what became of reverse chatter with these maps. We found that, while reverse chatter is indeed possible under Poisson’s hypothesis, it is not possible under Newton’s. In doing so, we established that, although the possibility of reverse chatter is sensitive to the impact termination condition used, it is not simply an artifact of Stronge’s hypothesis. Furthermore, we showed that the necessary conditions for reverse chatter with Poisson’s and Stronge’s hypotheses coincide in the case of perfectly elastic impacts, a somewhat reassuring result. Finally, in Chapter 7, we went on to investigate numerically what became of chatter-like behavior in the presence of a control scheme, developed by Tarn et al. in 1996, which attempts to keep the end effector of a robotic manipulator in sustained contact with its environment. We observed that, due to the finite bandwidth of the controller, chatter-like behavior is still possible, and can lead to a loss of contact. The subsequent damped oscillatory motion, involving a finite sequence of impacts, was seen to be qualitatively similar to that of chalk hopping on a blackboard. It was argued that reverse chatter may be responsible for this and similar phenom-
ena, though ultimately only experiment can decide. Fortunately, the results of this work indicate that reverse chatter occurs under easily achievable experimental conditions, setting the stage for it to be studied experimentally in the future.

Possible avenues for future research include the following. (i) As alluded to above, it should now be fairly straightforward to study reverse chatter experimentally, and to see whether it actually occurs under real-world conditions. Slight modifications to the numerical work done here can provide theoretical results to compare with experiment. (ii) It may be of interest to investigate the possibility of reverse chatter in multibody contact models, such as the time-marching approach used by Friedrich Pfeiffer and Chiostroph Glocker in [49] (which is fundamentally different from the single-impact, event-driven model used here), as the time-marching approach is often favored in industrial software. The latter may require modifications, if it turns out that reverse chatter is possible in real life but not in these models, or vice versa. (iii) If reverse chatter is observed in experiments, it will no doubt be of interest to identify or develop control schemes capable of mitigating or eliminating its effects entirely, when it is undesirable. (iv) Alternatively, there may be certain applications in which the occurrence of reverse chatter is actually desirable—perhaps in threshold sensors similar to those developed by Bryan Wilcox et al. in [50]. Such applications, if they exist, should be identified and exploited. Finally, it might be of theoretical interest to investigate the possibility of reverse chatter (v) for impact termination conditions other than those presented here (in particular, whether the necessary conditions for reverse chatter so derived coincide with Poisson and Stronge for perfectly elastic impacts, and, if so, whether this coincidence can be shown to be a general principle), (vi) between elastoplastic bodies (perhaps with the aid of finite element software), and (vii) when the coefficients of static and kinetic friction are not equal.
APPENDIX A

INDICIAL NOTATION

Given an orthonormal basis \( \{e_1, e_2, e_3\} \) for Euclidian space \( \mathbb{E}^3 \), any vector \( \mathbf{a} \) which lives in \( \mathbb{E}^3 \) can be written in component form as

\[
\mathbf{a} = a_i \, e_i, \tag{A.1}
\]

where \( a_i := \mathbf{a} \cdot e_i \) is called the component of \( \mathbf{a} \) in the direction of \( e_i \), and summation over repeated indices from 1 to 3 is implied. Similarly, any tensor \( \mathbf{T} \) of rank two which maps \( \mathbb{E}^3 \) to itself can be written in component form as

\[
\mathbf{T} = T_{ij} \, e_i \otimes e_j, \tag{A.2}
\]

where \( T_{ij} := e_i \cdot (\mathbf{T} \cdot e_j) \) is called the \( ij \)th component of \( \mathbf{T} \) in the given basis, and \( e_i \otimes e_j \) is the tensor product between \( e_i \) and \( e_j \), which is defined so that for any three vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3 \), \((\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})\).

In this *indicical notation*, the scalar product between two unit vectors in our basis is given by

\[
\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{A.3}
\]

where \( \delta_{ij} \) is the Kronecker delta, which takes the following values:

\[
\delta_{ij} := \begin{cases} 
+1, & i = j \\
0, & i \neq j 
\end{cases} \tag{A.4}
\]

It follows that the scalar product between any two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{E}^3 \) is simply

\[
\mathbf{a} \cdot \mathbf{b} = a_i \, b_j \, e_i \cdot e_j = a_i \, b_j \, \delta_{ij} = a_i \, b_i. \tag{A.5}
\]

The vector product between two unit vectors in our basis is given by

\[
\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \, e_k. \tag{A.6}
\]
Here $\epsilon_{ijk}$ is the Levi-Civita symbol, which takes the following values:

$$
\epsilon_{ijk} := \begin{cases} 
+1, & (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \\
-1, & (i, j, k) = (3, 2, 1), (2, 1, 3), (1, 3, 2) \\
0, & i = j \text{ or } j = k \text{ or } k = i 
\end{cases} \quad (A.7)
$$

It follows that the vector product between any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$ is simply

$$
\mathbf{a} \times \mathbf{b} = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} a_i b_j \mathbf{e}_k. \quad (A.8)
$$

Finally, since $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ for any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$, when a tensor $\mathbf{T}$ operates on a vector $\mathbf{a}$, the result is given by

$$
\mathbf{T} \cdot \mathbf{a} = T_{ij} a_k (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbf{e}_k = T_{ij} a_k \mathbf{e}_i (\mathbf{e}_j \cdot \mathbf{e}_k) = T_{ij} a_k \mathbf{e}_i \delta_{jk} = T_{ij} a_j \mathbf{e}_i. \quad (A.9)
$$
Listing B.1: Code used to generate Figures 7.2-7.5.

```matlab
function [T,Y] = chatter_without_control()

% [ y(1) ] [ x ]
% [ y(2) ] [ y ]
% y = [ y(3) ] = [ th ]
% [ y(4) ] [ vT ]
% [ y(5) ] [ vN ]
% [ y(6) ] [ w ]

global MU R status a b A B C LN LTstick

eps = 1e-5;
MU = 0.9;

R = 0;
Sy = -1;
Sx = 0.5;

a = @(t,y) Sx - 3*R*sin(y(3)) - y(6)^2*cos(y(3));
b = @(t,y) Sy - 3*R*cos(y(3)) + y(6)^2*sin(y(3));
A = @(t,y) 1 + 3*sin(y(3))^2;
B = @(t,y) 3*sin(y(3))*cos(y(3));
C = @(t,y) 1 + 3*cos(y(3))^2;
kNp = @(t,y) C(t,y) - MU*B(t,y);
kNm = @(t,y) C(t,y) + MU*B(t,y);
LN = @(t,y) subplus(-y(2))/eps^2;
LTstick = @(t,y) -(a(t,y)+B(t,y)*LN(t,y))/A(t,y);

tmin = 0;
tmax = 4;

th0 = 1.18;

Y0 = [0; 0; th0; 0; 0; 0];

yp = b(0,Y0)*eps^2/kNp(0,Y0);
ym = b(0,Y0)*eps^2/kNm(0,Y0);
```
\[ x_0 = 0; \]
\[ y_0 = y_p; \]
\[ w_0 = 0; \]
\[ v_{N0} = 0; \]
\[ v_{T0} = 0.1; \]

\[ Y_0 = [x_0; y_0; \theta_0; v_{T0}; v_{N0}; w_0]; \]

if \( Y_0(2) > 0 \)
\[ \text{status} = 1; \% \text{free flight} \]
elseif \( Y_0(4) > 0 \) || (\( Y_0(4) == 0 \) && \( \text{LTstick}(Y_0) < -\mu \cdot \text{LN}(Y_0) \))
\[ \text{status} = 2; \% \text{positive slip} \]
elseif \( Y_0(4) < 0 \) || (\( Y_0(4) == 0 \) && \( \text{LTstick}(Y_0) > -\mu \cdot \text{LN}(Y_0) \))
\[ \text{status} = 3; \% \text{negative slip} \]
else
\[ \text{status} = 4; \% \text{stick} \]
end

reltol = 3e-14;
abstol = 3e-14;
refinement = 30;

\[ T = []; \]
\[ Y = []; \]

while \( t_{\text{max}} - t_{\text{min}} > 0 \)

\[ \text{TSPAN} = [t_{\text{min}}, t_{\text{max}}]; \]

\[ \text{OPTIONS} = \text{odeset(‘RelTol’, reltol, ‘AbsTol’, abstol, ‘Events’, @EVENTS, ‘Refine’, refinement}); \]

\[ [Tn, Yn, ~, ~, IE] = \text{ode15s(}@f, \text{TSPAN}, Y0, \text{OPTIONS}); \]

if \( \text{status} == 4 \)
\[ \text{Yn}(:,1) = \text{repmat}(Y0(1), \text{length}(Tn), 1); \]
\[ \text{Yn}(:,4) = \text{zeros}(\text{length}(Tn), 1); \]
end

\[ T = [T; Tn]; \]
\[ Y = [Y; Yn]; \]

\[ \text{TE} = \text{Tn} \text{end}; \]
\[ \text{YE} = \text{Yn} \text{end}; \]

\[ t_{\text{min}} = \text{TE}; \]

if \( \text{TE} < \text{tmax} \)

\[ \text{for} ~ k = 1: \text{length(IE)} \]
\[ \text{eventindex} = \text{IE}(k); \]

\[ \text{switch} \text{ eventindex} \]
\[ \text{case} 1 \]
\[ \text{end} \]

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status = 1;
case 2
  if YE(4)>0 || (YE(4)==0 && LTstick(TE,YE)<=-MU*LN(TE,YE))
    status = 2;
  elseif YE(4)<0 || (YE(4)==0 && LTstick(TE,YE)>MU*LN(TE,YE))
    status = 3;
  else
    status = 4;
  end
end

Y0 = transpose(YE);

function [value, isterminal, direction] = EVENTS(t, y)
global MU status LN LTstick
switch status
  case 1
    value = [1; y(2); 1; 1; 1; 1];
    isterminal = [0; 1; 0; 0; 0; 0];
end
direction = [0; -1; 0; 0; 0];
case 2
value = [y(2); 1; 1; 1; 1; y(4)];
isternal = [1; 0; 0; 0; 0; 1];
direction = [1; 0; 0; 0; 0; -1];
case 3
value = [y(2); 1; 1; 1; y(4); 1];
isternal = [1; 0; 0; 0; 1; 0];
direction = [1; 0; 0; 0; 1; 0];
case 4
value = [y(2); 1; LTstick(t, y) + MU*LN(t, y); LTstick(t, y) - MU*LN(t, y); 1; 1; 1];
isternal = [1; 0; 1; 1; 0; 0];
direction = [1; 0; -1; 1; 0; 0];
end
end
function f = f(t, y)
global R a b A B C LN
f = [y(4);
y(5);
y(6);
a(t, y) + A(t, y)*LT(t, y) + B(t, y)*LN(t, y);
b(t, y) + B(t, y)*LT(t, y) + C(t, y)*LN(t, y);
3*R - 3*sin(y(3))*LT(t, y) - 3*cos(y(3))*LN(t, y)];
end
function LT = LT(t, y)
global status MU LN LTstick
switch status
case 1
LT = 0;
case 2
LT = -MU*LN(t, y);
case 3
LT = MU*LN(t, y);
case 4
LT = LTstick(t, y);
end
end
Listing B.2: Code used to generate Figures 7.7-7.10.

function \([t,y,te,ye,ym,c,s,sy,r,l] =\)
  chatter_with_control(fs,ff,N)

  % Written by John Sanders from 2012-2013
  %
  % Last update: 4/15/2013 @ 11:48 AM
  %
  % DESCRIPTION:
  %
  % Simulates chatter-like behavior in the presence of a
  % hybrid position/force control scheme.
  %
  % INPUTS:
  %
  % * fs - the sampling frequency (in Hz)
  % * ff - the filter cutoff frequency (in Hz)
  % * N - the refinement for postprocessing: outputs give every
  %   Nth point generated
  %
  % OUTPUTS:
  %
  % * t - nx1 matrix of time values, where n is the number of
  %   points computed
  % * y - nx8 matrix of corresponding state vector values
  % * te - ?x1 matrix of time values at which events occurred
  % * ye - ?x8 matrix of corresponding state vector values
  % * ym - nx8 matrix of 'measured' state vector values
  % * c - nx2 matrix of commanded acceleration values
  % * s - nx1 matrix of force integral values
  % * sy - nx1 matrix of unfiltered external normal force values
  % * r - nx1 matrix of unfiltered external torque values
  % * l - nx4 matrix of actual and measured tangential and
  %   normal force values
  %
  % The state vector:
  %
  % \[
  % \begin{bmatrix}
  % y(1) \\
  % y(2) \\
  % y(3) \\
  % y(4) \\
  % y(5) \\
  % y(6) \\
  % y(7) \\
  % y(8)
  % \end{bmatrix}
  % = \begin{bmatrix}
  % x \\
  % y \\
  % \theta \\
  % v_T \\
  % v_N \\
  % \omega \\
  % S_y \\
  % R
  % \end{bmatrix}
  %\]

  %%% Declare global variables %%%
  global MU fsw index status controlstatus Rtilde Sytilde wc a
  b A B C LN LTstick

  Rtilde = 0;
  Sytilde = 0;

  %%% Contact force constants %%%
eps = 1e-5; \% \epsilon
MU = 0.9; \% \mu

%%% Control constants %%%
fsw = 0.01; \% F_{sw}
Sx = 0.5; \% S_{x}
thd = 1.18; \% \theta^d
yd = -eps^2/(1+3*cos(thd)^2-2*3*MU*sin(thd)*cos(thd)); \% y^d
fd = yd/eps^2; \% f^d
zetay = 0.1; \% k_v^y/2*sqrt{k_p^y}
zetath = 0.5; \% k_v^\theta/2*sqrt{k_p^\theta}
kd = 15; \% k_d
kf = 0.06; \% k_f
ki = 0.8; \% k_I
alpha = 0.85; \% a
wc = 2*pi*ff; \% w_c

%%% Inline functions %%%
a = @(y) Sx - 3*y(8)*sin(y(3)) - y(6)^2*cos(y(3));
b = @(y) y(7) - 3*y(8)*cos(y(3)) + y(6)^2*sin(y(3));
A = @(y) 1 + 3*sin(y(3))^2;
B = @(y) 3*sin(y(3))*cos(y(3));
C = @(y) 1 + 3*cos(y(3))^2;
LN = @(y) subplus(-y(2,:))/eps^2;
LTstick = @(y) -(a(y) + B(y)*LN(y))/A(y);

%%% Check control constants for stability %%%
condition1 = kd*kf-ki*(1-alpha);
condition2 =
    kd*kf*(1-exp(-2*zetay*pi/sqrt(1-zetay^2)))-ki*(1-alpha);
if condition1<=0 || condition2<=0
    fprintf('\n Note: Tarn''s stability conditions are not met.\n');
end

%%% Initial value of the state vector and discrete variables %%%
index = 1;
x(index) = 0;
y(index) = yd;
th(index) = thd;
vT(index) = 0.1;
vN(index) = 0;
w(index) = 0;
cy(index) = 0;
cth(index) = 0;
s(index) = 0;
Rtilde(1) = 0;
Sytilde(1) = -1;

YO = [x(1); y(1); th(1); vT(1); vN(1); w(1); Sytilde(1);
    Rtilde(1)];
Ym = Y0;

%%% Check initial contact status %%%
% 1 = free flight
% 2 = positive slip
% 3 = negative slip
% 4 = sustained stick
if Y0(2)>0
    status = 1;
elseif Y0(4)>0 || (Y0(4)==0 && LTstick(Y0)<-MU*LN(Y0))
    status = 2;
elseif Y0(4)<0 || (Y0(4)==0 && LTstick(Y0)>MU*LN(Y0))
    status = 3;
else
    status = 4;
end

LNm = LN(Ym);
LTm = LT(Ym);

%%% Check initial control status %%%
% 1 = position control
% 2 = force control
controlstatus = checkcontrolstatus(Y0);

%%% Total time span for integration %%%
TMIN = 0;   % initial time
TMAX = 4;   % final time
TSTEP = 1/fs;   % time step (= 1/sampling frequency)

%%% Initial time span for integration %%%
tmin = TMIN;
tmax = TMIN + TSTEP;

%%% Integrator parameters %%%
abstol = 3e-14;   % The absolute tolerance
refinement = 40;   % The refinement

%%% Initialize output matrices %%%
T = [];
Y = [];
te = [];
ye = [];
YM = [];
CC = [];
S = [];
SY = [];
RR = [];
L = [];

%%% Open file for output %%%
fid=fopen('chatter_control_data.txt','wt');

%%% Run integration algorithm %%%
while tmax<=TMAX

while tmax-tmin>0

%%% Set the time span for the integrator %%%
TSPAN = [tmin,tmax];

fprintf(fid,'\%f  \%f  \%d  \%d\n',tmin,tmax,controlstatus,status);

%%% Set the integrator options %%%
OPTIONS = odeset('AbsTol',abstol,'Events',@EVENTS,'Refine',refinement);

%%% Run the integrator %%%
[Tn,Yn,~,~,IE] = ode15s(@f,TSPAN,Y0,OPTIONS);

for j=1:length(IE)
    fprintf(fid,'\%d\n',IE(j));
end

fprintf(fid,'\n');

%%% If sustained stick, correct \(x\) and \(vT\) values %%%
if status==4
    Yn(:,1) = Y(end,1)*ones(size(Yn(:,1)));
    Yn(:,4) = zeros(size(Yn(:,4)));
end

%%% Add new values to outputs %%%
T = [T;Tn];
Y = [Y;Yn];
YM = [YM;repmat(Ym.',length(Tn),1)];
CC = [CC;repmat([cy(index),cth(index)],length(Tn),1)];
S = [S;repmat(s(index),length(Tn),1)];
SY = [SY;repmat(Sytilde(index),length(Tn),1)];
RR = [RR;repmat(Rtilde(index),length(Tn),1)];
L = [L;LT(Yn.'),',repmat(LTm,length(Tn),1),LN(Yn.'),',repmat(LNm,length(Tn),1)];

%%% Determine the time and state at the end of integration %%%
TE = Tn(end);
YE = transpose(Yn(end,:));

%%% Set the initial time for the next integration %%%
tmin = TE;

%%% If tmax has yet to be reached, an event must have occurred %%%
if TE<tmax
    te = [te;TE];
end
ye = [ye;YE.'];

for k=1:length(IE)
    eventindex = IE(k);

    switch eventindex
        case 1
            status = 1;
            if YE(4)>abstol || (abs(YE(4))<=abstol && LTstick(YE)+MU*LN(YE)<-abstol)
                status = 2;
            elseif YE(4)<-abstol || (abs(YE(4))<=abstol && LTstick(YE)-MU*LN(YE)>abstol)
                status = 3;
            else
                status = 4;
            end
            if status ==4
                status = 2;
            end
        case 2
            if status ==4
                status = 3;
            end
        case 3
            if status==4
                status = 2;
            end
        case 4
            if status==4
                status = 3;
            end
        case 5
            if status^=1
                if LTstick(YE)+MU*LN(YE)<-abstol
                    status = 2;
                else
                    status = 4;
                end
            end
        case 6
            if status^=1
                if LTstick(YE)-MU*LN(YE)>abstol
                    status = 3;
                else

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status = 4;
end
end
end
end

Y0 = YE;
end

%%% Update discrete variables %%%
index = index + 1;
x(index) = Y0(1);
y(index) = Y0(2);
th(index) = Y0(3);
vT(index) = Y0(4);
vN(index) = Y0(5);
w(index) = Y0(6);

Ym = Y0;

LNm = LN(Ym);

switch status
    case 1
        LTm = 0;
    case 2
        LTm = -MU*LNm;
    case 3
        LTm = MU*LTm;
    case 4
        LTm = -(Sx-3*Ym(8)*sin(th(index))-w(index)^2*cos(th(index))+(3*cos(th(index))*sin(th(index)))*LNm)/(1+3*sin(th(index))^2);
end

s(index) = s(index-1) + (fd + LNm)*(1/fs);

%%% Check for position or force control %%%
if LNm<fsw
    controlstatus = 1;
else
    if controlstatus==1
        s(index) = 0;
    end
    controlstatus = 2;
end

%%% Update time span %%%
tmin = tmax;
tmax = tmin + TSTEP;

cth(index) = -kvth*w(index) + kpth*(thd - th(index));
switch controlstatus
    case 1
        cy(index) = -kvy*vN(index) + kpy*(yd - y(index));
    case 2
        cy(index) = alpha*cy(index - 1) - kd*vN(index) +
                    kf*(fd + LNm) + ki*s(index);
end

Rtilde(index) = (1/3)*cth(index) + sin(th(index))*LTm +
                cos(th(index))*LNm;
Sytilde(index) = cy(index) + 3*Rtilde(index)*cos(th(index)) -
                w(index)^2*sin(th(index)) -
                3*sin(th(index))*cos(th(index))*LTm -
                (1+3*cos(th(index))^2)*LNm;
end

%%% Postprocessing %%%
if N==1
    t = T;
    y = Y;
    ym = YM;
    c = CC;
    s = S;
    sy = SY;
    r = RR;
    l = L;
else
    m = 1;
    for k=1:N:length(T)
        t(m,:) = T(k,:);
        y(m,:) = Y(k,:);
        ym(m,:) = YM(k,:);
        c(m,:) = CC(k,:);
        s(m,:) = S(k,:);
        sy(m,:) = SY(k,:);
        r(m,:) = RR(k,:);
        l(m,:) = L(k,:);
        m = m + 1;
    end
end

%%% Close text file %%%
fclose(fid);
end

function [value,isterminal,direction] = EVENTS(~,y)
% Event Function Terminal? Direction
% 1 y Y Increasing
% 2 y Y Decreasing
% 3 LT+MU*LN Y Decreasing
% 4 LT-MU*LN Y Increasing
% 5 vT Y Increasing
% 6 vT Y Decreasing

global MU status controlstatus LN LTstick

switch controlstatus
    case 1
        switch status
            case 1
                value = [1;y(2);1;1;1;1];
                isterminal = [0;1;0;0;0;0];
                direction = [0;-1;0;0;0;0];
            case 2
                value = [y(2);1;1;1;y(4);y(4)];
                isterminal = [1;0;0;1;1;1];
                direction = [1;0;0;0;1;-1];
            case 3
                value = [y(2);1;1;1;y(4);y(4)];
                isterminal = [1;0;0;0;1;1];
                direction = [1;0;0;0;1;-1];
            case 4
                value = [y(2);1;LTstick(y)+MU*LN(y);
                        LTstick(y)-MU*LN(y);1;1];
                isterminal = [1;0;1;1;0;0];
                direction = [1;0;-1;1;0;0];
        end
    case 2
        switch status
            case 1
                value = [1;y(2);1;1;1;1];
                isterminal = [0;1;0;0;0;0];
                direction = [0;-1;0;0;0;0];
            case 2
                value = [y(2);1;1;1;y(4);y(4)];
                isterminal = [1;0;0;1;1;1];
                direction = [1;0;0;0;1;-1];
            case 3
                value = [y(2);1;1;1;y(4);y(4)];
                isterminal = [1;0;0;0;1;1];
                direction = [1;0;0;0;1;-1];
            case 4
                value = [y(2);1;LTstick(y)+MU*LN(y);
                        LTstick(y)-MU*LN(y);1;1];
                isterminal = [1;0;1;1;0;0];
                direction = [1;0;-1;1;0;0];
        end
    end
end

function f = f(~,y)
global wc Stilde index Rtilde LN a b A B C

f = [y(4);
y(5);
y(6);
a(y) + A(y)*LT(y) + B(y)*LN(y);
b(y) + B(y)*LT(y) + C(y)*LN(y);
3*y(8) - 3*sin(y(3))*LT(y) - 3*cos(y(3))*LN(y);
wc*(Stilde(index)-y(7));
wc*(Rtilde(index)-y(8))];

end

function controlstatus = checkcontrolstatus(y)

global fsw LN

if abs(LN(y))<fsw || (y(5) >0 && abs(LN(y))==fsw)
    controlstatus = 1; %position control
else
    controlstatus = 2; %force control
end

end

function LT = LT(y)

global status MU LN LTstick

switch status
    case 1
        LT = zeros(size(y(1,:)));
    case 2
        LT = -MU*LN(y);
    case 3
        LT = MU*LN(y);
    case 4
        LT = LTstick(y);
end
REFERENCES


