DESIGN OF MULTIDIMENSIONAL RADIOFREQUENCY PULSES IN MAGNETIC RESONANCE IMAGING

BY

CHAO MA

DISSERTATION

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ABSTRACT

Multidimensional radiofrequency (RF) pulses have wide applications in magnetic resonance imaging and spectroscopy experiments. These applications include reduced field-of-view (FOV) imaging of region-of-interest (ROI), $B_1$ inhomogeneity correction, and simultaneous spatial-spectral selective excitation. While an elegant method known as Shinnar-Le Roux (SLR) method has been developed in one-dimensional RF pulse design, multidimensional RF pulse design remains an open question. The primary challenges are: a) the nonlinearity of the governing Bloch equation, inaccurate treatment of which can cause excitation pattern distortions in the large-tip-angle regime; and b) the long duration of multidimensional RF pulses, which can cause signal loss and excitation pattern distortions. The research in this thesis has developed new RF pulse design methods to address these two challenges.

To address the first challenge, a novel approach is proposed to generalize the conventional SLR method to the multidimensional case. In the proposed method, the multidimensional RF pulse design problem is converted to a series of 1D polynomial design problem, each of which is efficiently solved as a convex optimization problem. The proposed method is able to accurately treat the nonlinearity of the Bloch equation, generate excitation patterns with equiripple errors, make explicit tradeoff of design parameters and allow fast computation. The proposed method is further generalized to correct $B_0$ inhomogeneity effects and to design spatial-spectral RF pulses. The effectiveness of the proposed method is demonstrated through representative design examples using simulation and experimental results.

To address the second challenge, two methods are proposed. In the first method, we propose a novel joint design of gradient and RF waveforms to shorten spoke trajectory based RF pulses in parallel excitation. The joint design problem is formulated as an optimal spoke selection problem and is solved using a greedy algorithm. Simulation and experimental re-
sults demonstrate that the proposed method is able to achieve superior excitation accuracy with high computational efficiency compared with conventional methods. In the second method, we propose a novel reduced FOV excitation pulse using second-order gradients and a spatial-spectral RF pulse. By leveraging the unique multidimensional spatial dependence of second-order gradients, the proposed method is able to achieve 3D spatial selectivity, i.e., a circular ROI in a thin slice, using a 2D spatial-spectral RF pulse. Simulation and experimental results demonstrate that the proposed method can significantly improve excitation profiles compared with conventional methods.

In addition, we have performed a perturbation analysis to investigate the effects of $B_1$ mapping errors on excitation accuracies in parallel excitation. A closed-form solution of the perturbations on the excitation patterns caused by $B_1$ mapping errors is derived by locally linearizing the Bloch equation. Through the perturbation analysis, we show that the excitation errors are increasingly sensitive to $B_1$ mapping errors as the tip-angle and reduction factor of an RF pulse increase.
To my family, for their love
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<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>1D</td>
<td>One dimensional</td>
</tr>
<tr>
<td>2D</td>
<td>Two dimensional</td>
</tr>
<tr>
<td>3D</td>
<td>Three dimensional</td>
</tr>
<tr>
<td>AFI</td>
<td>Actual flip-angle imaging</td>
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<td>BS</td>
<td>Bloch-Siegert</td>
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<tr>
<td>DAM</td>
<td>Double angle method</td>
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<td>EPI</td>
<td>Echo planar imaging</td>
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<tr>
<td>FIR</td>
<td>Finite impulse response</td>
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<tr>
<td>FOV</td>
<td>Field of view</td>
</tr>
<tr>
<td>FOX</td>
<td>Field of excitation</td>
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<tr>
<td>GRE</td>
<td>Gradient recalled echo</td>
</tr>
<tr>
<td>LCLTA</td>
<td>Linear class large tip angle</td>
</tr>
<tr>
<td>LTA</td>
<td>Large tip angle</td>
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<tr>
<td>LTI</td>
<td>Linear time invariant</td>
</tr>
<tr>
<td>LTV</td>
<td>Linear time varying</td>
</tr>
<tr>
<td>MRI</td>
<td>Magnetic resonance imaging</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
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<tr>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>MS</td>
<td>Magnus series</td>
</tr>
<tr>
<td>NAA</td>
<td>N-acetylaspartate</td>
</tr>
<tr>
<td>NMR</td>
<td>Nuclear magnetic resonance</td>
</tr>
<tr>
<td>PBS</td>
<td>Peano-Baker series</td>
</tr>
<tr>
<td>PexLoc</td>
<td>Parallel excitation using localized gradients</td>
</tr>
<tr>
<td>RF</td>
<td>Radiofrequency</td>
</tr>
<tr>
<td>ROI</td>
<td>Region of interest</td>
</tr>
<tr>
<td>SAR</td>
<td>Specific absorption rate</td>
</tr>
<tr>
<td>SE</td>
<td>Spin echo</td>
</tr>
<tr>
<td>SLR</td>
<td>Shinnar-Le Roux</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-noise ratio</td>
</tr>
<tr>
<td>SOLO</td>
<td>Sequential optimal selection of spokes</td>
</tr>
<tr>
<td>STA</td>
<td>Small tip angle</td>
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<tr>
<td>VERSE</td>
<td>Variable rate selective excitation</td>
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CHAPTER 1
INTRODUCTION

1.1 Problem Statement

This thesis addresses the problem of designing multidimensional spatially selective radiofrequency (RF) pulses (multidimensional RF pulses for short) in magnetic resonance imaging (MRI). The RF pulse design problem can be generally stated as:

*Find an RF pulse such that starting from a given initial state, the magnetization at the end of the pulse best approximates a desired pattern, while the dynamics of magnetization is governed by the Bloch equation:*

\[
\frac{dM(r,t)}{dt} = \gamma M(r,t) \times [(G(t) \cdot r)k + B_{1,x}(r,t)i + B_{1,y}(r,t)j]. \tag{1.1}
\]

In Eq. (1.1), \(M(r,t)\) denotes the magnetization at location \(r\) and time \(t\), \(\gamma\) is the gyromagnetic ratio, \(G(t)\) denotes gradient field, \(B_{1,x}(r,t)\) and \(B_{1,y}(r,t)\) denote RF field, and relaxations are ignored. In multidimensional RF pulses, the gradient field \(G(t)\) is time-varying along multiple directions, and the desired excitation pattern is also multidimensional.

There are two major challenges in designing multidimensional RF pulses:

- **Nonlinearity.** The Bloch equation is nonlinear in the input RF pulse. Inaccurate treatment of the nonlinearity of the Bloch equation can cause excitation pattern distortions in the large-tip-angle regime.

- **Long duration.** Multidimensional RF pulses can be much longer than one-dimensional RF pulses. Long pulses can cause signal loss and excitation pattern distortions, which largely limit their applications.
In this work, we develop new RF pulse design methods to address these two challenges.

1.2 Motivation

This work is motivated by the wide applications of multidimensional RF pulses in MR imaging and spectroscopy experiments. These applications include reduced field-of-view (FOV) imaging of region-of-interest (ROI) [1–6], $B_1$ inhomogeneity correction [7–11], simultaneous spatial-spectral selective excitation [12–16], motion tracking [17, 18], localized spectroscopic imaging [19, 20], and susceptibility induced signal loss recovery in functional MRI [21].

The one-dimensional (1D) RF pulse design problem [22–25] has been well studied for several decades, and an elegant solution known as the Shinnar-Le Roux (SLR) method [26] has been found. The multidimensional RF pulse design, however, remains an open question due to difficulties in accurate treatment of the nonlinearity of the Bloch equation. Methods based on the linear approximate solutions of the Bloch equation have been successfully applied to design small-tip-angle multidimensional RF pulses [27–29]. However, these methods can yield significant excitation errors in the large-tip-angle regime [30, 31], which limit their applications in designing saturation, inversion and refocusing pulses. Methods using optimal control have been proposed to design large-tip-angle multidimensional RF pulses, but are computationally inefficient and lack of performance characterization [31]. It is, therefore, desirable to develop new multidimensional RF pulse design methods that can efficiently and accurately account for the nonlinearity of the Bloch equation.

A primary limitation of multidimensional RF pulses is long pulse duration. Unlike 1D RF pulses that usually last a few milliseconds, multidimensional RF pulses can be as long as tens to hundreds of milliseconds [32–34]. Such long duration may lead to signal loss due to relaxation, and makes multidimensional RF pulses sensitive to $B_0$ inhomogeneity, chemical shift, and imperfections of gradients. Many methods have been proposed to reduce multidimensional RF pulse length, e.g., those using multi-shot multidimensional RF pulses [19, 33] and those using advanced excitation $k$-space trajectories [8, 11, 21, 34, 35]. Most recently, parallel excitation [36, 37] has emerged as a promising approach to reducing multidimensional RF pulse length. In parallel excitation, multiple transmit coils are driven simultaneously with
independently controlled RF pulse waveform generators and RF power amplifiers. However, the reduction factor of pulse length is limited by the number of transmit coils. Developing new methods to reduce pulse length is very important for using multidimensional RF pulses in practice.

1.3 Main Results

The main results of this dissertation are summarized as follows:

- We have proposed a novel approach to generalizing the conventional SLR method to multidimensional RF pulse design. The multidimensional RF pulse design problem is converted to a series of a 1D polynomial design problem, which is efficiently solved as a convex optimization problem. The proposed method is able to accurately treat the nonlinearity of the Bloch equation, generate excitation patterns with equiripple errors, make explicit tradeoff of design parameters and allow fast computation. The proposed method is further generalized to correct $B_0$ inhomogeneity effects and to design spatial-spectral RF pulses. The power of the proposed method is demonstrated through representative RF pulse design examples using simulation and experimental results.

- We have proposed a novel approach to jointly designing gradient and RF waveforms to shorten spoke trajectory based RF pulses in parallel excitation. The joint design problem is formulated as an optimal spoke selection problem. The cost function, which minimizes the excitation error with a regularization term for RF power control, is represented as a function only of spoke locations. The cost function is minimized using a greedy algorithm. Simulation and experimental results demonstrate that the proposed method is able to achieve superior excitation accuracy with high computational efficiency compared with conventional methods.

- We have proposed a novel approach to designing RF pulses for reduced FOV excitation using second-order gradients and spatial-spectral pulses. By leveraging the unique
multidimensional spatial dependence of second-order gradients, the proposed method is able to achieve 3D spatial selectivity, i.e., a circular or elliptical ROI in a thin slice, using a 2D spatial-spectral RF pulse. Simulation and experimental results demonstrate that the proposed method can significantly improve excitation profiles compared with the conventional 3D RF pulses using linear gradients.

- We have performed a perturbation analysis to investigate the effects of $B_1$ mapping errors on excitation accuracies in parallel excitation. A closed-form solution of the perturbations on the excitation patterns caused by $B_1$ mapping errors is derived by locally linearizing the Bloch equation. Through the perturbation analysis, we show that the excitation errors are increasingly sensitive to $B_1$ mapping errors as the tip-angle and reduction factor of an RF pulse increase.

1.4 Organization of the Dissertation

The remainder of the dissertation is organized as follows.

- Chapter 2 presents a review of spatially selective RF pulse design methods in MRI. After a brief introduction of spatially selective RF pulses, various forms of the Bloch equation and analytical approximate solutions of the Bloch equation are systematically derived. The most relevant previously proposed methods are reviewed. In addition, important practical issues in RF pulse design, including specific absorption rate (SAR) control and $B_1$ mapping, are discussed.

- Chapter 3 describes a novel multidimensional SLR RF pulse design method. Rigorous mathematical treatment is described to convert the multidimensional RF pulse design problem into a series of 1D polynomial design problems. The excitation error of the proposed method is characterized. The proposed method is further generalized to correct $B_0$ inhomogeneity effects and to design spatial-spectral RF pulses. Finally, a number of design examples are presented to demonstrate the effectiveness of the proposed method in designing large-tip-angle multidimensional RF pulses.
• Chapter 4 describes a novel method to jointly design the gradient and RF waveforms of spoke trajectory based RF pulses in parallel excitation. The problem is first formulated as an optimal spoke selection problem. An efficient greedy algorithm is described to solve the problem. The proposed method is further generalized to consider $B_0$ inhomogeneity effects. Simulation and experimental results are presented to demonstrate the superior performance of the proposed method both in excitation accuracy and computation efficiency compared with conventional methods.

• Chapter 5 describes a novel method to achieve reduced FOV excitation using second-order gradients and spatial-spectral pulses. The intuition behind the proposed is first described, followed by detailed description of the design procedure. Proof-of-concept experimental results on a commercial MRI scanner are presented to demonstrate the effectiveness and feasibility of the proposed method.

• Chapter 6 presents a perturbation analysis of the effects of $B_1$ mapping errors on excitation accuracies in parallel excitation. A new perturbation solution of the Bloch equation is derived. The perturbation solution is then used to investigate the effects of $B_1$ mapping errors as a function of tip-angle and reduced factor.

• Finally, Chapter 7 presents a brief discussion of future work.
CHAPTER 2

LITERATURE REVIEW

In this chapter, we present a literature review of spatially selective RF pulse design. Several important families of RF pulses, including spectral selective RF pulses, adiabatic RF pulses, and composite RF pulses are therefore beyond our scope. We further focus on discussing RF pulse design methods that assume pre-selected gradient waveforms. Joint design of RF and gradient waveforms is also beyond our scope.

2.1 Introduction

Mathematically, spatially selective RF pulse design can be generally formulated as

$$\arg\min_{b_1(t), \ldots, b_L(t)} C(M(r, T), D(r))$$

subject to

$$\begin{bmatrix}
\dot{M}_x(r, t) \\
\dot{M}_y(r, t) \\
\dot{M}_z(r, t)
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & G(t) \cdot r & -B_{1,y}(r, t) \\
-G(t) \cdot r & 0 & B_{1,x}(r, t) \\
B_{1,y}(r, t) & -B_{1,x}(r, t) & 0
\end{bmatrix}
\begin{bmatrix}
M_x(r, t) \\
M_y(r, t) \\
M_z(r, t)
\end{bmatrix}$$

$$B_1(r, t) = \sum_{l=1}^{L} s_l(r)b_l(t)$$

$$\int_0^T |b_l(t)|^2 dt \leq P, \ l = 1, \ldots, L,$$

(2.1)
In Eq. (2.1), the cost function \( C(M(r, T), D(r)) \) penalizes the distance between the actual and the desired excitation pattern at the end of an RF pulse. The distance is commonly measured using \( \ell_2 \)-norm, weighted \( \ell_2 \)-norm and infinity norm. The cost function also depends on the type of RF pulses to be designed. For instance, the transverse component of magnetization is of interest for excitation pulses, and the longitudinal component of magnetization is of interest for saturation and inversion pulses.

If \( G(t) \) and \( D(r) \) are along one dimension, the design problem in Eq. (2.1) defines a 1D RF pulse design problem. If \( G(t) \) and \( D(r) \) are along multiple dimensions, it defines a multidimensional RF pulse design problem. If the number of transmit channels is larger than one, it defines a multi-channel RF pulse design problem.

Although the formulation in Eq. (2.1) is most widely used in spatially selective RF pulse design, we note that there are other forms of formulation such as minimum length and minimum RF power/SAR formulation [30, 38]. The minimum length formulation is particularly interesting, since it attempts to design the “shortest” pulse. This formulation essentially leads to a joint design of gradient and RF waveforms, which is beyond the scope of this review.

Significant technical challenges arise toward solving the optimization problem in Eq. (2.1).

First, the Bloch equation does not have a closed-form solution in general. Without it, the Bloch equation appears as a first-order differential equation constraint in the optimization problem. This is a significant difference compared with a nonlinear optimization problem, which has equalities and inequalities as constraints. Mathematically, the optimization problem in Eq. (2.1) is an optimal control problem. The mathematical tool to solve an optimal control problem is calculus of variation rather than calculus for a nonlinear optimization problem.

Second, the problem in Eq. (2.1) is an optimal control problem of a nonlinear control system. If we consider magnetization \( M \) as the state of a control system and RF field as the control of the system, the Bloch equation can be rewritten as

\[
\dot{M}(r, t) = -\gamma (G(t) \cdot r) I_z M(r, t) - \gamma (I_x B_{1x}(r, t) + I_y B_{1y}(r, t)) M(r, t), \tag{2.2}
\]
where $I_x$, $I_y$, and $I_z$ are three skew-symmetric basis matrices:

$$
I_x = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
\end{bmatrix},
I_y = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
\end{bmatrix},
I_z = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
$$

Note that the general form of a linear control system is given by

$$
\dot{x} = A(t)x + B(t)u(t),
$$

where $x$ denotes system state, $u(t)$ denotes control, $A(t)$ and $B(t)$ are in general time-varying matrices. Comparing Eq. (2.2) with Eq. (2.4), it is clear that the Bloch equation represents a nonlinear control system. Notably, in the Bloch equation the control $B_{1,x}(r, t)$ and $B_{1,y}(r, t)$ is coupled with the system state $M$.

The nonlinearity of the Bloch equation in the input RF field raises one of the fundamental difficulties in RF pulse design. The difficulty can be understood from several perspectives. First, the linear approximations of the Bloch equation only hold good accuracy for small-tip-angle RF pulses. Significant distortions may occur when designing large-tip-angle RF pulses based on the linear approximations. Second, optimal control theory only provides necessary or sufficient conditions of local optimum for a nonlinear control system. Optimal control approach, therefore, only yields local optimal RF pulses. Third, hardware imperfections, e.g., $B_0$ and $B_1$ inhomogeneity, and simultaneous multi-channel excitation (parallel excitation) add significant complications to RF pulse design.

The rest of this chapter is organized as follows. Sections 2.2 to Section 2.4 are devoted to background materials. In Section 2.2, a brief introduction of spatially selective RF pulses is presented with emphasis on the concept, applications, and limitations of various RF pulses. In Section 2.3, the Bloch equation, which is the governing equation of RF pulse design, is reviewed with emphasis on deriving different forms of the Bloch equation. In Section 2.4, the analytical solutions of the Bloch equation are reviewed. Especially, the approximate solutions of the Bloch equation, which are the most relevant analytical solutions to spatially selective RF pulse design, are derived based on the series solutions of a general linear time-
varying system. In Sections 2.5 to 2.7, three major design methods: methods based on the linear approximate solutions of the Bloch equation, methods using optimal control, and methods based on pulse synthesis are reviewed. Practical issues, including SAR control and $B_1$ mapping, are reviewed in Section 2.8.

2.2 Spatially Selective RF Pulses

Spatially selective RF pulses are played concurrently with gradient fields in order to tip magnetization along one or multiple spatial directions. Spatially selective RF pulses can be categorized according to a variety of properties, such as tip-angle, waveform, spatial selectivity and function. Depending on tip-angle, they are classified as small-tip-angle pulses (tip-angle $\leq 30^\circ$) and large-tip-angle pulses (tip-angle $\geq 90^\circ$). Depending on function in an MRI sequence, they are classified as excitation pulses, saturation pulses, inversion pulses and refocusing pulses. Depending on spatial selectivity, they are classified as one-dimensional pulses and multidimensional pulses.

An excitation pulse is used to tip magnetization away from the direction of the main magnetic field. After the excitation pulse, magnetization rotates about the main magnetic field, and thus creates MRI signals. Typically, excitation pulses of $90^\circ$ tip-angle are used in spin-echo (SE) sequences, and excitation pulses of $5^\circ$ to $70^\circ$ tip-angle are used in gradient recalled echo (GRE) sequences [39].

A saturation pulse is essentially a $90^\circ$ excitation pulse. In RF saturation, first a $90^\circ$ saturation pulse is used to tip magnetization from the direction of the main magnetic field to the transverse plane. Spoiler gradients are followed to dephase the transverse magnetization. Before the longitudinal magnetization has sufficient time to recover, an excitation pulse is applied. As a result, the areas that have been excited by the first RF pulse will not contribute signals in images and thus are saturated. Saturation pulses are widely used to attenuate various undesired signals, e.g., signals from the abdominal wall for reducing respiratory artifacts and signals outside of FOV for reducing aliasing in the phase-encoding direction.

An inversion pulse is used to tip magnetization from the direction of the main magnetic field to the opposite direction. An inversion pulse has a nominal tip-angle of $180^\circ$. In-
version pulses are widely used for various contrast preparations, e.g., in short tau inversion recovery (STIR) [40], spectral inversion at lipids (SPECIAL) [41], fluid attenuated inversion recovery (FLAIR) [42], inversion recovery preparation (IR-Prep) [40] and magnetization preparation (MP) [43] sequences. Inversion pulses are also widely used for measuring the $T_1$ relaxation time of tissues.

A refocusing pulse is applied after an excitation pulse to tip magnetization about an axis in the transverse plane to create spin echoes. A refocusing pulse has a nominal tip-angle of 180°. Refocusing pulses are widely used in SE, rapid acquisition with relaxation enhancement (RARE) or fast spin echo (FSE), and spin-echo echo-planar imaging (SE EPI) sequences. Refocusing pulses are also widely used for measuring the $T_2$ relaxation time of tissues.

Among these pulses, saturation, inversion and refocusing pulses are large-tip-angle pulses, and excitation pulses can be either small-tip-angle or large-tip-angle pulses.

2.2.1 One-Dimensional Pulses

Concept

A 1D pulse achieves spatial selectivity along one spatial direction, i.e., the through-slice direction, and is also known as slice selective pulse [22–25]. A 1D pulse is played concurrently with a slice-selection gradient. The waveform of an RF pulse essentially defines its frequency selectivity. It is the slice-selection gradient that converts the frequency selectivity of an RF pulse into spatial selectivity. For a linear-phase 1D excitation pulse, a refocusing gradient lobe is necessary for refocusing phase dispersion across the slice [25].

Applications and limitations

One-dimensional pulses are the most widely used spatially selective pulses. They have been routinely used in slice selection in most imaging sequences. Compared with multidimensional pulses, a significant advantage of 1D pulses is short pulse length. As a result, 1D pulses are relatively less sensitive to $B_0$ inhomogeneity, chemical shift, and relaxation effects. However, 1D pulses have no in-slice selectivities, and are not capable of more advanced applications, such as reduced FOV excitation and $B_1$ inhomogeneity correction.
2.2.2 Multidimensional Pulses

Concept

A multidimensional pulse achieves spatial selectivity along multiple spatial directions [27, 44]. For instance, a 2D pulse can select a long cylinder and a 3D pulse can select a thin disk. A multidimensional pulse is played concurrently with time-varying gradients along multiple directions.

The concept of multidimensional pulses can be easily understood through the excitation $k$-space interpretation proposed by Pauly et al. [27]. More specifically, assume that the initial state of magnetization is at its equilibrium state and the tip-angle of an RF pulse is small, at the end of the RF pulse, the end state of magnetization is related to the RF pulse through a Fourier transform:

$$M_{xy}(r, T)/M_0(r) \approx i\gamma \int_0^T B_1(t)e^{ik(t) \cdot r} dt,$$  \hspace{1cm} (2.5)

where $M_{xy}(r, T)$ is the transverse component of magnetization, $M_0(r)$ is the equilibrium value of magnetization, and $k(t)$ describes a path through the excitation $k$-space and is determined by the gradients:

$$k(t) = -\gamma \int_t^T G(\tau) d\tau.$$  \hspace{1cm} (2.6)

One may readily realize the similarities between Eqs. (2.5) and (2.6) and an MRI imaging equation. Indeed, time-varying gradient waveforms commonly used in MRI data acquisition, e.g., spiral gradients and EPI gradients, are used to traverse the excitation $k$-space. RF pulses can be determined using Fourier transform with given excitation patterns [27].

Applications

Since the late 1980s, multidimensional pulses have been successfully used in a wide range of applications in MR imaging and spectroscopy. These applications include reduced FOV imaging of ROI [1–6], $B_1$ inhomogeneity correction [7–11], simultaneous spatial-spectral selective excitation [12–16], motion tracking [17, 18], localized spectroscopic imaging [19, 20], and susceptibility induced signal loss recovery in functional MRI [21]. Reduced FOV imaging of ROI, $B_1$ inhomogeneity corrections, and spatial-spectral selective excitation are
the primary applications of multidimensional pulses.

In reduced FOV imaging of ROI, a multidimensional excitation pulse is used to excite spins only in a ROI, and thus reduce the number of phase-encoding steps. For instance, a 2D excitation pulse can be designed for simultaneous slice selection and reduced FOV excitation along the phase-encoding direction. Reduced FOV imaging using 2D excitation pulses have been used in a variety of applications, including angiographic imaging [1], cardiac imaging [2], MR elastography [3], diffusion weighted imaging of the spinal cord [4], reducing echo-train length in EPI sequences [5] and improving temporal resolution in dynamic imaging [6]. Reducing FOV using a 3D excitation pulse is usually not necessary as reducing FOV along the frequency-encoding direction can be easily done by adjusting the cutoff frequency of the filter in the receiver chain. However, reducing FOV using a 3D excitation pulse can be desirable for sequences using non-Cartesian sampling trajectories, e.g., spiral trajectories and PROPELLER [45], or for sequences sampling multiple Cartesian lines in a single readout, e.g., EPI sequences.

In $B_1$ inhomogeneity correction, a multidimensional pulse is used to compensate an inhomogeneous transmit $B_1$ field. Transmit $B_1$ field becomes increasingly inhomogeneous as the field strength of an MRI scanner increases due to two factors. First, as the field strength increases, the wavelength of RF field decreases and is further shortened due to the dielectric properties of tissues. At high field (3.0 T) and ultra-high field (≥ 3.0 T, e.g., 7.0 T and 9.4 T) human MRI scanners, the wavelength of RF field becomes comparable and even shorter than the size of an imaging object. Therefore, wave effects play a dominate role in determining the spatial distributions of transmit $B_1$ fields. Second, the heterogeneous nature of tissues leads to complex patterns of the dielectric properties and conductive properties of an imaging object, which create even more complicated transmit $B_1$ field distributions. Transmit $B_1$ inhomogeneity can cause significant signal and contrast loss and lead to inaccurate quantitative analysis results at high or ultra high field MRI. Multidimensional pulses [7–11] have been proposed to address this issue. Note that different from reduced FOV excitation pulses, which usually need to excite patterns with sharp transitions, the multidimensional pulses designed for $B_1$ inhomogeneity corrections only need to excite patterns with mild spatial variations except the slice selection direction as transmit $B_1$ field is generally smooth.
In spatial-spectral selective excitation, a spatial-spectral selective pulse, which can be considered as a specific 2D pulse, is used to tip magnetization along the slice selection direction and the spectral direction (e.g., chemical shift) \[12\]. Spatial-spectral excitation pulses have been widely in suppressing lipid signals \[13\]. Spatial-spectral refocusing pulses on the other hand have been proposed to suppress water and/or lipid signals in MR spectroscopic imaging \[14–16\]. An advantage of spatial-spectral excitation pulses is that they are relatively less sensitive to $B_1$ inhomogeneities than other lipid and water suppression techniques \[13, 14\].

**Limitations**

A primary limitation of multidimensional pulses is long pulse duration. Unlike 1D pulses that are usually less than a few milliseconds, multidimensional pulses can be as long as tens to hundreds of milliseconds \[32–34\]. Such long duration leads to signal loss due to $T_2$ relaxation, and makes multidimensional pulses sensitive to $B_0$ inhomogeneity, chemical shift, and imperfections of gradients.

The long duration of multidimensional pulses can be understood from the perspective of excitation $k$-space. A 1D pulse needs a gradient field to traverse the 1D excitation $k$-space. However, a multidimensional pulse needs gradients to traverse the multidimensional excitation $k$-space. Therefore, the length of a multidimensional pulse is inherently limited by gradient hardware and peripheral nerve stimulation thresholds, which are the same limiting factors for MRI data acquisition. In addition, a multidimensional pulse generally requires larger RF power than a 1D pulse. The length of a multidimensional pulse is also limited by the maximum power of RF amplifiers, especially the maximum peak power.

### 2.2.3 Parallel Excitation

Many methods have been proposed to reduce multidimensional RF pulse length, e.g., those using multi-shot multidimensional RF pulses \[19, 33\], those using advanced excitation $k$-space trajectories \[8, 11, 21, 34, 35, 46\], and those using nonlinear gradients \[47\]. Most recently, parallel excitation \[36, 37\] has emerged as a promising approach to reducing the length of a multidimensional RF pulse.
Concept

In parallel excitation, multiple transmit coils are driven simultaneously with independently controlled RF pulse waveform generators and RF power amplifiers. Due to the linearity of the Maxwell equation, the synthesized $B_1$ field is a sum of the $B_1$ field generated by each transmit channel:

$$B_1(r, t) = \sum_{l=1}^{L} s_l(r) b_l(t), \quad (2.7)$$

where $s_l(r)$ and $b_l(t)$ are the sensitivity map and waveform of the $l$-th transmit channel, respectively.

Conceptually, parallel excitation is analogous to parallel imaging [48–50] in the imaging side. More specifically, substituting Eq. (2.7) to Eq. (2.5), yields the following design equation in the small-tip-angle regime:

$$M_{xy}(r, T)/M_0(r) \approx i\gamma \sum_{l=1}^{L} s_l(r) \int_0^T b_l(t) e^{i(k(t) \cdot r)} dt. \quad (2.8)$$

Similar to parallel imaging, the excitation $k$-space is undersampled in parallel excitation in order to reduce pulse length. The aliasing artifacts due to the undersampling are “resolved” through the distinguished sensitivity map of each transmit channel.

Applications and limitations

Since the initial proof-of-concept experiments [36, 37], significant technical advances in phased-array transmit coil and multi-channel transmit hardware have been achieved in parallel excitation. Parallel excitation is nowadays supported by major MRI scanner manufacturers. Applications in reducing multidimensional RF pulse length and improving excitation profiles have been successfully demonstrated.

However, compared to single channel excitation, parallel excitation requires significantly more complicated hardware. So far it is not widely available in research and clinical scanners. The reduction factor of pulse length is limited by the number of transmit coils. Parallel excitation requires knowledge of $B_1$ maps, which is challenging to be estimated accurately with fast data acquisition. Furthermore, parallel excitation significantly complicates the management and monitoring of SAR.
2.3 Bloch Equation

2.3.1 Bloch Equation in the Stationary Frame of Reference

The fundamental physical principle of MRI is the nuclear magnetic resonance (NMR) phenomenon in liquids and solids, which is independently discovered by Bloch and Purcell in 1946 who therewith shared the Nobel Prize in Physics in 1952. The NMR phenomenon is governed by the Bloch equation, which describes the ensemble behavior of a spin system of bulk matter in an external magnetic field. In its basic form, the Bloch equation can be written as [51, 52]

$$\frac{dM}{dt} = \gamma M \times B(t) - \frac{M_z i + M_y j}{T_2} - \frac{(M_z - M_0)k}{T_1} + D \nabla^2 M. \quad (2.9)$$

In Eq. (2.9), $M = M_x i + M_y j + M_z k$ is a magnetization vector, where $i$, $j$, and $k$ are the unit vectors codirectional with the $x$, $y$ and $z$ axes. $\gamma$ is the gyromagnetic ratio of the nucleus of interest. $M_0$ is the thermal equilibrium value of $M$. $T_1$ and $T_2$ denote longitudinal and transversal relaxation time constant. $D$ denotes diffusion coefficient, where isotropic diffusion is assumed for simplicity. $B$ denotes external magnetic field, which can be generally written as

$$B = (B_0 + G \cdot r)k + B_{1,x}i + B_{1,y}j, \quad (2.10)$$

where $B_0$ denotes main field that is assumed to be codirectional with the $z$-axis, $G(t) = [G_x, G_y, G_z]^T$ denotes gradient field (“$^T$” denotes transpose operation), $r = [x, y, z]^T$ denotes spatial location, and $B_{1,x}(t)$ and $B_{1,y}(t)$ are the $x$ and $y$ component of an RF field.

The tip-angle of an RF pulse is defined as the angle between the initial and end state of magnetization. The relaxation and diffusion terms in Eq. (2.9) cause additional attenuations. It is the tip-angle that of primary interest in most RF pulse design. The relaxation and
diffusion terms are thus often ignored, and the Bloch equation can be rewritten as

\[
\begin{bmatrix}
\dot{M}_x(r, t) \\
\dot{M}_y(r, t) \\
\dot{M}_z(r, t)
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & \mathbf{G}(t) \cdot \mathbf{r} + B_0 & -B_{1,y}(t) \\
-(\mathbf{G}(t) \cdot \mathbf{r} + B_0) & 0 & B_{1,x}(t) \\
B_{1,y}(t) & -B_{1,x}(t) & 0
\end{bmatrix}
\begin{bmatrix}
M_x(r, t) \\
M_y(r, t) \\
M_z(r, t)
\end{bmatrix},
\tag{2.11}
\]

where \(\dot{M}_{(x,y,z)}\) denotes time derivative.

Equation (2.11) describes the precession of magnetization about an external field. Especially, in the absence of gradient and the RF fields, the magnetization vector will precess about the main field at a frequency known as the Larmor frequency, which is linearly proportional to main field strength:

\[\omega_0 = \gamma B_0.\]  \(\tag{2.12}\)

### 2.3.2 Bloch Equation in the Rotating Frame of Reference

The main field of an MRI scanner (e.g., 0.3 ~ 3 T) is generally much stronger than the gradient (e.g., \(\leq 4 \text{ G/cm}, 1 \text{ G} = 10^{-4} \text{ T}\)) and RF fields (e.g., \(\leq 0.25 \text{ G at 3 T}\)). It is often convenient to “factor out” the precession caused by the main field when studying the effect of an RF pulse [53].

In order to do that, we introduce the following rotating frame of reference:

\[\mathbf{M} = M_{x,\text{rot}} \mathbf{i}_{\text{rot}} + M_{y,\text{rot}} \mathbf{j}_{\text{rot}} + M_{z,\text{rot}} \mathbf{k}_{\text{rot}},\]  \(\tag{2.13}\)

where \(\mathbf{i}_{\text{rot}}, \mathbf{j}_{\text{rot}}\) and \(\mathbf{k}_{\text{rot}}\) denote the unit directional vectors of the rotating frame, and are defined by the following differential equations:

\[
\begin{align*}
\frac{d\mathbf{i}_{\text{rot}}}{dt} &= \mathbf{\omega}_{\text{rot}}(t) \times \mathbf{i}_{\text{rot}}(t), & \mathbf{i}_{\text{rot}}(0) &= \mathbf{i}, \\
\frac{d\mathbf{j}_{\text{rot}}}{dt} &= \mathbf{\omega}_{\text{rot}}(t) \times \mathbf{j}_{\text{rot}}(t), & \mathbf{j}_{\text{rot}}(0) &= \mathbf{j}, \\
\frac{d\mathbf{k}_{\text{rot}}}{dt} &= \mathbf{\omega}_{\text{rot}}(t) \times \mathbf{k}_{\text{rot}}(t), & \mathbf{k}_{\text{rot}}(0) &= \mathbf{k}.
\end{align*}
\]  \(\tag{2.14}\)

If we choose

\[\mathbf{\omega}_{\text{rot}}(t) = -\omega_0 \mathbf{k},\]  \(\tag{2.15}\)

16
where \( \omega_0 \) is defined in Eq. (2.12), the solution of Eq. (2.14) is given by

\[
\begin{align*}
\mathbf{i}_{\text{rot}}(t) &= \cos(\omega_0 t) \mathbf{i} - \sin(\omega_0 t) \mathbf{j}, \\
\mathbf{j}_{\text{rot}}(t) &= \sin(\omega_0 t) \mathbf{i} + \cos(\omega_0 t) \mathbf{j}, \\
\mathbf{k}_{\text{rot}}(t) &= \mathbf{k}.
\end{align*}
\] (2.16)

The rotating frame defined in Eq. (2.16) is called the Larmor-rotating frame.

Substituting Eqs. (2.13) and (2.16) to Eq. (2.11), yields the following Bloch equation in the Larmor-rotating frame:

\[
\begin{bmatrix}
\dot{M}_{x,\text{rot}}(\mathbf{r}, t) \\
\dot{M}_{y,\text{rot}}(\mathbf{r}, t) \\
\dot{M}_{z,\text{rot}}(\mathbf{r}, t)
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & \mathbf{G}(t) \cdot \mathbf{r} & -B_{1,y,\text{rot}}(t) \\
-G(t) \cdot \mathbf{r} & 0 & B_{1,x,\text{rot}}(t) \\
B_{1,y,\text{rot}}(t) & -B_{1,x,\text{rot}}(t) & 0
\end{bmatrix}
\begin{bmatrix}
M_{x,\text{rot}}(\mathbf{r}, t) \\
M_{y,\text{rot}}(\mathbf{r}, t) \\
M_{z,\text{rot}}(\mathbf{r}, t)
\end{bmatrix},
\] (2.17)

where \( B_{1,x,\text{rot}}(t) \) and \( B_{1,y,\text{rot}}(t) \) are the \( x \) and \( y \) component of the RF field in the rotating frame and are given by

\[
\begin{bmatrix}
B_{1,x,\text{rot}}(t) \\
B_{1,y,\text{rot}}(t)
\end{bmatrix} = \begin{bmatrix}
\cos(\omega_0 t) & -\sin(\omega_0 t) \\
\sin(\omega_0 t) & \cos(\omega_0 t)
\end{bmatrix}
\begin{bmatrix}
B_{1,x}(t) \\
B_{1,y}(t)
\end{bmatrix}.
\] (2.18)

In the rest of this thesis, we mainly discuss the Bloch equation in the Larmor-rotating frame and drop the subscript “\( \text{rot} \)” for simplicity. The Bloch equation in Eq. (2.17) is then simply written as

\[
\begin{bmatrix}
\dot{M}_{x}(\mathbf{r}, t) \\
\dot{M}_{y}(\mathbf{r}, t) \\
\dot{M}_{z}(\mathbf{r}, t)
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & \mathbf{G}(t) \cdot \mathbf{r} & -B_{1,y}(t) \\
-G(t) \cdot \mathbf{r} & 0 & B_{1,x}(t) \\
B_{1,y}(t) & -B_{1,x}(t) & 0
\end{bmatrix}
\begin{bmatrix}
M_{x}(\mathbf{r}, t) \\
M_{y}(\mathbf{r}, t) \\
M_{z}(\mathbf{r}, t)
\end{bmatrix},
\] (2.19)

2.3.3 Spinor-Domain Bloch Equation

The solution of the Bloch equation in Eq. (2.19) can be always written as

\[
\mathbf{M}(t) = \mathbf{R}(t)\mathbf{M}(0),
\] (2.20)
where $R(t)$ is a $3 \times 3$ rotation matrix. A rotation matrix is real, orthogonal, and of determinant 1. The set of all $3 \times 3$ rotation matrices is a special group known as $SO(3)$.

Besides a $3 \times 3$ rotation matrix, rotation in three dimensions can be represented in multiple ways, including Euler axis and angle, Euler rotations, quaternion, and Cayley-Klein parameters. Representing rotation using the Cayley-Klein parameters leads to the spinor-domain Bloch equation and brings significant convenience in mathematics.

Using the Cayley-Klein parameters, a rotation in three dimensions is represented by a $2 \times 2$ unitary matrix:

$$Q = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}, \quad (2.21)$$

where $\alpha$ and $\beta$ are the Cayley-Klein parameters, and $"^*"$ denotes conjugate operation. The set of all the $2 \times 2$ unitary matrices in Eq. (2.21) is a special group known as $SU(2)$.

Suppose that the rotation axis is represented by a unit vector $n = [n_x, n_y, n_z]^T$ and the rotation angle is $\theta$, $\alpha$ and $\beta$ in Eq. (2.21) are given by [26]

$$\alpha = \cos \frac{\theta}{2} - in_z \sin \frac{\theta}{2}, \quad (2.22)$$

$$\beta = -i(n_x + in_y) \sin \frac{\theta}{2}, \quad (2.23)$$

where the sign of the rotation angle follows the right-handed rotation convention, i.e., a counterclockwise rotation has a positive rotation angle.

Given the initial state of the magnetization vector $[M_x^-, M_y^-, M_z^-]^T$, after the rotation in Eq. (2.21), the end state of the magnetization vector $[M_x^+, M_y^+, M_z^+]^T$ can be easily calculated:

$$\begin{bmatrix} M_{xy}^+ \\ M_{xy}^{+*} \\ M_z^+ \end{bmatrix} = \begin{bmatrix} (\alpha^*)^2 & -\beta^2 & 2\alpha^*\beta \\ -\beta^2 & \alpha^2 & 2\alpha^* \beta \\ -\alpha^*\beta^* & \alpha^* \beta & \alpha^2 - \beta^2 \end{bmatrix} \begin{bmatrix} M_{xy}^- \\ M_{xy}^{-*} \\ M_z^- \end{bmatrix}, \quad (2.24)$$

where $M_{xy} = M_x + iM_y$.

In order to describe the dynamics of the magnetization vector, it is sufficient and often convenient to track the change of the rotation matrix directly. In the following, we derive the differential equations of the Cayley-Klein parameters, which are referred to as the spinor-
domain Bloch equation.

Note that the rotation matrix in Eq. (2.21) can be rewritten as

\[ Q = e^{-\frac{-\vec{n} \cdot \vec{\sigma} \theta}{2}}, \]

\[ = \mathbf{I} \cos \left( \frac{\theta}{2} \right) - i(\vec{n} \cdot \vec{\sigma}) \sin \left( \frac{\theta}{2} \right). \]  (2.25)

In Eq. (2.25), \( \mathbf{I} \) is a \( 2 \times 2 \) identity matrix and \( \vec{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \), where \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) are the Pauli spin matrices [54]:

\[ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]  (2.26)

Equation (2.25) can be easily validated by expanding the exponential function of matrices:

\[ Q = 1 - i \left( \frac{\vec{n} \cdot \vec{\sigma}}{2} \right) \theta - \frac{1}{2!} \left( \frac{\vec{n} \cdot \vec{\sigma} \theta}{2} \right)^2 + i \frac{1}{3!} \left( \frac{\vec{n} \cdot \vec{\sigma} \theta}{2} \right)^3 + \cdots \]  (2.27)

Suppose that in the presence of an external field \( \vec{B}(t) = [B_{1,x}(t), B_{1,y}(t), G(t) \cdot \vec{r}]^T \), the rotation matrix \( Q(t) \) evolves in a short duration \( \delta t \) to \( Q(t+\delta t) \), the derivative of the matrix \( Q(t) \) can be calculated as

\[ \dot{Q}(t) = \lim_{\delta t \to 0} \frac{Q(t+\delta t) - Q(t)}{\delta t} \]

\[ = \lim_{\delta t \to 0} \left( e^{i\gamma \delta t \vec{B}(t) \cdot \vec{\sigma}} Q(t) - Q(t) \right) / \delta t \]

\[ = \lim_{\delta t \to 0} (e^{i\gamma \delta t \vec{B}(t) \cdot \vec{\sigma}} - \mathbf{I}) Q(t) / \delta t \]

\[ = i \gamma \vec{B}(t) \cdot \vec{\sigma} Q(t). \]  (2.28)

The spinor-domain Bloch equation is given by

\[
\begin{bmatrix}
\dot{\alpha}(\vec{r},t) \\
\dot{\beta}(\vec{r},t)
\end{bmatrix} = \frac{i \gamma}{2} \begin{bmatrix}
\mathbf{G}(t) \cdot \vec{r} & B_1^*(t) \\
B_1(t) & -\mathbf{G}(t) \cdot \vec{r}
\end{bmatrix} \begin{bmatrix}
\alpha(\vec{r},t) \\
\beta(\vec{r},t)
\end{bmatrix},
\]  (2.29)

where \( B_1(t) = B_{1,x}(t) + iB_{1,y}(t) \).
2.4 Solutions of the Bloch Equation

2.4.1 Closed-Form Solutions of the Bloch Equation

If the gradient field $G$ and RF field $B_1$ are constant, the dynamic system in Eq. (2.19) is a linear time-invariant (LTI) system. A closed-form solution of such a system exists:

$$M(r, t) = e^{-\gamma t \left( (G \cdot r) I_z + B_{1,x} I_x + B_{1,y} I_y \right)} M(r, 0),$$

(2.30)

where $I_x$, $I_y$, and $I_z$ are the three skew-symmetric basis matrices in Eq. (2.3).

The solution in Eq. (2.30) can be extended to the case where the gradient and RF field are piece-wise constant. Suppose that the pulse is decomposed into $n$ segments, and for each segment the gradient and the RF field are denoted as $G^{(j)}$ and $(B_{1,x}^{(j)}, B_{1,y}^{(j)})$, respectively, the solution of the Bloch equation is given by

$$M(r, T) = \left( \prod_{j=1}^{n} e^{-\gamma T_j \left( (G^{(j)} \cdot r) I_z + B_{1,x}^{(j)} I_x + B_{1,y}^{(j)} I_y \right)} \right) M(r, 0),$$

(2.31)

where $T_j$ is the duration of the $j$-th segment. Although not often directly used in RF pulse design, the solution in Eq. (2.31) is widely used to numerically solve the Bloch equation.

However, the gradient and RF field are time-varying in the most cases of spatially selective RF pulse design. The dynamic system in Eq. (2.19) then becomes a linear time-varying (LTV) system. There is no closed-form solution of such an LTV system with arbitrary gradient and RF field. Analytical solutions of the Bloch equation exist for RF pulses that can be represented by hypergeometric functions. They have been successfully used for adiabatic inversion pulse design [55, 56]. However, the restrictions on the RF pulse waveforms make them hard to be applied to spatially selective RF pulse design. Extensive studies, instead, have been done to find approximate solutions of the Bloch equation. As will be discussed in Section 2.5, a large family of RF pulse design methods are actually based on the linear approximate solutions of the Bloch equation.

In the rest of this section, we derive approximate solutions of the Bloch equation using the series solutions of a general linear time-varying system. The derivation is largely influenced
2.4.2 Series Solutions of an LTV System

An LTV system is generally written as

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \] (2.32)

where \( A(t) \) and \( B(t) \) are time-varying matrices, and \( u(t) \) is a control.

If \( A(t) \) and \( B(t) \) are piecewise-continuous functions of \( t \), then the solution of Eq. (2.32) is unique and is given by

\[ x(t) = \Phi(t, 0)x(0) + \Phi(t, 0) \int_0^t \Phi^{-1}(s, 0)B(s)u(s)ds, \] (2.33)

where \( x(0) \) is the initial state, and \( \Phi(t, 0) \) is the state transition matrix.

If \( A(t) \) is not commutative, which is the case for the Bloch equation, a general closed-form solution of \( \Phi(t, 0) \) does not exist. However, \( \Phi(t, 0) \) can be written as series. There are two types of series widely used in solving an LTV system: the Peano-Baker series (PBS) and the Magnus series (MS).

Using the PBS, the state transition matrix is written as

\[ \Phi(t, 0) = \sum_{n=0}^{\infty} \Phi_n(t), \] (2.34)

where,

\[ \Phi_0(t) = I, \] (2.35)

\[ \Phi_n(t) = \int_0^t A(s)\Phi_{n-1}(s)ds. \] (2.36)

Using the MS, the state transition matrix is written as

\[ \Phi(t, 0) = e^{\sum_{n=1}^{\infty} \Psi_n(t)}. \] (2.37)
In Eq. (2.37), the first two terms of $\Psi_n(t)$ are given by

$$\Psi_1(t) = \int_0^t A(s) ds,$$

$$\Psi_2(t) = \frac{1}{2} \int_0^t \left[ A(s), \int_0^s A(\tau) d\tau \right] ds,$$

where $[A, B] \equiv AB - BA$ denotes the matrix commutator of $A$ and $B$.

### 2.4.3 Peano-Baker Series Solution of the Bloch Equation

**The first-order PBS solution of the Bloch equation**

We first introduce a second rotating frame to further factor out the precession caused by the gradient field. The new rotating frame is defined as

$$\begin{align*}
&\dot{i}'(t) = \cos(\phi(r, t))i - \sin(\phi(r, t))j, \\
&\dot{j}'(t) = \sin(\phi(r, t))i + \cos(\phi(r, t))j, \\
&\dot{k}'(t) = k, \\
&\phi(r, t) = \gamma \int_0^t G(s) \cdot r ds,
\end{align*}$$

where $i'(t)$, $j'(t)$, and $k'(t)$ are the unit directional vectors in the new rotating frame, and $i$, $j$, and $k$ are the unit directional vectors in the Larmor-rotating frame.

Following the derivations in Section 2.3.2, the Bloch equation in the new rotating frame is given by

$$\begin{bmatrix}
\dot{M}'_x(r, t) \\
\dot{M}'_y(r, t) \\
\dot{M}'_z(r, t)
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & 0 & -B'_{1,y}(t) \\
0 & 0 & B'_{1,x}(t) \\
B'_{1,y}(t) & -B'_{1,x}(t) & 0
\end{bmatrix}
\begin{bmatrix}
M'_x(r, t) \\
M'_y(r, t) \\
M'_z(r, t)
\end{bmatrix}.$$  

In Eq. (2.41), $[M'_x(r, t), M'_y(r, t), M'_z(r, t)]^T$ denotes the magnetization vector in the new rotating frame, and $B'_1(t) = B'_{1,x}(t) + iB'_{1,y}(t)$ denotes the RF field in the new rotating frame:

$$B'_1(t) = B_1(t)e^{i\phi(r, t)}.$$
first-order PBS solution of the Bloch equation in Eq. (2.41) is given by

\[
M'(r, t) \approx \begin{bmatrix}
1 & 0 & -\gamma \int_0^t B'_{1,y}(s)ds \\
0 & 1 & \gamma \int_0^t B'_{1,x}(s)ds \\
\gamma \int_0^t B'_{1,y}(s)ds & -\gamma \int_0^t B'_{1,x}(s)ds & 1
\end{bmatrix} M'(r, 0). \tag{2.43}
\]

Especially, suppose that the initial state of the magnetization vector is at its equilibrium state, i.e., \(M'(r, 0) = [0, 0, M_0(r)]^T\), the solution in Eq. (2.43) can be written as

\[
M'_{xy}(r, t) \approx i\gamma M_0(r) \int_0^t B_1'(s)ds, \tag{2.44}
\]
\[
M_z'(r, t) \approx M_0(r). \tag{2.45}
\]

Representing the solution in Eqs. (2.44) and (2.45) in the Larmor-rotating frame yields

\[
M_{xy}(r, t) \approx i\gamma M_0(r)e^{-i\phi(r,t)} \int_0^t B_1(s)e^{i\phi(r,s)}ds, \tag{2.46}
\]
\[
= i\gamma M_0(r) \int_0^t B_1(s)e^{i\mathbf{k}(s)\cdot \mathbf{r}}ds, \tag{2.47}
\]
\[
M_z(r, t) \approx M_0(r), \tag{2.48}
\]

where \(k\) is known as the excitation \(k\)-space \[27\]:

\[
k(s) = -\gamma \int_s^t G(\tau)d\tau. \tag{2.49}
\]

This is the “small-tip-angle” approximate solution in \[27, 54, 58\].

The first-order PBS solution of the spinor-domain Bloch equation

The PBS can be used to derive approximate solutions of the spinor-domain Bloch equation as well. Similar to the above derivations, we first introduce a second rotating frame, which
is defined as

\[
\begin{bmatrix}
\alpha'(r, t) \\
\beta'(r, t)
\end{bmatrix} =
\begin{bmatrix}
e^{-i\phi(r,t)/2} & 0 \\
0 & e^{i\phi(r,t)/2}
\end{bmatrix}
\begin{bmatrix}
\alpha(r, t) \\
\beta(r, t)
\end{bmatrix},
\]

(2.50)

\[
\phi(r, t) = \gamma \int_0^t G(s) \cdot rds.
\]

(2.51)

Substituting Eqs. (2.50) and (2.51) to the spinor-domain Bloch equation in Eq. (2.29), yields

\[
\begin{bmatrix}
\dot{\alpha}'(r, t) \\
\dot{\beta}'(r, t)
\end{bmatrix} =
\begin{bmatrix}
0 & B_1^*(t)e^{-i\phi(r,t)} \\
B_1(t)e^{i\phi(r,t)} & 0
\end{bmatrix}
\begin{bmatrix}
\alpha'(r, t) \\
\beta'(r, t)
\end{bmatrix}.
\]

(2.52)

The first-order PBS solution of the spinor-domain Bloch equation in Eq. (2.52) is given by

\[
\begin{bmatrix}
\alpha'(r, t) \\
\beta'(r, t)
\end{bmatrix} \approx
\begin{bmatrix}
1 & i\gamma \int_0^t B_1^*(s)e^{-i\phi(r,s)} ds \\
i\gamma \int_0^t B_1(s)e^{i\phi(r,s)} ds & 1
\end{bmatrix}
\begin{bmatrix}
\alpha'(r, 0) \\
\beta'(r, 0)
\end{bmatrix}.
\]

(2.53)

Note that the initial state \([\alpha'(r, 0), \beta'(r, 0)]^T\) is always \([1, 0]^T\) meaning no rotation at \(t = 0\).

Representing the solution in Eq. (2.53) in the Larmor-rotating frame yields

\[
\alpha(r, t) \approx e^{-(i/2)k(0) \cdot r},
\]

(2.54)

\[
\beta(r, t) \approx \frac{i\gamma}{2}e^{-(i/2)k(0) \cdot r} \int_0^t B_1(s)e^{ik(s) \cdot r} ds.
\]

(2.55)

This is the “small-excitation” solution in [28]. Note that the Cayley-Klein parameters in [28] follow the left-handed rotation convention.

Suppose that the initial state of the magnetization vector is at its equilibrium state, the end state of the magnetization vector can be calculated by substituting Eqs. (2.54) and (2.55) to Eq. (2.24), which yields the “small-tip-angle” solution in Eqs. (2.47) and (2.48).

**Properties of the first-order PBS solution**

- The first-order PBS solution, i.e., Eqs. (2.47) and (2.48) for the Bloch equation and Eqs. (2.54) and (2.55) for the spinor-domain Bloch equation, provides a linear approximation of the Bloch equation. Especially, in the first-order PBS solution, the magnetization vector or the Cayley-Klein parameters linearly depends on the input
RF pulse, leading to a linear control system. For such a linear control system, the RF pulse design problem can be easily formulated and efficiently solved.

- The first-order PBS solution does not represent a rotation. Especially, suppose that the RF field only contains a $x$-component, i.e., $B_1(t) = B_{1,x}(t)$, starting from its equilibrium state, the magnetization vector of on-resonance spins is $M_{xy} = iM_0 \sin \theta(t)$, where $\theta(t) = \gamma \int_0^t B_{1,x}(s)ds$. However, the corresponding first-order PBS solution is $M_{xy} = iM_0 \theta(t)$, which is a good approximation when $\theta(t) < 30^\circ$ but can contribute significant errors when $\theta(t) > 90^\circ$.

2.4.4 Magnus Series Solution of the Bloch Equation

The first-order MS solution of the spinor-domain Bloch equation

We derive the MS solution using the spinor-domain Bloch equation in the second rotating frame, i.e., Eq. (2.52). According to the MS solution of a general LTV system, i.e., Eqs. (2.33), (2.37) and (2.38), the first-order MS solution of the spinor-domain Bloch equation is given by

$$
\begin{bmatrix}
\alpha'(r,t) \\
\beta'(r,t)
\end{bmatrix}
\approx
\exp\left(\frac{i\gamma}{2}
\begin{bmatrix}
0 & \int_0^t B_1^*(s)e^{-i\phi(r,s)}ds \\
\int_0^t B_1(s)e^{i\phi(r,s)}ds & 0
\end{bmatrix}\right)
\begin{bmatrix}
\alpha'(r,0) \\
\beta'(r,0)
\end{bmatrix}.
$$

(2.56)

Note that the initial state $[\alpha'(r,0), \beta'(r,0)]^T$ is $[1, 0]^T$. Expanding the matrix exponential yields

$$
\alpha'(r,t) \approx \cos \left(\frac{\theta(r,t)}{2}\right),
$$

(2.57)

$$
\beta'(r,t) \approx -i \frac{\theta(r,t)}{\theta(r,t)} \sin\left(\frac{\theta(r,t)}{2}\right),
$$

(2.58)

where,

$$
\theta(r,t) = -\gamma e^{-ik(0)\cdot r} \int_0^t B_1(s)e^{ik(s)\cdot r}ds.
$$

(2.59)
Representing the solution in Eqs. (2.57) and (2.58) in the Larmor-rotating frame yields

\[
\alpha(r, t) \approx e^{i\phi(r, t)/2} \cos \left( \frac{|\theta(r, t)|}{2} \right),
\]

\[
\beta(r, t) \approx -ie^{-i\phi(r, t)/2} \frac{\theta(r, t)}{|\theta(r, t)|} \sin \left( \frac{|\theta(r, t)|}{2} \right).
\]

(2.60)

(2.61)

This is the first-order MS solution of the spinor-domain Bloch equation.

**Hard-pulse approximation**

The first-order MS solution can be used to derive the hard-pulse approximation, which is a key step of the SLR method [26]. More specifically, suppose that \(B_1\) and \(G\) are constant over a short period \(\Delta t\). According to Eq. (2.59), we have

\[
\theta(r, t) = -\gamma e^{-i\mathbf{k}(0) \cdot \mathbf{r}} \int_0^{\Delta t} B_1 e^{i\mathbf{k}(s) \cdot \mathbf{r}} ds,
\]

\[
= -\gamma e^{i\gamma GT \cdot \mathbf{r}} \int_0^{\Delta t} B_1 e^{-i\gamma GT \cdot (\Delta t - s)} ds,
\]

\[
= -\gamma B_1 \int_0^{\Delta t} e^{i\gamma GT \cdot \mathbf{r}} ds,
\]

\[
\approx -\gamma B_1 \Delta t.
\]

(2.62)

The approximation in the last step is a good one as long as \(\Delta t\) is small, i.e., a “hard-pulse”.

Substituting Eq. (2.62) to Eqs. (2.60) and (2.61), yields

\[
\alpha(r, t) \approx e^{i\gamma GT \cdot \mathbf{r}/2} \cos \left( \frac{|\gamma B_1 \Delta t|}{2} \right),
\]

\[
\beta(r, t) \approx ie^{-i\gamma GT \cdot \mathbf{r}/2} e^{i\gamma B_1} \sin \left( \frac{|\gamma B_1 \Delta t|}{2} \right).
\]

(2.63)

(2.64)

Or, equivalently we have

\[
\begin{bmatrix}
\alpha & -\beta^* \\
\beta & \alpha^*
\end{bmatrix} = \begin{bmatrix}
e^{i\Delta GT \cdot \mathbf{r}/2} & 0 \\
0 & e^{-i\Delta GT \cdot \mathbf{r}/2}
\end{bmatrix} \begin{bmatrix}
\cos \left( \frac{|\gamma B_1 \Delta t|}{2} \right) & ie^{-i\gamma B_1} \sin \left( \frac{|\gamma B_1 \Delta t|}{2} \right) \\
ie^{i\gamma B_1} \sin \left( \frac{|\gamma B_1 \Delta t|}{2} \right) & \cos \left( \frac{|\gamma B_1 \Delta t|}{2} \right)
\end{bmatrix}.
\]

(2.65)

Equation (2.65) is the hard-pulse approximation in the SLR method [26], despite that the order of the rotation matrices is different. However, this difference is trivial, and will not
affect the accuracy of pulse synthesisization when approximating a soft pulse with a series of hard pulses.

**Properties of the first-order MS solution**

- Although the Cayley-Klein parameters in the first-order MS solution (Eqs. (2.60) and (2.61)) do not linearly depend on the input RF pulse, the tip-angle (Eq. (2.59)) linearly depends on the input RF pulse. Therefore, the first-order MS solution also provides a linear approximation of the Bloch equation but in the terms of tip-angle.

- The first-order MS solution indeed represents a rotation. For on-resonance spins, the first-order MS solution is accurate. This advantage over the first-order PBS solution significantly increases the approximation accuracy [59].

- If the tip-angle is small, i.e., \( \sin(|\theta|/2) \approx |\theta|/2 \) and \( \cos(|\theta|/2) \approx 1 \), when \( \theta \) is small, the first-order MS solution in Eqs. (2.60) and (2.61) is reduced to the first-order PBS solution of the spinor-domain Bloch equation in Eqs. (2.54) and (2.55).

2.4.5 Perturbation Solution of the Bloch Equation

The series solution of a general LTV system can be used to derive the perturbation solution of the Bloch equation as well. In the following, we derive the perturbation solution of the spinor-domain Bloch equation.

**The perturbation solution of the spinor-domain Bloch equation**

Given an RF field \( B_1(t) \) and a gradient field \( G(t) \), suppose the resulting Cayley-Klein parameter is \( \alpha(r,t) \) and \( \beta(r,t) \):

\[
\begin{bmatrix}
\dot{\alpha}(r,t) \\
\dot{\beta}(r,t)
\end{bmatrix} = \frac{i\gamma}{2} \begin{bmatrix}
G(t) \cdot r & B_1^*(t) \\
B_1(t) & -G(t) \cdot r
\end{bmatrix} \begin{bmatrix}
\alpha(r,t) \\
\beta(r,t)
\end{bmatrix}.
\] (2.66)

Suppose that there is a small perturbation \( \tilde{B}_1(t) \) imposed on the RF field, the resulting
Cayley-Klein parameters can be written as

\[
\begin{bmatrix}
\dot{\alpha}(r, t) + \dot{\alpha}(r, t) \\
\dot{\beta}(r, t) + \dot{\beta}(r, t)
\end{bmatrix} = \frac{i\gamma}{2} \begin{bmatrix}
G(t) \cdot r & B_1^*(t) + \tilde{B}_1(t) \\
B_1(t) + \tilde{B}_1(t) & -G(t) \cdot r
\end{bmatrix} \begin{bmatrix}
\alpha(r, t) + \tilde{\alpha}(r, t) \\
\beta(r, t) + \tilde{\beta}(r, t)
\end{bmatrix},
\]  
(2.67)

where \(\tilde{\alpha}(r, t)\) and \(\tilde{\beta}(r, t)\) are the perturbations on the Cayley-Klein parameters. Our goal is to find a closed-form solution of \(\tilde{\alpha}\) and \(\tilde{\beta}(r, t)\) as a function of \(\alpha(r, t), \beta(r, t), B_1(t), G(t)\) and \(\tilde{B}_1(t)\).

Subtracting Eq. (2.66) from (2.67), yields

\[
\begin{bmatrix}
\dot{\tilde{\alpha}}(r, t) \\
\dot{\tilde{\beta}}(r, t)
\end{bmatrix} \approx \frac{i\gamma}{2} \begin{bmatrix}
G(t) \cdot r & B_1^*(t) \\
B_1(t) & -G(t) \cdot r
\end{bmatrix} \begin{bmatrix}
\tilde{\alpha}(r, t) \\
\tilde{\beta}(r, t)
\end{bmatrix} + \frac{i\gamma}{2} \begin{bmatrix}
\tilde{B}_1(t)\beta(r, t) \\
\tilde{B}_1(t)\alpha(r, t)
\end{bmatrix},
\]  
(2.68)

where the high-order perturbation terms \(\tilde{B}_1(t)\beta(r, t)\) and \(\tilde{B}_1(t)\alpha(r, t)\) are ignored.

Equation (2.68) can be rewritten in the second rotating frame defined in Eqs. (2.50) and (2.51) as

\[
\begin{bmatrix}
\dot{\tilde{\alpha}}'(r, t) \\
\dot{\tilde{\beta}}'(r, t)
\end{bmatrix} = \frac{i\gamma}{2} \begin{bmatrix}
0 & B_1^*(t)e^{-i\phi(r, t)} \\
B_1(t)e^{i\phi(r, t)} & 0
\end{bmatrix} \begin{bmatrix}
\tilde{\alpha}'(r, t) \\
\tilde{\beta}'(r, t)
\end{bmatrix} + \frac{i\gamma}{2} \begin{bmatrix}
\tilde{B}_1^*(t)\beta(r, t)e^{-i\phi(r, t)/2} \\
\tilde{B}_1(t)\alpha(r, t)e^{i\phi(r, t)/2}
\end{bmatrix},
\]  
(2.69)

For the differential equation in Eq. (2.69), the PBS and MS can be used to approximate the state transition matrix as discussed before, and the solution can be calculated using Eq. (2.33). More specifically, if the zero-th order MS is used, i.e., \(\Phi(r, t) = I\), the solution is given by

\[
\begin{aligned}
\tilde{\alpha}'(r, t) &= \frac{i\gamma}{2} \int_0^t \tilde{B}_1^*(s)\beta(r, s)e^{-i\phi(r, s)/2}ds, \\
\tilde{\beta}'(r, t) &= \frac{i\gamma}{2} \int_0^t \tilde{B}_1(s)\alpha(r, s)e^{i\phi(r, s)/2}ds.
\end{aligned}
\]  
(2.70)
Representing the solution in Eqs. (2.70) and (2.71) in the Lamor-rotating frame yields

\[
\tilde{\alpha}(r, t) = \frac{i\gamma}{2} \int_0^t \tilde{B}_1^*(s) \beta(r, s) e^{-i(k(s) \cdot r/2)ds}, \quad (2.72)
\]

\[
\tilde{\beta}(r, t) = \frac{i\gamma}{2} \int_0^t \tilde{B}_1(s) \alpha(r, s) e^{i(k(s) \cdot r/2)ds}. \quad (2.73)
\]

This is the perturbation solution in [60]. Note that the Cayley-Klein parameters in [60] follow the left-handed rotation convention.

**Properties of the perturbation solution**

- In the perturbation solution, the perturbations of the Cayley-Klein parameters \(\tilde{\alpha}(r, t)\) and \(\tilde{\beta}(r, t)\), e.g., Eqs. (2.72) and (2.73), depend linearly on the perturbation of the RF field \(\tilde{B}_1(s)\).

- The linearity of the perturbation solution is due to ignoring the high-order perturbation terms in deriving Eq. (2.68), and thus does not depend on the specific form of the state transition matrix nor whether \(B_1(t)\) is small. In other words, as long as the perturbation of the RF field is small, the perturbation solution provides a good linear approximation of Bloch equation around an “operating point”. The “operating point” is determined by \(B_1(t)\), which can correspond to a large-tip-angle RF pulse.

### 2.5 RF Pulse Design Using Linear Approximate Solutions of the Bloch Equation

This type of methods include the small-tip-angle method [27, 29, 36, 37, 61, 62], linear-class large-tip-angle method [28, 63], and perturbation method [60, 64, 65].

#### 2.5.1 Small-Tip-Angle RF Pulse Design

As discussed in Section 2.4.3, assuming that the initial state of the magnetization is at its equilibrium state and the tip-angle of the designed RF pulse is small, i.e., the small-tip-angle
(STA) assumption, there is a simple inverse Fourier transform relation between the RF pulse and the excitation pattern:

\[
M_{xy}(r, T)/M_0(r) \approx i\gamma \int_0^T B_1(t)e^{ik(t)\cdot r} dt.
\]  
(2.74)

In the 1D case, assuming a constant slice selection gradient, the RF pulse to be designed is simply the Fourier transform of the desired excitation pattern. This leads to the well-known sinc pulse \([39]\). In the multidimensional case, if the excitation \(k\)-space trajectory is Cartesian, the RF pulse can be calculated using the Fourier transform as well. If the excitation \(k\)-space trajectory is non-Cartesian, however, sophisticated sampling density compensation should be done before performing the Fourier transform \([27, 61]\).

Alternatively, Yip et al. have proposed an iterative method to design multidimensional RF pulses using arbitrary excitation \(k\)-space trajectories \([29]\). Discretizing Eq. (2.74) in space and time, the STA RF pulse design problem can be formulated as \([29]\):

\[
\hat{b} = \arg\min_b ||Ab - d||_W^2 + \mu ||b||_A^2.
\]  
(2.75)

In Eq. (2.75), \(b\) is a vector containing the samples of the RF waveform. \(d\) is a vector containing the samples of the desired excitation pattern defined as \(M_{xy,d}(r)/M_0(r)\). \(A\) denotes a system matrix, the \((m,n)\)-th entry of which is given by:

\[
a_{m,n} = i\gamma e^{ik(t_n)\cdot r_m} \Delta t,
\]  
(2.76)

where \((t_n, r_m)\) defines a discrete grid in time and space, and \(\Delta t\) is the sampling period in time. The first term of the cost function penalizes a weighted \(\ell_2\)-norm error, i.e., \((Ab - d)^H W (Ab - d)\), where \(^H\) denotes conjugate operation of a vector, and \(W\) is a diagonal matrix used to impose weightings at different spatial locations. The second term of the cost function is a regularization term, which is used to control the average and/or peak RF power by penalizing a weighted \(\ell_2\)-norm of the designed RF pulse.

Solving an unconstrained optimization problem is computationally more efficient than solving a constrained one. However, the regularization parameter \(\mu\) and the weighting matrix...
Λ should be carefully chosen. In fact, the optimization problem in Eq. (2.75) is convex and has a closed-form solution:

$$\hat{b} = (A^HWA + \mu \Lambda)^{-1}A^Hd.$$  \hspace{1cm} (2.77)

The RF pulse can be calculated either using the closed-form solution in Eq. (2.77), or by solving the optimization problem in Eq. (2.75) directly, e.g., using a conjugate gradient method.

### 2.5.2 Small-Tip-Angle RF Pulse Design for Parallel Excitation

In parallel excitation, the synthesized $B_1$ field is a sum of the $B_1$ field generated by each transmit coil:

$$B_1(r, t) = \sum_{l=1}^{L} s_l(r)b_l(t), \hspace{1cm} (2.78)$$

where $s_l(r)$ and $b_l(t)$ are the sensitivity map and waveform of the $l$-th transmit channel, respectively.

Under the STA assumption, the Bloch equation is linear, and thus the synthesized excitation pattern is a sum of the excitation pattern generated by each transmit channel. More specifically, we substitute Eq. (2.78) to the STA solution in Eq. (2.74) and obtain

$$M_{xy}(r, T)/M_0(r) \approx i\gamma \sum_{l=1}^{L} s_l(r) \int_0^T b_l(t)e^{i k(t) \cdot r} dt.$$  \hspace{1cm} (2.79)

Based on Eq. (2.79), the RF pulse can be designed either in the excitation $k$-space domain [36] or in the spatial domain [37]. We focus our discussion on the extension of the more general iterative method [62]. Discretizing the design equation in Eq. (2.79) as discussed before, the RF pulse can be calculated by solving the following optimization problem [62]:

$$\hat{b}_{PE} = \arg\min_{\hat{b}_{PE}} ||A_{PE}\hat{b}_{PE} - d||^2_W + \mu ||\hat{b}_{PE}||^2_A.$$  \hspace{1cm} (2.80)

In Eq. (2.80), $b_{PE}$ is a vector concatenating the discretized RF pulse waveforms of all the
channels, and $A_{PE}$ is given by

$$A_{PE} = [S_1, S_2, \cdots, S_L]A,$$  \hspace{1cm} (2.81)$$

where $S_l$ is a diagonal matrix containing the sensitivity map of the $l$-th channel and $A$ is defined in Eq. (2.76).

2.5.3 Linear-Class Large-Tip-Angle RF Pulse Design

Large-tip-angle (LTA) RF pulses can also be designed based on a linear approximation of the Bloch equation, known as the linear-class large-tip-angle (LCLTA) method [28]. In the LCLTA method, it is assumed that the pulse can be decomposed into a sequence of STA subpulses, each of which consists of an inherently refocused gradient waveform and a Hermitian symmetric RF subpulse.

If the tip-angle of each subpulse is small, according to the “small-excitation” solution (Eq. (2.55) in Section 2.4.3), for the $j$-th subpulse we have [28]

$$\beta_j(r, T_j) = -i(n_{x,j} + i n_{y,j}) \sin \frac{\theta_j(r, T_j)}{2} \quad \text{(definition of the Cayley-Klein parameters)}$$

$$\approx \frac{i\gamma}{2} e^{-(i/2)k_j(T_{j-1}) \cdot r} \int_{T_{j-1}}^{T_j} B_1(t) e^{ik_j(t) \cdot r} dt \quad \text{(the “small-excitation” solution)}$$

$$= \frac{i\gamma}{2} \int_{T_{j-1}}^{T_j} B_1(t) e^{ik_j(t) \cdot r} dt \quad \text{(an inherently refocused gradient waveform)}$$

$$= \frac{i\gamma}{2} \int_{T_{j-1}}^{T_j} B_1(t) e^{ik_j(t) \cdot r} dt,$$

imaginary regardless of $r$

where $k_j(t) = -\gamma \int_t^{T_j} G(\tau) d\tau$, and $k_j(T_{j-1}) = 0$.

Therefore there is a simple inverse Fourier transform relation between the RF pulse and the tip-angle:

$$\theta_j(r, T_j) \approx -\gamma \int_{T_{j-1}}^{T_j} B_1(t) e^{ik_j(t) \cdot r} dt.$$  \hspace{1cm} (2.82)$$

Note that the resulting tip-angle of such a subpulse is about the $x$-axis regardless the
initial state of magnetization and spatial location. The total tip-angle of the RF pulse is simply a sum of the tip-angle of each subpulse [28]:

\[
\theta(r, T) \approx \sum_{j=1}^{n} \theta_j(r, T_j), \\
\approx -\gamma \int_{T_0}^{T} B_1(t)e^{i\mathbf{k}(t) \cdot r} dt, 
\]

(2.83)

where \( \mathbf{k}(t) = -\gamma \int_{t}^{T} \mathbf{G}(\tau)d\tau \).

Based on Eq. (2.83), the LCLTA pulse can be designed in a way similar to the STA method. Common choices of inherently refocused gradient waveforms include spiral and radial trajectories.

In parallel excitation, Xu et al. has proposed that Eq. (2.83) can be generalized as [63]

\[
\theta(r, T) \approx -\gamma \sum_{l=1}^{L} s_l(r) \int_{0}^{T} b_l(t)e^{i\mathbf{k}(t) \cdot r} dt. 
\]

(2.84)

The LCLTA RF pulse for parallel excitation is then designed in a way similar to the STA method for parallel excitation.

It has been reported that the LTA RF pulses designed by the LCLTA method generated superior excitation patterns than those designed by the STA method [63]. The superior performance of the LCLTA method over the STA method can be understood from another perspective. On the one hand, note that the same design equation (Eq. (2.83)) can be derived from the first-order MS solution of the spinor-domain Bloch equation (Eq. (2.59)) without requiring Hermitian symmetric RF pulses. On the other hand, the STA method is essentially based on the first-order PBS solution of the Bloch equation. Since the first-order MS solution of the Bloch equation generally provides better approximation accuracy than the first-order PBS solution [59], the RF pulse designed based on the first-order MS solution (the LCLTA method) should generate a more accurate excitation pattern than that designed based on the first-order PBS solution (the STA method).
2.5.4 Perturbation Method for Large-Tip-Angle RF Pulse Design

The perturbation method for LTA RF pulse design [60, 64, 65] is an iterative method. An initial RF pulse is first designed either by the STA or the LCLTA method. The RF pulse is then refined in each step by adding small perturbations to the pulse to better approximate the desired excitation pattern.

Suppose that in the \( j \)-th step, the RF pulse is \( B_1^{(j)}(t) \), and the resulting magnetization is \( M^{(j)}(r, t) \). In the \((j + 1)\)-th step, a perturbation \( \tilde{B}_1(t) \) is added. The perturbation is determined by solving the following optimization problem [60, 65]:

\[
\text{argmin}_b ||m^{(j)} + \tilde{m}(\tilde{b}) - d||_W^2 + \mu ||b^{(j)} + \tilde{b}||_A^2, \tag{2.85}
\]

where, \( b^{(j)} \) is a vector containing the samples of the RF pulse \( B_1^{(j)}(t) \), \( \tilde{b} \) is a vector containing the samples of the perturbation \( \tilde{B}_1(t) \), \( m^{(j)} \) is a vector containing the samples of the magnetization \( M^{(j)}(r, T) \), and \( \tilde{m}(\tilde{b}) \) is a vector containing the samples of the perturbation of the magnetization.

Assuming that the perturbation is small, \( \tilde{m}(\tilde{b}) \) can be approximated by the perturbation solution of the Bloch equation as discussed in Section 2.4.5. The perturbation \( \tilde{m}(\tilde{b}) \) then becomes a linear function of \( \tilde{b} \):

\[
\tilde{m}(\tilde{b}) = \tilde{A}(b^{(j)})\tilde{b}, \tag{2.86}
\]

where \( \tilde{A}(b^{(j)}) \) is a matrix depends on \( b^{(j)} \), the elements of which need to be calculated by solving the Bloch equation numerically. Substituting Eq. (2.86) to Eq. (2.85), the perturbation of the RF pulse is finally determined in a way similar to the STA method.

2.5.5 Extensions

A large variety of methods have been proposed to further improve the above discussed RF pulse design methods that use linear approximate solutions of the Bloch equation. Significant extensions and generalizations have been made in hardware imperfection correction, phase relaxing and fast computation. In the following, we briefly summarize these advances. SAR
control is another very important practical problem especially for parallel excitation, which will be discussed in details in Section 2.8.

**Hardware imperfection correction**

$B_0$ inhomogeneity is a significant limiting factor for a multidimensional RF pulse due to its long duration, especially for high field MRI. Parallel excitation can effective reduce pulse length, and thus make multidimensional RF pulses less sensitive to $B_0$ inhomogeneities. However, it is still important to take $B_0$ inhomogeneity into account while designing multidimensional RF pulses even in the case of parallel excitation. The effects of $B_0$ inhomogeneity can be easily factored into the RF pulse design equation, e.g., Eq. (2.74) and Eq. (2.79), without introducing extra difficulties in solving the resulting optimization problem, e.g., Eqs. (2.75) and (2.80), [29, 62].

Gradient imperfections should also be carefully calibrated or compensated in multidimensional RF pulse design, otherwise geometric distortions and blurring may occur [66–70]. Common sources of gradient imperfections include eddy-currents, time-delay, and nonlinear fields. Eddy-currents and time-delay affect linear gradient fields, and thus introduce distortions to the excitation $k$-space trajectory. The effects of eddy-currents and time-delay can be compensated using the actual excitation $k$-space trajectory in the RF pulse design [66–69]. The actual excitation $k$-space trajectory can be obtained either through experimental measurements [66–68] or using a calibrated eddy-current model [69]. Schneider et al. have proposed a more general and sophisticated approach to compensate gradient imperfections [70]. More specifically, the RF pulse design equation, i.e., Eq. (2.79), is generalized as

$$M_{xy}(r, T)/M_0(r) \approx i\gamma \sum_{l=1}^{L} s_l(r) \int_0^T b_l(t) e^{i\phi(r,t)} dt,$$

where $\phi(r,t)$ is the phase evolution at spatial location $r$, which is determined by the actual gradient fields, $B_0$ inhomogeneity, and nonlinear gradient fields. If ideal gradient fields are assumed, Eq. (2.87) is reduced to Eq. (2.79), i.e., $\phi(r,t) = k(t) \cdot r$. Sophisticated sequences are used to measure the phase evolution $\phi(r,t)$ with the gradients that will be applied during excitation. The RF pulse is finally designed using the measured phase evolution $\phi(r,t)$ based on Eq. (2.87).
Phase relaxing

In the conventional RF pulse design in parallel excitation, the desired excitation pattern is specified with a desired magnitude profile and a uniform phase profile (after rephasing). However, since it is the magnitude of an image that is displayed, the phase of an excitation pattern is often not of interest, especially for saturation and inversion pulses. For excitation pulses, requiring uniform phase is not necessary as long as the phase variation across an imaging pixel would not lead to notable signal loss. The uniform phase constraint can be a significant limiting factor in parallel excitation because the phase profiles of an phased-array transmit coils are intrinsically inhomogeneous. It has been shown that relaxing the uniform phase constraint can significantly improve the quality of the resulting excitation pattern in parallel excitation [71–73].

In order to relax the phase constraint, the STA method for parallel excitation, i.e., Eq. (2.80), can be reformulated as the following magnitude least square problem [72]:

\[
\hat{b}_{PE} = \arg\min_{b_{PE}} \| A_{PE} b_{PE} - d \|_W^2 + \mu \| b_{PE} \|_A^2,
\]

where \( d \) is the desired magnitude profile. The optimization problem in Eq. (2.88) is unfortunately not convex, and thus only a local minimum is obtained. It can be solved using a nonlinear conjugate gradient method [72] or by converting the magnitude least square problem into a least square problem while updating the tentative phase of the desired excitation pattern iteratively [71, 73]. In [73], the tentative phase of the desired excitation pattern is smoothed in each iteration step to prevent signal loss due to severe phase variation across an imaging pixel.

Fast computation

Although the RF pulse design methods that use linear approximate solutions of the Bloch equation are computationally efficient in general, computation time may still be a limiting factor for long multidimensional RF pulses with non-Cartesian excitation \( k \)-space trajectories. It is especially true for the on-line implementation of these methods for parallel excitation. That is because, in principle, RF pulses should be designed for each subject, since the transmit \( B_1 \) sensitivity maps intrinsically depend on imaging subject.
Take the STA method as an example, the RF pulse is designed by solving the optimization problem in Eq. (2.75). Due to the similarities between the RF pulse design equation, i.e., Eq. (2.74) and the MRI imaging equation, the optimization problem in Eq. (2.75) is very similar to the one in MRI image reconstruction using non-Cartesian sampling trajectories. Techniques based on gridding [74–76] and nonuniform fast Fourier transform (NUFFTs) [77, 78] have been proposed to accelerate multidimensional RF pulse design especially in parallel excitation [60, 79]. In addition, graphics processing units (GPUs), which have draw increasing attentions in accelerating MRI image reconstruction [80], have also been used to accelerate RF pulse design in parallel excitation [81]. An acceleration factor of 20 has been reported [81].

2.5.6 Characteristics

The properties of the RF pulse design method using the linear approximate solutions of the Bloch equation are summarized as follows.

+ It is guided by the significant insights from the linear approximate solutions of the Bloch equation. The excitation k-space interpretation reveals the profound analogies between RF excitation and MRI imaging.

+ It is flexible for a variety of extensions, e.g., incorporating $B_0$ inhomogeneity and gradient imperfection correction and phase relaxing in RF pulse design. Most importantly, it has been extended to the case of parallel excitation based on the linearity of the solutions.

+ It is in general computationally efficient. With a closed-form approximate solution, the differential equation constraint in Eq. (2.1) is eliminated, resulting in a nonlinear optimization problem, which is often convex and thus can be solved with global optimality and high computational efficiency.

− Although capable of designing large-tip-angle RF pulses, it is less accurate than the methods using optimal control or pulse synthesization.
2.6 RF Pulse Design Using Optimal Control

The RF pulse design method using the linear approximate solutions of the Bloch equation in Section 2.5 may cause significant distortions on the excitation patterns in the large-tip-angle region due to the nonlinearity of the Bloch equation [30, 31]. To address this issue, many methods have been proposed to treat the nonlinearity of the Bloch equation accurately, including the simulated annealing method [82], neural network method [83], evolutionary method [84], SPINCALC [85], optimal control approach [30, 31, 86], etc. Among them, the optimal control approach has been widely applied in designing one-dimensional and multidimensional RF pulses due to its flexibility in systematically imposing constraints and relative efficiency in computation.

2.6.1 Theory

We describe the basic theory of the optimal control approach for multidimensional RF pulse design. Extension to parallel excitation is conceptually straightforward, and has been done by Xu et al. [31]. In the optimal control approach, instead of solving the constrained optimization problem in Eq. (2.1), the RF pulse is designed by solving the following unconstrained optimization problem [30]:

\[
\{\hat{u}(t), \hat{v}(t)\} = \arg\min_{u(t), v(t)} \mathcal{J}(u(t), v(t))
\]

\[
= \arg\min_{u(t), v(t)} C(M_{\text{full}}(T), \mathbf{D}_{\text{full}}) + \int_0^T L(u(t), v(t), t)dt
\]

\[
+ \int_0^T \lambda^T(t) \{f[u(t), v(t), M_{\text{full}}(t)] - \dot{M}_{\text{full}}(t)\}dt. \tag{2.89}
\]

In Eq. (2.89),

- \(u(t)\) and \(v(t)\) are the real and imaginary part of the RF waveform to be designed.

- \(M_{\text{full}}(t)\) is a vector concatenating the magnetization vector along different spatial locations, and the initial state \(M_{\text{full}}(0)\) is given.
• $C(M_{\text{full}}(T), D_{\text{full}})$ penalizes the distance between the final state $M_{\text{full}}(T)$ and the desired one $D_{\text{full}}$:

$$C(M_{\text{full}}(T)) = \frac{1}{2} [M_{\text{full}}(T) - D_{\text{full}}]^T W [M_{\text{full}}(T) - D_{\text{full}}],$$

(2.90)

where $W$ is a diagonal matrix for imposing nonnegative weights.

• $L(u(t), v(t))$ penalizes the RF power:

$$L(u(t), v(t)) = \frac{1}{2} \mu (u(t)^2 + v(t)^2),$$

(2.91)

where $\mu$ is a regularization parameter.

• $\lambda(t)$ is a Lagrange multiplier.

• $f[u(t), v(t), M_{\text{full}}(t)]$ is given by

$$f[u(t), v(t), M_{\text{full}}(t)] = [A_z(t) + A_x(t)u(t) + A_y(t)v(t)]M_{\text{full}}(t),$$

(2.92)

where $A_z(t)$, $A_x(t)$, and $A_y(t)$ are block diagonal matrices that concatenating right-hand side of the Bloch equations (Eq. (2.19)) along different spatial locations. The entries corresponding to location $r$ are given by

$$A_z(r, t) = -\gamma G(t) \cdot r I_z,$$

(2.93)

$$A_x(r, t) = -\gamma I_x,$$

(2.94)

$$A_y(r, t) = -\gamma I_y.$$  

(2.95)

In order to derive the first-order necessary condition for a local minimum of the optimization problem in Eq. (2.89), we first integrate the $\lambda^T(t) \dot{M}_{\text{full}}(t)$ term in $J(u(t), v(t))$ by parts.
and obtain

\[ J(u(t), v(t)) = C(M_{\text{full}}(T), D_{\text{full}}) - \lambda^T(T)M_{\text{full}}(T) + \lambda^T(0)M_{\text{full}}(0) + \int_0^T L(u(t), v(t), t) + \lambda^T(t)f[u(t), v(t), M_{\text{full}}(t)] + \dot{\lambda}^T(t)M_{\text{full}}(t) \, dt. \quad (2.96) \]

Assuming a small variation \( \delta u(t) \) and \( \delta v(t) \) is added to the control \( u(t) \) and \( v(t) \), the variation of the cost function \( J \) can be written as

\[ \delta J = \left( \frac{\partial C}{\partial M_{\text{full}}} - \lambda^T \right) \delta M_{\text{full}} \bigg|_{t=T} + \int_0^T \left( \lambda^T(t) \frac{\partial f}{\partial M_{\text{full}}} + \dot{\lambda}^T(t) \right) \delta M_{\text{full}}(t) \, dt \]

\[ + \int_0^T \left[ \left( \frac{\partial L}{\partial u} + \lambda^T(t) \frac{\partial f}{\partial u} \right) \delta u + \left( \frac{\partial L}{\partial v} + \lambda^T(t) \frac{\partial f}{\partial v} \right) \delta v \right] \, dt, \quad (2.97) \]

where \( \delta M_{\text{full}}(t) \) is the corresponding variation of the magnetization.

If \( \lambda(t) \) is chosen to satisfy the following differential equation:

\[ \dot{\lambda}^T(t) = -\lambda^T(t) \frac{\partial f}{\partial M_{\text{full}}}, \quad (2.98) \]

\[ \lambda^T(T) = \left. \frac{\partial C}{\partial M_{\text{full}}} \right|_{t=T}, \quad (2.99) \]

the first two terms on the right-hand side of Eq. (2.97) will vanish.

At a stationary point, \( \delta J \) should be zero for arbitrary \( \delta u \) and \( \delta v \), requiring

\[ \frac{\partial L}{\partial u} + \lambda^T(t) \frac{\partial f}{\partial u} = 0, \quad (2.100) \]

\[ \frac{\partial L}{\partial v} + \lambda^T(t) \frac{\partial f}{\partial v} = 0. \quad (2.101) \]

According to Eqs. (2.98) to (2.101), the first-order necessary condition for a local minimum
is given by \( [30, 87] \)

\[
\dot{M}_{\text{full}}(t) = [A_z(t) + A_x(t)u(t) + A_y(t)v(t)]M_{\text{full}}(t),
\]

\( (2.102) \)

\[
M_{\text{full}}(0) = M_{\text{full},0},
\]

\( (2.103) \)

\[
\dot{\lambda}(t) = [A_z(t) + A_x(t)u(t) + A_y(t)v(t)]\lambda(t),
\]

\( (2.104) \)

\[
\lambda(T) = W(M_{\text{full}}(T) - D_{\text{full}}),
\]

\( (2.105) \)

\[
0 = \lambda^T(t)A_x(t)M_{\text{full}}(t) + \mu u(t),
\]

\( (2.106) \)

\[
0 = \lambda^T(t)A_y(t)M_{\text{full}}(t) + \mu v(t),
\]

\( (2.107) \)

where \( M_{\text{full},0} \) is a vector concatenating the initial magnetization at each spatial location. Note that solving the above equations is a two point boundary value problem, since the initial state of \( M_{\text{full}}(t) \) is given at \( t = 0 \) while the end state of \( \lambda(t) \) is determined by Eq. (2.105) at \( t = T \). This type of problem does not have a closed-form solution in general. Based on the first-order necessary condition for a local minimum in Eqs. (2.102) - (2.107), many numerical methods, including gradient decent method [30] and conjugate gradient method [86], have been used.

In the gradient decent method [30], in the \((j+1)\)-th step the control \( u^{(j+1)}(t) \) and \( v^{(j+1)}(t) \) is updated as follows:

\[
u^{(j+1)}(t) = u^{(j)}(t) - a\left(\frac{\partial L}{\partial u} + \lambda^T(t)\frac{\partial f}{\partial u}\right)\bigg|_{u=u^{(j)},v=v^{(j)},M_{\text{full}}=M_{\text{full}}^{(j)},\lambda=\lambda^{(j)}}
\]

\[
u^{(j+1)}(t) = v^{(j)}(t) - a\left(\frac{\partial L}{\partial v} + \lambda^T(t)\frac{\partial f}{\partial v}\right)\bigg|_{u=u^{(j)},v=v^{(j)},M_{\text{full}}=M_{\text{full}}^{(j)},\lambda=\lambda^{(j)}}
\]

\( (2.108) \)

\( (2.109) \)

where \( a \) denotes step size, \( M_{\text{full}}^{(j)}(t) \) is calculated by integrating the differential equation Eq. (2.102) forward with the control \( u^{(j)}(t) \) and \( v^{(j)}(t) \), and \( \lambda^{(j)}(t) \) is calculated by integrating the differential equation Eq. (2.104) backward with the control \( u^{(j)}(t) \) and \( v^{(j)}(t) \). It is easy to show that for a sufficient small \( a \), updating the control using Eqs. (2.108) and (2.109)
always decreases the value of the cost function. The initial control \(u^{(0)}(t)\) and \(v^{(0)}(t)\) can be determined using the RF pulse design methods discussed in Section 2.5.

2.6.2 Characteristics

The properties of the optimal control approach are summarized as follows.

+ It accurately treats the nonlinearity of the Bloch equation, since the Bloch equation is directly considered as a differential equation constraint in the optimization problem. It generates more accurate large-tip-angle RF pulses than the methods using the linear approximate solutions of the Bloch equation.

+ It is flexible for various extensions, e.g., incorporating SAR and peak RF power constraints in RF pulse design. Especially, it has been extended to the case of parallel excitation.

− It only achieves a local minimum of the cost function, and thus is sensitive to the initial guess.

− It is less computational efficient than the methods using the linear approximate solutions of the Bloch equation. Especially, the Bloch equation is numerically solved in each iterative step, which is computationally expensive.

2.7 RF Pulse Design Using Pulse Synthesization

There are two major methods belonging to this type of RF pulse design methods: \textit{inverse scattering method} [88–95] and \textit{SLR method} [26, 96–101]. Although the mathematical foundation of the two methods is fundamentally different, the practical implementation of the two methods [26, 93, 95] has some common features: (1) the hard-pulse approximation is used for recursively pulse synthesization; (2) the RF pulse design problem is converted to a polynomial design problem; (3) the polynomial is designed by solving a minimax approximation problem; (4) RF power control is achieved through control of auxiliary parameters;
and (5) the RF pulse is calculated using a recursive inversion transform from the designed polynomials. Our discussion will focus on the SLR method, because it is more familiar to the MRI community and more widely used in practice. We note that compared to the SLR method, the inverse scattering method has direct control of the phase of the achieved excitation pattern at cost of indirect control of the pulse length [93].

2.7.1 Theory

Forward SLR transform

In the SLR method, the gradient is assumed to be constant and the RF waveform is assumed to be piece-wise constant. The rotation of the pulse can be considered as a product of a sequence of rotations:

\[
\begin{bmatrix}
\alpha_n & -\beta_n^* \\
\beta_n & \alpha_n^*
\end{bmatrix} = \prod_{j=0}^{n} Q_j,
\]

(2.110)

where \( \alpha_n \) and \( \beta_n \) are the Cayley-Klein parameters representing the rotation of a pulse of \( n \) piece-wise segments, and \( Q_j \) represents the rotation of the \( j \)-th segment. Since the initial state is no rotation, \( Q_0 \) is an identity matrix.

The SLR method uses the hard-pulse approximation to represent the rotation of each segment. Especially, if the rotation of each segment is small, the rotation can be decomposed into two sequential rotations: the first rotation is precession by the gradient field; and the second rotation is nutation by the RF field. Suppose the gradient field is along the \( x \)-axis and the amplitude is \( G \), the RF field in the \( n \)-th segment is \( B_{1,n} \), and the duration of each segment is \( \Delta t \), according to the hard-pulse approximation [26], the rotation matrix \( Q_n \) can be approximated by

\[
Q_n \approx \begin{bmatrix} C_n & -S_n^* \\ S_n & C_n \end{bmatrix} \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix},
\]

(2.111)

where

\[
\begin{align*}
C_n &= \cos \left( \gamma |B_{1,n}| \Delta t / 2 \right), \\
S_n &= ie^{i\gamma B_{1,n}} \sin \left( \gamma |B_{1,n}| \Delta t / 2 \right), \\
z &= e^{i \gamma G x \Delta t}.
\end{align*}
\]

(2.112)
Substituting Eq. (2.111) to Eq. (2.110), the recursion of the Cayley-Klein parameters is given by
\[
\begin{bmatrix}
\alpha_n \\
\beta_n
\end{bmatrix} = z^{1/2} \begin{bmatrix}
C_n & -S_n^* \\
S_n & C_n
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & z^{-1}
\end{bmatrix} \begin{bmatrix}
\alpha_{n-1} \\
\beta_{n-1}
\end{bmatrix},
\]
which can be rewritten as
\[
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix} = \begin{bmatrix}
C_n & -S_n^* \\
S_n & C_n
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & z^{-1}
\end{bmatrix} \begin{bmatrix}
A_{n-1} \\
B_{n-1}
\end{bmatrix},
\]
where
\[
A_n = z^{-n/2} \alpha_n, \\
B_n = z^{-n/2} \beta_n.
\]
In fact, \(A_n\) and \(B_n\) can be written as \((n - 1)\)-th order polynomials of \(z\):
\[
A_n(z) = \sum_{j=0}^{n-1} a_j^{(n)} z^{-j},
\]
\[
B_n(z) = \sum_{j=0}^{n-1} b_j^{(n)} z^{-j}.
\]
Equations (2.114) - (2.117) map an RF pulse into two polynomials, and are known as the forward SLR transform.

**Inverse SLR transform**

Given two polynomials \(A_n\) and \(B_n\) in Eqs. (2.116) and (2.117), to represent a rotation, \(A_n\) and \(B_n\) should satisfy
\[
|A_n(z)|^2 + |B_n(z)|^2 = 1, \forall |z| = 1.
\]
The inverse recursion of the polynomials is given by
\[
\begin{bmatrix}
A_{n-1} \\
B_{n-1}
\end{bmatrix} = \begin{bmatrix}
C_n & S_n^* \\
-S_n z & C_n z
\end{bmatrix} \begin{bmatrix}
A_n \\
B_n
\end{bmatrix}.
\]
Note that $A_{n-1}$ and $B_{n-1}$ are two $(n-2)$-order polynomials of $z$, which requires

$$C_n a^{(n)}_{n-1} + S_n^* b^{(n)}_{n-1} = 0, \quad (2.120)$$
$$-S_n a^{(n)}_0 + C_n b^{(n)}_0 = 0. \quad (2.121)$$

These two equations are equivalent, if Eq. (2.118) is satisfied [26].

The RF field of the $n$-th segment is determined by

$$\frac{b^{(n)}_{1,n}}{a^{(n)}_0} = \frac{S_n}{C_n} \cdot \frac{i e^{i \gamma |B_{1,n}| \Delta t/2}}{\cos (\gamma |B_{1,n}| \Delta t/2)},$$

$$B_{1,n} = \frac{2}{\gamma \Delta t} e^{i \gamma (-i b^{(n)}_0 / a^{(n)}_0)} \arctan |b^{(n)}_0 / a^{(n)}_0|. \quad (2.122)$$

Equations (2.119) and (2.122) map the coefficients of two polynomials to an RF pulse, and are known as the inverse SLR transform.

**Polynomial design**

The forward and inverse SLR transform builds a bijective mapping between an RF pulse and two complex polynomials. The RF pulse design problem is then equivalent to a polynomial design problem.

Instead of designing the two polynomial $A_n$ and $B_n$ simultaneously, the SLR method only designs the polynomial $B_n$ that is directly related to the resulting excitation profile. The polynomial $A_n$ is then calculated from a properly designed polynomial $B_n$. More specifically, given a polynomial $B_n$, the magnitude of the polynomial $A_n$ is determined by Eq. (2.118):

$$|A_n(z)| = \sqrt{1 - |B_n(z)|^2}. \quad (2.123)$$

The phase of the polynomial $A_n$ is now a free parameter, and an additional constraint should be imposed to determine the polynomial $A_n$ uniquely. It is shown that the polynomial $A_n$ of minimum-phase leads to the minimum RF power, and is given by [26]

$$A_n(z) = |A_n(z)| \exp [i \mathcal{H}\{\log(|A_n(z)|)}], \quad (2.124)$$
where “\(\mathcal{H}\)” denotes the Hilbert transform.

The RF pulse design problem is now reduced to the design of the polynomial \(B_n\), which can be efficiently solved using the Parks-McClellan algorithm in finite impulse response (FIR) filter design. The Parks-McClellan algorithm designs a polynomial that minimizes the maximum ripple/error in the predetermined passband and stopband [102, 103]. More importantly, the tradeoff among conflicting design parameters, including ripple level, time-bandwidth product and transition bandwidth, is made explicitly [26]. Once the polynomial \(B_n\) and \(A_n\) are designed, the RF pulse is calculated using the inverse SLR transform.

### 2.7.2 Extension

Significant extensions of the SLR method have been developed, especially in designing nonlinear, nonminimum phase RF pulses [104–109]. The phase of the resulting excitation profile is sometimes not of interest, which is especially true for inversion and saturation pulses. Relaxing the phase constraint of the resulting excitation profile has been proved useful in reducing peak RF power [104, 106, 107] and increasing robustness to \(B_1\) inhomogeneities [109].

In the SLR method, the phase of the resulting excitation pattern is determined by the phase of the polynomial \(A_n(z)\) and \(B_n(z)\). As a minimum-phase polynomial \(A_n(z)\) will lead to the minimum RF power, it is more advantageous to modify the phase of the polynomial \(B_n(z)\) to design a nonlinear phase RF pulse. There are two major type methods to do that: zero-flipping and complex FIR filter design.

In the zero-flipping method, a linear-phase polynomial \(B_{2n-1}(z)\) of order \(2n - 1\) is first designed to determine the magnitude profile \(|B_n(z)|^2\). The zeros of \(|B_n(z)|^2\) come in pairs \((z_j, 1/(z_j)^*)\), where \(|z_j| > 1\). The polynomial \(B_n(z)\) can be written as

\[
B_n(z) = c \prod_{j=0}^{n-1} (z - z_j'), \quad z_j' = z_j \text{ or } 1/z_j^*.
\]

The phase of the polynomial \(B_n(z)\) can then be modified by flipping the zeros without affecting the magnitude profile of the resulting excitation pattern. Especially, if all the zeros of \(B_n(z)\) are chosen to be inside the unit circle, the resulting polynomial \(B_n(z)\) is a
minimum-phase polynomial.

However, it is not easy to determine which zeros should be flipped to generated an RF pulse of the minimum peak RF power. The problem itself is a combinatorial optimization problem, the computation complexity quickly become infeasible as the order of the polynomials increases. Genetic algorithms have been proposed to solve this problem [108]. Another issue of the zero-flipping method is that it can not achieve a specific phase pattern. This issue can be addressed by designing a complex polynomial $B_n(z)$ directly.

There are cases that it is beneficial to design the polynomial $B_n(z)$ with a specific phase pattern. For instance, a quadratic-phase polynomial has been proved to be advantageous in reducing peak RF power [106] and/or increasing robustness to $B_1$ inhomogeneities [109]. This type of polynomial can be designed with equiripple errors using the complex FIR filter design method [110, 111].

In the conventional equiripple FIR filter design, the FIR filter design problem is converted to an approximation problem, which attempts to find a set of real coefficients that approximates a target real function with real sinusoidal functions in terms of minimax error. The Remez exchange algorithm is used to solve the approximation problem. The core of the Remez exchange algorithm is the alternation property, which is a necessary and sufficient condition for an optimum [102]. In the complex case, the complex FIR filter design problem is converted to an approximation problem, which attempts to find a set of complex coefficients that approximates a target complex function with real sinusoidal functions in terms of minimax error. The complex Remez exchange algorithm is used to solve the approximation problem. The alternation property, however, is only a sufficient condition in the complex case [110]. Nevertheless, satisfactory complex FIR filter design is usually obtained in practice [111].

2.7.3 Characteristics

The properties of the SLR method are summarized as follows.

+ It accurately treats the nonlinearity of the Bloch equation, and thus generates accurate large-tip-angle RF pulses. Through pulse synthesisization, the nonlinearity of the Bloch
equation is well preserved. The hard-pulse approximation turns to be highly accurate as long as each hard-pulse is short.

+ It allows explicit tradeoff among conflicting design parameters, including ripple level, time-bandwidth product and transition bandwidth.

+ It achieves equiripple excitation error with global optimality and high computational efficiency. Furthermore, minimum RF power is always achieved.

− It is limited to design one-dimensional RF pulses. It has been very challenging to design multidimensional RF pulses using the SLR method. Chapter 3 of this thesis presents a significant extension of the conventional SLR method to the multidimensional case. However, it remains challenging to design multidimensional RF pulses using the SLR method with non-Cartesian excitation $k$-space trajectories and in the case of parallel excitation.

2.8 Practical Issues

2.8.1 Specific Absorption Rate Control

In MRI, RF energy is nonionizing, electromagnetic radiation. The main safety issue for RF pulses is thermal heating, since it causes increase in tissue temperature. Limitations or recommendations are imposed on the maximum temperature increase in MRI procedures. For instance, in the normal mode of operation, whole body temperature rise should be less than 0.5 °C. However, tissue temperature is subject to regularization of the complex thermoregulatory system of the individual. It is, unfortunately, not practical to monitor the tissue temperature of the individual in MRI. Instead, limitations are imposed on specific absorption rate (SAR), which is a dosimetric measure of the rate at which thermal energy is absorbed by the body when exposed to an RF field.
SAR is defined as:

\[
\text{SAR} = \frac{\text{Total RF energy dissipated in a sample (Joules)}}{\text{Exposure time (Seconds) } \times \text{Sample weight (Kg)}} \text{ (Watts/Kg)}.
\] (2.126)

Whole body SAR is a metric regarding to total RF energy dissipated in the body, while local SAR is a metric regarding to RF energy dissipated in a local region of the body. In the normal mode of operation, Food and Drug Administration requires whole body SAR not higher than 1.5 W/kg over any period of 15 minutes, head SAR not higher than 3.0 W/kg over any period of 10 minutes, and local tissue SAR not higher than 8.0 W/kg in the head over any period of 5 minutes [112].

SAR is a very important safety issue for RF pulse design in high field MRI, especially for parallel excitation for two reasons. First, SAR increases dramatically as the field strength increases. Furthermore, the spatial distribution of SAR is increasingly inhomogeneous, making it hard to predict the ratio of whole body SAR to local SAR. Second, in parallel excitation, SAR is a complicate function of a variety of parameters, including, coil, excitation pattern, tip-angle, excitation \(k\)-space trajectory, reduction factor, RF pulse design method, and so on. It has been shown through simulation that it is not uncommon for RF pulses in parallel excitation to exceed the limits on local SAR and whole body SAR, especially for those with large tip-angles and/or large reduction factors. Simulations also predict that local SAR is likely the limiting factor, which is harder to predict, calculate, or measure than whole body SAR.

In the following, we briefly review the techniques for SAR control in parallel excitation.

**The relation between SAR and RF pulse**

In MRI, whole body SAR during an RF pulse can be written as:

\[
\text{SAR}_{ave} = \frac{1}{T} \int_{0}^{T} \frac{1}{V} \int_{V} \frac{\sigma(r)}{2\rho(r)} |E(r, t)|^2 dr dt,
\] (2.127)

where \(\sigma(r)\) is tissue conductivity, \(\rho(r)\) is density, \(V\) denotes the volume of an imaging object, \(T\) is the length of the RF pulse, and \(E(r, t)\) is the electrical field induced by the RF pulse.

In MRI, the bandwidth of an RF pulse (\(10^2\) to \(10^3\) Hz) is relatively small compared to
resonance frequency ($10^6$ Hz). The electrical field $\mathbf{E}(\mathbf{r}, t)$ can thus be approximated as:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_u(\mathbf{r})b(t), \quad (2.128)$$

where $b(t)$ is the waveform of the RF pulse and $\mathbf{E}_u(\mathbf{r})$ denotes the electrical field induced by an RF pulse with a unit voltage.

Substituting Eq. (2.128) to (2.127), we obtain the following relation between whole body SAR and RF pulse:

$$\text{SAR}_{\text{ave}}(b(t)) = \frac{1}{V} \int \frac{\sigma(\mathbf{r})}{2\rho(\mathbf{r})}[\mathbf{E}_u^H(\mathbf{r}) \cdot \mathbf{E}_u(\mathbf{r})]d\mathbf{r} \frac{1}{T} \int_0^T b^*(t)b(t)dt. \quad (2.129)$$

Equation (2.129) states that whole body SAR is proportional to RF power.

Local SAR is a function of spatial location and is given by:

$$\text{SAR}_{\text{local}}(\mathbf{r}, b(t)) = \frac{\sigma(\mathbf{r})}{2\rho(\mathbf{r})}[\mathbf{E}_u^H(\mathbf{r}) \cdot \mathbf{E}_u(\mathbf{r})] \frac{1}{T} \int_0^T b^*(t)b(t)dt. \quad (2.130)$$

Equations (2.129) and (2.130) indicate that in the single-channel case, a minimum RF power pulse is also a minimum SAR pulse.

In parallel excitation, due to the linearity of the Maxwell equation, electrical field can be written as:

$$\mathbf{E}(\mathbf{r}, t) = \sum_{l=1}^{L} \mathbf{E}_{u,l}(\mathbf{r})b_l(t), \quad (2.131)$$

where $b_l(t)$ is the RF pulse waveform of the $l$-th channel, and $\mathbf{E}_{u,l}(\mathbf{r})$ denotes the electrical field induced by an RF pulse with a unit voltage in the $l$-th channel.

Substituting Eq. (2.131) to Eq. (2.127), yields:

$$\text{SAR}_{\text{ave}}(\mathbf{b}(t)) = \int_0^T \mathbf{b}(t)\mathbf{b}^H(t)\mathbf{b}(t)dt, \quad (2.132)$$

where

$$\mathbf{b}(t) = [b_1(t), b_2(t), \ldots, b_L(t)]^T, \quad (2.133)$$
and

\[ \Psi_{l,m}^{\text{ave}} = \int \frac{\sigma(r)}{2\rho(r)} [E_{u,l}^*(r) \cdot E_{u,m}(r)] dr. \]  

(2.134)

Accordingly, in parallel excitation, local SAR can be written as:

\[ \text{SAR}_{\text{local}}(b(t), r) = \int_0^T b^H(t) \Psi_{\text{local}}(r) b(t) dt, \]  

(2.135)

where

\[ \Psi_{l,m}^{\text{local}}(r) = \frac{\sigma(r)}{2\rho(r)} [E_{u,l}^*(r) \cdot E_{u,m}(r)]. \]  

(2.136)

**SAR control**

There are three complementary type of methods to control SAR in RF pulse design: (1) optimizing the gradient waveform or excitation \( k \)-space trajectory [113, 114]; (2) using regularization or imposing constraints in RF pulse design [37, 38, 79, 115–117]; and (3) adaptively manipulating both the gradient and RF pulse waveform based on the variable-rate selective excitation (VERSE) technique [68, 118, 119].

In the first type of methods, the excitation \( k \)-space trajectory, i.e., variable-density spiral trajectories, are optimized to reduce SAR or peak RF power. The variable-density spirals can be described by a parametric model. These parameters can be optimized to reduce SAR [114]. However, the slew rate of the corresponding gradient may exceed hardware limit. Xu et al. proposed a method to design variable slew rate spirals to address this issue directly [113].

In the second type of methods, SAR is controlled by introducing regularization terms or constraints in the formulation of RF pulse design. For instance, for the STA method, the RF pulse design problem (Eq. (2.80)) can be reformulated as the following unconstrained optimization problem [117]:

\[ \hat{b}_{PE} = \arg\min_{b_{PE}} ||A_{PE}b_{PE} - d||_W^2 + \mu ||b_{PE}||_A^2 + \sum_j \eta_j b_{PE}^H \Psi_{PE,j} b_{PE}, \]  

(2.137)

where \( \Psi_{PE,j} \) is defined based on Eq. (2.134) and Eq. (2.136) and is used to control whole body SAR and local SAR. Choosing regularization parameter value is not trivial. The
multi-shift conjugate gradient algorithm in [120] can be used to simultaneously design the RF pulse and determine the optimal regularization parameter value with high computation efficiency. Alternatively, the RF pulse design problem can be formulated as a constrained optimization problem. In [37,115], whole body SAR is minimized while excitation fidelity is treated as a constraint. In [79], excitation error is minimized while whole body SAR and local SAR is treated as constraints. For both the constrained and unconstrained formulations, efficiency issue may arise when controlling local SAR as the number of regularization terms or constraints may be large.

The third type of methods are based on the VERSE principle first proposed by Conolly et al. [121,122]. The VERSE principle states that for a given RF pulse if at any time point the RF field and gradient field are modified with the same ratio the resulting excitation pattern will keep unchanged. Let $B_1(r,t)$ and $G(t)$ denote the original RF field and gradient field, the VERSE adapted RF field and gradient field $B_1^v(r,t)$ and $G^v(t)$ are given by:

$$
B_1^v(r,t) = B_1(r,\tau(t))\dot{\tau}(t),
$$

$$
G^v(t) = G(\tau(t))\dot{\tau}(t),
$$

where $\tau(t)$ is called time-dilation function, and $\dot{\tau}(t) > 0$. VERSE is a post-processing technique that can locally modify the RF and gradient field to control SAR without changing the resulting excitation pattern.

While it is possible to directly minimize SAR or impose SAR constraints, in [68,118,119] SAR is controlled indirectly by imposing a peak $B_1$ constraint. More specifically, in [118,119]
a time-optimal pulse is designed by solving the following optimization problem:

\[
\begin{align*}
\min & \quad T^v \\
\text{s.t.} & \quad b^v_l(t) = b_l(\tau(t))\dot{\tau}(t), l = 1, 2, \ldots, L, \\
& \quad G^v(t) = G(\tau(t))\dot{\tau}(t), \\
& \quad |b^v_l(t)| < B_{1,\text{max}}, l = 1, 2, \ldots, L, \\
& \quad |G^v(t)| < G_{\text{max}}, \\
& \quad |G^v(t)| < S_{\text{max}}, \\
& \quad \tau(0) = 0, \tau(T^v) = T, \dot{\tau}(t) > 0,
\end{align*}
\]

(2.140)

where \(T^v\) is the RF pulse length after the VERSE operation, \(b_l(t)\) and \(b^v_l(t)\) are the RF pulse waveform of the \(l\)-the channel before and after the VERSE operation. The above time-optimal problem can be difficult to solve for complex curves. Instead, the problem is converted to the \(s\)-domain, where the problem is greatly simplified and efficiently solved [123].

A drawback of the VERSE technique is that the VERSE adapted RF pulse is sensitive to \(B_0\) inhomogeneity. It is often necessary to redesign the RF pulse with the VERSE adapted gradient waveform to compensate \(B_0\) inhomogeneity effects [68, 119].

**SAR prediction**

In order to directly control SAR, it is critical to be able to predict subject-specific SAR. Zhu et al. have proposed a method to predict subject-specific whole body SAR through experimental calibration [124].

According to Eq. (2.132), the total dissipated RF power at time \(t\) is given by:

\[
p(t) = b^H(t)\Psi^{\text{ave}}b(t),
\]

\[
= \sum_{l=1}^{L} \sum_{m=1}^{L} b^*_l(t)b_m(t)\Psi^{\text{ave}}_{l,m},
\]

(2.141)

where \(\Psi^{\text{ave}}\) is Hermitian symmetric and independent on time as defined in Eq. (2.134).

For a human MRI scanner, the RF power loss in the RF transmission chain is negligible compared to the RF power dissipated in a subject. Therefore, the total dissipated RF power
can be determined by measuring the forward and reflected power of the RF power amplifier of each channel:

\[ p(t) = \sum_{l=1}^{L} p_{\text{forward},l}(t) - \sum_{l=1}^{L} p_{\text{reflected},l}(t). \]  

(2.142)

The matrix \( \Psi^{\text{ave}} \) can be experimentally determined as the following. A set of preselected “test pulse” are used to drive phased-array transmit coils. The dissipated RF power is recorded based on Eq. (2.142). Entries of \( \Psi^{\text{ave}} \) are determined by Eq. (2.141). Once \( \Psi^{\text{ave}} \) is determined, whole body SAR of an arbitrary RF pulse can be predicted using Eq. (2.132), facilitating SAR management using the previously discussed SAR control methods.

However, predicting local SAR is still challenging. Although pioneering exploration has been done [125], there is no established method to directly measure electrical field in MRI. Alternatively, local SAR is so far mainly predicted through simulations using generic human body models [126].

2.8.2 \( B_1 \) Mapping

The majority of RF pulse design methods in parallel excitation require subject-specific \( B_1 \) maps of each channel as a priori information. An ideal \( B_1 \) mapping should be: (1) fast, especially for applications that require 3D \( B_1 \) maps since measurements should be done for each channel and each subject; (2) accurate for a large range of \( B_1 \) values, since each transmit coil is a surface coil and usually has a large dynamic range in the \( B_1 \) profile; and (3) robust to noise, especially in regions of low \( B_1 \) values, where the acquired images are expected to be of low signal-to-noise ratio.

A variety of \( B_1 \) mapping methods have been developed. These methods can be classified into two categories: methods based on image magnitude [127–140] and methods based on image phase [141–143]. In the following, we briefly review these methods.

\( B_1 \) mapping based on image magnitude

The image magnitude based methods in this category can be roughly classified into two types: those using a fixed repetition time but varying tip-angles [127–134,136,140] and those
In the first type of methods, methods based on gradient echo [129], spin echo [127, 132] and stimulated echo [128, 130, 140] have been developed. Among them, the double angle method (DAM), which is based on gradient echo signals, is most widely used for $B_1$ mapping in parallel excitation due to its good sensitivities in the small-tip-angle region [129].

In the basic form of DAM, two gradient echo images $I_1(r)$ and $I_2(r)$ are acquired with nominal tip-angle $\theta_1$ and $\theta_2$, respectively. The image magnitude of these two images are related to a $B_1$ map by:

$$\frac{|I_1(r)|}{|I_2(r)|} = \frac{\sin(s(r)\theta_1)f(\theta_1, T_1(r), T_R)}{\sin(s(r)\theta_2)f(\theta_2, T_1(r), T_R)},$$

(2.143)

where $S(r)$ denotes $B_1$ map, $T_R$ denotes repetition time and $f(\theta, T_1(r), T_R)$ is given by:

$$f(\theta, T_1(r), T_R) = \frac{1 - e^{-T_R/T_1(r)}}{1 - e^{-T_R/T_1(r)}\cos(S(r)\theta)}.$$  

(2.144)

If $T_R$ is much larger than $T_1(r)$, $f(\theta, T_1(r), T_R) \approx 1$, the $B_1$ map can be easily calculated from Eq. (2.143).

One of the major problems of DAM is its long repetition time for eliminating $T_1$ factor, which results in long data acquisition time. Compensation pulses have been proposed to remove the dependence on $T_1$ to enable fast imaging [131, 133, 135]. Alternatively, if more than two-tip angles are used, $B_1$ map can be jointly estimated with $T_1$ map with short $T_R$ [136].

The second type of methods are also known as the actual flip-angle imaging (AFI) method [137–139]. In AFI, the imaging sequence consists of two identical RF pulses with tip-angle $\theta$ followed by two delays $T_{R1}$ and $T_{R2}$ ($T_{R1} < T_{R2} < T_1(r)$). Gradient echoes are acquired after each RF pulse. Let $I_1(r)$ and $I_2(r)$ denote the corresponding images. Assume ideal spoiling, at the steady state, we have [137]:

$$a(r) = \frac{|I_2(r)|}{|I_1(r)|} = \frac{1 - e^{-T_{R1}/T_1(r)} + (1 - e^{-T_{R2}/T_1(r)})e^{-T_{R1}/T_1(r)}\cos(S(r)\theta)}{1 - e^{-T_{R2}/T_1(r)} + (1 - e^{-T_{R1}/T_1(r)})e^{-T_{R2}/T_1(r)}\cos(S(r)\theta)}.$$  

(2.145)
If $T_{R1}$ and $T_{R2}$ are much smaller than $T_1(r)$, Eq. (2.145) can by approximated by:

$$a(r) \approx \frac{1 + n \cos(s(r)\theta)}{n + \cos(s(r)\theta)},$$  \hspace{1cm} (2.146)$$

where $n = T_{R2}/T_{R1}$. And thus, the actual tip-angle $\theta$ can be calculated as:

$$s(r) \approx \frac{1}{\theta} \arccos \left( \frac{a(r)n - 1}{n - 1} \right).$$  \hspace{1cm} (2.147)$$

The AFI method makes use of short $T_R$, and therefore significantly accelerates imaging speed. However, the AFI method becomes less accurate at large tip-angles or in the presence of short $T_1$ components. In addition, the very long $T_1$ components will be heavily suppressed in the AFI method, leading to noisy estimation.

**B$_1$ mapping based on image phase**

The image phased based methods have been developed to remove $T_1$ dependence of the signal, leading to fast and accurate $B_1$ mapping. $B_1$ related image phase can be induced using composite RF pulses [141, 142, 144–147] or based on Bloch-Siegert (BS) shift [143]. Among them, the BS method is most widely used.

In the basic form of BS, an off-resonance RF pulse (BS pulse) is inserted after normal signal excitation followed by image encoding and data acquisition. Suppose the off-resonance frequency of the BS pulse is $\omega_{RF}$, which is much larger than its magnitude $\omega_{RF} \gg \gamma |B_1|$. The BS pulse contributes a small shift to the resonance frequency (Bloch-Siegert shift) [148, 149], which results in extra phase accumulation in image phase [143]:

$$\phi_{BS} = \int_0^r \frac{(\gamma |B_1(s)|)^2}{2\omega_{RF}(s)} ds.$$  \hspace{1cm} (2.148)$$

The BS pulse induced phase in Eq. (2.148) can be detected by subtracting image phase of two separate scans with $\pm \omega_{RF}(s)$ respectively to remove background phase. One additional advantage is that the resulting BS phase is independent of $B_0$ inhomogeneity to the first-order approximation [143]. The off-resonance frequency and shape of a BS pulse is chosen such that no spins in an imaging object will be excited. Note that the BS method estimates the magnitude of a $B_1$ map. The phase of a $B_1$ map can be estimated through a separate
scan.

The BS method is insensitive to $T_1$, and thus leads to fast $B_1$ mapping with short $T_R$ and fast data acquisition schemes [150–154]. However, SAR induced by BS pulses may limit the shortest $T_R$. Methods have been proposed to optimize BS pulses to increase sensitivity while reducing SAR [155–158].
CHAPTER 3
MULTIDIMENSIONAL SHINNAR-LE ROUX RF PULSE DESIGN

3.1 Introduction

Spatially selective RF pulses have found widespread use in MR imaging/spectroscopy experiments. One-dimensional RF pulse is most frequently used for slice selection [22–25]. Multidimensional RF pulses [27,44] can achieve more flexible spatial selectivity than 1D pulses. They have a wide range of applications, including reduced FOV imaging of ROI [1–6], simultaneous spatial-spectral selective excitation [12,13], localized spectroscopic imaging [19,20], and motion tracking [17,18].

The 1D RF pulse design problem has been well studied for several decades [26,30,58,82–85]. Among these methods, the SLR method [26,96–99] is often the “method-of-choice”, because it accurately treats the nonlinearity of the Bloch equation, allows explicit tradeoff among design parameters, and has a fast algorithm. The multidimensional RF pulse design, however, is still an open question largely due to the difficulties in accurately handling the nonlinearity of the Bloch equation. Methods based on the linear approximate solutions of the Bloch equation have been successfully used to design small-tip-angle multidimensional excitation pulses [27–29]. However, these methods can yield significant excitation errors for large-tip-angle pulses, which limit their applications in designing saturation, inversion and refocusing pulses. Methods using optimal control have been proposed to design large-tip-angle multidimensional RF pulses, but are generally inefficient in computation [31,60]. Therefore, it is very desirable to generalize the SLR method to design multidimensional RF pulses.

However, while generalizing the SLR method to the multidimensional case, significant technical difficulties arise mainly due to the obstacle in finding a unique invertible mapping
between a multidimensional RF pulse and the coefficients of multidimensional polynomials. Without such a mapping, it becomes hard to efficiently solve the multidimensional RF pulse design problem by solving a polynomial design problem as the SLR method [159]. Alternatively, a “separate design” method has been proposed for multidimensional RF pulse design [16]. In this method, while assuming an EPI gradient, a multidimensional RF pulse is decomposed into a series of 1D subpulses. The tip-angles and waveforms of the subpulses are then designed separately using the conventional SLR method. However, this method essentially assumes that the rotation of each subpulse is about an axis in the transverse plane. The assumption holds well when the tip-angle of each subpulse is small. However, distortions on excitation patterns may occur, especially when designing refocusing and inversion pulses. It is also lack of performance characterization.

This chapter presents a novel approach to generalizing the SLR method to the multidimensional RF pulse design. The multidimensional RF pulse design problem is first converted to a series of 1D RF pulse design problems. Each of the 1D RF pulse design problem is then mapped into a 1D polynomial design problem, which is formulated as a convex optimization problem and thus can be solved efficiently. The excitation pattern achieved by the proposed method demonstrates equiripple error. The tradeoff among ripple level, transition bandwidth and pulse length can be made explicitly. The proposed method is further generalized to compensate $B_0$ inhomogeneity effect, which is an important issue for a multidimensional RF pulse because of its usually long duration. Based on a similar treatment, the proposed method is extended to design spatial-spectral RF pulses as well.

3.2 Theory

For simplicity, the proposed method is described for designing a 2D spatially selective RF pulse. Assume that $B_1$ field is uniform, and that an EPI gradient is used as shown in Fig. 3.1a. The pulse in Fig. 3.1a can be decomposed into a series of segments in Fig. 3.1b. Each segment consists of two parts: a subpulse with an inherently refocused $z$-gradient that generates rotation by a tip-angle varying along the $z$-axis, and an $x$-gradient blip that generates precession about the main magnetic field by an angle varying along the $x$-axis.
Figure 3.1: A 2D RF pulse with an EPI gradient in (a) can be decomposed into a series of segments in (b). Note that the two refocusing z-gradient lobes in (b) will be canceled by neighboring segments.

In the following, we show that such a 2D RF pulse can be designed by solving two nested 1D problems: (1) determining the desired rotation of each subpulse such that the resulting excitation pattern of the whole pulse best approximates the desired pattern (Section 3.2.1); and (2) designing the waveform of each subpulse such that the resulting rotation of the subpulse best approximates the desired rotation determined in (1) (Section 3.2.2). We show that each problem is equivalent to a 1D polynomial design problem, which is further formulated as a convex optimization problem that can be solved efficiently. The design procedure of the proposed method is summarized in Section 3.2.3, followed by ripple analysis in Section 3.2.4, $B_0$ inhomogeneity effect analysis and correction in Section 3.2.5, and generalization to spatial-spectral RF pulse design in Section 3.2.6.

3.2.1 Determine the Desired Rotation of Each Subpulse

The rotation of each segment in Fig. 3.1b can be written in the form of the Cayley-Klein parameters as

$$Q_{n_x}(z, x) = \begin{bmatrix} C_{n_x}(z) & -S^*_{n_x}(z) \\ S_{n_x}(z) & C^*_{n_x}(z) \end{bmatrix} \begin{bmatrix} e^{i\omega_x/2} & 0 \\ 0 & e^{-i\omega_x/2} \end{bmatrix}.$$  (3.1)

In Eq. (3.1), $Q_{n_x}$ represents the rotation matrix of the $n_x$-th segment; the first matrix on the right-hand side represents the rotation by the $n_x$-th subpulse, where $C_{n_x}(z)$ and $S_{n_x}(z)$ are the corresponding Cayley-Klein parameters; and the second matrix on the right-hand side represents the precession by the $x$-gradient blip, where $\omega_x = \gamma x \int_0^\tau G_x(s)ds$ and $\tau$ is the duration of the blip. Note that the Cayley-Klein parameters are defined following the
right-handed rotation convention, i.e., a counterclockwise rotation has a positive rotation angle.

There is no $x$ gradient blip in the first segment, the rotation matrix of which is given by

$$Q_1(z) = \begin{bmatrix} C_1(z) & -S_1^*(z) \\ S_1(z) & C_1^*(z) \end{bmatrix}. \quad (3.2)$$

The total rotation of the first $n_x$ segments is given by

$$\begin{bmatrix} \alpha_{n_x}(z, x) & -\beta_{n_x}^*(z, x) \\ \beta_{n_x}(z, x) & \alpha_{n_x}^*(z, x) \end{bmatrix} = Q_{n_x} Q_{n_x-1} \cdots Q_1. \quad (3.3)$$

Equations (3.1) and (3.3) describe a rotation synthesis procedure that is similar to the conventional SLR method. Moreover, the synthesis does not depend on the hard-pulse approximation, and thus is accurate even when the tip-angle of each subpulse is large. In the following, an invertible transform is derived to uniquely map the rotation of each subpulse into the coefficients of two complex polynomials.

The forward transform is given by

$$\begin{bmatrix} A_{n_x}(z, x) \\ B_{n_x}(z, x) \end{bmatrix} = \begin{bmatrix} C_{n_x}(z) & -S_{n_x}^*(z) \\ S_{n_x}(z) & C_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\omega_x} \end{bmatrix} \begin{bmatrix} A_{n_x-1}(z, x) \\ B_{n_x-1}(z, x) \end{bmatrix}, \quad (3.4)$$

where

$$A_{n_x}(z, x) = e^{-i\omega_x(n_x-1)/2} \alpha_{n_x}(z, x),$$

$$B_{n_x}(z, x) = e^{-i\omega_x(n_x-1)/2} \beta_{n_x}(z, x). \quad (3.5)$$

$A_{n_x}(z, x)$ and $B_{n_x}(z, x)$ can be further written as polynomials in $e^{-i\omega_x}$ of degree $n_x - 1$ (at any $z$):

$$A_{n_x}(z, x) = \sum_{n=0}^{n_x-1} a_n^{(n_x)}(z) e^{-i\omega_x n}, \quad (3.6)$$

$$B_{n_x}(z, x) = \sum_{n=0}^{n_x-1} b_n^{(n_x)}(z) e^{-i\omega_x n}. \quad (3.7)$$
In order to derive the inverse transform, consider two polynomials \( A_{nx}(z, x) \) and \( B_{nx}(z, x) \) in Eq. (3.6) and Eq. (3.7), which represent a rotation at any spatial location:

\[
|A_{nx}(z, x)|^2 + |B_{nx}(z, x)|^2 = 1, \forall (z, x) \in \mathbb{R}^2. \tag{3.8}
\]

According to Eq. (3.4), the inverse recursion of the polynomials is given by

\[
\begin{bmatrix}
A_{nx-1}(z, x) \\
B_{nx-1}(z, x)
\end{bmatrix}
= \begin{bmatrix}
C_{nx}^*(z) & S_{nx}^*(z) \\
-e^{i\omega x}S_{nx}(z) & e^{i\omega x}C_{nx}(z)
\end{bmatrix}
\begin{bmatrix}
A_{nx}(z, x) \\
B_{nx}(z, x)
\end{bmatrix}. \tag{3.9}
\]

Following a derivation similar to the SLR method, making \( A_{nx-1}(z, x) \) and \( B_{nx-1}(z, x) \) polynomials of degree \( n_x - 2 \) yields

\[
\frac{S_{nx}(z)}{C_{nx}(z)} = \frac{b_0^{(nx)}(z)}{a_0^{(nx)}(z)}. \tag{3.10}
\]

\( C_{nx}(z) \) and \( S_{nx}(z) \) should also satisfy the following magnitude constraints to represent a rotation:

\[
|C_{nx}(z)|^2 + |S_{nx}(z)|^2 = 1, \forall z. \tag{3.11}
\]

Note that Eqs. (3.10) and (3.11) cannot uniquely determine the rotation of the \( n_x \)-th subpulse, as the phase of \( C_{nx}(z) \) can be freely chosen. We choose to make the phase of \( C_{nx}(z) \) depend on \( S_{nx}(z) \). More specifically, while satisfying Eqs. (3.10) and (3.11), we choose to determine \( C_{nx}(z) \) and \( S_{nx}(z) \) as the following:

\[
C_{nx}(z) = \sqrt{1 - |S_{nx}(z)|^2} e^{i\mathcal{H}\{\log \sqrt{1 - |S_{nx}(z)|^2}\}}, \tag{3.12}
\]

\[
\angle S_{nx}(z) = \mathcal{H}\{\log \sqrt{1 - |S_{nx}(z)|^2}\} + \angle\left(\frac{b_0^{(nx)}(z)}{a_0^{(nx)}(z)}\right), \tag{3.13}
\]

\[
|S_{nx}(z)| = \sin \left(\arctan \left(\frac{|b_0^{(nx)}(z)|}{|a_0^{(nx)}(z)|}\right)\right). \tag{3.14}
\]

As will be discussed in Section 3.2.2, this specific choice of the phase of \( C_{nx}(z) \) will lead to proper design of the RF subpulse with minimum RF power. Equations (3.9), (3.12) to
(3.14) define the inverse transform.

Suppose the number of subpulses is \( N \times x \), the problem of determining the desired rotation of each subpulse is then equivalent to a problem of designing the two polynomials \( A_{N_x}(z, x) \) and \( B_{N_x}(z, x) \). As the SLR method, only the polynomial \( B_{N_x}(z, x) \) needs to be designed, and the polynomial \( A_{N_x}(z, x) \) is calculated using spectral factorization:

\[
A_{N_x}(z, x) = \sqrt{1 - |B_{N_x}(z, x)|^2} e^{i\mathcal{H}\left(\log \sqrt{1-|B_{N_x}(z, x)|^2}\right)}, \tag{3.15}
\]

where \( \mathcal{H} \) denotes the Hilbert transform.

Suppose the desired profile of \( B_{N_x}(z, x) \) is \( d(z, x) \), which is discretized on a grid \( (d(z_l, x_m), l = 0, 1, \ldots, N_{sz} - 1, m = 0, 1, \ldots, N_{sx} - 1) \). At each \( z_l \), a set of polynomial coefficients \( b(z_l) = [b_0(z_l), b_1(z_l), \cdots, b_{N_x-1}(z_l)]^T \) are determined by solving the following optimization problem:

\[
\argmin_{b(z_l)} \| W(z_l)(p(z_l) - d(z_l)) \|_\infty \quad \text{s.t.} \quad p(z_l) = E b(z_l), \tag{3.16}
\]

where \( W(z_l) \) is a diagonal matrix containing non-negative weights for different spatial locations, \( p(z_l) \) is a complex vector of length \( N_{sx} \) concatenating the profile of \( B_{N_x}(z_l, x_m) \) along the \( x \)-axis, \( d(z_l) \) is a complex vector of length \( N_{sx} \) concatenating the desired profile \( d(z_l, x_m) \) along the \( x \)-axis, and \( E \) is an \( N_{sx} \times N_x \) complex matrix, the \( (m,n) \)-th entry of which is given by

\[
E_{m,n} = e^{-i(n-1)\gamma \int_0^1 G_x(s)ds} x_m. \tag{3.17}
\]

In Eq. (3.16), the maximum approximate error is minimized. The optimization problem in Eq. (3.16) is convex, and can be solved efficiently using a convex optimization solver, such as CVX [160].

By solving the polynomial design problem in Eq. (3.16) at each \( z_l \), the polynomial \( B_{N_x}(z_l, x) \) is designed, the polynomial \( A_{N_x}(z_l, x) \) is then calculated using Eq. (3.15), and finally the desired rotation of each subpulse is determined using the inverse transform in Eqs. (3.9), (3.12) to (3.14).
3.2.2 Design Each Subpulse to Approximate the Desired Rotation

Suppose for the \( n_x \)-th subpulse, the desired rotation determined by the procedure in Section 3.2.1 is represented by the Cayley-Klein parameters \( C_{d,n_x}(z) \) and \( S_{d,n_x}(z) \). In the following, we discuss how to design each subpulse to approximate the desired rotation.

Suppose each subpulse is approximated by \( N_z \) hard pulses. Performing the forward SLR transform, the Cayley-Klein parameters \( C_{n_x}(z) \) and \( S_{n_x}(z) \) in Eq. (3.1) can be written as two complex polynomials (for detailed derivations, please refer to Section 3.6.1):

\[
C_{n_x}(z) = \sum_{n_z=0}^{N_z-1} a_{n_x,n_z} e^{-i\omega_z^{(n_x)} n_z},
\]

\[
S_{n_x}(z) = \sum_{n_z=0}^{N_z-1} b_{n_x,n_z} e^{-i\omega_z^{(n_x)} (n_z-N_z/2)},
\]

where,

\[
\omega_z^{(n_x)} = \begin{cases} 
\gamma G_z z \Delta t, & \text{if } n_x \text{ is odd,} \\
-\gamma G_z z \Delta t, & \text{if } n_x \text{ is even.}
\end{cases}
\]

The polynomials \( C_{n_x}(z) \) and \( S_{n_x}(z) \) need to be designed to approximate the desired profiles \( C_{d,n_x}(z) \) and \( S_{d,n_x}(z) \). As the SLR method, we intend to only design the polynomial \( S_{n_x}(z) \), and determine the polynomial \( C_{n_x}(z) \) by spectral factorization, which leads to a pulse of minimum RF power [26]:

\[
C_{n_x}(z) = \sqrt{1 - |S_{n_x}(z)|^2} e^{i\mathcal{H}\left\{\log\sqrt{1-|S_{n_x}(z)|^2}\right\}}.
\]

However, by doing that, the polynomials \( C_{n_x}(z) \) and \( S_{n_x}(z) \) cannot be designed to approximate an arbitrary pair of \( C_{d,n_x}(z) \) and \( S_{d,n_x}(z) \), since the control of the phase of \( C_{n_x}(z) \) is lost. Fortunately, this does not impose any limitation on the proposed method, because the phase of \( C_{d,n_x}(z) \) is a free parameter as discussed in Section 3.2.1. According to Eqs. (3.12) to (3.14), if \( S_{n_x}(z) \) is designed to approximate \( S_{d,n_x}(z) \), the polynomial \( C_{n_x}(z) \) determined by Eq. (3.21) will automatically approximate \( C_{d,n_x}(z) \).

The problem of designing the polynomial \( S_{n_x}(z) \) can be formulated as the following convex
optimization problem:

\[
\arg\min_{b_{nx}} \| W(p_{nx} - S_{d, nx}) \|_\infty \\
\text{s.t. } p_{nx} = F^{(nx)}b_{nx},
\] (3.22)

where \( b_{nx} \) is a complex vector of length \( N_z \) concatenating the polynomial coefficients in Eq. (3.19), \( W \) is a diagonal matrix containing non-negative weights for different spatial locations, \( p_{nx} \) is a complex vector of length \( N_{sz} \) concatenating the profile of the polynomial \( S_{nx}(z) \) evaluated at \( z_l \), \( S_{d, nx} \) is a complex vector of length \( N_{sz} \) concatenating the desired profile \( S_{d, nx}(z) \) and evaluated at \( z_l \), and \( F^{(nx)} \) is a \( N_{sz} \times N_z \) complex matrix, the \((l, n)\)-th entry of which is given by

\[
F^{(nx)}_{l,n} = \begin{cases} 
  e^{-i\gamma G_z \Delta t_l (n-N_z/2)}, & n_x \text{ is odd,} \\
  e^{i\gamma G_z \Delta t_l (n-N_z/2)}, & n_x \text{ is even.}
\end{cases}
\] (3.23)

With the polynomial \( S_{nx}(z) \) designed by solving the optimization problem in Eq. (3.22), the polynomial \( C_{nx}(z) \) is calculated using Eq. (3.21). The \( n_x \)-th RF subpulse is then calculated using the inverse SLR transform.

### 3.2.3 Design Procedure

The design procedure of the proposed method is summarized as follows.

Step 1: At each \( z_l \) \((l = 0, 1, \cdots, N_{sz} - 1)\), design a 1D polynomial \( B_{Nx}(z_l, x) \) that best approximates the desired profile \( d(z_l, x_m) \) \((m = 0, 1, \cdots, N_{sx} - 1)\) by solving the convex optimization problem in Eq. (3.16). The polynomial \( A_{Nx}(z_l, x) \) is then determined using Eq. (3.15).

Step 2: For each \( n_x \) \((n_x = N_x, \cdots, 2)\), determine the desired rotation of each subpulse (i.e., \( C_{d, nx}(z_l) \) and \( S_{d, nx}(z_l) \)) using Eqs. (3.12) to (3.14), and then reduce the polynomial order by 1 using Eq. (3.9).

Step 3: For each \( n_x \) \((n_x = 1, \cdots, N_x)\), design the polynomial \( S_{nx}(z) \) that best approxi-
mates the desired profile $S_{d,n_s}(z_i)$ by solving the convex optimization problem in Eq. (3.22). The polynomial $C_{n_s}(x)$ is then determined using Eq. (3.21). The waveform of the $n_s$-th RF subpulse is finally determined using the inverse SLR transform.

### 3.2.4 Ripple Analysis

In the following, we discuss how to specify the ripple levels in designing the polynomial $B_{N_x}(z, x)$ and each subpulse to achieve given effective ripple levels of the resulting excitation pattern, i.e., the ripple levels of the transverse or longitudinal component of the magnetization vector.

We first investigate the relation between the ripples of the Cayley-Klein parameters of the resulting excitation pattern and the ripples in designing the polynomial $B_{N_x}(z, x)$ and each subpulse. For simplicity, suppose the desired excitation pattern is a rectangular ROI, the field-of-excitation (FOX) can be divided into four regions as shown in Fig. 3.2, where region 1 is the passband along both the $x$- and $z$-axes, region 2 is the stopband along the $x$-axis, region 3 is the stopband along the $z$-axis, and region 4 is the stopband along both the $x$- and $z$-axes. Suppose the desired flip angle is $\theta$, and the designed polynomial $B_{N_x}(z, x)$ approximates the desired excitation pattern with passband ripple $\delta_{p,x}$ and stopband ripple $\delta_{s,x}$. The polynomial $B_{N_x}(z, x)$ can be written as:

$$B_{N_x}(z, x) = \begin{cases} 
\sin \frac{\theta}{2} (1 + f_{p,x}(x)), & \text{region 1}, \\
\sin \frac{\theta}{2} f_{s,x}(x), & \text{region 2}, \\
0, & \text{regions 3 and 4}, 
\end{cases}$$

(3.24)

where $f_{p,x}(x)$ and $f_{s,x}(x)$ denote the approximation error and are bounded by $\delta_{p,x}$ and $\delta_{s,x}$, respectively.

Suppose that the $n_s$-th subpulse approximates the desired rotation with passband ripple $\delta_{p,z}$ and stopband ripple $\delta_{s,z}$. Assume the ripple and the tip-angle of each subpulse is small,
Figure 3.2: The passband, transition band and stopband for a rectangular excitation pattern. The region within the red box is the desired excitation pattern. The shaded area is the transition band. Region 1: passband along both the $x$- and $z$-axes. Region 2: stopband along the $x$-axis. Region 3: stopband along the $z$-axis. Region 4: stopband along both the $x$- and $z$-axes. 

$S_{n_s}(z)$ and $C_{n_x}(z)$ can be approximated by

$$
S_{n_s}(z) \approx \begin{cases} 
S_{d,n_s}(1 + f_{p,z}(z)), & \text{regions 1 and 2}, \\
S_{d,n_s}f_{s,z}(z), & \text{regions 3 and 4}, 
\end{cases} 
$$

(3.25)

$$
C_{n_x}(z) \approx \begin{cases} 
C_{d,n_x}, & \text{regions 1 and 2}, \\
1, & \text{regions 3 and 4}, 
\end{cases} 
$$

(3.26)

where $f_{p,z}(z)$ and $f_{s,z}(z)$ denote the approximation error and are bounded by $\delta_{p,z}$ and $\delta_{s,z}$, respectively.

Substituting Eqs. (3.25) and (3.26) into Eqs. (3.4) and (3.24), the polynomial achieved by the designed RF pulse can be approximated by

$$
\hat{B}_{N_s}(z,x) \approx \begin{cases} 
sin \frac{\theta}{2}(1 + f_{p,x}(x))(1 + f_{p,z}(z)), & \text{region 1}, \\
sin \frac{\theta}{2}f_{s,x}(x)(1 + f_{p,z}(z)), & \text{region 2}, \\
sin \frac{\theta}{2}f_{s,z}(z)(1 + f_{p,x}(x)), & \text{region 3}, \\
sin \frac{\theta}{2}f_{s,z}(z)f_{s,x}(x), & \text{region 4}. 
\end{cases} 
$$

(3.27)

According to Eq. (3.27), the ripple of the Cayley-Klein parameter of the resulting excita-
tion pattern, i.e., the ripple of the $\beta$ parameter, can be estimated as:

$$
\delta \approx \begin{cases} 
\delta_{p,x} + \delta_{p,z}, & \text{region 1,} \\
\delta_{s,x}, & \text{region 2,} \\
\delta_{s,z}, & \text{region 3,} \\
0, & \text{region 4,}
\end{cases}
$$

where the high-order terms of $\delta_{p,x}$, $\delta_{s,x}$, $\delta_{p,z}$, and $\delta_{s,z}$ are ignored.

Given the desired effective ripple levels of an excitation pattern, the desired ripple levels of the Cayley-Klein parameters can be determined based on the RF pulse type as the SLR method [26]. The desired ripple levels of the passband and stopband in designing the polynomial $B_{N_x}(z, x)$ and each subpulse can then be determined using Eq. (3.28). The transition bandwidth can be determined in a way similar to the SLR method as well, using the empirical relationship among passband and stopband ripple, transition bandwidth, and filter order in digital filter design [26].

### 3.2.5 $B_0$ Inhomogeneity Effect Analysis and Correction

Assume that the $B_0$ inhomogeneity is relatively small compared to the bandwidth of the subpulse, the rotation matrix of the $n_x$-th segment in the presence of $B_0$ inhomogeneity can be approximated by (for detailed derivations, please refer to Section 3.6.2)

$$
\tilde{Q}_{n_x}(z, x) = \begin{bmatrix} e^{i\frac{\psi_f}{2}} & 0 \\ 0 & e^{-i\frac{\psi_f}{2}} \end{bmatrix} Q_{n_x}(z, x') \begin{bmatrix} e^{-i\frac{\psi_f}{2}} & 0 \\ 0 & e^{i\frac{\psi_f}{2}} \end{bmatrix}.
$$

In Eq. (3.29), $Q_{n,x}$ is the rotation matrix in the absence of $B_0$ inhomogeneity as defined in Eq. (3.1), and $\omega_f$ and $x'$ are given by

$$
\psi_f = \pi f N_x \Delta t, \\
x' = x + \frac{2\pi f (N_x \Delta t + \tau)}{\gamma \int_0^\tau G_x(s)ds},
$$

68
Figure 3.3: A type I spatial-spectral RF pulse in (a) can be decomposed into a series of segments in (b).

where $\Delta f$ denotes the $B_0$ field inhomogeneity in Hertz.

Cascading the rotation of each segment as in Eq. (3.3), in the presence of the $B_0$ inhomogeneity the total rotation of the first $n_x$ segments is given by

$$
\begin{bmatrix}
\tilde{\alpha}_{nx}(z, x) & -\tilde{\beta}^*_{nx}(z, x) \\
\tilde{\beta}_{nx}(z, x) & \tilde{\alpha}^*_{nx}(z, x)
\end{bmatrix}
= 
\begin{bmatrix}
e^{i\frac{\omega_f}{2}} & 0 \\
0 & e^{-i\frac{\omega_f}{2}}
\end{bmatrix}
\begin{bmatrix}
\alpha_{nx}(z, x') & -\beta^*_{nx}(z, x') \\
\beta_{nx}(z, x') & \alpha^*_{nx}(z, x')
\end{bmatrix}
\begin{bmatrix}
e^{-i\frac{\omega_f}{2}} & 0 \\
0 & e^{i\frac{\omega_f}{2}}
\end{bmatrix}.
$$

Equation (3.32) shows that the major effect of the $B_0$ inhomogeneity on the resulting excitation pattern is a shift along the $x$-axis.

Assume the $B_0$ inhomogeneity map $\Delta f(z, x)$ is known, one can design an RF pulse using the proposed method to excite a pre-distorted excitation profile to compensate the distortion caused by $B_0$ inhomogeneity. Suppose the desired excitation profile is $d(z, x)$, the pre-distorted excitation profile $\tilde{d}(z, x)$ is given by

$$
\tilde{d}(z, x) = d\left(z, x - \frac{2\pi \Delta f(z, x)(N_z \Delta t + \tau)}{\gamma \int_0^1 G_x(s)ds}\right).
$$

### 3.2.6 Generalization to Spatial-Spectral RF Pulse Design

The proposed method can be extended to design spatial-spectral RF pulses as well. For simplicity, consider a type I spatial-spectral RF pulse as shown in Fig. 3.3a [161]. The pulse can also be decomposed into a series of segments as shown in Fig. 3.3b. Following a derivation similar to Section 3.2.5, the rotation matrix of the $n_f$-th segment can be written
as:

\[
Q_{n, f}(z, \Delta f) = \begin{bmatrix}
0 & e^{i\psi f / 2} & 0 & e^{-i\psi f / 2} \\
0 & e^{-i\psi f / 2} & 0 & e^{i\psi f / 2}
\end{bmatrix}
\begin{bmatrix}
C_{n, z}(z') & -S_{n, z}(z') \\
S_{n, z}(z') & C_{n, z}(z')
\end{bmatrix}
\begin{bmatrix}
0 & e^{-i\omega f / 2} & 0 & e^{i\omega f / 2} \\
0 & e^{i\omega f / 2} & 0 & e^{-i\omega f / 2}
\end{bmatrix},
\]

(3.34)

where

\[
z' = z + \frac{2\pi \Delta f}{\gamma G_z},
\]

(3.35)

\[
\omega_f = 2\pi \Delta f T,
\]

(3.36)

and \( T \) is the length of each segment as labeled in Fig. 3.3.

The first and last matrix in Eq. (3.34) are not critical as they will be canceled by neighboring segments. The similarity between Eq. (3.34) and Eq. (3.1) indicates that the proposed method can be easily extended to design spatial-spectral RF pulses.

3.3 Results

In this section, we present several representative design examples, including a 2D linear-phase 90° excitation pulse, a 2D linear-phase 180° refocusing pulse, a 2D linear-phase 180° spatial-spectral refocusing pulse, a 2D linear-phase 90° excitation pulse with \( B_0 \) inhomogeneity correction, and a 2D maximum-phase 90° saturation pulse. The proposed method was validated both through Bloch equation simulation and experiments. The experiments were performed on a Siemens 3.0 T MAGNETOM Trio scanner (Siemens, Erlangen, Germany) with a uniform spherical phantom of 18 cm diameter. All the gradient waveforms were designed with maximum gradient slew rate of 13000 G/cm/s.

3.3.1 Experiment I: A 2D Linear-Phase 90° Excitation Pulse

The “separate design” method in [16] has only been shown to design RF pulses to excite a separable excitation pattern, which can be expressed as a product of 1D profiles. In the first example, the proposed method was used to design a 2D linear-phase 90° excitation
Figure 3.4: A 2D linear-phase 90° excitation pulse was designed to excite a circular ROI of 3.5 cm radius in 20 cm × 20 cm FOX with 5% passband ripple and 5% stopband ripple. 

- **a:** Echo planar gradient (blue line: z-gradient; red line: x-gradient).
- **b:** RF pulse (blue line: real part, red line: imaginary part).
- **c:** Simulated and experimental excitation pattern (|m_{xy}|).
- **d** and **e:** Excitation profile plots along the dash lines in (c), where the red dash lines and the blue solid lines represent the simulated and experimental excitation pattern, respectively.

The designed RF pulse is shown in Fig. 3.4b. The simulated and experimental excitation pulse that excited a non-separable excitation pattern, i.e., a circular ROI of 3.5 cm radius in 20 cm × 20 cm field-of-excitation (FOX) with 5% ripples in both the passband and the stopband. The RF pulse was designed with an echo planar gradient of 11 lobes (Fig. 3.4a). The remaining parameters were: pulse length = 9.9 ms and time-bandwidth product of the subpulse = 5.

Nine images were first collected using a GRE sequence with nominal flip angles varying from 10° to 90°. The receiving inhomogeneity was estimated as a byproduct of B_1 mapping as in [46]. The GRE sequence was then modified by replacing the original sinc RF pulse with the designed multidimensional RF pulse for excitation. An image was collected using the modified GRE sequence, and the excitation pattern was finally determined by dividing the GRE image by the receiving inhomogeneity. These images were collected using the following parameters: TE = 10 ms, TR = 500 ms, matrix size = 64 × 64 × 18, and FOV = 18 cm × 18 cm × 18 cm.

The designed RF pulse is shown in Fig. 3.4b. The simulated and experimental excitation
pattern (the center slice) are shown in Fig. 3.4c. Two representative excitation profile plots are shown in Fig. 3.4d and Fig. 3.4e, where the red dash lines and the blue solid lines represent the simulated and experimental excitation pattern, respectively. For the simulated excitation pattern, the achieved passband ripple was 5.2%, which was very close to the desired value, while the achieved stopband ripple was 6.3%, which was higher than the desired value. For the experimental excitation pattern, the achieved passband ripple was 6.9%, and the achieved stopband ripple was 5.4%. The experimental excitation pattern matched closely with the simulation predicted one, and the slight mismatch was likely due to transmit $B_1$ inhomogeneity. Also note that the transition band along the $z$-axis (Fig. 3.4e) is narrower than that along the $x$-axis (Fig. 3.4d). This is because the time-bandwidth product of the subpulse that determines the transition bandwidth along the $z$-axis is larger than the effective time-bandwidth product of the envelope of the pulse that determines the transition bandwidth along the $x$-axis.

3.3.2 Experiment II: A 2D Linear-Phase 180° Refocusing Pulse

The “separate design” method in [16] and the proposed method were used to design a 2D linear-phase 180° pulse that refocused spins in an 8 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOX with 5% passband ripple and 1% stopband ripple. The RF pulses were designed with an echo planar gradient of nine lobes (Fig. 3.5a), where the center lobe was stretched to reduce the peak $B_1$ value of the designed RF pulse. The remaining parameters were: pulse length = 8.3 ms and time-bandwidth product of the subpulse = 3.

An SE sequence was modified to collect experimental data. The slice-selective excitation RF pulse of the SE sequence (along the $y$-axis) was kept unchanged, while the slice selective refocusing RF pulse was replaced with the designed multidimensional refocusing RF pulses. A reference image was first collected while turning off the refocusing RF pulse and crusher gradients. SE images were then collected while turning on the refocusing RF pulse and crusher gradients. The excitation patterns ($|\beta|^2$) were finally determined by normalizing the SE images to the reference image. These images were collected using the following parameters: TE = 30 ms, TR = 1000 ms, matrix size = 128 × 128, slice thickness = 5 mm,
Two-dimensional linear-phase 180° refocusing pulses were designed using the “separate design” method in [16] and the proposed method to refocus spins in an 8 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOX with 5% passband ripple and 1% stopband ripple.

**a:** Echo planar gradient (blue line: z-gradient; red line: x-gradient).

**b:** RF pulses designed by the “separate design” method (black dash line) and the proposed method (red solid line).

**c:** Simulated and experimental excitation patterns \(|\beta|^2\) achieved by the “separate design” method and the proposed method.

**d and e:** Excitation profile plots along the dash lines in (c), where the black dash lines and the red solid lines represent the excitation profiles achieved by the “separate design” method and the proposed method, respectively.
and FOV = 18 cm × 18 cm.

Figure 3.5b shows the RF pulses designed by the “separate design” (black dash line) method and the proposed method (red solid line). The simulated and experimental excitation patterns are shown in Fig. 3.5c. While the excitation pattern achieved by the “separate design” method has notable geometric distortions, the excitation pattern achieved by the proposed method approximates the desired one more accurately. Representative excitation profile plots are shown in Fig. 3.5d and Fig. 3.5e. For the simulated excitation pattern, the achieved ripples for the “separate design” method and the proposed method were 7.2% and 4.9% respectively in the passband, and 2.1% and 1.2% respectively in the stopband. For the experimental excitation pattern, the achieved ripples for the “separate design” method and the proposed method were 8.6% and 7.0% respectively in the passband, and 3.0% and 3.0% respectively in the stopband. In both cases, the proposed method achieved smaller ripples.

In the proposed method, two 1D minimax polynomial approximation problems (Eq. (3.16) and Eq. (3.22)) are solved sequentially to design a 2D RF pulse. Figure 3.6 shows the performance of the two polynomial approximations while designing the 2D refocusing pulse in Fig. 3.5. Figure 3.6a shows the performance of the first approximation, which is approximating the desired excitation profile while assuming ideal rotation of each subpulse. Figure 3.6b (the magnitude), Fig. 3.6c (the real part) and Fig. 3.6d (the imaginary part) show the performance of the second approximation, which is approximating the desired rotation (the $\beta$ parameter in Eqs. (3.13) and (3.14)) of the fifth subpulse. Both approximations show equal-ripple errors as expected.

3.3.3 Experiment III: A 2D Linear-Phase 180° Spatial-Spectral Refocusing Pulse

The proposed method was used to design a 2D linear-phase 180° spatial-spectral refocusing pulse that refocused spins in a 1 cm × 350 Hz ROI in 5 cm × 770 Hz FOX in the z-f plane with 10% passband ripple and 5% stopband ripple. The RF pulse was designed with echo planar gradient of 15 lobes (Fig. 3.7a). The remaining parameters were: pulse length = 19.1
ms and time-bandwidth production of the subpulse = 2.

In the experiments, the modified SE sequence in Experiment II was used. First, a reference image was collected while turning off the spatial-spectral refocusing pulse and crusher gradients. Second, a 180.7 $\mu$T/m $x$ gradient was added by modifying the value of the $x$ gradient of the shimming settings to mimic the spectral dimension of a 1.38 kHz width in the phantom. At last, a SE image was collected while turning on the spatial-spectral refocusing pulse and crusher gradients. The excitation pattern ($|\beta|^2$) was finally determined by normalizing the SE image to the reference image. These images were collected using the following parameters: TE = 38 ms, TR = 1000 ms, matrix size = 256 $\times$ 256, slice thickness = 5 mm, and FOV = 18 cm $\times$ 18 cm.

The designed RF pulse is shown in Fig. 3.7b. The simulated and experimental excitation pattern in the center 5 cm $\times$ 678 Hz ROI are shown in Fig. 3.7c. The experimental excitation pattern is very close to the simulation predicted one. The excitation profile plots along the frequency axis and the $z$-axis are shown in Fig. 3.7d and Fig. 3.7e, where the red dash lines and the blue solid lines represent the simulated and experimental excitation pattern,
Figure 3.7: A linear-phase 180° refocusing spatial-spectral pulse was designed to refocusing spins in a 1 cm × 350 Hz ROI in 5 cm × 770 Hz FOX in the z-f plane with 10% passband ripple and 5% stopband ripple. 

- **a:** Echo planar gradient (the z-gradient).
- **b:** RF pulse (blue line: real part, red line: imaginary part).
- **c:** Simulated and experimental excitation pattern ($|\beta|^2$) in the center 5 cm × 678 Hz ROI.
- **d** and **e:** Excitation profile plots along the frequency axis and the z-axis, where the red dash lines and the blue solid lines represent the simulated and experimental excitation pattern, respectively. In (d), the frequency is converted to ppm (123 Hz/ppm for our 3T scanner and assuming proper frequency shift of the transmitter). The downward-pointing triangles from left to right indicate the positions of water (4.7 ppm), Choline (3.22 ppm), Creatine (3.03 ppm), NAA (2.02 ppm) and lipid (the main peak around 1.2 ppm) signals.
respectively. In Fig. 3.7d, the frequency is converted to ppm (123 Hz/ppm for our 3T scanner and assuming proper frequency shift of the transmitter). The downward-pointing triangles from left to right indicate the positions of water (4.7 ppm), Choline (3.22 ppm), Creatine (3.03 ppm), NAA (2.02 ppm) and lipid (the main peak around 1.2 ppm) signals. Figure 3.7d indicates that the designed spatial-spectral refocusing pulse can be used in chemical shift-imaging for simultaneous water and fat suppression. For the simulated excitation pattern, the achieved passband and stopband ripple was 10.5% and 5.0%, respectively. While for the experimental excitation pattern, the achieved passband and stopband ripple was 11.9% and 5.3%, respectively. The simulation predicted and experimentally measured ripple levels closely matched with the design specification. We also designed a spatial-spectral refocusing pulse using the “separate design” method. Distortions on the achieved excitation patterns similar to those in Experiment II were found. The results are, therefore, not repeated here.

3.3.4 Experiment IV: A 2D Linear-Phase 90° Excitation Pulse with $B_0$ Inhomogeneity Correction

Two-dimensional linear-phase 90° excitation pulses were designed to excite a 6 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOX with 5% passband ripple and 5% stopband ripple. The $B_0$ inhomogeneity was created by introducing a -50 $\mu$T/m $z$ gradient, a 20 $\mu$T/m $x$ gradient and a 30 Hz center frequency offset. The RF pulses were then designed without and with $B_0$ inhomogeneity correction using the proposed method. The gradient was an echo planar gradient of 13 lobes as shown in Fig. 3.8a. The remaining parameters were: pulse length = 9.5 ms and time-bandwidth product of the subpulse = 3.

In the experiments, “good shimming” was first obtained using the automatic shimming of the scanner. An image was collected using the modified GRE sequence in Experiment I with the RF pulse designed without $B_0$ inhomogeneity correction. The $B_0$ inhomogeneity was then created as “bad shimming” by modifying the values of the $z$ gradient, $x$ gradient and center frequency offset of the shimming settings. Images were collected using the RF pulses designed without and with $B_0$ inhomogeneity correction. These images were collected using the following parameters: TE = 10 ms, TR = 500 ms, matrix size = $64 \times 64 \times 18$, and
Figure 3.8: Two-dimensional linear-phase 90° excitation pulses were designed to excite a 6 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOX with 5% passband ripple and 5% stopband ripple. 

- **a**: Echo planar gradient (blue line: z-gradient; red line: x-gradient).
- **b** and **c**: RF pulses designed without and with $B_0$ inhomogeneity correction, respectively (blue line: real part, red line: imaginary part).
- **d**: The $B_0$ map in Hz, which consists of a -50 µT/m z gradient, a 20 µT/m x gradient, and a 30 Hz offset.
- **e**: The simulated and experimental excitation patterns. Left column: excitation patterns achieved by the RF pulse in (**b**) without $B_0$ inhomogeneity. Middle column: excitation patterns achieved by the RF pulse in (**b**) with $B_0$ inhomogeneity. Right column: excitation patterns achieved by the RF pulse in (**c**) with $B_0$ inhomogeneity. The boxes of red dash line in (**d**) and (**e**) indicate the position of the desired ROI.
FOV = 18 cm × 18 cm × 18 cm. The experimental excitation patterns of the experiments were determined in the same way as Experiment I.

The RF pulses designed without and with $B_0$ inhomogeneity correction are shown in Fig. 3.8b and Fig. 3.8c, respectively. The resulting simulated and experimental excitation patterns (the center slice) are shown in Fig. 3.8e. In the presence of the $B_0$ inhomogeneity, the excitation patterns achieved by the RF pulse designed without $B_0$ inhomogeneity correction (the middle column of Fig. 3.8e) show significant distortions, i.e., the notable shearing along the $z$-axis and shrinkage along the $x$-axis. These distortions were effectively corrected by the RF pulse designed with $B_0$ inhomogeneity correction as shown in the right column of Fig. 3.8e.

3.3.5 Experiment V: A 2D Maximum-Phase 90° Saturation Pulse

A linear-phase and a maximum-phase 90° saturation pulse were designed to excite/saturate spins in a 6 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOX with 5% passband ripple and 1% stopband ripple. The RF pulse was designed with an echo planar gradient of 11 lobes (Fig. 3.9a). The remaining parameters were: pulse length = 7.6 ms and time-bandwidth product of the subpulse = 3.

When designing the linear-phase RF pulse, suppose linear-phase polynomials of length $N_x$ ($B_{N_x,\text{lin}}(z, x)$) were designed to approximate the excitation profiles along the $x$-axis with passband ripple $\delta_1$ and stopband ripple $\delta_2$. To design the maximum-phase RF pulse, linear-phase polynomials of length $2N_x - 1$ ($B_{2N_x-1,\text{lin}}(z, x)$) were first designed to approximate the excitation profiles along the $x$-axis with passband ripple $2\delta_1$ and stopband ripple $\delta_2^2/2$. The magnitudes of the maximum-phase polynomials ($B_{N_x,\text{max}}(z, x)$) were the square-root of the profiles of the linear-phase polynomials $B_{2N_x-1,\text{lin}}(z, x)$. The maximum-phase polynomials were finally determined using spectral factorization. The rest design procedure for the maximum-phase RF pulse was the same as that for the linear-phase RF pulse.

In the experiments, a GRE sequence was modified to test the designed saturation pulses. In the sequence, the designed saturation pulse was first applied, immediately followed by spoiler gradients in three axes. A slice-selective excitation pulse (along the $y$-axis) and
Figure 3.9: A linear-phase and a maximum-phase 90° saturation pulse were designed to excite/saturate spins in a 6 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOX with 5% passband ripple and 1% stopband ripple.  

a: Echo planar gradient (blue line: $z$-gradient; red line: $x$-gradient).  
b: RF waveforms (black dash line: linear-phase RF pulse, red solid line: maximum-phase RF pulse).  
c: Simulated and experimental excitation patterns ($|m_z|$) achieved by the linear-phase RF pulse and the maximum-phase RF pulse.  
d and e: Excitation profile plots along the horizontal (the $x$-axis ) directions and the vertical (the $z$-axis) in (c), where the black dash lines and the red solid lines represent the excitation profiles achieved by the linear-phase RF pulse and the maximum-phase RF pulse, respectively.
encoding gradients were then applied for imaging. A reference image was first collected while turning off the saturation RF pulse. Images were then collected while turning on the linear-phase and the maximum-phase saturation pulse, respectively. The excitation patterns ($|m_z|$) were finally determined by dividing the images by the reference image. These images were collected using the following parameters: $\text{TE} = 5\,\text{ms}$, $\text{TR} = 500\,\text{ms}$, matrix size $= 256 \times 256$, slice thickness $= 5\,\text{mm}$, and $\text{FOV} = 18\,\text{cm} \times 18\,\text{cm}$.

Figure 3.9b shows the RF waveforms of the linear-phase (black dash line) and the maximum-phase (red solid line) saturation pulse. The simulated and experimental excitation patterns ($m_z$) are shown in Fig. 3.9c. Two representative excitation profiles are shown in Fig. 3.9d and Fig. 3.9e. The maximum-phase pulse achieved sharper transition bandwidth along the $x$-axis. To achieve similar transition bandwidth, a 10.4 ms linear-phase saturation pulse will be needed. This is 36.8% longer than the maximum-phase pulse, which is a significant difference for generally long multidimensional RF pulses.

3.4 Discussion

In this chapter, a novel method is proposed to extend the conventional SLR method to the multidimensional case. Assuming echo planar gradients and uniform $B_1$ field, rigorous mathematical treatments are presented to convert the multidimensional RF pulse design problem into a series of 1D polynomial design problem. Each of the 1D polynomial design problem is convex and can be solved efficiently. The proposed method preserves almost all the desirable features of the SLR method. More specifically, similar to the SLR method, the proposed method can accurately handle the nonlinearity of the Bloch equation, since only hard-pulse approximation is used to map an RF pulse to polynomials. The proposed method achieves equiripple excitation error, and allows explicit tradeoff among ripple level, transition bandwidth and pulse length. The proposed method is computational efficient as well. For a typical 2D RF pulse design, the proposed method takes less than 30 seconds on a laptop of a 2.4 GHz dual core CPU. Furthermore, the proposed method is extended to compensate $B_0$ inhomogeneity effects and to design spatial-spectral RF pulses.

In the proposed method, each of the 1D polynomial design problem is formulated as mini-
mizing the maximum approximation error. The optimization problem is convex, and thus can be solved efficiently with global optimality. However, note that the final multidimensional polynomial designed by the proposed method is not necessarily a global optimum in approximating the desired multidimensional excitation pattern, because the proposed method is not the only way to construct a multidimensional polynomial. This is a significant difference from the conventional SLR method, which always designs a globally optimal 1D polynomial for 1D RF pulse design guaranteed by the Chebyshev alternation theorem [103].

In the proposed method, the desired profiles are defined on discrete grids. One-dimensional polynomials are designed to approximate the desired profiles by solving a minmax optimization problem using a general convex optimization solver, e.g., CVX [160]. This approach is very flexible when designing polynomials to approximate general profiles. This is especially convenient when designing a subpulse to approximate its desired rotation, since it can be difficult to analytically express the phase of $S_{d,n_x}(z)$. Increasing the number of discrete points will improve approximation accuracy at cost of computation efficiency. According to our experience, good RF pulse design was always achieved when the number of discrete points was about 15 times of polynomial order. As comparison, in the conventional SLR method, i.e., the Parks-McClellan method [103], the desired profile is defined in continuous regions, and the Remez exchange algorithm is used for polynomial design with high efficiency in computation. Fortunately, for a modern PC, the proposed method is already very efficient in computation.

It is important to study the RF power of a multidimensional RF pulse designed by the proposed method. Suppose two polynomials $B_{N_x}(z,x)$ and $A_{N_x}(z,x)$ are designed to approximate a 2D excitation pattern, and the corresponding desired rotation profiles are $C_{d,n_x}(z)$ and $S_{d,n_x}(z)$. If we ignore the phase of $C_{d,n_x}(z)$, using arguments similar to the conventional SLR method [26], we can show that determining the polynomial $A_{N_x}(z,x)$ using Eq. (3.15) will result in a set of subpulses with minimum tip-angles. More specifically, suppose the desired profile of tip-angle of each subpulse is $\theta'_{d,n_x}(z)$, $\theta'_{d,n'_x}(z)$ then maximizes

$$\prod_{n_x=1}^{N_x} \cos\left(\frac{\theta'_{d,n_x}(z)}{2}\right) \approx 1 - \frac{1}{8} \sum_{n_x=1}^{N_x} |\theta'_{d,n_x}(z)|^2,$$  

(3.37)
among all the choices that generate the same excitation pattern. We can further show that modifying the phase of $C_{d,n_x}(z)$ as in Eq. (3.12) and accordingly the phase of $S_{d,n_x}(z)$ as in Eq. (3.13) will not require extra RF energy to approximate such a rotation profile. Intuitively, it is because only the magnitude of $S_{d,n_x}(z)$ determines the minimum amount of RF power to approximate a rotation profile (for rigorous arguments, please refer to Section 3.6.3). Each of the designed subpulses is a minimum energy 1D RF pulse, as it is designed similar to the conventional SLR method. However, also note that the resulting 2D RF pulse is not necessarily a minimum energy 2D RF pulse. This is another significant difference from the conventional SLR method, which always results in a minimum energy 1D RF pulse.

Compared to the “separate design” method in [16], the proposed method assumes the same gradient waveform and uses a similar design procedure. However, the proposed method does differ significantly from the “separate design” method. In the “separate design” method, the desired profile of tip-angle of each subpulse is first designed. Then each subpulse is designed to approximate such a profile of tip-angle. However, the underlying assumption is that the rotation of each subpulse is about an axis in the transverse plane. The assumption holds well when the tip-angle of each subpulse is small. However, when designing large-tip-angle pulses, e.g., refocusing and inversion pulses, the tip-angle of each subpulse may be large, in which case the assumption breaks down and can result in distortions in the achieved excitation pattern as shown in Fig. 3.5. In the proposed method, on the other hand, it is the desired profile of rotation of each subpulse that is designed in the first step. More importantly, the desired profile of rotation, especially of the phase of the Cayley-Klein parameter $C_{d,n_x}(z)$, is designed such that it is feasible for a minimum RF power subpulse. In fact, when the tip-angle of each subpulse is small, the Cayley-Klein parameter $C_{d,n_x}(z)$ can be considered as real, meaning a rotation around an axis in the transverse plane. The proposed method is then reduced to the “separate design” method. When the tip-angles of subpulses are large, the phase of $C_{d,n_x}(z)$ cannot be ignored (an example is shown in Fig. 3.6), otherwise distortions on the resulting excitation pattern may occur.

The NUSLR method in [159] also extends the conventional SLR method to the multidimensional case. For instance of 2D RF pulse design, the NUSLR method directly constructs and designs a two-dimensional polynomial $B(z,x)$ to approximate the desired excitation
pattern. The polynomial $A(z,x)$ is not explicitly constructed but designed jointly with the polynomial $B(z,x)$. However, the resulting optimization problem is so far time-consuming to solve due to the highly nonlinear dependence of the polynomial $B(z,x)$ on the design parameters and the lack of computationally efficient way to control RF power. The major advantage of the proposed method is computation efficiency. For a typical 2D RF pulse design, the proposed method only takes tens of seconds compared to a few hours using the NUSLR method reported in [159]. However, the proposed method is limited to design multidimensional RF pulses with echo planar gradients. The NUSLR method can design multidimensional RF pulses with arbitrary time-varying gradients, e.g., spiral trajectories.

The optimal control approach [31, 60] is the only other method so far that can design large tip-angle multidimensional RF pulses while accurately handling the nonlinearity of the Bloch equation. Compared to the optimal control approach, the proposed method has significant advantages in achieving equiripple excitation error, allowing explicit tradeoff among design parameters and fast computation. However, the proposed method is limited to design multidimensional RF pulses with echo planar gradients, and is not capable of designing multidimensional RF pulses for parallel excitation [31, 60].

3.5 Conclusions

In this chapter, a novel method to extend the conventional SLR method to the multidimensional case is described. The proposed method achieves equiripple excitation error, allows explicit tradeoff among design parameters, and is computational efficient. The proposed method is further generalized for $B_0$ inhomogeneity correction and spatial-spectral RF pulse design. The proposed method was validated through Bloch equation simulation and experiments on a 3.0 T MRI scanner. The proposed method may prove useful especially for designing multidimensional refocusing and inversion pulses, e.g., for water/fat suppression in MR spectroscopic imaging.
3.6 Miscellaneous Derivations

3.6.1 Derivation of $C_{n_x}(z)$ and $S_{n_x}(z)$ in Eqs. (3.18) and (3.19)

The rotation matrix of the $n_x$-th segment is given in Eq. (3.1), repeated here for convenience:

$$Q_{n_x}(z, x) = \begin{bmatrix} C_{n_x}(z) & -S_{n_x}^*(z) \\ S_{n_x}(z) & C_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} e^{i\omega_z/2} & 0 \\ 0 & e^{-i\omega_z/2} \end{bmatrix}, \quad (3.38)$$

where

$$\omega_z = \gamma x \int_0^\tau G_z(s)ds. \quad (3.39)$$

The first rotation matrix on the right-hand side in Eq. (3.38) represents the rotation of the $n_x$-th subpulse with an inherently refocused $z$ gradient as shown in Fig. 3.1b. It can be decomposed as:

$$\begin{bmatrix} C_{n_x}(z) & -S_{n_x}^*(z) \\ S_{n_x}(z) & C_{n_x}^*(z) \end{bmatrix} = \begin{bmatrix} e^{-i\psi_{n_x}/2} & 0 \\ 0 & e^{i\psi_{n_x}/2} \end{bmatrix} \begin{bmatrix} \mu_{n_x}(z) & -\nu_{n_x}^*(z) \\ \nu_{n_x}(z) & \mu_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} e^{-i\psi_{n_x}/2} & 0 \\ 0 & e^{i\psi_{n_x}/2} \end{bmatrix}, \quad (3.40)$$

where the first and the third rotation matrix on the right-hand side represent the precession by the refocusing $z$ gradient lobes, and the second rotation matrix represents the rotation of the $n_x$-th subpulse with a constant $z$ gradient.

Suppose each RF subpulse is approximated by $N_z$ hard pulses and the duration of each hard pulse is $\Delta t$. Performing the forward SLR transform, $\mu_{n_x}(z)$ and $\nu_{n_x}(z)$ can be written as:

$$\mu_{n_x}(z) = e^{i\omega_z^{(n_x)}N_z/2} \sum_{n_z=0}^{N_z-1} a_{n_x,n_z} e^{-i\omega_z^{(n_x)}n_z}, \quad (3.41)$$

$$\nu_{n_x}(z) = e^{i\omega_z^{(n_x)}N_z/2} \sum_{n_z=0}^{N_z-1} b_{n_x,n_z} e^{-i\omega_z^{(n_x)}n_z}, \quad (3.42)$$

where

$$\omega_z^{(n_x)} = \begin{cases} \gamma G_z\Delta tz, & n_x \text{ is odd,} \\ -\gamma G_z\Delta tz, & n_x \text{ is even.} \end{cases} \quad (3.43)$$
For the refocusing \( z \) gradient lobe, the angle of the precession \( \psi_{n_x} \) in Eq. (3.40) is chosen to be:

\[
\psi_{n_x} = \begin{cases} 
-\gamma G_z \Delta t z N_z/2, & n_x \text{ is odd,} \\
\gamma G_z \Delta t z N_z/2, & n_x \text{ is even.}
\end{cases}
\] (3.44)

Substituting Eqs. (3.41) to (3.44) to Eq. (3.40), \( C_{n_x}(z) \) and \( S_{n_x}(z) \) can be written as:

\[
C_{n_x}(z) = \sum_{n_z=0}^{N_z-1} a_{n_x,n_z} e^{-i\tilde{\omega}_x(n_x)n_z},
\] (3.45)

\[
S_{n_x}(z) = \sum_{n_z=0}^{N_z-1} b_{n_x,n_z} e^{-i\tilde{\omega}_x(n_x-N_z/2)}.
\] (3.46)

3.6.2 The Rotation Matrix of Each Segment in the Presence of \( B_0 \) Inhomogeneity

In the presence of \( B_0 \) inhomogeneity, \( \omega_x \) in Eq. (3.39) and \( \omega_z^{(n_x)} \) in Eq. (3.43) becomes:

\[
\tilde{\omega}_x = \gamma x \int_0^\tau G_x(s) \, ds + 2\pi \Delta f \tau,
\] (3.47)

\[
\tilde{\omega}_z^{(n_x)} = \begin{cases} 
\gamma G_z \Delta t(z + \frac{2\pi \Delta f}{\gamma G_z}), & n_x \text{ is odd,} \\
-\gamma G_z \Delta t(z - \frac{2\pi \Delta f}{\gamma G_z}), & n_x \text{ is even,}
\end{cases}
\] (3.48)

where \( \Delta f \) denotes the \( B_0 \) field inhomogeneity in Hertz.

Replacing \( \omega_x \) in Eq. (3.38) with \( \tilde{\omega}_x \) in Eq. (3.47) and \( \omega_z^{(n_x)} \) in Eqs. (3.41) and (3.42) with \( \tilde{\omega}_z^{(n_x)} \) in Eq. (3.48), the rotation matrix of the \( n_x \)-th segment in the presence of \( B_0 \) inhomogeneity is then given by:

\[
\tilde{Q}_{n_x}(z,x) = \begin{bmatrix} e^{i\frac{\psi_f}{2}} & 0 \\ 0 & e^{-i\frac{\psi_f}{2}} \end{bmatrix} \begin{bmatrix} \tilde{C}_{n_x}(z) & -\tilde{S}_{n_x}^*(z) \\ \tilde{S}_{n_x}(z) & \tilde{C}_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} e^{i\frac{\omega_{x'}}{2}} & 0 \\ 0 & e^{-i\frac{\omega_{x'}}{2}} \end{bmatrix} \begin{bmatrix} e^{-i\frac{\psi_f}{2}} & 0 \\ 0 & e^{i\frac{\psi_f}{2}} \end{bmatrix}.
\] (3.49)
where $ψ_f$, $ω'_x$, $\tilde{C}_{nx}(z)$ and $\tilde{S}_{nx}(z)$ are given by:

$$ψ_f = \pi \Delta f N_z \Delta t,$$

(3.50)

$$ω'_x = \gamma \int_0^\tau G_x(s)ds \left[ x + \frac{2\pi \Delta f (N_z \Delta t + \tau)}{\gamma \int_0^\tau G_x(s)ds} \right],$$

(3.51)

$$\tilde{C}_{nx}(z) = \sum_{n_x=0}^{N_z-1} a_{n_x,n_z} e^{-i \tilde{ω}_x^{(n_x)} n_z},$$

(3.52)

$$\tilde{S}_{nx}(z) = \sum_{n_x=0}^{N_z-1} b_{n_x,n_z} e^{-i \tilde{ω}_x^{(n_x)} (n_z-N_z/2)}.$$

(3.53)

Assume that the $B_0$ inhomogeneity is relatively small compared with the bandwidth of the subpulse, $\tilde{C}_{nx}(z)$ and $\tilde{S}_{nx}(z)$ can be approximated by $C_{nx}(z)$ and $S_{nx}(z)$ in Eqs. (3.45) and (3.46), and thus $\tilde{Q}_{nx}(z, x)$ can be approximated by:

$$\tilde{Q}_{nx}(z, x) = \begin{bmatrix} e^{i \psi_f/2} & 0 \\ 0 & e^{-i \psi_f/2} \end{bmatrix} Q_{nx}(z, x') \begin{bmatrix} e^{-i \psi_f/2} & 0 \\ 0 & e^{i \psi_f/2} \end{bmatrix},$$

(3.54)

where $Q_{nx}$ is the rotation matrix of the $n_x$-th segment in the absence of $B_0$ inhomogeneity as defined in Eq. (3.38), and $x'$ is given by:

$$x' = x + \frac{2\pi \Delta f (N_z \Delta t + \tau)}{\gamma \int_0^\tau G_x(s)ds}.$$

(3.55)

3.6.3 RF Power

In the case of 2D RF pulse design, suppose two polynomials $B_{N_z}(z, x)$ and $A_{N_z}(z, x)$ are designed to approximate a 2D excitation pattern. According to Eqs. (3.9), (3.10), and
(3.11), the desired rotation profile of each pulse can be determined by:

\[ C'_{d,nx}(z) = \sqrt{1 - |S'_{d,nx}(z)|^2}, \quad (3.56) \]

\[ \angle S'_{d,nx}(z) = \angle \left( \frac{b_0^{(nx)}(z)}{a_0^{(nx)}(z)} \right), \quad (3.57) \]

\[ |S'_{d,nx}(z)| = \left| \arctan \left( \frac{|b_0^{(nx)}(z)|}{|a_0^{(nx)}(z)|} \right) \right|. \quad (3.58) \]

Note that \( C'_{d,nx}(z) \) in Eq. (3.56) is real.

Suppose the corresponding tip-angle of \( C'_{d,nx}(z) \) and \( S'_{d,nx}(z) \) is \( \theta'_{d,nx}(z) \), using arguments similar to the conventional SLR method [26], \( \theta'_{d,nx}(z) \) then maximizes

\[ \prod_{n_x=1}^{N_x} \cos \left( \frac{\theta'_{d,nx}(z)}{2} \right) \approx 1 - \frac{1}{8} \sum_{n_x=1}^{N_x} |\theta'_{d,nx}(z)|^2, \quad (3.59) \]

among all the choices that generate the same excitation pattern. In other words, we obtain a set of subpulses with minimum tip-angles. This is desirable, since the tip-angle of an RF pulse is directly related to RF power.

However, as discussed in Section 3.2, a minimum RF power subpulse designed using the SLR method can only approximate \( S'_{d,nx}(z) \) and the magnitude of \( C'_{d,nx}(z) \). The zero-phase of \( C'_{d,nx}(z) \) is in fact not achievable. To address this issue, we use Eq. (3.12) to Eq. (3.14) to define the desired rotation \( C_{d,nx}(z) \) and \( S_{d,nx}(z) \). We can show that modifying the phase of \( C_{d,nx}(z) \) as in Eq. (3.12) and accordingly the phase of \( S_{d,nx}(z) \) as in Eq. (3.13) will not require extra RF energy to approximate such a rotation profile. Intuitively, it is because for any \( n_x \), \( S_{d,nx}(z) \) has the same magnitude profile as \( S'_{d,nx}(z) \). It is the magnitude of \( S_{d,nx}(z) \) that determines the minimum amount of RF power to approximate a rotation profile. Mathematically, the above statement is guaranteed by the following lemma.

**Lemma** Given two polynomials \( B_{N_x}(z,x) \) and \( A_{N_x}(z,x) \), suppose a set of polynomials \( B_{nx}(z,x) \) and \( A_{nx}(z,x) \) and desired rotation profiles \( C_{d,nx}(z) \) and \( S_{d,nx}(z) \), are determined using Eq. (3.9), (3.12) to (3.14). Suppose another set of polynomials \( B'_{nx}(z,x) \) and \( A'_{nx}(z,x) \) and desired rotation profiles \( C'_{d,nx}(z) \) and \( S'_{d,nx}(z) \) are determined using Eq. (3.9), (3.56) to
The following holds for any \( n_x, n_x = N_x, \cdots, 1 \):

\[
\begin{align*}
A_{n_x}(z, x) &= e^{-i\phi_{n_x}(z)}A'_{n_x}(z, x), \\
B_{n_x}(z, x) &= e^{i\phi_{n_x}(z)}B'_{n_x}(z, x), \\
C_{d,n_x}(z) &= e^{i\psi_{n_x}(z)}C'_{d,n_x}(z), \\
S_{d,n_x}(z) &= e^{i[2\phi_{n_x}(z) + \psi_{n_x}(z)]}S'_{d,n_x}(z), \\
\phi_{n_x-1}(z) &= \phi_{n_x}(z) + \psi_{n_x}(z), \\
\psi_{n_x}(z) &= \mathcal{H}\{\log \sqrt{1 - |S_{d,n_x}(z)|^2}\},
\end{align*}
\]

where \( \phi_{N_x}(z) = 0 \).

The proof is straightforward using mathematical induction.
CHAPTER 4

JOINT DESIGN OF SPOKE TRAJECTORIES AND RF PULSES FOR PARALLEL EXCITATION

4.1 Introduction

The spoke trajectory [11] (also known as “fast-$k_z$” [8] or echo-volumnar (EV) trajectory [21] in the 3D case and EPI trajectory in the 2D case [1]) is an often-used $k$-space trajectory in designing multidimensional RF pulses for applications that require thin slice selection and in-slice modulation simultaneously. It typically consists of several identical sub-trajectories along the $k_z$ direction (through-slice axis) placed at distinct ($k_x$, $k_y$) locations (in-slice axes). These sub-trajectories are often called “spokes”, which are straight lines designed for extensive coverage of the $k_z$ axis to achieve thin slice selection. The spoke trajectory based RF pulses have been applied to $B_1$ inhomogeneity correction at high field MRI [8–11], reduced FOV imaging [1, 5], and susceptibility induced signal loss recovery in functional MRI [21].

Ideally, to achieve a given in-slice excitation pattern of high spatial resolution, a large number of spokes should be used to ensure sufficient coverage of the $k_x$-$k_y$ plane. In practice, however, only a small number of spokes can be used due to limitation of RF pulse length. Consequently, the locations of these spokes are critical to the performance of the RF pulse and should be, in principle, optimized for a given excitation pattern and $B_1$ sensitivity maps. Therefore, different from the conventional RF pulse design that optimizes an RF pulse based on a predetermined $k$-space trajectory, the spoke trajectory based RF pulse design is inherently a joint design of the RF pulse and the $k$-space trajectory, i.e., the spoke locations.

Several methods have been proposed to address this joint design problem. In [21], spoke locations are chosen based on the discrete Fourier transform of the desired in-slice excitation pattern. Locations in the $k_x$-$k_y$ plane with largest Fourier transform coefficients are
chosen to be the spoke locations. In the single-channel case with a uniform $B_1$ map, the optimality of this method is guaranteed by the Parseval’s theorem. However, for the case of parallel excitation [36, 37] where multiple coils with distinct $B_1$ maps are simultaneously used for excitation, this method does not lead to the optimal spoke locations because the $B_1$ map information is not considered. The sparsity enforced method [11, 162] formulates the joint design problem as an optimization problem where the excitation error along with an $\ell_1$-norm regularization term (to promote sparsity) is minimized. However, the sparsity enforced method has two weaknesses: (1) it is computationally inefficient; and (2) choosing an appropriate regularization parameter is not trivial and may make this method even more time consuming. The computational inefficiency of the sparsity enforced method may limit its applications in many in vivo MRI experiments. More recently, several other methods [35, 163–165] based on modified orthogonal matching pursuit (OMP) methods used in compressed sensing [166, 167] have been proposed. In [168], a similar joint design problem is considered and solved by a generic optimization method.

In this chapter, we present a new method, termed Sequential Optimal seLection of spOkes (SOLO), to jointly design the spoke locations and the RF pulse to minimize the excitation error with an RF power penalty term. The joint design problem is first formulated as an optimal spoke selection problem based on the small-tip-angle RF pulse design [27]. A sequential selection based algorithm [169, 170] is then used to solve the problem with high computational efficiency. In each step, the cost function is evaluated recursively, and an optimized spoke location is added to previously selected spoke locations by solving a series of small size problems. SOLO is applicable to both parallel excitation and single-channel excitation. In addition to optimizing the spoke locations in the spoke trajectory based 3D RF pulse design, SOLO can also be used to optimize the “phase encoding” locations for the echo-planar trajectory based 2D RF pulse design. Through Bloch equation simulations and experimental results on a 3 Tesla GE Excite scanner equipped with a two-channel parallel excitation system, we demonstrate that SOLO can produce significantly smaller excitation error and/or shorter RF pulses than conventional designs with high computational efficiency, e.g., a few seconds on a typical desktop for a typical design in two-channel parallel excitation.
4.2 Theory

4.2.1 Optimal Spoke Selection Problem

When a spoke trajectory is used for designing a 3D RF pulse to achieve thin slice selection along the $z$ axis and modulation in the $x$-$y$ plane, due to the inherent separability of the desired excitation profile, under the small-tip-angle approximation, the RF pulse design can be simplified into designing a 1D RF pulse to achieve thin slice selection and a set of weights to generate the in-plane modulation [8,11,21]. The 3D pulse can then be decomposed into a series of subpulses. Each subpulse has the shape of the designed 1D RF pulse and is scaled by the corresponding weight. It is easy to design a 1D RF pulse to achieve thin slice selection. The remaining problem is to design the weights of subpulses to generate the desired in-plane excitation pattern.

Suppose the number of spokes is $N$ and the number of transmit coils is $L$. Based on the small-tip-angle approximation the in-plane excitation pattern $d(x, y)$ is given by:

$$d(x, y) = \sum_{l=1}^{L} \sum_{n=1}^{N} s_l(x, y) \beta_l(n) e^{i(k_x(n)x+k_y(n)y)}, \quad (4.1)$$

where $s_l(x, y)$ is the $B_1$ map of the $l$-th coil, $\beta_l(n)$ is the RF subpulse weight for the $l$-th coil and the $n$-th spoke, and $\mathbf{k}(n) = (k_x(n), k_y(n))$ denotes the $k$-space location of the $n$-th spoke. Please note that we ignore $T_2^*$ relaxation and $B_0$ inhomogeneity terms in Eq. (4.1) for simplicity. Generalizing the proposed method to compensate $B_0$ effects will be discussed in Section 4.2.3.

Discretizing the spatial coordinate $(x, y)$ on a 2D grid $\{(x_m, y_m)\}_{m=1}^{M}$, where $M$ is the total number of pixels, yields the following linear equation:

$$\mathbf{d} = \begin{bmatrix} \mathbf{A}^{(1)} & \cdots & \mathbf{A}^{(N)} \end{bmatrix} \begin{bmatrix} \beta^{(1)} \\ \vdots \\ \beta^{(N)} \end{bmatrix} = \mathbf{A} \beta, \quad (4.2)$$

where $\mathbf{d}$ is an $M \times 1$ vector containing the samples of the excitation pattern, $\beta^{(n)}$ is an $L \times 1$
vector containing the subpulse weights of \( L \) coils for the \( n \)-th spoke \((n = 1, \ldots, N)\), and \( \mathbf{A}^{(n)} \) is an \( M \times L \) matrix determined by the \( B_1 \) maps and the spoke location, the \((m,l)\)-th entry of which is given by:

\[
\alpha_{m,l}^{(n)} = s_l(x_m, y_m) e^{i(k_x(n)x_m + k_y(n)y_m)}. \tag{4.3}
\]

Denote \( \mathbb{K}_N = \{k(n)\}_{n=1}^N \) as a set of spoke locations and \( \mathbb{K}_N^* = \{k^*(n)\}_{n=1}^N \) as the optimal set of spoke locations. We use \( \mathbf{A}(\mathbb{K}_N) \) to explicitly denote the dependence of matrix \( \mathbf{A} \) on the spoke location set \( \mathbb{K}_N \). Assume that we seek optimal spoke locations on a discrete grid in the \( k_x-k_y \) plane. The problem of jointly designing the RF pulse (i.e., subpulse weights) and the \( k \)-space trajectory (i.e., spoke locations) can be formulated as:

\[
(\mathbb{K}_N^*, \beta^*) = \arg\min_{\mathbb{K}_N \subset \mathbb{K}, \beta} C(\mathbb{K}_N, \beta)
\]

\[
= \arg\min_{\mathbb{K}_N \subset \mathbb{K}, \beta} \| \mathbf{A}(\mathbb{K}_N)\beta - d \|^2_2 + \lambda \| \beta \|^2_2, \tag{4.4}
\]

where \( \mathbb{K} \) denotes a set of \( N_c \) candidate locations, the first term penalizes the excitation error, the second term penalizes RF power, and \( \lambda \) is a Lagrangian multiplier to trade off these two terms.

For a given set of spoke locations \( \mathbb{K}_N \), a closed-form solution of the optimal subpulse weights exists:

\[
\beta^*(\mathbb{K}_N) = [\mathbf{A}(\mathbb{K}_N)^H \mathbf{A}(\mathbb{K}_N) + \lambda \mathbf{I}]^{-1} \mathbf{A}(\mathbb{K}_N)^H d. \tag{4.5}
\]

Substituting the closed-form solution in Eq. (4.5) to Eq. (4.4), the optimal spoke selection problem can then be formulated as:

\[
\mathbb{K}_N^* = \arg\max_{\mathbb{K}_N \subset \mathbb{K}} J(\mathbb{K}_N)
\]

\[
= \arg\max_{\mathbb{K}_N \subset \mathbb{K}} \{d^H \mathbf{A}(\mathbb{K}_N) [\mathbf{A}(\mathbb{K}_N)^H \mathbf{A}(\mathbb{K}_N) + \lambda \mathbf{I}]^{-1} \mathbf{A}(\mathbb{K}_N)^H d \}. \tag{4.6}
\]

However, the problem in Eq. (4.6), which reads choosing \( N \) best spoke locations out of
$N_c$ candidate spoke locations, is a combinatorial optimization problem. The computational complexity of seeking the global optimum is not acceptable for practical problems. In the following, we present a greedy algorithm [169,170] to solve the problem with high computational efficiency.

4.2.2 Sequential Optimal Selection of Spokes (SOLO) Algorithm

SOLO starts from an empty set and sequentially add spoke locations. At each step, the previously selected spokes and a tentative new spoke are combined and the cost function is calculated. Among all the remaining candidate spokes, the one that maximizes the cost function in Eq. (4.6) is selected. After $N$ steps, all $N$ optimal spokes are selected, and the corresponding RF pulse can be calculated using Eq. (4.5). The SOLO algorithm is summarized as follows:

\[
\text{for } n = 0 : N - 1 \quad \text{end},
\]

\[
k^o(n) = \arg\max_{k \in K, k \notin K_n} J(K_n \cup k);
\]

\[
K_n = K_n \cup k^o(n);
\]

where $K_n^o$ denotes the set of spokes selected at the $n$-th step and $K_0^o = \emptyset$.

The above algorithm converts the combinatorial optimization problem in Eq. (4.6) with prohibitive computational complexity into a problem with reasonable computational complexity. However, the algorithm can still be slow for practical problems. The major burden of computation comes from calculating $[A(K_n^o \cup k)^H A(K_n^o \cup k) + \lambda I]^{-1}$ while calculating the cost function $J(K_n^o \cup k)$, which will be evaluated for all the candidate locations at each step. Note that at the $n$-th step, the new matrix $A(K_n^o \cup k)$ with a tentative spoke location $k$ is built by adding $L$ new columns to $A(K_n^o)$ that is known form the previous step. Taking advantage of this structure, we can compute $J(K_n^o \cup k)$ in a recursive and efficient way:

\[
J(K_n^o \cup k) = J(K_n^o) + [u(k)^H v - \alpha(k)]^H Q_n(k) [u(k)^H v - \alpha(k)], \quad (4.7)
\]
where the first term \( J(K^n) \) is known from the previous step and the second term only involves multiplication and inversion of a small matrix. For detailed derivations, see Section 4.6.

4.2.3 \( B_0 \) Inhomogeneity Effect Compensation

When \( B_0 \) inhomogeneity maps \( \Delta B_0(x, y) \) are available, the proposed method can be generalized to compensate \( B_0 \) inhomogeneity effects on excitation patterns. In the presence of \( B_0 \) inhomogeneity, Eq. (4.1) can be rewritten as:

\[
d(x, y) = \sum_{l=1}^{L} \sum_{n=1}^{N} s_l(x, y) \beta_l(n) e^{i(k_x(n)x+k_y(n)y)} e^{i\Delta\omega(x, y)((n-1)T_s-T)},
\]

where \( \Delta\omega(x, y) = \gamma \Delta B_0(x, y) \) accounts for \( B_0 \) inhomogeneity, \( T_s \) is the length of RF sub-pulses, and \( T \) is the total length of the RF pulse.

All the remaining derivations of the proposed method will keep the same as those in Section 4.2.1 except the entry of matrix \( A \) in Eq. (4.3) is now given by:

\[
a^{(n)}_{m,l} = s_l(x_m, y_m) e^{i(k_x(n)x_m+k_y(n)y_m)} e^{i\Delta\omega(x, y)((n-1)T_s-T)}.
\]

Note that in this generalized case the spoke location visiting order is fixed. That is because at each step the spoke location is optimized in a formulation including \( B_0 \) inhomogeneity caused phase accumulation, which depends on the spoke visiting order.

4.3 Results

In this section, we present several representative design examples, which include RF pulses for 3D ROI excitation, \( B_1 \) inhomogeneity correction and 2D reduced FOV excitation. For all examples, the proposed method was compared with conventional methods using experimental results (some with Bloch equation simulation results as well) on a 3T GE Excite scanner (GE Healthcare, Milwaukee, WI, USA) with a GRE sequence modified for parallel excitation. Two exciter boards were used and synchronized to generate independent RF
waveforms, which were amplified by two MKS power amplifiers (MKS Instruments, Andover, MA, USA) each with 35 kW power. Additional attenuators (about 2.4 dB) were used to limit the per channel power to 20 kW. I and Q channels of a standard product body coil were split and used for two-channel excitation. In all experiments, careful $B_0$ shimming was performed to reduce $B_0$ inhomogeneity effects. The delay between gradient and RF waveforms was carefully adjusted to minimize effects of eddy currents from the slice selective gradients. The RF pulse design and the spoke selection algorithm were implemented with MATLAB (MathWorks, Inc., Natick, MA) and run on an Intel Xeon based desktop at 2.66 GHz with 2 GB of RAM.

The $B_1$ maps were the required inputs for all the RF pulse design in our experiments. To estimate $B_1$ maps, we used the two-channel body coil for both excitation and receiving. A parametric fitting method [11] was used for $B_1$ mapping. Images were collected using a modified GRE sequence, which was capable of turning on different transmit channels and looping through different flip angles during RF excitations. To estimate the $B_1$ map of the $l$-th channel ($l = 1, 2$), nine images with nominal flip angles varying from $10^\circ$ to $90^\circ$ were collected by only driving the $l$-th channel with a slice selective RF pulse (leaving the other channel with zero input). Then, image intensities of these nine images at each location $r$ were fit to the following model:

$$|I_{l,j}(r)| = c |R(r)| \frac{\sin(V_j|S_l(r)|)}{1 - e^{-TR/T_1(r)} \cos(V_j|S_l(r)|)},$$

(4.10)

where $|I_{l,j}(r)|$ is the image intensity corresponding to the $j$-th nominal flip angle when only the $l$-th channel was used for excitation, $c$ is a reconstruction scaling constant, $|R(r)|$ is the magnitude of the receiving sensitivity, $|S_l(r)|$ is the magnitude of the $B_1$ map of the $l$-th channel (normalized), and $V_j = 10^\circ, \ldots, 90^\circ$ is the nominal flip angle. We used the `fit` function in MATLAB to solve for the magnitude of $|S_l(r)|$ (and $|cR(r)|$ as a by-product). We chose the phase of the image collected with a nominal flip angle of $60^\circ$, i.e., the phase of $I_{l,6}(r)$, as the phase of $|S_l(r)|$. The $T_1$ value of the phantoms used in our experiments was 120 ms. A typical data acquisition time of a 2D $B_1$ map ($64 \times 64$ image matrix size) was about 2 minutes with a 100 ms TR. Please note that the chosen TR was relatively
short because the $T_1$ maps ($T_1(r)$ in Eq. (4.10)) of imaging objects were known in our experiments. In practice, however, $T_1$ maps are not always available. In this case, we may use either longer TR to minimize $T_1$ relaxation effects (which could significantly increase the acquisition time), or more advanced $B_1$ mapping techniques. For instance, Kerr has proposed a multi-angle volumeric $B_1$ mapping technique that remove the long TR restriction using a RESET component [71].

4.3.1 Experiment I: 3D ROI Excitation

In the first example, the RF pulse was designed to excite a thin disk within a uniform sphere phantom. We compared the performance of the proposed method with the following methods:

1. Cartesian-spiral (C-Spiral) method: A fixed Cartesian spiral trajectory.

2. Fourier method: The discrete Fourier transform of the desired in-slice excitation pattern was first calculated. The spoke locations with the largest Fourier transform coefficients were the selected.

3. Inversion method [11]: The RF pulse was first designed with all spokes on a fully sampled $k$-space grid (denoted as $\mathbb{K}$) by solving the following optimization problem:

$$
\beta_{\text{inv}} = \arg \min_\beta \| A(\mathbb{K})\beta - d \|_2^2 + \lambda \| \beta \|_2^2.
$$

(4.11)

Spoke locations with the largest weights were then selected.

4. Sparsity enforced method [11]: The RF pulse was first designed with all spokes on a fully sampled $k$-space grid by solving the following optimization problem:

$$
\beta_{\text{sparse}} = \arg \min_\beta \| A(\mathbb{K})\beta - d \|_2^2 + \lambda \| \beta \|_s,
$$

(4.12)

where the so-called simultaneous sparsity norm $\| \beta \|_s = \sum_n \| \beta^{(n)} \|_2$ was used as a regularization term to promote the sparsity of the corresponding subpulse weights.
Spoke locations with the largest weights were then selected. The SeDuMi package [171] was used to solve the optimization problem.

The slice thickness was 10 mm. The desired in-slice excitation pattern was a circular ROI of a 7.9 cm radius in a 36 cm \( \times \) 36 cm field-of-excitation (FOX), which was defined on a 32 \( \times \) 32 Cartesian grid. The time-bandwidth product of the slice selective RF subpulse was 2.5, resulting in a subpulse length of 0.4 ms. The \( G_z \) gradient waveform was triangular with the maximum amplitude of 30 mT/m and the maximum slew rate of 150 T/m/s. The spoke selection task was to choose 15 optimal spoke locations from a Nyquist-spaced 18 \( \times \) 18 grid, which led to a 7 ms RF pulse. The data acquisition parameters were: flip angle = 30°, TE = 10 ms, TR = 100 ms and data acquisition matrix size = 256 \( \times \) 256.

In the spoke trajectory and the RF pulse design, weighted \( \ell_2 \)-norm was used to penalize the excitation errors. The weights were 1 for the pass band (magnitude > 0.5 of the peak magnitude) and the stop band (magnitude < 0.05 of the peak magnitude). The errors in the transition band were ignored, and therefore its weights were assigned to be zero. It has been reported that the accuracy of the magnitude excitation profile can be improved by appropriately imposing predetermined phase variations or relaxing phase constraints on the desired excitation pattern [31, 71]. Since the extra phase variations allowed are usually smooth, they should not create significant imaging problems (say signal cancellation within a voxel) if only magnitude images are used. In our experiments, we chose the phase of the excitation pattern to be the phase profile when the body coil worked in the quadrature mode, which worked reasonably well according to our experience.
Comparison of different spoke selection methods

Figure 4.1 shows the magnitude profiles of the $B_1$ maps of the two-channel transmit body coil. The Bloch equation simulation results of the compared methods are summarized in Fig. 4.2, where the relative excitation error is plotted as a function of the number of spokes. The proposed method produced the smallest excitation error. Alternatively, the proposed method can achieve reduced RF pulse length for a given excitation accuracy. For instance, suppose the excitation error is required to be below 1%. Using the proposed method 13 spokes or a 5.3 ms RF pulse are needed, compared with 23 spokes or a 9.7 ms RF pulse for the sparsity enforced method, and 30 spokes or a 12.1 ms RF pulse for the Fourier method. The proposed method has much higher computation efficiency than the sparsity enforced method. The computation time of the proposed method is about 3.7 seconds, compared with about 4 minutes for the sparsity enforced method. The comparison results are summarized in Table 4.1.

The experimental results using 15-spoke trajectories designed by different methods are summarized in Fig. 4.3. The cartoon image in Fig. 4.3a indicates the locations of the sphere phantom and the ROI (within the red circle in Fig. 4.3a). The proposed method (Fig. 4.3b) and sparsity-enforced methods (Fig. 4.3c) produced comparable excitation accuracy.
Table 4.1: Comparison of the pulse length and the computation time for different methods that are used to design 3D ROI excitation pulses with 1% excitation error.

<table>
<thead>
<tr>
<th>Method</th>
<th>Pulse length (ms)</th>
<th>Computation time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>5.3</td>
<td>3.7</td>
</tr>
<tr>
<td>Sparsity enforced</td>
<td>9.7</td>
<td>241.7</td>
</tr>
<tr>
<td>Inversion</td>
<td>9.7</td>
<td>2.9</td>
</tr>
<tr>
<td>Fourier</td>
<td>12.1</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Figure 4.3: Experimental images of 3D ROI excitation using different methods. The maximum intensity is scaled to be the same by post processing. The cartoon image in (a) indicates the locations of the phantom and the ROI (within the red circle).

The remaining three methods (Fig. 4.3d to Fig. 4.3f) produced larger excitation errors. Notably the excitation area extends outside the desired ROI due to lack of spokes at higher frequencies (see Fig. 4.4).

Spoke locations selected by different methods are shown in Fig. 4.4. The proposed method was able to place spokes at higher spatial frequencies than all the other methods. Also note that for the proposed method the spacing between spokes is roughly 2x Nyquist spacing, which matches our intuition in the sense that we would expect two-fold undersampling of the $k_x-k_y$ plane for two-channel parallel excitation. The sparsity enforced method placed
Figure 4.4: Spoke locations selected by different methods for the 3D ROI excitation.

Spokes at slightly higher spatial frequencies than the inversion method and the Fourier method, which matches results in [11]. The inversion method placed spokes around the DC component with Nyquist spacing.

**Sensitivity to $B_1$ mapping errors**

Accurate $B_1$ mapping is desirable for the design of optimal spoke trajectories. We chose to use the parametric fitting method (instead of the conventional double angle method [129], for example) mainly because it is more robust to noise and less sensitive to $T_1$ effects. The main weakness of the $B_1$ mapping method used lies in the fact that it requires relatively long data acquisition time and $T_1$ map information. As a 2D acquisition scheme, the $B_1$ mapping method used may also be affected by a non-ideal excitation profile across the slice [172].

It is important to study the sensitivity of the performance of the proposed method to $B_1$ mapping errors. We performed a sensitivity study of the proposed method through experiments. Two methods were used to estimate the $B_1$ maps: the parametric fitting approach and the double angle method (DAM). In the DAM method, the two flip angles were chosen to be $30^\circ$ and $60^\circ$ and TR was chosen to be 100 ms.

A representative result is shown in Fig. 4.5. We observe larger estimation errors in the $B_1$ map estimated by DAM (Fig. 4.5b) than the parametric fitting method (Fig. 4.5a), especially in the low sensitivity region. Figure 4.5c and Fig. 4.5d show the experimental images with RF pulses designed by the proposed method using $B_1$ maps in Fig. 4.5a and
Fig. 4.5b respectively. The images in Fig. 4.5c and Fig. 4.5d demonstrate comparable excitation accuracy, which suggests that the excitation pattern produced by the proposed method is reasonably robust to moderate $B_1$ mapping errors. An interesting observation is the significant difference between the spokes locations selected by the proposed method using $B_1$ maps of different accuracy (Fig. 4.5e). This difference is probably because there exist many suboptimal sets of spoke locations with similar performance and based on $B_1$ maps of different estimation accuracies the proposed method can select different suboptimal solutions. Please also note that although the difference in Fig. 4.5e looks like a random shift, an arbitrary random shift gives worse results according to our Bloch equation simulation study (results not shown).

**Choosing $\lambda$**

We studied the sensitivity of the proposed method to the choice of $\lambda$ (in Eq. (4.4)) by Bloch equation simulations. RF pulses and spoke trajectories were designed by the proposed method with different $\lambda$ values and numbers of spokes. The values of $\lambda$ were in the range of $[0.1, 10]$ and equally spaced in the logarithmic scale.

Both the excitation error and the RF power are functions of $\lambda$. As $\lambda$ decreases (meaning weaker constraint on RF power and hence more flexibility in designing RF pulses), the excitation error will decrease, probably at the expense of RF power. Figure 4.6a shows the relative excitation error as a function of $\lambda$, which suggests that $\lambda$ can be chosen over a wide range without significantly affecting the excitation error. In our experiments, we fixed $\lambda$ to be 1. Figure 4.6b is a redrawn version of Fig. 4.6a, where the relative excitation error is plotted against the RF power, and $\lambda$ is an implicit parameter that takes different values at different points (decreasing from 10 to 0.1 from left to right). Figure 4.6b demonstrates the tradeoff between excitation error and RF power for different choices of $\lambda$, and suggests that $\lambda$ should be carefully selected if we have strong constraints on excitation error and RF power. For instance, we can choose $\lambda$ by choosing the “turning-point” of the L-shape curve as in the L-curve method [173].
Figure 4.5: $B_1$ mapping error sensitivity study for the proposed method (a): The $B_1$ magnitude map (channel 1) estimated using the parametric fitting approach. (b): The $B_1$ magnitude map (channel 1) estimated using the double flip angle method. (c) and (d): Experimental results achieved by RF pulses designed using (a) and (b) respectively, where the red circles indicate the boundary of the desired excitation pattern. (e): Spoke locations selected by the proposed method using (a) (“o”) and (b) (“+”) respectively.
4.3.2 Experiment II: $B_1$ Inhomogeneity Correction

The proposed method was used to design RF pulses for $B_1$ inhomogeneity correction using a uniform torso phantom. The slice thickness was 5 mm. The in-slice excitation pattern, which was homogeneous in magnitude within the phantom, was defined on a $64 \times 64$ Cartesian grid in a $36 \text{ cm} \times 36 \text{ cm}$ FOX. The $G_z$ gradient waveform was trapezoidal with the maximum amplitude of 20 mT/m and the maximum slew rate of 150 T/m/s. The time bandwidth product of the slice selective RF subpulse was 3, resulting in a subpulse length of 0.85 ms. The spoke selection task was to choose three optimal spoke locations from a Nyquist-spaced $18 \times 18$ grid, which led to a 3.2 ms RF pulse. In the spoke trajectory and the RF pulse design, weighted $\ell_2$-norm was used to penalize the excitation errors. The weights were 1 for regions within the phantom and 0 for regions outside the phantom. Bloch equation simulations and experiments were done to compare the proposed method with a heuristic method that placed spokes in the center of $k$-space. In the experiments, an eight channel cardiac coil was used for receiving. The GRE sequence was used for imaging, again with flip angle = 30$^\circ$, TE = 10 ms, TR = 100 ms and data acquisition matrix size = $256 \times 256$.

Figure 4.7 shows the magnitude profiles of the $B_1$ maps of the two-channel body coil using the uniform torso phantom. Figure 4.8 shows the Bloch equation simulation results of different spoke placements, where the standard deviation of the magnitude of the excitation
Figure 4.7: The $B_1$ maps (magnitude) of the two-channel body coil with a torso phantom at 3T.

Pattern, the RF power (normalized to that of the RF pulse with a single spoke), and the cost function value in Eq. (4.4) are also shown. When only one spoke is used (i.e., RF shimming, Fig. 4.8a), significant $B_1$ shading appears near the upper and lower boundary of the torso phantom (Fig. 4.8e, $std = 0.17$). When three spokes are heuristically selected in the center of $k$-space along the $k_x$ axis (Fig. 4.8b), there are remaining shading effects on the excitation pattern (Fig. 4.8c, $std = 0.14$). The excitation homogeneity could be further improved by placing three spokes in the center of $k$-space along the $k_y$ axis (Fig. 4.8c and Fig. 4.8d, $std = 0.12$), which matches our intuition because we need modulations along $y$ to compensate the $B_1$ shading according to the $B_1$ inhomogeneity pattern in Fig. 4.7. However, by doing this the RF power is more than three times higher than that of the RF pulse with a single spoke. The proposed method, which roughly places the spokes along $y$ but with larger distance between spokes (Fig. 4.8d), achieves the most homogeneous excitation pattern both visually and numerically (Fig. 4.8h, $std = 0.11$) with a mild increase of the RF power (1.1 times of that of the RF pulse with a single spoke).

The experimental results on the torso phantom are shown in Fig. 4.9. The experimental excitation patterns were retrieved by dividing the original experimental images by the receiving sensitivity map, which was a by-product of the $B_1$ mapping technique we used with a uniform phantom. Figures 4.9a to 4.9c show the experimental excitation patterns obtained by the RF pulses designed with one spoke in the center (Fig. 4.8a), three spokes heuristically selected in the center of $k$-space along the $k_x$ axis (Fig. 4.8b), and three spokes selected by proposed method (Fig. 4.8d). The RF pulse designed by the proposed method achieves the best excitation homogeneity ($std = 0.08$). Notably the $B_1$ shading effects near the upper and lower boundary of the phantom are significantly reduced. There is good agreement between
Figure 4.8: Bloch equation simulation results of $B_1$ inhomogeneity correction. Columns are results of four different methods, from left to right: one spoke in the center, three spokes heuristically selected in the center of $k$-space along the $k_x$ axis, three spokes heuristically selected in the center of $k$-space along the $k_y$ axis, three spokes optimally selected by proposed method. Top row: spoke locations. Bottom row: normalized simulated excitation pattern. The standard deviation of the magnitude of the excitation pattern, the RF power (normalized to that of the RF pulse with a single spoke), and the cost function value defined in Eq. (4.4) are also shown.
the experimental (Fig. 4.9a, Fig. 4.9b, and Fig. 4.9c) and simulated (Fig. 4.8e, Fig. 4.8f, and Fig. 4.8h) excitation patterns. The standard deviation of the experimental excitation patterns \((std = 0.13, 0.11, \text{ and } 0.08 \text{ corresponding to Fig. 4.9a, Fig. 4.9b and Fig. 4.9c respectively})\) also match well with the values predicted by the simulations \((std = 0.17, 0.14, \text{ and } 0.11 \text{ corresponding to Fig. 4.8e, Fig. 4.8f, and Fig. 4.8h respectively})\). This close match between the experimental and simulated results provides strong support to the presented results. Figure 4.9d shows the slice profile for the proposed method, demonstrating that a thin slice selection profile is well achieved. The RF and gradient waveforms for achieving Fig. 4.9c and Fig. 4.9d are shown in Fig. 4.9e and Fig. 4.9f.

Figure 4.9: Experimental results of \(B_1\) inhomogeneity correction. (a) to (c): Normalized excitation patterns obtained by RF pulses with one spoke in the center (Fig. 4.8a), three spokes heuristically selected in the center of \(k\)-space along the \(k_x\) axis (Fig. 4.8b), and three spokes optimally selected by proposed method (Fig. 4.8d), respectively. The standard deviation of the magnitude of the excitation pattern is also shown. (d): The slice profile generated by the proposed method. (e) and (f): The gradient waveforms and the RF waveforms for the RF pulse designed by the proposed method.
4.3.3 Experiment III: 2D Reduced FOV Excitation

The proposed method was also used to design RF pulses for 2D reduced FOV excitation. In the 2D reduced FOV excitation, only a small rectangular region of interest within the slice plane is imaged. Unlike Experiment I where a 3D RF pulse is used, the rectangular reduced FOV can be achieved by 2D selective excitation along the through-slice direction and the in-slice phase encoding direction plus the ability to limit FOV along the in-slice frequency encoding direction by adjusting the cutoff frequency of the analog filter in the receiver chain.

The uniform sphere phantom in Experiment I was used in this experiment. The slice thickness was 10 mm. The desired excitation profile in the phase encoding direction ($k_y$ direction) was a smoothed rectangular-like profile with width of 7.2 cm in a 36 cm × 36 cm FOV, which was defined on a 64 × 64 Cartesian grid (the same resolution as the estimated $B_1$ maps). The phase of the excitation pattern was chosen to be the phase profile when the body coil works in the quadrature mode. The $G_z$ gradient waveform was triangular with the maximum amplitude of 30 mT/m and the maximum slew rate of 150 T/m/s. The time bandwidth product of the slice selective RF subpulse was 2.5, resulting in a subpulse length of 0.4 ms. The spoke selection task was to choose 10 optimal spoke locations from 32 Nyquist-spaced spokes placed on the $k_y$ axis (i.e., $k_z = 0$), which led to a 5 ms RF pulse.

In the spoke trajectory and the RF pulse design, weighted $\ell_2$-norm was used to penalize the excitation errors. The weights were 1 for the pass band (magnitude $> 0.5$ of the peak amplitude) and 20 for the stop band (magnitude $< 0.05$ of the peak amplitude) to suppress side lobes. The errors in the transition band and regions outside the predetermined FOV in the frequency encoding direction were ignored, and therefore its weights were assigned to be zero. Bloch equation simulations and experiments were done to compare the proposed method with a heuristic method that placed spokes in the center of $k$-space. For the GRE based phantom experiments, the two-channel body coil was also used for both excitation and receiving. The data acquisition parameters were: flip angle = 30°, TE = 10 ms, TR = 100 ms and data acquisition matrix size = 256 × 256.

Experimental results of 2D reduced FOV excitation are summarized in Fig. 4.10. The cartoon image in Fig. 4.10a indicates the locations of the sphere phantom and the ROI area.
(within the red rectangular in Fig. 4.10a) in a 36 cm × 36 cm FOV. The slice selection direction is perpendicular to the paper in Fig. 4.10a, the phase encoding direction is the vertical direction (y in Fig. 4.10a), and the frequency encoding direction is the horizontal direction (x in Fig. 4.10a). Using RF pulses designed by the proposed method and the heuristic method that Nyquist samples the center of the k-space, 2D reduced FOV images (Fig. 4.10c and Fig. 4.10d) are collected with a 14.4 cm × 14.4 cm FOV. Since the matrix size of the image is maintained, the spatial resolution is increased from 1.4 mm to 0.56 mm. Alternatively, if the spatial resolution is maintained, the imaging speed would be accelerated by 2.5 times. No aliasing artifact is observed in either of the image, but the proposed method (Fig. 4.10d) achieves a wider and flatter “pass band” and a sharper “transition band” than the heuristic method (Fig. 4.10c), which is shown more clearly in the corresponding excitation profile (Fig. 4.10b) produced by projecting images along the frequency encoding direction. The spoke locations for the heuristic method and the proposed method are also shown in Fig. 4.10c and Fig. 4.10d, where the proposed method places spokes at higher spatial frequencies.

4.4 Discussion

In the proposed method, the spoke trajectory based RF pulse design problem is formulated as a problem of jointly designing the RF pulse and the k-space trajectory based on a given excitation pattern and $B_1$ maps. The proposed SOLO method solves the joint design problem in a greedy fashion with high computational efficiency, which is enabled by the recursive evaluation of the cost function. Only suboptimal solution is achieved due to the greedy procedure, however, based on our experience, the proposed method almost always outperforms conventional methods. The Cartesian spiral method, which selects the spoke location along the fixed Cartesian spiral trajectory, is not desirable because neither the excitation pattern or the $B_1$ map information is considered. The Fourier method is optimal in single-channel excitation with a uniform $B_1$ map, but leads to suboptimal solutions in parallel excitation because the $B_1$ map information is not involved in the design. Although the inversion method incorporates the $B_1$ information into the design, the resulting spoke
Figure 4.10: Experimental results for 2D reduced FOV excitation. The phase encoding direction is the vertical direction (y-axis). The cartoon image in (a) indicates the locations of the phantom and the ROI area in a 36 cm × 36 cm FOV. With RF pulses using spoke locations designed by the proposed method and the heuristic method, 2D reduced FOV images are collected with a 14.4 cm × 14.4 cm FOV. The excitation profiles (projected along the frequency encoding direction) are shown in (b): the dash line for the heuristic method and the solid line for the proposed method. The reduced FOV images and the spoke locations are shown in (c) (the heuristic method) and (d) (the proposed method).
location design turns out to be still in the low frequency regions of the $k$-space and thus leads to less accurate excitation patterns. The optimality conditions of the sparsity enforced method are hard to check, and may or may not be satisfied in practice. Therefore, the sparsity enforced method may also lead to suboptimal solutions. Based on our experience, the proposed method achieves similar or better excitation accuracy compared with the sparsity enforced method. However, the sparsity enforced method is less computationally efficient because both solving the $\ell_1$ regularization problem and choosing appropriate regularization parameter are computational expensive.

For the sensitivity of the proposed method to $B_1$ mapping errors, an interesting observation is that considering the fairly comparable performance of the resulting excitation patterns (Fig. 4.5c and Fig. 4.5d) there is significant difference between the spokes locations selected by the proposed method using $B_1$ maps of different accuracy (Fig. 4.5e). This is probably because there exist many suboptimal sets of spoke locations with similar performance and based on slightly different $B_1$ maps the proposed method can select different suboptimal solutions. This observation may deserve more advanced experiments to further study the sensitivity of the proposed method to $B_1$ mapping errors. In the proposed method, the parameter $\lambda$ is used to trade off the excitation error and the RF power. As shown in Fig. 4.6a, the excitation error of the designed RF pulse is robust to the choice of $\lambda$, and hence in our implementation we chose fixed $\lambda$. In cases, where a good tradeoff between the excitation error and the RF power is important, we can choose $\lambda$ based on the L-curve method as illustrated in Fig. 4.6b. The performance of the proposed method can be further improved by “cycling.” That is once a predetermined number of spokes are selected, the method attempts to replace one selected spoke location by a spoke location from the remaining candidate spoke locations to maximize the cost function, while keeping the other selected spoke locations fixed. The method then cycles through the different spoke locations until a certain stopping criterion, e.g., the maximum iteration number, is satisfied. However, by doing this, the computation time also increases, and a practitioner should trade off the performance and the computational efficiency.

The proposed method offers high computational efficiency, which is enabled by the recursive evaluation of the cost function. The computational complexity of the proposed method
is on the order of $O(N_cML^2N^3)$ without recursive evaluation and $O(N_cML^2N^2)$ with recursive evaluation, i.e., using Eq. (4.7), where $N_c$ is the number of candidate spoke locations, $M$ is the number of pixels of the discretized desired excitation pattern, $L$ is the number of excitation channels, and $N$ is the predetermined number of spokes. Therefore, the computational complexity is reduced significantly by using recursive evaluation as the number of spokes increases. In the current implementation, using MATLAB on a desktop with an Intel Xeon 2.66GHz CPU and 2GB RAM, a typical design of the spoke trajectory and the RF pulse for two-channel parallel excitation takes only a few seconds. When the proposed method is used in eight-channel parallel excitation, a typical design requires computation time less than one minute. Note also for very large size problems, e.g., with large matrix size and a large number of excitation channels, the proposed method can be further accelerated by parallel computation.

In our current implementation, once the spoke locations are selected, another greedy algorithm is used to determine the visiting order of these spoke locations to minimize the length of the spoke trajectory, where the starting point is chosen to be the farthest spoke location from the origin of $k$-space and the ending point is enforced to be the origin. However, the “shortest-path” criterion may not be the best one in determining the visiting order of these spokes. One interesting problem is to determine an appropriate visiting order of the selected spokes to minimize the effects of gradient imperfections, i.e., eddy currents, which is important in practice and is still an open question in research.

4.5 Conclusions

In this chapter, a new method to jointly design the spoke trajectory and the RF pulse is described. The joint design problem is formulated as an optimal spoke selection problem based on the small-tip-angle RF pulse design. A sequential selection algorithm with recursive evaluation of the cost function is proposed to solve this optimization problem with high computational efficiency. The proposed method has been validated through both Bloch equation simulations and experiments on a 3T scanner with a two-channel parallel excitation system, and compared with the Fourier method and the sparsity enforced method. Our
results showed that: (a) compared to the sparsity enforced method, the proposed method produced RF pulses with similar or larger improvement in excitation accuracy over the Fourier method; and (b) compared to the sparsity enforced method, the proposed method achieved about ten-fold reduction in computation time. The proposed method may prove useful especially for designing spoke trajectory based RF pulses for high field applications where large field inhomogeneities and $T_2^*$ effects are significant.

4.6 Miscellaneous Derivations

In the following, we derive the recursive evaluation of the cost function in Eq. (4.7). Suppose the number of pixels of the discretized excitation pattern is $M$ and the number of excitation channels is $L$. We use the following notations for simplicity:

$$
\begin{align*}
A_n &\equiv A(\mathbb{K}_n^\circ), \in \mathbb{C}^{M \times nL}, \\
A_n(k) &\equiv A(\mathbb{K}_n^\circ \cup k), \in \mathbb{C}^{M \times (n+1)L}, \\
B_n &\equiv A_n^H A_n + \lambda I, \in \mathbb{C}^{nL \times nL}, \\
B_n(k) &\equiv A_n(k)^H A_n(k) + \lambda I, \in \mathbb{C}^{(n+1)L \times (n+1)L}.
\end{align*}
$$ (4.13)

At the $n$-th step, the cost function $J(\mathbb{K}_n^\circ \cup k)$ (as defined in Eq. (4.6) ) corresponding to the previous spoke selection $\mathbb{K}_n^\circ$ and the tentative spoke location $k$ is then calculated by:

$$
J(\mathbb{K}_n^\circ \cup k) = d^H A_n(k) \left[ A_n(k)^H A_n(k) + \lambda I \right]^{-1} A_n(k)^H d
$$

$$
= d^H A_n(k) \left[ B_n(k) \right]^{-1} A_n(k)^H d,
$$ (4.14)

where the major computation burden comes from the large matrix multiplication $A_n(k)^H A_n(k)$ and the matrix inversion $\left[ A_n(k)^H A_n(k) + \lambda I \right]^{-1}$. 

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First note the recursive relation between \( B_n \) and \( B_n(k) \):

\[
B_n(k) = \begin{bmatrix}
A_n^H \\
A(k)^H \\
A(k)^HA_n \\
A(k)^HA + \lambda I
\end{bmatrix} \begin{bmatrix}
A_n \\
A(k)
\end{bmatrix} + \lambda I.
\]

That is, with the tentative spoke location \( k \), only \( L \) columns and \( L \) rows corresponding to \( k \) are added to the matrix \( B_n \) to build the matrix \( B_n(k) \).

The recursive relation in Eq. (4.15) results in the following recursive relation between \( B_{n-1} \) and \( [B_n(k)]^{-1} \):

\[
[B_n(k)]^{-1} = \begin{bmatrix}
B_{n-1} + B_{n-1}u(k)Q_n(k)u(k)^HB_{n-1} & -B_{n-1}u(k)Q_n(k)
\\
-Q_n(k)u(k)^HB_{n-1} & Q_n(k)
\end{bmatrix},
\]

(4.16)

where \( B_{n-1} \) is known from the previous step; \( u(k) = A_n^HA(k) \) is an \( nL \times L \) matrix; and \( Q_n(k) = [A(k)^HA(k) + \lambda I - u(k)^HB_{n-1}^{-1}u(k)]^{-1} \) is the inversion of a small \( L \times L \) matrix, which can be calculated efficiently.

Equation (4.16) can be easily verified by multiplying Eq. (4.16) with Eq. (4.15) to finally get the identity matrix.

Substituting Eq. (4.16) into Eq. (4.14) yields the following recursive formula of \( J(\mathbb{K}_n^o \cup k) \):

\[
J(\mathbb{K}_n^o \cup k) = J(\mathbb{K}_n^o) + \begin{bmatrix} u(k)^Hv - \alpha(k) \end{bmatrix}^H Q_n(k) \begin{bmatrix} u(k)^Hv - \alpha(k) \end{bmatrix},
\]

(4.17)

where \( J(\mathbb{K}_n^o) \) is known from the previous step; \( \alpha(k) = A(k)^Hd \), is an \( L \times 1 \) vector, which can be calculated and stored in advance; and \( v = B_n^{-1}A_n^Hd \), is an \( nL \times 1 \) vector, which only needs to be calculated once at each step.
CHAPTER 5

REDUCED FOV EXCITATION USING SECOND-ORDER GRADIENTS

5.1 Introduction

A key limitation of multidimensional RF pulses is their long duration needed to traverse the multidimensional excitation k-space. Recently, parallel excitation techniques have been proposed to shorten multidimensional RF pulses [36, 37], although parallel excitation hardware is not yet widely available on commercial MRI scanners.

High-order, nonlinear gradients are another powerful tool that can be exploited to shorten multidimensional RF pulses. Cho et al. first proposed to use pulsed second-order gradients and frequency-selective RF pulses for localized volume excitation [174–176]. This method requires only two 1D RF pulses to excite a circular ROI in a thin slice for reduced FOV imaging. However, this method is limited to spin-echo sequences and difficult to be extended to multislice imaging. More recently, PexLoc (parallel excitation using localized gradients) has been proposed to use time-varying high-order gradients for multidimensional RF pulse design [177, 178]. In PexLoc, conventional multidimensional RF pulse design methods are extended to the case of nonlinear, nonbijective spatial encoding fields, and parallel excitation is used to resolve the inherent encoding ambiguities of such encoding fields. Potential advantages include locally increased spatial resolution [177, 179], faster switching time of high-order gradients and thus shorter RF pulses [178]. However, the PexLoc method requires strong and fast-switching high-order gradient fields, which are not available on commercial MRI scanners [180, 181].

In this chapter, we present a new method to leverage the multidimensional spatial dependence of second-order gradients to design multidimensional RF pulses for reduced FOV imaging. In the proposed method, a spatial-spectral pulse provides a spatial selectivity in
the slice-direction as well as a spectral selectivity that is transferred into a spatial selectivity by the spatial dependence of the resonance frequency introduced by the second-order gradients. As a result, 3D spatial selectivity (a circular ROI in a thin slice) is achieved by a 2D spatial-spectral RF pulse in the presence of second-order gradients, which can significantly shorten RF pulses and/or improve excitation accuracy compared to conventional methods using 3D RF pulses and linear gradients.

5.2 Methods

In source-free regions, magnetostatic fields are governed by the Laplace equation [182]:

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2} = 0,$$

(5.1)

where $B_z$ is the $z$ component of the magnetic field. Mathematically, gradient fields (or gradients) are solutions of the Laplace equation in the form of spherical harmonics. The first-order solutions are linear gradients. The second-order solutions/gradients are usually given in the following forms:

$$z^2 - (x^2 + y^2)/2, x^2 - y^2, xy, yz, \text{ and } xz,$$

(5.2)

which can be available either from specially designed second-order gradient coils [180, 181] or more often from second-order shim coils on commercial MRI scanners [183]. The Laplace equation limits the choice of a desired nonlinear gradient. For instance, a hypothetical gradient in the form of $x^2 + y^2$ along with a frequency-selective RF pulse would excite a pencil beam. However, such a field does not satisfy the Laplace equation. In the following, we describe a new method to use second-order gradients and spatial-spectral pulses to excite a circular ROI in a thin slice for reduced FOV imaging.

The basic idea of the proposed method can be illustrated through the following example. Suppose that the target excitation pattern is a circular ROI of radius $R$ in a thin slice of
Figure 5.1: Reduced FOV excitation using a second-order gradient and a spatial-spectral RF pulse. 

**a:** The spatial dependence of a 0.9 mT/m\(^2\) Z2 gradient (in Hz). 

**b:** The excitation pattern (\(|M_{xy}|\)) achieved by a 30° frequency-selective RF pulse of 138 Hz bandwidth in the presence of the Z2 gradient in (a). 

**c:** The excitation pattern achieved by a 30° spatial-spectral RF pulse that selects a 0.7 cm slice with a 138 Hz bandwidth in the presence of the Z2 gradient in (a). Note that the field in (a) and the excitation patterns in (b) and (c) are all cylindrically symmetric around the z axis.

thickness \(d\) with a 30° flip angle:

\[
|M_{xy}|(x, y, z) = \begin{cases} 
0.5, & \text{if } x^2 + y^2 \leq R^2 \text{ and } |z| \leq \frac{d}{2}; \\
0, & \text{otherwise}.
\end{cases}
\]  

(5.3)

To achieve such an excitation pattern, the conventional approach is to use 3D RF pulses and time-varying linear gradients to cover the 3D excitation k-space, along, for example, spoke trajectories [8, 162], which could result in long pulses. In the proposed method, a constant second-order gradient in the form of \(z^2 - (x^2 + y^2)/2\) (Z2 gradient) is applied during excitation. Such a Z2 gradient establishes a unique relationship between resonance frequency and spatial location (shown in Fig. 5.1a). Notably, in the target slice (\(|z| \leq \frac{d}{2}\)), the resulting excitation pattern of a frequency selective RF pulse is a circular region, and the radius of which (\(R\)) can be controlled by the bandwidth of the frequency selective RF pulse (\(BW\)):

\[
R \approx \sqrt{\frac{BW}{\gamma G_{Z2}}},
\]

(5.4)

where \(G_{Z2}\) is the magnitude of the Z2 gradient. Equation (5.4) is valid if the radius of the
excitation region is much larger than the slice thickness, i.e., $R \gg d$, which is valid for most reduced FOV imaging applications. However, using the frequency-selective RF pulse and the $Z2$ gradient alone also results in undesired excitation outside the target slice as shown in Fig. 5.1b. This problem is overcome in the proposed method using the simultaneous frequency and slice selectivity of a 2D spatial-spectral RF pulse as shown in Fig. 5.1c. Therefore, by leveraging the unique spatial dependence of the $Z2$ gradient, a 2D spatial-spectral pulse can achieve 3D spatial selectivity! This dimension reduction means that, with the same pulse length and the same slice selection specification, the proposed method can achieve larger excitation $k$-space coverage than the conventional methods using 3D RF pulses and linear gradients, which can lead to shorter pulses and/or better excitation accuracy compared to the conventional method.

We next describe the design of the proposed pulse that consists of a spatial-spectral RF pulse, an oscillating linear gradient in the slice-selection axis ($z$ axis), and a constant $Z2$ gradient, as illustrated in Fig. 5.2a. First, we express the waveform of the spatial-spectral pulse as [12, 39]:

$$b_1(t) = b_{\text{spat}}(t)b_{\text{spec}}(t),$$

(5.5)

where $b_{\text{spat}}(t)$ is a chain of subpulses used for slice selection, and $b_{\text{spec}}(t)$ describes the envelope of the spatial-spectral pulse and defines its spectral response. Note that the design equation Eq. (5.5) is based on small-tip-angle approximation [27], and thus has the same well-known limitations of small-tip-angle pulses.

Second, we propose to determine the amplitude of the $Z2$ gradient $G_{Z2}$, the oscillating gradient $G_z(t)$ and the RF pulse waveform $b_1(t)$ based on the following considerations/procedures:

1. The oscillating gradient $G_z(t)$ of period $T$ and the slice-selective subpulse of $b_{\text{spat}}(t)$ (e.g., a sinc/SLR pulse) are designed using the conventional method to select the desired thin slice ($|z| \leq \frac{d}{2}$).

2. The excitation pattern of the spatial-spectral pulse has a periodic pattern separated
by \( \frac{1}{T} \) along the frequency axis (shown in Fig. 5.2b). To avoid exciting the replicates, the amplitude of the \( Z_2 \) gradient is chosen to satisfy the following relation:

\[
f_{\text{max}} \leq f_{\text{rep}} - \frac{BW}{2},
\]

where \( f_{\text{max}} = \gamma |G_{Z2}| \frac{L^2}{2} \) is the maximum resonance frequency offset of the imaging object of radius \( L \) in the selected slice, \( BW = \gamma |G_{Z2}| R^2 \) is the bandwidth of the spectral-selective pulse envelope \( b_{\text{spec}}(t) \) for exciting a circular ROI of radius \( R \), and \( f_{\text{rep}} = \frac{1}{T} \) is the position of the center of the first excitation replicate that has width approximately equal to \( BW \). While satisfying the Eq. (5.6), a stronger second-order gradient allows higher time-bandwidth-product of the spectral-selective pulse envelope \( b_{\text{spec}}(t) \), which can improve the in-plane spatial resolution and/or shorten the proposed RF pulse.

3. The spectral-selective pulse envelope \( b_{\text{spec}}(t) \) (e.g., a sinc/SLR pulse) of length \( N \times T \) (resulting in an oscillating gradient of \( N \) periods) is designed to define a spectral response of bandwidth \( BW \). The RF pulse \( b_1(t) \) is then calculated using Eq. (5.5).

The discussion so far is focused on exciting a circular ROI in the center of the FOV. Shifting the excitation pattern to an arbitrary position can be done using a combination of first-order and second-order gradients plus frequency offset. Specifically, to excite a circular ROI centered at \((x_0, y_0)\) in a thin slice at \( z_0 \) (\(|z - z_0| \leq d/2\)), we can use the following gradients:

\[
G_{Z2}[(z - z_0)^2 - [(x - x_0)^2 + (y - y_0)^2]/2] \\
= G_{Z2}[z^2 - (x^2 + y^2)/2] \quad \text{(second-order gradient)} \\
- 2G_{Z2}z_0 z + G_{Z2}x_0 x + G_{Z2}y_0 y \quad \text{(linear gradient)} \\
+ G_{Z2}z_0^2 - (x_0^2 + y_0^2)/2 \quad \text{(frequency offset)}.
\]

(5.7)

The oscillating linear gradient remains the same with added frequency modulations of the subpulse \( b_{\text{spat}}(t) \) [161]. The design procedure follows. More generally, it can be shown that an elliptical ROI at an arbitrary location in an oblique slice can be excited with an appropriate
Figure 5.2: The proposed pulse consists of a spatial-spectral RF pulse, an oscillating linear gradient, and a constant Z2 gradient as shown in (a), where $T$ is the period of the oscillating linear gradient. In this specific example, the spatial-spectral RF pulse selects a 0.7 cm slice with a 138 Hz bandwidth and a 30° flip angle; the maximum slew rate of the oscillating linear gradient is 200 T/m/s; and the magnitude of the Z2 gradient is 0.9 mT/m². The contour plot of the corresponding excitation pattern in the $z$-$f$ plane is shown in (b), where $BW$ is the bandwidth of the spectral response, and $f_{\text{rep}}$ is the position of the first replicate.
combination of frequency offset, three linear and five second-order gradients [175,183].

5.3 Results

The proposed method has been validated using Bloch equation simulations and phantom experiments. In the Bloch equation simulation study, we used the proposed method to design an RF pulse (shown in Fig. 5.2a) that excited a thin disk of 6 cm radius and 0.7 cm thickness in a cylindrical phantom of 15 cm radius with a 30° flip angle (shown in Fig. 5.3a). The slice-selection gradient waveform was an oscillating triangular waveform of 13 periods, resulting in a 12.8 ms RF pulse. The maximum amplitude and the maximum slew rate in the slice-selection gradient design were chosen to be 50 mT/m and 200 T/m/s, respectively. The slice-selective subpulse was a sinc pulse of a time-bandwidth-product of 3. The magnitude of the Z2 gradient was 0.9 mT/m². The spectral-selective pulse envelope was chosen to be a linear-phase SLR pulse of 138 Hz bandwidth and with 0.1 ripples in the passband and 0.01 ripples in the stopband. For comparison, a conventional RF pulse using spoke trajectory was designed using a recently proposed method that jointly designed the RF pulse and the spoke locations [46]. The time-bandwidth-product of the slice-selective subpulse was also 3. Twenty four spokes were chosen in a 32 × 32 k-space grid, resulting in a 12.8 ms RF pulse.

The accompanying Z2 gradient is shown in Fig. 5.1a, and the proposed RF pulse is shown in Fig. 5.2a. Good slice selectivity was achieved as shown in the excitation pattern of the proposed pulse in the x-z plane (Fig. 5.1c) and in the z-f plane (Fig. 5.2b). The in-plane excitation profile is shown in Fig. 5.3. Compared to the conventional RF pulse using spoke trajectory (Fig. 5.3b), an excitation pattern of much sharper transition on the edge of the circular ROI and smaller side lobes (especially near the object boundary) was achieved by the proposed pulse (Fig. 5.3c). The impact in the transition bands can be better seen in the excitation profile plot in Fig. 5.3d. The through-plane excitation profiles of both methods (results not shown) are very close to each other, since the same slice-selective subpulses were used.

A preliminary experimental study of the proposed method was carried out on a 3.0 T GE whole-body MRI scanner. Since the scanner was not equipped with the capability to dynam-
Figure 5.3: Excitation patterns in the slice plane achieved by the proposed pulse and a conventional pulse using spoke trajectory. **a**: The desired excitation pattern (indicated by a red circle). **b**: The excitation pattern by the conventional pulse. **c**: The excitation pattern by the proposed RF pulse. **d**: The excitation profiles along the dashed lines (red dashed line: the desired profile; blue solid line: the proposed method; black dash-dotted line: the conventional pulse).

To adjust the Z2 gradient strength, the proposed pulse was implemented using a static Z2 gradient, which was controlled by changing the shimming current value. To calibrate the Z2 gradient strength, $B_0$ mapping was performed using a multi-slice gradient-recall echo sequence. For a given shimming current value, the measured $B_0$ map was fitted using a field model with up to second-order gradient terms. Although spin-echo imaging would be better for reducing the effects of the static Z2 gradient during data acquisition, we chose an EPI sequence for our experiments because the sequence has a built-in spatial-spectral RF pulse (that was originally designed for fat suppression at 3.0 T). An image-domain post-processing method [184] with the $B_0$ map as input was used to reduce the distortions of the EPI images caused by the Z2 gradient and the background $B_0$ inhomogeneity.

The Z2 gradient strength used in the experiments was 0.7 mT/m$^2$, while the maximum available Z2 gradient of the scanner was about 2.0 mT/m$^2$. The parameters of the spatial-spectral pulse used in the experiments were: pulse length 9 ms, main-lobe bandwidth 400 Hz, positions of the first excitation replicates ± 800 Hz, and slice thickness 8 mm. The radius of the resulting excitation pattern was 11.5 cm. The position of the first excitation
Figure 5.4: Experimental results from a phantom consisting of multiple tubes: a) without reduced FOV excitation, b) with reduced FOV excitation using the proposed method, and c) 1D profiles along the dashed lines (black dot-dashed line: the desired ROI; blue dashed line: without reduced FOV excitation; red solid line: with reduced FOV excitation).

side-lobe was at \( r = 23.1 \) cm. The imaging FOV was \( 48 \text{ cm} \times 48 \text{ cm} \). The experiments were carried out using a phantom consisting of nine tubes containing copper sulfate. Figure 5.4 shows the experimental results with and without reduced FOV excitation, respectively, which demonstrate that reduced FOV excitation was successfully achieved.

5.4 Discussion

Through Bloch equation simulations and phantom experiments on a commercial MRI scanner, we have demonstrated that a second-order gradient can be used along with a spatial-spectral RF pulse to excite a circular/elliptical ROI in a thin slice. By taking advantage of the unique multidimensional spatial dependence of second-order gradients, we can effectively reduce the RF pulse length required to achieve multidimensional selectivity. Compared to the PexLoc method described in [177, 178], which uses second-order gradients as strong as linear gradients, the proposed method requires only relatively weak second-order gradients. Our preliminary experimental studies demonstrate that reduced FOV excitation can be achieved using second-order gradients readily available on a commercial MRI scanner without requiring specially designed second-order gradient coils and/or amplifiers. However, stronger second-order gradients would make it possible to excite smaller ROIs, increase spatial resolution, and/or shorten the pulse using the proposed method. This is because, for
a given pulse length, stronger second-order gradients increase the time-bandwidth-product of the spectral-selective pulse envelope, thus leading to better in-plane spatial selectivity of the proposed pulse.

The unique spatial dependence of second-order gradients is especially suitable for exciting a circular/elliptical ROI. In conventional reduced FOV imaging, 2D RF pulses are designed to excite a rectangular ROI that limits the FOV in the slice and phase-encoding directions [2–4]. The FOV in the frequency-encoding direction can be reduced by changing the cutoff frequency of the lowpass filter in the receiver chain. While the rectangular FOV is desirable for Cartesian sampling trajectories, the circular/elliptical FOV achieved by the proposed method is more suitable for non-Cartesian sampling trajectories, e.g., radial, spiral and PROPELLER [45] trajectories. Recently, a multi-coil technique has been proposed to generate more complex gradients that are beyond second-order gradients through a set of circular individual coils [185, 186]. When combined with the multi-coil technique, the proposed method can excite more flexible ROIs.

Note that the second-order gradient will also introduce phase dispersion among the spins in the excited region. If the second-order gradient is fully controllable, the problem can be easily addressed by adding a rephasing second-order gradient lobe after excitation (as is done in conventional slice selective gradients). If the second-order gradient is static (stays on for the entire imaging period), which is the case for current commercial MRI scanners, it can be treated as an additional term of background $B_0$ inhomogeneity. Its major effects will be in-plane image distortion, which could be mitigated using large readout gradient and be corrected by $B_0$ inhomogeneity compensation techniques [78, 184] and gradient nonlinearity compensation techniques [187].

5.5 Conclusions

This chapter presents a new method to design RF pulses for reduced FOV excitation using second-order gradients and spatial-spectral pulses. By leveraging the unique multidimensional spatial dependence of second-order gradients, the proposed method achieves 3D spatial selectivity, i.e., a circular/elliptical ROI in a thin slice, using a 2D spatial-spectral RF
pulse. The proposed method is validated using Bloch equation simulation, which shows that the RF pulse dimension reduction achieved by the proposed method can lead to significantly improved excitation accuracy and/or much shorter pulses compared to conventional 3D RF pulses using linear gradients. Preliminary experimental results further demonstrate that with second-order gradients readily available on a commercial MRI scanner reduced FOV excitation can be successfully achieved using the proposed method.
CHAPTER 6

PERTURBATION ANALYSIS OF THE EFFECTS OF 
$B_1$ MAPPING ERRORS ON EXCITATION 
ACCURACIES IN PARALLEL EXCITATION

6.1 Introduction

The length of a multidimensional RF pulse is limited by the finite strength and slewrate of gradients that are designed to traverse the multidimensional excitation $k$-space [27]. Parallel excitation [36, 37] has recently emerged as a promising approach to shortening multidimensional RF pulses. In parallel excitation, each transmit coil presents a different sensitivity pattern, also known as $B_1$ map, and is driven by an independently controlled RF waveform generator. By taking advantage of the distinguished $B_1$ map of each coil, the RF waveform of each coil can be designed to enable undersampling of the excitation $k$-space, and thus shorten the pulse length.

The majority of RF pulse design methods in parallel excitation require knowledge of the $B_1$ maps of transmit coils [31, 36, 37, 62, 63]. An ideal method that measures the $B_1$ maps of a transmit coil array should be: (1) fast, especially for applications that require 3D $B_1$ maps since the measurement should be done for each transmit coil and each imaging object; (2) accurate for a large range of $B_1$ values, since each transmit coil is essentially a surface coil and usually has a large dynamic range in the $B_1$ profile; and (3) of high SNR, especially for regions of low $B_1$ values, where the acquired images are expected to be of low SNR. To address this challenging problem, a variety of $B_1$ mapping methods have been developed. These methods can be classified into two categories: methods based on image magnitude [127–140] and methods based on image phase [141–143]. However, despite these advanced $B_1$ mapping techniques, the accuracy of the measured $B_1$ map is often limited by SNR, resolution, modeling error, and practical issues/imperfections such as $B_0$ inhomogeneity, nonlinearity of large-tip-angle slice selective RF pulses, motion and so
on. It is, therefore, important to characterize the effects of $B_1$ mapping errors on parallel excitation.

The effects of $B_1$ mapping errors on parallel excitation can be analyzed in two aspects. First, $B_1$ mapping errors can affect the RF pulses designed based on the inaccurate $B_1$ maps. Furthermore, since SAR is directly related to the designed RF pulses, $B_1$ mapping errors can also affect SAR, which could be an important safety issue to investigate. Second, $B_1$ mapping errors can affect the accuracy of the resulting excitation pattern, which is the topic of this chapter. More specifically, suppose RF pulses are designed for each coil of a transmit coil array based on the estimated $B_1$ maps, and a designed excitation pattern is calculated by solving the Bloch equation. The question addressed in this chapter is: *How much the designed excitation pattern will derivate from the actual excitation pattern, which is the solution of the Bloch equation with the same RF pulses but with the actual $B_1$ maps?*

The problem is trivial if the Bloch equation has a closed-form solution. However, this is not the case for a general RF pulse. One could also perform the analysis using numerical methods, e.g., solving the Bloch equation numerically and study the effects of noise in the $B_1$ maps by Monte Carlo simulation. However, this approach is computationally exhausting and does not provide useful insights. For small-tip-angle RF pulses, one can use the closed-form linear approximate solution of the Bloch equation to perform the analysis. However, the extension to large-tip-angle RF pulses could fail due to the nonlinearity of the Bloch equation. In this chapter, we present a perturbation analysis approach to analyzing the effects of $B_1$ mapping errors on excitation accuracies in parallel excitation. The proposed approach can be used for analysis in both cases of small-tip-angle and large-tip-angle RF pulses. The proposed approach was validated numerically by Monte Carlo simulation. The effects of $B_1$ mapping errors on excitation accuracies were analyzed with respected to reduction factor and tip-angle.

### 6.2 Perturbation Analysis

Suppose there are an array of $L$ transmit coils, let $\{S_{l,d}(\mathbf{r})\}_{l=1}^L$ and $\{S_{l,a}(\mathbf{r}) = S_{l,d}(\mathbf{r}) + \Delta S_l(\mathbf{r})\}_{l=1}^L$ be the estimated and actual $B_1$ maps of the transmit coil array. Suppose a
gradient waveform $G(t)$ and a target excitation pattern $D(r)$ are preselected, and a set of RF pulses for each coil $\{b_{l,i}(t)\}_{l=1}^{L}$ are designed based on the estimated $B_1$ maps. The designed excitation pattern can then be calculated by solving the Bloch equation:

$$
\begin{align*}
\dot{M}_x(r, t) &= \gamma \begin{bmatrix} 0 & G(t) \cdot r & -B_{1d,y}(r, t) \\ -G(t) \cdot r & 0 & B_{1d,x}(r, t) \\ B_{1d,y}(r, t) & -B_{1d,x}(r, t) & 0 \end{bmatrix} \begin{bmatrix} M_x(r, t) \\ M_y(r, t) \\ M_z(r, t) \end{bmatrix}, \tag{6.1}
\end{align*}
$$

where $B_{1d}(r, t) = B_{1d,x}(r, t) + iB_{1d,y}(r, t) = \sum_{l=1}^{L} S_{l,d}(r)b_{1,l}(t)$ is the designed $B_1$ field. Denote the designed excitation pattern as $M_d(r)$. The actual excitation pattern can be calculated by replacing the designed $B_1$ field $B_{1d}(r, t)$ with the actual $B_1$ field $B_{1a}(r, t) = \sum_{l=1}^{L} S_{l,a}(r)b_{1,l}(t)$ in Eq. (6.1) and is denoted as $M_a(r)$. The goal of the perturbation analysis is to establish a transparent relationship between the $B_1$ mapping error $\Delta S_i(r)$ and the excitation error $\Delta M(r) = M_a(r) - M_d(r)$.

For the convenience of analysis, we make use of the spinor-domain Bloch equation. More specifically, with the designed $B_1$ field, the corresponding Cayley-Klein parameter $\alpha_d(r, t)$ and $\beta_d(r, t)$ are calculated by solving the following spinor-domain Bloch equation:

$$
\begin{align*}
\dot{\alpha}_d(r, t) &= \frac{i\gamma}{2} \begin{bmatrix} G(t) \cdot r, & B_{1d}^*(r, t) \end{bmatrix} \begin{bmatrix} \alpha_d(r, t) \\ \beta_d(r, t) \end{bmatrix}, \\
\dot{\beta}_d(r, t) &= \frac{i\gamma}{2} \begin{bmatrix} B_{1d}(r, t), & -G(t) \cdot r \end{bmatrix} \begin{bmatrix} \alpha_d(r, t) \\ \beta_d(r, t) \end{bmatrix}. \tag{6.2}
\end{align*}
$$

With the actual $B_1$ field, the corresponding Cayley-Klein parameter $\alpha_a(r, t)$ and $\beta_a(r, t)$ are calculated by solving the following spinor-domain Bloch equation:

$$
\begin{align*}
\dot{\alpha}_a(r, t) &= \frac{i\gamma}{2} \begin{bmatrix} G(t) \cdot r, & B_{1a}^*(r, t) \end{bmatrix} \begin{bmatrix} \alpha_a(r, t) \\ \beta_a(r, t) \end{bmatrix}, \\
\dot{\beta}_a(r, t) &= \frac{i\gamma}{2} \begin{bmatrix} B_{1a}(r, t), & -G(t) \cdot r \end{bmatrix} \begin{bmatrix} \alpha_a(r, t) \\ \beta_a(r, t) \end{bmatrix}. \tag{6.3}
\end{align*}
$$

Equation (6.3) can be equivalently rewritten as:

$$
\begin{align*}
\dot{\alpha}_d(r, t) + \dot{\alpha}(r, t) &= \frac{i\gamma}{2} \begin{bmatrix} G(t) \cdot r, & B_{1d}^*(r, t) + \Delta B_1^*(r, t) \end{bmatrix} \begin{bmatrix} \alpha_d(r, t) + \tilde{\alpha}(r, t) \\ \beta_d(r, t) + \tilde{\beta}(r, t) \end{bmatrix}, \\
\dot{\beta}_d(r, t) + \dot{\beta}(r, t) &= \frac{i\gamma}{2} \begin{bmatrix} B_{1d}(r, t) + \Delta B_1(r, t), & -G(t) \cdot r \end{bmatrix} \begin{bmatrix} \alpha_d(r, t) + \tilde{\alpha}(r, t) \\ \beta_d(r, t) + \tilde{\beta}(r, t) \end{bmatrix}, \tag{6.4}
\end{align*}
$$

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where $\Delta B_1(r,t) = \sum_{i=1}^{L} \Delta S_i(r) b_{1i}(t)$ denotes the difference between the actual $B_1$ field and the designed $B_1$ field, and $\tilde{\alpha}(r,t)$ and $\tilde{\beta}(r,t)$ denote the resulting perturbations on the Cayley-Klein parameters.

Subtracting Eq. (6.2) from Eq. (6.4), we obtain the following differential equations to calculate the perturbations on the Cayley-Klein parameters:

\[
\dot{\tilde{\alpha}}(r,t) = \frac{i\gamma}{2} \left[ (G(t) \cdot r) \tilde{\alpha}(r,t) + B_{1d}(r,t) \tilde{\beta}(r,t) + \Delta B_1^*(r,t) \beta_d(r,t) + \Delta B_1^*(r,t) \tilde{\beta}(r,t) \right],
\]

\[
\dot{\tilde{\beta}}(r,t) = \frac{i\gamma}{2} \left( B_{1d}(r,t) \tilde{\alpha}(r,t) + \Delta B_1(r,t) \alpha_d(r,t) + \Delta B_1(r,t) \tilde{\alpha}(r,t) - (G(t) \cdot r) \tilde{\beta}(r,t) \right),
\]

where the initial conditions are $\tilde{\alpha}(r,0) = 0$ and $\tilde{\beta}(r,0) = 0$.

Unfortunately, Eqs. (6.5) and (6.6) in general do not have a closed-form solution, and therefore provides few useful insights for analysis. To get an approximate closed-form solution of the perturbations of the Cayley-Klein parameters, we assume the difference between the actual and designed $B_1$ field, is relatively small compared to the designed $B_1$ field, i.e., $|\Delta B_1(r,t)| \ll |B_{1d}(r,t)|$. We then ignore all the second-order terms in Eqs. (6.5) and (6.6), and obtain the following differential equations:

\[
\begin{bmatrix}
\dot{\tilde{\alpha}}(r,t) \\
\dot{\tilde{\beta}}(r,t)
\end{bmatrix} = \frac{i\gamma}{2} \begin{bmatrix}
G(t) \cdot r & B_{1d}^*(r,t) \\
B_{1d}(r,t) & -G(t) \cdot r
\end{bmatrix} \begin{bmatrix}
\tilde{\alpha}(r,t) \\
\tilde{\beta}(r,t)
\end{bmatrix} + \frac{i\gamma}{2} \begin{bmatrix}
\Delta B_1^*(r,t) \beta_d(r,t) \\
\Delta B_1(r,t) \alpha_d(r,t)
\end{bmatrix}.
\]

We further define a rotation frame as

\[
\begin{bmatrix}
\tilde{\alpha}(r,t) \\
\tilde{\beta}(r,t)
\end{bmatrix} = \begin{bmatrix}
e^{\frac{i\phi(r,t)}{2}} & 0 \\
0 & e^{-\frac{i\phi(r,t)}{2}}
\end{bmatrix} \begin{bmatrix}
\tilde{\alpha}_{rot}(r,t) \\
\tilde{\beta}_{rot}(r,t)
\end{bmatrix},
\]

where $\phi(r,t) = \gamma \int_0^t G(\tau) d\tau \cdot r$. 

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Substituting Eq. (6.8) to Eq. (6.7), yields

\[
\begin{bmatrix}
\dot{\alpha}_{\text{rot}}(r, t) \\
\dot{\beta}_{\text{rot}}(r, t)
\end{bmatrix} = \frac{i\gamma}{2} \begin{bmatrix}
0 & B_{1d}^*(r, t)e^{-i\phi(r,t)} \\
B_{1d}(r, t)e^{i\phi(r,t)} & 0
\end{bmatrix} \begin{bmatrix}
\alpha_{\text{rot}}(r, t) \\
\beta_{\text{rot}}(r, t)
\end{bmatrix} + \frac{i\gamma}{2} v(r, t),
\] (6.9)

where

\[
v(r, t) = \begin{bmatrix}
\Delta B_1^*(r, t)\beta_d(r, t)e^{-i\phi(r,t)} \\
\Delta B_1(r, t)\alpha_d(r, t)e^{i\phi(r,t)}
\end{bmatrix}.
\] (6.10)

In Eq. (6.9), the perturbations on the Cayley-Klein parameters linearly depend on the perturbations on the \(B_1\) field. That is, Eq. (6.9) presents a linear approximation of the Bloch equation around an operating point determined by the designed \(B_1\) field \(B_{1d}(r, t)\) and the preselected gradient \(G(t)\). Equation (6.9) is an inhomogeneous linear differential equation with time varying coefficients and zero initial conditions. The solution of Eq. (6.9) is

\[
\begin{bmatrix}
\alpha_{\text{rot}}(r, t) \\
\beta_{\text{rot}}(r, t)
\end{bmatrix} = \frac{i}{2} \int_0^t U(t, \tau)v(r, \tau)d\tau = \frac{i}{2} U(t, 0) \int_0^t U^{-1}(\tau, 0)v(r, \tau)d\tau,
\] (6.11)

where \(U(t, 0)\) is the state transition matrix.

The state transition matrix \(U(t, 0)\) of Eq. (6.11) does not have a closed-form solution. We use the first-order Magnus series to approximate the state transition matrix, which is given by

\[
U(t, 0) \approx \begin{bmatrix}
\cos \frac{|Q(r, t)|}{2} & i\frac{Q^*(r, t)}{|Q(r, t)|} \sin \frac{|Q(r, t)|}{2} \\
-\frac{iQ(r, t)}{|Q(r, t)|} \sin \frac{|Q(r, t)|}{2} & \cos \frac{|Q(r, t)|}{2}
\end{bmatrix}.
\] (6.12)

In Eq. (6.12), \(Q(r, t)\) is given by

\[
Q(r, t) = e^{-i(k(0):r)}\theta(r, t),
\] (6.13)

\[
\theta(r, t) = \gamma \int_0^t B_{1d}(r, \tau)e^{i(k(t):r)}d\tau,
\] (6.14)

where \(k(t) = -\gamma \int_0^T G(\tau)d\tau\) is the excitation k-space trajectory and \(T\) is the pulse length.

Substituting the approximate state transition matrix in Eq. (6.12) to Eq. (6.11), and
transform the solution back to the original rotation frame based on Eq. (6.8), we obtain
the following closed-form approximate solution of the perturbations on the Cayley-Klein
parameters:

\[
\tilde{\alpha}(r, T) \approx \sum_{l=1}^{L} [\Delta S_l(r)c_{\alpha,l}(r, T) + \Delta S^*_l(r)d_{\alpha,l}(r, T)],
\]

\[
\tilde{\beta}(r, T) \approx \sum_{l=1}^{L} [\Delta S_l(r)d_{\beta,l}(r, T) + \Delta S^*_l(r)c_{\beta,l}(r, T)],
\]

where

\[
c_{\alpha,l}(r, T) = -\frac{\gamma}{2} \int_0^T q(r, t)\alpha_d(r, t)b_{1,l}(t)e^{i\Phi_{k(t)}r} dt,
\]

\[
d_{\alpha,l}(r, T) = \frac{i\gamma}{2} \int_0^T p(r, t)\beta_d(r, t)b_{1,l}^*(t)e^{-i\Phi_{k(t)}r} dt,
\]

\[
c_{\beta,l}(r, T) = -\frac{\gamma}{2} \int_0^T q^*(r, t)\beta_d(r, t)b_{1,l}^*(t)e^{-i\Phi_{k(t)}r} dt,
\]

\[
d_{\beta,l}(r, T) = \frac{i\gamma}{2} \int_0^T p^*(r, t)\alpha_d(r, t)b_{1,l}(t)e^{i\Phi_{k(t)}r} dt,
\]

\[
p(r, t) = \cos \frac{\theta(r, T)}{2} \cos \frac{\theta(r, t)}{2} + \frac{\theta^*(r, T)}{2} \theta(r, t) \sin \frac{\theta(r, T)}{2} \sin \frac{\theta(r, t)}{2},
\]

\[
q(r, t) = \frac{\theta^*(r, T)}{2} \sin \frac{\theta(r, T)}{2} \cos \frac{\theta(r, t)}{2} - \frac{\theta^*(r, t)}{2} \cos \frac{\theta(r, T)}{2} \sin \frac{\theta(r, t)}{2}.
\]

The perturbations on the excitation pattern can be calculated based on the following
relationship between the magnetization vector and the Cayley-Klein parameters:

\[
\begin{bmatrix}
M_{xy}(T) \\
M^*_{xy}(T) \\
M_z(T)
\end{bmatrix} =
\begin{bmatrix}
(\alpha^*)^2 & -\beta^2 & 2\alpha^*\beta \\
-(\beta^*)^2 & \alpha^2 & 2\alpha\beta^* \\
-\alpha^*\beta^* & \alpha\beta & \alpha\alpha^* - \beta\beta^*
\end{bmatrix}
\begin{bmatrix}
M_{xy}(0) \\
M^*_x(0) \\
M_z(0)
\end{bmatrix}.
\]

For an excitation RF pulse, the initial state of the magnetization vector is \([0, 0, 1]^T\). The
interested excitation pattern is the transversal component of the magnetization vector. Ig-
noring the second-order terms, the perturbations on the excitation pattern can be approxi-
\[
\begin{align*}
\Delta M_{xy}(r) & \approx 2[\beta_d(r, T)\tilde{\alpha}(r, T) + \alpha_d^*(r, T)\tilde{\beta}(r, T)] \\
& \approx 2 \sum_{l=1}^{L} \left\{ \Delta S_l(r)[\beta_d(r, T)d_{\alpha,l}^*(r, T) + \alpha_d^*(r, T)d_{\beta,l}(r, T)] \\
& + \Delta S_l^*(r)[\beta_d(r, T)c_{\alpha,l}^*(r, T) + \alpha_d^*(r, T)c_{\beta,l}(r, T)] \right\}.
\end{align*}
\]

(6.24)

For a saturation or an inversion RF pulse, the initial state of the magnetization vector is \([0, 0, 1]^T\). The interested excitation pattern is the longitudinal component of the magnetization vector. Ignoring the second-order terms, the perturbations on the excitation pattern can be approximated by:

\[
\begin{align*}
\Delta M_z(r) & \approx -2[\beta_d(r, T)\tilde{\beta}(r, T) + \beta_d^*(r, T)\tilde{\beta}(r, T)] \\
& \approx -2 \sum_{l=1}^{L} \left\{ \Delta S_l(r)[\beta_d(r, T)c_{\beta,l}^*(r, T) + \beta_d^*(r, T)d_{\beta,l}(r, T)] \\
& + \Delta S_l^*(r)[\beta_d(r, T)c_{\beta,l}^*(r, T) + \beta_d^*(r, T)c_{\beta,l}(r, T)] \right\}.
\end{align*}
\]

(6.25)

For a refocusing RF pulse, suppose the initial state of the magnetization vector is along the \(y\)-axis, i.e., \([0, 1, 0]^T\). Assuming sufficiently big crusher gradients are used, the resulting excitation pattern is \(M_{xy} = i\beta^2\). Ignoring the second-order terms, the perturbations on the excitation pattern can be approximated by:

\[
\begin{align*}
\Delta M_{xy}(r) & \approx 2i\beta_d(r, T)\tilde{\beta}(r, T) \\
& \approx 2i \sum_{l=1}^{L} \left\{ \Delta S_l(r)\beta_d(r, T)d_{\beta,l}(r, T) + \Delta S_l^*(r)\beta_d(r, T)c_{\beta,l}(r, T) \right\}.
\end{align*}
\]

(6.26)

6.3 Results

6.3.1 Validation by Monte Carlo Simulation

The accuracy of the perturbation analysis replies on how well the two approximations we made in the derivation of Eqs. (6.15) and (6.16) are satisfied. First, we assumed that the difference between the actual \(B_1\) field and the designed \(B_1\) field caused by \(B_1\) mapping errors is relatively small compared to the designed \(B_1\) field. We then ignored the second-order terms
in Eqs. (6.5) and (6.6). Second, we used the first-order Magnus series to approximate the state transition matrix, i.e., Eq. (6.12). The accuracy of the first-order Magnus series would decrease as the tip-angle of the designed RF pulse increases. We studied the accuracy of the perturbation analysis with respect to $B_1$ mapping error and flip angle by Monte Carlo simulation.

More specifically, we performed the simulation using an eight-channel transmit coil array. The $B_1$ map of each channel that were collected on a 3T GE scanner is shown in Fig. 6.1. Two-dimensional RF pulses were designed to excite a circular region-of-interest (as shown in Fig. 6.2a) using the linear class large-tip-angle design method in [63] with a 12-turn spiral-in gradient. Since the validation of the perturbation analysis does not rely on specific distributions of $B_1$ mapping errors, for the convenience of result comparison, we assumed that the actual $B_1$ maps were the estimated $B_1$ maps plus i.i.d Gaussian noise with a standard deviation $\sigma$.

Solving the Bloch equation with the designed RF pulses and the estimated $B_1$ maps, we obtained the designed excitation pattern. Solving the Bloch equation with the designed RF pulses and the actual $B_1$ maps, we obtained the actual excitation pattern and the corresponding excitation error. Through Monte Carlo simulation, we calculated the standard deviation of the excitation error, which we treated as the gold standard. The standard deviation of the excitation error can be also be estimated based on the perturbation analysis, i.e., Eqs. (6.24) to (6.26). For an excitation pulse, the estimated standard deviation of the excitation error is given by,

$$
\sigma_{M_{xy}}^2(r) \approx 4\sigma^2 \sum_{l=1}^{L} \left\{ |\beta_d(r, T)d_{\alpha,l}(r, T) + \alpha_d(r, T)d_{\beta,l}(r, T)|^2 \\
+ |\beta_d(r, T)c_{\alpha,l}(r, T) + \alpha_d(r, T)c_{\beta,l}(r, T)|^2 \right\}.
$$

(6.27)

The root-mean-square (RMS) error of the standard deviation of the excitation error estimated using Eq. (6.27) is shown in Fig. 6.2. As shown in Fig. 6.2a, while increasing with the tip-angles as expected, the RMS errors are less than 10 % for tip-angles from 10° to 90°, which indicates the perturbation analysis is applicable to both small-tip-angle and large-tip-angle RF pulses. Figure 6.2b shows that the RMS error changes a little as the
Figure 6.1: The magnitude profiles of $B_1$ maps of an eight-channel transmit coil array at 3T.

Figure 6.2: Verification of the perturbation analysis using Monte Carlo simulation. **a:** The designed excitation pattern achieved by a 90° 2D RF pulse with a 12-turn spiral-in gradient and reduction factor of 1. **b:** RMS error of the estimated standard deviation of the excitation error as a function of tip-angle. **c:** RMS error of the estimated standard deviation of the excitation error as a function of the lowest SNR of the estimated $B_1$ maps.
lowest SNR of the estimated $B_1$ maps changes from 0 to 20 dB, which indicates that the accuracy of the perturbation analysis is not sensitive to the magnitude of $B_1$ mapping errors in a large range.

6.3.2 The Effects of $B_1$ Mapping Errors

We first studied the effects of $B_1$ mapping errors on excitation accuracies with respect to tip-angle. More specifically, 2D RF pulses were designed to excitation a circular region-of-interest (as shown in Fig. 6.2a) with a reduction factor of 2, i.e., a six-turn spiral-in gradient, and tip-angles varying from $10^\circ$ to $90^\circ$. We assumed that the actual $B_1$ maps were the estimated $B_1$ maps plus i.i.d Gaussian noise with a standard deviation $\sigma$ that made the lowest SNR of the estimate $B_1$ maps be 10. The standard deviation of the excitation error were estimated by Eq. (6.27), and was normalized by the tip-angle $\theta$ the standard deviation of the injected Gaussian noise:

$$\hat{\sigma}_{\Delta M_{xy}} = \frac{\sigma_{\Delta M_{xy}}}{\sigma \sin(\theta)}. \quad (6.28)$$

The normalized standard deviations of the excitation errors for the $10^\circ$ and $90^\circ$ RF pulse are shown in Fig. 6.3a and Fig. 6.3b, respectively. Comparing Fig. 6.3b with Fig. 6.3a, the spatial distribution of the excitation error in the passband, i.e., the bright circular region, is very similar, however, the spatial distribution of the excitation error in the stopband is significantly different, which reflects the effects of the nonlinearity of the Bloch equation.

Figure 6.3c and Fig. 6.3d show the average standard deviation of the excitation error in the passband and stopband as a function of tip-angle, which is further normalized with the average standard deviation of $10^\circ$ tip-angle. While almost keeping constant in the passband, the average standard deviation of the excitation error in the stopband increases significantly with the tip-angle, which indicates that the excitation accuracy, especially in the stopband, is more sensitive to $B_1$ mapping errors for large-tip-angle RF pulses.

Second, we studied the effects of $B_1$ mapping errors on excitation accuracies with respect to reduction factor. The simulation setup is similar to the previous case, expect that $90^\circ$ RF pulses are designed with reduction factors of 1, 2, and 3. The normalized standard deviations
Figure 6.3: The normalized standard deviations of the excitation errors for 2D RF pulses with a reduction factor of 2. a and b: The normalized standard deviation of the excitation error for the RF pulses with tip-angles of 10° and 90° respectively. c: The normalized average standard deviation of the excitation error in the passband as a function of tip-angle. d: The normalized average standard deviation of the excitation error in the stopband as a function of tip-angle.

of the excitation errors for the RF pulses with different reduction factors are shown in Fig. 6.4a to Fig. 6.4c. The normalized average standard deviation of the excitation error in the passband (blue-line) and the stopband (red-line) are shown in Fig. 6.4d as functions of reduction factor. Notably, the standard deviation of the excitation error in the stopband increase significantly as the reduction factor increases.

Figure 6.4: The normalized standard deviations of the excitation errors for 2D RF pulses with 90° tip-angle. a to c: The normalized standard deviation of the excitation error for the RF pulses with reduction factors of 1, 2, and 3, respectively. d: The normalized average standard deviation of the excitation error in the passband (blue-line) and stopband (red-line) as a function of reduction factor.
6.4 Conclusions

In this chapter, we have presented a perturbation analysis approach to analyzing the effects of $B_1$ mapping errors on excitation accuracies in parallel excitation. We show that the excitation errors are increasingly sensitive to $B_1$ mapping errors as the tip-angle and reduction factor of an RF pulse increases. The current analysis models the $B_1$ mapping errors as uniform Gaussian noises. Ongoing work is to consider more systematic $B_1$ mapping errors including those caused by resolution loss, motion, modeling error in $B_1$ map estimation.
CHAPTER 7

FUTURE WORK

There are several important issues worth further investigation.

- **Design of $B_1$ insensitive multidimensional SLR RF pulse**: The $B_1$ inhomogeneity increases as the main field strength increases. It is important to take $B_1$ inhomogeneity into account while using multidimensional RF pulses at high field or ultra high field scanners. One approach to addressing this issue is to explicitly factor the $B_1$ inhomogeneity into the formulation of RF pulse design. Another approach is to design $B_1$ insensitive RF pulses. One significant advantage of this approach is that it does not require the knowledge of subject-dependent $B_1$ maps. It has been shown that imposing quadratic phase to the resulting excitation pattern of a 1D RF pulse can significantly improve its robustness to $B_1$ inhomogeneity [109]. It is, therefore, of great interest to design $B_1$ insensitive multidimensional SLR RF pulses by imposing a nonlinear phase to the resulting excitation pattern. The proposed multidimensional SLR RF pulse design method (Chapter 3) is fully capable of designing nonlinear-phase RF pulses. The remaining issue is to design phase patterns to optimize the robustness to $B_1$ inhomogeneity.

- **Applications of reducing FOV excitation using second-order gradients**: Chapter 5 presents a novel reduced FOV excitation method using second-order gradients and a spatial-spectral RF pulse. The proposed method is able to significantly reduce pulse length and/or improve excitation profiles in applications, where the desired ROI is a circular or elliptical disk. However, only proof-of-concept experiments have been done on a commercial scanner with a phantom. It is of great interest to explore applications of the proposed method. The proposed method especially fits in applications using
non-Cartesian $k$-space trajectories that result in non-rectangular FOV. Potential applications include imaging of the liver and prostate and diffusion tensor imaging of the brain and spinal cord.

- *Comparison of commonly used $B_1$ mapping methods:* Since the introduction of parallel excitation, significant advances have been achieved in developing fast and accurate $B_1$ mapping methods, including the fast double angle method [133], actual flip-angle imaging method [137], and Bloch-Siegert method [143]. The proposed perturbation analysis in Chapter 6 provides a systematic approach to comparing these $B_1$ mapping methods in the terms of their effects on excitation accuracies. In order to do that, the bias and variations of the compared $B_1$ mapping methods should be first derived. The proposed perturbation analysis can then be applied to investigate the effects of the $B_1$ mapping errors of the compared methods on excitation accuracies with different tip-angles and reduction factor.
REFERENCES


