UNIFORMLY RIGID HOMEOMORPHISMS

BY

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DISSERTATION

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Abstract

In this dissertation we are interested in the study of dynamical systems that display rigidity and weak mixing. We are particularly interested in the topological analogue of rigidity, called uniform rigidity. A map T defined on a topological space X is called *uniformly rigid* if there exists an increasing sequence of natural numbers (n_m) such that (T^{n_m}) converges to the identity uniformly on X and is called *weakly mixing* if there exists a sequence (s_m) of density one such that $\mu(T^{s_m}A \cap B)$ converges to $\mu(A)\mu(B)$ for every A, B of positive μ-measure (the sequence (s_m) is called a *mixing sequence*). Uniform rigidity and weak mixing are two properties of a dynamical system that are very different, though not exclusive. Rigidity implies that at certain times the image of an interval is close to the interval, while weak mixing implies that at other times the images of intervals are evenly distributed. Observe that the rigidity times for a weakly mixing map have density zero. This dissertation attempts to better understand the interplay between weak mixing and uniform rigidity.

The underlying theme of this dissertation has two threads: (1) to determine how the topology of a space affects dynamical properties of maps that are defined there and (2) to characterize the structure of uniform rigidity sequences for weakly mixing maps. The work in this dissertation has involved several projects that were designed to provide a better understanding of these maps and their uniform rigidity sequences, thereby yielding information about the dynamical properties that are compatible with certain spaces and information about the structure of those sequences.

For my family, new and old.

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Table of Contents

Chapter 1

Introduction

1.1 History

Even though ergodic theory originated out of considerations in statistical mechanics, that is only a small part of the theory that has developed. Nevertheless, that is where we begin. Consider n gas particles in space, where each particle is represented by its position and momentum. Thus, each gas particle has six numbers associated with it and the collection of gas particles can be thought of as a point in $\mathbb{R}^{6n}$. That is, a point in $\mathbb{R}^{6n}$ represents the state of the system at a moment in time and the whole space is the phase space or collection of possible states.

The laws of motion can be represented by a map $T : \mathbb{R}^{6n} \to \mathbb{R}^{6n}$ where, if x is a state, then $T(x)$ describes what the system will look like after one unit of time. According to the laws of classical mechanics, the past, present, and future of the system can be determined once we know one instantaneous state. However, in the real world we rarely have enough information to determine the entire trajectory.

In statistical mechanics, the idea is to think about what will probably happen and leave the deterministic viewpoint behind. Instead of trying to figure out what the system will look like at a specific time, it is better to think about what the probability is that a state of the system will belong to a specific subset of the phase space.

These new questions led Liouville to investigate what happens to subsets of the phase space as time changes continuously according the the laws of motion, which are given by Hamilton's equations. Liouville discovered that the Lebesgue measure of a subset of the phase space is the same as the Lebesgue measure of the subset after some amount of time. This is known as an *invariance principle*.

These types of questions and discoveries led to the development of ergodic theory. For more information see [23] or [33]

1.2 Terminology

1.2.1 Ergodic Theory

Let (X, β, μ) be a measure space where β is the σ-algebra of μ measurable subsets of X. In addition, suppose that (X, β, μ) is a *standard Lebesgue space*, which means that it is measure-theoretically isomorphic to the unit interval with Lebesgue measure. In particular, (X, β, μ) is nonatomic and $L_2(X, \mu)$ is separable.

Suppose $T : X \to X$ is a *measure-preserving transformation*, i.e. $\mu(T^{-1}A) = \mu(A)$ for all $A \in \beta$. Furthermore, assume that T is invertible. In this case, we will call (X, β, μ, T) a *dynamical system*.

Definition 1.2.1. A transformation T is *ergodic* if every T-invariant set A satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Let $\langle f \mid g \rangle$ denote the standard inner-product on $L_2(X, \mu)$ given by

$$\langle f \mid g \rangle = \int_X f \cdot \overline{g} \, d\mu.$$

Let U_T be the unitary operator on $L_2(X, \mu)$ induced by T, called the *Koopman operator*, defined by $U_T(f) = f \circ T$ for all $f \in L_2(X, \mu)$. An equivalent form of ergodicity follows, along with the definitions of weak mixing and strong mixing.

Proposition 1.2.2 ([33])**.** A transformation T is ergodic if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle U_T^n f|g\rangle - \langle f|1\rangle\langle 1|g\rangle = 0$$

for all $f, g \in L_2(X, \mu)$.

Definition 1.2.3. A transformation T is *weakly mixing* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n f|g\rangle - \langle f|1\rangle\langle 1|g\rangle| = 0$$

for all $f, g \in L_2(X, \mu)$.

Definition 1.2.4. A transformation T is *strongly mixing* if

$$\lim_{n\to\infty} |\langle U_T^n f|g\rangle - \langle f|1\rangle\langle 1|g\rangle| = 0$$

for all $f, g \in L_2(X, \mu)$.

From the above definitions it is easy to see that strong mixing $\implies$ weak mixing $\implies$ ergodic. However the reverse implications are not true. To see that ergodic does not imply weak mixing consider an irrational rotation of the circle, and to see that weak mixing does not imply strong mixing consider Chacón's transformation [**?**].

One of the most celebrated theorems in ergodic theory is the Birkhoff ergodic theorem, also known as the pointwise ergodic theorem (see [33]). This theorem says that if the transformation is ergodic, then the time average converges to the space average almost everywhere.

Theorem 1.2.5 (Birkhoff Ergodic Theorem)**.** Let (X, β, μ, T) be an ergodic dynamical system. If $f \in L_1(X, \mu)$ then

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to \int_X f \, d\mu$$

for almost every $x \in X$.

1.2.2 Topological Dynamics

Let X be a topological space. In addition, assume it is a compact metric space with metric d and $T : X \to X$ a homeomorphism. In this case, we call (X, T) a *flow*. Flows, in particular minimal flows, are the primary object of study in topological dynamics.

Definition 1.2.6. A flow (X, T) is *minimal* if every compact T-invariant subset is either the whole space or the empty set.

Similar to ergodic theory, there are different types of mixing in topological dynamics.

Definition 1.2.7. A flow (X, T) is *topologically transitive* if for every pair U, V of nonempty open subsets of X, there exists a time n with

$$T^n U \cap V \neq \emptyset.$$

The above definition is sometimes referred to as *topological ergodicity*. It is also equivalent to the existence of a point with a dense orbit. That is, there exists $x \in X$ with $\{T^n x : n \in \mathbb{Z}\}$ dense in X.

Proposition 1.2.8 ([16])**.** A flow (X, T) is minimal if and only if every point $x \in X$ has a dense orbit.

This proposition is standard and is sometimes taken as the definition of a minimal flow. From this, it is also easy to see that minimality is stronger than topological transitivity.

Definition 1.2.9. A flow (X, T) is *topologically weakly mixing* if for every collection U_1, U_2, V_1, V_2 of nonempty open subsets of X, there exists a time n with

$$T^n U_1 \cap V_1 \neq \emptyset \quad \textit{and} \quad T^n U_2 \cap V_2 \neq \emptyset.$$

Remark 1.2.10. The above definition is equivalent to $X \times X$ being topologically transitive.

Definition 1.2.11. A flow (X, T) is *topologically strongly mixing* if for every pair U, V of nonempty open subsets of X, there exists a time N such that for all $n \geq N$

$$T^n U \cap V \neq \emptyset.$$

In a similar fashion as before, topological strong mixing $\implies$ topological weak mixing $\implies$ topological transitivity. To see that topological transitivity does not imply topological weak mixing consider an irrational rotation of the circle, and to see that topological weak mixing does not imply topological strong mixing consider the example constructed by Blanchard in [8].

1.3 Analogies Between Measurable and Topological Dynamics

The interplay between measurable and topological dynamics is vast. We will focus on a few key analogies: recurrence, weak mixing, higher order mixing, and rigidity.

1.3.1 Recurrence

Poincaré's recurrence theorem is one of the oldest theorems in ergodic theory. It is a very powerful theorem with an elegant proof that relies on the finiteness of the measure space (see [33] for more details).

Theorem 1.3.1 (Poincaré Recurrence)**.** Let (X, β, μ, T) be a dynamical system and $A \in \beta$. Then, μ-almost-every point of A returns to A.

A stronger formulation can also be proven, which says that μ-a.e point of A returns to A infinitely often.

Definition 1.3.2. Suppose X is a metric space. A point $x \in X$ is a *recurrent point* for the map T if there exists an increasing sequence (n_m) such that

$$T^{n_m} x \to x$$

as $m \to \infty$.

Now suppose that X is a separable metric space in addition to (X, β, μ, T) being a dynamical system. Then by covering the space with countably many $\frac{\epsilon}{2}$ balls and using Poincaré's recurrence theorem, we see that almost every point returns to within ϵ of itself. Hence, almost every point of X is a recurrent point.

There is also a topological road to recurrence and a topological proof of the existence of a recurrent point for flows. This theory was developed by Birkhoff and is known as the Birkhoff recurrence theorem [7].

Theorem 1.3.3 (Birkhoff Recurrence)**.** Let (X, T) be a flow. Then there exists a recurrent point in X.

Birkhoff gave a purely topological proof of the above result. In his argument compactness played a similar role to the assumption of finite measure by Poincaré.

1.3.2 Weak Mixing

We have already seen the definitions of weak mixing and topological weak mixing. We will now discuss equivalent definitions for each; this will make it more clear how the notions of weak mixing are related. Recall that the Koopman operator U_T is defined on $L_2(X, \mu)$ by $U_T(f) = f \circ T$. We say that $f \in L_2(X, \mu)$ is an eigenfunction of U_T if $U_T(f) = \lambda f$ for some eigenvalue λ.

Proposition 1.3.4 (see [33])**.** Supoose (X, β, μ, T) is a dynamical systems. The automorphism T is weakly mixing if and only if the associated Koopman operator has no non-constant eigenfunctions.

Keynes and Robertson proved an analogous result for minimal flows.

Proposition 1.3.5 ([29])**.** Suppose (X, T) is a minimal flow. The homeomorphism T is topologically weakly mixing if and only if the associated Koopman operator has no non-constant continuous eigenfunctions.

The method that Keynes and Robertson used to prove the above proposition relies heavily on the fact that if your flow is minimal, then there exists an invariant probability measure on the space with full support.

1.3.3 Higher Order Mixing

An old and still open problem in ergodic theory is whether strong mixing implies mixing of all orders.

Definition 1.3.6. Suppose (X, β, μ, T) is a dynamical system. The automorphism T is *strongly 3-mixing* if for every $A, B, C \in \beta$

$$\mu(T^n A \cap T^{n+m} B \cap C) \to \mu(A)\mu(B)\mu(C)$$

as $n, m \to \infty$.

From the above definition, it is clear that strong 2-mixing is the same as strong mixing. The following is known as Rokhlin's problem in ergodic theory:

Question 1.3.7. Suppose (X, β, μ, T) is a dynamical system. Does strong 2-mixing imply strong 3-mixing?

The above problem is still wide open. A similar problem can be stated in topological dynamics and in this setting there is an answer.

Definition 1.3.8. Suppose (X, T) is a flow. The homeomorphism T is *strongly 3-topologically mixing* if for all nonempty open subsets U, V, W of X there exists a time N such that for all $n, m \geq M$

$$T^n U \cap T^{n+m} V \cap W \neq \emptyset.$$

In this setting, Rokhlin's problem was solved by Goodman and Marcus [18]. They provided two examples of flows that are strongly 2-topologically mixing, but not strongly 3-topologically mixing. One of the examples is a symbolic dynamical system that has an ergodic invariant measure, with respect to which the dynamical system is weakly mixing, but not strongly mixing. The second example in their work is a homeomorphism of the two-torus called a Stepanoff flow.

1.3.4 Rigidity

The notion of rigidity, like recurrence, weak mixing, and higher order mixing, appears in ergodic theory and topological dynamics. We will explore various aspects of rigidity later, but here we begin with the definitions.

Definition 1.3.9. Suppose (X, β, μ, T) is a dynamical system. The automorphism T is *rigid* if there exists an increasing sequence of natural number (n_m) such that the powers T^{n_m} converge to the identity in the strong operator topology. That is,

$$\|f \circ T^{n_m} - f\|_2 \to 0$$

as $m \to \infty$ for all $f \in L_2(X, \mu)$.

The above definition can also be formulated as there exists an increasing sequence of natural numbers (n_m) such that $\mu(T^{n_m} A \triangle A) \to 0$ as $m \to \infty$ for any measurable set A.

In topological dynamics there are a two notions of rigidity; *topological rigidity* and *uniform rigidity* (see [17] for more information). For now, we will concentrate on uniform rigidity.

Definition 1.3.10. Suppose (X, T) is a flow. The homeomorphism T is *uniformly rigid* if there exists an increasing sequence of natural numbers (n_m) such that the powers T^{n_m} converge to the identity uniformly on the space X.

In the next chapter we will see that uniform rigidity is strictly stronger than rigidity, expect in the case where the space is the unit interval or circle.

Chapter 2

Background and Relevant Results

In this chapter we provide the background needed to understand the results of this thesis and the way they fit into the larger framework. We also provide selected proofs of some relevant results to display the flavor and style of proofs in ergodic theory and topological dynamics.

2.1 Generic Properties

What does the "typical" map look like and what dynamical properties does it display? We will investigate that question in this section with respect to dynamical systems and with respect to flows.

2.1.1 $Auto(X, \mu)$

Suppose (X, β, μ) is a standard Lebesgue space and let $Auto(X, \mu)$ denote the set of automorphisms (measurable bijections) from X to X which preserve the probability measure μ. The space $Auto(X, \mu)$ is equipped with the weak topology. In this topology, a sequence of automorphisms (T_i) converge to T if $\mu(T_i A \triangle TA) \to 0$ as $i \to \infty$ for every $A \in \beta$ where $\triangle$ denotes symmetric difference. Let $\{E_i\}$ be a countable dense subset of measurable sets. The distance between two automorphisms S, T in the weak topology is defined by

$$d_w(S,T) = \sum_{i=1}^{\infty} \frac{1}{2^i}(\mu(TE_i \triangle SE_i) + \mu(T^{-1}E_i \triangle S^{-1}E_i)).$$

The space $Auto(X, \mu)$ is a complete metric space and thus we can discuss subsets of different sizes in terms of category. Let $L_2^0(X, \mu)$ denote the space of $L_2(X, \mu)$ functions defined on (X, μ) that have L_2 norm one and are mean zero. When the underlying measure space is clear from context we will drop the (X, μ) and just write L_2^0.

The first category result says that the set of weakly mixing automorphisms is generic. This proof uses the fact that there exists a weakly mixing automorphism and is therefore not sufficient in its own right to prove existence. Before we proceed to the proof, we turn to Halmos' conjugacy lemma [23], which is used to prove that the weakly mixing automorphisms are generic. An automorphism is said to be *antiperiodic* if the set of periodic points has measure zero.

Lemma 2.1.1 (Conjugacy Lemma [23])**.** In the weak topology the conjugacy class of each antiperiodic automorphism is everywhere dense in $Auto(X, \mu)$.

The power of this lemma lies in the fact that it allows you to approximate any automorphism with the conjugation of an ergodic automorphism. For example, the conjugates of an irrational rotation will be dense in $Auto(X, \mu)$ with respect to the weak topology. We will see later that this is not the case when you only allow conjugation by homeomorphisms and use a finer topology.

Theorem 2.1.2 ([23])**.** In the weak topology the set of all weakly mixing automorphisms is an everywhere dense G_δ subset of $Auto(X, \mu)$.

Proof. To show that the set of weakly mixing automorphisms is a dense G_δ subset, we will rely heavily on Halmos' conjugacy lemma. Recall that there exist a weakly mixing automorphism of $Auto(X, \mu)$ and that the property of weak mixing is invariant under conjugation by automorphisms. Thus by Halmos' conjugacy lemma the set of conjugates of this automorphism are dense. All that remains to be shown is that weak mixing is a G_δ property.

To that end, let $\{\phi_i\}$ be a dense subset of L_2^0 and define

$$R_{\phi_i} = \{T \in Auto(X,\mu) : \textit{there exists } n \textit{ with } |\langle U_T^n \phi_i | \phi_i \rangle| < 0.99\}.$$

Let $\mathcal{R} = \bigcap_{i=1}^{\infty} R_{\phi_i}$. Notice that R_{ϕ_i} is an open condition and thus it suffices to show that $\mathcal{R}$ is the set of weakly mixing automorphisms.

Recall that T is weakly mixing if and only if for all $\phi \in L_2^0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n \phi | \phi \rangle| = 0.$$

So, if T is weakly mixing, then for every ϕ_i there exists n such that $|\langle U_T^n \phi_i | \phi_i \rangle| < 0.99$.

Now suppose that $T \in \mathcal{R}$. To show that T is weakly mixing it suffices to show that T has no nonconstant eigenfunctions. Suppose for a contradiction that $\phi \in L_2^0$ is a nonconstant eigenfunction for T. In this case, there exists $\lambda \in \mathbb{C}$ of modulus one such that $U_T \phi = \lambda \phi$. Let ϕ_i be such that $\|\phi - \phi_i\|_2 < 0.001$.

For all n

$$|\langle U_T^n \phi | \phi \rangle| = |\lambda|^n |\langle \phi | \phi \rangle| = \|\phi\|_2^2 = 1.$$

Since $T \in R_{\phi_i}(0.99)$ for some n the following holds:

$$\begin{aligned} 1 &= |\langle U_T^n \phi | \phi \rangle| \\ &= |\langle U_T^n(\phi - \phi_i) + U_T^n \phi_i | (\phi - \phi_i) + \phi_i \rangle| \\ &\leq |\langle U_T^n \phi_i | \phi_i \rangle| + 3\|\phi - \phi_i\|_2 \\ &< 0.993. \end{aligned}$$

This is a contradiction. Therefore T has no nonconstant eigenfunctions and is weakly mixing.

□

The second category result says that the set of strongly mixing automorphisms is contained in a set of first category. Before we state and prove this result, we will need an approximation theorem, known as the weak approximation theorem. For simplicity, we will use $Auto([0,1],\lambda)$ where λ is Lebesgue measure on the unit interval.

An interval of the form $\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right)$ is called a *dyadic interval* of order m where $k = 0, \ldots, 2^m - 1$. If m is fixed, then a union of such intervals is called a *dyadic set* of order m. A *dyadic permutation* is simply a translation of the dyadic intervals. A dyadic neighborhood of the automorphism T is of the form

$$\{S \in Auto([0,1],\lambda) \ : \ \lambda(SD \triangle TD) < \epsilon\}$$

for some ϵ and dyadic set D.

Theorem 2.1.3 (Weak Approximation Theorem [23])**.** Every dyadic neighborhood contains cyclic permutations of arbitrarily high orders.

The weak approximation theorem will appear again when we are dealing with homeomorphisms and will be proved in that context. This theorem will help us prove that the strongly mixing autormorphisms are not generic.

Theorem 2.1.4 ([23])**.** In the weak topology the set of all strongly mixing automorphisms is contained in a set of first category.

Proof. Our goal is to show that the set of strongly mixing automorphisms in $Auto([0,1],\lambda)$ is contained in a countable union of nwd (nowhere dense) sets.

First recall that the set of dyadic permutations of arbitrarily high order are dense in the space of automorphisms by the weak approximation theorem. Let

$$P_k = \{T \in Auto([0,1],\lambda) \ : \ T \textit{ is a permutation of order } k\}.$$

Note that if $T \in P_k$ then $T^k = Id$. Thus for all n, the set $\bigcup_{n \geq k} P_k$ is dense.

Let $A = [0, \frac{1}{2^m}]$ be a dyadic interval of order m and

$$M_k = \left\{ T \in Auto([0,1],\lambda) \ : \ \left| \lambda(T^k A \cap A) - \frac{1}{2^{2m}} \right| \leq \frac{1}{2^{4m}} \right\}.$$

Notice that in the above statement $\frac{1}{2^{2m}} = \lambda(A)^2$. We will show that $\bigcup_{n\geq 0} \bigcap_{k\geq n} M_k$ contains all of the strongly mixing automorphisms. Suppose for a contradiction that there exists T strongly mixing such that for all $n \geq 0$ there exists $k \geq n$ such that $T \notin M_k$, i.e.

$$\left| \lambda(T^k A \cap A - \lambda(A)^2 \right| > \frac{1}{2^{4m}}.$$

This contradicts that fact that T is strongly mixing.

It remains to show that $\bigcap_{k\geq n} M_k$ is nwd. To see this, note that $P_k \cap M_k = \emptyset$ and therefore $\left(\bigcup_{k\geq n} P_k\right) \cap \left(\bigcap_{k\geq n} M_k\right) = \emptyset$. Since $\bigcup_{k\geq n} P_k$ is dense and M_k is closed, $\bigcap_{k\geq n} M_k$ has empty interior and is nwd.

□

Rigidity is another property that is generic in $Auto(X, \mu)$. The proof of this result uses very similar techniques as the previous theorem. It relies heavily on the density of dyadic permutations of arbitrarily high order.

Theorem 2.1.5. In the weak topology the set of rigid automorphisms is an everywhere dense G_δ subset of $Auto(X, \mu)$.

In particular this means that the typical automorphism is rigid and weak mixing. So, even though these two behaviors of a dynamical system are quite different, they are not exclusive.

2.1.2 $Homeo(X, \mu)$

We now turn to the dynamics of volume preserving homeomorphisms of compact metric spaces, or flows that preserve a volume measure. Let $Homeo(X, \mu)$ denote the set of homeomorphisms from a compact metric space X to itself that preserve a probability measure μ. This set is equipped with the topology of uniform convergence of homeomorphisms and their inverses. If d is the metric on X and S, T are two homeomorphisms, define the uniform distance to be

$$d_u(S,T) = \sup_{x \in X} d(S(x), T(x)) + \sup_{x \in X} d(S^{-1}(x), T^{-1}(x)).$$

With this metric, $Homeo(X, \mu)$ is a complete metric space.

Sometimes we will need to discuss the distance between a homeomorphism and an automorphism, and view $Homeo(X, \mu)$ as a subset of $Auto(X, \mu)$. In this case we will use the following metric (which also induces the uniform topology on $Homeo(X, \mu)$)

$$\|S - T\| = ess\ sup_{x \in X} d(S(x), T(x)).$$

Recall that $ess\ sup\ f = \inf\{a : \mu(\{x : f(x) > a\}) = 0\}$.

In 1941 Oxtoby and Ulam [32] proved that ergodicity is generic for measure-preserving homeomorphisms of compact manifolds of dimension two or greater. This work proceeded Halmos' category results that were previously discussed and the techniques are very different. In the late 1970's Alpern (see [3]) unified the two themes, showing that when you translate the study from topological dynamics to ergodic theory, no new generic properties appear. We will formulate these theorems in a precise way later. For the moment, let us restrict our attention to the unit square and homeomorphisms that preserve the natural volume measure (or Lebesgue measure). There are many beautiful approximation theorems which appear in this setting.

Similar to before, we will be working with dyadic cubes, the two-dimensional analogue of dyadic intervals. We will be considering permutations of these dyadic cubes, which are simply translations of the cubes. Note that such a permutation is not an element of $Homeo([0,1]^2, \mu)$ where μ is the product Lebesgue measure, but does belong to $Auto([0,1]^2, \mu)$. We will prove a theorem of Lax (see [3]) that states that any homeomorphism can be approximated by such a permutation.

Theorem 2.1.6 (Lax's Theorem [3])**.** Let $T \in Homeo([0,1]^2, \mu)$ and $\epsilon > 0$. Then there exists a dyadic permutation σ such that $\|T - \sigma\| < \epsilon$.

This proof makes use of the marriage lemma in combinatorics, which says that if every group of girls of size k collectively know more than k boys, then there is a matching that allows every girl to be paired with a boy that she knows. To be more precise, the marriage lemma is stated below.

Lemma 2.1.7. Let E, F be two finite sets and $\approx$ a relation between the elements of E and F. If for every $E' \subset E$ we have

$$|E'| \leq |\{f \in F \ : \ \textit{there exists } e \textit{ in } E' \textit{ with } e \approx f\}|$$

then there exists a one-to-one map $\phi : E \to F$ such that for all $e \in E$, $e \approx \phi(e)$.

Proof of Lax's Theorem. Let $\mathcal{D}_m$ be a dyadic subdivision of $[0,1]^2$ of order m. Suppose $T \in Homeo([0,1]^2, \mu)$ and $\epsilon > 0$. We will be using the marriage lemma, so the first thing to do in this case is to define a relation. If c and c' are two dyadic cubes of $\mathcal{D}_m$, then define $c \approx c'$ if and only if $T(c) \cap c' \neq \emptyset$.

Since T is measure-preserving, the image of any collection of k dyadic cubes from $\mathcal{D}_m$ under T, intersects at least k cubes from $\mathcal{D}_m$. This means that we may apply the marriage lemma to our setup. Hence, there exists a permutation σ of $\mathcal{D}_m$ such that for every cube $c \in \mathcal{D}_m$ we have $c \approx \sigma(c)$. This implies that for any cube $c \in \mathcal{D}_m$, $T(c) \cap \sigma(c) \neq \emptyset$.

Therefore

$$\begin{aligned}
\|T - \sigma\| &\leq diam(c) + \sup_{c \in \mathcal{D}_m} diam(T(c)) \\
&\leq \frac{\sqrt{2}}{2^m} + \sup_{c \in \mathcal{D}_m} diam(T(c))
\end{aligned}$$

$$< \epsilon.$$

if we make m sufficently large. □

The above theorem can be strengthened so that the approximating dyadic permutations are cyclic without too much trouble. This was accomplished by Alpern in 1976 [3].

Lemma 2.1.8 ([3])**.** Given any permutation ρ of $\mathcal{I} = \{1, \ldots, N\}$ there is a cyclic dyadic permutation σ of $\mathcal{I}$ with $d(\rho(i), \sigma(i)) \leq 2$ for all $i \in \mathcal{I}$.

Theorem 2.1.9 ([3])**.** Let $T \in Homeo([0,1]^2, \mu)$ and $\epsilon > 0$. Then there exists a cyclic dyadic permutation σ such that $\|T - \sigma\| < \epsilon$.

For an immediate application of this approximation theorem, we will examine the set of transitive homeomorphisms in $Homeo([0,1]^2, \mu)$. First recall the definition of topological transitivity and a theorem regarding the extension of finite maps.

Definition 2.1.10. Let (X, T) be a flow. The homeomorphism T is *topologically transitive* if for any two open subsets U, V of X there exists n such that $T^n U \cap V \neq \emptyset$.

Theorem 2.1.11 ([3])**.** Let $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$ be two lists of N distinct interior points of $[0,1]^2$ with the property that $d(p_i, q_i) < \epsilon$ for each i. Then there is a homeomorphism $\phi \in Homeo([0,1]^2, \mu)$, with $\|\phi - Id\| < \epsilon$ and equal to the identity on the boundary of $[0,1]^2$, which for each i maps some neighborhood of p_i by simple translation onto a neighborhood of q_i, and p_i into q_i.

Theorem 2.1.12 ([3])**.** In the uniform topology the set of topologically transitive homeomorphisms is a dense G_δ subset of $Homeo([0,1]^2, \mu)$.

Remark 2.1.13. The above theorem is not true if the measure-preserving assumption is dropped.

Proof. To begin let c_i, $i = 1, 2, \ldots$ be an enumeration of all the open dyadic cubes of $[0,1]^2$ of all orders. Define

$$T_{i,j} = \left\{ T \in Homeo([0,1]^2, \mu) \ : \ \textit{there exists } k \textit{ such that } T^k(c_i) \cap c_j \neq \emptyset \right\}.$$

Then $\bigcap_{i,j>0} T_{i,j}$ is the set of topologically transitive homeomorphisms in $Homeo([0,1]^2, \mu)$. Since $T_{i,j}$ is an open condition, it suffices to show that this set is dense in $Homeo([0,1]^2, \mu)$.

Fix $i, j > 0$ and $\epsilon > 0$. Suppose that h is a homeomorphism in $Homeo([0,1]^2, \mu)$. We will show that there exists $h' \in T_{i,j}$ such that $\|h' - h\| < \epsilon$. Let σ be a cyclic dyadic permutation of sufficiently large order (larger than the orders of c_i and c_j) such that $\|h - \sigma\| < \epsilon$. The previous statement can be accomplished by using the stengthened version of Lax's theorem. Number the centers of the dyadic cubes $p_1, \ldots, p_N$ according to the action of σ so that $\sigma(p_m) = p_{m+1}$ where the indices are taken modulo N. Then for each m we have $d(h(p_m), p_{m+1}) = d(h(p_m), \sigma(p_m)) < \epsilon$.

Now consider the two lists of numbers $h(p_1), \ldots, h(p_{N-1}), h(p_N)$ and $p_2, \ldots, p_N, p_1$. By the extension of finite maps theorem, there exists a homeomorphism $\phi \in Homeo([0,1]^2, \mu)$ that is the identity on the boundary, such that $\|\phi - Id\| < \epsilon$ and $\phi(h(p_m)) = p_{m+1}$. Let $h' = \phi \circ h$. It is clear that h' is a homeomorphism that cyclically permutes the centers of the dyadic cubes, p_m, and hence $h' \in T_{i,j}$. Finally notice that $\|h' - h\| = \|\phi \circ h - h\| = \|\phi - Id\| < \epsilon$.

□

Techniques similar to the ones used in the above proof can be used to establish genericity of topological weak mixing.

Theorem 2.1.14 ([3]). In the uniform topology the set of topologically weakly mixing homeomorphisms is a dense G_δ subset of $Homeo([0,1]^2, \mu)$.

We now return to a result of Alpern which states that a property that is generic in $Auto(X, \mu)$ is also generic in $Homeo(X, \mu)$.

Definition 2.1.15. The measure μ on X is an OU (Oxtoby-Ulam) measure if it is a Borel probability measure that satisfies the following three conditions:

1. μ is nonatomic
2. μ is locally positive (nonempty open sets have positive measure)
3. $\mu(\partial X) = 0$.

Theorem 2.1.16 ([3])**.** Let μ be an OU measure on a compact connected manifold of dimension at least 2. Let $\mathcal{P}$ be a conjugate invariant, dense G_δ subset of $Auto(X, \mu)$ with respect to the weak topology. Then $\mathcal{P} \cap Homeo(X, \mu)$ is a dense G_δ subset of $Homeo(X, \mu)$ with respect to the uniform topology.

Previously we proved that weakly mixing automorphisms are generic in $Auto(X, \mu)$. Since weak mixing is a property that is invariant under conjugation by automorphisms, the above theorem of Alpern implies that the weak mixing property is also generic inside $Homeo(X, \mu)$. Katok and Stepin [26] had previously shown this result, however the above approach is much simpler.

2.2 More on Rigidity

2.2.1 Rigid vs. Uniformly Rigid

As we saw in Section 2.1.1, the typical automorphism is rigid and weakly mixing. What happens if we now consider the notion of uniform rigidity? In some cases weak mixing and uniform rigidity are compatible notions. In the next chapter we will explore this idea further. However, in some cases they are incompatible. For example in 2009 Silva et. al. [25] showed that weak mixing and uniform rigidity are incompatible notions when dealing with maps of a Cantor space.

Recall the definition of uniform rigidity.

Definition 2.2.1. Let (X, T) be a flow. The homeomorphism T is *uniformly rigid* if there exists an increasing sequence (n_m) such that T^{n_m} converges to the identity uniformly on the space X.

Definition 2.2.2. Suppose (X, β, μ, T) is a dynamical system. The automorphism T is *totally ergodic* if T^n is ergodic for all powers n.

Observe that if an automorphism is weakly mixing, then it is also totally ergodic.

Theorem 2.2.3 ([25])**.** Let C denote Cantor space, equipped with a Borel probability measure. There exists no totally ergodic, measure-preserving system that is uniformly rigid in any metric compatible with the topology, except in the case where all the measure is concentrated at one point.

Proof. Recall that a Cantor space is homeomorphic to the space of sequences of 0's and 1's, i.e. the space $2^{\mathbb{N}} = \{(x_n) : x_n \in \{0,1\} \textit{ and } n \in \mathbb{N}\}$. Thus C is equipped with the p-adic metric on $2^{\mathbb{N}}$ where the distance between two sequences (x_n) and (y_n) is $d((x_n),(y_n)) = \frac{1}{k}$ where k is the smallest index such that $x_k \neq y_k$. We will prove the result for this metric, but one can do the same for any metric compatible with the topology.

Suppose for a contradiction that T is uniformly rigid and totally ergodic. Our plan is to produce an invariant ball. Fix $r > 0$ and let (n_m) be the uniform rigidity sequence for T. If $x \in C$, let $B_r(x)$ denote the ball of radius r centered at x. Since the p-adic metric satisfies the ultrametric inequality, i.e. $d(x,y) \leq \max(d(x,z), d(z,y))$, there exists M such that $T^{n_M}y \in B_r(x)$ if and only if $y \in B_r(x)$. Thus each of the balls of radius r is an invariant ball and, as such, the measure must be concentrated in one ball since T is totally ergodic. Iterating this argument reveals that the measure must be concentrated at one point, which is a contradiction.

□

This type of argument carries over for any compact metric space whose metric satisfies the ultrametric inequality. A corollary of this theorem is that any weakly mixing dynamical system is isomorphic to a system that is not uniformly rigid. This fact follows from the Jewett-Krieger theorem which states that any ergodic transformation has a realization as a homeomorphism of a Cantor set [33]. An interesting question is

Question 2.2.4. For which spaces are weak mixing and uniform rigidity compatible?

In this section we would also like to display the fact that rigidity and uniform rigidity coincide in very special cases, but in general are not the same. A simple argument from [25] shows that rigidity and uniform rigidity are the same on the circle or unit interval. But first, we will examine an example.

Example 2.2.5. Let $X = \mathbb{S}^1$, μ be the measure of an arc (normalized so that $\mu(\mathbb{S}^1) = 1$), and $T : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $T(e^{2\pi i x}) = e^{2\pi i(x+\alpha)}$ where $\alpha \in (0,1)$ is irrational. Let $z = e^{2\pi i x}$ and $a = 2\pi\alpha$.

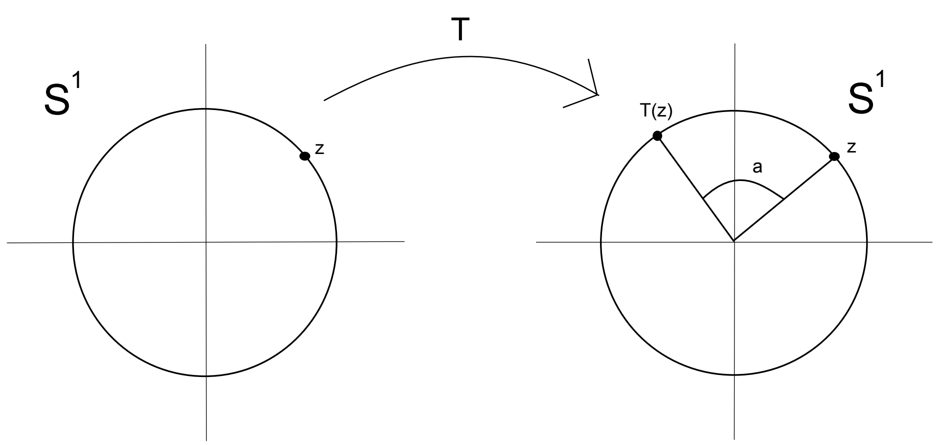

Figure 2.1: Irrational Rotation

The map T is called an irrational rotation. We will show that T is uniformly rigid and explore the uniform rigidity sequence. Consider the continued fraction expansion for α where the convergents are denoted by $\frac{p_n}{q_n}$. Recall that the convergents of a continued fraction provide an excellent rational approximation. Since $\frac{p_n}{q_n}$ are the convergents of α, the following inequality is satisfied

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$$

where $|\cdot|$ denotes absolute value or distance to the nearest integer (the distinction should be clear from context). Thus $|q_n\alpha| \to 0$ as $n \to \infty$ and we have

$$T^{q_n}(e^{2\pi ix}) = e^{2\pi i(x+q_n\alpha)} \to e^{2\pi ix}$$

as $n \to \infty$. The above calculation is independent from the choice of x and so T is uniformly rigid.

Lemma 2.2.6 ([25])**.** Let T be a continuous map of the circle or unit interval that preserves an OU measure. Then rigidity and uniform rigidity are identical notions.

Proof. Clearly uniform rigidity implies rigidity. So, let us concentrate on the opposite implication. Suppose T defined on the unit interval is rigid with rigidity sequence is (n_m).

For fixed $\delta > 0$ there exists $r > 0$ such that the ball of radius δ centered at a point x, $B_\delta(x)$, has measure greater than r for any x, i.e. $\mu(B_\delta(x)) > r$. To see this cut the interval into finitely many disjoint half-open balls of size less than or equal to $\frac{\delta}{3}$. Consider the measure of each of these balls, each of which is nonzero by the assumption that μ is an OU measure. Let r be the least measure of these balls. Now, if B is any ball of radius δ then B contains one such ball and hence has measure greater that r.

Let B be a ball of radius δ. Since T is continuous, $B \triangle T^{n_m}B$ has at most two connected components. As $m \to \infty$, the measure of each connected component goes to zero. Thus the diameter also goes to zero since we are in one dimension and are dealing with an OU measure. Hence $T^{n_m} \to Id$ uniformly as $m \to \infty$.

□

The above argument is unique to the one-dimensional case. In higher dimensions it is not always true that small measure implies small diameter. In fact, rigidity and uniform rigidity are not the same if you go up one dimension to the two-torus. To see this, we'll consider an example.

Example 2.2.7. In [6] Bergelson, del Junco, Lemańczyk, and Rosenblatt use the Gaussian Measure Space construction (for details regarding this construction see [12] page 188) to build an example of a weakly mixing transformation that is rigid along powers of 2. Since rigid transformations have zero measure-theoretic entropy, we may apply a result of Lind and Thouvenot [30] to obtain a hyperbolic toral automorphism of $\mathbb{T}^2$ that preserves an OU measure. Hyperbolic toral automorphisms of the two-torus are given by 2×2 integer matrices with determinant ± 1. Since weak mixing and rigidity are spectral properties they carry through measure-theoretic isomorphisms. Thus this hyperbolic total automorphism is weakly mixing and rigid along powers of 2. Hyperbolic total automorphisms have been very well studied and it is known that they are strongly mixing with respect to Lebesgue measure. That means that they cannot be uniformly rigid.

The above example illustrates that rigidity does not imply uniform rigidity. A question that remains open, is:

Question 2.2.8. In which situations do rigidity and uniform rigidity coincide?

2.2.2 Purely Topological Results

Now we will focus on different aspects of rigidity in topological dynamics only. Thus, there will be no underlying measure space assumptions and we will only be dealing with flows. We begin with two different types of topological rigidity, one of which we have already seen.

Definition 2.2.9. Let (X, T) be a flow. The homeomorphism T is *topologically rigid* if there exists an increasing sequence (n_m) such that $T^{n_m}x \to x$ as $m \to \infty$ for all x in X.

To put it a different way, topologically rigid maps converge to the identity point-wise along their rigidity sequence, while uniformly rigid maps converge uniformly to the identity along their rigidity sequence. It's clear from the definitions that uniform rigidity implies rigidity. To see that the opposite implication is not true, we will look at an example.

Example 2.2.10. Consider a family of rotating circle that are all rotating at different rates. To be more precise, let

$$X = \left\{ re^{2\pi i\theta} \ : \ 0 \leq \theta \leq 1, \ r = 1 - \frac{1}{2^n} \ for \ n = 1, 2, \ldots, \ and \ r = 1 \right\}$$

where $T(z) = ze^{2\pi i(\frac{1}{2^n})}$ when $|z| = 1 - \frac{1}{2^n}$ and $T(z) = z$ when z is on the unit circle. Then $n_m = 2^m$ is a topological rigidity sequence, but not a uniform rigidity sequence for this example.

Now we will examine uniform rigidity in more depth. Let (X, T) be a minimal flow that is topologically rigid with respect to (n_m). Define the following relation

$$\overline{N} = \{(x, y) \ : there\ exists\ a\ subsequence\ (n'_m)\ of\ (n_m)\ and\ sequences\ x_m \to x,$$
$$y_m \to y\ such\ that\ d(T^{n'_m} x_m, T^{n'_m} y_m) \to 0\}.$$

The flow (X, T) is uniformly rigid if and only if $\overline{N} = \triangle$ where $\triangle$ is the diagonal of $X \times X$.

Definition 2.2.11. The flow (X, T) is *distal* if for every pair of points $x \neq y$ we have $\inf_{n \in \mathbb{N}} d(T^n x, T^n y) > 0$.

Let (X, T) be a minimal flow and define the relation

$$P = \{(x, y) \ : \ \triangle \subset \overline{Orb(x, y)}\}$$

where $\overline{Orb(x, y)}$ is the closure of the orbit of (x, y) under the map T. In this case, the flow (X, T) is distal if and only if $P = \triangle$. To see this, notice that clearly $\triangle \subset P$. For the other direction, suppose $(x, y) \in P$. Then for any $z \in X$ we have $(z, z) \in \overline{Orb(x, y)}$. Thus there exists a sequence (n_m) such that $T^{n_m}x \to z$ and $T^{n_m}y \to z$ as $m \to \infty$. Since our flow is distal, $x = y$ and $P \subset \triangle$.

Theorem 2.2.12 ([17])**.** A minimal flow that is distal and topologically rigid is uniformly rigid.

Proof. Suppose (X, T) is a minimal flow that is distal and topologically rigid. Let (n_m) be the rigidity sequence. In this case $P = \triangle$. In order to show that T is uniformly rigid, we must show that $\overline{N} = \triangle$. Clearly $\triangle \subset \overline{N}$. To show that $\overline{N} \subset \triangle$ we will show that $\overline{N} \subset P$.

For this define a new relation

$$N = \{(x, y) \ : \textit{there exists a subsequence } (n'_m) \textit{ of } (n_m) \textit{ and } x_m \to x \\ \textit{with } T^{n'_m}x_m \to y\}.$$

Note that $N \subset \overline{N}$. Hence it suffices to show that $N \subset P$, since if so, $\overline{N} \subset N \circ N^{-1} \subset P \circ P^{-1} = \triangle \circ \triangle^{-1} = \triangle$ where $R \circ S = \{(x, y) \ : \ \textit{there exists } z \textit{ with } (x, z) \in R \textit{ and } (z, y) \in S\}$. Suppose for a contradiction that $(x, y) \in N \setminus P$. Let $\delta = \inf_{n \in \mathbb{N}} d(T^n x, T^n y) > 0$ since T is distal. There exists a subsequence (n'_m) of (n_m) and $x_m \to x$ such that $T^{n'_m}x_m \to y$ as $m \to \infty$ since $(x, y) \in N$. Let U be any open set and ℓ be such that $T^{\ell}x \in U$. Note that we can do this since our flow is minimal. Then

$$T^{n'_m}T^{\ell}x_m = T^{\ell}T^{n'_m}x_m \to T^{\ell}y$$

and

$$T^{\ell}x_m \to T^{\ell}x \in U.$$

Let M be large enough so that $T^{\ell}x_M \in U$ and $d(T^{n'_M}T^{\ell}x_M, T^{\ell}x_M) > \frac{\delta}{2}$.

Define

$$V_M = \{z \ : \ \textit{there exists } m > M \textit{ with } d(T^m z, z) > \frac{\delta}{2}\}.$$

Then by the above argument V_M is dense and also open. Let $V = \bigcap_{m=1}^{\infty} V_m$. Since V is a dense G_δ subset of X there are plenty of points in V. Let $z \in V$. By the topologically rigid assumption on (X, T), $T^{n_m} z \to z$ which contradicts the fact that $z \in V$.

□

The last result from Glasner and Moan [17] that we want to mention is regarding topologically rigid factors of topologically strongly mixing systems. First recall the definition of topological strong mixing.

Definition 2.2.13. The flow (X, T) is *topologically strongly mixing* if for any open subsets U, V of X there exists N such that for all $n \geq N$ we have $T^n U \cap V \neq \emptyset$.

Theorem 2.2.14 ([17])**.** Suppose (X, T) is a minimal, topologically strongly mixing flow. Then (X, T) admits only trivial topologically rigid factors.

Proof. Suppose the flow (X, T) is minimal and topologically strongly mixing. Since every factor has the same properties, it suffices to show that if (X, T) is also topologically rigid then X is trivial.

Suppose (X, T) is topologically rigid with respect to (n_m) and suppose $(x, y) \in X \times X$ where $x \neq y$. Let $z \in X$. Let U be an open neighborhood of z and V_1, V_2 be open neighborhoods of x, y respectively. There exists M_1 such that for every $m \geq M_1$, $T^{n_m} V_1 \cap U \neq \emptyset$ and there exists M_2 such that for every $m \geq M_2$, $T^{n_m} V_2 \cap U \neq \emptyset$. Let M be the larger of M_1 and M_2. Then the n_M-th iterate of both x and y gets close to z. Continue shrinking the neighborhoods of x, y, z in order to produce a subsequence (n'_m) such that $T^{n'_m} x \to z$ and $T^{n'_m} y \to z$ as $m \to \infty$. However, since (n_m) is a topological rigidity sequence for (X, T), $x = z = y$. This is a contradiction and therefore X is trivial.

□

The above theorem shows that in the purely topological framework, strong mixing and rigidity are not compatible notions. Similar results hold for strong mixing and rigidity in the measurable framework. That is, a transformation cannot be strongly mixing and rigid with respect to the same measure (this is clear from the definitions).

2.3 Spectral Properties

In this section we will examine rigidity and weak mixing as spectral properties. First recall the definitions of rigid and weak mixing for a dynamical system (X, β, μ, T).

Definition 2.3.1. An automorphism T is *rigid* if there exists a sequence (n_m) such that the powers T^{n_m} converge to the identity in the strong operator topology. That is,

$$\|f \circ T^{n_m} - f\| \to 0$$

as $m \to \infty$ for all $f \in L_2$.

In this case, (n_m) is called a rigidity sequence for T.

Definition 2.3.2. An automorphism T is *weakly mixing* if for all $f, g \in L_2$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n f | g \rangle - \langle f | 1 \rangle \langle 1 | g \rangle| \to 0$$

as $N \to \infty$.

To see that these two properties are spectral properties, we begin with some basics of spectral theory. For a fixed transformation T and $f \in L_2$, let $b(n) = \langle f \circ T^n | f \rangle$. This function is a positive definite function and therefore can be represented by the Fourier transform of a positive Borel measure on the circle. We will call this unique measure associated to f, the *spectral measure* and denote it by ν_f^T. So, ν_f^T is determined by $\widehat{\nu_f^T}(n) = \langle f \circ T^n | f \rangle$ for all integers n and ν_f^T is a positive measure that satisfies $\nu_f^T(\mathbb{T}) = \|f\|_2^2$.

From the definition of weak mixing, we see that T is weakly mixing if for all functions $f \in L_2$ that are mean zero, $\frac{1}{N} \sum_{n=0}^{N-1} |\langle f \circ T^n | f \rangle| \to 0$ as $N \to \infty$. Wiener's Lemma says that if ν is a positive Borel measure on the circle then

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\widehat{\nu}(n)|^2 = \sum_{x\in\mathbb{T}} \nu^2(\{x\}).$$

Thus the measure ν is continuous if and only if $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\widehat{\nu}(n)|^2 = 0$. Putting these facts together tells us that T is weakly mixing if and only if the spectral measure ν_f^T is continuous for each mean zero $f \in L_2$.

Now we turn to rigidity. Observe that $f \circ T^{n_m} \to f$ as $m \to \infty$ in L_2 if and only if $\langle f \circ T^{n_m}, f\rangle \to \|f\|_2^2$ as $m \to \infty$. By the definition of spectral measure, this means that $\widehat{\nu_f^T}(n_m) \to \|f\|_2^2$. Thus T is rigid if there exists a sequence (n_m) such that $\widehat{\nu_f^T}(n_m) \to 1$ as $m \to \infty$ for all $f \in L_2$ of norm one.

From the above discussion, we see that both rigidity and weak mixing are spectral properties of the transformation. A theorem from "Rigidity and Nonrecurrence Along Sequences" by Bergelson, del Junco, Lemańczyk, and Rosenblatt [6] brings together the ideas above.

Theorem 2.3.3 ([6])**.** The sequence (n_m) is a rigidity sequence for some weakly mixing transformation if and only if there is a continuous Borel probability measure ν on the circle such that $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$.

2.4 Measurable Sensitivity

Measurable sensitivity was one of the main ideas that motivated the following question:

Question 2.4.1. Does there exist a homeomorphism of a compact metric space that is weakly mixing and uniformly rigid.

Measurable sensitivity, which appears in [24], is the measure-theoretic version of sensitive dependence on initial conditions in chaos theory (for more on this see [22]). Sensitive dependence on initial conditions is a topological notion and depends on the metric that you are using, while measurable sensitivity is a measurable notion. Let us examine some definitions to compare the two theories.

Definition 2.4.2. A flow (X, T) has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in X$ and $\epsilon > 0$, there exists $n \in \mathbb{N}$ and $y \in B_\epsilon(x)$ such that $d(T^n x, T^n y) > \delta$.

Definition 2.4.3. Let (X, β, μ, T) be a dynamical system. The metric d on X is *μ-compatible* if μ is positive on all nonempty, open d-balls.

Furthermore, given two nonempty sets A, B in X, we define

$$d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

Definition 2.4.4. Let (X, β, μ, T) be a dynamical system. The automorphism T exhibits *measurable sensitivity* if whenever a dynamical system $(X_1, \beta_1, \mu_1, T_1)$ is measure-theoretically isomorphic to (X, β, μ, T) and d is a μ_1-compatible metric on X_1, then there exists $\delta > 0$ such that for all $x \in X_1$ and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\mu_1\left(\{y \in B_\epsilon(x) \ : \ d(T^n x, T^n y) > \delta\}\right) > 0.$$

Definition 2.4.5. Let (X, β, μ, T) be a dynamical system. The automorphism T exhibits *strong measurable sensitivity* if whenever a dynamical system $(X_1, \beta_1, \mu_1, T_1)$ is measure-theoretically isomorphic to (X, β, μ, T) and d is a μ_1-compatible metric on X_1, then there exists $\delta > 0$ such that for all $x \in X_1$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all integers $n \geq N$

$$\mu_1\left(\{y \in B_\epsilon(x) \ : \ d(T^n x, T^n y) > \delta\}\right) > 0.$$

We will show that a homeomorphism of a compact metric space that is uniformly rigid and weakly mixing is an example of a system that is measurably sensitive, but not strongly measurably sensitive.

Proposition 2.4.6 ([24])**.** If (X, β, μ, T) is a weakly mixing dynamical system, then T exhibits measurable sensitivity.

Proof. Suppose that (X, β, μ, T) is a weakly mixing dynamical system. Recall that this property is equivalent to $X \times X$ being ergodic. Now, let $(X_1, \beta_1, \mu_1, T_1)$ be a dynamical system that is measure-theoretically isomorphic to (X, β, μ, T). Note that this means that $(X_1, \beta_1, \mu_1, T_1)$ is also weakly mixing and therefore $X_1 \times X_1$ is ergodic.

Let d be a μ_1-compatible metric on X_1. Let A, C be subsets of X_1 with positive μ_1 measure such that $d(A, C) > 0$. Take $\delta = \frac{d(A,C)}{2}$. Let $x \in X_1$ and $\epsilon > 0$. Since our system is weakly mixing, there exists $n \in \mathbb{N}$ such that $\mu_1(T_1^{-n} A \cap B_\epsilon(x)) > 0$ and $\mu_1(T_1^{-n} C \cap B_\epsilon(x)) > 0$. This means that

$$\mu_1 \left(\{ y \in B_\epsilon(x) \ : \ T^n y \in A \} \right) > 0$$

and

$$\mu_1 \left(\{ y \in B_\epsilon(x) \ : \ T^n y \in C \} \right) > 0.$$

By our choice of δ,

$$\mu_1 \left(\{ y \in B_\epsilon(x) \ : \ d(T_1^n x, T_1^n y) > \delta \} \right) > 0.$$

□

Proposition 2.4.7. If (X, β, μ, T) is a uniformly rigid dynamical system with respect to a μ-compatible metric d, then T does not exhibit strong measurable sensitivity.

Proof. Suppose that (X, β, μ, T) is a uniformly rigid dynamical system with respect to a μ-compatible metric d and (n_m) is the uniform rigidity sequence. Suppose for a contradiction that this dynamical system is also strongly measurably sensitive. Let $\delta > 0$ be such that if $x \in X$ and $\epsilon > 0$ then there exists N where for all integers $n \geq N$,

$$\mu \left(\{ y \in B_\epsilon(x) \ : \ d(T^n x, T^n y) > \delta \} \right) > 0.$$

Let $x \in X$ and fix $\epsilon < \frac{\delta}{3}$. Take M large enough so that $n_M > N$, $d(T^{n_M}x, x) < \frac{\delta}{3}$, and $d(T^{n_M}y, y) < \frac{\delta}{3}$ for any $y \in B_\epsilon(x)$. Then for all such y,

$$\begin{aligned} d(T^{n_M}x, T^{n_M}y) &\leq d(T^{n_M}x, x) + d(x, y) + d(y, T^{n_M}y) \\ &< \frac{\delta}{3} + \epsilon + \frac{\delta}{3} \\ &< \delta \end{aligned}$$

which is a contradiction.

□

Chapter 3

Category Results

In the previous chapter we saw that, in the realm of dynamical systems, the typical automorphism is rigid and weak mixing. In this chapter we will explore similar properties of typical flows on the two-torus and the Klein bottle. The rigidity that we will consider in this chapter is uniform rigidity. Since the two-torus and Klein bottle are both compact manifolds, the first natural tool that comes to mind to prove a generic statement in the realm of homeomorphisms is Alpern's theorem. However this naive approach does not work since uniform rigidity is not invariant under conjugation by automorphism. Thus different tools will be needed.

3.1 The Two-Torus

In this section, we will restrict our attention to the two-torus and consider both measurable and topological questions. This is possible by viewing homeomorphisms of the two-torus as dynamical systems and flows simultaneously. The contents of this section are taken from [35].

As we have discussed, it is well known that if you consider the group of all measure-preserving automorphisms of a Lebesgue probability space equipped with the weak topology, then the set of weakly mixing transformations and the set of rigid transformations each form a dense G_δ subset (see [23]). It is possible to prove a similar result for homeomorphisms of certain compact spaces.

Silva et. al. posed the following question in [25]:

Question 3.1.1. Does there exist a measure-preserving homeomorphism which is both weakly mixing and uniformly rigid?

Silva et. al. proved that weak mixing and uniform rigidity are not compatible properties on a Cantor space. But, what happens for other compact metric spaces?

In [19], Glasner and Weiss showed that there is a large family of weakly mixing homeomorphisms on the infinite torus that are strictly ergodic. This result, coupled with an earlier result regarding uniform rigidity in [17], gives a positive answer to the above posed question. We will use their method of proof to show that you can do the same for the two torus. In fact, let $\mathcal{O}$ be the closure of the set of conjugations of an aperiodic rotation by Lebesgue measure-preserving homeomorphisms of $\mathbb{T}^2$. We will prove the following theorem:

Theorem 3.1.2. There exists a dense G_δ subset $\mathcal{R}$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}$, $(\mathbb{T}^2, T, \mu)$ is weakly mixing, uniformly rigid, and strictly ergodic.

We will be considering measure-preserving homeomorphisms defined from X to X. If S, T are two measure-preserving homeomorphisms from X to X, then in the sup metric the distance from S to T is defined by $\sup_{x \in X} d(S(x), T(x))$. When endowed with this metric, which induces the topology of uniform convergence, the group becomes a topological group that is not complete. To see this, notice that you can construct a sequence of measure-preserving homeomorphisms which converge uniformly to a continuous function with no inverse. Some of the issues that we will be considering are generic issues and therefore we need our space to be complete. To that end, we define a new metric where the uniform distance is given by

$$d_u(S, T) = \sup_{x \in X} d(S(x), T(x)) + \sup_{x \in X} d(S^{-1}(x), T^{-1}(x)).$$

The topology induced by d_u is still the topology of uniform convergence and with this metric the group of measure-preserving homeomorphisms on X is a complete metric space. We will also call this the topology of uniform convergence of homeomorphisms and their inverses. To simplify notation, if S, T are two homeomorphisms defined on X, let

$$\overline{d}(S,T) = \sup_{x \in X} d(S(x), T(x)).$$

With this new notation,

$$d_u(S,T) = \overline{d}(S,T) + \overline{d}(S^{-1}, T^{-1}).$$

Notice that even though d_u is not right-invariant, $\overline{d}$ is right-invariant.

Before we proceed with our category argument, we require a few more definitions and some notation.

Definition 3.1.3. A homeomorphism T is called *uniquely ergodic* if there is only one T-invariant probability measure μ on (X, T).

Definition 3.1.4. A homeomorphism T is *strictly ergodic* if it is uniquely ergodic and the unique T-invariant probability measure on X has full topological support (ie. $supp(\mu) = X$).

Remark 3.1.5. A homeomorphism T is strictly ergodic if and only if it is uniquely ergodic and minimal.

Let $C(X)$ be the continuous functions defined on X and let $C^0(X)$ be those functions that are mean zero and L_2-norm one. That is, if $f \in C^0(X)$ then $\int_X f(x) d\mu = 0$ and $\left(\int_X |f(x)|^2 \, d\mu\right)^{\frac{1}{2}} = 1$.

We will show, using a category argument, that there is a large family of homeomorphisms on the two torus $\mathbb{T}^2$ where each homeomorphism is weakly mixing, uniformly rigid, and strictly ergodic (thus minimal).

Throughout this section we will be working on the two torus $\mathbb{T}^2$. We will be using the model of $\mathbb{T}^2$ where it is viewed as $[0,1)^2$ and the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context). Note that $\mathbb{T}^2$ is a compact monothetic group. Thus we may choose $\alpha = (\alpha_1, \alpha_2)$ such that the set $\{n\alpha \ : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$. Let $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be a rotation homeomorphism defined by $\sigma(x_1, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$. Suppose that $\mathbb{T}$ is equipped with the usual Lebesgue measure and μ is the corresponding product measure on $\mathbb{T}^2$. Note that σ preserves μ.

Let $\mathcal{H}(\mathbb{T}^2)$ be the set of measure-preserving homeomorphisms of $\mathbb{T}^2$. Define the set $O(\sigma)$ as follows:

$$O(\sigma) = \{G \circ \sigma \circ G^{-1} : G \in \mathcal{H}(\mathbb{T}^2)\}.$$

Throughout this section we will be considering $O(\sigma)$ as a subset of all homeomorphisms of $\mathbb{T}^2$ with the topology of uniform convergence of homeomorphisms and their inverses.

From now on we will be considering generic properties inside $\mathcal{O} = \overline{O(\sigma)}$.

3.1.1 Weak Mixing

The goal of this subsection is to prove the following theorem:

Theorem 3.1.6. There exists a dense G_δ subset $\mathcal{R}_1$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_1$, $(\mathbb{T}^2, T, \mu)$ is weakly mixing.

We will need several lemmas to prove this theorem. The first lemma below will help us write the set of weakly mixing homeomorphisms in $\mathcal{O}$ as a G_δ set.

Lemma 3.1.7. Let T be a measure-preserving homeomorphism of X. Then T is weakly mixing if and only if there exists a dense subset $\{\phi_i\}$ of $C^0(X)$ such that for all i there exists n with

$$|\langle U_T^n \phi_i | \phi_i \rangle| < 0.99.$$

Proof. First notice that T is weakly mixing if and only if for all $\phi \in C^0(X)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n \phi | \phi \rangle - \langle \phi | 1 \rangle \langle 1 | \phi \rangle| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n \phi | \phi \rangle| = 0.$$

Now suppose that T is weakly mixing. Let $\{\phi_i\}$ be a dense subset of $C^0(X)$. By the above statement, for every i we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n \phi_i | \phi_i \rangle| = 0$$

and therefore for every i there exists n such that $|\langle U_T^n \phi_i | \phi_i \rangle| < 0.99$.

Now suppose that there exists a dense subset $\{\phi_i\}$ of $C^0(X)$ such that for all i there exists n with $|\langle U_T^n \phi_i | \phi_i \rangle| < 0.99$. To show that T is weakly mixing, it suffices to show that there are no nonconstant eigenfunctions. Observe that all constant functions are trivially eigenfunctions of T.

Suppose for a contradiction that $\phi \in C^0(X)$ is a nonconstant eigenfunction. Since $\{\phi_i\}$ is dense in $C^0(X)$ there exists $\phi_i \in C^0(X)$ such that $\|\phi - \phi_i\|_2 < 0.001$. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$ and $U_T \phi = \lambda \phi$. Thus for all n

$$|\langle U_T^n \phi | \phi \rangle| = |\lambda|^n \, |\langle \phi | \phi \rangle| = \|\phi\|_2^2 = 1.$$

For some n we have

$$\begin{aligned} 1 &= |\langle U_T^n \phi | \phi \rangle| \\ &= |\langle U_T^n (\phi - \phi_i) + U_T^n \phi_i | (\phi - \phi_i) + \phi_i \rangle| \\ &\leq |\langle U_T^n \phi_i | \phi_i \rangle| + 3 \, \|\phi - \phi_i\|_2 \\ &< 0.993. \end{aligned}$$

This is a contradiction. Therefore T has no nonconstant eigenfunctions and is weakly mixing.

□

Let $\phi \in C^0(\mathbb{T}^2)$ and $0 < \eta < 1$. Define

$$R_\phi(\eta) = \{T \in \mathcal{O} : \textit{there exists } n \textit{ with } |\langle U_T^n \phi | \phi \rangle| < \eta\}.$$

Lemma 3.1.8. Let $\phi(x_1, x_2) = \sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ be an element of $L_2(\mathbb{T}^2, \mu)$ with $\|\phi\|_2 = 1$ (ie. $\sum |c_{\ell_1,\ell_2}|^2 = 1$). Suppose that there exists an η, $0 < \eta < 1$, such that for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have $|c_{\ell_1,\ell_2}| < \eta$. Then $\sigma \in R_\phi(\eta)$.

Proof. Let $\phi(x_1, x_2) = \sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ be an element of $L_2(\mathbb{T}^2, \mu)$ with $\|\phi\|_2 = 1$. Suppose that there exists an η, $0 < \eta < 1$, such that for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have $|c_{\ell_1,\ell_2}| < \eta$. To show that $\sigma \in R_\phi(\eta)$ we must show that there exists n such that $|\langle U_\sigma^n \phi | \phi \rangle| < \eta$.

Let $t \in \mathbb{Z}$. Then

$$\begin{aligned}
\left|\langle U_\sigma^t \phi | \phi \rangle\right| &= \left| \int_{[0,1)^2} \phi(\sigma^t(x)) \overline{\phi(x)} d\mu \right| \\
&= \left| \int_{[0,1)^2} \phi(x_1 + t\alpha_1, x_2 + t\alpha_2) \overline{\phi(x_1, x_2)} dx_1 dx_2 \right| \\
&= \left| \int_{[0,1)^2} \left(\sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_1 t\alpha_1 + \ell_2 x_2 + \ell_2 t\alpha_2)} \right) \left(\sum \overline{c}_{\ell_1',\ell_2'} e^{-2\pi i(\ell_1' x_1 + \ell_2' x_2)} \right) dx_1 dx_2 \right| \\
&= \left| \sum c_{\ell_1,\ell_2} \overline{c}_{\ell_1',\ell_2'} \int_{[0,1)^2} e^{2\pi i(\ell_1 x_1 + \ell_1 t\alpha_1 + \ell_2 x_2 + \ell_2 t\alpha_2)} e^{2\pi i(-\ell_1' x_1 - \ell_2' x_2)} dx_1 dx_2 \right| \\
&= \left| \sum c_{\ell_1,\ell_2} \overline{c}_{\ell_1',\ell_2'} e^{2\pi i(\ell_1 t\alpha_1 + \ell_2 t\alpha_2)} \int_{[0,1)^2} e^{2\pi i(\ell_1 - \ell_1')x_1} e^{2\pi i(\ell_2 - \ell_2')x_2} dx_1 dx_2 \right| \\
&= \left| \sum |c_{\ell_1,\ell_2}|^2 e^{2\pi i(\ell_1 t\alpha_1 + \ell_2 t\alpha_2)} \right|
\end{aligned}$$

Let ν be the probability measure on $\mathbb{T}$ defined by

$$\nu = \sum |c_{\ell_1,\ell_2}|^2 \delta_{\ell_1\alpha_1 + \ell_2\alpha_2}.$$

Observe

$$\left|\langle U_\sigma^t \phi | \phi \rangle\right| = \left| \int_0^1 e^{2\pi i t x} d\nu \right| = \left| \hat{\nu}(t) \right|.$$

By Wiener's theorem

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} |\hat{\nu}(t)|^2 &= \sum_{x\in[0,1)} \nu^2(\{x\}) \\
&= \sum |c_{\ell_1,\ell_2}|^4 < \eta^2 \sum |c_{\ell_1,\ell_2}|^2 \\
&= \eta^2.
\end{aligned}$$

Therefore there exists n with $|\hat{\nu}(n)|^2 < \eta^2$ and hence there exists n with $|\langle U_\sigma^n \phi|\phi\rangle| < \eta$.

□

Lemma 3.1.9. Let $G \in \mathcal{H}(\mathbb{T}^2)$ and $\phi \in C^0(\mathbb{T}^2)$. Then

$$G^{-1} R_\phi(\eta) G = R_{\phi\circ G}(\eta).$$

Proof. Suppose $T \in R_{\phi\circ G}(\eta)$. Let n be such that $|\langle U_T^n(\phi\circ G)|\phi\circ G\rangle| < \eta$. Notice that

$$\begin{aligned}
\langle U_T^n(\phi\circ G)|\phi\circ G\rangle &= \int_{[0,1)^2} (\phi\circ G)(T^n(x))\overline{\phi\circ G(x)} d\mu \\
&= \int_{[0,1)^2} \phi(G\circ T^n \circ G^{-1}(x))\overline{\phi(x)} d\mu \\
&= \langle U^n_{G\circ T\circ G^{-1}} \phi|\phi\rangle.
\end{aligned}$$

Thus $G\circ T\circ G^{-1} \in R_\phi(\eta)$ and therefore $R_{\phi\circ G}(\eta) \subseteq G^{-1}R_\phi(\eta)G$. The other containment is proved in the same way. □

Lemma 3.1.10. Let g be a continuous real-valued function on $\mathbb{T}$ which is twice differentiable and assume that g'' has only finitely many zeros. Then there exists $K \in \mathbb{N}$ such that for all

M we have

$$\left| \int_0^1 e^{2\pi i [Kg(x)+Mx]} dx \right| < 0.3.$$

Remark 3.1.11. In the above lemma, K does not depend on M.

Proof of Lemma 3.1.10. This follows from Van der Corput's Lemma for the second derivative (see [27] page 220). A similar lemma is proved in Chapter 4.

□

We are now prepared to prove Theorem 3.1.6.

Proof of Theorem 3.1.6. Let $\{\phi_i\}$ be a dense subset of $C^0(\mathbb{T}^2)$. We will show that

$$\mathcal{R}_1 = \bigcap_{i=1}^{\infty} R_{\phi_i}(0.99)$$

is our desired dense G_δ set. From Lemma 3.1.7 we see that $\mathcal{R}_1$ is the set of weakly mixing homeomorphisms of $\mathbb{T}^2$ inside $\mathcal{O}$. Let $\phi \in C^0(\mathbb{T}^2)$. It suffices to show that $R_\phi(0.99)$ is open and dense in $\mathcal{O}$.

To see that $R_\phi(0.99)$ is open in $\mathcal{O}$, notice that the set

$$\{T \in \mathcal{H}(\mathbb{T}^2) : \textit{there exists } n \textit{ with } \ |\langle U_T^n \phi | \phi \rangle| < 0.99\}$$

is open in $\mathcal{H}(\mathbb{T}^2)$. Therefore this set restricted to $\mathcal{O}$, which is $R_\phi(0.99)$, is open in $\mathcal{O}$.

The bulk of this proof is showing that $R_\phi(0.99)$ is dense in $\mathcal{O}$. Since $R_\phi(0.99) \subseteq \mathcal{O}$ and $\mathcal{O}$ is closed, it suffices to show that if $G_0 \in \mathcal{H}(\mathbb{T}^2)$ then $G_0 \circ \sigma \circ G_0^{-1} \in \overline{R_\phi(0.99)}$.

Suppose that (G_m) is a sequence in $\mathcal{H}(\mathbb{T}^2)$ such that $d_u(G_m \circ \sigma \circ G_m^{-1}, \sigma) \to 0$ as $m \to \infty$ and for all m, $G_m \circ \sigma \circ G_m^{-1} \in R_{\phi \circ G_0}(0.99)$. Since G_0, G_0^{-1} are continuous, $d_u(G_0 \circ G_m \circ \sigma \circ G_m^{-1} \circ G_0^{-1}, G_0 \circ \sigma \circ G_0^{-1}) \to 0$ as $m \to \infty$. Thus we can write

$$G_0 \circ \sigma \circ G_0^{-1} = \lim_{m \to \infty} (G_0 \circ G_m \circ \sigma \circ G_m^{-1} \circ G_0^{-1}) \in \overline{G_0 R_{\phi \circ G_0}(0.99) G_0^{-1}} = \overline{R_\phi(0.99)}.$$

Therefore we have reduced the rest of the proof of Theorem 3.1.6 to the following lemma:

Lemma 3.1.12. Let $\epsilon > 0$ and $\phi \in C^0(\mathbb{T}^2)$. Then there exists $G \in \mathcal{H}(\mathbb{T}^2)$ such that the following two properties hold:

1. $d_u(G \circ \sigma \circ G^{-1}, \sigma) < \epsilon$

2. $G \circ \sigma \circ G^{-1} \in R_\phi(0.99)$ (ie. $\sigma \in R_{\phi\circ G}(0.99)$).

Remark 3.1.13. The ϕ that appears in the above lemma is $\phi \circ G_0$ from the proof of Theorem 3.1.6.

Proof of Lemma 3.1.12. Let $\sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ be the Fourier expansion of $\phi \in C^0(\mathbb{T}^2)$. Recall that $\sum |c_{\ell_1,\ell_2}|^2 = 1$.

If for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have $|c_{\ell_1,\ell_2}| < 0.99$ then by Lemma 3.1.8, $\sigma \in R_\phi(0.99)$. In this case, if we take G to be the identity we have the lemma.

Otherwise, there is exactly one index, call it $c_0 = c_{N_1,N_2}$, such that $|c_0| \geq 0.99$. Let $\phi_1 = c_0 e^{2\pi i(N_1 x_1 + N_2 x_2)}$ and $\phi_2 = \phi - \phi_1$. Then $1 = \|\phi\|_2^2 = \|\phi_1\|_2^2 + \|\phi_2\|_2^2$. Since $\|\phi_1\|_2^2 = |c_0|^2 \geq 0.9801$ we have $\|\phi_2\|_2^2 < 0.02$ and therefore $\|\phi_2\|_2 < 0.2$. For all n and $G \in \mathcal{H}(\mathbb{T}^2)$ we have the following:

$$\begin{aligned}
|\langle U_\sigma^n(\phi \circ G)|\phi \circ G\rangle| &= |\langle U_\sigma^n((\phi_1 + \phi_2) \circ G)|(\phi_1 + \phi_2) \circ G\rangle| \\
&= |\langle U_\sigma^n(\phi_1 \circ G) + U_\sigma^n(\phi_2 \circ G)|(\phi_1 \circ G) + (\phi_2 \circ G)\rangle| \\
&\leq |\langle U_\sigma^n(\phi_1 \circ G)|\phi_1 \circ G\rangle| + |\langle U_\sigma^n(\phi_1 \circ G)|\phi_2 \circ G\rangle| \\
&+ |\langle U_\sigma^n(\phi_2 \circ G)|\phi_1 \circ G\rangle| + |\langle U_\sigma^n(\phi_2 \circ G)|\phi_2 \circ G\rangle| \\
&\leq |\langle U_\sigma^n(\phi_1 \circ G)|\phi_1 \circ G\rangle| + \|U_\sigma^n(\phi_1 \circ G)\|_2 \|\phi_2 \circ G\|_2 \\
&+ \|U_\sigma^n(\phi_2 \circ G)\|_2 \|\phi_1 \circ G\|_2 + \|U_\sigma^n(\phi_2 \circ G)\|_2 \|\phi_2 \circ G\|_2 \\
&< |\langle U_\sigma^n(\phi_1 \circ G)|\phi_1 \circ G\rangle| + 3\|\phi_2\|_2
\end{aligned}$$

$$< |\langle U_\sigma^n(\phi_1 \circ G)|\phi_1 \circ G\rangle| + 0.6.$$

From the above, we see that it suffices to find n and $G \in \mathcal{H}(\mathbb{T}^2)$ such that

$$|\langle U_\sigma^n(\phi_1 \circ G)|\phi_1 \circ G\rangle| < 0.3.$$

Let $\phi_0 = \frac{\phi_1}{c_0}$. Since $|c_0| \leq 1$ it suffices to find n and $G \in \mathcal{H}(\mathbb{T}^2)$ such that $|\langle U_\sigma^n(\phi_0 \circ G)|\phi_0 \circ G\rangle|$ < 0.3 (ie. $\sigma \in R_{\phi_0 \circ G}(0.3)$).

Let $\psi = \phi_0 \circ G$. Our goal is to show that there exists $G \in \mathcal{H}(\mathbb{T}^2)$ such that $\sigma \in R_\psi(0.3)$. By Lemma 3.1.8 it suffices to show that for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$, the absolute value of the coefficients in the Fourier expansion of ψ are less than 0.3. Hence, we need to show that for every $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have

$$\left|\langle \psi | e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}\rangle\right| = \left|\langle \phi_0 \circ G | e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}\rangle\right| < 0.3.$$

Recall that $\phi_0 = e^{2\pi i(N_1 x_1 + N_2 x_2)}$ where N_1 and N_2 cannot both be equal to zero since $\phi \in C^0(\mathbb{T}^2)$.

Suppose that $N_1 \neq 0$. In this case let $G \in \mathcal{H}(\mathbb{T}^2)$ have the form

$$G(x_1, x_2) = (x_1 + f(x_2), x_2)$$

where f is a continuous function. The inverse of G is easily defined as well and is equal to $G^{-1}(x_1, x_2) = (x_1 - f(x_2), x_2)$. For all $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have

$$\begin{aligned}
\left|\langle \phi_0 \circ G | e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}\rangle\right| &= \left|\langle e^{2\pi i(N_1 x_1 + N_1 f(x_2) + N_2 x_2)} | e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}\rangle\right| \\
&= \left|\int_{[0,1)^2} e^{2\pi i[N_1 x_1 + N_1 f(x_2) + N_2 x_2]} e^{-2\pi i[\ell_1 x_1 + \ell_2 x_2]} dx_1 dx_2\right| \\
&= \left|\int_{[0,1)^2} e^{2\pi i(N_1 - \ell_1) x_1} e^{2\pi i[N_1 f(x_2) + (N_2 - \ell_2) x_2]} dx_1 dx_2\right|
\end{aligned}$$

$$= \delta_0 \left| \int_0^1 e^{2\pi i[N_1 f(x_2)+(N_2-\ell_2)x_2]} dx_2 \right|$$

where δ_0 is 0 when $N_1 \neq \ell_1$ and 1 otherwise.

Let $g : \mathbb{T} \to \mathbb{R}$ be continuous, twice differentiable, and such that g'' has finitely many zeros. The same properties hold for $N_1 g$ and thus by Lemma 3.1.10 there exists $K \in \mathbb{N}$ such that for all M we have

$$\left| \int_0^1 e^{2\pi i[KN_1 g(x)+Mx]} dx \right| < 0.3.$$

Hence for any $\ell_2 \in \mathbb{Z}$

$$\delta_0 \left| \int_0^1 e^{2\pi i[KN_1 g(x_2)+(N_2-\ell_2)x_2]} dx_2 \right| < 0.3.$$

Any G of the above form where $f(x_2) = Kg(x_2)$ will satisfy (2). However, we need to be a little more careful since we also need G to satisfy (1). To that end, let $\delta > 0$ be such that if $|x - x'| < \delta$ then $|Kg(x) - Kg(x')| < \frac{\epsilon}{2}$. Let $q \in \mathbb{Z} \setminus \{0\}$ be such that $\left|\alpha_2 - \frac{p}{q}\right| < \frac{1}{q^2}$ and $\frac{1}{q} < \delta$ where $p \in \mathbb{Z}$.

Now we will define the formula for G explicitly. Let $f(x_2) = Kg(qx_2)$ on $\mathbb{T}$. Then $G(x_1, x_2) = (x_1 + Kg(qx_2), x_2)$. In order to estimate the uniform distance between $G \circ \sigma \circ G^{-1}$ and σ observe the following:

$$G \circ \sigma \circ G^{-1}(x_1, x_2) = (x_1 + \alpha_1 + f(x_2 + \alpha_2) - f(x_2), x_2 + \alpha_2)$$

and

$$G \circ \sigma^{-1} \circ G^{-1}(x_1, x_2) = (x_1 - \alpha_1 + f(x_2 - \alpha_2) - f(x_2), x_2 - \alpha_2).$$

For our estimate we will need the fact that $|q\alpha_2| < \delta$. Now

$$|f(x_2 + \alpha_2) - f(x_2)| = |Kg(qx_2 + q\alpha_2) - Kg(qx_2)| < \frac{\epsilon}{2}$$

and

$$|f(x_2-\alpha_2)-f(x_2)| = |Kg(qx_2-q\alpha_2)-Kg(qx_2)| < \frac{\epsilon}{2}.$$

Therefore $d_u(G\circ\sigma\circ G^{-1},\sigma) = \overline{d}(G\circ\sigma\circ G^{-1},\sigma)+\overline{d}(G\circ\sigma^{-1}\circ G^{-1},\sigma^{-1}) < \epsilon$ and (1) is proved.

To verify (2), notice that

$$\int_0^1 e^{2\pi i[KN_1 g(x_2)+(N_2-\ell_2)x_2]}dx_2$$

is a Fourier coefficient of the function $e^{2\pi i[KN_1 g(x_2)]}$. Since the set of nonzero Fourier coefficients of $e^{2\pi i[KN_1 g(x_2)]}$ is the same as that of $e^{2\pi i[N_1 Kg(qx_2)]} = e^{2\pi i N_1 f(x_2)}$ we have

$$\delta_0\left|\int_0^1 e^{2\pi i[N_1 f(x_2)+(N_2-\ell_2)x_2]}dx_2\right| < 0.3$$

proving (2).

In the case that $N_2 \neq 0$ we follow a very similar argument except we skew G in the second variable. If $s\in\mathbb{Z}\setminus\{0\}$ is chosen so that $\left|\alpha_1-\frac{r}{s}\right| < \frac{1}{s^2}$ and $\frac{1}{s}<\delta$ where $r\in\mathbb{Z}$ then let $G(x_1,x_2) = (x_1, x_2+Kg(sx_1))$. It is easy to see that this G satisfies (1) and (2). □

3.1.2 Uniform Rigidity

In this subsection we will show that uniform rigidity is generic in the set $\mathcal{O}$. Specifically we will prove the following theorem:

Theorem 3.1.14. There exists a dense G_δ subset $\mathcal{R}_2$ of $\mathcal{O}$ such that for every $T\in\mathcal{R}_2$, $(\mathbb{T}^2, T)$ is uniformly rigid.

Proof. Recall that $\{n\alpha : n\in\mathbb{Z}\}$ where $\alpha=(\alpha_1,\alpha_2)$, is dense in $\mathbb{T}^2$. If $\sigma:\mathbb{T}^2\to\mathbb{T}^2$ is defined by $\sigma(x_1,x_2) = (x_1+\alpha_1, x_2+\alpha_2)$ then σ is uniformly rigid. This follows easily from the above statement regarding the density of $\{n\alpha : n\in\mathbb{Z}\}$. Let (n_m) be the uniform rigidity sequence for σ.

Let

$$R_{i,\epsilon} = \{T \in \mathcal{O} \ : \ there\ exists\ n_m \geq i\ with\ d_u(T^{n_m}, Id) < \epsilon\}.$$

We will show that

$$\mathcal{R}_2 = \bigcap_{i=1}^{\infty} R_{i,\frac{1}{i}}$$

is our desired dense G_δ set. By the definition of uniform rigidity, the set $\mathcal{R}_2$ is the set of uniformly rigid homeomorphisms in $\mathcal{O}$ with respect to a subsequence of (n_m). Since $R_{i,\epsilon}$ is clearly open in $\mathcal{O}$, it suffices to show that $R_{i,\epsilon}$ is dense in $\mathcal{O}$.

To show that $R_{i,\epsilon}$ is dense in $\mathcal{O}$ we will show that $O(\sigma) \subseteq R_{i,\epsilon}$. To do this we will rely on

$$d_u(\sigma^{n_m}, Id) \to 0$$

as $m \to \infty$. Let $G \in \mathcal{H}(\mathbb{T}^2)$. Our goal is to show that there exists $n_m \geq i$ such that $d_u(G \circ \sigma^{n_m} \circ G^{-1}, Id) < \epsilon$. Recall that we can write

$$d_u(G \circ \sigma^{n_m} \circ G^{-1}, Id) = \overline{d}(G \circ \sigma^{n_m} \circ G^{-1}, Id) + \overline{d}(G \circ \sigma^{-n_m} \circ G^{-1}, Id)$$

and so we need to find $n_m \geq i$ such that $\overline{d}(G(\sigma^{n_m}), G(Id)) < \frac{\epsilon}{2}$ and $\overline{d}(G(\sigma^{-n_m}), G(Id)) < \frac{\epsilon}{2}$. Since $d_u(\sigma^{n_m}, Id) \to 0$ we can find a large enough n_m for our purpose. Therefore $G^{-1} \circ \sigma \circ G \in R_{i,\epsilon}$.

□

Remark 3.1.15. If (n_m) is the uniform rigidity sequence for σ, then for every $T \in \mathcal{R}_2$, $(\mathbb{T}^2, T)$ is uniformly rigid with respect to a subsequence of (n_m).

3.1.3 Strict Ergodicity

The goal of this subsection is to prove the following theorem:

Theorem 3.1.16. There exists a dense G_δ subset $\mathcal{R}_3$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_3$, $(\mathbb{T}^2, T)$ is strictly ergodic.

The following lemma is standard in ergodic theory.

Lemma 3.1.17 ([16] page 99)**.** The following are equivalent:

1. (X, T) is uniquely ergodic.

2. For every $\phi \in C(X)$ the sequence

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(T^n x)$$

   converges to a constant independent of x.

3. For every $\phi \in C(X)$ the sequence

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(T^n x)$$

   converges uniformly to a constant.

4. The convergence in (2) holds for every ϕ in a dense subset of $C(X)$.

Proof. First we will show $(1 \Rightarrow 2)$. Suppose (X, T, μ) is uniquely ergodic. Let $M(X)$ be the space of Borel probability measures on X and $M^T(X)$ be the space of T-invariant Borel probability measures on X. Since (X, T, μ) is uniquely ergodic, $M^T(X) = \{\mu\}$. Now consider the constant sequence of measures δ_x. Then

$$\frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n x}$$

has a weak* limit in $M^T(X)$ since $M^T(X)$ is weak* compact. Since $M^T(X) = \{\mu\}$, for all $\phi \in C(X)$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n x) = \int_X \phi(x)d\mu.$$

Now we will show that $(2 \Rightarrow 1)$.

Suppose that $\mu, \nu \in M^T(X)$. Let $\phi \in C(X)$. Then by assumption we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n x) = c.$$

By the Dominated Convergence Theorem we have

$$\begin{aligned}\int_X \phi(x)d\mu &= \lim_{N\to\infty}\int_X \frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n x)d\mu \\ &= \int_X \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n x)d\mu = c\end{aligned}$$

and by the same argument $\int_X \phi(x)d\nu = c$. Thus $\mu = \nu$.

Next we will show that $(1 \Rightarrow 3)$.

Suppose that $M^T(X) = \{\mu\}$. By the above argument we see that the constant in (2) only depends on the function and is in fact, the integral of that function over X with respect to μ. For a contradiction, assume that the convergence in (2) is not uniform. Thus there exists $\phi \in C(X)$ and $\epsilon > 0$ such that for all N_0 there exists $N > N_0$ and a point $x_i \in X$ with

$$\left|\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n x_i) - \int_X \phi(x)d\mu\right| \geq \epsilon.$$

Let $\mu_N = \frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n x_i}$. Then

$$\left|\int_X \phi(x)d\mu_N - \int_X \phi(x)d\mu\right| \geq \epsilon.$$

Notice that $\mu_N \in M^T(X)$ and $M^T(X)$ is weak* compact. Thus by passing to a subsequence if necessary, we have that

$$\mu_N \to \nu$$

in $M^T(X)$. Thus

$$\left|\int_X \phi(x)d\nu - \int_X \phi(x)d\mu\right| \geq \epsilon$$

and $\nu \neq \mu$ which is a contradiction.

The implication $(3 \Rightarrow 4)$ is clear and the implication $(4 \Rightarrow 1)$ follows a similar argument as $(2 \Rightarrow 1)$.

□

Lemma 3.1.18. Suppose $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ is defined by $\sigma(x_1, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$. Then σ is strictly ergodic.

Proof. We will show that property (4) of Lemma 3.1.17 is satisfied. Since $(\mathbb{T}^2, \sigma)$ is minimal, this is sufficient to show that it is strictly ergodic by Remark 3.1.5.

First let us consider the case where $\phi(x_1, x_2) = e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ is a character. In this case we have the following:

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1}\phi(\sigma^n(x_1,x_2)) &= \frac{1}{N}\sum_{n=0}^{N-1}\phi(x_1 + n\alpha_1, x_2 + n\alpha_2) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i(\ell_1 x_1 + \ell_1 n\alpha_1 + \ell_2 x_2 + \ell_2 n\alpha_2)} \\
&= \frac{1}{N} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)} \sum_{n=0}^{N-1}\left(e^{2\pi i(\ell_1\alpha_1 + \ell_2\alpha_2)}\right)^n
\end{aligned}$$

$$= \begin{cases} 1 & \text{if } \ell_1 = \ell_2 = 0, \\ \frac{1}{N} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)} \left(\frac{1 - e^{2\pi i N(\ell_1 \alpha_1 + \ell_2 \alpha_2)}}{1 - e^{2\pi i(\ell_1 \alpha_1 + \ell_2 \alpha_2)}} \right) & \text{otherwise.} \end{cases}$$

Now observe

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi(\sigma^n(x_1, x_2)) &= \begin{cases} 1 & \text{if } \ell_1 = \ell_2 = 0, \\ 0 & \text{otherwise.} \end{cases} \\ &= \int_{[0,1)^2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)} dx_1 dx_2 \\ &= \int_{[0,1)^2} \phi(x) d\mu. \end{aligned}$$

By linearity the above statement holds when ϕ is a trigonometric polynomial. Since the set of trigonometric polynomials are dense in $C(\mathbb{T}^2)$, Lemma 3.1.17 tells us that σ is strictly ergodic.

□

We are now ready to prove Theorem 3.1.16.

Proof. First notice that σ is strictly ergodic. If $G \in \mathcal{H}(\mathbb{T}^2)$ then $G \circ \sigma \circ G^{-1}$ is also strictly ergodic. Hence $O(\sigma)$ is a subset of the strictly ergodic homeomorphisms. Let $\phi \in C(\mathbb{T}^2)$ and $\epsilon > 0$. Consider the open set $R_{\phi,\epsilon}$ defined by $T \in \mathcal{O}$ such that there exists N and c with $\left| \frac{1}{N} \sum_{n=0}^{N-1} \phi(T^n(x)) - c \right| < \epsilon$ for all $x \in \mathbb{T}^2$. Let $\{\phi_i\}$ be dense in $C(\mathbb{T}^2)$. We will show that

$$\mathcal{R}_3 = \bigcap_{i,j=1}^{\infty} R_{\phi_i, \frac{1}{j}}$$

is our desired dense G_δ set.

By the definition of unique ergodicity, the set $\mathcal{R}_3$ is the set of uniquely ergodic homeomorphisms in $\mathcal{O}$. The unique ergodic measure is a product of Lebesgue measures and therefore has full topological support. Thus, $\mathcal{R}_3$ is the set of strictly ergodic homeomorphisms in $\mathcal{O}$

by Remark 3.1.5. Clearly, $R_{\phi_i, \frac{1}{i}}$ is open and dense in $\mathcal{O}$.

$\square$

3.1.4 Main Result

In this subsection we put all of the previous theorems together to give a positive answer to the question stated at the beginning of this section, proving Theorem 3.1.2.

Proof of Theorem 3.1.2. Let

$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3.$$

$\square$

3.2 The Klein Bottle

While presenting results from the previous section in a seminar, Alica Miller asked if it would be possible to obtain similar generic results on the Klein bottle. This is an interesting question since constructing with specified topological properties has always presented difficulties. For example, the question of constructing a minimal homeomorphism of the Klein bottle presented mathematicians with trouble for some time, until 1965 when Robert Ellis produced such a map [14].

In this section, whose contents are taken from [36], we will be working with flows defined on the Klein bottle. We will produce a large family of topologically weakly mixing homeomorphisms that are uniformly rigid on the Klein bottle. We were not able to prove measure weak mixing, which was the original goal. Our approach is to view the Klein bottle as the quotient of $\mathbb{T}^2$ by an appropriate group action and produce homeomorphisms of $\mathbb{T}^2$ that are topologically weakly mixing, uniformly rigid, and equivariant with respect to the group action. Since these maps are equivariant and the desired properties are compatible with the projection to the Klein bottle, the maps induce homeomorphisms on the Klein bottle that are topologically weakly mixing and uniformly rigid.

Let $\mathcal{P}$ be the closure of the set of conjugations of an aperiodic rotation by homeomorphisms of the Klein Bottle. The main result of this section is the following:

Theorem 3.2.1. There exists a dense G_δ subset $\mathcal{S}$ of $\mathcal{P}$ such that every $T \in \mathcal{S}$ is topologically weakly mixing and uniformly rigid.

3.2.1 Set-theoretic Klein bottle

In this subsection we will show how one obtains the Klein bottle from the quotient of the two-torus by a group action. For this discussion, let X be a topological space and H a discrete group. Suppose H acts on the space X on the left by $(x, h) \mapsto h.x$ where $x \in X$ and $h \in H$. We will be considering the quotient X/H.

The H-orbit of a point $x \in X$ is the set $\{h.x : h \in H\}$. The quotient map $\pi : X \to X/H$ sends $x \in X$ to the H-orbit of x. This means that we can think of X/H as the space X with the H-orbits collapsed to points.

Definition 3.2.2. Suppose f is a function from X to X. The function f is H*-equivariant* if $f(h.x) = h.f(x)$ for all $h \in H$ and $x \in X$.

If $f : X \to X$ is H-equivariant, then f carries the H-orbits of x to the H-orbits of $f(x)$. Thus f induces a well-defined map on the quotient X/H. Let $\overline{f} : X/H \to X/H$ be the map induced by f. In this case $\overline{f}$ commutes with the quotient map, ie. $\pi \circ f = \overline{f} \circ \pi$.

Definition 3.2.3. The left action of H on X is *continuous* if for each $h \in H$ the map $x \mapsto h.x$ is continuous and is *free* if for each $x \in X$ the subgroup $\{h \in H : h.x = x\}$ is trivial.

Definition 3.2.4. The left action of H on X is *properly discontinuous* if it is continuous and for every $x \in X$ there exists an open neighborhood U_x of x such that the H-translates $h.U_x$ meet U_x for only finitely many $h \in H$.

We will be studying actions that are free and properly discontinuous. In this case, if X is a locally Hausdorff space, then for every $x \in X$ we can find an open neighborhood U_x such that $U_x \cap h.U_x = \emptyset$ for all $h \in H \setminus \{1\}$. This means that when we identify points that lie in the same H-orbit to form X/H we are not squashing the space. Also, there is a unique topology on X/H called the *quotient topology* such that $\pi : X \to X/H$ is a continuous map that is a local homeomorphism. In the quotient topology, a subset Y of X/H is open if and only if its preimage under π is open in X. Finally, two points in the quotient X/H are close if the corresponding H-orbits in X contain points that are close.

To form the Klein bottle, let $X = \mathbb{T}^2$ where $\mathbb{T}^2$ is viewed as $[0,1)^2$ modulo one in each coordinate. Let $H = \{\mathbf{1}, -\mathbf{1}\}$ be a discrete group of order two and the action of H on $\mathbb{T}^2$ be defined by

$$\mathbf{1}.(x,y) = (x,y)$$

and

$$-\mathbf{1}.(x,y) = \left(x + \frac{1}{2}, 1 - y\right)$$

for all $(x,y) \in \mathbb{T}^2$. This action is easily seen to be free and properly discontinuous since you can think of it as rotation by π in the first coordinate and complex conjugation in the second. The quotient $\mathbb{T}^2/H$ will be called the set-theoretic Klein bottle and be denoted by $\mathbb{K}^2$.

3.2.2 Topological Weak Mixing

In this subsection we will show that there is a large family of topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid. In [20] Glasner and Weiss produced a large family of topologically weakly mixing homeomorphisms of the two-torus that were uniformly rigid. We use their maps on the two-torus as inspiration for our constructions that eventually get pushed down to the Klein bottle.

Our plan is to produce homeomorphisms of $\mathbb{T}^2$ that are topologically weakly mixing, uniformly rigid, and H-equivariant. Since these maps are H-equivariant, they will induce homeomorphisms on $\mathbb{K}^2$. Notice that topological weak mixing and uniform rigidity are compatible with the projection to the Klein bottle. Thus, these induced homeomorphisms of $\mathbb{K}^2$ will be topologically weakly mixing and uniformly rigid.

The model of $\mathbb{T}^2$ that we will be using is the unit interval model where it is viewed as $[0,1)^2$ and the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context). Let $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $\sigma(x,y) = (x+\alpha, y)$ where α is irrational and $\{n\alpha : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$. In this case it is easy to see that σ is H-equivariant.

We will reuse and abuse some of the notation from the previous section on the two-torus. In this section, let $\mathcal{H}(X)$ be the set of homeomorphisms of a compact metric space X (not necessarily measure-preserving). Define the set $O(\sigma)$ as follows:

$$O(\sigma) = \left\{G^{-1} \circ \sigma \circ G \ : \ G \in \mathcal{H}(\mathbb{T}^2) \text{ and } G \text{ is } H-\text{equivariant}\right\}.$$

We will be considering $O(\sigma)$ as a subset of all homeomorphisms of $\mathbb{T}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. Let $\mathcal{O} = \overline{O(\sigma)}$ with the closure taken in the above topology.

In a similar fashion, define the set $P(\sigma)$ as follows:

$$P(\sigma) = \left\{G^{-1} \circ \overline{\sigma} \circ G \ : \ G \in \mathcal{H}(\mathbb{K}^2)\right\}.$$

Recall that $\overline{f}$ denotes the induced map on $\mathbb{K}^2$ of the original map $f : \mathbb{T}^2 \to \mathbb{T}^2$. The set $P(\sigma)$ will be considered as a subset of all homeomorphisms of $\mathbb{K}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. Let $\mathcal{P} = \overline{P(\sigma)}$ with the closure taken in the above topology.

The main goal of this subsection is to prove the following theorem and corollary:

Theorem 3.2.5. There exists a dense G_δ subset $\mathcal{R}_1$ of $\mathcal{O}$ such that every $T \in \mathcal{R}_1$ is topologically weakly mixing.

Corollary 3.2.6. There exists a dense G_δ subset $\mathcal{S}_1$ of $\mathcal{P}$ such that every $T \in \mathcal{S}_1$ is topologically weakly mixing.

Proof of Theorem 3.2.5.

Let $\{U_i\}$ be a countable basis for $\mathbb{T}^2$. Define the set R_{U_i, U_j, U_l, U_m} as follows:

$$R_{U_i, U_j, U_l, U_m} = \{T \in \mathcal{O} \ : \ \textit{there exists an integer } k \textit{ with } T^k(U_i \times U_j) \cap (U_l \times U_m) \neq \emptyset\}.$$

We will show that $\mathcal{R}_1 = \bigcap_{i, j, l, m} R_{U_i, U_j, U_l, U_m}$ is our desired dense G_δ subset of $\mathcal{O}$.

From the definition of topological weak mixing, we see that $\mathcal{R}_1$ is precisely the set of topologically weakly mixing homeomorphisms in $\mathcal{O}$. Also, clearly $\mathcal{R}_1$ is open in $\mathcal{O}$. Thus it remains to show that each R_{U_i, U_j, U_l, U_m} is dense. For simplicity of notation, we will show that R_{U_1, U_2, U_3, U_4} is dense in $\mathcal{O}$. Since $R_{U_1, U_2, U_3, U_4} \subseteq \mathcal{O}$ and $\mathcal{O}$ is closed, it suffices to show that if $G_0 \in \mathcal{H}(\mathbb{T}^2)$ and G_0 is H-equivariant then $G_0^{-1} \circ \sigma \circ G_0 \in \overline{R_{U_1, U_2, U_3, U_4}}$.

Suppose that (G_m) is a sequence in $\mathcal{H}(\mathbb{T}^2)$ of H-equivariant homeomorphisms such that $d_u(G_m^{-1} \circ \sigma \circ G_m, \sigma) \to 0$ as $m \to \infty$ and for all m, $G_m^{-1} \circ \sigma \circ G_m \in R_{G_0U_1, G_0U_2, G_0U_3, G_0U_4}$. Notice that

$$G_0 R_{U_1, U_2, U_3, U_4} G_0^{-1} = R_{G_0U_1, G_0U_2, G_0U_3, G_0U_4}.$$

Since G_0, G_0^{-1} are continuous, $d_u(G_0^{-1} \circ G_m^{-1} \circ \sigma \circ G_m \circ G_0, G_0^{-1} \circ \sigma \circ G_0) \to 0$ as $m \to \infty$. Thus we can write

$$\begin{aligned} G_0^{-1} \circ \sigma \circ G_0 &= \lim_{m \to \infty} (G_0^{-1} \circ G_m^{-1} \circ \sigma \circ G_m \circ G_0) \\ &\in \overline{G_0^{-1} R_{G_0U_1, G_0U_2, G_0U_3, G_0U_4} G_0} \\ &= \overline{R_{U_1, U_2, U_3, U_4}}. \end{aligned}$$

Therefore we have reduced the rest of the proof of Theorem 3.2.5 to the following lemma:

Lemma 3.2.7. Let $\epsilon > 0$ and U_1, U_2, U_3, U_4 be open sets in $\mathbb{T}^2$. Then there exists an H-equivariant $G \in \mathcal{H}(\mathbb{T}^2)$ such that the following two properties hold:

1. $d_u(G^{-1} \circ \sigma \circ G, \sigma) < \epsilon$

2. $G^{-1} \circ \sigma \circ G \in R_{U_1,\, U_2,\, U_3,\, U_4}$ (ie. $\sigma \in R_{GU_1,\, GU_2,\, GU_3,\, GU_4}$).

Remark 3.2.8. The U_1, U_2, U_3, U_4 that appear in the above lemma are $G_0U_1, G_0U_2, G_0U_3, G_0U_4$ from the proof of Theorem 3.2.5.

Proof of Lemma 3.2.7. Let $\epsilon > 0$ and U_1, U_2, U_3, U_4 be open sets in $\mathbb{T}^2$. Let $p : \mathbb{T}^2 \to \mathbb{T}$ be a projection onto the second coordinate. Let h_1, h_2 be homeomorphisms of $\mathbb{T}$ such that the following two properties hold:

- $h_1(pU_1) \cap (pU_3) \neq \emptyset$ and $h_2(pU_2) \cap (pU_4) \neq \emptyset$

- $h_i(1-y) = 1 - h_i(y)$ for $i = 1, 2$.

The figure at the top of the next page shows how such a h_1 can be constructed.

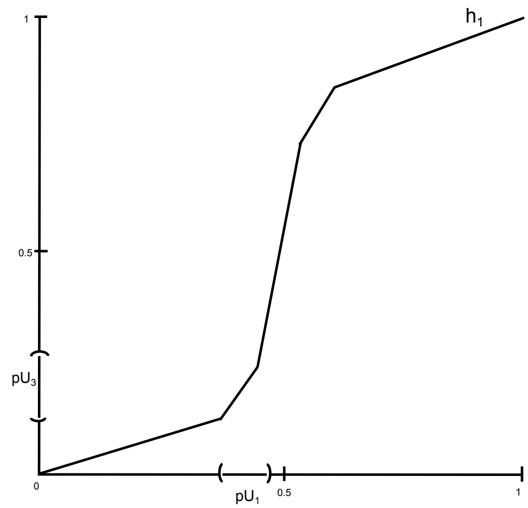

Figure 3.1: Example of h_1

Now choose points in the above intersections such that

- $y_i \in pU_i$ for $i = 1, 2, 3, 4$

- $h_1(y_1) = y_3$ and $h_2(y_2) = y_4$

- $y_1, y_2, y_3, y_4, 1 - y_1, 1 - y_2, 1 - y_3, 1 - y_4$ are all distinct points.

Also choose distinct x_i such that the point (x_i, y_i) belongs to U_i for $i = 1, 2, 3, 4$.

We are now ready to start building our desired function G. Let $x \to g_x$ be a continuous function from $[0, 1)$ to $\mathcal{H}(\mathbb{T})$ such that $g_0, g_{\frac{1}{4}}, g_{\frac{1}{2}}, g_{\frac{3}{4}} = Id$, $g_{\frac{1}{8}}, g_{\frac{5}{8}} = h_1$, and $g_{\frac{3}{8}}, g_{\frac{7}{8}} = h_2$ with linear interpolation in between. By the choice of the y_i's above, we know that y_i and $1 - y_i$ are distinct. Let V_i and V_{-i} be pairwise distinct, symmetric neighborhoods around y_i and $1 - y_i$ respectively that are all equal in length. Define continuous bump functions b_i, b_{-i} on $\mathbb{T}$ such that

- b_i is symmetric about y_i on V_i.

- $b_i(V_i^c) = 0$ and $b_{-i}(V_{-i}^c) = 0$

- $b_i(y_i) = 1$, $b_{-i}(1 - y_i) = 1$, and $b_i(y) = b_{-i}(1 - y)$ for all $y \in \mathbb{T}$

Let $\eta > 0$ be such that if $|y - y'| < \eta$ then $\max_{1 \le i \le 4} |b_i(y) - b_i(y')| < \frac{\epsilon}{32}$. Let $\delta > 0$ be such that if $|x - x'| < \delta$ then $\overline{d}(g_x^{-1} g_{x'}, Id) < \min(\eta, \frac{\epsilon}{4})$. Now we will use a rational approximation of α. Choose $q \in \mathbb{Z} \backslash \{0\}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$ with $\frac{1}{q} < \delta$ for some $p \in \mathbb{Z}$.

Define $c_i \in [0, 1)$ such that

$$x_1 + c_1 = \frac{1}{8q}, \;\; x_2 + c_2 = \frac{3}{8q}, \;\; x_3 + c_3 = \frac{1}{4q}, \;\; x_4 + c_4 = \frac{1}{2q}$$

all taken modulo one. Let f be defined by $f(y) = \sum_{i=1}^{4} [c_i b_i(y) + c_i b_{-i}(y)]$. Then it is easy to see that $f(1 - y) = f(y)$. Now let $G : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G(x, y) = \left(x + f(y), g_{q(x+f(y))}(y)\right).$$

Notice that

$$G^{-1}(x, y) = (x - f(g_{qx}^{-1}(y)), g_{qx}^{-1}(y))$$

and

$$G^{-1} \circ \sigma \circ G(x, y) = (x + \alpha + f(y) - f(y_*), y_*)$$

where

$y_* = g^{-1}_{q(x+\alpha+f(y))} g_{q(x+f(y))}(y)$. We claim that this is our desired $G \in \mathcal{H}(\mathbb{T}^2)$. The first thing to check is that G is H-equivariant. To see this observe the following:

$$\begin{aligned}
G\left(-\mathbf{1}.(x,y)\right) &= G\left(x+\frac{1}{2}, 1-y\right) \\
&= \left(x+\frac{1}{2}+f(1-y), g_{q(x+\frac{1}{2}+f(1-y))}(1-y)\right) \\
&= \left(x+\frac{1}{2}+f(y), g_{q(x+f(y))+\frac{q}{2}}(1-y)\right) \\
&= \left(x+\frac{1}{2}+f(y), g_{q(x+f(y))}(1-y)\right) \\
&= \left(x+\frac{1}{2}+f(y), 1-g_{q(x+f(y))}(y)\right) \\
&= -\mathbf{1}.G\left(x,y\right).
\end{aligned}$$

Thus G is H-equivariant.

Now we need to check that (2) is verified, that is $\sigma \in R_{GU_1,\,GU_2,\,GU_3,\,GU_4}$. To see this observe

$$G\left(x_1,y_1\right) = \left(x_1+c_1, g_{q(x_1+c_1)}(y_1)\right) = \left(\frac{1}{8q}, y_3\right)$$

$$G\left(x_2,y_2\right) = \left(x_2+c_2, g_{q(x_2+c_2)}(y_2)\right) = \left(\frac{3}{8q}, y_4\right)$$

$$G\left(x_3,y_3\right) = \left(x_3+c_3, g_{q(x_3+c_3)}(y_3)\right) = \left(\frac{1}{4q}, y_3\right)$$

$$G\left(x_4,y_4\right) = \left(x_4+c_4, g_{q(x_4+c_4)}(y_4)\right) = \left(\frac{1}{2q}, y_4\right).$$

Thus $G\left(x_3,y_3\right) - G\left(x_1,y_1\right) = \left(\frac{1}{8q}, 0\right)$ and $G\left(x_4,y_4\right) - G\left(x_2,y_2\right) = \left(\frac{1}{8q}, 0\right)$. Hence, there exists $k \in \mathbb{Z}$ such that $\sigma^k\left(GU_1\right) \cap \left(GU_3\right) \neq \emptyset$ and $\sigma^k\left(GU_2\right) \cap \left(GU_4\right) \neq \emptyset$. Therefore $\sigma \in R_{GU_1,\,GU_2,\,GU_3,\,GU_4}$ as desired.

It remains to check that (1) is verified, that is $d_u(G^{-1} \circ \sigma \circ G, \sigma) < \epsilon$. To begin notice that

$$G^{-1} \circ \sigma \circ G\left(x,y\right) - \sigma\left(x,y\right) = \left(f(y) - f(y_*), y_* - y\right).$$

Since $|q(x+\alpha+f(y))-q(x+f(y))|=|q\alpha|<\frac{1}{q}<\delta$ we must have $\overline{d}\left(g^{-1}_{q(x+\alpha+f(y))}g_{q(x+f(y))}, Id\right)<\min\left(\eta,\frac{\epsilon}{4}\right)$. This implies that $|y_*-y|<\eta$ and therefore

$$\begin{aligned}|f(y)-f(y_*)| &= \left|\sum_{i=1}^{4}[c_i(b_i(y)-b_i(y_*))+c_i(b_{-i}(y)-b_{-i}(y_*))]\right| \\ &\le \sum_{i=1}^{4} c_i\,|b_i(y)-b_i(y_*)|+\sum_{i=1}^{4} c_i\,|b_{-i}(y)-b_{-i}(y_*)| \\ &< 4\left(\frac{\epsilon}{32}\right)+4\left(\frac{\epsilon}{32}\right) \\ &= \frac{\epsilon}{4}.\end{aligned}$$

Thus $\overline{d}\,(G^{-1}\circ\sigma\circ G,\sigma)<\frac{\epsilon}{2}$. In a similar fashion, $\overline{d}\,(G^{-1}\circ\sigma^{-1}\circ G,\sigma^{-1})<\frac{\epsilon}{2}$. Therefore, $d_u(G^{-1}\circ\sigma\circ G,\sigma)<\epsilon$.

□

Proof of Corollary 3.2.6.

Let $\{U_i\}$ be a countable basis for $\mathbb{T}^2$. We may assume, that for each U_i we have $U_i\cap h.U_i=\emptyset$ for all $h\in H$. Let $\overline{U_i}$ be the image of U_i under the quotient map π. Then $\{\overline{U_i}\}$ is a countable basis for $\mathbb{K}^2$. Define the set $S_{\overline{U_i},\,\overline{U_j},\,\overline{U_l},\,\overline{U_m}}$ as follows:

$$S_{\overline{U_i},\,\overline{U_j},\,\overline{U_l},\,\overline{U_m}}=\{T\in\mathcal{P}\ :\ \textit{there exists an integer } k \textit{ with } T^k(\overline{U_i}\times\overline{U_j})\cap(\overline{U_l}\times\overline{U_m})\neq\emptyset\}.$$

We will show that $\mathcal{S}_1=\bigcap_{i,\,j,\,l,\,m} S_{\overline{U_i},\,\overline{U_j},\,\overline{U_l},\,\overline{U_m}}$ is our desired dense G_δ subset of $\mathcal{P}$.

From the definition of topological weak mixing we see that $\mathcal{S}_1$ is precisely the set of topologically weakly mixing homeomorphisms in $\mathcal{P}$. Also, clearly $\mathcal{S}_1$ is open in $\mathcal{P}$. Thus it remains to show that each $S_{\overline{U_i},\,\overline{U_j},\,\overline{U_l},\,\overline{U_m}}$ is dense. As in the previous theorem, everything reduces to proving a key lemma.

Lemma 3.2.9. Let $\epsilon>0$ and $\overline{U_1},\overline{U_2},\overline{U_3},\overline{U_4}$ be open sets in $\mathbb{K}^2$. Then there exists $G\in\mathcal{H}(\mathbb{K}^2)$ such that the following two properties hold:

1. $d_u(G^{-1} \circ \overline{\sigma} \circ G, \overline{\sigma}) < \epsilon$

2. $G^{-1} \circ \overline{\sigma} \circ G \in S_{\overline{U_1},\, \overline{U_2},\, \overline{U_3},\, \overline{U_4}}$ (ie. $\overline{\sigma} \in S_{G\overline{U_1},\, G\overline{U_2},\, G\overline{U_3},\, G\overline{U_4}}$).

Proof. First you apply Lemma 3.2.7 to obtain $H \in \mathcal{H}(\mathbb{T}^2)$ that is H-equivariant and satisfies

- $d_u(H^{-1} \circ \sigma \circ H, \sigma) < \epsilon$

- $H^{-1} \circ \sigma \circ H \in R_{U_1,\, U_2,\, U_3,\, U_4}$ (ie. $\sigma \in R_{HU_1,\, HU_2,\, HU_3,\, HU_4}$).

Then $\overline{H}$ is the induced map of H on $\mathbb{K}^2$. Take $G = \overline{H}$.

□

3.2.3 Uniform Rigidity

In this subsection we will show that uniform rigidity is generic in the set $\mathcal{O}$ and in the set $\mathcal{P}$. Similar to a theorem found in [17] we have

Theorem 3.2.10 ([17])**.** There exists a dense G_δ subset $\mathcal{R}_2$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_2$, $(\mathbb{T}^2, T)$ is uniformly rigid.

A simply corollary is the following:

Corollary 3.2.11. There exists a dense G_δ subset $\mathcal{S}_2$ of $\mathcal{P}$ such that for every $T \in \mathcal{S}_2$, $(\mathbb{K}^2, T)$ is uniformly rigid.

3.2.4 Main Result

We now put the previous two theorems together to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let

$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2.$$

□

Chapter 4

Rate Results

In this chapter we will explore the structure of rigidity and uniform rigidity sequences for weakly mixing maps. We will discuss examples of sequences and also rates of growth.

4.1 Rigidity Sequences

We will begin by analyzing rigidity sequences for weakly mixing transformations. To that end, in this section let (X, β, μ, T) be a dynamical system.

Definition 4.1.1. Suppose (X, β, μ, T) is a dynamical system. We say that T has *discrete spectrum* if the eigenfunctions of the Koopman operator associated to T span $L_2(X, \mu)$.

One of the first things to notice is that ergodic transformations with discrete spectrum are rigid. To see this, just think of the ergodic transformation as a rotation of a compact monothetic group since each ergodic transformation with discrete spectrum is conjugate to a rotation of a compact monothetic group by the Halmos-von Neumann theorem [23]. Moreover, the ergodic transformations with discrete spectrum are completely determined by their rigidity sequence.

Proposition 4.1.2 ([6])**.** Assume that (X, β_X, μ_X, T) is an ergodic transformation with discrete spectrum, and (Y, β_Y, μ_Y, S) is ergodic and has the same rigidity sequence as T. Then S is measure-theoretically isomorphic to T.

A question that was asked in [6] that they were unable to answer is the following:

Question 4.1.3. Which rigidity sequences of an ergodic transformation with discrete spectrum can be rigidity sequences for some weakly mixing transformation?

Recently, Adams [1] was able to answer the above question.

Proposition 4.1.4 ([6])**.** Given any ergodic measure-preserving transformations T on a Lebesgue probability space with discrete spectrum, and a rigidity sequence (n_m) for T, there exists a weakly mixing transformation R with rigidity sequence (n_m).

4.1.1 Examples

In this subsection we will explore examples of sequences that can be realized as rigidity sequences for weakly mixing transformations. The first two examples construct a continuous Borel probability measure on the circle ν such that $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$. While this approach is interesting, it is not as constructive as the third example. In the third example, the transformation is constructed using a technique called cutting and stacking.

Proposition 4.1.5 ([6])**.** Suppose (n_m) is an increasing sequence that satisfies $\frac{n_{m+1}}{n_m} \to \infty$ as $m \to \infty$. Then (n_m) is a rigidity sequence for some weakly mixing transformation.

Proposition 4.1.6 ([6])**.** Suppose (n_m) is an increasing sequence that satisfies $\frac{n_{m+1}}{n_m} = a$ for some nonzero whole number a. Then (n_m) is a rigidity sequence for some weakly mixing transformation.

Proposition 4.1.7 ([6])**.** Suppose $n_{m+1} = q_m n_m + r_m$ where $0 \leq r_m < n_m$, $q_m \to \infty$, $\sum_{m=1}^{\infty} \frac{r_m}{n_{m+1}} < \infty$, and $r_m \neq 0$ infinitely often. Then there is a finite measure-preserving transformation that has (n_m) as a rigidity sequence, and is weakly mixing and rank one.

We will describe the cutting and stacking construction that is used in the previous proposition to get a feel for the cutting and stacking technique. The way these constructions are produced is to take an interval, cut it into a certain number of pieces, and then stack the pieces on top of one another, sometimes adding spacers. The procedure is inductive and the transformation acts by moving up the stack. To define what happens on the top stack, you much perform the procedure again.

Sketch of Construction. For the previous proposition, suppose you have towers τ_m of height $h_m = n_m$. We will describe how to construct the tower τ_{m+1} from tower τ_m. There are two cases.

Case 1. Suppose $r_m = 0$. In this case, cut the tower τ_m into q_m columns, creating $q_m h_m$ news pieces. Then stack the pieces on top of one another to create the tower τ_{m+1} of height $q_m h_m = q_m n_m = n_{m+1} = h_{m+1}$.

Case 2. Suppose $r_m \neq 0$. In this case, cut the tower τ_m into q_m columns, creating $q_m h_m$ new pieces. Let $a_m = \lfloor \frac{q_m}{3} \rfloor$. Now, stack in the following order: $a_m h_m$ pieces, 1 spacer, $(q_m - a_m)h_m$ pieces, $r_m - 1$ spacers. The height of the new tower τ_{m+1} is

$$a_m h_m + 1 + (q_m - a_m)h_m + r_m - 1 = q_m h_m + r_m = n_{m+1} = h_{m+1}.$$

To see that this is in fact a finite measure-preserving system, let S_m be the spacers that were added to tower τ_m to produce tower τ_{m+1}. Then,

$$\frac{\mu(S_m)}{\mu(\tau_{m+1})} = \frac{r_m \mu(\tau_m)/(q_m h_m)}{h_{m+1}\mu(\tau_m)/(q_m h_m)} = \frac{r_m}{h_{m+1}}.$$

By assumption, $\sum_{m=1}^{\infty} \frac{r_m}{n_{m+1}} < \infty$ and thus the system has finite measure.

4.1.2 Non-examples

In this subsection we will explore a few non-examples, that is sequences that cannot be rigidity sequences for weakly mixing transformations.

Proposition 4.1.8 ([6])**.** Suppose (n_m) is an increasing sequence such that $(n_m x \textit{ mod } 1)$ is uniformly distributed for all but a countable set of x values. Then (n_m) cannot be a rigidity sequence for a weakly mixing transformation.

Proof. Suppose that $(n_m x \mod 1)$ is uniformly distributed for all but a countable set of x values. In particular, this means that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i n_m x} = 0$$

for all but countably many x. Let ν be a continuous measure on $\mathbb{T}$. Then,

$$\int_{\mathbb{T}} \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i n_m x} d\nu \to 0$$

as $N \to \infty$. Note that (n_m) being a rigidity sequence for some transformation is equivalent to $e^{2\pi i n_m x} \to 1$ in measure as $m \to \infty$ (with respect to the spectral measure). Since the spectral measure for a weakly mixing transformation is continuous, this cannot occur.

□

Proposition 4.1.9 ([6])**.** Suppose $n_m = p(m)$ where p is a nonzero polynomial with integer coefficients. Then (n_m) cannot be a rigidity sequence for a weakly mixing transformation.

Proof. We will prove the above proposition when $p(m) = m^2$. Notice that

$$n_{m+2} - 2n_{m+1} + n_m = (m+2)^2 - 2(m+1)^2 + m^2 = 2.$$

Suppose ν is a continuous Borel probability measure on the circle such that $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$. Then as we have previously seen, this means that $z^{n_m} \to 1$ in measure with respect to ν as $m \to \infty$. Thus $z^2 = z^{n_{m+2}-2n_{m+1}+n_m} \to 1$ in measure with respect to ν as $m \to \infty$. This tells us that ν must be supported on the 2nd roots of unity, which cannot be for a weakly mixing transformation.

□

A similar argument can also be used to show that $2^m + 1$ is not a rigidity sequence for a weakly mixing transformation, even though 2^m is.

4.2 Uniform Rigidity Sequences for Weakly Mixing Homeomorphisms

In this section we will explore the structure of uniform rigidity sequences for weakly mixing homeomorphisms. The contents of this section are taken from [35]. We have seen in the previous chapter that there is a large family of such homeomorphisms defined on the two-torus. Now, we will give results in the direction of the following question posed by the authors of [25]:

Question 4.2.1. Which zero density sequences occur as uniform rigidity sequences for an ergodic transformation?

We were not able to answer this question in full, but we were able to prove that given a sufficient growth rate, the existence of a weakly mixing (ergodic) homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to your sequence is guaranteed. Specifically we prove the following two theorems:

Theorem 4.2.2. If (n_m) is an increasing sequence of natural numbers satisfying

$$\lim_{m\to\infty} \frac{n_{m+1}}{n_m} = \infty$$

there exists an ergodic homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m).

Theorem 4.2.3. Let $\psi(x) = x^{x^3}$. If (n_m) is an increasing sequence of natural numbers satisfying

$$\frac{n_{m+1}}{n_m} \geq \psi(n_m)$$

there exists a weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m).

Recall that rigidity sequences for weakly mixing transformations have gaps that tend to infinity and have zero density (see [6]). Since uniform rigidity implies rigidity, this is also the case for uniform rigidity sequences. Even though this is the case, we give a direct proof here.

Proposition 4.2.4. Let (n_m) be an increasing sequence of natural numbers and (X, β, μ, T) a dynamical system. Let T be a weakly mixing homeomorphism that is uniformly rigid with respect to (n_m) (ie. $d_u(T^{n_m}, Id) \to 0$ as $m \to \infty$). Then the sequence (n_m) has gaps tending to infinity and has zero density.

Proof. Our goal is to show that (n_m) has gaps that tend to infinity. Thus we want to show that $n_{m+1} - n_m \to \infty$. Suppose for a contradiction that there exists $d \geq 1$ such that $d = n_{m+1} - n_m$ infinitely often. Thus infinitely often we have

$$\begin{aligned} d_u(T^{n_{m+1}}, Id) &= d_u(T^{d+n_m}, Id) \\ &= d_u(T^{n_m}(T^d), Id) \\ &= \overline{d}(T^{n_m}(T^d), Id) + \overline{d}(T^{-d}(T^{-n_m}), Id) \\ &= \overline{d}(T^{n_m}, T^{-d}) + \overline{d}(T^{-n_m}, T^d) \\ &= d_u(T^{n_m}, T^{-d}) \end{aligned}$$

Thus $T^d = Id$ since $d_u(T^{n_{m+1}}, Id) \to 0$. Let $f \in L_2(X, \mu)$ be nonconstant. Then $f \circ T^d = f$ which is a contradiction since T is weakly mixing and hence totally ergodic. Therefore, (n_m) has gaps that tend to infinity and hence zero density.

□

4.2.1 Construction

In this subsection we are interested in sequences that are uniform rigidity sequences for weakly mixing (ergodic) homeomorphisms of $\mathbb{T}^2$. We will show that if a sequence satisfies a certain growth rate, then we can construct a weakly mixing (ergodic) homeomorphism of

$\mathbb{T}^2$ for which the given sequence is a uniform rigidity sequence. This gives a result in the direction of Question 4.2.1.

The goal of this subsection is then to prove Theorems 4.2.2 and 4.2.3. We will use the structure involved in the category argument of the previous chapter to construct the desired homeomorphisms. Throughout this subsection we will be working on the two torus $\mathbb{T}^2$. We will be using the model of $\mathbb{T}^2$ where it is viewed as $[0,1)^2$ and the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context). Also, let $\{\cdot\}$ denote the fractional part. If $0 \leq x \leq \frac{1}{2}$ then $|x| = \{x\}$ and if $\frac{1}{2} < x < 1$ then $|x| = 1 - \{x\}$. Suppose that $\mathbb{T}$ is equipped with the usual Lebesgue measure and μ is the corresponding product measure on $\mathbb{T}^2$.

The first step in the construction is to choose an irrational rotation that we will then conjugate. In [13] Eggleston showed that if an increasing sequence of natural numbers (n_m) is such that $\lim_{m\to\infty} \frac{n_{m+1}}{n_m} = \infty$, then $\lim_{m\to\infty} |n_m x| = 0$ holds for an uncountable set of x values. In the following lemma we use a similar argument.

Lemma 4.2.5. Let $\psi(x) = x^{x^3}$ and suppose (n_m) is an increasing sequence of natural numbers satisfying $\frac{n_{m+1}}{n_m} \geq \psi(n_m)$. Then there exists $\alpha = (\alpha_1, \alpha_2)$ such that $\{n\alpha : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$ and

$$\frac{1}{4(n_m)^2} < |n_m \alpha_i| < \frac{1}{2(n_m)^2}$$

for $i = 1, 2$.

Proof. Our goal is to build a Cantor set using some of the n_m-th roots of unity. From this Cantor set we will be able to select α_1, α_2 irrational and rationally independent such that the desired bounds hold.

Let $h_m = 2n_m^2$. In this case $\frac{1/h_m}{n_m/n_{m+1}} \to \infty$ as $m \to \infty$. Let M be large enough so that for all $m \geq M$ we have $n_{m+1} \geq 10 n_m h_m$.

Now we will build our Cantor set inductively. Suppose $m \geq M$. As part of the construction put two intervals close to some of the n_m-th roots of unity (determined as part of the induction) such that any point in either of the intervals is at most $\frac{1}{n_m h_m}$ away from the n_m-th root of unity and at least $\frac{1}{2n_m h_m}$ away. In this stage of the construction note that each n_m-th root of unity that appears above has two symmetric intervals close to it, one on either side, each of length $\frac{1}{2n_m h_m}$. Call the union of this collection of intervals C_m.

Since $\frac{1}{n_{m+1}}$ is much smaller than $\frac{1}{2n_m h_m}$, there are many points of the form $\frac{j}{n_{m+1}}$ in each symmetric interval around the above mentioned n_m-th roots of unity. Now select in C_m pairs of symmetric intervals, each of size at least $\frac{1}{2n_{m+1}h_{m+1}}$, close to each of the n_{m+1}-th roots of unity inside C_m in the same way as above. Call the union of this collection of intervals C_{m+1}.

Continue on in this manner and let the Cantor set C be defined as

$$C = \bigcap_{m=M}^{\infty} C_m.$$

For each point $x \in C$ we have that $n_m x$ is at most $\frac{1}{h_m}$ away from an integer and at least $\frac{1}{2h_m}$ away from an integer. That is,

$$\frac{1}{2h_m} < |n_m x| < \frac{1}{h_m}.$$

Hence, if $x \in C$ then $|n_m x| \to 0$ as $m \to \infty$.

Note that C is uncountable. Thus there exists $\alpha_1 \in C$ that is irrational. Since the set of all irrational numbers in C that are rationally dependent to α_1 is countable, there exists $\alpha_2 \in C$ that is irrational and rationally independent with respect to α_1. Hence, $\alpha = (\alpha_1, \alpha_2)$ has the desired properties.

□

We can now prove Theorem 4.2.2.

Proof of Theorem 4.2.2. First use Eggleston's result in [13] to obtain $\alpha = (\alpha_1, \alpha_2)$ such that

$|n_m\alpha_i| \to 0$ as $m \to \infty$ for $i = 1, 2$. Let $\sigma(x_1, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$ be an irrational rotation of $\mathbb{T}^2$. Then $\sigma^{n_m}(x_1, x_2) = (x_1 + n_m\alpha_1, x_2 + n_m\alpha_2)$ and $d_u(\sigma^{n_m}, Id) \to 0$ as $m \to \infty$. Thus σ is an ergodic homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m). $\square$

We will also need the following lemma regarding an oscillatory integral estimate in our proof of the main theorem of this subsection.

Lemma 4.2.6. Suppose N is a given constant, $g(y) = \sin(2\pi y)$, and $\frac{1}{4} < A < \frac{3}{4}$. Then there exists $K \in \mathbb{N}$, which only depends on N, such that for all M we have

$$\left| \int_0^1 e^{2\pi i[K(g(y+A)-g(y))+My]} dy \right| < \frac{0.3}{N^3}.$$

Moreover, if $N \geq \pi$, then any integer larger than $260N^9$ will suffice for K.

Proof. To obtain this estimate we will use Van der Corput's lemma (see [27] page 220) with respect to the second derivative.

Let

$$\psi(y) = 2\pi[Kg(y + A) - Kg(y) + My] = 2\pi[K \sin(2\pi(y + A)) - K \sin(2\pi y) + My].$$

Then,

$$\psi''(y) = -8\pi^3 K[\sin(2\pi(y + A)) - \sin(2\pi y)].$$

The proof will now be split into cases depending on the location of A inside the interval $(\frac{1}{4}, \frac{3}{4})$.

For our first case, suppose that $\frac{1}{4} < A < \frac{1}{2}$. Then the zeros of ψ'' are $a_1 = \frac{1}{4} - \frac{A}{2}$ and $a_2 = \frac{3}{4} - \frac{A}{2}$. This can be seen algebraically or graphically. Also note that $\psi''(0)$ is negative. Let $\epsilon = \frac{0.01}{N^3}$.

Suppose that $2\epsilon < a_1$ and $|\psi''(y)| \geq \lambda > 0$ on $[\epsilon, a_1 - \epsilon]$. Then

$$\begin{aligned}\left|\int_0^1 e^{i\psi(y)} dy\right| &\leq 6\epsilon + \left|\int_\epsilon^{a_1-\epsilon} e^{i\psi(y)} dy\right| + \left|\int_{a_1+\epsilon}^{a_2-\epsilon} e^{i\psi(y)} dy\right| + \left|\int_{a_2+\epsilon}^{1-\epsilon} e^{i\psi(y)} dy\right| \\ &\leq 6\epsilon + 3\left(\frac{6}{\lambda^{\frac{1}{2}}}\right).\end{aligned}$$

Upon inspecting the graph we see that we can take $\lambda = \psi''(a_1 + \epsilon) > 0$ where

$$\psi''(a_1 + \epsilon) = -8\pi^3 K[\sin(2\pi(a_1 + \epsilon + A)) - \sin(2\pi(a_1 + \epsilon))].$$

Thus we need to find $K \in \mathbb{N}$ such that

$$6\epsilon + 3\left(\frac{6}{(-8\pi^3 K[\sin(2\pi(a_1 + \epsilon + A)) - \sin(2\pi(a_1 + \epsilon))])^{\frac{1}{2}}}\right) < \frac{0.3}{N^3}.$$

In particular we need to find $K \in \mathbb{N}$ such that

$$K > \frac{5,625N^6}{-8\pi^3[\sin(2\pi(a_1 + \epsilon + A)) - \sin(2\pi(a_1 + \epsilon))]}.$$

Upon close inspection we see that,

$$\begin{aligned}&\frac{5625N^6}{-8\pi^3[\sin(2\pi(a_1 + \epsilon + A)) - \sin(2\pi(a_1 + \epsilon))]} \\ &\qquad = \frac{-703.125N^6}{\pi^3[\sin((\frac{\pi}{2} + \frac{\pi}{50N^3}) + A\pi) - \sin((\frac{\pi}{2} + \frac{\pi}{50N^3}) - A\pi)]} \\ &\qquad = \frac{-703.125N^6}{2\pi^3 \cos(\frac{\pi}{2} + \frac{\pi}{50N^3}) \sin(A\pi)} \\ &\qquad \leq \frac{-703.125N^6}{\sqrt{2}\pi^3 \cos(\frac{\pi}{2} + \frac{\pi}{50N^3})} \\ &\qquad = \frac{703.125N^6}{\sqrt{2}\pi^3 \sin(\frac{\pi}{50N^3})}\end{aligned}$$

Thus as long as K is an integer greater than $\frac{703.125N^6}{\sqrt{2}\pi^3\sin(\frac{\pi}{50N^3})}$ we have the claim.

If we use the underestimate

$$\sin\left(\frac{\pi}{50N^3}\right) \approx \left(\frac{\pi}{50N^3}\right) - \frac{1}{6}\left(\frac{\pi}{50N^3}\right)^3$$

and suppose that $N \geq \pi$, we obtain

$$\begin{aligned}
\frac{703.125N^6}{\sqrt{2}\pi^3\sin(\frac{\pi}{50N^3})} &\leq \frac{703.125N^6}{\sqrt{2}\pi^3((\frac{\pi}{50N^3}) - \frac{1}{6}\left(\frac{\pi}{50N^3}\right)^3)} \\
&\leq \frac{3847300N^{15}}{15000N^6 - \pi^2} \\
&\leq \frac{3847300N^{15}}{15000N^6 - N^6} \\
&\leq 260N^9
\end{aligned}$$

Hence in the case $N \geq \pi$ we can take $K = 260N^9$.

Suppose $2\epsilon \geq a_1$ and $|\psi''(y)| \geq \lambda > 0$ on $[a_1 + \epsilon, a_2 - \epsilon]$. Then

$$\begin{aligned}
\left|\int_0^1 e^{i\psi(y)}dy\right| &\leq 6\epsilon + \left|\int_{a_1+\epsilon}^{a_2-\epsilon} e^{i\psi(y)}dy\right| + \left|\int_{a_2+\epsilon}^{1-\epsilon} e^{i\psi(y)}dy\right| \\
&\leq 6\epsilon + 2\left(\frac{6}{\lambda^{\frac{1}{2}}}\right).
\end{aligned}$$

In this case, λ is the same as before and the same value for K is sufficient.

For our second case, suppose that $\frac{1}{2} < A < \frac{3}{4}$. Then the zeros of ψ'' are $a_1 = \frac{3}{4} - \frac{A}{2}$ and $a_2 = \frac{5}{4} - \frac{A}{2}$. Also note that $\psi''(0)$ is positive. This case is similar to the above (here $\lambda = \psi''(a_1 - \epsilon)$) and yields the same K value.

The final case is when $A = \frac{1}{2}$. In this case,

$$\psi''(y) = -8\pi^3 K[\sin(2\pi(y + \frac{1}{2})) - \sin(2\pi y)] = 16\pi^3 K \sin(2\pi y)$$

and the zeros of ψ'' are 0 and $\frac{1}{2}$. This case is also very similar to the first one and yields a K value of $181N^9$.

After we put all of the cases together we see that the K value from case 1 is sufficient for all cases.

□

Before we give the proof of Theorem 4.2.3 we need to discuss a quantitative aspect of estimating a function in $L_2(\mathbb{T}^2, \mu)$ by it's partial Fourier sum. Let $S_N f$ be the N-th partial Fourier sum of some function $f \in L_2(\mathbb{T}^2, \mu)$. Let $F_N(x_1, x_2)$ be the Fejér kernel defined on $\mathbb{T}^2$ by

$$F_N(x_1, x_2) = \sum_{|\ell_i| \leq N} \left(1 - \frac{|\ell_1|}{N}\right)\left(1 - \frac{|\ell_2|}{N}\right) e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)} = F_N(x_1) F_N(x_2)$$

where $F_N(x_i)$ is the usual Fejér kernel on $\mathbb{T}$. When discussing the Fejér kernel, it is easier to view $\mathbb{T}^2$ as $[\frac{-1}{2}, \frac{1}{2}]^2$ under addition modulo one in each coordinate.

Recall that F_N is an approximate identity and that $\int_{[\frac{-1}{2}, \frac{1}{2}]^2} F_N(x_1, x_2) dx_1 dx_2 = 1$. Another important fact that we will need about F_N is the following:

$$F_N(x_1, x_2) \leq \min\left(\frac{1}{(N x_1^2)(N x_2^2)}, (N+1)^2\right).$$

This follows from the representation of F_N where

$$F_{N-1}(x_1, x_2) = \left(\frac{1}{N}\right)^2 \left(\frac{\sin(N\pi x_1)}{2\pi x_1}\right)^2 \left(\frac{\sin(N\pi x_2)}{2\pi x_2}\right)^2.$$

Lemma 4.2.7. Suppose f is a function in $C^0(\mathbb{T}^2)$ with modulus of continuity $\omega_f(\delta) \leq M\delta$ for some constant M. If $N \geq 2^{20} M$ then

$$\|S_N f - f\|_2 < 0.01.$$

Proof. Let $\delta = \frac{1}{1{,}000M}$ and suppose $N \geq 2^{20}M$. Our goal is to show that $\|S_N f - f\|_2 < 0.01$. The N-th partial Fourier sum of f is the best L_2-approximation of f by trigonometric polynomials of degree N. Since $f * F_N$ is a trigonometric polynomial of degree N, $\|S_N f - f\|_2 \leq \|f * F_N - f\|_2$. Thus it suffices to show that $\|f * F_N - f\|_2 < 0.01$. Notice that $N > \frac{1}{\delta}$. To this end, observe

$$
\begin{aligned}
\|f * F_N - f\|_2 &= \left\| \int_{[\frac{-1}{2},\frac{1}{2}]^2} f(x_1 - y_1, x_2 - y_2) F_N(y_1, y_2) dy_1 dy_2 - f(x_1, x_2) \right\|_2 \\
&= \left\| \int_{[\frac{-1}{2},\frac{1}{2}]^2} (f(x_1 - y_1, x_2 - y_2) - f(x_1, x_2)) F_N(y_1, y_2) dy_1 dy_2 \right\|_2 \\
&\leq \sup_{|y_i| < \delta} \left(\|f(x_1 - y_1, x_2 - y_2) - f(x_1, x_2)\|_2 \right) \\
&+ 4 \|f(x_1, x_2)\|_2 \int_{\substack{|y_1| < \delta \\ |y_2| > \delta}} |F_N(y_1, y_2)| \, dy_1 dy_2 \\
&+ 2 \|f(x_1, x_2)\|_2 \int_{|y_i| > \delta} |F_N(y_1, y_2)| \, dy_1 dy_2 \\
&< M\delta + 4 \int_{|y_2| > \delta} |F_N(y_2)| \, dy_2 + \frac{2(2 + \frac{1}{\delta})^2}{N^2} \\
&\leq M\delta + \frac{4(2 + \frac{1}{\delta})}{N} + \frac{2(2 + \frac{1}{\delta})^2}{N^2} \\
&\leq 0.001 + 0.004 + 0.004 \\
&< 0.01.
\end{aligned}
$$

□

Remark 4.2.8. In the previous proof we used the Fejér kernel to obtain an estimate for $\|S_N f - f\|_2$. Alternately, you could use the uniform estimate found on the bottom of page 115 of [38].

Lemma 4.2.9. Let $\phi \in C^0(\mathbb{T}^2)$, $G \in \mathcal{H}(\mathbb{T}^2)$, and σ be a rotation of $\mathbb{T}^2$. Suppose that $\phi_0 \in C^0(\mathbb{T}^2)$ is such that $\|\phi - \phi_0\|_2 < 0.01$. If $\sigma \in R_{\phi_0 \circ G}(0.9)$ then $\sigma \in R_{\phi \circ G}(0.93)$.

Proof.

Let $t \in \mathbb{N}$. Now it suffices to observe the following:

$$\begin{aligned}
\left|\langle \phi \circ G(\sigma^t) | \phi \circ G \rangle\right| &= \left|\langle (\phi - \phi_0) \circ G(\sigma^t) + \phi_0 \circ G(\sigma^t) | (\phi - \phi_0) \circ G + \phi_0 \circ G \rangle\right| \\
&\leq \left|\langle \phi_0 \circ G(\sigma^t) | \phi_0 \circ G \rangle\right| + \left|\langle (\phi - \phi_0) \circ G(\sigma^t) | (\phi - \phi_0) \circ G \rangle\right| \\
&+ \left|\langle (\phi - \phi_0) \circ G(\sigma^t) | \phi_0 \circ G \rangle\right| + \left|\langle \phi_0 \circ G(\sigma^t) | (\phi - \phi_0) \circ G \rangle\right| \\
&\leq \left|\langle \phi_0 \circ G(\sigma^t) | \phi_0 \circ G \rangle\right| + \left\|\phi - \phi_0\right\|_2^2 + 2 \left\|\phi - \phi_0\right\|_2 \left\|\phi_0\right\|_2 \\
&< \left|\langle \phi_0 \circ G(\sigma^t) | \phi_0 \circ G \rangle\right| + 3(0.01).
\end{aligned}$$

□

We are now ready to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Let (n_m) be a sequence of natural numbers satisfying

$$n_{m+1} \geq \psi(n_m) n_m$$

where $\psi(x) = x^{x^3}$. From Lemma 4.2.5 we obtain irrationals α_1, α_2 such that

$$\frac{1}{4(n_m)^2} < |n_m \alpha_i| < \frac{1}{2(n_m)^2}$$

for $i = 1, 2$. We will need both of these bounds later in the proof. Let $\alpha = (\alpha_1, \alpha_2)$ and $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be a rotation defined by $\sigma(x_1, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$. By the nature of our choice of α, (n_m) is a uniform rigidity sequence for σ.

Let $\{\phi_i\}$ be a sequence of trigonometric polynomials that are a countable dense subset of $C^0(\mathbb{T}^2)$. Let $M_i > 0$ be the modulus of continuity of ϕ_i, that is

$$\omega_{\phi_i}(\delta) = \sup_{d(x,x')<\delta} |\phi_i(x) - \phi_i(x')| \leq M_i \delta.$$

Suppose that the ϕ_i are ordered so that the corresponding sequence, (M_i), is nondecreasing.

Going back to the previous chapter, let $\mathcal{H}(\mathbb{T}^2)$ be the set of measure-preserving homeomorphisms of $\mathbb{T}^2$ and the set $O(\sigma)$ be defined as

$$O(\sigma) = \{G \circ \sigma \circ G^{-1} : G \in \mathcal{H}(\mathbb{T}^2)\}.$$

Let $\mathcal{O} = \overline{O(\sigma)}$.

Recall that the set of weakly mixing homeomorphisms in $\mathcal{O} = \overline{O(\sigma)}$ is a dense G_δ set and is equal to

$$\bigcap_{i=1}^{\infty} R_{\phi_i}(0.99)$$

where $R_{\phi_i}(0.99)$ is the set of $T \in \mathcal{O}$ such that there exists $t \in \mathbb{N}$ with the property that

$$\left|\langle U_T^t(\phi_i)|\phi_i\rangle\right| < 0.99.$$

We are going to show that successive conjugations of σ converge to a weakly mixing homeomorphism that is uniformly rigid with respect to (n_m). We will form a nested sequence of closed balls B_i such that each $B_i \subseteq R_{\phi_i}(0.99)$. Then $\bigcap_{i=1}^{\infty} B_i$ will contain a homeomorphism T_0 that is weakly mixing. The center of each B_i will be a conjugation of σ and will be chosen carefully so that in the end, T_0 will be the uniform limit of these conjugations and (n_m) will be a uniform rigidity sequence for T_0. We will use Lemma 4.2.6 to help us form this nested sequence of closed balls. This will be an inductive construction.

To begin let $0 < \epsilon_1 < 1$. The first step is to find $G_1 \in \mathcal{H}(\mathbb{T}^2)$ such that

1. $d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) < \epsilon_1$

2. $G_1 \circ \sigma \circ G_1^{-1} \in R_{\phi_1}(0.93)$ (ie. $\sigma \in R_{\phi_1 \circ G_1}(0.93) \subseteq R_{\phi_1 \circ G_1}(0.99)$)

The homeomorphism G_1 will have a similar form as the homeomorphism G in the generic argument in Section 3.1.1. However, this construction is more technical because we need explicit constants in order to use the given growth rate to form our first closed ball B_1. In constructing G_1 we will use Lemma 4.2.6 to ensure that our oscillatory integral is small. To that end, let $g(y) = \sin(2\pi y)$ be defined on $\mathbb{T}$. The Lipschitz constant for g is 2π and from here on will be denoted by C (ie. $C = 2\pi$). Therefore the modulus of continuity of g is

$$\omega_g(\delta) = \sup_{|x-x'|<\delta} |g(x) - g(x')| \leq C\delta.$$

Suppose the trigonometric polynomial ϕ_1 is of the form

$$\phi_1(x_1, x_2) = \sum_{|\ell_i|\leq N_1} c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}.$$

Recall from the category argument that the homeomorphism G that we define in Section 3.1.1 is skewed in one variable. Similarly, the homeomorphism G_1 that we construct here will be skewed in one variable. To decide which variable to skew, notice that at least one of the following must occur:

1. $\sum_{|\ell_2|\leq N_1} |c_{0,\ell_2}|^2 < 0.6$

2. $\sum_{|\ell_1|\leq N_1} |c_{\ell_1,0}|^2 < 0.6$

WLOG suppose $\sum_{|\ell_2|\leq N_1} |c_{0,\ell_2}|^2 < 0.6$. This means that if we skew G_1 in the first variable we can use the oscillation to ensure that ϕ_1 is not an eigenfunction.

Let $K_1 = 260N_1^9$ (note that this is the K value from Lemma 4.2.6), $C_1 = 2 + K_1 C$, and $\delta_1 = \frac{\epsilon_1}{K_1 C}$. If $|x - x'| < \delta_1$, then $|K_1 g(x) - K_1 g(x')| < \epsilon_1$. Since (n_m) is an increasing sequence, there exists M such that $n_M > \max(\frac{1}{\delta_1}, C_1, (17334\pi)2^{180} M_2^9)$. WLOG suppose that $n_1 > \max(\frac{1}{\delta_1}, C_1, (17334\pi)2^{180}M_2^9)$. Let $G_1 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G_1(x_1, x_2) = (x_1 + f_1(x_2), x_2)$$

where

$$f_1(x_2) = K_1 g(n_1 x_2) = 260 N_1^9 \sin(2\pi n_1 x_2).$$

The modulus of continuity of G_1 satisfies $\omega_{G_1}(\delta) \leq C_1 n_1 \delta$.

Now we need to show that $G_1 \in \mathcal{H}(\mathbb{T}^2)$ satisfies (1) (ie. $d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) < \epsilon_1$). Consider,

$$G_1 \circ \sigma \circ G_1^{-1}(x_1, x_2) - \sigma(x_1, x_2) = (f_1(x_2 + \alpha_2) - f_1(x_2), 0)$$

and

$$G_1 \circ \sigma^{-1} \circ G_1^{-1}(x_1, x_2) - \sigma^{-1}(x_1, x_2) = (f_1(x_2 - \alpha_2) - f_1(x_2), 0).$$

If we use the two statements above, we see that

$$\begin{aligned} d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) &= \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, \sigma) + \overline{d}(G_1 \circ \sigma^{-1} \circ G_1^{-1}, \sigma^{-1}) \\ &= |f_1(x_2 + \alpha_2) - f_1(x_2)| + |f_1(x_2 - \alpha_2) - f_1(x_2)| \\ &= |K_1 g(n_1 x_2 + n_1 \alpha_2) - K_1 g(n_1 x_2)| + |K_1 g(n_1 x_2 - n_1 \alpha_2) - K_1 g(n_1 x_2)|. \end{aligned}$$

Recall that α_2 was chosen to satisfy $|n_1 \alpha_2| < \frac{1}{2(n_1)^2} < \frac{1}{2n_1} < \frac{\delta_1}{2}$. Thus $d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) < \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1$ and we have (1).

At this point our first conjugation of σ remains close to σ. Now we need to check that the conjugation of σ we chose belongs to $R_{\phi_1}(0.93)$, that is $\sigma \in R_{\phi_1 \circ G_1}(0.93)$. Our goal is to find $t_1 \in \mathbb{N}$ such that $|\langle U_\sigma^{t_1}(\phi_1 \circ G_1) | \phi_1 \circ G_1 \rangle| < 0.93$. Recall that in our generic argument in Section 3.1.1, we only showed the existence of such a t_1 using Wiener's Lemma. For this proof we need to explicitly calculate t_1. This is where the upper and lower bounds on $|n_m \alpha_2|$ come into play.

Consider $t_1 = n_1^2$. Then $\frac{1}{4} < |t_1 n_1 \alpha_2| < \frac{1}{2}$ and $\frac{1}{4} < \{t_1 n_1 \alpha_2\} < \frac{3}{4}$ where $\{\cdot\}$ denotes fractional part.

We can now use Lemma 4.2.6 with $N = N_1$ and $A = \{t_1 n_1 \alpha_2\}$ to show that

$$\left|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle\right| < 0.93.$$

First consider

$$\begin{aligned}
\left|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle\right| &= \left|\langle(\phi_1 \circ G_1)(\sigma^{t_1})|\phi_1 \circ G_1\rangle\right| \\
&= \left|\int_{[0,1)^2} \phi_1 \circ G_1(x_1 + t_1\alpha_1, x_2 + t_1\alpha_2)\overline{\phi_1 \circ G_1(x_1, x_2)}dx_1dx_2\right| \\
&\leq \sum_{|\ell_i| \leq N_1, \left|\ell_i'\right| \leq N_1} \left|c_{\ell_1,\ell_2}\bar{c}_{\ell_1',\ell_2'}e^{2\pi i(\ell_1 t_1 \alpha_1 + \ell_2 t_1 \alpha_2)}\right| \\
&\cdot \left|\int_{[0,1)^2} e^{2\pi i[(\ell_1 - \ell_1')x_1 + \ell_1 f_1(x_2 + t_1\alpha_2) - \ell_1' f_1(x_2) + (\ell_2 - \ell_2')x_2]}dx_1dx_2\right| \\
&\leq \sum_{|\ell_i| \leq N_1, \left|\ell_2'\right| \leq N_1} \left|c_{\ell_1,\ell_2}\right| \left|\bar{c}_{\ell_1,\ell_2'}\right| \\
&\cdot \left|\int_0^1 e^{2\pi i[\ell_1 K_1(g(n_1x_2 + A) - g(n_1x_2)) + (\ell_2 - \ell_2')x_2]}dx_2\right| \\
&\leq \sum_{\ell_1 \neq 0, |\ell_i| \leq N_1, \left|\ell_2'\right| \leq N_1} \left|\int_0^1 e^{2\pi i[\ell_1 K_1(g(n_1x_2 + A) - g(n_1x_2)) + (\ell_2 - \ell_2')x_2]}dx_2\right| \\
&+ \sum_{|\ell_2| \leq N_1} \left|c_{0,\ell_2}\right|^2 \\
&< \sum_{\ell_1 \neq 0, |\ell_i| \leq N_1, \left|\ell_2'\right| \leq N_1} \left|\int_0^1 e^{2\pi i[\ell_1 K_1(g(n_1x_2 + A) - g(n_1x_2)) + (\ell_2 - \ell_2')x_2]}dx_2\right| \\
&+ 0.6.
\end{aligned}$$

Notice that

$$\int_0^1 e^{2\pi i[\ell_1 K_1(g(n_1x_2 + A) - g(n_1x_2)) + (\ell_2 - \ell_2')x_2]}dx_2$$

is a Fourier coefficient of the function $e^{2\pi i[\ell_1 K_1(g(n_1x_2+A)-g(n_1x_2))]}$. Since the set of nonzero Fourier coefficients of $e^{2\pi i[\ell_1 K_1(g(n_1x_2+A)-g(n_1x_2))]}$ is the same as that of $e^{2\pi i[\ell_1 K_1(g(x_2+A)-g(x_2))]}$ and by Lemma 4.2.6 $\left|\int_{[0,1)^2} e^{2\pi i[\ell_1 K_1(g(x_2+A)-g(x_2))+(\ell_2-\ell_2')x_2]}dx_2\right| < \frac{0.3}{N_1^3}$ we have

$$\left|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle\right| < N_1^3\left(\frac{0.3}{N_1^3}\right) + 0.6 < 0.93.$$

Thus (2) is satisfied (ie. $G_1 \circ \sigma \circ G_1^{-1} \in R_{\phi_1}(0.93)$).

Now that we have $G_1 \circ \sigma \circ G_1^{-1} \in R_{\phi_1}(0.93)$ we proceed with finding a closed ball, which we will call B_1, centered at $G_1 \circ \sigma \circ G_1^{-1}$ such that $B_1 \subseteq R_{\phi_1}(0.99)$. We need to explicitly calculate the radius of B_1 to ensure that $B_1 \subseteq R_{\phi_1}(0.99)$. Let

$$\kappa_1 = \frac{0.06}{2n_1(C_1n_1)^{2n_1^2-1}}$$

and

$$B_1 = \left\{T \in \mathcal{O} : d_u(G_1 \circ \sigma \circ G_1^{-1}, T) \leq \kappa_1\right\}.$$

Notice that for any $n \in \mathbb{N}$ and $T \in \mathcal{O}$ we have $d_u(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) = \overline{d}(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) + \overline{d}(T^{-n}, G_1 \circ \sigma^{-n} \circ G_1^{-1})$. Consider the following:

$$\begin{aligned}
\overline{d}(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) &= \overline{d}((G_1 \circ \sigma \circ G_1^{-1})(G_1 \circ \sigma^{n-1} \circ G_1^{-1}), T(T^{n-1})) \\
&\leq \overline{d}((G_1 \circ \sigma \circ G_1^{-1})(G_1 \circ \sigma^{n-1} \circ G_1^{-1}), (G_1 \circ \sigma \circ G_1^{-1})(T^{n-1})) \\
&+ \overline{d}((G_1 \circ \sigma \circ G_1^{-1})(T^{n-1}), T(T^{n-1})) \\
&\leq \omega_{G_1\circ\sigma\circ G_1^{-1}}(\overline{d}(G_1 \circ \sigma^{n-1} \circ G_1^{-1}, T^{n-1})) + \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T) \\
&\leq \sum_{i=0}^{n-1} \omega^i_{G_1\circ\sigma\circ G_1^{-1}}(\overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T)) \\
&\leq \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T)\sum_{i=0}^{n-1}[(C_1n_1)^2]^i \\
&= \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T)\frac{(C_1n_1)^{2n} - 1}{(C_1n_1)^2 - 1}
\end{aligned}$$

$$\leq \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T)(C_1 n_1)^{2n-1}$$

where $\omega^0 = Id$. A similar calculation can be carried out to yield $\overline{d}(T^{-n}, G_1 \circ \sigma^{-n} \circ G_1^{-1}) \leq \overline{d}(G_1 \circ \sigma^{-1} \circ G_1^{-1}, T^{-1})(C_1 n_1)^{2n-1}$. Thus

$$d_u(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) \leq d_u(G_1 \circ \sigma \circ G_1^{-1}, T)(C_1 n_1)^{2n-1}.$$

We will show that $B_1 \subseteq R_{\phi_1}(0.99)$. Let $T \in B_1$. In this case

$$\begin{aligned} \left|\langle U_T^{t_1}(\phi_1)|\phi_1\rangle\right| &= \left|\langle[\phi_1(T^{t_1}) - \phi_1(G_1 \circ \sigma^{t_1} \circ G_1^{-1})] + \phi_1(G_1 \circ \sigma^{t_1} \circ G_1^{-1})|\phi_1\rangle\right| \\ &\leq \left\|\phi_1(T^{t_1}) - \phi_1(G_1 \circ \sigma^{t_1} \circ G_1^{-1})\right\|_2 \|\phi_1\|_2 + \left|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle\right| \\ &< M_1 d_u(G_1 \circ \sigma \circ G_1^{-1}, T)(C_1 n_1)^{2t_1 - 1} + 0.93 \\ &\leq n_1 \kappa_1 (C_1 n_1)^{2n_1^2 - 1} + 0.93 \\ &< 0.99. \end{aligned}$$

Hence we have the desired result (ie. $B_1 \subseteq R_{\phi_1}(0.99)$).

Thus far we have constructed the closed ball B_1 centered at $G_1 \circ \sigma \circ G_1^{-1}$ such that $B_1 \subseteq R_{\phi_1}(0.99)$. The next step in our inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1 \circ G_2 \circ \sigma \circ G_2^{-1} \circ G_1^{-1} \in R_{\phi_2}(0.99) \cap B_1$ and then construct the closed ball B_2 centered at $G_1 \circ G_2 \circ \sigma \circ G_2^{-1} \circ G_1^{-1}$ such that $B_2 \subseteq R_{\phi_2}(0.99) \cap B_1$. To that end, let $\epsilon_2 = \frac{\kappa_1}{2C_1 n_1} < \epsilon_1$. Now similar to before we want to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that

1. $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$

2. $G_2 \circ \sigma \circ G_2^{-1} \in R_{\phi_2 \circ G_1}(0.93)$ (ie. $\sigma \in R_{\phi_2 \circ G_1 \circ G_2}(0.93) \subseteq R_{\phi_2 \circ G_1 \circ G_2}(0.99)$)

To proceed as before, let $\phi_2' = \phi_2 \circ G_1$ and suppose that the Fourier expansion has the form:

$$\phi_2'(x_1, x_2) = \sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}.$$

Now we need to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$ and $\sigma \in R_{\phi_2' \circ G_2}(0.93)$. The first step is to use an approximation of ϕ_2' in our calculations. We will use Lemma 4.2.7 and Lemma 4.2.9 to aid us in this step. Let $N_2 = 2^{20} M_2 C_1 n_1$ (this is the N value from Lemma 4.2.7) and

$$\widetilde{\phi_2'}(x_1, x_2) = S_{N_2}\phi_2'(x_1, x_2) = \sum_{|\ell_i| \leq N_2} c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}.$$

By Lemma 4.2.7 $\left\|\widetilde{\phi_2'} - \phi_2'\right\|_2 < 0.01$ and therefore it suffices to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $\sigma \in R_{\widetilde{\phi_2'} \circ G_2}(0.9)$ by Lemma 4.2.9.

The next step is to decide which variable to skew. As before, notice that at least one of the following must occur:

1. $\sum_{|\ell_2| \leq N_2} |c_{0,\ell_2}|^2 < 0.6$

2. $\sum_{|\ell_1| \leq N_2} |c_{\ell_1,0}|^2 < 0.6$

WLOG suppose $\sum_{|\ell_1| \leq N_2} |c_{\ell_1,0}|^2 < 0.6$. This means that if we skew G_2 in the second variable we can use the oscillation to ensure that ϕ_2 is not an eigenfunction.

Let $K_2 = 260N_2^9$ (note that this is the K value from Lemma 4.2.6), $C_2 = 2 + K_2 C$, and $\delta_2 = \frac{\epsilon_2}{K_2 C}$. If $|x - x'| < \delta_2$, then $|K_2 g(x) - K_2 g(x')| < \epsilon_2$. If $n_2 \geq \max(C_2, (17334\pi)2^{180} M_3^9)$ then we proceed similar to before by letting $G_2(x_1, x_2) = (x_1, x_2 + f_2(x_1))$ where $f_2(x_1) = K_2 g(n_2 x_1) = 260 N_2^9 \sin(2\pi n_2 x_1)$. The modulus of continuity of G_2 satisfies $\omega_{G_2}(\delta) \leq C_2 n_2 \delta$. If this is not the case then we use ϕ_1 in place of ϕ_i at each stage of our induction until a term of the sequence (n_m) exceeds $\max(C_2, (17334\pi)2^{180} M_3^9)$.

We now proceed with the induction under the assumption that $n_2 \geq \max(C_2, (17334\pi) 2^{180} M_3^9)$. Similar to before, to show that $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$ we need to check that $\frac{1}{2n_2} < \frac{\delta_2}{2}$ or $n_2 > \frac{1}{\delta_2}$. Observe

$$\begin{aligned}
\frac{1}{\delta_2} &= \frac{260 N_2^9 C(2C_1 n_1)}{\kappa_1} \\
&\leq ((17334\pi) 2^{180} M_2^9)(C_1 n_1)^{10}(2n_1)(C_1 n_1)^{2n_1^2 - 1} \\
&\leq n_1 (2n_1)(C_1 n_1)^{2n_1^2 + 9} \\
&< n_1^{n_1^3} \cdot n_1 \\
&= \psi(n_1) n_1 \\
&\leq n_2.
\end{aligned}$$

Therefore $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$.

Next we need to show that $\sigma \in R_{\widetilde{\phi_2'} \circ G_2}(0.9)$. Our goal is now to find $t_2 \in \mathbb{N}$ such that $\left| \langle U_\sigma^{t_2}(\widetilde{\phi_2'} \circ G_2) | \widetilde{\phi_2'} \circ G_2 \rangle \right| < 0.9$. Let $t_2 = n_2^2$. It follows that $\frac{1}{4} < |t_2 n_2 \alpha_1| < \frac{1}{2}$ and $\frac{1}{4} < \{t_2 n_2 \alpha_1\} < \frac{3}{4}$. We can now use Lemma 4.2.6 with $N = N_2$ and $A = \{t_2 n_2 \alpha_1\}$. Consider

$$\begin{aligned}
\left| \langle U_\sigma^{t_2}(\widetilde{\phi_2'} \circ G_2) | \widetilde{\phi_2'} \circ G_2 \rangle \right| &= \left| \langle (\widetilde{\phi_2'} \circ G_2)(\sigma^{t_2}) | \widetilde{\phi_2'} \circ G_2 \rangle \right| \\
&= \left| \int_{[0,1)^2} \widetilde{\phi_2'} \circ G_2(x_1 + t_2\alpha_1, x_2 + t_2\alpha_2) \overline{\widetilde{\phi_2'} \circ G_2(x_1, x_2)} dx_1 dx_2 \right| \\
&\leq \sum_{|\ell_i| \leq N_2, \left|\ell_i'\right| \leq N_2} \left| c_{\ell_1,\ell_2} \overline{c}_{\ell_1',\ell_2'} e^{2\pi i(\ell_1 t_2 \alpha_1 + \ell_2 t_2 \alpha_2)} \right| \\
&\quad \cdot \left| \int_{[0,1)^2} e^{2\pi i[(\ell_1 - \ell_1')x_1 + \ell_2 f_2(x_1 + t_2\alpha_1) - \ell_2' f_2(x_1) + (\ell_2 - \ell_2')x_2]} dx_1 dx_2 \right| \\
&\leq \sum_{|\ell_i| \leq N_2, \left|\ell_1'\right| \leq N_2} \left| c_{\ell_1,\ell_2} \right| \left| \overline{c}_{\ell_1',\ell_2} \right| \\
&\quad \cdot \left| \int_0^1 e^{2\pi i[\ell_2 K_2(g(n_2 x_1 + A) - g(n_2 x_1)) + (\ell_1 - \ell_1')x_1]} dx_1 \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\ell_2\neq 0, |\ell_i|\leq N_2, \left|\ell_1'\right|\leq N_2} \left| \int_0^1 e^{2\pi i[\ell_2 K_2(g(n_2x_1+A)-g(n_2x_1))+(\ell_1-\ell_1')x_1]}dx_1 \right| \\
&+ \sum_{|\ell_1|\leq N_2} \left|c_{\ell_1,0}\right|^2 \\
&< \sum_{\ell_2\neq 0, |\ell_i|\leq N_2, \left|\ell_1'\right|\leq N_2} \left| \int_0^1 e^{2\pi i[\ell_2 K_2(g(n_2x_1+A)-g(n_2x_1))+(\ell_1-\ell_1')x_1]}dx_1 \right| \\
&+ 0.6.
\end{aligned}
$$

Similar to before using Lemma 4.2.6

$$
\left|\langle U_\sigma^{t_2}(\widetilde{\phi_2'}\circ G_2)|\widetilde{\phi_2'}\circ G_2\rangle\right| < N_2^3\left(\frac{0.3}{N_2^3}\right) + 0.6 = 0.9.
$$

Therefore $\sigma \in R_{\widetilde{\phi_2'}\circ G_2}(0.9)$ which implies that $\sigma \in R_{\phi_2\circ G_1\circ G_2}(0.93)$. Thus (2) is satisfied (ie. $G_2\circ\sigma\circ G_2^{-1} \in R_{\phi_2\circ G_1}(0.93)$).

Recall that our goal for the second step in the inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1} \in R_{\phi_2}(0.99)\cap B_1$. Thus far we have constructed $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1} \in R_{\phi_2}(0.99)$. We now need to check that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1} \in B_1$. To that end, observe

$$
\begin{aligned}
\overline{d}(G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1}, G_1\circ\sigma\circ G_1^{-1}) &= \overline{d}(G_1(G_2\circ\sigma\circ G_2^{-1}), G_1(\sigma)) \\
&\leq C_1 n_1\left(\frac{\epsilon_2}{2}\right) \\
&= C_1 n_1\left(\frac{\kappa_1}{4C_1 n_1}\right) \\
&= \frac{\kappa_1}{4}.
\end{aligned}
$$

Similarly $\overline{d}(G_1\circ G_2\circ\sigma^{-1}\circ G_2^{-1}\circ G_1^{-1}, G_1\circ\sigma^{-1}\circ G_1^{-1}) \leq \frac{\kappa_1}{4}$ and $d_u(G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1}, G_1\circ\sigma\circ G_1^{-1}) \leq \frac{\kappa_1}{2}$ which implies that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1} \in B_1 \subseteq R_{\phi_1}(0.99)$.

Let $\overline{G_2} := G_1\circ G_2$. With this new notation we have shown that $\overline{G_2}\circ\sigma\circ(\overline{G_2})^{-1} \in$

$R_{\phi_2}(0.99) \cap B_1$. Now we need to find a closed ball, call it B_2, centered at $\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}$ that is a subset of $R_{\phi_2}(0.99) \cap B_1$. Let

$$\kappa_2 = \frac{0.06}{2^2 n_2 (C_1 C_2 n_1 n_2)^{2n_2^2-1}}$$

and

$$B_2 = \left\{T \in \mathcal{O} : d_u(\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}, T) \leq \kappa_2\right\}.$$

We will first show that $B_2 \subseteq R_{\phi_2}(0.99)$. Let $T \in B_2$ and consider

$$\begin{aligned}
\left|\langle U_T^{t_2}(\phi_2)|\phi_2\rangle\right| &= \left|\langle[\phi_2(T^{t_2}) - \phi_2(\overline{G_2} \circ \sigma^{t_2} \circ (\overline{G_2})^{-1})] + \phi_2(\overline{G_2} \circ \sigma^{t_2} \circ (\overline{G_2})^{-1})|\phi_2\rangle\right| \\
&\leq \left\|\phi_2(T^{t_2}) - \phi_2(\overline{G_2} \circ \sigma^{t_2} \circ (\overline{G_2})^{-1})\right\|_2 \|\phi_2\|_2 + \left|\langle U_\sigma^{t_2}(\phi_2 \circ \overline{G_2})|\phi_2 \circ \overline{G_2}\rangle\right| \\
&< M_2 d_u(\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}, T)(C_1 C_2 n_1 n_2)^{2t_2-1} + 0.93 \\
&\leq n_2 \kappa_2 (C_1 C_2 n_1 n_2)^{2n_2^2-1} + 0.93 \\
&< 0.99.
\end{aligned}$$

Hence we have the desired result, that is $B_2 \subseteq R_{\phi_2}(0.99)$.

Next we will show that $B_2 \subseteq B_1$. Let $T \in B_2$ and consider,

$$\begin{aligned}
d_u(T, G_1 \circ \sigma \circ G_1^{-1}) &\leq d_u(T, \overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}) \\
&\quad + d_u(\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}, G_1 \circ \sigma \circ G_1^{-1}) \\
&\leq \kappa_2 + \frac{\kappa_1}{2} \\
&< \frac{\kappa_1}{2} + \frac{\kappa_1}{2} \\
&= \kappa_1.
\end{aligned}$$

Therefore $B_2 \subseteq R_{\phi_2}(0.99) \cap B_1$.

Thus far in our inductive procedure we have constructed two closed nested balls $B_1 \supseteq B_2$

centered at conjugations of σ such that $B_1 \subseteq R_{\phi_1}(0.99)$ and $B_2 \subseteq R_{\phi_2}(0.99)$. The general inductive step can be carried out in the same way and for brevity we don't include it here.

In the end, this inductive procedure produces a nested sequence of closed balls (B_m) and a sequence $(\overline{G_m})$ of homeomorphisms where each G_m is of the form

$$G_m(x_1, x_2) = (x_1 + f_m(x_2), x_2)$$

or

$$G_m(x_1, x_2) = (x_1, x_2 + f_m(x_1).$$

After the m-th stage of the construction has been completed we have a homeomorphism G_m in one of the above forms that satisfies:

1. $d_u(G_m \circ \sigma \circ G_m^{-1}, \sigma) < \epsilon_m$ where $\epsilon_m = \frac{\kappa_{m-1}}{2C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}$

2. $G_m \circ \sigma \circ G_m^{-1} \in R_{\phi_m \circ G_1 \circ \cdots \circ G_{m-1}}(0.93)$ or $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1} \in R_{\phi_m}(0.93)$.

At the end of this stage we also have a closed ball B_m centered at $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1}$ with radius

$$\kappa_m = \frac{0.06}{2^m n_m \left(C_1 \cdots C_m n_1 \cdots n_m\right)^{2n_m^2 - 1}}$$

such that $B_m \subseteq R_{\phi_m}(0.99)$. Recall that we are working in a complete metric space. Let $T_0 = \bigcap_{m=1}^{\infty} B_m$. Therefore T_0 is weakly mixing. Also, $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1}$ converges uniformly to T_0 since $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1}$ is the center of B_m.

Now that we have T_0 which is weakly mixing, we need to show that it is uniformly rigid with respect to (n_m). To do this, we need to make a preliminary estimate. First notice that, depending on which variable is skewed, we have either

$$G_m \circ \sigma^{n_m} \circ G_m^{-1}(x_1, x_2) - \sigma^{n_m}(x_1, x_2) = (K_m g(n_m x_2 + n_m^2 \alpha_2) - K_m g(n_m x_2), 0)$$

or

$$G_m \circ \sigma^{n_m} \circ G_m^{-1}(x_1, x_2) - \sigma^{n_m}(x_1, x_2) = (0, K_m g(n_m x_1 + n_m^2 \alpha_1) - K_m g(n_m x_1)).$$

In either case

$$\left|n_m^2 \alpha_i\right| < \frac{1}{2n_m} < \frac{\delta_m}{2}$$

and we can conclude that $d_u(G_m \circ \sigma^{n_m} \circ G_m^{-1}, \sigma^{n_m}) < \epsilon_m$. Now observe the following:

$$\begin{aligned}
\overline{d}(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) &= \overline{d}(\overline{G_{m-1}}(G_m \circ \sigma^{n_m} \circ G_m^{-1}), \overline{G_{m-1}}(\sigma^{n_m})) \\
&\leq (C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}) \left(\frac{\epsilon_m}{2}\right) \\
&= \frac{\kappa_{m-1}}{4}.
\end{aligned}$$

Hence $d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \leq \frac{\kappa_{m-1}}{2}$.

The final estimate will show that T_0 is uniformly rigid with respect to (n_m). Observe

$$\begin{aligned}
d_u(T_0^{n_m}, Id) &\leq d_u(T_0^{n_m}, \overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}) \\
&+ d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \\
&+ d_u(\overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}, Id) \\
&= d_u(T_0^{n_m}, \overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}) \\
&+ d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \\
&+ \overline{d}(\overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}, Id) \\
&+ \overline{d}(\overline{G_{m-1}} \circ \sigma^{-n_m} \circ (\overline{G_{m-1}})^{-1}, Id) \\
&= d_u(T_0^{n_m}, \overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}) \\
&+ d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \\
&+ \overline{d}(\overline{G_{m-1}}(\sigma^{n_m}), \overline{G_{m-1}}(Id))
\end{aligned}$$

$$
\begin{aligned}
&+ \overline{d}(\overline{G_{m-1}}(\sigma^{-n_m}), \overline{G_{m-1}}(Id)) \\
&\leq \kappa_m (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m - 1} + \frac{\kappa_{m-1}}{2} \\
&+ 2\left(\frac{C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}{n_m^2}\right) \\
&\leq \left(\frac{0.06}{2^m n_m (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m^2 - 1}}\right)(C_1 \cdots C_m n_1 \cdots n_m)^{2n_m - 1} \\
&+ \frac{\kappa_{m-1}}{2} + 2\left(\frac{n_1 n_2 \cdots n_{m-1}}{n_m}\right)^2 \\
&\leq \frac{1}{2^m} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{\psi(n_{m-1})}{n_m}\right)^2 \\
&\leq \frac{1}{2^m} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{1}{n_{m-1}}\right)^2 .
\end{aligned}
$$

Thus $d_u(T_0^{n_m}, Id) \to 0$ as $m \to \infty$ and T_0 is uniformly rigid with respect to (n_m). Therefore we have constructed a weakly mixing homeomorphism that is uniformly rigid with respect to (n_m).

□

4.3 Uniform Rigidity Sequences for Topologically Weakly Mixing Homeomorphisms

We will now turn to constructing homeomorphisms of the two-torus that are topologically weakly mixing and uniformly rigid. The contents of this section are taken from [37].

The idea of constructing maps with varied behavior by conjugating rotations is due to Anosov and Katok in their seminal paper [4]. This idea of conjugating a rotation is exploited in [20] to produce a large family of topologically weakly mixing homeomorphisms of the two-torus.

Let α be an irrational number between 0 and 1 and σ be a homeomorphism of the two-torus defined as irrational rotation by α in the first coordinate and the identity in the second coordinate. Throughout this section let $\mathcal{O}$ be the closure of the set of conjugations of σ by homeomorphisms of the two-torus (this closure is taken with respect to the topology of uniform convergence of homeomorphisms and their inverses) and $\mathcal{H}(\mathbb{T}^2)$ the set of homeomorphisms defined on $\mathbb{T}^2$. The result of Glasner and Weiss in [20] discussed above can be stated precisely as:

Theorem 4.3.1 ([20])**.** There exists a dense G_δ subset $\mathcal{R}$ of $\mathcal{O}$ such that every $T \in \mathcal{R}$ is topologically weakly mixing and uniformly rigid.

We will use the above category argument by Glasner and Weiss to obtain information about the structure of uniform rigidity sequences for topologically weakly mixing homeomorphisms, in the same spirit as the previous section. The main result of this section is the following theorem.

Theorem 4.3.2. Suppose (n_m) is an increasing sequence of odd natural numbers and $\psi(n_m) = n_m^{m(4n_m^2+2)}$. If (n_m) satisfies

$$\frac{n_{m+1}}{n_m} \geq \psi(n_m)$$

then there exists a topologically weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m).

In the previous section we studied uniform rigidity sequences for weakly mixing homeomorphisms of the two-torus equipped with Lebesgue measure. In that context, the required growth rate is faster and a weakly mixing homeomorphism is produced, as opposed to a topologically weakly mixing homeomorphism.

4.3.1 Construction

In [20] Glasner and Weiss produced a large family of homeomorphisms of the two-torus that are topologically weakly mixing. We will use the inherent structure of their category argument to determine a sufficient growth rate for a sequence of natural numbers that guarantees the existence of a topologically weakly mixing homeomorphism of the two-torus that is uniformly rigid with respect to the given sequence.

Throughout this subsection we will be working on the two torus $\mathbb{T}^2$. We will be using the model of $\mathbb{T}^2$ where it is viewed as $[0,1)^2$ and the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context).

The first step in our construction is to choose an irrational rotation that we will then conjugate. The proof of the following lemma is the same as the proof of the corresponding lemma from the previous section.

Lemma 4.3.3. Suppose (n_m) is an increasing sequence of natural numbers and let $\psi(n_m) = n_m^{m(4n_m^2+2)}$. If (n_m) satisfies $\frac{n_{m+1}}{n_m} \geq \psi(n_m)$ and (h'_m) is an increasing sequence of natural numbers satisfying $\frac{1/h'_m}{n_m/n_{m+1}} \to \infty$ as $m \to \infty$ where $h'_m > n_m^2$ then there exists α such that

$$\frac{1}{h'_m} < |n_m \alpha| < \frac{1}{2(n_m)^2}.$$

Proof of Theorem 4.3.2.

Let (n_m) be a sequence of odd natural numbers satisfying

$$n_{m+1} \geq \psi(n_m) n_m$$

where $\psi(n_m) = n_m^{m(4n_m^2+2)}$. Let (h'_m) be a sequence that satisfies the conditions of Lemma 4.3.3 (this sequence will be easier to point out at each stage of our construction). From Lemma 4.3.3 we obtain an irrational α such that

$$\frac{1}{2h'_m} < |n_m \alpha| < \frac{1}{2(n_m)^2}.$$

We will need both of these bounds later in the proof. Let $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$\sigma(x,y) = (x+\alpha, y)$. By the nature of our choice of α, (n_m) is a uniform rigidity sequence for σ.

Define the set $O(\sigma)$ as

$$O(\sigma) = \left\{ G^{-1} \circ \sigma \circ G \;:\; G \in \mathcal{H}(\mathbb{T}^2) \right\}.$$

This set will be considered as a subset of all homeomorphisms of $\mathbb{T}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. Let $\mathcal{O} = \overline{O(\sigma)}$ with the closure taken in the above topology.

Before we proceed, we need to define the set of topologically weakly mixing homeomorphisms of $\mathcal{O}$ as a dense G_δ set. Consider the countable collection of open dyadic cubes in $\mathbb{T}^2$, where a dyadic cube of order i has the form $\left(\frac{\ell}{2^i}, \frac{\ell+1}{2^i}\right) \times \left(\frac{m}{2^i}, \frac{m+1}{2^i}\right)$ where $\ell, m \in \{0, 1, \ldots, 2^i - 1\}$. Now, select open dyadic cubes $U_1^1, U_2^1, U_3^1, U_4^1$ such that each U_j^1 has order 1. For the second step select open dyadic cubes $U_1^2, U_2^2, U_3^2, U_4^2$ such that each cube still has order 1 and $U_1^1 \times U_2^1 \times U_3^1 \times U_4^1 \neq U_1^2 \times U_2^2 \times U_3^2 \times U_4^2$ as a subset of $\mathbb{T}^8$. We continue in this manner until we have exhausted all selections of four open dyadic cubes of order 1 and then proceed to dyadic cubes of order 2. In this way define $U_1^i, U_2^i, U_3^i, U_4^i$ for $i \geq 1$.

Define the set R_i as

$$R_i = \left\{ T \in \mathcal{O} \;:\; \textit{there exists an integer } t \textit{ with } T^t(U_1^i \times U_2^i) \cap (U_3^i \times U_4^i) \neq \emptyset \right\}.$$

Note that we are using shorthand notation when we write $T^t(U_1^i \times U_2^i)$. It is clear that $\mathcal{R} = \bigcap_i^\infty R_i$ is the set of topologically weakly mixing homeomorphisms of $\mathcal{O}$. Recall that in [20] Glasner and Weiss showed that this set is a dense G_δ subset of $\mathcal{O}$.

We are going to show that successive conjugations of σ converge to a topologically weakly mixing homeomorphism that is uniformly rigid with respect to (n_m). We will form a nested sequence of closed balls B_i such that each $B_i \subseteq R_i$. Then, $\bigcap_{i=1}^{\infty} B_i$ will contain a homeomorphism T_0 that is topologically weakly mixing. The center of each B_i will be a conjugation of σ and will be chosen carefully so that in the end, T_0 will be the uniform limit of these conjugations and (n_m) will be a uniform rigidity sequence for T_0. We will use Lemma 4.3.3 to help us form this nested sequence of closed balls. This will be an inductive construction very similar in nature to the construction in the previous section.

To begin, let $0 < \epsilon_1 < 1$. Let $U_j^1 = \left(\frac{a_j}{2}, \frac{a_j+1}{2}\right) \times \left(\frac{b_j}{2}, \frac{b_j+1}{2}\right)$, where $a_j, b_j \in \{0,1\}$ for $j = 1, 2, 3, 4$. Notice that if $G \in \mathcal{H}(\mathbb{T}^2)$ then

$$GR_iG^{-1} = \left\{T \in \mathcal{O} \ : \ \textit{there exists an integer } t \textit{ with } T^t(GU_1^i \times GU_2^i) \cap (GU_3^i \times GU_4^i) \neq \emptyset\right\}$$

where GU_j^i should be interpreted as $G\left(U_j^i\right)$. Thus for notational purposes, if $G \in \mathcal{H}(\mathbb{T}^2)$ define

$$R_{G\circ i} = \left\{T \in \mathcal{O} \ : \ \textit{there exists an integer } t \textit{ with } T^t(GU_1^i \times GU_2^i) \cap (GU_3^i \times GU_4^i) \neq \emptyset\right\}.$$

Then, if $G \in \mathcal{H}(\mathbb{T}^2)$ we have $GR_iG^{-1} = R_{G\circ i}$. The first step is to find $G_1 \in \mathcal{H}(\mathbb{T}^2)$ such that

1. $d_u(G_1^{-1} \circ \sigma \circ G_1, \sigma) < \epsilon_1$

2. $G_1^{-1} \circ \sigma \circ G_1 \in R_1$.

The homeomorphism G_1 will have a similar form as the homeomorphism G in the generic argument in [20]. However, this construction is more technical because we need explicit constants in order to use the given growth rate to form our first closed ball B_1.

Let y_1 be a point in $\left(\frac{32b_1+9}{64}, \frac{32b_1+11}{64}\right)$ and choose β_1 irrational such that $y_3 := y_1 + \beta_1 \in \left(\frac{32b_3+13}{64}, \frac{32b_3+15}{64}\right)$. Define $h_1 : \mathbb{T} \to \mathbb{T}$ by $h_1(y) = y + \beta_1$.

Similarly, let y_2 be a point in $\left(\frac{32b_2+17}{64}, \frac{32b_2+19}{64}\right)$ and choose β_2 irrational such that $y_4 := y_2 + \beta_2 \in \left(\frac{32b_4+21}{64}, \frac{32b_4+23}{64}\right)$. Define $h_2 : \mathbb{T} \to \mathbb{T}$ by $h_2(y) = y + \beta_2$. Without loss of generality, assume $\beta_1 > \beta_2$.

Now choose $x_1 \in \left(\frac{32a_1+9}{64}, \frac{32a_1+11}{64}\right)$, $x_2 \in \left(\frac{32a_2+17}{64}, \frac{32a_2+19}{64}\right)$, $x_3 \in \left(\frac{32a_3+13}{64}, \frac{32a_3+15}{64}\right)$, and $x_4 \in \left(\frac{32a_4+21}{64}, \frac{32a_4+23}{64}\right)$.

We are now ready to start building our desired function $G_1 \in \mathcal{H}(\mathbb{T}^2)$. Let $x \to g_x^1$ be a continuous function from $[0,1)$ to $\mathcal{H}(\mathbb{T})$ such that $g_0^1, g_{\frac{3}{4}}^1, g_1^1 = Id$, $g_{\frac{1}{4}}^1 = h_1$, and $g_{\frac{1}{2}}^1 = h_2$ with linear interpolation in between. Thus

$$\begin{aligned}
&g_x^1(y) = 4x\beta_1 + y \ ; \ 0 \le x \le \frac{1}{4} \\
&g_x^1(y) = \beta_1(2-4x) + \beta_2(4x-1) + y \ ; \ \frac{1}{4} \le x \le \frac{1}{2} \\
&g_x^1(y) = \beta_2(3-4x) + y \ ; \ \frac{1}{2} \le x \le \frac{3}{4} \\
&g_x^1(y) = y \ ; \ \frac{3}{4} \le x \le 1.
\end{aligned}$$

The modulus of continuity of g^1 is $\omega_{g^1}(\delta) = \sup_{|x-x'|<\delta} d_u(g_x^1, g_{x'}^1) \le 8\beta_1\delta$.

By the choice of the y_j's above we know that they are all distinct. Thus we may place non-overlapping tent maps around each y_j. To that end, let p_1 be a tent map such that $p_1(y_1) = 1$ and $p_1\left(\left(\frac{8b_1+2}{16}, \frac{8b_1+3}{16}\right)^c\right) = 0$. Similarly, define p_2, p_3, p_4 by

$$\begin{aligned}
p_2(y_2) = 1 \ ; \ p_2\left(\left(\frac{8b_2+4}{16}, \frac{8b_2+5}{16}\right)^c\right) = 0 \\
p_3(y_3) = 1 \ ; \ p_3\left(\left(\frac{8b_3+3}{16}, \frac{8b_3+4}{16}\right)^c\right) = 0 \\
p_4(y_4) = 1 \ ; \ p_4\left(\left(\frac{8b_4+5}{16}, \frac{8b_4+6}{16}\right)^c\right) = 0.
\end{aligned}$$

In this case the modulus of continuity of each p_j is $\omega_{p_j}(\delta) \le 32 \cdot 2^1\delta$. Let $M_1 = 32 \cdot 2^1$ and $C_1 = 26M_1$.

Let $\eta_1 = \frac{\epsilon_1}{16M_1}$ and $\delta_1 = \frac{\eta_1}{16}$. Then, if $|x - x'| < \delta_1$ we have $d_u(g_x^1, g_{x'}^1) < \frac{\eta_1}{2}$. Since (n_m) is an increasing sequence, there exists M such that $n_M > \max(\frac{1}{\delta_1}, 8192 \cdot 32 \cdot 2^{15} C_1)$. WLOG, suppose that $n_1 > \max(\frac{1}{\delta_1}, 8192 \cdot 32 \cdot 2^{15} C_1)$.

Define $c_j \in [0,1)$ such that

$$x_1 + c_1 = \frac{1}{4n_1}, \quad x_2 + c_2 = \frac{3}{4n_1}, \quad x_3 + c_3 = \frac{1}{4n_1} + \frac{1}{2}, \quad x_4 + c_4 = \frac{3}{4n_1} + \frac{1}{2}$$

all taken modulo one. Let f_1 be defined by

$$f_1(y) = \sum_{j=1}^{4} c_j p_j(y).$$

Then, if $|y - y'| < \eta_1$ we have

$$|f_1(y) - f_1(y')| \le \sum_{j=1}^{4} c_j \left| p_j(y) - p_j(y') \right| < 4M_1\eta_1 = \frac{\epsilon_1}{4}.$$

Now we are ready to define G_1. Let $G_1 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G_1(x,y) = \left(x + f_1(y), g^1_{n_1(x+f_1(y))}(y)\right).$$

Then,

$$G_1^{-1}(x,y) = \left(x - f_1((g^1_{n_1x})^{-1}(y)), (g^1_{n_1x})^{-1}(y)\right)$$

and

$$G_1^{-1} \circ \sigma \circ G_1(x,y) = (x + \alpha + f_1(y) - f_1(y_*), y_*)$$

where $y_* = (g^1_{n_1(x+\alpha+f_1(y))})^{-1} g^1_{n_1(x+f_1(y))}(y)$. The modulus of continuity of G_1 is given by $\omega_{G_1}(\delta) \le C_1 n_1 \delta$. It should also be noted that the modulus of continuity of G_1^{-1} is bounded by the same number.

We will first check that condition (1) is satisfied, that is $d_u(G_1^{-1} \circ \sigma \circ G_1, \sigma) < \epsilon_1$. To begin, notice that

$$G_1^{-1}\circ\sigma\circ G_1\,(x,y)-\sigma\,(x,y)=\left(f_1(y)-f_1(y_*),y-y_*\right).$$

Since $|n_1(x+\alpha+f_1(y))-n_1(x+f_1(y))|=|n_1\alpha|<\frac{1}{2(n_1)^2}<\frac{1}{n_1}<\delta_1$, we must have

$$\overline{d}\left(\left(g^1_{n_1(x+\alpha+f_1(y))}\right)^{-1}g^1_{n_1(x+f_1(y))},Id\right)<\frac{\eta_1}{2}.$$

This implies that $|y-y_*|<\eta_1$ and therefore, $|f(y)-f(y_*)|<\frac{\epsilon_1}{4}$. Thus $\overline{d}\left(G_1^{-1}\circ\sigma\circ G_1,\sigma\right)<\frac{\epsilon_1}{2}$. In a similar fashion, $\overline{d}\left(G_1^{-1}\circ\sigma^{-1}\circ G_1,\sigma^{-1}\right)<\frac{\epsilon_1}{2}$. Therefore, $d_u(G_1^{-1}\circ\sigma\circ G_1,\sigma)<\epsilon_1$ and (1) is verified.

At this point our first conjugation of σ remains close to σ. Now we need to check that the conjugation of σ we chose belongs to R_1. Our goal is to find $t_1\in\mathbb{N}$ such that

$$G_1^{-1}\circ\sigma^{t_1}\circ G_1\left(\left(\frac{a_1}{2},\frac{a_1+1}{2}\right)\times\left(\frac{b_1}{2},\frac{b_1+1}{2}\right)\right)\cap\left(\left(\frac{a_3}{2},\frac{a_3+1}{2}\right)\times\left(\frac{b_3}{2},\frac{b_3+1}{2}\right)\right)\neq\emptyset$$
$$G_1^{-1}\circ\sigma^{t_1}\circ G_1\left(\left(\frac{a_2}{2},\frac{a_2+1}{2}\right)\times\left(\frac{b_2}{2},\frac{b_2+1}{2}\right)\right)\cap\left(\left(\frac{a_4}{2},\frac{a_4+1}{2}\right)\times\left(\frac{b_4}{2},\frac{b_4+1}{2}\right)\right)\neq\emptyset.$$

Recall that in the generic argument in [20] the existence of such a t_1 is shown. For this proof we need however to explicitly calculate t_1. This is where the upper and lower bounds on $|n_m\alpha|$ come into play.

We have chosen α so that

$$\frac{512n_1-1}{1024n_1^3}<|n_1\alpha|<\frac{1}{2n_1^2}.$$

Note that $h_1'=\frac{1024n_1^3}{512n_1-1}$. Let $t_1=n_1^2$. In this case, $\frac{512n_1-1}{1024n_1}<|t_1n_1\alpha|<\frac{1}{2}$ and $\left|t_1n_1\alpha-\frac{1}{2}\right|<\frac{1}{1024n_1}$. Notice that

$$G_1\left(x_1,y_1\right)=\left(x_1+c_1,g^1_{n_1(x_1+c_1)}(y_1)\right)=\left(\frac{1}{4n_1},y_3\right)$$

and

$$G_1^{-1}\circ\sigma^{t_1}\circ G_1(x_1,y_1)=\left(\frac{1}{4n_1}+t_1\alpha-f_1\left(\left(g^1_{n_1\left(\frac{1}{4n_1}+t_1\alpha\right)}\right)^{-1}(y_3)\right),\left(g^1_{n_1\left(\frac{1}{4n_1}+t_1\alpha\right)}\right)^{-1}(y_3)\right).$$

Therefore,

$$\begin{aligned}G_1^{-1}\circ\sigma^{t_1}\circ G_1(x_1,y_1)-(x_3,y_3)&\\ &=\left(\frac{1}{4n_1}+t_1\alpha-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3,\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)-y_3\right).\end{aligned}$$

Consider the first coordinate above:

$$\begin{aligned}&\frac{1}{4n_1}+t_1\alpha-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3\\ &\quad=\left(\frac{1}{4n_1}+\frac{1}{2}\right)+\left(t_1\alpha-\frac{1}{2}\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3\\ &\quad=c_3+\left(t_1\alpha-\frac{1}{2}\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)\\ &\quad=\left(t_1\alpha-\frac{1}{2}\right)+f_1\left(\left(g^1_{n_1\left(\frac{1}{4n_1}+\frac{1}{2}\right)}\right)^{-1}(y_3)\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)\end{aligned}$$

Since $\left|t_1n_1\alpha-\frac{1}{2}\right|<\frac{1}{1024n_1}$, we have

$$\left|\left(g^1_{n_1\left(\frac{1}{4n_1}+\frac{1}{2}\right)}\right)^{-1}(y_3)-\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right|<4\beta_1\left(\frac{1}{1024n_1}\right)<\frac{4}{1024n_1},$$

which implies

$$\left|f_1\left(\left(g^1_{n_1\left(\frac{1}{4n_1}+\frac{1}{2}\right)}\right)^{-1}(y_3)\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)\right|<4M_1\left(\frac{4}{1024n_1}\right)=\frac{M_1}{64n_1}.$$

Since n_1 is odd, we have

$$\left|\frac{1}{4n_1}+t_1\alpha - f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3\right| < \frac{1}{1024}+\frac{M_1}{64n_1} < \frac{1}{64}+\frac{1}{64}=\frac{1}{32}.$$

Similarly, for the second coordinate we obtain

$$\left|\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)-y_3\right| < \frac{4}{1024n_1} < \frac{1}{32}.$$

Thus, $G_1^{-1}\circ\sigma^{t_1}\circ G_1\left(\left(\frac{a_1}{2},\frac{a_1+1}{2}\right)\times\left(\frac{b_1}{2},\frac{b_1+1}{2}\right)\right)\cap\left(\left(\frac{a_3}{2},\frac{a_3+1}{2}\right)\times\left(\frac{b_3}{2},\frac{b_3+1}{2}\right)\right)\neq\emptyset$. In a similar manner, we obtain $G_1^{-1}\circ\sigma^{t_1}\circ G_1\left(\left(\frac{a_2}{2},\frac{a_2+1}{2}\right)\times\left(\frac{b_2}{2},\frac{b_2+1}{2}\right)\right)\cap\left(\left(\frac{a_4}{2},\frac{a_4+1}{2}\right)\times\left(\frac{b_4}{2},\frac{b_4+1}{2}\right)\right)\neq\emptyset$. Therefore (2) is satisfied.

Now that we have $G_1^{-1}\circ\sigma\circ G_1\in R_1$, we proceed with finding a closed ball, which we will call B_1, centered at $G_1^{-1}\circ\sigma\circ G_1$ such that $B_1\subseteq R_1$. We need to explicitly calculate the radius of B_1 to ensure that $B_1\subseteq R_1$. Let

$$\kappa_1=\frac{1}{16\cdot 2^1(C_1n_1)^{2n_1^2-1}}$$

and

$$B_1=\left\{T\in\mathcal{O}: d_u(G_1^{-1}\circ\sigma\circ G_1,T)\leq\kappa_1\right\}.$$

Notice that for any $n\in\mathbb{N}$ and $T\in\mathcal{O}$, we have $d_u(T^n,G_1^{-1}\circ\sigma^n\circ G_1)=\overline{d}(T^n,G_1^{-1}\circ\sigma^n\circ G_1)+\overline{d}(T^{-n},G_1^{-1}\circ\sigma^{-n}\circ G_1)$. Consider the following:

$$\begin{aligned}\overline{d}(T^n,G_1^{-1}\circ\sigma^n\circ G_1)&=\overline{d}((G_1^{-1}\circ\sigma\circ G_1)(G_1^{-1}\circ\sigma^{n-1}\circ G_1),T(T^{n-1}))\\&\leq\overline{d}((G_1^{-1}\circ\sigma\circ G_1)(G_1^{-1}\circ\sigma^{n-1}\circ G_1),(G_1^{-1}\circ\sigma\circ G_1)(T^{n-1}))\\&+\overline{d}((G_1^{-1}\circ\sigma\circ G_1)(T^{n-1}),T(T^{n-1}))\\&\leq\omega_{G_1^{-1}\circ\sigma\circ G_1}(\overline{d}(G_1^{-1}\circ\sigma^{n-1}\circ G_1,T^{n-1}))+\overline{d}(G_1^{-1}\circ\sigma\circ G_1,T)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n-1} \omega^i_{G_1^{-1}\circ\sigma\circ G_1}(\overline{d}(G_1^{-1}\circ\sigma\circ G_1, T)) \\
&\leq \overline{d}(G_1^{-1}\circ\sigma\circ G_1, T)\sum_{i=0}^{n-1}[(C_1 n_1)^2]^i \\
&= \overline{d}(G_1^{-1}\circ\sigma\circ G_1, T)\frac{(C_1 n_1)^{2n}-1}{(C_1 n_1)^2-1} \\
&\leq \overline{d}(G_1^{-1}\circ\sigma\circ G_1, T)(C_1 n_1)^{2n-1}
\end{aligned}
$$

where $\omega^0 = Id$. A similar calculation can be carried out to yield $\overline{d}(T^{-n}, G_1^{-1}\circ\sigma^{-n}\circ G_1) \leq \overline{d}(G_1^{-1}\circ\sigma^{-1}\circ G_1, T^{-1})(C_1 n_1)^{2n-1}$. Thus,

$$
d_u(T^n, G_1^{-1}\circ\sigma^n\circ G_1) \leq d_u(G_1^{-1}\circ\sigma\circ G_1, T)(C_1 n_1)^{2n-1}.
$$

We will show that $B_1 \subseteq R_1$. Let $T \in B_1$. In this case,

$$
\begin{aligned}
d\left(T^{t_1}(x_1,y_1),(x_3,y_3)\right) &\leq d\left(T^{t_1}(x_1,y_1), G_1^{-1}\circ\sigma^{t_1}\circ G_1(x_1,y_1)\right) \\
&\quad + d\left(G_1^{-1}\circ\sigma^{t_1}\circ G_1(x_1,y_1),(x_3,y_3)\right) \\
&\leq d_u(G_1^{-1}\circ\sigma\circ G_1, T)(C_1 n_1)^{2t_1-1} + \frac{1}{16} \\
&\leq \kappa_1 (C_1 n_1)^{2t_1-1} + \frac{1}{16} \\
&= \frac{1}{32} + \frac{1}{16} \\
&< \frac{1}{8}.
\end{aligned}
$$

Thus $T^{t_1}\left(\left(\frac{a_1}{2},\frac{a_1+1}{2}\right)\times\left(\frac{b_1}{2},\frac{b_1+1}{2}\right)\right)\cap\left(\left(\frac{a_3}{2},\frac{a_3+1}{2}\right)\times\left(\frac{b_3}{2},\frac{b_3+1}{2}\right)\right)\neq\emptyset$. In a similar manner, $T^{t_1}\left(\left(\frac{a_2}{2},\frac{a_2+1}{2}\right)\times\left(\frac{b_2}{2},\frac{b_2+1}{2}\right)\right)\cap\left(\left(\frac{a_4}{2},\frac{a_4+1}{2}\right)\times\left(\frac{b_4}{2},\frac{b_4+1}{2}\right)\right)\neq\emptyset$. Hence, we have the desired result i.e. $B_1 \subseteq R_1$.

Thus far we have constructed the closed ball B_1 centered at $G_1^{-1}\circ\sigma\circ G_1$ such that

$B_1 \subseteq R_1$. The next step in our inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in R_2 \cap B_1$ and then construct the closed ball B_2 centered at $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1$ such that $B_2 \subseteq R_2 \cap B_1$. Notice that in the second step of the induction, the dyadic cubes still have order 1. To that end, let $\epsilon_2 = \frac{\kappa_1}{2C_1 n_1} < \epsilon_1$. Now similar to before, we want to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that

1. $d_u(G_2^{-1} \circ \sigma \circ G_2, \sigma) < \epsilon_2$

2. $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$.

Let $U_j^2 = \left(\frac{a_j}{2}, \frac{a_j+1}{2}\right) \times \left(\frac{b_j}{2}, \frac{b_j+1}{2}\right)$, where $a_j, b_j \in \{0,1\}$ for $j = 1,2,3,4$. Let $U_j^{2'}$ be an open dyadic sub-cube of U_j^2 such that any point in $U_j^{2'}$ is at least $\frac{1}{8}$ from the boundary of U_j^2. Since we need to construct $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$, we will consider dyadic cubes inside each $G_1 U_j^{2'}$ and repeat a similar argument.

Observe that G_1 is a bi-Lipschitz map such that

$$\frac{1}{C_1 n_1} d\left(\left(x,y\right),\left(x',y'\right)\right) \leq d\left(G_1\left(x,y\right), G_1\left(x',y'\right)\right) \leq C_1 n_1 d\left(\left(x,y\right),\left(x',y'\right)\right).$$

Let k_1 be the smallest integer such that $n_1 \leq 2^{k_1}$. Then each $G_1 U_j^{2'}$ contains a dyadic cube of order $15 + k_1$. To see this, use the bi-Lipschitz property of G_1 to obtain a lower bound on the size ball that each $G_1 U_j^{2'}$ contains and then place a dyadic cube inside the ball. Now let $\left(\frac{c_j}{2^{15+k_1}}, \frac{c_j+1}{2^{15+k_1}}\right) \times \left(\frac{d_j}{2^{15+k_1}}, \frac{d_j+1}{2^{15+k_1}}\right)$, where $c_j, d_j \in \{0, 1, \ldots, 2^{15+k_1} - 1\}$, denote the dyadic cube inside $G_1 U_j^{2'}$ for $j = 1,2,3,4$.

Now we will pick new points x_j, y_j and new functions h_1, h_2 for the second step in the induction. We abuse notation here to avoid excessive use of superscripts. Let y_1 be a point in $\left(\frac{32d_1+9}{2^{20+k_1}}, \frac{32d_1+11}{2^{20+k_1}}\right)$ and choose β_1 irrational such that $y_3 := y_1 + \beta_1 \in \left(\frac{32d_3+13}{2^{20+k_1}}, \frac{32d_3+15}{2^{20+k_1}}\right)$. Define $h_1 : \mathbb{T} \to \mathbb{T}$ by $h_1(y) = y + \beta_1$.

Similarly, let y_2 be a point in $\left(\frac{32d_2+17}{2^{20+k_1}}, \frac{32d_2+19}{2^{20+k_1}}\right)$ and choose β_2 irrational such that $y_4 := y_2+\beta_2 \in \left(\frac{32d_4+21}{2^{20+k_1}}, \frac{32d_4+23}{2^{20+k_1}}\right)$. Define $h_2 : \mathbb{T} \to \mathbb{T}$ by $h_2(y) = y+\beta_2$. Without loss of generality, assume $\beta_1 > \beta_2$.

Now choose $x_1 \in \left(\frac{32c_1+9}{2^{20+k_1}}, \frac{32c_1+11}{2^{20+k_1}}\right)$, $x_2 \in \left(\frac{32c_2+17}{2^{20+k_1}}, \frac{32c_2+19}{2^{20+k_1}}\right)$, $x_3 \in \left(\frac{32c_3+13}{2^{20+k_1}}, \frac{32c_3+15}{2^{20+k_1}}\right)$, and $x_4 \in \left(\frac{32c_4+21}{2^{20+k_1}}, \frac{32c_4+23}{2^{20+k_1}}\right)$.

We are now ready to start building our desired function $G_2 \in \mathcal{H}(\mathbb{T}^2)$. Let $x \to g^2_x$ be a continuous function from $[0,1)$ to $\mathcal{H}(\mathbb{T})$ such that $g^2_0, g^2_{\frac{3}{4}}, g^2_1 = Id$, $g^2_{\frac{1}{4}} = h_1$, and $g^2_{\frac{1}{2}} = h_2$ with linear interpolation in between. Thus as before, the modulus of continuity of g^2 is $\omega_{g^2}(\delta) = \sup_{|x-x'|<\delta} d_u(g^2_x, g^2_{x'}) \leq 8\beta_1\delta$.

By the choice of the y_j's above we know that they are all distinct. Thus, we may place non-overlapping tent maps p_j around each y_j as before, where the modulus of continuity of each p_j is $\omega_{p_j}(\delta) \leq 32 \cdot 2^{15+k_1}\delta$. Let $M_2 = 32 \cdot 2^{15+k_1}$ and $C_2 = 26M_2$.

Let $\eta_2 = \frac{\epsilon_2}{16M_2}$ and $\delta_2 = \frac{\eta_2}{16}$. Then, if $|x - x'| < \delta_2$ we have $d_u(g^2_x, g^2_{x'}) < \frac{\eta_2}{2}$. If $n_2 \geq 8192 \cdot 32 \cdot 2^{40} C_2$ then we proceed similar to before and define new $c_j \in [0,1)$ such that

$$x_1 + c_1 = \frac{1}{4n_2}, \;\; x_2 + c_2 = \frac{3}{4n_2}, \;\; x_3 + c_3 = \frac{1}{4n_2} + \frac{1}{2}, \;\; x_4 + c_4 = \frac{3}{4n_2} + \frac{1}{2}$$

all taken modulo one. Let f_2 be defined by

$$f_2(y) = \sum_{j=1}^{4} c_j p_j(y).$$

Then, if $|y - y'| < \eta_2$ we have

$$|f_2(y) - f_2(y')| \leq \sum_{j=1}^{4} c_j \left|p_j(y) - p_j(y')\right| < 4M_2\eta_2 = \frac{\epsilon_2}{4}.$$

Now we are ready to define G_2. Let $G_2 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G_2(x,y) = \left(x + f_2(y), g^2_{n_2(x+f_2(y))}(y)\right).$$

Then,

$$G_2^{-1}(x,y) = \big(x - f_2((g^2_{n_2 x})^{-1}(y)), (g^2_{n_2 x})^{-1}(y)\big)$$

and

$$G_2^{-1} \circ \sigma \circ G_2(x,y) = (x + \alpha + f_2(y) - f_2(y_*), y_*)$$

where $y_* = (g^2_{n_2(x+\alpha+f_2(y))})^{-1} g^2_{n_2(x+f_2(y))}(y)$. The modulus of continuity of G_2 is given by $\omega_{G_2}(\delta) \leq C_2 n_2 \delta$, where $C_2 = 26M_2$. It should also be noted that the modulus of continuity of G_2^{-1} is bounded by the same number.

If it is not the case that $n_2 \geq 8192 \cdot 2^2 C_2^2$, then we use U_j^1 in place of U_j^i and $G_i = Id$ at each stage of the induction until a term of the sequence (n_m) exceeds $8192 \cdot 32 \cdot 2^{40} C_2$.

We now proceed with the induction under the assumption that $n_2 \geq 8192 \cdot 32 \cdot 2^{40} C_2$. Similar to before, to show that $d_u(G_2^{-1} \circ \sigma \circ G_2, \sigma) < \epsilon_2$ we need to check that $n_2 > \frac{1}{\delta_2}$. Observe

$$\begin{aligned}
\frac{1}{\delta_2} &= \frac{16}{\eta_2} \\
&= \frac{256 M_2}{\epsilon_2} \\
&= \frac{512\,(C_1 n_1)\,M_2}{\kappa_1} \\
&= 512\left(16 \cdot 2^1\right) M_2\,(C_1 n_1)^{2n_1^2} \\
&\leq 8192 C_1\,(32) \cdot 2^{15} n_1^2\,(n_1)^{4n_1^2} \\
&< (n_1)^{4n_1^2+2} \cdot n_1 \\
&\leq \psi\,(n_1)\,n_1.
\end{aligned}$$

Therefore, $d_u(G_2^{-1} \circ \sigma \circ G_2, \sigma) < \epsilon_2$.

Next we need to show that $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$. Our goal is to find $t_2 \in \mathbb{N}$ such that

$$G_2^{-1} \circ \sigma^{t_2} \circ G_2 \left(\left(\frac{c_1}{2^{15+k_1}}, \frac{c_1+1}{2^{15+k_1}} \right) \times \left(\frac{d_1}{2^{15+k_1}}, \frac{d_1+1}{2^{15+k_1}} \right) \right) \\ \cap \left(\left(\frac{c_3}{2^{15+k_1}}, \frac{c_3+1}{2^{15+k_1}} \right) \times \left(\frac{d_3}{2^{15+k_1}}, \frac{d_3+1}{2^{15+k_1}} \right) \right) \neq \emptyset$$

and

$$G_2^{-1} \circ \sigma^{t_2} \circ G_2 \left(\left(\frac{c_2}{2^{15+k_1}}, \frac{c_2+1}{2^{15+k_1}} \right) \times \left(\frac{d_2}{2^{15+k_1}}, \frac{d_2+1}{2^{15+k_1}} \right) \right) \\ \cap \left(\left(\frac{c_4}{2^{15+k_1}}, \frac{c_4+1}{2^{15+k_1}} \right) \times \left(\frac{d_4}{2^{15+k_1}}, \frac{d_4+1}{2^{15+k_1}} \right) \right) \neq \emptyset.$$

We have chosen α so that

$$\frac{\left(16 \cdot 2^{18+k_1}\right) n_2 - 1}{\left(16 \cdot 2^{19+k_1}\right) n_2^3} < |n_2 \alpha| < \frac{1}{2n_2^2}.$$

Note that $h_2' = \frac{\left(16 \cdot 2^{19+k_1}\right) n_2^3}{\left(16 \cdot 2^{18+k_1}\right) n_2 - 1}$. Let $t_2 = n_2^2$. It follows that $\frac{\left(16 \cdot 2^{18+k_1}\right) n_2 - 1}{\left(16 \cdot 2^{19+k_1}\right) n_2} < |t_2 n_2 \alpha| < \frac{1}{2}$ and $\left| t_2 n_2 \alpha - \frac{1}{2} \right| < \frac{1}{\left(16 \cdot 2^{19+k_1}\right) n_2}$. Similar to the earlier calculation, we obtain

$$G_2^{-1} \circ \sigma^{t_2} \circ G_2(x_1, y_1) - (x_3, y_3) \\ = \left(\frac{1}{4n_2} + t_2 \alpha - f_2 \left(\left(g^2_{\frac{1}{4} + t_2 n_2 \alpha} \right)^{-1} (y_3) \right) - x_3, \left(g^2_{\frac{1}{4} + t_2 n_2 \alpha} \right)^{-1} (y_3) - y_3 \right)$$

and

$$\left| \frac{1}{4n_2} + t_2 \alpha - f_2 \left(\left(g^2_{\frac{1}{4} + t_2 n_2 \alpha} \right)^{-1} (y_3) \right) - x_3 \right| < \frac{1}{16 \cdot 2^{19+k_1}} + \frac{M_2}{2^{19+k_1} n_2} < \frac{1}{2^{18+k_1}}.$$

Similarly, for the second coordinate we obtain

$$\left| \left(g^2_{\frac{1}{4}+t_2 n_2 \alpha} \right)^{-1} (y_3) - y_3 \right| < \frac{4}{(16 \cdot 2^{19+k_1})\, n_2} < \frac{1}{2^{18+k_1}}.$$

Thus, $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$.

Recall that our goal for the second step in the inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in R_2 \cap B_1$. Thus far we have constructed $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in R_2$. We now need to check that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in B_1$. To that end, observe

$$\begin{aligned} \overline{d}(G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1, G_1^{-1} \circ \sigma \circ G_1) &= \overline{d}(G_1^{-1}(G_2^{-1} \circ \sigma \circ G_2), G_1^{-1}(\sigma)) \\ &\leq C_1 n_1 \left(\frac{\epsilon_2}{2} \right) \\ &= C_1 n_1 \left(\frac{\kappa_1}{4 C_1 n_1} \right) \\ &= \frac{\kappa_1}{4}. \end{aligned}$$

Similarly, $\overline{d}(G_1^{-1} \circ G_2^{-1} \circ \sigma^{-1} \circ G_2 \circ G_1, G_1^{-1} \circ \sigma^{-1} \circ G_1) \leq \frac{\kappa_1}{4}$ and $d_u(G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1, G_1^{-1} \circ \sigma \circ G_1) \leq \frac{\kappa_1}{2}$, which implies that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in B_1 \subseteq R_1$.

Let $\overline{G_2} := G_2 \circ G_1$. With this new notation we have shown that $(\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2} \in R_2 \cap B_1$. Now we need to find a closed ball, call it B_2, centered at $(\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}$ that is a subset of $R_2 \cap B_1$. Let

$$\kappa_2 = \frac{1}{16 \cdot 2^2 (C_1 C_2 n_1 n_2)^{2n_2^2 - 1}}$$

and

$$B_2 = \left\{ T \in \mathcal{O} : d_u((\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}, T) \leq \kappa_2 \right\}.$$

We will first show that $B_2 \subseteq R_2$. Let $\left(x'_j, y'_j\right) = G_1^{-1}\left(x_j, y_j\right) \in U_j^{2'}$ for $j = 1, 2, 3, 4$. Let

$T \in B_2$ and consider

$$\begin{aligned}
d\left(T^{t_2}\left(x_1', y_1'\right), \left(x_3', y_3'\right)\right) &\leq d\left(T^{t_2}\left(x_1', y_1'\right), G_1^{-1} \circ G_2^{-1} \circ \sigma^{t_2} \circ G_2 \circ G_1\left(x_1', y_1'\right)\right) \\
&+ d\left(G_1^{-1} \circ G_2^{-1} \circ \sigma^{t_2} \circ G_2 \circ G_1\left(x_1', y_1'\right), \left(x_3', y_3'\right)\right) \\
&\leq d_u(G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1, T)(C_1 C_2 n_1 n_2)^{2t_2 - 1} \\
&+ d\left(G_1^{-1} \circ G_2^{-1} \circ \sigma^{t_2} \circ G_2\left(x_1, y_1\right), G_1^{-1}\left(x_3, y_3\right)\right) \\
&\leq \kappa_2\left(C_1 C_2 n_1 n_2\right)^{2t_2 - 1} + C_1 n_1\left(\frac{1}{2^{17+k_1}}\right) \\
&\leq \frac{1}{32} + \frac{1}{32} \\
&= \frac{1}{16}.
\end{aligned}$$

Hence, we have the desired result, that is $B_2 \subseteq R_2$.

Next we will show that $B_2 \subseteq B_1$. Let $T \in B_2$ and consider

$$\begin{aligned}
d_u(T, G_1^{-1} \circ \sigma \circ G_1) &\leq d_u(T, (\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}) \\
&+ d_u((\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}, G_1^{-1} \circ \sigma \circ G_1) \\
&\leq \kappa_2 + \frac{\kappa_1}{2} \\
&< \frac{\kappa_1}{2} + \frac{\kappa_1}{2} \\
&= \kappa_1.
\end{aligned}$$

Therefore, $B_2 \subseteq R_2 \cap B_1$.

Thus far in our inductive procedure, we have constructed two closed nested balls $B_1 \supseteq B_2$ centered at conjugations of σ such that $B_1 \subseteq R_1$ and $B_2 \subseteq R_2$. The general inductive step can be carried out in the same way.

In the end, this inductive procedure produces a nested sequence of closed balls (B_m) and a sequence $(\overline{G_m})$ of homeomorphisms where each $\overline{G_m} = G_m \circ G_{m-1} \circ \cdots \circ G_1$ and each G_m is of the form

$$G_m(x,y) = (x + f_m(y), g^m_{n_m(x+f_m(y))}(y)).$$

After the m-th stage of the construction has been completed, we have a homeomorphism G_m that satisfies

1. $d_u(G_m^{-1} \circ \sigma \circ G_m, \sigma) < \epsilon_m$ where $\epsilon_m = \frac{\kappa_{m-1}}{2C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}$

2. $G_m^{-1} \circ \sigma \circ G_m \in R_{G_1 \circ \cdots \circ G_{m-1} \circ m}$.

At the end of this stage we also have a closed ball B_m centered at $\left(\overline{G_m}\right)^{-1} \circ \sigma \circ \overline{G_m}$ with radius

$$\kappa_m = \frac{1}{16 \cdot 2^m \left(C_1 \cdots C_m n_1 \cdots n_m\right)^{2n_m^2 - 1}}$$

such that $B_m \subseteq R_m$. Recall that we are working in a complete metric space. Let $T_0 = \bigcap_{m=1}^{\infty} B_m$. Therefore, T_0 is topologically weakly mixing. Also, $\left(\overline{G_m}\right)^{-1} \circ \sigma \circ \overline{G_m}$ converges uniformly to T_0 since $\left(\overline{G_m}\right)^{-1} \circ \sigma \circ \overline{G_m}$ is the center of B_m.

Now that we have T_0 which is topologically weakly mixing, we need to show that it is uniformly rigid with respect to (n_m). To do this, we need to make a preliminary estimate. First notice that

$$G_m^{-1} \circ \sigma^{n_m} \circ G_m(x,y) - \sigma^{n_m}(x,y) = (x + f_m(y) - f_m(y_*), y_*)$$

where $y_* = \left(g^m_{n_m(x+n_m\alpha+f_m(y))}\right)^{-1} g^m_{n_m(x+f_m(y))}(y)$. In either case

$$\left|n_m^2 \alpha\right| < \frac{1}{n_m} < \delta_m$$

and we can conclude that $d_u(G_m^{-1} \circ \sigma^{n_m} \circ G_m, \sigma^{n_m}) < \epsilon_m$. Now observe the following:

$$\begin{aligned}
&\overline{d}\left(\left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}, \left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}\right) = \\
&\overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}\left(G_m^{-1} \circ \sigma^{n_m} \circ G_m\right), \left(\overline{G_{m-1}}\right)^{-1}\left(\sigma^{n_m}\right)\right) \\
&\leq \left(C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}\right)\left(\frac{\epsilon_m}{2}\right) \\
&= \frac{\kappa_{m-1}}{4}.
\end{aligned}$$

Hence, $d_u(\left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}, \left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}) \leq \frac{\kappa_{m-1}}{2}$.

The final estimate will show that T_0 is uniformly rigid with respect to (n_m). Indeed

$$\begin{aligned}
d_u\left(T_0^{n_m}, Id\right) \leq\ & d_u\left(T_0^{n_m}, \left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}\right) \\
&+ d_u\left(\left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}, \left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}\right) \\
&+ d_u\left(\left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}, Id\right) \\
&= d_u\left(T_0^{n_m}, \left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}\right) \\
&+ d_u\left(\left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}, \left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}\right) \\
&+ \overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}, Id\right) \\
&+ \overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{-n_m} \circ \overline{G_{m-1}}, Id\right) \\
&= d_u\left(T_0^{n_m}, \left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}\right) \\
&+ d_u\left(\left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}, \left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}\right) \\
&+ \overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}(\sigma^{n_m}), \left(\overline{G_{m-1}}\right)^{-1}(Id)\right) \\
&+ \overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}(\sigma^{-n_m}), \left(\overline{G_{m-1}}\right)^{-1}(Id)\right) \\
&\leq \kappa_m (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m - 1} + \frac{\kappa_{m-1}}{2}
\end{aligned}$$

$$
\begin{aligned}
&+ 2\left(\frac{C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}{n_m^2}\right) \\
&\leq \left(\frac{1}{16 \cdot 2^m (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m^2 - 1}}\right)(C_1 \cdots C_m n_1 \cdots n_m)^{2n_m - 1} \\
&+ \frac{\kappa_{m-1}}{2} + 2\left(\frac{n_1 n_2 \cdots n_{m-1}}{n_m}\right)^2 \\
&\leq \frac{1}{2^m} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{\psi(n_{m-1})}{n_m}\right)^2 \\
&\leq \frac{1}{2^m} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{1}{n_{m-1}}\right)^2.
\end{aligned}
$$

Thus $d_u(T_0^{n_m}, Id) \to 0$ as $m \to \infty$ and T_0 is uniformly rigid with respect to (n_m). Therefore we have constructed a topologically weakly mixing homeomorphism that is uniformly rigid with respect to (n_m).

□

Chapter 5

Future Directions

In this final chapter we will discuss open questions that are related to the previous chapters and future research projects.

5.1 Uniform Rigidity for Connected Compact Metric Spaces:

We have already seen that the topology of a space can affect the dynamical properties defined there. As mentioned before, there are no weakly mixing homeomorphisms of a Cantor space that are uniformly rigid, however on the two-torus there is a large family of such maps. Also, there is a large family of topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid. The work on the two-torus and Klein bottle use very different techniques. We would like to develop a unified approach and answer the following question:

Question 5.1.1. Does every connected compact metric space admit a large family of (topologically) weakly mixing homeomorphisms that are uniformly rigid?

The answer to this question would shed some light on how the topology of a space affects the dynamical properties defined there.

There are two aspects of this question that we would like to explore. The first is restricting the space to compact manifolds.

Question 5.1.2. Let X be a compact manifold of dimension at least 2 and μ be the volume measure. Are the weakly mixing, uniformly rigid homeomorphisms generic in $Homeo(X, \mu)$

with respect to the uniform topology?

Since the weakly mixing homeomorphisms are generic in $Homeo(X, \mu)$ with respect to the uniform topology (with (X, μ) as stated in the question), the above question amounts to proving genericity of uniform rigidity.

Another aspect of the first question it to consider a space, like the solenoid, that is a connected compact metric space, but also has many characteristics in common with a Cantor space. There are many ways to describe the solenoid. Visually, the solenoid can be thought of as an intersection of a nested sequence of solid tori that are wrapped into the previous one two times. Each cross-section of a solenoid is a Cantor space. The solenoid, $\mathbb{S}$, can also be described as an inverse limit as follows:

$$S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 2} \cdots$$

$$\mathbb{S} = \left\{ (z_1, z_2, \ldots) \in \prod_{i=1}^{\infty} S^1 \ : \ z_i = z_{i+1}^2 \ for \ i = 1, 2, \ldots \right\}$$

Question 5.1.3. Does there exist a weakly mixing measure-preserving homeomorphism of $\mathbb{S}$ that is uniformly rigid with respect to any metric compatible with the topology?

5.2 Interval Exchange Transformations:

Given $L = \{\ell_1, \ldots, \ell_d\}$ where each $\ell_i > 0$ and $\sum_{i=0}^{d} \ell_i = 1$ define subintervals of $[0, 1)$ by $I_1 = [0, \ell_1), I_2 = [\ell_1, \ell_1 + \ell_2), \ldots, I_d = [\ell_1 + \cdots + \ell_{d-1}, 1)$. A permutation π on $\{1, \ldots, d\}$, together with L, defines a *d-Interval-Exchange-Transformation (IET)* $T : [0, 1) \to [0, 1)$ that exchanges the intervals I_i according to π. This notation is taken from [11]. In 1984 William Veech proved that almost every IET is rigid [34] and then in 2007 Artur Avila and Giovanni Forni proved that almost every IET that is not a rotation is weakly mixing [5].

Some of the ideas from topological dynamics still make sense in the IET framework, even though the space is no longer compact and IET's have finitely many discontinuities. For example, the property of uniform rigidity can still be considered in this context. Recently, Jon Chaika asked the following question:

Question 5.2.1. Does there exist a weakly mixing IET that is uniformly rigid?

This is a natural question since almost every IET is weakly mixing and rigid. The first place to look for such an IET would be in the class of 3-IET's that are induced maps of rotations. Specifically, the only permutation to consider would be $\pi : \{1,2,3\} \to \{1,2,3\}$ such that $\pi(1) = 3$, $\pi(2) = 2$, and $\pi(3) = 1$ since all of the other permutations reduce to exchanges on two intervals. Let $A(\ell_1, \ell_2) = \frac{1-\ell_1}{1+\ell_2}$ and $B(\ell_1, \ell_2) = \frac{1}{1+\ell_2}$. In this case, the map T is induced by a rotation of the circle by angle $A(\ell_1, \ell_2)$. In [15] Sébastien Ferenczi, Charles Holton, and Luca Zamboni characterized when T satisfies the *infinite distinct orbit condition (i.d.o.c.)* of Keane [28], based solely on properties of $A(\ell_1, \ell_2)$ and $B(\ell_1, \ell_2)$. The i.d.o.c says that the negative trajectories of the discontinuities are infinite and disjoint sets. If T satisfies the i.d.o.c then T is minimal and uniquely ergodic with the invariant probability measure being Lebesgue measure. In [15] they also produced specific conditions for when T is weakly mixing. These conditions are relatively easy to check and would give verification that a targeted map is weakly mixing and uniquely ergodic, thus leaving only uniform rigidity to contend with.

Another topological property of IET's that is not entirely understood is topological strong mixing. Recall that a map T of a topological space X is called *topologically strongly mixing* if for every pair of open subsets U, V of X there exists a natural number N such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. Notice that topological strong mixing is the opposite of uniform rigidity, that is a map cannot be uniformly rigid and topologically strongly mixing. Michael Boshernitzan and Jon Chaika showed that no 3-IET is topologically strongly mixing [9]. However, Jon Chaika (personal communication) has recently shown that a residual set of 4-IETs in the Rauzy class (4321) are topologically strongly mixing. This leads to the following questions:

Question 5.2.2. Is it true that, for more general permutations, the set of 4-IET's that are topologically strongly mixing is residual?

Question 5.2.3. Is almost every 4-IET topologically strongly mixing?

5.3 Local Rokhlin Property for the Two-Torus:

Let X be a compact topological space and $\mathcal{H}(X)$ be the set of homeomorphisms of X with the topology of uniform convergence. The space X is said to have the *Rokhlin property* if $\mathcal{H}(X)$ is the closure of a single conjugacy class. For example, in 2008 Glasner and Weiss showed that the Cantor set has the Rokhlin property [21]. For the circle this is no longer true, but a similar local property holds. The rotation number of a homeomorphism of the circle is conjugacy invariant and any homeomorphism with irrational rotation number α lies in the closure of the conjugacy class of irrational rotation by α [2]. Thus the circle has what is called the *local Rokhlin property.* The idea of rotation number has been extended to $\mathbb{T}^2$ by Micha Misiurewicz in [31].

Question 5.3.1. Does the two-torus have the local Rokhlin property?

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