DISTRIBUTION OF SOME ARITHMETIC SEQUENCES

BY

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DISSERTATION

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Abstract

In this thesis, we first study the correlations of some arithmetic sequences. We prove the existence of the limiting pair correlations of fractions with prime and power denominators, and give the explicit pair correlation density functions. Next, we study the higher level correlations of these fractions, and construct an arithmetic sequence with showing the independence of its different level correlations.

We also study the distribution of angles between common tangents of Ford circles, which is a special case of Apollonian circle packing. We provide the limiting distribution functions of these angles in different situations.

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Chapter 1

Introduction to the correlations of arithmetic sequences

The study of correlations of sequences is more subtle than the study of the classical Weyl uniform distribution to measure the distribution of sequences. It was initiated by physicists, in order to understand the spectra of high energies. After the revolutionary work by Montgomery [19], Hehjal [13], and Rudnick and Sarnak [22] on the correlations of imaginary parts of zeros of L-functions, the use of random matrix theory [16] has drawn a lot of attention from number theorists.

Let $k \geq 2$ be an integer and F be a finite set of N elements in $[0, 1]$. For any box $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_{k-1}, b_{k-1}] \subset \mathbb{R}^{k-1}$, the *k-level correlation measure* $\mathcal{R}_F^{(k)}(\mathcal{B})$ is defined as

$$\frac{1}{N} \#\Big\{(x_1, \ldots, x_k) \in F^k : x_i \text{ distinct, } (x_1 - x_2, x_2 - x_3, \ldots, x_{k-1} - x_k) \in \frac{1}{N}\ \mathcal{B} + \mathbb{Z}^{k-1}\Big\}.$$

Especially, when $k = 2$, the *pair correlation measure* of an interval $I \subset \mathbb{R}$ is

$$\mathcal{R}_F^{(2)}(I) = \frac{1}{N} \#\Big\{(x, y) \in F^2 : x \neq y \text{ and } x - y \in \frac{1}{N}\ I + \mathbb{Z}\Big\}.$$

Suppose that $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ is an increasing sequence of finite subsets of $[0, 1]$ and that

$$\mathcal{R}^{(k)}(\mathcal{B}) = \lim_{n \to \infty} \mathcal{R}_{F_n}^{(k)}(\mathcal{B})$$

exists for every box $\mathcal{B} \subset \mathcal{R}^{k-1}$. Then $\mathcal{R}^{(k)}$ is called the *limiting k-level correlation measure of F_n*. The measure $\mathcal{R}^{(2)}$ is called the *limiting pair correlation measure of F_n*.

For randomly distributed sequences, the k-level correlation is expected to be Poissonian, that is, for every box $\mathcal{B} \subset \mathbb{R}^{k-1}, \mathcal{R}^{(k)}(\mathcal{B}) = vol(\mathcal{B}) = \prod\limits_{i=1}^{k-1} (b_i - a_i)$.

Frequently, we can simplify the problem by setting $\mathcal{B} = \mathcal{B}_\Lambda = (0, \Lambda_1) \times \cdots \times (0, \Lambda_{k-1})$, and $\mathcal{R}^{(k)}$ is Poissonian if for every box $\mathcal{B}_\Lambda \subset \mathbb{R}^{k-1}, \mathcal{R}^{(k)}(\mathcal{B}_\Lambda) = \prod_{i=1}^{k-1} \Lambda_i$.

There are very few sequences of arithmetic interest for which one can establish the existence of correlation measures, and many of them are conditional. For interesting results, see [23] [24] for fractional parts of irrational numbers, [17] [18] for quadratic forms, and [4] for Farey fractions.

Chapter 2

Correlations of fractions with prime denominators

2.1 Statement of main results

For each integer Q, let B_Q be a subset of prime numbers, $B_1 \subset B_2 \subset \cdots \subset B_Q \subset \cdots$

Let $S_Q = \{\frac{a}{q}, 1 \leq a \leq q-1, q \in B_Q\}$ be all the fractions between [0,1] with the denominators in B_Q. In this chapter, we will consider the limiting correlations of the set S_Q.

Let $N_Q = N = |S_Q| = \sum_{p \in B_Q} (p-1) = \sum_{p \in B_Q} p - |B_Q|$, and $M_Q = \sum_{p \in B_Q} p^2$.

Theorem 2.1. *When $Q \to \infty$, the limiting pair correlation of S_Q is Poissonian if and only if $M_Q = o(N_Q^2)$, i.e. $\sum_{p \in B_Q} p^2 = o((\sum_{p \in B_Q} p)^2)$.*

From Theorem 2.1 and the prime number theorem, we immediately deduce

Corollary 2.2. *The pair correlation of all the fractions in [0,1] with prime denominators is Poissonian.*

Corollary 2.3. *The pair correlation of all the fractions in [0,1] with prime denominators which lie in an arithmetic progression is Poissonian.*

If $M_Q = o(N_Q^2)$, then for every $p \in B_Q$, $p = o(N_Q)$, this condition is quite 'natural' since otherwise the subsequence $\{\frac{a}{p}, 1 \leq a \leq p-1\}$ will dominate the whole sequence S_Q, and it will be of less interest.

Now, for the higher level correlation, we have

Theorem 2.4. *For $k \geq 3$, if for every $p \in B_Q$, $p = o(N_Q)$, then when $Q \to \infty$, the limiting k-level correlation of S_Q equals the number of solutions of the following linear modular equation system*

$$\lim_{Q\to\infty} \mathcal{R}^{(k)}_{S_Q}(\mathcal{B}) = \lim_{Q\to\infty} \frac{1}{N_Q} \# \left\{ \begin{matrix} p_1, p_2, \ldots, p_k \in B_Q \ \text{ distinct} \\ z_1, z_2, \ldots, z_{k-1} \in \mathbb{Z}^+ \end{matrix} \middle| \begin{matrix} p_i | z_{i+1} p_{i-1} + z_{i-1} p_{i+1}, 2 \leq i \leq k-1 \\ z_j < \frac{p_{j-1} p_j}{N_Q} \Lambda_j, 1 \leq j \leq k-1 \end{matrix} \right\}.$$

Generally, the number of solutions for the linear equation system in Theorem 2.4 is still not easy to estimate, and many times for fixed $(p_1, \ldots, p_k)$, there is no solution for $(z_1, \ldots, z_{k-1})$. However, if $p_1, p_2, \ldots, p_k$ forms an arithmetic progression, we can just take $z_1 = z_2 = \cdots = z_{k-1} = 1, 2, \ldots,$ to get several solutions.

And this is assured by the famous Green-Tao theorem [10] about the existence of arbitrarily long arithmetic progressions of prime numbers.

Now, combining Theorems 2.1 and 2.4, and using the Green-Tao theorem, we can prove

Theorem 2.5. *There are infinitely many sequences with Poissonian pair correlation, but none of their higher correlations is Poissonian.*

As far as the author knows, this is the first construction of such examples.

If we use the methods in [11], we may be able to obtain more concrete numerical results for Theorem 2.4.

2.2 Preliminaries

In this section, we introduce some notations and basic settings; see also [4].

For a fixed integer $k \geq 2$, we take a smooth real-valued function $H \in C^\infty(\mathbb{R}^{k-1})$ such that $\mathrm{supp}(H) \subset \mathcal{B}_\Lambda = (0, \Lambda_1) \times \cdots \times (0, \Lambda_{k-1})$.

Construct the $\mathbb{Z}^{k-1}$-periodic function f

$$f(y) = f_Q(y) = \sum_{r \in \mathbb{Z}^{k-1}} H\big(N(y + r)\big), \qquad y \in \mathbb{R}^{k-1},$$

and let the smooth k-level correlation sum be

$$\mathcal{S}_k = R^{(k)}(Q, H) = \frac{1}{N} \sum_{\gamma_1, \ldots, \gamma_k \in S_Q \text{ distinct}} f(\gamma_1 - \gamma_2, \gamma_2 - \gamma_3, \ldots, \gamma_{k-1} - \gamma_k). \tag{2.2.1}$$

Approximate H by the characteristic function $\chi_{\mathcal{B}_\Lambda}$; $\mathcal{S}_k$ will be exactly the k-level correlation measure.

Express f by the Fourier series

$$f(y) = \sum_{r \in \mathbb{Z}^{k-1}} c_r e(r \cdot y).$$

The Fourier coefficients of f will be

$$\begin{aligned} c_r &= \int\limits_{[0,1)^{k-1}} f(y)e(-r\cdot y)dy = \int\limits_{[0,1)^{k-1}} e(-r\cdot y)\sum_{n\in\mathbb{Z}^{k-1}} H\big(N(y+n)\big)dy \\ &= \sum_{n\in\mathbb{Z}^{k-1}} \int\limits_{[0,1)^{k-1}} e(-r\cdot y)H\big(N(y+n)\big)dy \\ &= \sum_{n\in\mathbb{Z}^{k-1}} \int\limits_{n+[0,1)^{k-1}} e\big(-r\cdot(u-n)\big)H(Nu)du \\ &= \int\limits_{\mathbb{R}^{k-1}} e(-r\cdot u)H(Nu)du = \frac{1}{N^{k-1}}\int\limits_{\mathbb{R}^{k-1}} e\bigg(-\frac{r\cdot y}{N}\bigg)H(y)dy \\ &= \frac{1}{N^{k-1}}\widehat{H}\bigg(\frac{1}{N}\,r\bigg), \end{aligned} \tag{2.2.2}$$

where $\widehat{H}(x)$ is the Fourier transform of H

$$\widehat{H}(x) = \int\limits_{\mathbb{R}^{k-1}} H(y)e(-x\cdot y)dy, \qquad x\in\mathbb{R}^{k-1}.$$

Since H is supported on $\mathcal{B}_\Lambda$ we can remove in (2.2.1), for Q large enough such that $N > |\Lambda|$, the condition that $\gamma_1,\dots,\gamma_k$ are distinct, and obtain

$$\begin{aligned} \mathcal{S}_k &= \frac{1}{N}\sum_{\substack{\gamma_1,\dots,\gamma_k\in\mathcal{F}_Q \\ r_1,\dots,r_{k-1}\in\mathbb{Z}}} H\big(N(r_1+\gamma_1-\gamma_2, r_2+\gamma_2-\gamma_3,\dots,r_{k-1}+\gamma_{k-1}-\gamma_k)\big) \\ &= \frac{1}{N}\sum_{\gamma_1,\dots,\gamma_k\in\mathcal{F}_Q} f(\gamma_1-\gamma_2,\gamma_2-\gamma_3,\dots,\gamma_{k-1}-\gamma_k) \\ &= \frac{1}{N}\sum_{\substack{\gamma_1,\dots,\gamma_k\in\mathcal{F}_Q \\ r_1,\dots,r_{k-1}\in\mathbb{Z}}} c_r\, e\big(r\cdot(\gamma_1-\gamma_2,\gamma_2-\gamma_3,\dots,\gamma_{k-1}-\gamma_k)\big) \\ &= \frac{1}{N}\sum_{\substack{\gamma_1,\dots,\gamma_k\in\mathcal{F}_Q \\ r_1,\dots,r_{k-1}\in\mathbb{Z}}} c_r\, e(r_1\gamma_1)e\big((r_2-r_1)\gamma_2\big)\cdots e\big((r_{k-1}-r_{k-2})\gamma_{k-1}\big)e(r_{k-1}\gamma_k). \end{aligned} \tag{2.2.3}$$

2.3 Proof of Theorem 2.1

From (2.2.1) and (2.2.3), the pair correlation sum equals

$$
\begin{aligned}
PC &= \frac{1}{N}\sum_{x,y\in S_Q} h(x-y) = \frac{1}{N}\sum_{m\in Z} c_m \Big|\sum_{x\in S_Q} e(mx)\Big|^2 \\
&= \frac{1}{N}\sum_{m\in Z} c_m (\sum_{p\in B_Q, p|m} p - |B_Q|)^2 \\
&= \frac{1}{N}\sum_{m\in Z} c_m (\sum_{\substack{p,q\in B_Q \\ p|m, q|m}} pq + |B_Q|^2 - 2|B_Q| \sum_{p\in B_Q, p|m} p) \\
&= \frac{1}{N}\sum_{p,q\in B_Q} pq \sum_{l\in Z} c_{[p,q]l} + \frac{1}{N}(|B_Q|)^2 \sum_{m\in Z} c_m - \frac{2}{N}|B_Q| \sum_{p\in B_Q} p \sum_{l\in Z} c_{pl}.
\end{aligned}
\tag{2.3.1}
$$

By the Poisson summation formula,

$$
\begin{aligned}
&\sum_{l\in\mathbb{Z}} c_{[p,q]l} = \sum_{l\in\mathbb{Z}} \frac{1}{N}\widehat{H}(\frac{[p,q]l}{N}) = \sum_{l\in\mathbb{Z}} \frac{1}{N}\frac{N}{[p,q]} H(\frac{lN}{[p,q]}) = \sum_{l\in Z} \frac{1}{[p,q]} H(\frac{lN}{[p,q]}), \\
&\sum_{l\in\mathbb{Z}} c_{pl} = \sum_{l\in\mathbb{Z}} \frac{1}{N}\widehat{H}(\frac{pl}{N}) = \sum_{l\in\mathbb{Z}} \frac{1}{N}\frac{N}{p} H(\frac{lN}{p}) = \sum_{l\in Z} \frac{1}{p} H(\frac{lN}{p}), \\
&\sum_{m\in\mathbb{Z}} c_m = \sum_{m\in\mathbb{Z}} \frac{1}{N}\widehat{H}(\frac{m}{N}) = \sum_{m\in\mathbb{Z}} H(mN).
\end{aligned}
\tag{2.3.2}
$$

So (2.3.1) becomes

$$
\begin{aligned}
PC &= \frac{1}{N}\sum_{p,q\in B_Q} pq \sum_{l\in Z} \frac{1}{[p,q]} H(\frac{lN}{[p,q]}) + \frac{1}{N}(|B_Q|)^2 \sum_{m\in Z} H(mN) - \frac{2}{N}|B_Q| \sum_{p\in B_Q} p \sum_{l\in Z} \frac{1}{p} H(\frac{lN}{p}) \\
&= \frac{1}{N}\sum_{p\neq q\in B_Q} \sum_{l\in Z} H(\frac{lN}{pq}) + \frac{1}{N}\sum_{p\in B_Q} pH(\frac{lN}{p}) + 0 - \frac{2}{N}|B_Q| \sum_{p\in B_Q}\sum_{l\in Z} H(\frac{lN}{p}).
\end{aligned}
\tag{2.3.3}
$$

If $M_Q = \sum_{p\in B_Q} p^2 = o(N^2)$, then for every $p \in B_Q$, $p^2 = o(N^2)$, so $p = o(N)$ and when $Q \to \infty$, $H(\frac{lN}{p}) = 0$.

Now, when Q is large enough, the pair correlation is equal to

$$\begin{aligned}
PC &= \frac{1}{N}\sum_{p\neq q\in B_Q}\sum_{l\in Z} H(\frac{lN}{pq}) \\
&= \frac{1}{N}\sum_{p\neq q\in B_Q}(\frac{pq}{N}\int_R H(x)dx + O_{\Lambda,H}(1)) \\
&= \frac{1}{N^2}\sum_{p\neq q\in B_Q} pq\int_R H(x)dx + \frac{1}{N}O_{\Lambda,H}(|B_Q|) \\
&= \frac{1}{N^2}\sum_{p,q\in B_Q} pq\int_R H(x)dx - \frac{1}{N^2}\sum_{p\in B_Q}p^2\int_R H(x)dx + O_{\Lambda,H}(\frac{|B_Q|^2}{N}) \\
&= (1+o(1))\int_R H(x)dx + O_{\Lambda,H}(\frac{|B_Q|^2}{N}).
\end{aligned} \tag{2.3.4}$$

Suppose $|B_Q| = k$. Then $N \gg$ sum of the first k-th primes, i.e, write out in summation notation $\gg \sum_{t=1}^{k} t\log t \gg k^2\log k$, when $k\to\infty$. So $|B_Q|^2 = o(N)$, and hence the limiting pair correlation is Poissonian.

For the reverse direction, if $M_Q \neq o(N^2)$, we approximate H by the characteristic function $\chi_{(0,\Lambda)}$. Now from the last equality in (2.3.3),

$$\begin{aligned}
PC &= \frac{1}{N}\sum_{p\neq q\in B_Q}\left[\frac{pq\Lambda}{N}\right] + \frac{1}{N}\sum_{p\in B_Q} p\left[\frac{p\Lambda}{N}\right] - \frac{2}{N}|B_Q|\sum_{p\in B_Q}\left[\frac{p\Lambda}{N}\right] \\
&= \frac{1}{N^2}\sum_{p\neq q\in B_Q} pq\Lambda + O(\frac{|B_Q|^2}{N}) + \frac{1}{N^2}\sum_{p\in B_Q}p^2\Lambda - \frac{1}{N}\sum_{p\in B_Q}p\left\{\frac{p\Lambda}{N}\right\} \\
&\quad -\frac{2}{N}|B_Q|(\sum_{p\in B_Q}\frac{p\Lambda}{N} + O(|B_Q|)) \\
&= \frac{1}{N^2}\sum_{p,q\in B_Q} pq\Lambda + o(1) - \frac{1}{N}\sum_{p\in B_Q}p\left\{\frac{p\Lambda}{N}\right\} - o(1) + O(\frac{|B_Q|^2}{N}) \\
&= \Lambda + o(1) - \frac{1}{N}\sum_{p\in B_Q}p\left\{\frac{p\Lambda}{N}\right\}.
\end{aligned}$$

If for every $p\in B_Q$, $p = o(N)$, then when $Q\to\infty$

$$\begin{aligned}
\lim_{Q\to\infty}\frac{1}{N}\sum_{p\in B_Q}p\left\{\frac{p\Lambda}{N}\right\} &= \lim_{Q\to\infty}\frac{1}{N_Q}\sum_{p\in B_Q}p\frac{p\Lambda}{N_Q} \\
&= \lim_{Q\to\infty}\Lambda\frac{M_Q}{N_Q^2}\neq 0.
\end{aligned}$$

If there exists a $p \in B_Q$, $p \neq o(N)$, then when $Q \to \infty$

$$\lim_{Q\to\infty} \frac{1}{N} \sum_{p\in B_Q} p\left\{\frac{p\Lambda}{N}\right\} \geq \lim_{Q\to\infty} \frac{1}{N} p\left\{\frac{p\Lambda}{N}\right\} \neq 0. \tag{2.3.5}$$

In either case, the limiting pair correlation is not Poissonian, so the proof is complete.

2.4 Proof of Theorem 2.4

For $k \geq 3$, equality (2.2.3) further yields

$$\begin{aligned}
\mathcal{S}_k &= \frac{1}{N} \sum_{r=(r_1,\ldots,r_{k-1})\in\mathbb{Z}^{k-1}} c_{r_1,\ldots,r_{k-1}} \sum_{\substack{p_1|r_1\\ p_2|r_2-r_1\\ \cdots\\ p_{k-1}|r_{k-1}-r_{k-2}\\ p_k|r_{k-1}}} (p_1 - |B_Q|)\cdots(p_k - |B_Q|)\\
&= \frac{1}{N} \sum_{p_1,\ldots,p_k\in B_Q} (p_1 - |B_Q|)\cdots(p_k - |B_Q|) \sum_{\substack{r\in\mathbb{Z}^{k-1}\\ p_1|r_1\\ p_2|r_2-r_1\\ \cdots\\ p_{k-1}|r_{k-1}-r_{k-2}\\ p_k|r_{k-1}}} c_{r_1,\ldots,r_{k-1}}\\
&= \frac{1}{N} \sum_{p_1,\ldots,p_k\in B_Q \text{distinct}} p_1\cdots p_k \sum_{\substack{r\in\mathbb{Z}^{k-1}\\ p_1|r_1\\ p_2|r_2-r_1\\ \cdots\\ p_{k-1}|r_{k-1}-r_{k-2}\\ p_k|r_{k-1}}} c_{r_1,\ldots,r_{k-1}} + E,
\end{aligned} \tag{2.4.1}$$

and the term E in the last equality is

$$\begin{aligned}
E =& \frac{1}{N} \sum_{\substack{p_1,\ldots,p_k\in B_Q\\ p_i=p_j}} p_1\cdots p_k \sum_{\substack{r\in\mathbb{Z}^{k-1}\\ p_1|r_1\\ p_2|r_2-r_1\\ \cdots\\ p_{k-1}|r_{k-1}-r_{k-2}\\ p_k|r_{k-1}}} c_{r_1,\ldots,r_{k-1}} +\\
& \frac{1}{N} \sum_{\substack{p_{i_1},\ldots,p_{i_m}\in B_Q\\ 0\leq m\leq k-1}} p_{i_1}\ldots p_{i_m}(-|B_Q|)^{k-m} \sum_{\substack{r_{i_1},r_{i_2},\ldots,r_{i_m}\in\mathbb{Z}\\ p_{i_j}|r_{i_j}-r_{i_j-1}, 1\leq j\leq m\\ r_{i_j-1}=0 \text{ if } i_j=1\\ r_{i_j}=0 \text{ if } i_j=k}} c_{r_1,\ldots,r_{k-1}}.
\end{aligned} \tag{2.4.2}$$

We claim that the first term in the last line of equation (2.4.1) will contribute the main term in the estimation of $\mathcal{S}_k$, and E will consist of the minor terms.

Rewrite the divisibility conditions $p_1|r_1$, $p_2|r_2 - r_1$, $p_3|r_3 - r_2$,…,$p_{k-1}|r_{k-1} - r_{k-2}$, $p_k|r_{k-1}$ as

$$\begin{aligned} &r_1 = p_1 d_1, \\ &r_2 = p_1 d_1 + p_2 d_2, \\ &\dots\dots\dots\dots\dots\dots \\ &r_{k-1} = p_1 d_1 + \dots + p_{k-1} d_{k-1} = p_k d_k, \end{aligned} \tag{2.4.3}$$

for some integers $d_1, \dots, d_k$.

We want to solve the last equation, where $d_1, \dots, d_{k-2}$ can be arbitrary integers, and then

$$p_{k-1} d_{k-1} \equiv -(p_1 d_1 + \dots + p_{k-2} d_{k-2}) \pmod{p_k}.$$

Choose $s, t \in \mathbb{Z}$, so that $p_{k-1} t + p_k s = 1$. Then

$$d_{k-1} \equiv -t(p_1 d_1 + \dots + p_{k-2} d_{k-2}) \pmod{p_k}.$$

Now we can express $(r_1, \dots, r_{k-1})$ in $k-1$ free variables $(l_1, \dots, l_{k-1})$,

$$\begin{aligned} r_1 &= p_1 l_1, \\ r_2 &= p_1 l_1 + p_2 l_2, \\ &\dots\dots\dots\dots\dots\dots \\ r_{k-2} &= p_1 l_1 + \dots + p_{k-2} l_{k-2} \\ r_{k-1} &= p_1 l_1 + \dots + p_{k-2} l_{k-2} - t p_{k-1}(p_1 l_1 + \dots + p_{k-2} l_{k-2}) + p_{k-1} p_k l_{k-1} \\ &= s p_k (p_1 l_1 + \dots + p_{k-2} l_{k-2}) + p_{k-1} p_k l_{k-1}. \end{aligned} \tag{2.4.4}$$

Now, the inner sum in the last line of equation (2.4.1) will be

$$
\begin{aligned}
\sum_{r_1,\dots,r_{k-1}\in\mathbb{Z}} c_r &= \sum_{r_1,\dots,r_{k-1}\in\mathbb{Z}} \frac{1}{N^{k-1}}\widehat{H}\left(\frac{r_1}{N},\dots,\frac{r_{k-1}}{N}\right)\\
&= \frac{1}{N^{k-1}}\sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}}\int_{\mathbb{R}^{k-1}} H(x_1,\dots,x_{k-1})e(-\frac{p_1l_1}{N}x_1-\frac{p_1l_1+p_2l_2}{N}x_2-\dots\\
&\qquad -\frac{p_1l_1+\dots+p_{k-2}l_{k-2}}{N}x_{k-2}\\
&\qquad -\frac{sp_k(p_1l_1+\dots+p_{k-2}l_{k-2})+p_{k-1}p_kl_{k-1}}{N}x_{k-1})dx_1\dots dx_{k-1}\\
&= \frac{1}{N^{k-1}}\sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}}\int_{\mathbb{R}^{k-1}} H(x_1,\dots,x_{k-1})e(-\frac{p_1x_1+\dots+p_1x_{k-2}+sp_1p_kx_{k-1}}{N}l_1\\
&\qquad -\frac{p_2x_2+\dots+p_2x_{k-2}+sp_2p_kx_{k-1}}{N}l_2-\dots\\
&\qquad -\frac{p_{k-2}l_{k-2}+sp_{k-2}p_kx_{k-1}}{N}l_{k-2}-\frac{p_{k-1}p_kx_{k-1}}{N}l_{k-1})dx_1\dots dx_{k-1}.
\end{aligned}
\tag{2.4.5}
$$

Define

$$
\begin{aligned}
y_1 &= \frac{1}{N}(p_1x_1+\dots+p_1x_{k-2}+sp_1p_kx_{k-1})\\
y_2 &= \frac{1}{N}(p_2x_2+\dots+p_2x_{k-2}+sp_2p_kx_{k-1})\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
y_{k-2} &= \frac{1}{N}(p_{k-2}x_{k-2}+sp_{k-2}p_kx_{k-1})\\
y_{k-1} &= \frac{1}{N}p_{k-1}p_kx_{k-1},
\end{aligned}
\tag{2.4.6}
$$

then

$$
\begin{aligned}
x_1 &= \frac{N}{p_1}y_1-\frac{N}{p_2}y_2\\
x_2 &= \frac{N}{p_2}y_2-\frac{N}{p_3}y_3\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
x_{k-3} &= \frac{N}{p_{k-3}}y_{k-3}-\frac{N}{p_{k-2}}y_{k-2}\\
x_{k-2} &= \frac{N}{p_{k-2}}y_{k-2}-\frac{Ns}{p_{k-1}}y_{k-1}\\
x_{k-1} &= \frac{N}{p_{k-1}p_k}y_{k-1}
\end{aligned}
\tag{2.4.7}
$$

and

$$
\frac{\partial(x_1,\dots,x_{k-1})}{\partial(y_1,\dots,y_{k-1})}=\frac{N^{k-1}}{p_1p_2\dots p_k}.
$$

Define

$$\begin{aligned}G(y_1,\ldots,y_{k-1}) &= H(x_1,\ldots,x_{k-1})\\ &= H(\frac{N}{p_1}y_1-\frac{N}{p_2}y_2,\frac{N}{p_2}y_2-\frac{N}{p_3}y_3,\ldots,\\ &\qquad \frac{N}{p_{k-3}}y_{k-3}-\frac{N}{p_{k-2}}y_{k-2},\frac{N}{p_{k-2}}y_{k-2}-\frac{Ns}{p_{k-1}}y_{k-1},\frac{N}{p_{k-1}p_k}y_{k-1}).\end{aligned}$$

Then

$$\begin{aligned}\sum_{r_1,\ldots,r_{k-1}\in\mathbb{Z}} c_r &= \frac{1}{p_1\ldots p_k}\sum_{l_1,\ldots,l_{k-1}\in\mathbb{Z}}\int_{\mathbb{R}^{k-1}} G(y_1,\ldots,y_{k-1})e(-y_1l_1-\cdots-y_{k-1}l_{k-1})dy_1\ldots dy_{k-1}\\ &= \frac{1}{p_1\ldots p_k}\sum_{l_1,\ldots,l_{k-1}\in\mathbb{Z}}\widehat{G}(l_1,\ldots,l_{k-1})\\ &= \frac{1}{p_1\ldots p_k}\sum_{l_1,\ldots,l_{k-1}\in\mathbb{Z}} G(l_1,\ldots,l_{k-1}),\end{aligned}$$

by the Poisson summation formula.

So the first part of the last equality in (2.4.1) becomes

$$\begin{aligned}&\frac{1}{N}\sum_{p_1,\ldots,p_k\in B_Q\ \text{distinct}} p_1\cdots p_k \sum_{\substack{r\in\mathbb{Z}^{k-1}\\ p_1|r_1\\ p_2|r_2-r_1\\ \cdots\\ p_{k-1}|r_{k-1}-r_{k-2}\\ p_k|r_{k-1}}} c_{r_1,\ldots,r_{k-1}}\\ =&\frac{1}{N}\sum_{p_1,\ldots,p_k\in B_Q\ \text{distinct}}\ \sum_{l_1,\ldots,l_{k-1}\in\mathbb{Z}} H(\frac{N}{p_1}l_1-\frac{N}{p_2}l_2,\frac{N}{p_2}l_2-\frac{N}{p_3}l_3,\ldots,\\ &\qquad \frac{N}{p_{k-3}}l_{k-3}-\frac{N}{p_{k-2}}l_{k-2},\frac{N}{p_{k-2}}l_{k-2}-\frac{Ns}{p_{k-1}}l_{k-1},\frac{N}{p_{k-1}p_k}l_{k-1}).\end{aligned}$$

Approximate H by $\chi_{\mathcal{B}_\Lambda}$; the last double sum equals the number of integral solutions

$$\{p_1,\ldots,p_k\in B_Q\ \text{distinct}, l_1,\ldots,l_{k-1}\in\mathbb{Z}\},$$

such that

$$\begin{cases} 0<\frac{N}{p_1}l_1-\frac{N}{p_2}l_2<\Lambda_1,\\ 0<\frac{N}{p_2}l_2-\frac{N}{p_3}l_3<\Lambda_2,\\ \cdots\cdots\cdots\cdots\\ 0<\frac{N}{p_{k-3}}l_{k-3}-\frac{N}{p_{k-2}}l_{k-2}<\Lambda_{k-3},\\ 0<\frac{N}{p_{k-2}}l_{k-2}-\frac{Ns}{p_{k-1}}l_{k-1}<\Lambda_{k-2},\\ 0<\frac{N}{p_{k-1}p_k}l_{k-1}<\Lambda_{k-1}.\end{cases} \tag{2.4.8}$$

Let $z_{k-1} = l_{k-1}$, we have $z_{k-1} \in \mathbb{Z}^+$ and $z_{k-1} < \frac{p_{k-1}p_k}{N}\Lambda_{k-1}$.

Next, the second last inequality of (3.2.3) becomes

$$\frac{Ns}{p_{k-1}}z_{k-1} < \frac{N}{p_{k-2}}l_{k-2} < \frac{Ns}{p_{k-1}}z_{k-1} + \Lambda_{k-2},$$

i.e.

$$\frac{sp_{k-2}}{p_{k-1}}z_{k-1} < l_{k-2} < \frac{sp_{k-2}}{p_{k-1}}z_{k-1} + \frac{p_{k-2}}{N}\Lambda_{k-2}.$$

Suppose $sp_{k-2}z_{k-1} \equiv -z_{k-2} (\text{mod } p_{k-1})$, $0 < z_{k-2} \le p_{k-1}$. Then

$$p_{k-2}z_{k-1} + p_k z_{k-2} \equiv p_k(sp_{k-2}z_{k-1} + z_{k-2}) \equiv 0 \qquad (\text{mod } p_{k-1}),$$

so there exists $I \in \mathbb{Z}$ such that

$$I - \frac{z_{k-2}}{p_{k-1}} < l_{k-2} < I - \frac{z_{k-2}}{p_{k-1}} + \frac{p_{k-2}}{N}\Lambda_{k-2}.$$

Since $p_{k-2} = o(N)$, when $Q \to \infty$, $0 < \frac{p_{k-2}}{N}\Lambda_{k-2} < 1$. So there is at most 1 integral value for l_{k-2}, which exists if and only if $-\frac{z_{k-2}}{p_{k-1}} + \frac{p_{k-2}}{N}\Lambda_{k-2} > 0$, i.e. $0 < z_{k-2} < \frac{p_{k-2}p_{k-1}}{N}\Lambda_{k-2}$. We already have $p_{k-1} | p_{k-2}z_{k-1} + p_k z_{k-2}$, and under such conditions, $l_{k-2} = \dfrac{sp_{k-2}z_{k-1} + z_{k-2}}{p_{k-1}}$.

Next, consider l_{k-3}. We have

$$\frac{N}{p_{k-2}}l_{k-2} < \frac{N}{p_{k-3}}l_{k-3} < \frac{N}{p_{k-2}}l_{k-2} + \Lambda_{k-3},$$

i.e.

$$\frac{p_{k-3}}{p_{k-2}}l_{k-2} < l_{k-3} < \frac{p_{k-3}}{p_{k-2}}l_{k-2} + \frac{p_{k-3}}{N}\Lambda_{k-3}.$$

Let $p_{k-3}l_{k-2} \equiv -z_{k-3} (\text{mod } p_{k-2})$, $0 < z_{k-3} \le p_{k-2}$. Then the integer l_{k-3} exists if and only if $0 < z_{k-3} < \frac{p_{k-3}p_{k-2}}{N}\Lambda_{k-3}$, and

$$-p_{k-1}z_{k-3} \equiv p_{k-1}p_{k-3}l_{k-2} = p_{k-3}(sp_{k-2}z_{k-1} + z_{k-2}) \equiv p_{k-3}z_{k-2} (\text{ mod } p_{k-2}),$$

i.e.

$$p_{k-2} | p_{k-1}z_{k-3} + p_{k-3}z_{k-2}.$$

Continuing the above construction for $z_{k-4}, \ldots, z_1$, we find that the number of solutions of (3.2.3) is equal to the number of solutions satisfying

$$\{p_1, \ldots, p_k \in B_Q \text{ distinct}, z_1, \ldots, z_{k-1} \in \mathbb{Z}^+\},$$

under the conditions

$$z_j < \frac{p_{j-1}p_j}{N_Q}\Lambda_j, 1 \le j \le k-1,$$

and

$$p_i | z_{i+1}p_{i-1} + z_{i-1}p_{i+1}, 2 \le i \le k-1.$$

Thus, we obtain the main term of the last equality of (2.4.1).

Now consider the error term E in (2.4.2); it contains two types of sums.

The first type is

$$\frac{1}{N} \sum_{\substack{p_{i_1},\ldots,p_{i_m} \in B_Q \\ 0 \le m \le k-1}} p_{i_1} \ldots p_{i_m} (-|B_Q|)^{k-m} \sum_{\substack{r_{i_1}, r_{i_2}, \ldots, r_{i_m} \in \mathbb{Z} \\ p_{i_j} | r_{i_j} - r_{i_j - 1}, 1 \le j \le m \\ r_{i_j - 1} = 0 \text{ if } i_j = 1 \\ r_{i_j} = 0 \text{ if } i_j = k}} c_{r_1, \ldots, r_{k-1}}, \tag{2.4.9}$$

which implies that at least one of the following conditions is missing

$$\begin{cases} p_1 | r_1 \\ p_2 | r_2 - r_1 \\ \ldots \\ p_{k-1} | r_{k-1} - r_{k-2} \\ p_k | r_{k-1}. \end{cases}$$

If the first missing divisibility condition is $p_n | r_n - r_{n-1}$, $2 \le n \le k-1$, then $(r_1, \ldots, r_{k-1})$ can be expressed as

$$\begin{cases} r_1 = p_1 l_1, \\ r_2 = p_1 l_1 + p_2 l_2, \\ \ldots\ldots\ldots \\ r_{n-1} = p_1 l_1 + \cdots + p_{n-1} l_{n-1}, \\ r_n = l_n, \\ \ldots\ldots\ldots \end{cases}$$

Now, the inner sum of (2.4.9) becomes

$$
\begin{aligned}
\sum c_r &= \sum \frac{1}{N^{k-1}} \widehat{H}\left(\frac{r_1}{N}, \dots, \frac{r_{n-1}}{N}, \frac{r_n}{N}, \dots \frac{r_{k-1}}{N}\right) \\
&= \frac{1}{N^{k-1}} \sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}} \int_{\mathbb{R}^{k-1}} H(x_1,\dots,x_{k-1}) e(-\frac{p_1 l_1}{N}x_1 - \frac{p_1 l_1 + p_2 l_2}{N} x_2 - \dots \\
&\quad - \frac{p_1 l_1 + \dots + p_{n-1} l_{n-1}}{N} x_{n-1} - \frac{l_n}{N} x_n - \dots) dx_1 \dots dx_{k-1} \\
&= \frac{1}{N^{k-1}} \sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}} \int_{\mathbb{R}^{k-1}} H(x_1,\dots,x_{k-1}) e(-\frac{p_1 x_1 + \dots + p_1 x_{n-1}}{N} l_1 \\
&\quad - \frac{p_2 x_2 + \dots + p_2 x_{n-1}}{N} l_2 - \dots - \frac{p_{n-1} x_{n-1}}{N} l_{n-1} - \dots) dx_1 \dots dx_{k-1}.
\end{aligned}
$$

Since there appears a free term $\frac{p_{n-1}x_{n-1}}{N} l_{n-1}$, the last equality becomes

$$
\begin{aligned}
\sum c_r &= \sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}} \frac{1}{N^{k-1}} \widehat{H}\left(\dots, \frac{N}{p_{n-1}} l_{n-1}, \dots\right) \\
&= \sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}} \frac{1}{N^{k-1}} H\left(\dots, \frac{N}{p_{n-1}} l_{n-1}, \dots\right),
\end{aligned}
$$

by the Poisson summation formula.

Now approximating H by $\chi_{\mathcal{B}_\Lambda}$, we must have $0 < \frac{N}{p_{n-1}} l_{n-1} < \Lambda_{n-1}$, i.e. $0 < l_{n-1} < \frac{p_{n-1}}{N}\Lambda_{n-1}$. Since $p_{n-1} = o(N)$, when $Q \to \infty$, there is no integral value for l_{n-1}. So in this case (2.4.9) vanishes.

If the condition $p_1|r_1$ or $p_k|r_{k-1}$ is missing, by symmetry, we may assume the latter is missing. Then r_{k-1} is a free variable , or it can be expressed as $r_{k-1} = p_{k-1}l_{k-1} + \cdots$, so that l_{k-1} is a free variable. For either case, by using the same deduction as above with the Poisson summation formula, we have the inner sum

$$
\sum c_r = \sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}} \frac{1}{N^{k-1}} H\left(\dots, N l_{k-1}\right) \text{ or } \sum_{l_1,\dots,l_{k-1}\in\mathbb{Z}} \frac{1}{N^{k-1}} H\left(\dots, \frac{N}{p_{k-1}} l_{k-1}\right)
$$

and they both vanish as $Q \to \infty$.

The second type of sum in E is of the form

$$
\frac{1}{N} \sum_{\substack{p_1,\dots,p_k \in B_Q \\ p_i = p_j}} p_1 \cdots p_k \sum_{\substack{r\in\mathbb{Z}^{k-1} \\ p_1|r_1 \\ p_2|r_2-r_1 \\ \cdots \\ p_{k-1}|r_{k-1}-r_{k-2} \\ p_k|r_{k-1}}} c_{r_1,\dots,r_{k-1}}. \tag{2.4.10}
$$

Thus, at least two prime factors are the same. We consider two cases.

First, if p_k is different from any of $p_1, \dots p_{k-1}$, then our deductions of (3.2.2), (2.4.5), (2.4.6), (2.4.7), (3.2.3) remain valid. Now we must have $p_i = p_j$, $1 \le i, j \le k-1$, and by (3.2.3),

$$0 < \frac{N}{p_i} l_i - \frac{N}{p_j} l_j < \Lambda_i + \cdots + \Lambda_{j-1},$$

so

$$0 < l_i - l_j < \frac{p_i}{N}(\Lambda_i + \cdots + \Lambda_{j-1}).$$

Since $p_i = o(N)$, when $Q \to \infty$; l_i, l_j do not exist. So the whole sum (2.4.10) vanishes.

Second, if $p_k = p_i$, for some $i < k$, then return to (2.4.3); the divisibility condition

$$p_k | p_1 l_1 + \cdots + p_i l_i + \cdots + p_{k-1} l_{k-1},$$

produces a solution lattice that contains l_i as a free variable. So (2.4.10) still vanishes. This completes the proof of Theorem 2.4.

2.5 Proof of Theorem 2.5

We construct the sequence B_Q inductively, B_1 can be arbitrarily chosen.
If $B_1 \subset B_2 \subset \cdots \subset B_{l-1}$ are decided, such that $\frac{M_{l-1}}{N_{l-1}^2} = O(\frac{1}{l-1})$. We choose an arithmetic progression of primes of $t = 10M_{l-1}l$ elements $r_1, \ldots, r_t$, and take $q_1, \ldots q_l$ to be the last l elements, $q_1 = r_{t-l+1}, q_2 = r_{t-l+2}, \ldots, q_l = r_t$. We add these l primes into B_{l-1} to make B_l.

Now $q_1 > 9M_{l-1}l$, so

$$\begin{aligned}\frac{M_l}{N_l^2} &= \frac{M_{l-1} + q_1^2 + \cdots + q_l^2}{(N_{l-1} + q_1 + \cdots + q_l - l)^2} \\ &< \frac{(l+1)q_l^2}{(lq_1)^2} < \frac{2}{l}\frac{q_l^2}{q_1^2} \\ &< \frac{2}{l}(1 + \frac{q_l - q_1}{q_1})^2 < \frac{2}{l}(1 + \frac{l}{9l})^2 < \frac{3}{l}.\end{aligned}$$

From the construction of B_Q, $\frac{M_Q}{N_Q^2} = O(\frac{1}{Q}) = o(1)$. Thus, by using Theorem 2.1; we immediately see that the limiting pair correlation of S_Q is Poissonian.

For the higher correlations, we appeal to Theorem 2.4. Under this condition, we can find quite a number of solutions for the linear equation system, for $l > k$. We fix the box $\mathcal{B} = (0, \Lambda)^{k-1}$.

If $(p_1, p_2, \ldots, p_k) = (q_1, q_2 \ldots, q_k)$, then $z_1 = z_2 = \cdots = z_{k-1} = 1, 2, 3, \ldots, [\frac{q_1 q_2}{N}\Lambda]$ will all satisfy the linear equations, and so we have $[\frac{q_1 q_2}{N}\Lambda]$ solutions.

Similarly,

$$(p_1,\ldots,p_k)=(q_2,\ldots,q_{k+1}),(q_3,\ldots,q_{k+2}),\ldots,(q_{l-k+1},\ldots,q_l),$$

will provide, respectively,

$$[\frac{q_2q_3}{N}\Lambda],[\frac{q_3q_4}{N}\Lambda],\ldots,[\frac{q_{l-k+1}q_{l-k+2}}{N}\Lambda],$$

solutions.

We may then take the gap of the arithmetic progression to be 2, and let

$$(p_1,p_2,\ldots,p_k)=(q_1,q_3,\ldots,q_{2k-1}),(q_2,q_4,\ldots,q_{2k}),\ldots,(q_{l-2k+2},q_{l-2k+4},\ldots,q_l),$$

to obtain, respectively,

$$[\frac{q_1q_3}{N}\Lambda],[\frac{q_2q_4}{N}\Lambda],\ldots,[\frac{q_{l-2k+2}q_{l-2k+4}}{N}\Lambda],$$

additional solutions.

Taking the gaps of lengths to be $3,4,\ldots,$ until $[\frac{l-1}{k}]$, we obtain correspondingly

$$[\frac{q_1q_4}{N}\Lambda],[\frac{q_2q_5}{N}\Lambda],\ldots,[\frac{q_{l-3k+3}q_{l-3k+4}}{N}\Lambda],$$

$$[\frac{q_1q_5}{N}\Lambda],[\frac{q_2q_6}{N}\Lambda],\ldots,[\frac{q_{l-4k+4}q_{l-4k+5}}{N}\Lambda],$$

$$\cdots\cdots\cdots\cdots$$

$$[\frac{q_1q_{1+[\frac{l-1}{k}]}}{N}\Lambda],[\frac{q_2q_{2+[\frac{l-1}{k}]}}{N}\Lambda],\ldots,[\frac{q_{l-[\frac{l-1}{k}](k-1)}q_{l-[\frac{l-1}{k}](k-1)+1}}{N}\Lambda],$$

solutions.

Now we obtain at least

$$\sum_{j=1}^{[\frac{l-1}{k}]}\sum_{i=1}^{l-j(k-1)}\left[\frac{q_iq_{i+j}}{N}\Lambda\right]>\sum_{j=1}^{[\frac{l-1}{k}]}(l-j(k-1))q_1^2\frac{\Lambda}{N}>\frac{k}{4}l^2q_1^2\frac{\Lambda}{N}$$

solutions.

Thus, the k-level correlation

$$\mathcal{R}_{S_Q}^{(k)}(\mathcal{B})>\frac{1}{N}\frac{k}{4}l^2q_1^2\frac{\Lambda}{N}=\frac{kl^2q_1^2}{(l(q_1+q_l))^2}\Lambda>\frac{k\Lambda}{16},$$

and so when Λ is small enough, $\Lambda^{k-1}<\dfrac{k\Lambda}{16}<\mathcal{R}_{S_Q}^{(k)}(\mathcal{B})$. Thus, all the limiting k-level correlations are non-Poissonian for $k \geq 3$.

Chapter 3

Correlations of fractions with power denominators

3.1 Statement of main results

In [4], the pair correlation of Farey fractions is proved to exist. The Farey fractions of order Q are defined as the set of reduced fractions in the closed interval [0,1] with denominators $\leq Q$ listed in increasing order of magnitude, and it is denoted as F_Q.

We can continue to consider the fractions with power denominators. Let

$$F_{Q,k} = \{\frac{a}{q^k} : 1 \leq q \leq Q, 1 \leq a \leq q^k - 1\}.$$

Theorem 3.1. *When $Q \to \infty$, for $k \geq 2$, the limiting pair correlation of $F_{Q,k}$ is always Poissonian.*

We can generalize Theorem 3.1 to fractions with denominators as k^{th} powers of elements in an arithmetic progression. Let

$$F_{Q,k}^{m,b} = \{\frac{a}{q^k} : 1 \leq q \leq Q, q \equiv b \pmod{m}, 1 \leq a \leq q^k - 1\}.$$

Theorem 3.2. *When $Q \to \infty$, the limiting pair correlation of $F_{Q,n}^{m,b}$ exists.*

If $n \geq 2$, it is always Poissonian.

If $n = 1$ and $(m, b) = 1$, the pair correlation density function is given by

$$g_m(\lambda) = \frac{1}{L(2,\chi_0)} \frac{1}{\lambda^2} \sum_{T < 2m\lambda} \psi_m(T) \log \frac{2m\lambda}{T},$$

where χ_0 is the principal character mod m, and

$$\psi_m(T) = \frac{1}{m^2} \sum_{(\frac{T}{l}, m)=1} l.$$

If $n = 1$ and $(m, b) = d > 1$, the pair correlation function is $g_{\frac{m}{d}}(\lambda)$.

Since there are multiplicities in the sequence $F_{Q,k}^{m,b}$, we have

Theorem 3.3. *If $\nu \geq k+2$, then when $Q \to \infty$, the limiting ν-level correlation of $F_{Q,k}^{m,b}$ doesn't exist.*

If $m = 1$, we have a way to remove the multiplicities. For an integer n, we say n is k^{th}-powerfree if n has no factor as a k^{th} power. Let

$$F_Q^k = \{\frac{a}{q^k} : 1 \leq a \leq q^k, 1 \leq q \leq Q, (a, q^k) \text{ is } k^{th} - \text{powerfree}\}.$$

It is easy to see every two elements in F_Q^k are nonequal.

When $k = 1$, F_Q^k is just the sequence of Farey fractions of order Q. In [4], its pair correlation density function is proved to be

$$g_2(\lambda) = \frac{6}{\pi^2\lambda^2} \sum_{1 \leq k < \frac{\pi^2\lambda}{3}} \varphi(k) \log \frac{\pi^2\lambda}{3k}. \tag{3.1.1}$$

However, for $k \geq 2$ we have different results.

Theorem 3.4. *If $k \geq 2$, then when $Q \to \infty$, the limiting pair correlation of F_Q^k is Poissonian.*

3.2 Preliminaries

We choose a smooth real-valued function H such that $\text{supp}(H) \subset (0, \Lambda)$, The Fourier transform of H is defined by

$$\widehat{H}(x) = \int_{\mathbb{R}} H(y)e(-x \cdot y)dy, \qquad x \in \mathbb{R}.$$

Consider the periodic function f given by

$$f(y) = f_Q(y) = \sum_{r \in \mathbb{Z}} H\big(N(y+r)\big), \qquad y \in \mathbb{R}$$

and the smooth pair correlation sum defined by

$$PC = PC(Q, H) = \frac{1}{N} \sum_{\gamma_1 \neq \gamma_2 \in F_{Q,k}^{m,b}} f(\gamma_1 - \gamma_2).$$

We will show that $PC(Q, H)$ has a limit as $Q \to \infty$ and that there exists a continuous function $g : \mathbb{R} \to [0, \infty)$ such that

$$\lim_{Q \to \infty} PC(Q, H) = \int_{\mathbb{R}} H(x)g(x)dx.$$

The Fourier coefficients in the Fourier series

$$f(y) = \sum_{r \in \mathbb{Z}} c_r e(ry)$$

of f are given by

$$\begin{aligned} c_r &= \int_0^1 f(y)e(-ry)dy = \int_0^1 e(-ry) \sum_{n \in \mathbb{Z}} H\big(N(y+n)\big)dy \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 e(-ry) H\big(N(y+n)\big)dy \\ &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} e\big(-r(u-n)\big)H(Nu)du \\ &= \int_{\mathbb{R}} e(-ru)H(Nu)du = \frac{1}{N}\int_{\mathbb{R}} e\left(-\frac{ry}{N}\right)H(y)dy \\ &= \frac{1}{N}\widehat{H}\left(\frac{1}{N}\, r\right). \end{aligned} \tag{3.2.1}$$

For each $y > 0$, consider the function

$$H_y(x) = \frac{1}{y} H\left(\frac{x}{y}\right), \qquad x \in \mathbb{R}.$$

Then

$$\widehat{H_y}(z) = \widehat{H}(yz).$$

By Poisson's summation formula,

$$\sum_{r \in \mathbb{Z}} \widehat{H_y}(r) = \sum_{r \in \mathbb{Z}} H_y(r). \tag{3.2.2}$$

So we have

$$\sum_{r \in \mathbb{Z}} \widehat{H}(yr) = \sum_{r \in \mathbb{Z}} H_y(r) = \sum_{r \in \mathbb{Z}} \frac{1}{y} H(\frac{r}{y}). \tag{3.2.3}$$

We also need some results for counting lattice points in a bounded region.

Let $\Gamma_{m,i,j}$ be the lattice points (a, b) on the plane such that $a \equiv i \pmod m, b \equiv j \pmod m$, Ω be a bounded region with rectifiable boundary $\partial\Omega$, and f be a C^1 function on Ω. Let

$$S = \sum_{(a,b) \in \Omega \cap \Gamma_{m,i,j}} f(a, b).$$

Lemma 3.5. *Suppose that Ω and f are as above. Then*

$$\left| S - \frac{1}{m^2} \iint_{\Omega} f(x,y)dxdy \right| \ll_m \left(\left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial f}{\partial y} \right\|_\infty \right) Area(\Omega) + \|f\|_\infty (1 + length(\partial\Omega)).$$

Let

$$S' = \sum_{\substack{(a,b) \in \Omega \cap \Gamma_{m,i,j} \\ (a,b)=1}} f(a,b).$$

Lemma 3.6. *In addition to the hypotheses of Lemma 3.5, suppose $(m,i) = (m,j) = 1$, and $\Omega \subset [1,R] \times [1,R]$. Then*

$$\left| S' - \frac{1}{L(2,\chi_0)} \frac{1}{m^2} \iint_{\Omega} f(x,y)dxdy \right| \ll_m \left(\left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial f}{\partial y} \right\|_\infty \right) Area(\Omega) \log R$$
$$+ \|f\|_\infty (R + length(\partial\Omega) \log R).$$

The proofs of Lemma 3.5 and Lemma 3.6 are similar to those for Lemma 5.6 and Lemma 5.7 in section 5.3.

3.3 Proof of Theorem 3.2

The cardinality of the set $F_{Q,k}^{m,b}$ is

$$N = N_Q = \sum_{\substack{q \le Q \\ q \equiv b (\bmod m)}} (q^k - 1) = \frac{Q^{k+1}}{m(k+1)} + O_m(Q^k). \tag{3.3.1}$$

For $l \in \mathbb{Z}$, consider the exponential sum

$$\sum_{\gamma \in F_{Q,k}^{m,b}} e(l \cdot \gamma) = \sum_{\substack{q \le Q \\ q \equiv b (\bmod m)}} \sum_{a=1}^{q^k - 1} e\left(\frac{la}{q^k}\right). \tag{3.3.2}$$

The inner sum of the right side of (3.3.2) is

$$\sum_{a=1}^{q^k-1} e\left(\frac{la}{q^k}\right) = \begin{cases} q^k - 1, & q^k | l \\ -1, & q^k \nmid l \end{cases}.$$

So (3.3.2) becomes

$$\sum_{\gamma\in F_{Q,k}^{m,b}} e(l\cdot\gamma) = \sum_{\substack{q^k|l,q\le Q\\ q\equiv b(\text{mod } m)}} q^k - \left[\frac{Q+m-b}{m}\right]. \tag{3.3.3}$$

Next, we first consider the case $(m,b)=1$, and later we will see that it can simplify some computation.

By the same notation in last chapter, the pair correlation sum of $F_{Q,k}^{m,b}$ is

$$\begin{aligned} PC &= \frac{1}{N}\sum_{r\in\mathbb{Z}} c_r \sum_{\gamma_1,\gamma_2\in\mathcal{F}_Q} e\big(r(\gamma_2-\gamma_1)\big) = \frac{1}{N}\sum_{r\in\mathbb{Z}} c_r \left|\sum_{\gamma\in\mathcal{F}_Q} e(r\gamma)\right|^2 \\ &= \frac{1}{N}\sum_{r\in Z} c_r\Bigg(\sum_{\substack{q_1^k|r,q_2^k|r\\ q_1,q_2\le Q\\ q_1\equiv q_2\equiv b(\text{mod } m)}} q_1^k q_2^k - 2\left[\frac{Q+m-b}{m}\right]\sum_{\substack{q^k|r,q\le Q\\ q\equiv b(\text{mod } m)}} q^k + \left[\frac{Q+m-b}{m}\right]^2\Bigg) \\ &= \frac{1}{N}\Bigg(\sum_{\substack{q_1,q_2\le Q\\ q_1\equiv q_2\equiv b(\text{mod } m)}} q_1^k q_2^k \sum_{[q_1^k,q_2^k]|r} c_r - 2\left[\frac{Q+m-b}{m}\right]\sum_{\substack{q\le Q\\ q\equiv b(\text{mod } m)}} q^k \sum_{q^k|r} c_r \\ &\qquad + \left[\frac{Q+m-b}{m}\right]^2 \sum_{r\in\mathbb{Z}} c_r\Bigg). \end{aligned} \tag{3.3.4}$$

By (3.2.3), we have

$$\begin{aligned} \sum_{[q_1^k,q_2^k]|r} c_r &= \sum_{l\in\mathbb{Z}} \frac{1}{N}\widehat{H}\left(\frac{[q_1^k,q_2^k]}{N} l\right) = \sum_{l\in\mathbb{Z}} \frac{1}{[q_1^k,q_2^k]} H\left(\frac{lN}{[q_1^k,q_2^k]}\right), \\ \sum_{q^k|r} c_r &= \sum_{l\in\mathbb{Z}} \frac{1}{N}\widehat{H}\left(\frac{q^k}{N} l\right) = \sum_{l\in\mathbb{Z}} \frac{1}{q^k} H\left(\frac{lN}{q^k}\right), \\ \sum_{r\in Z} c_r &= \sum_{l\in\mathbb{Z}} H(lN). \end{aligned} \tag{3.3.5}$$

Since $supp(H)\subset(0,\Lambda)$, when Q is large enough, we have

$$\sum_{q^k|r} c_r = \sum_{r\in\mathbb{Z}} c_r = 0. \tag{3.3.6}$$

Thus, the pair correlation sum becomes

$$PC = \frac{1}{N}\sum_{\substack{q_1,q_2\le Q\\ q_1\equiv q_2\equiv b(\text{mod } m)}} (q_1,q_2)^k \sum_{l\in\mathbb{Z}} H\left(\frac{lN}{[q_1,q_2]^k}\right). \tag{3.3.7}$$

Let $(q_1,q_2)=\delta, q_1=\delta t_1, q_2=\delta t_2, (t_1,t_2)=1$. Since $\text{supp}(H)\subset(0,\Lambda)$, we only need to consider l such that

$$0<\frac{lN}{[q_1,q_2]^k}=\frac{lN}{\delta^k t_1^k t_2^k}<\Lambda,$$

i.e.

$$l\delta^k<(\delta t_1)^k(\delta t_2)^k\Lambda/N<m(k+1)\Lambda Q^{k-1}.$$

If we define $c=m(k+1)\Lambda$, then

$$PC=\frac{1}{N}\sum_{l\delta^k<cQ^{k-1}}\delta^k\sum_{\substack{\delta t_1\equiv\delta t_2\equiv b(\text{mod } m)\\ t_1,t_2\le Q/\delta\\ (t_1,t_2)=1}}H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right). \tag{3.3.8}$$

We can use Dirichlet's characters to remove the divisibility conditions. Let χ_i, $i=0,1,2,\dots,\phi(m)-1$ be all Dirichlet's characters mod m, in which χ_0 is the principal character. It is well known that

$$\frac{1}{\phi(m)}\left(\sum_{j=0}^{\phi(m)-1}\chi_j(xb^{-1})\right)=\begin{cases}1, & x\equiv b \pmod m,\\ 0, & else.\end{cases} \tag{3.3.9}$$

Thus,

$$\begin{aligned}PC&=\frac{1}{N}\sum_{\substack{l\delta^k<cQ^{k-1}\\(\delta,m)=1}}\delta^k\sum_{\substack{t_1,t_2\le Q/\delta\\(t_1,t_2)=1}}H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\frac{1}{\phi(m)}\left(\sum_{i=0}^{\phi(m)-1}\chi_i(\delta t_1 b^{-1})\right)\frac{1}{\phi(m)}\left(\sum_{j=0}^{\phi(m)-1}\chi_j(\delta t_2 b^{-1})\right)\\
&=\frac{1}{N\phi^2(m)}\sum_{\substack{l\delta^k<cQ^{k-1}\\(\delta,m)=1}}\delta^k\sum_{\substack{t_1,t_2\le Q/\delta\\(t_1,t_2)=1}}H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_0(t_1)\chi_0(t_2)+\\
&\sum_{\substack{0\le i,j\le\phi(m)-1\\ i\ne0 \text{ or } j\ne0}}\frac{1}{N\phi^2(m)}\chi_i(\delta b^{-1})\chi_j(\delta b^{-1})\sum_{\substack{l\delta^k<cQ^{k-1}\\(\delta,m)=1}}\delta^k\sum_{\substack{t_1,t_2\le Q/\delta\\(t_1,t_2)=1}}H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_i(t_1)\chi_j(t_2).\end{aligned} \tag{3.3.10}$$

We will show that the first part of the right side of (3.3.10) contributes the main term and the second part of the right side of (3.3.10) is contained in the error term.

Without loss of generality, suppose $j \neq 0$. The inner sum of the second part of (3.3.10) is

$$\begin{aligned}
&\sum_{\substack{l\delta^k < cQ^{k-1} \\ (\delta,m)=1}} \delta^k \sum_{\substack{t_1,t_2 \le Q/\delta \\ (t_1,t_2)=1}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_i(t_1)\chi_j(t_2) \\
=&\sum_{\substack{l\delta^k < cQ^{k-1} \\ (\delta,m)=1}} \delta^k \sum_{t_1,t_2 \le Q/\delta} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_i(t_1)\chi_j(t_2) \sum_{d|(t_1,t_2)} \mu(d) \\
=&\sum_{\substack{l\delta^k < cQ^{k-1} \\ (\delta,m)=1}} \delta^k \sum_{d \le Q/\delta} \mu(d) \sum_{s_1,s_2 \le Q/d\delta} H\left(\frac{lN}{\delta^k d^{2k} s_1^k s_2^k}\right)\chi_i(ds_1)\chi_j(ds_2) \\
=&\sum_{\substack{l\delta^k < cQ^{k-1} \\ (\delta,m)=1}} \delta^k \sum_{d \le Q/\delta} \mu(d) \sum_{s_1 \le Q/d\delta} \chi_i(ds_1)\chi_j(d) \sum_{s_2 \le Q/d\delta} H\left(\frac{lN}{\delta^k d^{2k} s_1^k s_2^k}\right)\chi_j(s_2).
\end{aligned} \tag{3.3.11}$$

Let $A(t) = \sum_{s_2 \le t} \chi_j(s_2)$, by partial summation, the inner sum of the last line of (3.3.11) is

$$\begin{aligned}
&\sum_{s_2 \le Q/d\delta} H\left(\frac{lN}{\delta^k d^{2k} s_1^k s_2^k}\right)\chi_j(s_2) \\
=&H\left(\frac{lN}{\delta^k d^{2k} s_1^k (Q/d\delta)^k}\right)A(Q/d\delta) - \int_1^{Q/d\delta} A(t)H'\left(\frac{lN}{\delta^k d^{2k} s_1^k t^k}\right)\frac{lN}{\delta^k d^{2k} s_1^k t^k}\frac{1}{k+1}\frac{1}{t}dt \\
\ll&\|H\|_\infty \phi(m) + \phi(m)\|H'\|_\infty \Lambda \frac{1}{k+1}\log Q \\
\ll&_{m,k,\Lambda,H} \log Q.
\end{aligned} \tag{3.3.12}$$

Now (3.3.11) can be estimated as

$$\begin{aligned}
&\sum_{\substack{l\delta^k < cQ^{k-1} \\ (\delta,m)=1}} \delta^k \sum_{\substack{t_1,t_2 \le Q/\delta \\ (t_1,t_2)=1}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_i(t_1)\chi_j(t_2) \\
\ll_{m,k,\Lambda,H}& \sum_{l\delta^k < cQ^{k-1}} \delta^k \sum_{d \le Q/\delta} |\mu(d)| \sum_{s_1 \le Q/d\delta} |\chi_i(ds_1)\chi_j(d)| \log Q \\
\ll_{m,k,\Lambda,H}& \sum_{l\delta^k < cQ^{k-1}} \delta^k \sum_{d \le Q/\delta} \frac{Q}{d\delta} \log Q \\
\ll_{m,k,\Lambda,H}& \sum_{\delta \le c^{1/k}Q^{1-1/k}} \delta^k \sum_{l \le cQ^{k-1}/\delta^k} \frac{Q}{\delta} \log^2 Q \\
\ll_{m,k,\Lambda,H}& \sum_{\delta \le c^{1/k}Q^{1-1/k}} \frac{Q^k}{\delta} \log^2 Q \\
\ll_{m,k,\Lambda,H}& Q^k \log^3 Q.
\end{aligned} \tag{3.3.13}$$

Therefore, the second part of the right side of (3.3.10) is estimated as

$$
\begin{aligned}
&\sum_{\substack{0\le i,j\le \phi(m)-1\\ i\neq 0 \text{ or } j\neq 0}} \frac{1}{N\phi^2(m)}\chi_i(\delta b^{-1})\chi_j(\delta b^{-1}) \sum_{\substack{l\delta^k<cQ^{k-1}\\ (\delta,m)=1}} \delta^k \sum_{\substack{t_1,t_2\le Q/\delta\\ (t_1,t_2)=1}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_i(t_1)\chi_j(t_2)\\
&\ll_{m,k,\Lambda,H} (\phi^2(m)-1)\frac{1}{Q^{k+1}\phi^2(m)}Q^k\log^3 Q\\
&\ll_{m,k,\Lambda,H} \frac{\log^3 Q}{Q}\\
&= o_{m,k,\Lambda,H}(1).
\end{aligned}
\tag{3.3.14}
$$

Next, we consider the first part of the right side of (3.3.10), which can be simplified to

$$
\begin{aligned}
&\frac{1}{N\phi^2(m)} \sum_{\substack{l\delta^k<cQ^{k-1}\\ (\delta,m)=1}} \delta^k \sum_{\substack{t_1,t_2\le Q/\delta\\ (t_1,t_2)=1}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right)\chi_0(t_1)\chi_0(t_2)\\
&\quad = \frac{1}{N\phi^2(m)} \sum_{\substack{l\delta^k<cQ^{k-1}\\ (\delta,m)=1}} \delta^k \sum_{\substack{t_1,t_2\le Q/\delta\\ (t_1,m)=(t_2,m)=1\\ (t_1,t_2)=1}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right).
\end{aligned}
\tag{3.3.15}
$$

Let R_m be the reduced residue class mod m, the right side of (3.3.15) can be written as

$$
\frac{1}{N\phi^2(m)} \sum_{i,j\in R_m} \sum_{\substack{l\delta^k<cQ^{k-1}\\ (\delta,m)=1}} \delta^k \sum_{\substack{t_1,t_2\le Q/\delta\\ t_1\equiv i(\text{mod } m),t_2\equiv j(\text{mod } m)\\ (t_1,t_2)=1,t_1t_2>(\frac{lN}{\delta^k\Lambda})^{1/k}}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right). \tag{3.3.16}
$$

We will show that for every pair (i,j), the above inner sums contribute same main terms.

By Lemma 3.6, the inner sum of (3.3.16) can be written as

$$
\sum_{\substack{t_1,t_2\le Q/\delta\\ t_1\equiv i(\text{mod } m),t_2\equiv j(\text{mod } m)\\ (t_1,t_2)=1,t_1t_2>(\frac{lN}{\delta^k\Lambda})^{1/k}}} H\left(\frac{lN}{\delta^k t_1^k t_2^k}\right) = \frac{1}{L(2,\chi_0)}\frac{1}{m^2}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{\frac{Q}{\delta}}}^{\frac{Q}{\delta}}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{t_1}}^{\frac{Q}{\delta}} H(\frac{lN}{\delta^k t_1^k t_2^k})dt_1dt_2 + E_{l,\delta,i,j}, \tag{3.3.17}
$$

where the error term $E_{l,\delta,i,j}$ is estimated as

$$
\begin{aligned}
E_{l,\delta,i,j} &\ll_m (\|\frac{\partial H}{\partial x}\|\frac{lN}{\delta^k t_2^k t_1^{k+1}} + \|\frac{\partial H}{\partial y}\|\frac{lN}{\delta^k t_1^k t_2^{k+1}})\frac{Q^2}{\delta^2}\log Q + \|H\|(\log Q + \frac{Q}{\delta}\log Q)\\
&\ll_{m,H} (\frac{1}{t_1}+\frac{1}{t_2})\frac{Q^2\log Q}{\delta^2} + \frac{Q\log Q}{\delta}.
\end{aligned}
\tag{3.3.18}
$$

Since $t_1, t_2 \le Q/\delta$, and $t_1 t_2 > (\frac{lN}{\delta^k \Lambda})^{1/k}$, we have $1/t_1, 1/t_2 < \frac{Q}{\delta}/(\frac{lN}{\delta^k\Lambda})^{1/k}$. Thus, from (3.3.18),

$$E_{l,\delta,i,j} \ll_{m,H,\Lambda} \frac{Q^{2-1/k}\log Q}{\delta^2 l^{1/k}} + \frac{Q\log Q}{\delta}. \tag{3.3.19}$$

Therefore, the sum of all these error terms $E_{l,\delta,i,j}$ in (3.3.16) will be

$$\begin{aligned} &\frac{1}{N\phi^2(m)} \sum_{i,j\in R_m} \sum_{\substack{l\delta^k<cQ^{k-1}\\(\delta,m)=1}} \delta^k \cdot E_{l,\delta,i,j} \\ &\ll_{m,H,\Lambda} \frac{1}{N} \sum_{l\delta^k<cQ^{k-1}} \delta^k \big(\frac{Q^{2-1/k}\log Q}{\delta^2 l^{1/k}} + \frac{Q\log Q}{\delta}\big) \\ &\ll_{m,k,H,\Lambda} \frac{\log^2 Q}{Q}. \end{aligned} \tag{3.3.20}$$

Returning to (3.3.17), let $\lambda = \frac{lN}{\delta^k t_1^k t_2^k}$. Then $dt_2 = (\frac{lN}{\delta^k t_1^k})^{1/k}(-1/k)\lambda^{-1-1/k}d\lambda$. The main term of (3.3.17) becomes

$$\begin{aligned} &\frac{1}{L(2,\chi_0)}\frac{1}{m^2}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{\frac{Q}{\delta}}}^{\frac{Q}{\delta}} \int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{t_1}}^{\frac{Q}{\delta}} H(\frac{lN}{\delta^k t_1^k t_2^k})dt_1 dt_2 \\ =&\frac{1}{L(2,\chi_0)}\frac{1}{m^2}\int_{(\frac{lN}{\Lambda})^{1/k}\frac{1}{Q}}^{\frac{Q}{\delta}} \int_{\Lambda}^{\frac{lN}{t_1^k Q^k}} -H(\lambda)\frac{1}{k}\lambda^{-1-1/k}\frac{(lN)^{1/k}}{\delta t_1} d\lambda dt_1 \\ =&\frac{1}{L(2,\chi_0)}\frac{1}{m^2}\int_{\frac{lN}{Q^k}/(Q/\delta)^k}^{\Lambda} \int_{(\frac{lN}{\lambda})^{1/k}\frac{1}{Q}}^{\frac{Q}{\delta}} \frac{H(\lambda)(lN)^{1/k}}{\lambda^{1+1/k}k\delta t_1} dt_1 d\lambda \\ =&\frac{1}{L(2,\chi_0)}\frac{1}{m^2}\int_{\frac{l\delta^k N}{Q^{2k}}}^{\Lambda} \frac{H(\lambda)(lN)^{1/k}}{\lambda^{1+1/k}k\delta} \log(\frac{Q^2}{\delta}(\frac{\lambda}{lN})^{1/k})d\lambda. \end{aligned} \tag{3.3.21}$$

Now combine (3.3.14), (3.3.20) and (3.3.21), and let $T = l\delta^k$. The pair correlation sum (3.3.10) becomes

$$\begin{aligned} PC = &\frac{1}{N\phi^2(m)}\phi^2(m) \sum_{\substack{T=l\delta^k<cQ^{k-1}\\(\delta,m)=1}} \delta^k \frac{1}{L(2,\chi_0)}\frac{1}{m^2} \\ &\sum_{1\le T<cQ^{k-1}} \int_{\frac{TN}{Q^{2k}}}^{\Lambda} \frac{H(\lambda)}{\lambda^{1+1/k}k} N^{1/k}\frac{l^{1/k}}{\delta}\log\frac{Q^2\lambda^{1/k}}{T^{1/k}N^{1/k}}d\lambda + o_{m,k,H,\Lambda}(1) \\ = &\frac{1}{L(2,\chi_0)}\frac{1}{Nm^2} \sum_{1\le T<cQ^{k-1}} \int_{\frac{TN}{Q^{2k}}}^{\Lambda} \frac{H(\lambda)}{\lambda^{1+1/k}k} N^{1/k}\log\frac{Q^2\lambda^{1/k}}{T^{1/k}N^{1/k}} \sum_{\substack{l\delta^k=T\\(\delta,m)=1}} \delta^{k-1}l^{1/k}d\lambda + o(1). \end{aligned} \tag{3.3.22}$$

If $k=1$, then

$$\begin{aligned} PC &= \frac{1}{L(2,\chi_0)}\frac{1}{N}\sum_{1\le T<c}\int_{\frac{TN}{Q^2}}^{\Lambda}\frac{H(\lambda)}{\lambda^2}N\log\frac{Q^2\lambda}{TN}\frac{1}{m^2}\sum_{\substack{l\delta=T\\(\delta,m)=1}} l d\lambda + o(1) \\ &= \frac{1}{L(2,\chi_0)}\sum_{1\le T<c}\int_{\frac{TN}{Q^2}}^{\Lambda}\frac{H(\lambda)}{\lambda^2}\log\frac{Q^2\lambda}{TN}\psi_m(T)d\lambda + o(1) \\ &= \frac{1}{L(2,\chi_0)}\int_0^{\Lambda}\frac{H(\lambda)}{\lambda^2}\sum_{T<2m\lambda}\psi_m(T)\log\frac{Q^2\lambda}{TN}d\lambda + o(1) \\ &= \int_0^{\Lambda}H(\lambda)\frac{1}{L(2,\chi_0)}\frac{1}{\lambda^2}\sum_{T<2m\lambda}\psi_m(T)\log\frac{2m\lambda}{T}d\lambda + o(1). \end{aligned} \tag{3.3.23}$$

For $k\ge 2$,

$$\begin{aligned} PC &= \frac{1}{L(2,\chi_0)}\frac{1}{Nm^2}\sum_{1\le T<cQ^{k-1}}\int_{\frac{TN}{Q^{2k}}}^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k^2}(TN)^{1/k}\log\frac{Q^{2k}\lambda}{TN}\sum_{\substack{\delta^k|T\\(\delta,m)=1}}\delta^{k-2}d\lambda + o(1) \\ &= \frac{1}{L(2,\chi_0)}\frac{1}{Nm^2}\int_0^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k^2}\sum_{1\le T<Q^{2k}\lambda/N}(TN)^{1/k}\log\frac{Q^{2k}\lambda}{TN}\sum_{\substack{\delta^k|T\\(\delta,m)=1}}\delta^{k-2}d\lambda + o(1) \\ &= \frac{1}{L(2,\chi_0)}\frac{1}{Nm^2}\int_0^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k^2}N^{1/k}\sum_{\substack{\delta<Q^2(\frac{\lambda}{N})^{1/k}\\(\delta,m)=1}}\delta^{k-2}\sum_{\substack{1\le T\le\frac{Q^{2k}\lambda}{N}\\ \delta^k|T}}T^{1/k}\log\frac{Q^{2k}\lambda}{TN}d\lambda + o(1) \\ &= \frac{1}{L(2,\chi_0)}\frac{1}{Nm^2}\int_0^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k^2}N^{1/k}\sum_{\substack{\delta<Q^2(\frac{\lambda}{N})^{1/k}\\(\delta,m)=1}}\delta^{k-2}(\frac{k}{k+1})^2\delta(\frac{Q^{2k}\lambda}{N\delta^k})^{1+1/k}d\lambda + o(1). \end{aligned} \tag{3.3.24}$$

Here we get some cancelation, and

$$\begin{aligned} PC &= \frac{1}{L(2,\chi_0)}\int_0^{\Lambda}H(\lambda)\frac{1}{N^2}\frac{Q^{2k+2}}{(k+1)^2m^2}L(2,\chi_0)d\lambda + o(1) \\ &= \int_0^{\Lambda}H(\lambda)d\lambda + o(1). \end{aligned} \tag{3.3.25}$$

Approximating H by the characteristic function $\chi_{(0,\Lambda)}$ in (3.3.23) and (3.3.25), we prove the case $(m,b)=1$.

Next, we consider the case $(m,b)=d>1$. The pair correlation sum is

$$PC = \frac{1}{N}\#\{(x_1,x_2)\in F_{Q,k}^{m,b}\times F_{Q,k}^{m,b} : x_1\ne x_2, x_1-x_2\in\frac{1}{N}(0,\Lambda)+\mathbb{Z}\}. \tag{3.3.26}$$

Consider the sequence $G_{Q,k}^{m,b} = \{y_i\} = \{d^k x_i, x_i \in F_{Q,k}^{m,b}\} \subseteq (0, d^k)$. Then

$$G_{Q,k}^{m,b} = \bigcup_{j=0}^{d^k-1} (F_{Q/d,k}^{m/d,b} + j).$$

Thus,

$$\begin{aligned}
PC &= \frac{1}{N}\#\{(y_1,y_2) \in G_{Q,k}^{m,b} \times G_{Q,k}^{m,b} : y_1 \neq y_2, y_1 - y_2 \in \frac{1}{N}(0, d^k\Lambda) + d^k\mathbb{Z}\} \\
&= \frac{1}{N}\sum_{j=0}^{d^k-1} \#\{(y_1,y_2) : y_1, y_2 \in F_{Q/d,k}^{m/d,b} + j, y_1 \neq y_2, y_1 - y_2 \in \frac{1}{N}(0, d^k\Lambda) + \mathbb{Z}\} + o(1) \\
&= \frac{1}{N} d^k \#\{(y_1,y_2) : y_1, y_2 \in F_{Q/d,k}^{m/d,b}, y_1 \neq y_2, y_1 - y_2 \in \frac{1}{N/d^k}(0, \Lambda) + \mathbb{Z}\} + o(1) \qquad (3.3.27) \\
&= \frac{1}{N} d^k \frac{N}{d^k}(\int_0^\Lambda g_{\frac{m}{d}}(\lambda) d\lambda + o(1)) + o(1) \\
&= \int_0^\Lambda g_{\frac{m}{d}}(\lambda) d\lambda + o(1).
\end{aligned}$$

3.4 Proof of Theorem 3.3

There are multiplicities in $F_{Q,k}^{m,b}$, and they may appear very frequently.

Choose a prime p, such that $(p,m) = 1$. Then there exist $l = [\frac{Q}{pm}]$ integers $\{y_i, i = 1, \ldots, l\}$ such that $0 < y_i \leq Q$ and

$$\begin{cases} y_i \equiv b \pmod m, \\ y_i \equiv 0 \pmod p. \end{cases}$$

If $z_i = \dfrac{y_i^k/p}{y_i^k}$, then $z_i \in F_{Q,k}^{m,b}$ and $z_i = \dfrac{1}{p}, i = 1, \ldots, l$.

The $k+2$ level correlation of $F_{Q,k}^{m,b}$ is defined as

$$\begin{aligned}
\mathcal{S}_{k+2} = \quad & \tfrac{1}{N}\#\{(x_1, \ldots, x_{k+2}) : x_i \in F_{Q,k}^{m,b}, x_i \text{ distinct}, (x_1 - x_2, x_2 - x_3, \ldots, x_{k+1} - x_{k+2}) \\
& \in \tfrac{1}{N}\ \mathcal{B} + \mathbb{Z}^{k+1}\}.
\end{aligned}$$

Now if $0 \in \mathcal{B}$, we can choose x_i from $\{z_j\}$, and there are C_l^{k+2} choices. So

$$\mathcal{S}_{k+2} > \frac{1}{N} C_l^{k+2} \gg \frac{(Q/pm)^{k+2}}{Q^{k+1}} \to \infty.$$

when $Q \to \infty$.

3.5 Proof of Theorem 3.4

The cardinality of the set F_Q^k is

$$\begin{aligned} N = \# F_Q^k &= \sum_{q=1}^{Q} q^k \prod_{p|q}(1-\frac{1}{p^k}) \\ &= \sum_{q=1}^{Q}\sum_{d|q}\mu(d)\frac{q^k}{d^k} \\ &= \frac{1}{(k+1)\zeta(k+1)}Q^{k+1} + O(Q^k). \end{aligned} \tag{3.5.1}$$

Define the generalized Mobius function

$$\mu_k(d) = \begin{cases} 1, & d \ is \ k^{th} - powerfree, \\ 0, & else. \end{cases}$$

By a result on Ramanujan sum (see [12]),

$$\sum_{\substack{1\le a\le n \\ (a,n)=1}} e(\frac{la}{n}) = \sum_{d|(l,n)} d\mu(\frac{n}{d}). \tag{3.5.2}$$

We have

$$\begin{aligned} \sum_{\substack{1\le a\le q^k \\ (a,q^k)k^{th}-powerfree}} e(\frac{la}{q^k}) &= \sum_{d|q^k}\mu_k(d)\sum_{e|(l,\frac{q^k}{d})} e\mu(\frac{q^k}{de}) \\ &= \sum_{e|(l,q^k)}\sum_{de|q^k} e\mu_k(d)\mu(\frac{q^k}{de}) \\ &= \sum_{e|(l,q^k)} e\sum_{d|\frac{q^k}{e}}\mu(\frac{q^k}{de})\mu_k(d). \end{aligned} \tag{3.5.3}$$

If e is not a k^{th}-power, then $\dfrac{q^k}{e}$ can be written in the form $a^k p_1^{\alpha_1}\cdots p_s^{\alpha_s}$, $s \ge 1$, $1 \le \alpha_i \le k-1$. The inner sum is

$$\sum_{d|\frac{q^k}{e}} \mu(\frac{q^k}{de})\mu_k(d) = \sum_{d|\frac{q^k}{e}=a^k p_1^{\alpha_1}\cdots p_s^{\alpha_s}} \mu(\frac{a^k p_1^{\alpha_1}\cdots p_s^{\alpha_s}}{d})\mu_k(d)$$
$$= \sum_{\substack{d=a^{k-1}d_1 \\ d_1|p_1^{\alpha_1}\cdots p_s^{\alpha_s}}} \mu(\frac{a^k p_1^{\alpha_1}\cdots p_s^{\alpha_s}}{a^{k-1}d_1})\mu_k(a^{k-1}d_1) \tag{3.5.4}$$
$$= 0.$$

So the exponential sum of all the elements in F_Q^k is

$$\sum_{\gamma\in F_Q^k} e(l\gamma) = \sum_{q=1}^{Q} \sum_{\substack{1\le a\le q^k \\ (a,q^k)k^{th}-powerfree}} e(\frac{la}{q^k})$$
$$= \sum_{q=1}^{Q} \sum_{e_1^k|(l,q^k)} e_1^k \sum_{d|\frac{q^k}{e_1^k}} \mu(\frac{q^k}{e_1^k d})\mu_k(d). \tag{3.5.5}$$

The term $\mu(\frac{q^k}{e_1^k d})\mu_k(d)$ does not vanish only if $d = \dfrac{q^{k-1}}{e^{k-1}}$, so

$$\sum_{\gamma\in F_Q^k} e(l\gamma) = \sum_{e_1^k|(l,q^k)} e_1^k \sum_{q=1}^{Q} \mu(\frac{q}{e_1})$$
$$= \sum_{d^k|l} d^k M(\frac{Q}{d}), \tag{3.5.6}$$

where

$$M(x) = \sum_{n\le x} \mu(n).$$

Proceeding as in Section 3.3, we find that the pair correlation sum of F_Q^k is

$$PC = \frac{1}{N}\sum_{r\in Z} c_r(\sum_{\substack{1\le d\le Q \\ d^k|r}} d^k M(\frac{Q}{d}))^2$$
$$= \frac{1}{N}\sum_{1\le d_1,d_2\le Q} d_1^k d_2^k M(\frac{Q}{d_1})M(\frac{Q}{d_2})\sum_{l\in Z} c_{l[d_1^k,d_2^k]}$$
$$= \frac{1}{N^2}\sum_{1\le d_1,d_2\le Q} d_1^k d_2^k M(\frac{Q}{d_1})M(\frac{Q}{d_2})\sum_{l\in Z} H_{\frac{[d_1^k,d_2^k]}{N}}(l) \tag{3.5.7}$$
$$= \frac{1}{N^2}\sum_{1\le d_1,d_2\le Q} d_1^k d_2^k \sum_{\substack{1\le r_1\le Q/d_1 \\ 1\le r_2\le Q/d_2}} \mu(r_1)\mu(r_2)\sum_{l\in Z}\frac{N}{[d_1^k,d_2^k]}H(\frac{lN}{[d_1^k,d_2^k]}).$$

If we switch the order of sums in (3.5.7), we find that

$$\begin{aligned} PC &= \frac{1}{N}\sum_{1\le r_1,r_2\le Q}\mu(r_1)\mu(r_2)\sum_{\substack{1\le d_1\le Q/r_1\\1\le d_2\le Q/r_2}}(d_1^k,d_2^k)\sum_{l\in Z}H(\frac{lN}{[d_1^k,d_2^k]})\\ &= \frac{1}{N}\sum_{1\le r_1,r_2\le Q}\mu(r_1)\mu(r_2)\sum_{1\le\delta\le Q/max\{r_1,r_2\}}\delta^k\sum_{\substack{1\le q_1\le Q/\delta r_1\\1\le q_2\le Q/\delta r_2\\(q_1,q_2)=1}}\sum_{l\in Z}H(\frac{lN}{\delta^k q_1^k q_2^k}). \end{aligned} \tag{3.5.8}$$

Now,the value of H is nontrivial only if $0 < lN/\delta^k q_1^k q_2^k < \Lambda$, i.e. $0 < l < \delta^k q_1^k q_2^k/N$. Letting $c = (k+1)\zeta(k+1)$, we have

$$0 < l\delta^k r_1^k r_2^k < (\delta r_1 q_1)^k(\delta r_2 q_2)^k\Lambda/N \le Q^k Q^k\Lambda/N \le c\Lambda Q^{k-1}.$$

When Q is large enough, say $Q > Q(\Lambda)$, (3.5.8) becomes

$$PC = \frac{1}{N}\sum_{l(\delta r_1 r_2)^k < c\Lambda Q^{k-1}}\mu(r_1)\mu(r_2)\delta^k\sum_{\substack{1\le q_1\le Q/\delta r_1\\1\le q_2\le Q/\delta r_2\\(q_1,q_2)=1\\q_1q_2>(\frac{lN}{\delta^k\Lambda})^{1/k}}}H(\frac{lN}{\delta^k q_1^k q_2^k}). \tag{3.5.9}$$

We can still use Lemma 3.6 to estimate the inner sum

$$\sum_{\substack{1\le q_1\le Q/\delta r_1\\1\le q_2\le Q/\delta r_2\\(q_1,q_2)=1\\q_1q_2>(\frac{lN}{\delta^k\Lambda})^{1/k}}}H(\frac{lN}{\delta^k q_1^k q_2^k}) = \frac{6}{\pi^2}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{\frac{Q}{\delta r_2}}}^{\frac{Q}{\delta r_1}}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{q_1}}^{\frac{Q}{\delta r_2}}H(\frac{lN}{\delta^k q_1^k q_2^k})dq_2dq_1 + E_{l,\delta,r_1,r_2}. \tag{3.5.10}$$

The error term E_{l,δ,r_1,r_2} is estimated as

$$\begin{aligned} E_{l,\delta,r_1,r_2} &\ll (\|\frac{\partial H}{\partial x}\|\frac{lN}{\delta^k q_2^k q_1^{k+1}} + \|\frac{\partial H}{\partial y}\|\frac{lN}{\delta^k q_1^k q_2^{k+1}})\frac{Q^2}{\delta^2 r_1 r_2}\log Q + \|H\|(\log Q + \frac{Q}{\delta}\log Q)\\ &\ll_H (\frac{1}{q_1}+\frac{1}{q_2})\frac{Q^2\log Q}{\delta^2 r_1 r_2} + \frac{Q\log Q}{\delta}\\ &\ll_H \frac{Q^{2-1/k}\log Q}{\delta^2 r_1 r_2 l^{1/k}} + \frac{Q\log Q}{\delta}. \end{aligned} \tag{3.5.11}$$

Therefore, the sum of all error terms in (3.5.9) can be estimated as

$$\begin{aligned}
&\frac{1}{N}\sum_{l(\delta r_1r_2)^k<c\Lambda Q^{k-1}}\mu(r_1)\mu(r_2)\delta^k\cdot E_{l,\delta,r_1,r_2}\\
&\ll_H \frac{1}{N}\sum_{l(\delta r_1r_2)^k<c\Lambda Q^{k-1}}\delta^k\Big(\frac{Q^{2-1/k}\log Q}{\delta^2 r_1r_2l^{1/k}}+\frac{Q\log Q}{\delta}\Big) \qquad (3.5.12)\\
&\ll_H Q^{-1+1/k}\log Q=o(1),
\end{aligned}$$

when $k\geq 2$.

Returning to (3.5.10), let $\lambda=\dfrac{lN}{\delta^k q_1^k q_2^k}$. Then $dq_2=(\frac{lN}{\delta^k q_1^k})^{1/k}(-1/k)\lambda^{-1-1/k}d\lambda$, and the main term of (3.5.10) becomes

$$\begin{aligned}
&\frac{6}{\pi^2}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{\frac{Q}{\delta r_2}}}^{\frac{Q}{\delta r_1}}\int_{\frac{(\frac{lN}{\delta^k\Lambda})^{1/k}}{q_1}}^{\frac{Q}{\delta r_2}}H\Big(\frac{lN}{\delta^k q_1^k q_2^k}\Big)dq_2dq_1\\
&=\frac{6}{\pi^2}\int_{(\frac{lN}{\Lambda})^{1/k}\frac{r_2}{Q}}^{\frac{Q}{\delta r_1}}\int_{\Lambda}^{\frac{lNr_2^k}{q_1^kQ^k}}-H(\lambda)\frac{1}{k}\lambda^{-1-1/k}\frac{(lN)^{1/k}}{\delta q_1}d\lambda dq_1\\
&=\frac{6}{\pi^2}\int_{\frac{lNr_2^k}{Q^k}/(Q/\delta r_1)^k}^{\Lambda}\int_{(\frac{lN}{\lambda})^{1/k}\frac{r_2}{Q}}^{\frac{Q}{\delta r_1}}\frac{H(\lambda)(lN)^{1/k}}{\lambda^{1+1/k}k\delta q_1}dq_1d\lambda \qquad (3.5.13)\\
&=\frac{6}{\pi^2}\int_{\frac{l(\delta r_1r_2)^kN}{Q^{2k}}}^{\Lambda}\frac{H(\lambda)(lN)^{1/k}}{\lambda^{1+1/k}k\delta}\log\frac{Q}{\delta r_1}\frac{Q}{r_2}\Big(\frac{\lambda}{lN}\Big)^{1/k}d\lambda.
\end{aligned}$$

Let $M=l(\delta r_1r_2)^k$, by (3.5.12) and (3.5.13), when $k\geq 2$, the pair correlation sum (3.5.9) becomes

$$\begin{aligned}
PC&=\frac{6}{\pi^2}\frac{1}{N}\sum_{1\leq M<c\Lambda Q^{k-1}}\int_{\frac{MN}{Q^{2k}}}^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k}N^{1/k}\log\frac{Q^2\lambda^{1/k}}{M^{1/k}N^{1/k}}\cdot\\
&\qquad\sum_{l(\delta r_1r_2)^k=M}\mu(r_1)\mu(r_2)\delta^{k-1}l^{1/k}d\lambda+o(1) \qquad (3.5.14)\\
&=\frac{6}{\pi^2}\frac{1}{N}\sum_{1\leq M<c\Lambda Q^{k-1}}\int_{\frac{MN}{Q^{2k}}}^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k^2}(MN)^{1/k}\log\frac{Q^{2k}\lambda}{MN}\cdot\\
&\qquad\sum_{(\delta r_1r_2)^k|M}\frac{\mu(r_1)\mu(r_2)\delta^{k-1}}{r_1r_2\delta}d\lambda+o(1).
\end{aligned}$$

Switching the order of sums, we have

$$\begin{aligned}
PC&=\frac{6}{\pi^2}\frac{1}{N}\int_0^{\Lambda}\frac{H(\lambda)}{\lambda^{1+1/k}k^2}\sum_{1\leq M<Q^{2k}\lambda/N}(MN)^{1/k}\log\frac{Q^{2k}\lambda}{MN}\cdot\\
&\qquad\sum_{(\delta r_1r_2)^k|M}\frac{\mu(r_1)\mu(r_2)\delta^{k-1}}{r_1r_2\delta}d\lambda+o(1). \qquad (3.5.15)
\end{aligned}$$

Letting $t = r_1 r_2 \delta$, we find that

$$
\begin{aligned}
PC &= \frac{6}{\pi^2}\frac{1}{N}\int_0^{\Lambda} \frac{H(\lambda)}{\lambda^{1+1/k}k^2} N^{1/k} \sum_{1\le t<Q^2(\frac{\lambda}{N})^{1/k}} \frac{\mu*\mu*id^{k-1}(t)}{t} \cdot \\
&\quad \sum_{\substack{1\le M<\frac{Q^{2k}\lambda}{N} \\ t^k|M}} M^{1/k}\log\frac{Q^{2k}\lambda}{MN} d\lambda + o(1) \\
&= \frac{6}{\pi^2}\frac{1}{N}\int_0^{\Lambda} \frac{H(\lambda)}{\lambda^{1+1/k}k^2} N^{1/k} \sum_{1\le t<Q^2(\frac{\lambda}{N})^{1/k}} \frac{\mu*\mu*id^{k-1}(t)}{t} \cdot \\
&\quad t(\frac{k}{k+1})^2(\frac{Q^{2k}\lambda}{Nt^k})^{1+1/k} d\lambda + o(1) \\
&= \int_0^{\Lambda} H(\lambda) \sum_{1\le t<Q^2(\frac{\lambda}{N})^{1/k}} \frac{\mu*\mu*id^{k-1}(t)}{t^{k+1}} \frac{1}{(k+1)^2}\frac{Q^{2k+2}}{N^2}\frac{6}{\pi^2} d\lambda + o(1).
\end{aligned}
\tag{3.5.16}
$$

We have

$$\sum_{t=1}^{\infty} \frac{\mu*\mu*id^{k-1}(t)}{t^{k+1}} = \frac{1}{\zeta(k+1)}\frac{1}{\zeta(k+1)}\zeta(2).$$

and

$$\lim_{Q\to\infty} \frac{Q^{2k+2}}{N^2} = c^2 = ((k+1)\zeta(k+1))^2.$$

So (3.5.16) becomes

$$
\begin{aligned}
PC &= \int_0^{\Lambda} H(\lambda)\frac{1}{\zeta(k+1)}\frac{1}{\zeta(k+1)}\zeta(2)\frac{1}{(k+1)^2}c^2\frac{6}{\pi^2} d\lambda \\
&= \int_0^{\Lambda} H(\lambda) d\lambda.
\end{aligned}
\tag{3.5.17}
$$

when $Q \to \infty$.

Chapter 4

Pair correlations of fractions with prime power denominators

4.1 Statement of results

In the last two chapters, we discussed the correlation of fractions with prime denominators and power denominators. We can also consider the fractions with prime power denominators.

In this chapter p is always a prime number.

Let $n \geq 1$, $P_Q^n = \{\frac{a}{p^l} : 1 \leq a \leq p^l - 1, (a, p^l) = 1, l \leq n, 1 \leq p \leq Q\}$ be the set of all fractions in $[0, 1]$, where the denominators are n^{th}-powers of prime numbers.

Theorem 4.1. *When $Q \to \infty$, the limiting pair correlation of P_Q^n is always Poissonian.*

When $n = 1$, Theorem 4.1 is just Corollary 2.2.

Theorem 4.1 can be generalized to primes in arithmetic progressions.
Let $P_Q^{n,k,b} = \{\frac{a}{p^l} : 1 \leq a \leq p^l, (a, p^l) = 1, l \leq n, 1 \leq p \leq Q, p \equiv b \pmod{k}\}$.

Theorem 4.2. *When $Q \to \infty$, the limiting pair correlation of $P_Q^{n,k,b}$ is always Poissonian.*

When $n = 1$, Theorem 4.2 is just Corollary 2.3.

4.2 Proof of Theorem 4.2

The cardinality of $P_Q^{n,k,b}$ is

$$\begin{aligned} N = \#P_Q^{n,k,b} &= \sum_{\substack{p \leq Q \\ p \equiv b (\text{mod } k)}} (p^n - 1) \\ &= \frac{Q^{n+1}}{\phi(k)(n+1)\log Q} + O_k(\frac{Q^{n+1}}{\log^2 Q}). \end{aligned} \tag{4.2.1}$$

The exponential sum of all the elements in $P_Q^{n,k,b}$ is

$$\begin{aligned}\sum_{\gamma\in P_Q^{n,k,b}} e(l\gamma) &= \sum_{\substack{p\le Q\\ p\equiv b(\bmod\ k)}} \sum_{a=1}^{p^n-1} e(\frac{la}{p^n})\\ &= \sum_{\substack{p\le Q\\ p\equiv b(\bmod\ k)\\ p^n|l}} p^n - \pi(Q;b,k),\end{aligned} \tag{4.2.2}$$

where $\pi(Q;b,k)$ counts the number of primes $\le Q$ in the arithmetic progression $kn+b$, $n=0,1,2,\dots$

Proceeding as in the last chapter, we see that the pair correlation sum of $P_Q^{n,k,b}$ equals

$$\begin{aligned}PC &= \frac{1}{N}\sum_{r\in\mathbb{Z}} c_r \left|\sum_{\gamma\in\mathcal{F}_Q} e(r\gamma)\right|^2\\ &= \frac{1}{N}\sum_{r\in\mathbb{Z}} c_r(\sum_{\substack{p,q\le Q\\ p^n|r,q^n|r\\ p\equiv q\equiv b(\bmod\ k)}} p^nq^n - 2\pi(Q;b,k)\sum_{\substack{p\le Q\\ p\equiv b(\bmod\ k)\\ p^n|r}} p^n + \pi^2(Q;b,k))\\ &= \frac{1}{N}(\sum_{\substack{p\ne q\le Q\\ p\equiv q\equiv b(\bmod\ k)}} p^nq^n\sum_{p^nq^n|r} c_r + \sum_{\substack{p\le Q\\ p\equiv b(\bmod\ k)}} p^{2n}\sum_{p^n|r} c_r - 2\pi(Q;b,k)\sum_{p^n|r} c_r\\ &\quad + \pi^2(Q;b,k)\sum_{r\in\mathbb{Z}} c_r).\end{aligned} \tag{4.2.3}$$

By the Poisson summation formula,

$$\begin{aligned}&\sum_{p^nq^n|r} c_r = \sum_{l\in\mathbb{Z}} H_{p^nq^n}(l) = \sum_{l\in\mathbb{Z}} \frac{1}{p^nq^n}H(\frac{lN}{p^nq^n}),\\ &\sum_{p^n|r} c_r = \sum_{l\in\mathbb{Z}} H_{p^n}(l) = \sum_{l\in\mathbb{Z}} \frac{1}{p^n}H(\frac{lN}{p^n}),\\ &\sum_{r\in\mathbb{Z}} c_r = \sum_{l\in\mathbb{Z}} H(l) = \sum_{l\in\mathbb{Z}} H(lN).\end{aligned} \tag{4.2.4}$$

Since $supp(H)\subset(0,\Lambda)$, when Q is large enough, we have

$$\sum_{p^n|r} c_r = \sum_{r\in\mathbb{Z}} c_r = 0. \tag{4.2.5}$$

Thus, the pair correlation sum can be simplified to

$$
\begin{aligned}
PC &= \frac{1}{N} \sum_{\substack{p \neq q \leq Q \\ p \equiv q \equiv b (\bmod\ k)}} \sum_{l \in \mathbb{Z}} H(\frac{lN}{p^n q^n}) \\
&= \frac{1}{N} \sum_{\substack{p \neq q \leq Q \\ p \equiv q \equiv b (\bmod\ k)}} (\frac{p^n q^n}{N} \int_0^{\Lambda} H(x)dx + O_H(1)) \\
&= \frac{1}{N^2} \int_0^{\Lambda} H(x)dx \sum_{\substack{p \neq q \leq Q \\ p \equiv q \equiv b (\bmod\ k)}} p^n q^n + \frac{1}{N} O_H(\frac{Q^2}{\log^2 Q}) \\
&= \frac{1}{N^2} \int_0^{\Lambda} H(x)dx(\sum_{\substack{p \leq Q \\ p \equiv b (\bmod\ k)}} p^n \sum_{\substack{q \leq Q \\ q \equiv b (\bmod\ k)}} q^n - \sum_{\substack{p \leq Q \\ p \equiv b (\bmod\ k)}} p^{2n}) + o(1) \\
&= \frac{1}{N^2} \int_0^{\Lambda} H(x)dx((N + o(N))(N + o(N)) - o(N^2)) + o(1) \\
&= \int_0^{\Lambda} H(x)dx + o(1).
\end{aligned}
\tag{4.2.6}
$$

Chapter 5

Apollonian circle packing

5.1 Introduction

Recently, the Apollonian circle packing has drawn a lot of attention. For instance, see [5], [8], [9], [15], [21], [25], [26].

By conformal mapping, we can always map the out circle of an Apollonian circle packing to the real axis and the other inside circles to the upper half plane $\mathbb{H}$ of the complex plane.

Among all such packings, Ford circles have special relations with Farey fractions. The Farey fractions of order Q are defined as the set of reduced fractions in the closed interval [0,1] with denominators $\leq Q$ listed in increasing order of magnitude, and it is denoted as F_Q.

For a reduced rational number h/k, the Ford circle attached to this fraction is denoted by $C(h,k)$, and it is the circle in the complex plane centered at the point $h/k + i/(2k^2)$ with radius $1/(2k^2)$.

It is easy to see that two Ford circles $C(h_1,k_1), C(h_2,k_2)$ are either tangent to each other or do not intersect at all. And they are tangent if and only if $h_1k_2 - h_2k_1 = 1$, in particular, Ford circles of consecutive Farey fractions are tangent to each other.

Now, consider three consecutive Farey fractions $h_1/k_1 < h_2/k_2 < h_3/k_3$, and the corresponding Ford circles C_1, C_2, C_3. For $i = 1, 2$, C_i and C_{i+1} are tangent, and we denote the tangent line by l_i, the angle θ_i between l_i and the positive real axis by θ_i, $0 < \theta_i < \pi$, and the angle between l_1 and l_2 by $\theta = \theta_2 - \theta_1$. Now for each Farey fraction $\gamma = h_2/k_2$, we can define an angle θ_γ. In this chapter, we consider the distribution of these angles, and it will give a special case of the distribution of angles of consecutive tangents of Apollonian circle packings.

For three consecutive Ford circles C_1, C_2, C_3 with corresponding centers O_1, O_2, O_3 and tangent points F_1, F_2, F_3 with the real axis, see Figure 5.1. Let T_i be the tangent point of circles C_i and C_{i+1}, $i = 1, 2$. Let G_2 be the other intersection point of C_2 and the line O_2F_2. We can easily check that F_1, T_1, G_2 are collinear, and G_2, T_2, F_3 are collinear. Let $\phi = \angle F_1G_2F_3$, $\phi_1 = \angle F_1G_2F_2$, $\phi_2 = \angle F_2G_2F_3$. Then $\theta_2 = \angle F_2O_2T_2 = 2\phi_2$, $\theta_1 = \pi - \angle F_2O_2T_1 = \pi - 2\phi_1$, and we have $\theta = \theta_2 - \theta_1 = 2(\phi_1 + \phi_2) - \pi = 2\phi - \pi$. So we can study the distribution of ϕ instead of θ, and later we will see that it can simplify the computation.

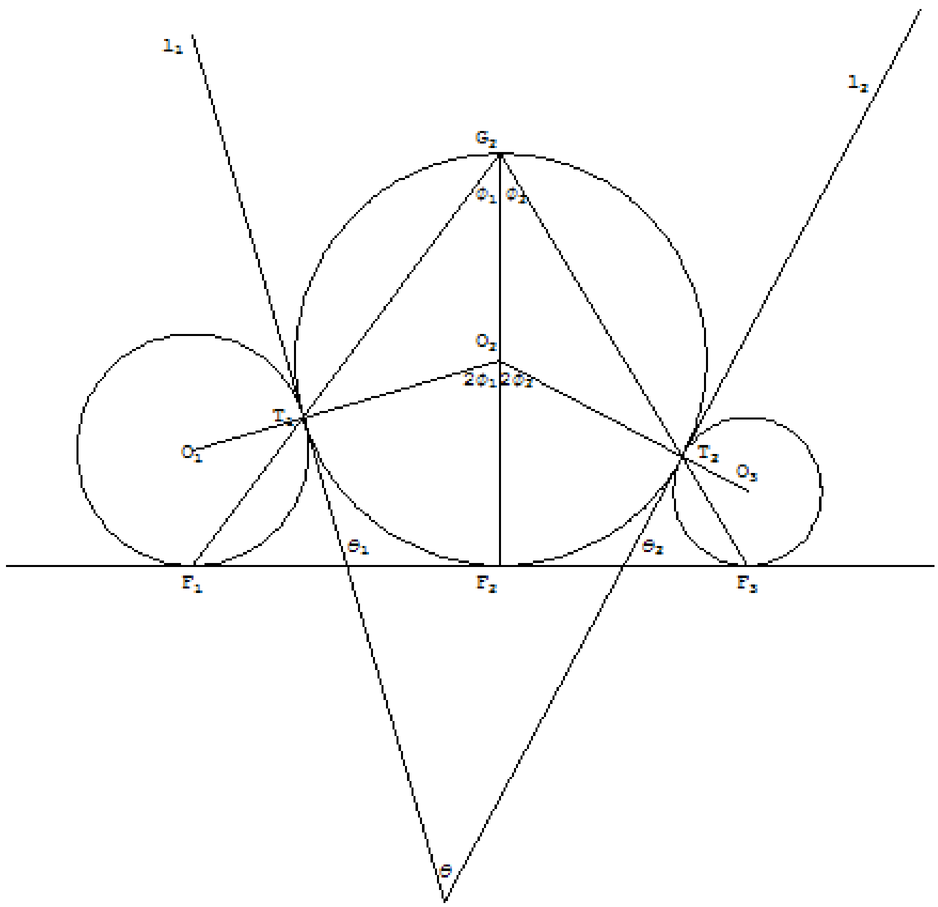

Figure 5.1: Ford Circles

Theorem 5.1. *For $Q \to \infty$, the ϕ_γ has a limiting distribution for the Farey fractions F_Q. Specifically, for*

$0 < \beta < \pi$,

$$\lim_{Q\to\infty} \frac{\#\{\gamma : 0 < \phi_\gamma < \beta < \pi\}}{\#\{\gamma \in F_Q\}} =$$
$$\begin{cases} 1, \quad \text{if } \beta > \cot^{-1}(-1) = \frac{3}{4}\pi; \\ 1 - \dfrac{(1-t)^2}{3-t}, t = \sqrt{4|\cot\beta| - 3}, \text{ if } \cot^{-1}(-\dfrac{3}{4}) < \beta \le \cot^{-1}(-1); \\ \dfrac{2}{3}, \quad \text{if } \cot^{-1}(-\dfrac{1}{8}) < \beta \le \cot^{-1}(-\dfrac{3}{4}); \\ \dfrac{2}{3} - \dfrac{8(1-t)^2}{3(4-t)(2+t)}, t = \sqrt{8|\cot\beta|}, \text{ if } \dfrac{\pi}{2} \le \beta \le \cot^{-1}(-\dfrac{1}{8}); \\ \dfrac{4}{v(v+1)}, \quad \text{if } \beta < \dfrac{\pi}{2}, \text{and } \dfrac{v^2-2v-3}{4v-4} < \cot\beta \le \dfrac{v^2-5}{4v}, v \in \mathbb{Z}; \\ \dfrac{4}{v(v+1)} - \dfrac{8(1-t)^2}{(v+1)(v+t)(v+2-t)}, t = \sqrt{v^2 - 4v\cot\beta - 4}, \\ \qquad \text{if } \beta < \frac{\pi}{2}, \text{and } \frac{v^2-5}{4v} < \cot\beta \le \frac{v^2-4}{4v}, v \in \mathbb{Z}. \end{cases} \tag{5.1.1}$$

The distribution function is shown in Figure 5.2.

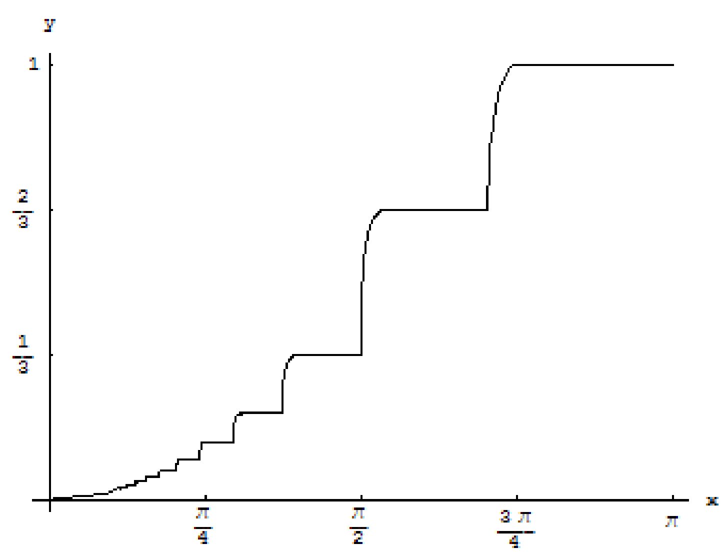

Figure 5.2: Distribution function of ϕ_γ

From Theorem 5.1 with the same notation, we immediately deduce that:

Corollary 5.2. *For Farey fractions $\gamma \in F_Q$, when $Q \to \infty$, 1/3 of the angles ϕ_γ are acute and 2/3 of them are obtuse.*

Corollary 5.3. *For Farey fractions $\gamma \in F_Q$, when $Q \to \infty$, 1/3 of the angles θ_γ are negative (traversed clockwise) and 2/3 of them are positive (traversed counterclockwise).*

By estimating certain type of Kloosterman sums, we can extend Theorem 5.1 to an arbitrary interval between $[0, 1]$.

Theorem 5.4. *For an arbitrary interval $I \subset [0, 1]$, when $Q \to \infty$,*

$$\frac{\#\{\gamma : 0 < \phi_\gamma < \beta < \pi\}}{\#\{\gamma \in F_Q \cap I\}}$$

has the same limiting distribution as in Theorem 5.1. The corresponding Corollaries 5.2 and 5.3 also remain true.

5.2 Some properties of the Farey triangle

In this section, we introduce some basic settings of the Farey triangle; see also [3].

For $(x, y) \in \mathcal{T} = \{(x, y)\,;\, 0 < x, y \le 1,\ x + y > 1\}$, we define the map

$$T(x, y) = \left(y, \left[\frac{1+x}{y}\right]y - x\right). \tag{5.2.1}$$

Since $0 < y \le 1$, we have $\left[\frac{1+x}{y}\right] = \left[\frac{x}{y} + \frac{1}{y}\right] \ge \left[\frac{x}{y} + 1\right] > \frac{x}{y}$. Hence,

$$\left[\frac{1+x}{y}\right]y - x > 0. \tag{5.2.2}$$

Moreover, the inequality $\frac{1+x}{y} \ge \left[\frac{1+x}{y}\right]$ yields

$$\left[\frac{1+x}{y}\right]y - x \le 1. \tag{5.2.3}$$

On the other hand, $1 + \left[\frac{1+x}{y}\right] > \frac{1+x}{y}$. Hence,

$$y + \left[\frac{1+x}{y}\right]y - x > 1. \tag{5.2.4}$$

The inequalities (5.2.2), (5.2.3) and (5.2.4) show that (5.2.1) defines a map $T : \mathcal{T} \to \mathcal{T}$.

Lemma 5.5. *The map T is a bijective area-preserving transformation of $\mathcal{T}$ with inverse*

$$T^{-1}(x,y) = \left(\left[\frac{1+y}{x}\right]x - y, x\right), \qquad (x,y) \in \mathcal{T}. \tag{5.2.5}$$

Proof. First, we can check as above that T^{-1} defines a transformation of $\mathcal{T}$. It is easily seen that $TT^{-1} = T^{-1}T = id_{\mathcal{T}}$.

The Farey triangle $\mathcal{T}$ can be decomposed as $\cup_{k\geq 1}\mathcal{T}_k$, where

$$\mathcal{T}_k = \left\{(x,y) \in \mathcal{T}\,;\, \left[\frac{1+x}{y}\right] = k\right\} = \{(x,y) \in \mathcal{T}\,;\, ky - x \leq 1 < (k+1)y - x\}. \tag{5.2.6}$$

Since $T(x,y) = (y, ky - x)$, $(x,y) \in \mathcal{T}_k$; the map $T\big|_{\mathcal{T}_k}$ has Jacobian equal to 1. Hence $T\big|_{\mathcal{T}_k}$ is area-preserving for all $k \geq 1$. Since T is bijective, it follows that T is area-preserving. $\square$

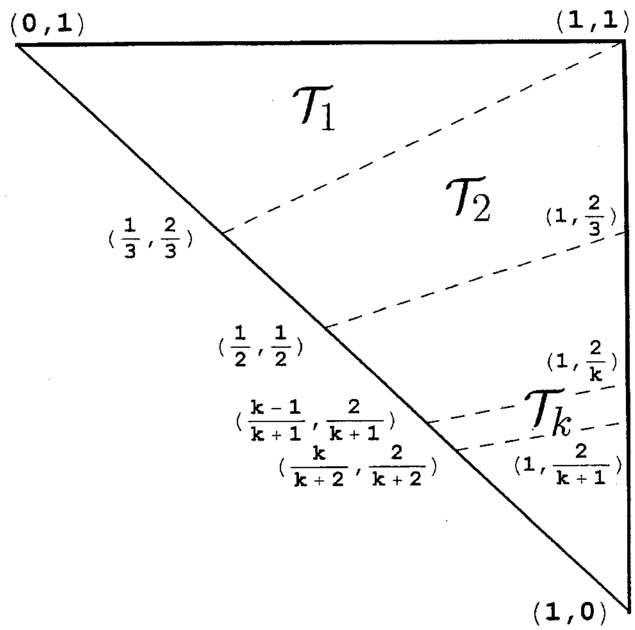

Figure 5.3: Farey Triangle

5.3 Some useful lemmas for counting lattice points in a domain

In this section, we prove some lemmas for counting lattice points, see also [3].

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded region with rectifiable boundary $\partial\Omega$ and assume that f is a C^1 function on Ω.

We denote $\|f\|_\infty = \sup_{(x,y)\in\Omega} |f(x,y)|$ and set

$$S = S(f,\Omega) = \sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b).$$

This sum is estimated in the following lemma.

Lemma 5.6. *Suppose that Ω and f are as above. Then*

$$\left| S - \iint\limits_{\Omega} f(x,y)\,dx\,dy \right| \ll \left(\left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial f}{\partial y} \right\|_\infty \right) \mathrm{Area}(\Omega) + \|f\|_\infty \big(1 + \mathrm{length}(\partial\Omega)\big).$$

Proof. We approximate S by a similar sum with a more convenient domain of summation. For, we divide the plane into squares $\mathcal{R}_{a,b} = [a,a+1]\times[b,b+1]$, $a,b\in\mathbb{Z}$, then set

$$\mathcal{D} = \bigcup_{\substack{a,b\in\mathbb{Z}\\ \mathcal{R}_{a,b}\subseteq\overline{\Omega}}} \mathcal{R}_{a,b} \qquad \text{and} \qquad E(\Omega) = \#\{(a,b)\in\mathbb{Z}^2\,;\, \mathring{\mathcal{R}}_{a,b}\cap\partial\Omega \neq \emptyset\}.$$

It is well-known (for a proof see e.g. Thm. 5.9 in [20]) that

$$E(\Omega) \ll 1 + \mathrm{length}(\partial\Omega). \tag{5.3.1}$$

The previous estimate gives

$$\begin{aligned} \left| \iint\limits_{\Omega} f(x,y)\,dx\,dy - \iint\limits_{\mathcal{D}} f(x,y)\,dx\,dy \right| &= \left| \iint\limits_{\Omega\setminus\mathcal{D}} f(x,y)\,dx\,dy \right| \ll \|f\|_\infty \mathrm{Area}(\Omega\setminus\mathcal{D}) \\ &\le \|f\|_\infty E(\Omega) \ll \|f\|_\infty\big(1+\mathrm{length}(\partial\Omega)\big). \end{aligned} \tag{5.3.2}$$

We may write $S = S_1 + S_2$, where

$$S_1 = \sum_{(a,b)\in\mathcal{D}} f(a,b) \qquad \text{and} \qquad S_2 = \sum_{(a,b)\in(\Omega\setminus\mathcal{D})\cap\mathbb{Z}^2} f(a,b).$$

Estimating S_2 trivially and using (5.3.1), we gather

$$|S_2| \le \|f\|_\infty \#\big((\Omega\setminus\mathcal{D})\cap\mathbb{Z}^2\big) \le \|f\|_\infty E(\Omega) \ll \|f\|_\infty\big(1+\mathrm{length}(\partial\Omega)\big). \tag{5.3.3}$$

Letting

$$\tilde{f}(x,y) = f([x],[y]), \qquad (x,y) \in \mathcal{D},$$

we clearly find that

$$\left| \iint\limits_{\mathcal{D}} f(x,y)\,dx\,dy - S_1 \right| = \left| \iint\limits_{\mathcal{D}} \Big(f(x,y) - \tilde{f}(x,y) \Big)\,dx\,dy \right| \leq \text{Area}(\mathcal{D}) \|f - \tilde{f}\|_\infty. \tag{5.3.4}$$

It is easy to see that

$$\|f - \tilde{f}\|_\infty \leq \Big\| \frac{\partial f}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f}{\partial y} \Big\|_\infty.$$

Inserting this into (5.3.4), we arrive at

$$\left| \iint\limits_{\mathcal{D}} f(x,y)\,dx\,dy - S_1 \right| \leq \text{Area}(\mathcal{D}) \bigg(\Big\| \frac{\partial f}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f}{\partial y} \Big\|_\infty \bigg). \tag{5.3.5}$$

The lemma follows now from (5.3.2), (5.3.3) and (5.3.5). □

Actually we need an estimate for a sum like S, but with sums only over points with coprime coordinates. More precisely, we set

$$S' = S'(f,\Omega) = {\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2}}' f(a,b) = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ \gcd(a,b)=1}} f(a,b)$$

and estimate such sums in the next lemma.

Lemma 5.7. *In addition to the hypotheses of Lemma* 5.6 *suppose that* $\Omega \subseteq [1,R] \times [1,R]$. *Then*

$$\left| S' - \frac{6}{\pi^2} \iint\limits_{\Omega} f(x,y)\,dx\,dy \right| \ll \bigg(\Big\| \frac{\partial f}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f}{\partial y} \Big\|_\infty \bigg) \text{Area}(\Omega) \log R + \|f\|_\infty \big(R + \text{length}(\partial\Omega) \log R \big).$$

Proof. Removing the coprimality condition by Möbius summation, we write

$$\begin{aligned} S' &= \sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b) \sum_{d \mid \gcd(a,b)} \mu(d) = \sum_{d=1}^{R} \mu(d) \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ d|a, d|b}} f(a,b) \\ &= \sum_{d=1}^{R} \mu(d) \sum_{(a_1,b_1)\in\Omega_d\cap\mathbb{Z}^2} f_d(a_1,b_1), \end{aligned} \tag{5.3.6}$$

where $f_d(x,y) = f(dx, dy)$ and $\Omega_d = \frac{1}{d}\Omega$. Now, Lemma 5.6 provides

$$\begin{aligned}
&\Bigg| \sum_{(a_1,b_1)\in\Omega_d\cap\mathbb{Z}^2} f_d(a_1,b_1) - \iint\limits_{\Omega_d} f_d(x,y)\,dx\,dy \Bigg| \\
&\qquad \ll \left(\Big\| \frac{\partial f_d}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f_d}{\partial y} \Big\|_\infty \right) \mathrm{Area}(\Omega_d) + \|f_d\|_\infty \big(1 + \mathrm{length}(\partial\Omega_d)\big) \qquad (5.3.7) \\
&\qquad = \frac{1}{d} \left(\Big\| \frac{\partial f}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f}{\partial y} \Big\|_\infty \right) \mathrm{Area}(\Omega) + \|f\|_\infty \Big(1 + \frac{1}{d}\,\mathrm{length}(\partial\Omega)\Big).
\end{aligned}$$

A change of variables gives

$$\iint\limits_{\Omega_d} f_d(x,y)\,dx\,dy = \frac{1}{d^2} \iint\limits_{\Omega} f(x,y)\,dx\,dy,$$

which combined with (5.3.6) and (5.3.7) leads to

$$\begin{aligned}
\Bigg| S' - \sum_{d=1}^{R} \frac{\mu(d)}{d^2} \iint\limits_{\Omega} f(x,y)\,dx\,dy \Bigg| &\ll \left(\Big\| \frac{\partial f}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f}{\partial y} \Big\|_\infty \right) \mathrm{Area}(\Omega) \log R \\
&\quad + \|f\|_\infty \big(R + \mathrm{length}(\partial\Omega) \log R\big).
\end{aligned}$$

We conclude the proof employing $\sum_{d=1}^{R} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O\big(\frac{1}{R}\big)$. □

We will need the following particular form of Lemma 5.7.

Lemma 5.8. *Suppose that Ω satisfies the hypotheses of Lemma 5.7 and in addition is convex. Then*

$$\Bigg| S' - \frac{6}{\pi^2} \iint\limits_{\Omega} f(x,y)\,dx\,dy \Bigg| \ll \left(\Big\| \frac{\partial f}{\partial x} \Big\|_\infty + \Big\| \frac{\partial f}{\partial y} \Big\|_\infty \right) \mathrm{Area}(\Omega) \log R + \|f\|_\infty R \log R.$$

5.4 Proof of Theorem 5.1

Consider three consecutive Ford circles C_1, C_2, C_3 which correspond to consecutive Farey fractions $\frac{h_1}{k_1} < \frac{h_2}{k_2} < \frac{h_3}{k_3}$. From Figure 5.1, we see that $\tan\phi_1 = \frac{1/k_1k_2}{1/k_2^2} = \frac{k_2}{k_1}$, $\tan\phi_2 = \frac{1/k_2k_3}{1/k_2^2} = \frac{k_2}{k_3}$, and so

$$\cot\phi = \cot(\phi_1 + \phi_2) = \frac{1 - \tan\phi_1 \tan\phi_2}{\tan\phi_1 + \tan\phi_2} = \frac{k_1k_3 - k_2^2}{k_2(k_1 + k_3)} = \frac{\frac{k_1}{k_2}\frac{k_3}{k_2} - 1}{\frac{k_1+k_3}{k_2}}. \qquad (5.4.1)$$

Let $v = \dfrac{k_1 + k_3}{k_2}$ be the index of the Farey fraction $\dfrac{h_2}{k_2}$, and $u = \dfrac{k_1}{k_2}$, then

$$\cot\phi = \frac{u(v-u)-1}{v}. \tag{5.4.2}$$

For fixed $0 < \beta < \pi$, $0 < \phi < \beta$ if and only if $\cot\phi > \cot\beta$, and so we just need to count the number of pairs (k_1, k_2) satisfying this inequality. We consider the following two cases:

Case 1: $0 < \beta \leq \pi/2$

In this case, $\cot\beta = \alpha \geq 0$. Then $\cot\phi > \cot\beta$, if and only if $u(v-u) - 1 > \alpha v$, i.e.

$$u^2 - vu + 1 + \alpha v < 0. \tag{5.4.3}$$

For fixed v, u exists in (5.4.3) if and only if $v^2 - 4(1+\alpha v) > 0$, and we can see that it only happens for $v > 2$. Let $t = \sqrt{v^2 - 4(1+\alpha v)}$. Then $t > 0$, and u must satisfy $\dfrac{v-t}{2} < u < \dfrac{v+t}{2}$.

We use the same notation of Farey triangles as in Section 5.2, and let $(x = k_1/Q, y = k_2/Q)$. Then $(x, y) \in \mathcal{T}_v$, and $\dfrac{2}{v+t} < \dfrac{y}{x} = \dfrac{1}{u} < \dfrac{2}{v-t}$, so (x, y) belongs to the intersection of $\mathcal{T}_v$ and the wedge $\mathcal{W}_v = \{(x,y) | \dfrac{2}{v+t}x < y < \dfrac{2}{v-t}x, x, y > 0\}$. The two sets of boundary lines may intersect at four points as follows:

The two lines $\dfrac{2}{v+t}x = y$ and $vy - x = 1$ intersect at the point $(\dfrac{v+t}{v-t}, \dfrac{2}{v-t})$, since $\dfrac{v+t}{v-t} > 1$, the intersection point is always outside $\mathcal{T}_v$.

The two lines $\dfrac{2}{v-t}x = y$ and $(v+1)y - x = 1$ intersect at the point $(\dfrac{v-t}{v+2+t}, \dfrac{2}{v+2+t})$. Since $\dfrac{v-t}{v+2+t} + \dfrac{2}{v+2+t} < 1$, the intersection point is below the line $x + y = 1$, and it is always outside $\mathcal{T}_v$.

The two lines $\dfrac{2}{v-t}x = y$ and $vy - x = 1$ intersect at the point $A = (\dfrac{v-t}{v+t}, \dfrac{2}{v+t})$. If $t > 1$ then $\dfrac{v-t}{v+t} + \dfrac{2}{v+t} < 1$, and the intersection point is outside $\mathcal{T}_v$. If $t \leq 1$ then the intersection point lies on the boundary of $\mathcal{T}_v$.

The two lines $\dfrac{2}{v+t}x = y$ and $(v+1)y - x = 1$ intersect at the point $B = (\dfrac{v+t}{v+2-t}, \dfrac{2}{v+2-t})$. Since $\dfrac{v+t}{v+2-t} + \dfrac{2}{v+2-t} > 1$, the intersection point is above the line $x + y = 1$. If $t > 1$ then $\dfrac{v+t}{v+2-t} > 1$, so the intersection point is outside $\mathcal{T}_v$. If $t \leq 1$ then the intersection point lies on the boundary of $\mathcal{T}_v$.

We define $\mathcal{S}_v = \mathcal{T}_v \cap \mathcal{W}_v$. From the above discussion, we can see that if $t > 1$ then all of $\mathcal{T}_v$ is contained in the wedge $\mathcal{W}_v$, so $\mathcal{S}_v = \mathcal{T}_v$; see Figure 5.4. If $0 < t \leq 1$, then only part of $\mathcal{T}_v$ is contained in the wedge $\mathcal{W}_v$; see Figure 5.5.

Next, we want to find the area of $\mathcal{S}_v$.

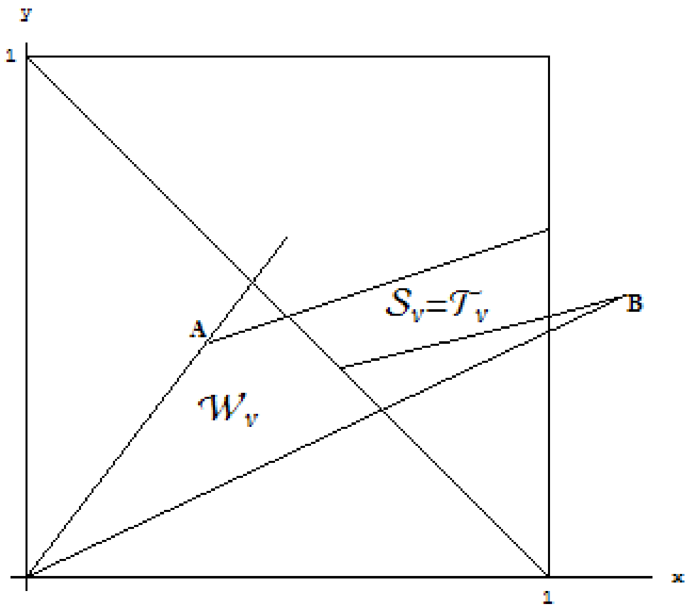

Figure 5.4: $v > 2, t > 1$

If $t > 1$, then $area(\mathcal{S}_v) = area(\mathcal{T}_v) = \frac{4}{v(v+1)(v+2)}$.
For $t \leq 1$, the two lines $\frac{2}{v-t}x = y$ and $x + y = 1$ intersect at the point $C = (\frac{v-t}{v+2-t}, \frac{2}{v+2-t})$. The two lines $\frac{2}{v+t}x = y$ and $x = 1$ intersect at the point $D = (1, \frac{2}{v+t})$. The points $E = (\frac{v-1}{v+1}, \frac{2}{v+1})$, $F = (1, \frac{2}{v+1})$ are two boundary points of $\mathcal{T}_v$. Let $\triangle_1$ be the triangle ACE, and $\triangle_2$ be the triangle BDF. Then

$$area(\mathcal{S}_v) = area(\mathcal{T}_v) - area(\triangle_1) - area(\triangle_2),$$

we have

$$area(\triangle_1) = \frac{1}{2}\left|\begin{matrix} 1 & \frac{v-t}{v+t} & \frac{2}{v+t} \\ 1 & \frac{v-1}{v+1} & \frac{2}{v+1} \\ 1 & \frac{v-t}{v+2-t} & \frac{2}{v+2-t} \end{matrix}\right| = \frac{2(1-t)^2}{(v+1)(v+t)(v+2-t)},$$

$$area(\triangle_2) = \frac{1}{2}\left|\begin{matrix} 1 & 1 & \frac{2}{v+t} \\ 1 & 1 & \frac{2}{v+1} \\ 1 & \frac{v+t}{v+2-t} & \frac{2}{v+2-t} \end{matrix}\right| = \frac{2(1-t)^2}{(v+1)(v+t)(v+2-t)},$$

so

$$area(\mathcal{S}_v) = \frac{4}{v(v+1)(v+2)} - \frac{4(1-t)^2}{(v+1)(v+t)(v+2-t)}.$$

Next we consider the sum of areas of $\mathcal{S}_v$ for all possible values v in (5.4.3). Let $\mathcal{S} = \bigcup_{v>0} \mathcal{S}_v$. Since

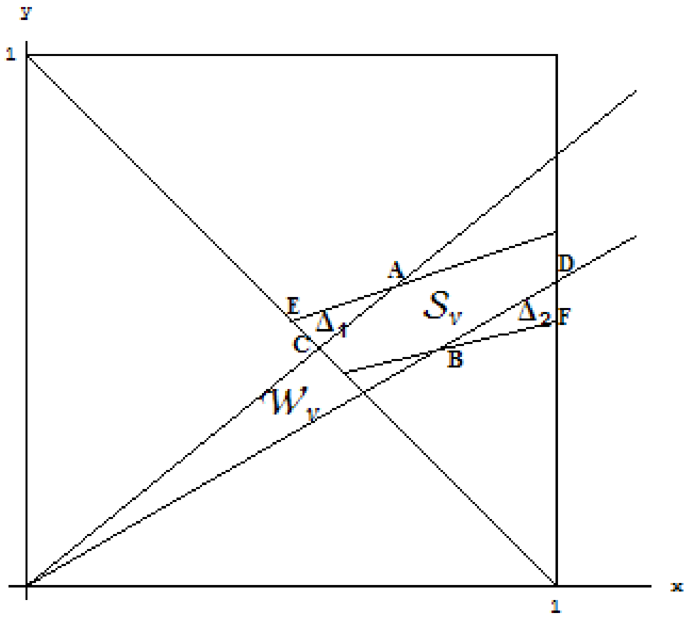

Figure 5.5: $v > 2, 0 < t \leq 1$

$\mathcal{S}_v = 0$ when $v^2 - 4\alpha v - 4 \leq 0$, we can only consider $v > 2\alpha + \sqrt{4\alpha^2+4}$.

Now $t \leq 1$ if and only if $v^2 - 4\alpha v - 5 \leq 0$, i.e. $v \leq 2\alpha + \sqrt{4\alpha^2+5}$, so there is at most one integral value v for such t. We have the two following cases:

(i) There exists one integer v, say v_0, such that $t \leq 1$. Then we must have

$$\lceil 2\alpha + \sqrt{4\alpha^2+4} \rceil = v_0 < \lceil 2\alpha + \sqrt{4\alpha^2+5} \rceil,$$

i.e.

$$\frac{v_0^2 - 5}{4v_0} < \alpha \leq \frac{v_0^2-4}{4v_0}.$$

In this situation, $\mathcal{S}$ will have two connected parts: $\mathcal{S}_{v_0}$ and the union of $\mathcal{T}_v$, $v > v_0$. The area of $\mathcal{S}$ is

$$\begin{aligned} area(\mathcal{S}) &= \sum_{v > 2\alpha + \sqrt{4\alpha^2+4}} area(\mathcal{S}_v) \\ &= area(\mathcal{S}_{v_0}) + \sum_{v > v_0} area(\mathcal{T}_v) \qquad (5.4.4) \\ &= \frac{2}{v_0(v_0+1)} - \frac{4(1-t)^2}{(v_0+1)(v_0+t)(v_0+2-t)}. \end{aligned}$$

(ii) There is no integer v to make $t \leq 1$. Then we must have an integer v_0, such that

$$\lceil 2\alpha + \sqrt{4\alpha^2 + 4} \rceil = v_0 = \lceil 2\alpha + \sqrt{4\alpha^2 + 5} \rceil,$$

i.e.

$$\frac{v_0^2 - 2v_0 - 3}{4v_0 - 4} < \alpha \leq \frac{v_0^2 - 5}{4v_0}.$$

Now $\mathcal{S}$ is just the union of $\mathcal{T}_v$ for all $v \geq v_0$, $\mathcal{S}$ is connected, and its area equals

$$\begin{aligned} area(\mathcal{S}) &= \sum_{v > 2\alpha + \sqrt{4\alpha^2+4}} area(\mathcal{S}_v) \\ &= \sum_{v > 2\alpha + \sqrt{4\alpha^2+5}} area(\mathcal{T}_v) \\ &= \sum_{v \geq v_0} \frac{4}{v(v+1)(v+2)} = \frac{2}{v_0(v_0+1)}. \end{aligned} \tag{5.4.5}$$

Finally, we can obtain the distribution for $0 < \phi < \beta < \pi/2$ by counting the lattice points (k_1, k_2) in (5.4.1). Since $(k_1, k_2) = (Qx, Qy) \subset \bigcup Q\mathcal{S}_v = Q\mathcal{S}$, by Lemma 5.8, there are $Q^2 \frac{6}{\pi^2} area(\mathcal{S}) + O(Q \log Q)$ lattice points (k_1, k_2) ($\mathcal{S}$ has at most two connected components, so the error term is assured), and there are $\frac{3}{\pi^2} Q^2 + O(Q \log Q)$ Farey fractions of order Q between $[0, 1]$. By taking the quotient we get the expressions in Theorem 5.1.

Case 2: $\pi/2 < \beta < \pi$

Let $\cot\beta =: -\alpha < 0$. Then $\cot\phi > \cot\beta$ if and only if $u(v-u) - 1 < -\alpha v$, i.e.

$$u^2 - vu + 1 - \alpha v < 0. \tag{5.4.6}$$

For fixed v, u exists in (5.4.6) if $v^2 - 4(1 - \alpha v) > 0$, i.e. $v > 2\sqrt{\alpha^2+1} - 2\alpha$. And we have $\dfrac{v - \sqrt{v^2 + 4\alpha v - 4}}{2} < u < \dfrac{v + \sqrt{v^2 + 4\alpha v - 4}}{2}$. Since u is always positive, and $u = \frac{k_1}{k_2} < \frac{k_1+k_3}{k_2} = v$, when $\alpha v \geq 1$, the bounds of u can be reduced to $0 < u < v$. We consider the following cases of α and use the same notation in Case 1.

(i) $\alpha \geq 1$.

In this case $2\sqrt{\alpha^2+1} - 2\alpha < 1$, so v can take all integral values ≥ 1, and we always have $\alpha v \geq 1$, so we can just consider $0 < u < v$. Now $(x, y) = (k_1/Q, k_2/Q)$ belongs to the intersection of $\mathcal{T}_v$ and the wedge $\mathcal{W}_v = \{(x, y) | vy - x > 0, x, y > 0\}$. The two lines $vy - x = 0$ and $(v+1)y - x = 1$ intersect at the point

$(v, 1)$, which is always outside $\mathcal{T}_v$, so $\mathcal{S}_v = \mathcal{T}_v \cap \mathcal{W}_v$ is exactly $\mathcal{T}_v$. We have

$$area(\mathcal{S}) = \sum_{v > 2\sqrt{\alpha^2+1}-2\alpha} area(\mathcal{S}_v) = \sum_{v \geq 1} area(\mathcal{T}_v) = 1/2, \tag{5.4.7}$$

and $\mathcal{S}$ is connected.

(ii) $3/4 < \alpha < 1$.

In this case $2\sqrt{\alpha^2 + 1} - 2\alpha < 1$, so v can take all values ≥ 1.

If $v \geq 2$, then $\alpha v > 1$, and by the same deduction in (i), $\mathcal{S}_v = \mathcal{T}_v$.

If $v = 1$, let $t = \sqrt{v^2 + 4\alpha v - 4} = \sqrt{4\alpha - 3}$. Then $0 < t < 1$, and $\frac{2}{1+t} < \frac{y}{x} = \frac{1}{u} < \frac{2}{1-t}$, so (x, y) belongs to the intersection of $\mathcal{T}_1$ and the wedge $\mathcal{W}_1 = \{(x, y) | y - \frac{2}{1+t}x > 0, y - \frac{2}{1-t}x < 0, x, y > 0\}$. The line $y - \frac{2}{1+t}x = 0$ intersects the boundary of $\mathcal{T}_1$ at points $A = (\frac{1+t}{2}, 1)$, $B = (\frac{t+1}{3-t}, \frac{2}{3-t})$. The line $y - \frac{2}{1-t}x = 0$ intersect the boundary of $\mathcal{T}_1$ at the points $C = (\frac{1-t}{2}, 1)$ and $D = (\frac{1-t}{3-t}, \frac{2}{3-t})$. The points $E = (1, 1)$, $F = (0, 1)$ are two boundary points of $\mathcal{T}_1$. Let $\mathcal{S}_1 = \mathcal{T}_1 \cap \mathcal{W}_1$, $\triangle_1$ be the triangle ABE, and $\triangle_2$ be the triangle CDF; see Figure 5.6.

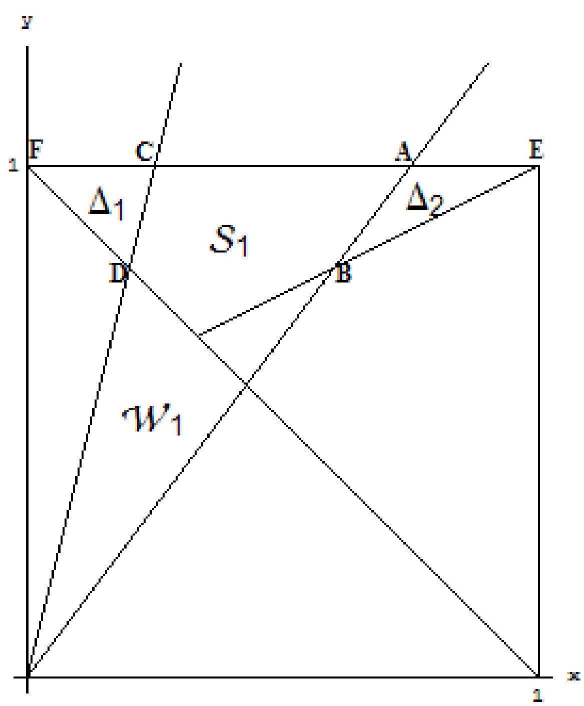

Figure 5.6: $v = 1$

We have

$$area(\triangle_1) = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 1 & \frac{1-t}{2} & 1 \\ 1 & \frac{1-t}{3-t} & \frac{2}{3-t} \end{vmatrix} = \frac{(1-t)^2}{4(3-t)},$$

$$area(\triangle_2) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \frac{1+t}{2} & 1 \\ 1 & \frac{1+t}{3-t} & \frac{2}{3-t} \end{vmatrix} = \frac{(1-t)^2}{4(3-t)},$$

so

$$area(\mathcal{S}_1) = area(\mathcal{T}_1) - area(\triangle_1) - area(\triangle_2) = \frac{1}{6} - \frac{(1-t)^2}{2(3-t)}.$$

Now $\mathcal{S}$ has two connected components, and

$$\begin{aligned} area(\mathcal{S}) &= \sum_{v>2\sqrt{\alpha^2+1}-2\alpha} area(\mathcal{S}_v) \\ &= area(\mathcal{S}_1) + \sum_{v\geq 2} area(\mathcal{T}_v) \\ &= \frac{1}{2} - \frac{(1-t)^2}{2(3-t)}. \end{aligned} \tag{5.4.8}$$

(iii) $\alpha \leq 3/4$.

If $v \geq 1/\alpha$, then by the same deduction above, we have $\mathcal{S}_v = \mathcal{T}_v$.

Otherwise, if $2\sqrt{\alpha^2+1} - 2\alpha < v < 1/\alpha$, let $t = \sqrt{v^2 + 4\alpha v - 4}$, then (x, y) belongs to the intersection of $\mathcal{T}_v$ and the wedge $\mathcal{W}_v = \{(x,y) | \frac{2}{v+t}x < y < \frac{2}{v-t}x, x, y > 0\}$. Here we can use the results in Case 1:

Let $\mathcal{S}_v = \mathcal{T}_v \cap \mathcal{W}_v$. If $t > 1$, then $\mathcal{S}_v = \mathcal{T}_v$.

If $t \leq 1$, then $v \leq \sqrt{4\alpha^2+5} - 2\alpha$. Since $v > 2\sqrt{\alpha^2+1} - 2\alpha$, there is at most one integral value of such v. If it does exist, say v_0, then $v_0 = \lceil 2\sqrt{\alpha^2+1} - 2\alpha \rceil = 2$, and $2 \leq \sqrt{4\alpha^2+5} - 2\alpha$, which implies $\alpha \leq 1/8$. We discuss the following two subcases:

(iii.a) $1/8 < \alpha \leq 3/4$.

In this case $\mathcal{S}_v = \mathcal{T}_v$ for every v, so $\mathcal{S}$ is connected, and

$$area(\mathcal{S}) = \sum_{v>2\sqrt{\alpha^2+1}-2\alpha} area(\mathcal{S}_v) = \sum_{v\geq 2} area(\mathcal{T}_v) = 1/3. \tag{5.4.9}$$

(iii.b) $0 < \alpha \leq 1/8$,

In this case $\mathcal{S}_v = \mathcal{T}_v$, for $v \geq 3$. We only need to further consider the case $v = 2$. See Figure 5.7.

When $v = 2$, $t = \sqrt{8\alpha} \leq 1$. The two lines $\frac{2}{2-t}x = y$ and $2y - x = 1$ intersect at the point $A =$

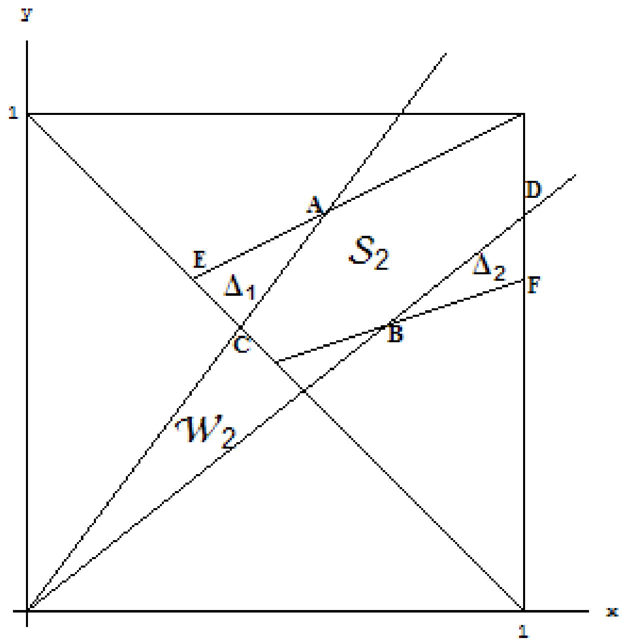

Figure 5.7: $v = 2$

$(\frac{2-t}{2+t}, \frac{2}{2+t})$, which lies on the boundary of $\mathcal{T}_2$. The two lines $\frac{2}{2+t}x = y$ and $3y - x = 1$ intersect at the point $B = (\frac{2+t}{4-t}, \frac{2}{4-t})$, which lies on the boundary of $\mathcal{T}_2$.

The two lines $\frac{2}{2-t}x = y$ and $x + y = 1$ intersect at the point $C = (\frac{2-t}{4-t}, \frac{2}{4-t})$. The two lines $\frac{2}{2+t}x = y$ and $x = 1$ intersect at the point $D = (1, \frac{2}{2+t})$. The points $E = (\frac{1}{3}, \frac{2}{3})$, $F = (1, \frac{2}{3})$ are two boundary points of $\mathcal{T}_2$.

Let $\triangle_1$ be the triangle ACE, and $\triangle_2$ be the triangle BDF. Then

$$area(\mathcal{S}_2) = area(\mathcal{T}_2) - area(\triangle_1) - area(\triangle_2),$$

we have

$$area(\triangle_1) = \frac{1}{2}\begin{vmatrix} 1 & \frac{2-t}{2+t} & \frac{2}{2+t} \\ 1 & \frac{1}{3} & \frac{2}{3} \\ 1 & \frac{2-t}{4-t} & \frac{2}{4-t} \end{vmatrix} = \frac{2(1-t)^2}{3(2+t)(4-t)},$$

$$area(\triangle_2) = \frac{1}{2}\begin{vmatrix} 1 & 1 & \frac{2}{2+t} \\ 1 & 1 & \frac{2}{3} \\ 1 & \frac{2+t}{4-t} & \frac{2}{4-t} \end{vmatrix} = \frac{2(1-t)^2}{3(2+t)(4-t)},$$

so

$$area(\mathcal{S}_2) = \frac{1}{6} - \frac{4(1-t)^2}{3(2+t)(4-t)}.$$

Now $\mathcal{S}$ has two connected components, and

$$\begin{aligned} area(\mathcal{S}) &= \sum_{v>2\sqrt{\alpha^2+1}-2\alpha} area(\mathcal{S}_v) \\ &= area(\mathcal{S}_2) + \sum_{v\geq 3} area(\mathcal{T}_v) \\ &= \frac{1}{3} - \frac{4(1-t)^2}{3(2+t)(4-t)}. \end{aligned} \tag{5.4.10}$$

Finally, we have $(k_1, k_2) = (Qx, Qy) \subset \bigcup Q\mathcal{S}_v = Q\mathcal{S}$, and $\mathcal{S}$ has at most two connected components. By Lemma 5.8, there are exactly $Q^2 \frac{6}{\pi^2} area(\mathcal{S}) + O(Q \log Q)$ lattice points for (k_1, k_2), and there are $\frac{3}{\pi^2}Q^2 + O(Q \log Q)$ Farey fractions of order Q between [0,1]. Taking the quotient and by the above discussion (5.4.7), (5.4.8), (5.4.9), (5.4.10) we get the rest four cases in Theorem 5.1.

5.5 The short interval version of the problem

The proof of Theorem 5.4 will proceed along the same lines as that of Theorem 5.1. However, the presence of the additional condition $\gamma_j = \frac{a_j}{q_j} \in I = (\alpha, \beta]$ for a given interval $I \subseteq (0, 1]$ might yet produce some technical difficulties; see also [3].

By a well-known property of the Farey sequence, if $\frac{a_1}{q_1}$ and $\frac{a_2}{q_2}$ are consecutive Farey fractions, then $a_2q_1 - a_1q_2 = 1$, so $a_1 = -\overline{q_2} \pmod{q_1}$. Then $\overline{q_2} = q_1 - a_1$ and the condition $\frac{a_1}{q_1} \in I$ is equivalent to

$$\overline{q_2} \in I_{q_1} = \big[q_1(1-\beta), q_1(1-\alpha)\big).$$

This means that we should impose the additional condition $\overline{q_{j+1}} \in I_{q_j}$ in all summations over γ_j from the proof of Theorem 5.1. Accordingly, we shall replace $\mathcal{M}_{\mathbf{k}}$ by its subset

$$\mathcal{M}_{\mathbf{k},I} = \{(a, b) \in \mathcal{M}_{\mathbf{k}} \, ; \, \bar{b} \in I_a\}.$$

Instead of Lemma 5.7 and Lemma 5.5, we need to estimate sums of the type

$$S_I' = S_I'(f,\Omega) = \sideset{}{'}\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ \bar{b}\in I_a}} f(a,b) = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ \gcd(a,b)=1 \\ \bar{b}\in I_a}} f(a,b), \tag{5.5.1}$$

where $\Omega \subset [1,R]\times[1,R]$ is a region with rectifiable boundary in the plane and f is a C^1 function on Ω. Here $\bar{b}$ denotes the unique integer with the property that $0<\bar{b}<a$ and $b\bar{b}=1 \pmod{a}$. We prove first the following result.

Lemma 5.9. *Suppose in addition that Ω is convex. Then*

$$\left| S_I' - \frac{6|I|}{\pi^2} \iint_{\Omega} f(x,y)\,dx\,dy \right| \ll_\varepsilon \left(\Big\|\frac{\partial f}{\partial x}\Big\|_\infty + \Big\|\frac{\partial f}{\partial y}\Big\|_\infty \right) \operatorname{Area}(\Omega)\log R$$

$$+ \|f\|_\infty \big(R + \operatorname{length}(\partial\Omega)\big)\log R + m\|f\|_\infty R^{3/2+\varepsilon},$$

where $m=m_f$ is an upper bound for the number of intervals of monotonicity of each of the maps $y\mapsto f(x,y)$.

The proof relies on some estimates for weighted incomplete Kloosterman sums

$$S_{f,J}(l,a) = \sideset{}{'}\sum_{b\in J} f(a,b)\, e\!\left(\frac{l\bar{b}}{a}\right) = \sum_{\substack{b\in J \\ \gcd(b,a)=1}} f(a,b)\, e\!\left(\frac{l\bar{b}}{a}\right) \tag{5.5.2}$$

associated with an interval $J\subset\mathbb{R}$ and a function f. A result of this kind is contained in the next statement.

Lemma 5.10. *Let $S_{f,J}(l,a)$ be as in* (5.5.2)*, where J is a bounded interval. Then*

$$|S_{f,J}(l,a)| \ll_\varepsilon m\|f\|_\infty \big(|J|a^{-1/2+\varepsilon} + a^{1/2+\varepsilon}\big)\gcd(l,a)^{1/2},$$

where $m=m_f$ is an upper bound for the number of intervals of monotonicity of the map $J\ni x\mapsto f(a,x)$.

Proof. For any interval $J_0\subseteq[1,a]$, a Weil type inequality [27] of Hooley provides ([14], see also [7])

$$|S_{J_0}(l,a)| \ll_\varepsilon a^{1/2+\varepsilon}\gcd(l,a)^{1/2},$$

where $S_{J_0}(l,a) = S_{1,J_0}(l,a)$ denotes the ordinary incomplete Kloosterman sum. The previous inequality gives immediately

$$|S_J(l,a)| \ll_\varepsilon \left(\frac{|J|}{a}+1\right)a^{1/2+\varepsilon}\gcd(l,a)^{1/2}. \tag{5.5.3}$$

The statement now follows from (5.5.3) and from Abel's summation formula

$$\sum_{k=1}^{n} a_k b_k = a_1 \sum_{k=1}^{n} b_k + (a_2 - a_1) \sum_{k=2}^{n} b_k + (a_3 - a_2) \sum_{k=3}^{n} b_k + \cdots + (a_{n-2} - a_{n-3}) \sum_{k=n-2}^{n} b_k$$
$$+ (a_{n-1} - a_{n-2}) \sum_{k=n-1}^{n} b_k + (a_n - a_{n-1}) b_n.$$

□

Proof of Lemma 5.9 . We start the proof by writing S'_I as

$$S'_I = \sideset{}{'}\sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ \bar{b}\in I_a}} f(a,b) = \sideset{}{'}\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b) \sum_{x\in I_a} \frac{1}{a} \sum_{l=1}^{a} e\Big(\frac{l(\bar{b}-x)}{a}\Big) = S_1 + S_2, \tag{5.5.4}$$

where S_1 is the sum of terms with $l = a$ and S_2 the sum of the remaining terms. Then, noting that $|I_a| = a|I|$, we find that

$$\begin{aligned} S_1 &= \sideset{}{'}\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b) \sum_{x\in I_a} \frac{1}{a} = \sideset{}{'}\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b) \cdot \frac{1}{a} \big(|I_a| + O(1)\big) \\ &= |I| S' + O\big(\|f\|_\infty R \log R\big), \end{aligned} \tag{5.5.5}$$

where we have denoted $S' = S'_{(0,1]}$. By definition we also see that

$$S_2 = \sideset{}{'}\sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b) \sum_{x\in I_a} \frac{1}{a} \sum_{l=1}^{a-1} e\Big(\frac{l(\bar{b}-x)}{a}\Big) = \sum_{a\in pr_1(\Omega)} \frac{1}{a} \sum_{l=1}^{a-1} \bigg(\sum_{x\in I_a} e\Big(-\frac{lx}{a}\Big) \bigg) S_{f,I'_a}(l,a),$$

where $I'_a = I_a \cap \{b\,;\,(a,b)\in\Omega\}$ is an interval for each a in the projection $pr_1(\Omega)$ of Ω on the first coordinate. As a geometric progression, the inner sum is $\ll \frac{a}{l}$. Thus Lemma 5.10 and $|I'_a| \le |I_a| \le R$ provide

$$|S_2| \ll \sum_{1\le a\le R} \frac{1}{a} \sum_{l=1}^{a-1} \frac{a}{l} \, |S_{f,I'_a}(l,a)| \ll_\varepsilon m\|f\|_\infty \sum_{1\le a\le R} (Ra^{-1/2+\varepsilon} + a^{1/2+\varepsilon}) \sum_{l=1}^{a-1} \frac{\gcd(l,a)^{1/2}}{l}\,. \tag{5.5.6}$$

We combine (5.5.6) with

$$\sum_{l=1}^{a} \frac{\gcd(l,a)^{1/2}}{l} = \sum_{d|a} \sum_{\substack{1\le l\le a \\ \gcd(l,a)=d}} \frac{d^{1/2}}{l} \le \sum_{d|a} d^{1/2} \sum_{m=1}^{[a/d]} \frac{1}{dm} \le \sum_{d|a} d^{-1/2} \log a \ll_\varepsilon a^\varepsilon, \tag{5.5.7}$$

to deduce that

$$|S_2| \ll_\varepsilon m\|f\|_\infty \sum_{1\le a\le R} (Ra^{-1/2+3\varepsilon} + a^{1/2+3\varepsilon}) \ll m\|f\|_\infty R^{3/2+3\varepsilon}. \tag{5.5.8}$$

The statement follows from (5.5.4), (5.5.5), (5.5.8) and Lemma 5.7. □

Proof of Theorem 5.4.

By following the same procedure as in the last section, and by Lemma 5.9, we find that there are exactly $Q^2 \frac{6}{\pi^2}|I| area(\mathcal{S}) + O(Q^{3/2+\epsilon})$ lattice points for (k_1, k_2), and there are $|I|\frac{3}{\pi^2}Q^2 + O(Q \log Q)$ Farey fractions of order Q in the interval I. Taking the quotient we get the same distribution as in Theorem 5.1.

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