PHASE VELOCITIES IN RECTANGULAR WAVEGUIDE PARTIALLY FILLED WITH DIELECTRIC.

by

WALTER L. WEEKS

20 December 1957

Contract No. AF33(616)-3220
Project No. 6(7-4600) Task 40572
WRIGHT AIR DEVELOPMENT CENTER

ELECTRICAL ENGINEERING RESEARCH LABORATORY
ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS
ANTENNA LABORATORY

Technical Report No. 28

PHASE VELOCITIES IN RECTANGULAR WAVEGUIDE
PARTIALLY FILLED WITH DIELECTRIC

by

Walter L. Weeks

20 December 1957

Contract AF33(616)-3220
Project No. 6(7-4600) Task 40572
WRIGHT AIR DEVELOPMENT CENTER

Electrical Engineering Research Laboratory
Engineering Experiment Station
University of Illinois
Urbana, Illinois
CONTENTS

Abstract ii
List of Symbols iii
1. Introduction iv
2. Field Theory 1
   2.1 PM Modes \( (H = 0) \) 3
   2.2 PE Modes \( (E^y = 0) \) 7
3. Computation 9
4. Note Added in Proof 63
5. Acknowledgement 63
References 64
Distribution List
A compilation is made of the phase velocity of the dominant mode of propagation in a rectangular waveguide which is partially loaded with a dielectric, in the case that the dielectric interface is parallel to the broad wall of the waveguide. Curves are presented which show the phase velocity ratio \( (c/v) \) as a function of the wavelength to guide width ratio \( \left( \frac{\lambda}{2a} \right) \) for seven values of dielectric constant in the range from 1.6 to 13.7; cut-off information on three other important modes is also given. The roots of the transcendental equation for the propagation constants are also tabulated. Enough of the field theory is presented to allow intelligent application of the data.
LIST OF SYMBOLS

\( \begin{align*}
E & \quad \text{electric field intensity} \\
H & \quad \text{magnetic field intensity} \\
\end{align*} \)

\( \begin{align*}
a & \quad \text{waveguide width} \\
b & \quad \text{waveguide height} \\
c & \quad \text{speed of light in free space} \\
d & \quad \text{thickness of dielectric material} \\
j & \quad \text{imaginary unit} \\
k & \quad \text{propagation constant in free space} \left( \frac{2\pi}{\lambda} \right) \\
k_x & \quad \text{propagation constant in } x \text{ direction in waveguide} \\
k_y & \quad \text{propagation constant in } y \text{ direction in waveguide} \\
k_z & \quad \text{propagation constant in } z \text{ direction in waveguide} \\
v & \quad \text{phase velocity in waveguide} \\
\hat{x} \quad \hat{y} \quad \hat{z} & \quad \text{unit vectors in coordinate directions} \\
\epsilon & \quad \text{permittivity of dielectric} \\
\lambda & \quad \text{free space wavelength} \\
\mu & \quad \text{permeability of dielectric} \\
\omega & \quad \text{radian frequency} \left( \text{radians/sec} \right) \\
\end{align*} \)
1. INTRODUCTION

There is considerable current interest in the production of guided electromagnetic waves having phase velocities equal to or less than the speed of light in free space (for example, in the design of traveling wave antennas and of devices involving electron-traveling-wave interactions). A convenient way to obtain such phase velocities is to partially load a rectangular waveguide with a dielectric material. In antenna work particularly, because of the field configurations, it is usually desirable to place the dielectric interface parallel to the broad wall of the waveguide, as indicated in Fig. 1. This problem has

![Diagram of partially dielectric loaded waveguide showing coordinate system and dimension designations]

FIGURE 1. PARTIALLY DIELECTRIC LOADED WAVEGUIDE SHOWING COORDINATE SYSTEM AND DIMENSION DESIGNATIONS
been considered by others, $^{1-3}$ and there is published information on some of the cut-off frequencies, $^3$ but (since in this case there is no convenient relationship between the cut-off frequencies and the propagation constants) there has been little detailed information available on the phase velocities as a function of waveguide geometry and dielectric material. This report presents such information. Main consideration is given to the dominant (hybrid) mode under the assumption of zero energy dissipation, but cut-off data for the next three higher order modes are included. Results are presented for most of the common solid dielectric materials which have a small loss tangent. The University of Illinois digital computer (ILLIAC) was employed, first to solve the transcendental equations, and then to calculate the c/v ratios. The results were spot checked with desk calculators.
2. FIELD THEORY

The elimination of one or the other of the field variables from the Maxwell equations with harmonic time variation results in the equation

\[ \nabla \times \nabla \times \vec{\mathcal{E}} - \omega^2 \mu \varepsilon \vec{\mathcal{H}} = 0 \]  

(1)

where \( \mathcal{E} \) is either \( \mathcal{E} \) or \( \mathcal{H} \). The mathematical problem is thus to find solutions of the vector Helmholtz equation (1) which satisfy the boundary conditions and Maxwell's equations. It is convenient to represent the fields in terms of a pair of scalars. The customary representation, in which the scalars are related to the longitudinal components of the fields, with the consequent representation in terms of TM and TE modes, is not particularly appropriate since ordinarily the simplest field configuration which can propagate in a waveguide as shown in Fig. 1 is neither TM nor TE. Consequently, we will employ the alternative representation in which the modes are separated into those for which there is no y-component of magnetic field \( (H_y = 0) \) and those for which there is no y-component of electric field \( (E_y = 0) \). For convenience, we will designate these modes as PM (for parallel magnetic, i.e., \( H \) is parallel to the dielectric interface) and PE (\( E \) is parallel to the dielectric interface).

2.1 PM Modes \( (H_y = 0) \)

We first look for solutions such that

\[ H = \nabla \times f \hat{y} \]  

(2)

(carat symbol designates unit vector).

In this case we find from Eqs (1) and (2) the equation

\[ \nabla \times (\nabla \times \nabla \times f \hat{y} - \omega^2 \mu \varepsilon f \hat{y}) = 0 \]

(3)

from which it follows that

\[ \nabla \times \nabla \times f \hat{y} - \omega^2 \mu \varepsilon f \hat{y} = \nabla U, \]

(4)

where \( U \) is an arbitrary scalar. If we take \( U = \partial f / \partial y \) we find that \( f \) must satisfy the equation

\[ \nabla^2 f + \omega^2 \mu \varepsilon f = 0. \]

(5)
We are looking for the fields which propagate in the $z$ direction, hence we let

$$f = F(x,y)e^{kz}$$

and find the equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + (\omega^2 \mu \epsilon + k_z^2) F = 0.$$  \hspace{1cm} (7)

To proceed, we divide the waveguide into two regions—in each of which the values of $\epsilon$ is constant. The solution consists of those functions $F_1$ and $F_2$ which satisfy (7) and the following boundary conditions.

$$\hat{x} \cdot H = 0 \quad (=- \frac{\partial f}{\partial z}) \text{ at } x = 0 \text{ and } x = a$$

$$\hat{x} \cdot H_1 = \hat{x} \cdot H_2 \text{ at } y = d$$

$$\hat{z} \cdot H_1 = \hat{z} \cdot H_2 \text{ at } y = d \quad (\hat{z} \cdot H = \frac{\partial f}{\partial x})$$

$$\hat{x} \cdot E = 0 \quad (= \frac{1}{j\omega \epsilon} \frac{\partial^2 f}{\partial x \partial y}) \text{ at } y = 0 \text{ and } y = b$$

$$\hat{x} \cdot E_1 = \hat{x} \cdot E_2 \text{ at } y = d$$

$$\hat{y} \cdot E = 0 \quad (= \frac{1}{j\omega \epsilon} (\frac{\partial^2 f}{\partial y^2} + \omega^2 \mu \epsilon f)) \text{ at } x = 0 \text{ and } x = a$$

$$\hat{z} \cdot E = 0 \quad (= \frac{1}{j\omega \epsilon} \frac{\partial^2 f}{\partial z \partial y}) \text{ at } x = 0 \text{ and } x = a \quad \text{ and at } y = 0 \text{ and } y = b$$

$$\hat{z} \cdot E_1 = \hat{z} \cdot E_2 \text{ at } y = d.$$  \hspace{1cm} (15)

It is clear from (9), for example, that the propagation constant $k_z$ must be the same in both regions. Assuming a product solution of (7) gives solutions of the type

$$F = \sin (k_x x + \phi) \cosh (k_y y + \theta)$$  \hspace{1cm} (16)
and applying the conditions (8), (9), and (14) gives the functions for the regions as follows:

\[ F_1 = A_1 \sin \frac{n\pi x}{a} \cosh k_{y_1} y \]

(17)

\[ F_2 = A_2 \sin \frac{n\pi x}{a} \cosh k_{y_2} (y-b). \]

(18)

Then to satisfy (10) we must have the equation

\[ A_1 \cosh k_{y_1} d = A_2 \cosh k_{y_2} (d-b) \]

(19)

and to satisfy (15) we must have

\[ \frac{1}{\epsilon_1} A_1 k_{y_1} \sinh k_{y_1} d = \frac{1}{\epsilon_2} A_2 k_{y_2} \sinh k_{y_2} (d-b). \]

(20)

These latter two equations combine to give the transcendental equation

\[ \frac{k_{y_1}}{\epsilon_1} \tanh k_{y_1} d = \frac{k_{y_2}}{\epsilon_2} \tanh k_{y_2} (d-b) \]

(21)

which can be used to solve for the propagation constants since by (7)

\[ -k_x^2 + k_y^2 + (\omega^2 \mu_1 \epsilon_1 + k_z^2) = 0 \]

(22)

so that

\[ k_{y_1}^2 + \omega^2 \mu_1 \epsilon_1 = k_{y_2}^2 + \omega^2 \mu_2 \epsilon_2. \]

(23)

If we put \( \mu_1 = \mu_2 \) and let medium 2 be vacuum, then

\[ k_{y_2}^2 = k_{y_1}^2 + \left( \frac{2\pi}{\lambda} \right)^2 (\epsilon_r - 1) \]

(24)

where \( \epsilon_r \) is the relative permittivity of medium 1.

The phase velocity ratio \( \frac{c}{v} = \frac{k}{k} \) can thus be found by solving (21)
subject to (24) and using (22) to find

\[
\frac{c}{v} = \sqrt{\frac{\epsilon}{r} - \frac{n_1^2 \lambda^2}{2a} + \left( \frac{k_{y1} \lambda}{2\pi} \right)^2}.
\]  

(25)

A careful examination discloses that equation (21) has no solutions if \(k_y\) is pure real, but there is no such limitation on \(k_y\) for the dominant mode \((a>b)\), \(k_y\) is real.

From Eq. (2), with (6), (17), and (18), we find the expressions for the magnetic field components

\[
H_{x1} = -A_k \sin \frac{n_1 \pi x}{a} \cosh k_y \frac{1}{y_1} e^{k_z z}  
\]

(26)

\[
H_{z1} = A \frac{n_1 \pi}{a} \cos \frac{n_1 \pi x}{a} \cosh k_y \frac{1}{y_1} e^{k_z z} 
\]

(27)

with similar expressions for region two, except with the \(\cosh\) argument changed to \(k_y(y-b)\).

From the Maxwell equation

\[
E = \frac{1}{\omega \epsilon} (\nabla \times H),
\]

we find that the electric field components are as follows:

\[
E_{x1} = \frac{A}{j \omega \epsilon} \frac{n_1 \pi}{a} k_y \cos \frac{n_1 \pi x}{a} \sinh k_y \frac{1}{y_1} e^{k_z z} 
\]

(28)

\[
E_{y1} = \frac{A}{j \omega \epsilon} \left[ \frac{(n_1 \pi)^2}{a} - k_z^2 \right] \sin \frac{n_1 \pi x}{a} \cosh k_y \frac{1}{y_1} e^{k_z z} 
\]

(29)

\[
E_{z1} = \frac{A}{j \omega \epsilon} k_y k_z \sin \frac{n_1 \pi x}{a} \sinh k_y \frac{1}{y_1} e^{k_z z} 
\]

(30)

with an obvious change of subscripts and argument to characterize the fields in region 2.
2.2 PE Modes \((E_y = 0)\)

To delineate the PE modes, we proceed in a fashion analogous to that above under PM modes. Thus we look for solutions \((E_y = 0)\) for which

\[
\mathbf{E} = \nabla \times g \hat{y}.
\]  

(31)

As before, we look for solutions of the type for which

\[
g = G(x,y) e^{k_y y}
\]  

(32)

and we find the equation

\[
\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + (\omega^2 \mu \epsilon + k_y^2) G = 0.
\]  

(33)

The boundary conditions (8) to (14), (except for the parentheses) apply with the additional condition,

\[
\hat{y} \cdot \mathbf{H} = 0 \quad (= - \frac{1}{j \omega \mu} \frac{\partial^2 g}{\partial y^2} + \omega^2 \mu \epsilon g) \text{ at } y = 0 \text{ and } y = b.
\]  

(34)

The conditions on \(g\) and \(G\) can be found by interchanging the roles of \(\mathbf{E}\) and \(\mathbf{H}\). (In particular

\[
\hat{x} \cdot \mathbf{E} = - \frac{\partial g}{\partial z}, \quad \hat{z} \cdot \mathbf{E} = \frac{\partial x}{\partial x}
\]

\[
\hat{x} \cdot \mathbf{H} = - \frac{1}{j \omega \mu} \frac{\partial^2 g}{\partial x \partial y}, \quad \hat{z} \cdot \mathbf{H} = - \frac{1}{j \omega \mu} \frac{\partial^2 g}{\partial z \partial y}.
\]

Application of these boundary conditions gives the result

\[
G_1 = B_1 \cos \frac{n \pi x}{a} \sinh \frac{k_y y}{y_1}
\]

\[
G_2 = B_2 \cos \frac{n \pi x}{a} \sinh \frac{k_y (y-b)}{y_2}
\]

and the transcendental equation

\[
\frac{\mu_1 \tanh \frac{k_y d}{y_1}}{k_y y_1} = \frac{\mu_2 \tanh \frac{k_y (d-b)}{y_2}}{k_y y_2}
\]
The field components are found in straightforward fashion to be

\[ E_x = -B_1 k_z \cos \frac{n\pi x}{a} \sinh \frac{k_y}{a} \ e^{k_z z} \]

\[ E_z = -B_1 \frac{n\pi}{a} \sin \frac{n\pi x}{a} \sinh k_y \ e^{k_z z} \]

\[ H_x = \frac{B_1}{j\omega_1} \frac{n\pi}{a} \sin \frac{n\pi x}{a} \cosh k_y \ e^{k_z z} \]

\[ H_y = -\frac{B_1}{j\omega_1} \left( \left( \frac{n\pi}{a} \right)^2 - k_z^2 \right) \cos \frac{n\pi x}{a} \sinh k_y \ e^{k_z z} \]

\[ H_z = -\frac{B_1}{j\omega_1} k_z \frac{n\pi}{a} \sinh k_y \ e^{k_z z} \]

Note that \( n = 0 \) gives a permissible PE solution, and these modes are also TE.
The computational difficulty lies in the fact that the quantities $k_1$ and $k_2$ which appear in the equations of the foregoing section depend upon the parameters $\varepsilon_r$, b, and d in such a way that a change in any one of these requires a new numerical solution to the transcendental equation (21). Thus, any extensive tabulation of results make the use of a high speed computing machine almost imperative. The availability of the ILLIAC made the present compilation feasible.

In selecting the parameters for calculation, an effort was made to obtain values for those situations most likely to be of interest. Thus, data are available for seven values of dielectric constant in the range from 1.6 to 13.7, with fillings $(d/b)$ ranging from 10 to 90 percent, and waveguide aspect ratios varying from 0.1 to 1.0. Selections from these data are presented here in graphical form. The results from the digital computer were printed out with eight significant figures; however, for most applications such accuracy is neither warranted nor realistic. The program was set up to instruct the computer to calculate $c/v$ ratios in small steps of $\lambda/2a$ until it reached a value for which $(c/v)^2$ was negative. Consequently, the cut-off dimension lies between the last calculated point and the next higher regular step. Table 1 is a facsimile of the form of the data as obtained from the computer.

It is convenient to distinguish the modes of propagation by a pair of numerical subscripts. The first subscript will specify the root of the transcendental equation which appears in the solution (roots numbered from the smallest), while the second subscript specifies the integer $n$ in the argument of the trigonometric functions $(n\pi/a)$. Thus, the mode $PM_{11}$ is the dominant mode if $a > b$. The $c/v$ ratios for this mode are plotted in detail in the following figures. The cut-off dimensions for the $PE_{10}$ mode and the $PM_{21}$ mode were calculated by the ILLIAC and are plotted on the curves for values of $d/b = .2, .4, .5, .6, .8$. The symbols are as follows: a circle (o) for the $PE_{10}$ mode, and a triangle (Δ) for the $PM_{21}$ mode. The cut-off dimension for the $PM_{12}$ mode was obtained (less accurately) by extrapolation. This information also appears in the graphs, with the square symbol (□)
designating the $PM_{12}$ mode. The cut-off dimensions for the modes analogous to these latter three modes in the limiting case of $d/b$ equal to zero or one are as follows:

$$PM_{12} \left( \square \right) \frac{\lambda}{2a} = \frac{\sqrt{\varepsilon_r}}{2} \text{ at cut-off}$$

$$PE_{10} \left( O \right) \frac{\lambda}{2a} = \frac{b}{a} \sqrt{\varepsilon_r} \text{ at cut-off}$$

$$PM_{21} \left( \Delta \right) \frac{\lambda}{2a} = \frac{b}{a} \sqrt{\frac{\varepsilon_r}{1 + (b/a)^2}} \text{ at cut-off}.$$  

Figures 2 through 43 contain the phase velocity information plotted so as to be immediately useful in the design problem in which the object is to prescribe the dimensions in a waveguide for single mode propagation when the phase velocity at a given frequency is specified.

An indication of other ways of presenting the same information to increase its utility is given in Figs. 44 through 48. For example, Fig. 44 shows the variation of phase velocity with dielectric constant for particular waveguide geometries. To facilitate interpolation between the $d/b$ and $b/a$ ratios used in the calculation, the data can be presented as in Figs. 45 through 48.

Tables 2 through 8 give the roots of the transcendental equation (21) for different dielectric constants and waveguide geometries. The results can be used directly in Eq.(25) to make specific calculations for particular designs.
FIGURE 2. PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

$\varepsilon_r = 1.6$  $\frac{b}{a} = 0.25$

$\lambda_{cut-off} \frac{\lambda}{2a}$

<table>
<thead>
<tr>
<th>$\frac{d}{b}$</th>
<th>$\Delta$</th>
<th>$o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.247</td>
<td>2.54</td>
</tr>
<tr>
<td>0.4</td>
<td>2.266</td>
<td>2.74</td>
</tr>
<tr>
<td>0.5</td>
<td>2.278</td>
<td>2.87</td>
</tr>
<tr>
<td>0.6</td>
<td>2.289</td>
<td>2.99</td>
</tr>
<tr>
<td>0.8</td>
<td>3.02</td>
<td>3.13</td>
</tr>
</tbody>
</table>
FIGURE 3. PHASE VELOCITY RATIO OF THE PM_{1,1} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[
\frac{d}{b} = 1
\]

\[
\frac{d}{b} = 0.8
\]

\[
\frac{d}{b} = 0.6
\]

\[
\frac{d}{b} = 0.5
\]

\[
\frac{d}{b} = 0.4
\]

\[
\frac{d}{b} = 0.3
\]

\[
\frac{d}{b} = 0.2
\]

\[
\frac{d}{b} = 0.1
\]

\[
\frac{d}{b} = 0
\]

\[
\epsilon_r = 1.6 \quad b/a = 0.4
\]

\[
\text{cut-off } \lambda/2a
\]

\[
\frac{d}{b} \quad \square \quad 0
\]

\[
0.2 \quad 0.380 \quad 0.406
\]

\[
0.4 \quad 0.408 \quad 0.438
\]

\[
0.5 \quad 0.425 \quad 0.459
\]

\[
0.6 \quad 0.440 \quad 0.478
\]

\[
0.8 \quad 0.460 \quad 0.501
\]
FIGURE 4. PHASE VELOCITY RATIO OF THE $P_{11}^m$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \frac{d}{d} = 1 \]

\[ \varepsilon_r = 1.6 \quad b/a = 0.45 \]

\[ \text{cut-off } \lambda_{2a} \]

\[ d/b \quad \Delta \]

\[ 0.2 \quad 0.421 \]

\[ 0.4 \quad 0.451 \]

\[ 0.5 \quad 0.469 \]

\[ 0.6 \quad 0.484 \]

\[ 0.8 \quad 0.507 \]
FIGURE 5. PHASE VELOCITY RATIO OF THE PM₁₁ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \varepsilon_r = 1.6 \quad \frac{b}{a} = 0.5 \]

cut-off \( \frac{\lambda}{2a} \)

\[ \frac{d}{b} \quad \Delta \]

0.2  0.459
0.4  0.492
0.5  0.510
0.6  0.527
0.8  0.551
FIGURE 6. PHASE VELOCITY RATIO OF THE $p_{m_11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 7. PHASE VELOCITY RATIO OF THE PM\textsubscript{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 8. PHASE VELOCITY RATIO OF THE PM
MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

<table>
<thead>
<tr>
<th>$\varepsilon_r$ = 2.1</th>
<th>$b/a = 0.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{ul-off}$</td>
<td>$\lambda_{2a}$</td>
</tr>
<tr>
<td>$d/b$</td>
<td>$0.2$</td>
</tr>
<tr>
<td>$0.4$</td>
<td>$0.294$</td>
</tr>
<tr>
<td>$0.5$</td>
<td>$0.316$</td>
</tr>
<tr>
<td>$0.6$</td>
<td>$0.335$</td>
</tr>
<tr>
<td>$0.7$</td>
<td>$0.358$</td>
</tr>
</tbody>
</table>
FIGURE 9. PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 10. PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \frac{\varepsilon_r}{\varepsilon_0} = 2.1 \]

\[ b/a = 0.45 \]
FIGURE 11. PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 12. PHASE VELOCITY RATIO OF THE PM\textsubscript{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 13. PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 14. PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN
FIGURE 1.

$\varepsilon_r = 2.54$

$b/a = 0.25$

d/b cut-off $\frac{\lambda}{2a}$

<table>
<thead>
<tr>
<th>d/b</th>
<th>cut-off</th>
<th>$\frac{\lambda}{2a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.261</td>
<td>0.254</td>
</tr>
<tr>
<td>0.4</td>
<td>0.311</td>
<td>0.302</td>
</tr>
<tr>
<td>0.5</td>
<td>0.340</td>
<td>0.328</td>
</tr>
<tr>
<td>0.6</td>
<td>0.364</td>
<td>0.349</td>
</tr>
<tr>
<td>0.8</td>
<td>0.393</td>
<td>0.376</td>
</tr>
</tbody>
</table>
FIGURE 15. PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 16. PHASE VELOCITY RATIO OF THE $P_{m1}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

$\varepsilon_r = 2.54$

$\frac{b}{a} = .45$
FIGURE 17. PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 18. PHASE VELOCITY RATIO OF THE PM\textsubscript{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
Figure 19. Phase velocity ratio of the $PM_{11}$ mode in the waveguide as shown in Figure 1.
FIGURE 20. PHASE VELOCITY RATIO OF THE $\text{PM}_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

<table>
<thead>
<tr>
<th>$d/b$</th>
<th>0</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.271</td>
<td>0.265</td>
</tr>
<tr>
<td>0.4</td>
<td>0.358</td>
<td>0.347</td>
</tr>
<tr>
<td>0.5</td>
<td>0.401</td>
<td>0.384</td>
</tr>
<tr>
<td>0.6</td>
<td>0.436</td>
<td>0.414</td>
</tr>
<tr>
<td>0.8</td>
<td>0.479</td>
<td>0.453</td>
</tr>
</tbody>
</table>

$b/a = 0.25$
$\varepsilon_r = 3.78$
FIGURE 21 PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

$\varepsilon_r = 3.78$

$\frac{b}{a} = .4$
FIGURE 22 PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \frac{d}{\lambda} = 1 \]

\[ \varepsilon_r = 3.78 \]

\[ \frac{b}{a} = .45 \]
FIGURE 23 PHASE VELOCITY RATIO OF THE PM11 MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \frac{\lambda}{2a} \]

\[ \varepsilon_r = 3.78 \]

\[ b/a = 0.5 \]
FIGURE 24  PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 25 PHASE VELOCITY RATIO OF THE PM\textsubscript{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 26 PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \varepsilon_r = 5.75 \]
\[ \frac{b}{a} = 0.25 \]

\[ \text{cut-off } \frac{\lambda}{2a} \]

<table>
<thead>
<tr>
<th>$d/b$</th>
<th>$\theta$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.290</td>
<td>0.285</td>
</tr>
<tr>
<td>0.4</td>
<td>0.424</td>
<td>0.408</td>
</tr>
<tr>
<td>0.5</td>
<td>0.484</td>
<td>0.458</td>
</tr>
<tr>
<td>0.6</td>
<td>0.531</td>
<td>0.497</td>
</tr>
<tr>
<td>0.8</td>
<td>0.588</td>
<td>0.548</td>
</tr>
</tbody>
</table>
FIGURE 27  PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE
WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 28 PHASE VELOCITY RATIO OF THE PM₁₁ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 29 PHASE VELOCITY RATIO OF THE PM\textsubscript{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE I.
FIGURE 30 PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 31. PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 31 PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 33  PHASE VELOCITY RATIO OF THE $\text{PM}_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 34  PHASE VELOCITY RATIO OF THE PM11 MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 35 PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 36 PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 37 PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 38 PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 39  PHASE VELOCITY RATIO OF THE PM\textsubscript{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \frac{d}{b} = 1 \]

\[ \varepsilon_r = 13.7 \]

\[ \frac{b}{a} = .4 \]
FIGURE 40 PHASE VELOCITY RATIO OF THE PN_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 41 PHASE VELOCITY RATIO OF THE PM₁₁ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 42 PHASE VELOCITY RATIO OF THE PM₁₁ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 43  PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \varepsilon_r = 13.7 \]
\[ b/a = 1.0 \]
\[ \text{cut-off } \lambda/2a \]

<table>
<thead>
<tr>
<th>( d/b )</th>
<th>0</th>
<th>.2</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
<th>.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\lambda}{2a} )</td>
<td>.151</td>
<td>.251</td>
<td>.291</td>
<td>.324</td>
<td>.362</td>
<td></td>
</tr>
</tbody>
</table>

\[ \frac{a}{b} = 1 \]

Graph showing phase velocity ratio as a function of \( \frac{\lambda}{2a} \) for different values of \( d/b \).
FIGURE 44 PHASE VELOCITY RATIO OF THE PM11 MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.
FIGURE 45 PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \frac{\lambda}{2\delta} = 0.5, \quad \frac{\lambda}{2\delta} = 0.8, \quad \frac{\lambda}{2\delta} = 1.0, \quad \frac{\lambda}{2\delta} = 1.2 \]

\[ \varepsilon_1 = 2.54 \]

\[ b/a = 0.45 \]
FIGURE 46 PHASE VELOCITY RATIO OF THE PM$_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

$\varepsilon_1 = 5.75$

$b/a = 0.5$
FIGURE 47 PHASE VELOCITY RATIO OF THE PM_{11} MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

\[ \varepsilon_{1} = 2.54 \]

\[ \frac{b}{a} = 0.45 \]
FIGURE 48 PHASE VELOCITY RATIO OF THE $PM_{11}$ MODE IN THE WAVEGUIDE AS SHOWN IN FIGURE 1.

$\varepsilon_i = 2.54$

$d/b = 0.6$

\[ \frac{\gamma}{\gamma_0} = 0.5 \]
\[ \frac{\gamma}{\gamma_0} = 0.8 \]
\[ \frac{\gamma}{\gamma_0} = 1.0 \]
\[ \frac{\gamma}{\gamma_0} = 1.2 \]
<table>
<thead>
<tr>
<th>( \frac{d}{b} )</th>
<th>( \lambda )</th>
<th>( \left( k_1 y_1 \right) / j )</th>
<th>( \frac{c}{v} ); ( n = 1 )</th>
<th>( \frac{c}{v} ); ( n = 2 )</th>
<th>( \lambda )</th>
<th>( \left( k_1 y_1 \right) / j )</th>
<th>( \frac{c}{v} ); ( n = 1 )</th>
<th>( \frac{c}{v} ); ( n = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>.6</td>
<td>.1015515</td>
<td>.90109266</td>
<td>.21937371</td>
<td>.5</td>
<td>.2685139</td>
<td>1.17837188</td>
<td>.7990277</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
<td>.10175515</td>
<td>.83653633</td>
<td>.70819803</td>
<td>.6</td>
<td>.5095631</td>
<td>1.10674231</td>
<td>.38052913</td>
</tr>
<tr>
<td>.7</td>
<td>.7</td>
<td>.10121167</td>
<td>.75901511</td>
<td>.61780461</td>
<td>.7</td>
<td>.5958161</td>
<td>1.03001196</td>
<td>.91279189</td>
</tr>
<tr>
<td>.8</td>
<td>.8</td>
<td>.1011551</td>
<td>.63333332</td>
<td>.59720601</td>
<td>.8</td>
<td>.6117968</td>
<td>.83875969</td>
<td>.70906867</td>
</tr>
<tr>
<td>.9</td>
<td>.9</td>
<td>.10119162</td>
<td>.59720601</td>
<td>.53371235</td>
<td>1.0</td>
<td>.6677816</td>
<td>.53371235</td>
<td>.5900345</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>.1013521</td>
<td>.23840561</td>
<td>.22109587</td>
<td>1.2</td>
<td>.9118092</td>
<td>.22109587</td>
<td>.9118092</td>
</tr>
<tr>
<td>.5</td>
<td>.6</td>
<td>.1334928</td>
<td>.94171360</td>
<td>.38351141</td>
<td>.5</td>
<td>.7155939</td>
<td>1.22254238</td>
<td>.86287946</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
<td>.1666299</td>
<td>.88639105</td>
<td>.15718189</td>
<td>.6</td>
<td>.9783802</td>
<td>1.08573938</td>
<td>.50900345</td>
</tr>
<tr>
<td>.7</td>
<td>.7</td>
<td>.1860759</td>
<td>.80064151</td>
<td>.70970144</td>
<td>.7</td>
<td>.1257617</td>
<td>.78964118</td>
<td>.70970144</td>
</tr>
<tr>
<td>.8</td>
<td>.8</td>
<td>.2031608</td>
<td>.5611108</td>
<td>.53703144</td>
<td>.8</td>
<td>.2188538</td>
<td>.53703144</td>
<td>.6085292</td>
</tr>
<tr>
<td>.9</td>
<td>.9</td>
<td>.2086315</td>
<td>.35155429</td>
<td>.22212323</td>
<td>1.0</td>
<td>.3370651</td>
<td>.22212323</td>
<td>.9118092</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>.1334928</td>
<td>.94171360</td>
<td>.38351141</td>
<td>1.2</td>
<td>.9118092</td>
<td>.22109587</td>
<td>.9118092</td>
</tr>
<tr>
<td>.5</td>
<td>.6</td>
<td>.76529241</td>
<td>1.00391835</td>
<td>.5077913</td>
<td>.5</td>
<td>3.1928156</td>
<td>1.2615810</td>
<td>.91718632</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
<td>.8512832</td>
<td>.93121283</td>
<td>.1356993</td>
<td>.6</td>
<td>3.318191</td>
<td>1.2028108</td>
<td>.6085292</td>
</tr>
<tr>
<td>.7</td>
<td>.7</td>
<td>.8973761</td>
<td>.85383628</td>
<td>.70970144</td>
<td>.7</td>
<td>3.586929</td>
<td>.78964118</td>
<td>.70970144</td>
</tr>
<tr>
<td>.8</td>
<td>.8</td>
<td>5.9252097</td>
<td>.75541768</td>
<td>.53703144</td>
<td>.8</td>
<td>3.683239</td>
<td>.86936011</td>
<td>.53703144</td>
</tr>
<tr>
<td>.9</td>
<td>.9</td>
<td>.933766</td>
<td>.62680878</td>
<td>.22212323</td>
<td>1.0</td>
<td>3.7173681</td>
<td>.7382310</td>
<td>.7382310</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>.9559978</td>
<td>.44883903</td>
<td>.12212323</td>
<td>1.2</td>
<td>.3985899</td>
<td>.12212323</td>
<td>.12212323</td>
</tr>
<tr>
<td>.5</td>
<td>.6</td>
<td>.3057336</td>
<td>1.06696913</td>
<td>.62202037</td>
<td>.5</td>
<td>3.1928156</td>
<td>1.2615810</td>
<td>.91718632</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
<td>.597852</td>
<td>.99213275</td>
<td>.1356993</td>
<td>.6</td>
<td>3.318191</td>
<td>1.2028108</td>
<td>.6085292</td>
</tr>
<tr>
<td>.7</td>
<td>.7</td>
<td>.5518273</td>
<td>.91179617</td>
<td>.70970144</td>
<td>.7</td>
<td>3.586929</td>
<td>.78964118</td>
<td>.70970144</td>
</tr>
<tr>
<td>.8</td>
<td>.8</td>
<td>.5953565</td>
<td>.81667512</td>
<td>.53703144</td>
<td>.8</td>
<td>3.683239</td>
<td>.86936011</td>
<td>.53703144</td>
</tr>
<tr>
<td>.9</td>
<td>.9</td>
<td>.6286285</td>
<td>.69821188</td>
<td>.22212323</td>
<td>1.0</td>
<td>3.7173681</td>
<td>.7382310</td>
<td>.7382310</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>.6515533</td>
<td>.53939868</td>
<td>.12212323</td>
<td>1.2</td>
<td>.7174873</td>
<td>.53939868</td>
<td>.7174873</td>
</tr>
<tr>
<td>.5</td>
<td>.6</td>
<td>.66806182</td>
<td>.27607863</td>
<td>.22212323</td>
<td>1.3</td>
<td>3.7956506</td>
<td>.27607863</td>
<td>.27607863</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
<td>.7927033</td>
<td>1.12612742</td>
<td>.71983537</td>
<td>.5</td>
<td>2.1776056</td>
<td>1.30172906</td>
<td>.97185316</td>
</tr>
<tr>
<td>.7</td>
<td>.7</td>
<td>.057015</td>
<td>1.05333139</td>
<td>.15905213</td>
<td>.6</td>
<td>2.5561612</td>
<td>1.25322898</td>
<td>.7001785</td>
</tr>
<tr>
<td>.8</td>
<td>.8</td>
<td>.1281151</td>
<td>.97153225</td>
<td>.12911878</td>
<td>.7</td>
<td>2.6503559</td>
<td>.19573233</td>
<td>.19573233</td>
</tr>
<tr>
<td>.9</td>
<td>.9</td>
<td>.2031666</td>
<td>.87982314</td>
<td>.95234737</td>
<td>.8</td>
<td>2.7031683</td>
<td>.12911878</td>
<td>.12911878</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>.2534766</td>
<td>.76805094</td>
<td>.83339012</td>
<td>1.0</td>
<td>2.7603371</td>
<td>.58008898</td>
<td>.58008898</td>
</tr>
<tr>
<td>.5</td>
<td>.6</td>
<td>.287951</td>
<td>.62586152</td>
<td>.41835377</td>
<td>1.1</td>
<td>2.7921512</td>
<td>.4593343</td>
<td>.4593343</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
<td>.31281</td>
<td>.41835377</td>
<td>.22212323</td>
<td>1.2</td>
<td>2.902630</td>
<td>.4593343</td>
<td>.4593343</td>
</tr>
</tbody>
</table>
TABLE II. VALUES OF \((k_{y_1} \lambda)/\jmath\) WHICH SATISFY EQUATION (21), RELATIVE DIELECTRIC CONSTANT \(\epsilon_1 = 1.60\)

<table>
<thead>
<tr>
<th>(\lambda/\beta)</th>
<th>(\text{d/\beta})</th>
<th>0.2</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>4.1047</td>
<td>2.8501</td>
<td>2.4309</td>
<td>2.1046</td>
<td>1.5995</td>
<td></td>
</tr>
<tr>
<td>1.20</td>
<td>4.2212</td>
<td>3.1712</td>
<td>2.7412</td>
<td>2.3874</td>
<td>1.7985</td>
<td></td>
</tr>
<tr>
<td>1.60</td>
<td>4.3797</td>
<td>3.5781</td>
<td>3.1705</td>
<td>2.7911</td>
<td>2.0769</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>4.6402</td>
<td>3.7791</td>
<td>3.5080</td>
<td>3.0346</td>
<td>2.2131</td>
<td></td>
</tr>
<tr>
<td>2.40</td>
<td>4.6693</td>
<td>3.8832</td>
<td>3.5398</td>
<td>3.1757</td>
<td>2.3451</td>
<td></td>
</tr>
<tr>
<td>3.00</td>
<td>4.4912</td>
<td>3.9629</td>
<td>3.6650</td>
<td>3.2931</td>
<td>2.4338</td>
<td></td>
</tr>
<tr>
<td>4.00</td>
<td>4.5071</td>
<td>4.0207</td>
<td>3.7234</td>
<td>3.3837</td>
<td>2.5059</td>
<td></td>
</tr>
<tr>
<td>5.00</td>
<td>4.5142</td>
<td>4.0461</td>
<td>3.7583</td>
<td>3.4219</td>
<td>2.5399</td>
<td></td>
</tr>
<tr>
<td>6.40</td>
<td>4.5189</td>
<td>4.0632</td>
<td>3.7819</td>
<td>3.4531</td>
<td>2.5638</td>
<td></td>
</tr>
<tr>
<td>8.00</td>
<td>4.5215</td>
<td>4.0726</td>
<td>3.7950</td>
<td>3.4688</td>
<td>2.5773</td>
<td></td>
</tr>
<tr>
<td>8.80</td>
<td>----</td>
<td>----</td>
<td>3.7990</td>
<td>3.4737</td>
<td>2.5815</td>
<td></td>
</tr>
</tbody>
</table>
### Table III. Values of \((k_1 \lambda)/j\) Which Satisfy Equation (21). Relative Dielectric Constant \(\epsilon_1 = 2.10\)

<table>
<thead>
<tr>
<th>(\lambda / b)</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
<th>.7</th>
<th>.8</th>
<th>.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>6.2120</td>
<td>5.1511</td>
<td>4.0502</td>
<td>3.2631</td>
<td>2.7153</td>
<td>2.3186</td>
<td>2.0169</td>
<td>1.7705</td>
<td>1.5256</td>
</tr>
<tr>
<td>1.333</td>
<td>6.3296</td>
<td>5.7185</td>
<td>4.8363</td>
<td>4.0372</td>
<td>3.4178</td>
<td>2.9124</td>
<td>2.5631</td>
<td>2.2336</td>
<td>1.8694</td>
</tr>
<tr>
<td>2.00</td>
<td>6.3996</td>
<td>6.1736</td>
<td>5.7403</td>
<td>5.2604</td>
<td>4.7325</td>
<td>4.2034</td>
<td>3.8136</td>
<td>3.4691</td>
<td>3.1091</td>
</tr>
<tr>
<td>5.00</td>
<td>6.1174</td>
<td>6.2086</td>
<td>5.9559</td>
<td>5.6516</td>
<td>5.2880</td>
<td>4.8558</td>
<td>4.3371</td>
<td>3.8636</td>
<td>3.5603</td>
</tr>
<tr>
<td>6.00</td>
<td>6.1188</td>
<td>6.2152</td>
<td>5.9717</td>
<td>5.6803</td>
<td>5.3311</td>
<td>4.9118</td>
<td>4.3986</td>
<td>3.7711</td>
<td>3.5224</td>
</tr>
<tr>
<td>7.00</td>
<td>6.1979</td>
<td>6.2191</td>
<td>5.9810</td>
<td>5.6971</td>
<td>5.3568</td>
<td>4.9919</td>
<td>4.4351</td>
<td>3.7873</td>
<td>3.5316</td>
</tr>
<tr>
<td>8.00</td>
<td>6.2033</td>
<td>6.2216</td>
<td>5.9869</td>
<td>5.7077</td>
<td>5.3729</td>
<td>4.9660</td>
<td>4.4591</td>
<td>3.7959</td>
<td>3.5243</td>
</tr>
<tr>
<td>12.00</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>15.00</td>
<td>6.1217</td>
<td>6.2275</td>
<td>6.0009</td>
<td>5.7331</td>
<td>5.4114</td>
<td>5.0166</td>
<td>4.5165</td>
<td>3.8189</td>
<td>3.5857</td>
</tr>
<tr>
<td>20.00</td>
<td>6.1219</td>
<td>6.2282</td>
<td>6.0026</td>
<td>5.7360</td>
<td>5.4159</td>
<td>5.0226</td>
<td>4.5234</td>
<td>3.8552</td>
<td>3.6859</td>
</tr>
</tbody>
</table>

### Table IV. Values of \((k_1 \lambda)/j\) Which Satisfy Equation (21). Relative Dielectric Constant \(\epsilon_1 = 2.54\)

<table>
<thead>
<tr>
<th>(\lambda / b)</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
<th>.7</th>
<th>.8</th>
<th>.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>7.3313</td>
<td>5.7931</td>
<td>4.3683</td>
<td>3.6475</td>
<td>2.8361</td>
<td>2.0502</td>
<td>2.0851</td>
<td>1.8329</td>
<td>1.6032</td>
</tr>
<tr>
<td>4.00</td>
<td>7.6229</td>
<td>7.3961</td>
<td>7.0998</td>
<td>6.7199</td>
<td>6.2525</td>
<td>5.7014</td>
<td>5.0802</td>
<td>4.4342</td>
<td>3.8462</td>
</tr>
<tr>
<td>5.00</td>
<td>7.6263</td>
<td>7.4317</td>
<td>7.1861</td>
<td>6.8076</td>
<td>6.3923</td>
<td>5.8894</td>
<td>5.2838</td>
<td>4.5282</td>
<td>3.9198</td>
</tr>
<tr>
<td>12.00</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
</tbody>
</table>
TABLE V. VALUES OF \((k, \lambda)/j\) WHICH SATISFY EQUATION (21).  
RELATIVE DIELECTRIC CONSTANT \(\varepsilon_1 = 3.78\)

<table>
<thead>
<tr>
<th>(\frac{\lambda}{b})</th>
<th>(.2)</th>
<th>(.4)</th>
<th>(.5)</th>
<th>(.6)</th>
<th>(.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.067</td>
<td>7.1015</td>
<td>3.9063</td>
<td>3.1721</td>
<td>2.6687</td>
<td>2.0221</td>
</tr>
<tr>
<td>1.60</td>
<td>9.0827</td>
<td>5.6157</td>
<td>4.6115</td>
<td>3.9073</td>
<td>2.9588</td>
</tr>
<tr>
<td>2.00</td>
<td>9.6732</td>
<td>6.7122</td>
<td>5.6037</td>
<td>4.7665</td>
<td>3.5073</td>
</tr>
<tr>
<td>2.40</td>
<td>9.8900</td>
<td>7.6709</td>
<td>6.4881</td>
<td>5.5568</td>
<td>4.1952</td>
</tr>
<tr>
<td>3.00</td>
<td>10.0116</td>
<td>8.5859</td>
<td>7.5110</td>
<td>6.5332</td>
<td>4.9128</td>
</tr>
<tr>
<td>4.00</td>
<td>10.0804</td>
<td>9.1971</td>
<td>8.4713</td>
<td>7.6262</td>
<td>5.8191</td>
</tr>
<tr>
<td>5.00</td>
<td>10.1066</td>
<td>9.4025</td>
<td>8.8165</td>
<td>8.1101</td>
<td>6.3328</td>
</tr>
<tr>
<td>6.00</td>
<td>-----</td>
<td>9.1910</td>
<td>9.0155</td>
<td>8.3954</td>
<td>6.5316</td>
</tr>
<tr>
<td>9.60</td>
<td>-----</td>
<td>9.6008</td>
<td>9.2118</td>
<td>8.7004</td>
<td>7.0171</td>
</tr>
</tbody>
</table>

TABLE VI. VALUES OF \((k, \lambda)/j\) WHICH SATISFY EQUATION (21).  RELATIVE DIELECTRIC CONSTANT \(\varepsilon_1 = 5.75\)

<table>
<thead>
<tr>
<th>(\frac{\lambda}{b})</th>
<th>(.2)</th>
<th>(.4)</th>
<th>(.5)</th>
<th>(.6)</th>
<th>(.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>8.5898</td>
<td>4.5297</td>
<td>3.6514</td>
<td>3.0618</td>
<td>2.3105</td>
</tr>
<tr>
<td>1.60</td>
<td>10.8068</td>
<td>5.9484</td>
<td>4.8176</td>
<td>4.0451</td>
<td>3.0561</td>
</tr>
<tr>
<td>2.00</td>
<td>12.2759</td>
<td>7.3065</td>
<td>5.9172</td>
<td>5.0059</td>
<td>3.7828</td>
</tr>
<tr>
<td>2.40</td>
<td>12.9082</td>
<td>8.5829</td>
<td>7.0314</td>
<td>5.9381</td>
<td>4.1847</td>
</tr>
<tr>
<td>3.00</td>
<td>13.1906</td>
<td>10.2502</td>
<td>8.5167</td>
<td>7.2614</td>
<td>5.1762</td>
</tr>
<tr>
<td>4.00</td>
<td>13.3123</td>
<td>11.9092</td>
<td>10.5215</td>
<td>9.1399</td>
<td>6.9082</td>
</tr>
<tr>
<td>5.00</td>
<td>13.3519</td>
<td>12.4619</td>
<td>11.5680</td>
<td>10.4122</td>
<td>7.9975</td>
</tr>
<tr>
<td>6.00</td>
<td>-----</td>
<td>12.6695</td>
<td>12.0200</td>
<td>11.1128</td>
<td>8.7480</td>
</tr>
<tr>
<td>8.00</td>
<td>13.3867</td>
<td>12.8212</td>
<td>12.3513</td>
<td>11.8886</td>
<td>9.5613</td>
</tr>
<tr>
<td>9.60</td>
<td>-----</td>
<td>12.8699</td>
<td>12.4539</td>
<td>11.8699</td>
<td>9.8717</td>
</tr>
</tbody>
</table>
TABLE VII. VALUES OF \((k_1 \lambda) / j\) WHICH SATISFY EQUATION (21).

RELATIVE DIELECTRIC CONSTANT \(\epsilon_1 = 10.0\)

<table>
<thead>
<tr>
<th>(\frac{\lambda}{b})</th>
<th>(d = )</th>
<th>(0.2)</th>
<th>(0.4)</th>
<th>(0.5)</th>
<th>(0.6)</th>
<th>(0.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>9.0946</td>
<td>4.6633</td>
<td>3.7216</td>
<td>3.1082</td>
<td>2.3374</td>
<td></td>
</tr>
<tr>
<td>1.60</td>
<td>11.9152</td>
<td>6.1453</td>
<td>4.9397</td>
<td>4.1289</td>
<td>3.1075</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>14.5080</td>
<td>7.6327</td>
<td>6.1515</td>
<td>5.1411</td>
<td>3.8718</td>
<td></td>
</tr>
<tr>
<td>3.00</td>
<td>18.0837</td>
<td>11.2235</td>
<td>9.0920</td>
<td>7.6283</td>
<td>5.7164</td>
<td></td>
</tr>
<tr>
<td>4.00</td>
<td>18.1618</td>
<td>14.1516</td>
<td>11.8839</td>
<td>10.0186</td>
<td>7.5117</td>
<td></td>
</tr>
<tr>
<td>5.00</td>
<td>18.5381</td>
<td>16.6706</td>
<td>14.3254</td>
<td>12.2242</td>
<td>9.2125</td>
<td></td>
</tr>
<tr>
<td>6.00</td>
<td>------</td>
<td>17.5394</td>
<td>16.0369</td>
<td>14.0790</td>
<td>10.7010</td>
<td></td>
</tr>
<tr>
<td>8.00</td>
<td>18.5926</td>
<td>17.9823</td>
<td>17.3075</td>
<td>15.1831</td>
<td>12.9273</td>
<td></td>
</tr>
<tr>
<td>9.50</td>
<td>------</td>
<td>18.0875</td>
<td>17.5921</td>
<td>16.7954</td>
<td>13.9811</td>
<td></td>
</tr>
<tr>
<td>11.20</td>
<td>------</td>
<td>------</td>
<td>17.7216</td>
<td>17.0743</td>
<td>14.5972</td>
<td></td>
</tr>
</tbody>
</table>

TABLE VIII. VALUES OF \((k_1 \lambda) / j\) WHICH SATISFY EQUATION (21). RELATIVE DIELECTRIC CONSTANT \(\epsilon_1 = 13.7\)

<table>
<thead>
<tr>
<th>(\frac{\lambda}{b})</th>
<th>(d = )</th>
<th>(0.2)</th>
<th>(0.4)</th>
<th>(0.5)</th>
<th>(0.6)</th>
<th>(0.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>9.2268</td>
<td>4.6657</td>
<td>3.7402</td>
<td>3.1210</td>
<td>2.3147</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>15.0466</td>
<td>7.7200</td>
<td>6.1991</td>
<td>5.1781</td>
<td>3.8830</td>
<td></td>
</tr>
<tr>
<td>3.00</td>
<td>20.7905</td>
<td>11.4549</td>
<td>9.2264</td>
<td>7.7190</td>
<td>5.8083</td>
<td></td>
</tr>
<tr>
<td>4.00</td>
<td>21.9512</td>
<td>13.0381</td>
<td>12.1811</td>
<td>10.2132</td>
<td>7.6836</td>
<td></td>
</tr>
<tr>
<td>5.00</td>
<td>22.0841</td>
<td>18.2105</td>
<td>15.0013</td>
<td>12.6295</td>
<td>9.1989</td>
<td></td>
</tr>
<tr>
<td>6.00</td>
<td>------</td>
<td>20.3523</td>
<td>17.5165</td>
<td>14.9018</td>
<td>11.2215</td>
<td></td>
</tr>
<tr>
<td>6.10</td>
<td>22.1369</td>
<td>20.7837</td>
<td>18.3608</td>
<td>15.7145</td>
<td>11.8811</td>
<td></td>
</tr>
<tr>
<td>8.00</td>
<td>22.1591</td>
<td>21.1496</td>
<td>20.3759</td>
<td>18.1506</td>
<td>14.2105</td>
<td></td>
</tr>
<tr>
<td>11.20</td>
<td>------</td>
<td>------</td>
<td>21.2453</td>
<td>20.4336</td>
<td>17.2425</td>
<td></td>
</tr>
</tbody>
</table>
4. NOTE ADDED IN PROOF

In looking over the curves in proof, it was apparent that some accuracy was lost in the reproduction of the graphs. For example, in Figs. 2-43, the c/v curves for the cases $d/b = 0$ and $d/b = 1$ (drawn in for reference) are circles centered at the origin with radii unity (in units of $\sqrt{\frac{\lambda}{2a}}$) and $\sqrt{\epsilon'}$, respectively. These circles were accurately drawn on the original graphs; therefore, an estimate of the error in reproduction can be made by drawing these circles on the graphs and comparing. If great accuracy is required, the values in Tables II-VIII can be used with equation (25) (substitute the negative of the square of the table entry for the quantity $(k, \lambda)^2$ in equation (25)).

5. ACKNOWLEDGEMENT

The author was helped and encouraged by discussions with P.E. Mayes and some unpublished notes of V.H. Rumsey. A special acknowledgement is due to Mrs. Judy Blankfield, who set up the ILLIAC program, and to Mr. G. Mochel, who plotted most of the graphs.
REFERENCES


4. After the method described in:
   Morse, P. and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill Book Co., Inc., New York, 1953, Chap. XIII.
DISTRIBUTION LIST FOR REPORTS ISSUED UNDER CONTRACT AF33(616)-3220

One copy each unless otherwise indicated

Armed Services Technical Information Agency
Knott Building
4th & Main Streets 4 and 1 repro.
Dayton 2, Ohio
ATTN: TIC-SC
(Excluding Top Secret and Restricted Data) (Reference AFR 205-43)

Commander
Wright Air Development Center
Wright-Patterson Air Force Base
Ohio 3 copies
ATTN: WCLRS-6, Mr. W.J. Fortune

Commander
Wright Air Development Center
Wright-Patterson Air Force Base
Ohio
ATTN: WCLN, Mr. N. Draganjac

Commander
Wright Air Development Center
Wright-Patterson Air Force Base
Ohio
ATTN: WCOSI, Library

Director
Evans Signal Laboratory
Belmar, New Jersey
ATTN: Technical Document Center

Commander
U.S. Naval Test Center
ATTN: ET-315
Antenna Section
Patuxent River, Maryland

Chief
Bureau of Aeronautics
Department of the Navy
ATTN: Aer-EL-931

Chief
Bureau of Ordnance
Department of the Navy
ATTN: Mr. C.H. Jackson, Code Re 9a
Washington 25, D.C.

Commander
Hq. A.F. Cambridge Research Center
Air Research and Development Command
Laurence G. Hanscom Field
Bedford, Massachusetts
ATTN: CRRD, R.E. Hiatt

Commander
Air Force Missile Test Center
Patrick Air Force Base
Florida
ATTN: Technical Library

Director
Ballistics Research Lab.
Aberdeen Proving Ground
Maryland
ATTN: Ballistics Measurement Lab.

Office of the Chief Signal Officer
ATTN: SIGNET-5
Eng. & Technical Division
Washington 25, D.C.

Commander
Rome Air Development Center
ATTN: RCERA-1D. Mather
Griffiss Air Force Base
Rome, N.Y.

Airborne Instruments Lab., Inc.
ATTN: Dr. E.G. Fubini
Antenna Section
160 Old Country Road
Mineola, New York
M/F Contract AF33(616)-2143

Andrew Alford Consulting Engrs.
ATTN: Dr. A. Alford
299 Atlantic Ave.
Boston 10, Massachusetts
M/F Contract AF33(038)-23700

Bell Aircraft Corporation
ATTN: Mr. J.D. Shantz
Buffalo 5, New York
M/F Contract W-33(038)-14169
Chief
Bureau of Ships
Department of the Navy
ATTN: Code 838D, L.E. Shoemaker
Washington 25, D.C.

Director
Naval Research Laboratory
ATTN: Dr. J.I. Bohnert
Anacostia
Washington 25, D.C.

National Bureau of Standards
Department of Commerce
ATTN: Dr. A.G. McNish
Washington 25, D.C.

Director
U.S. Navy Electronics Lab.
Point Loma
San Diego 52, California

Commander
USA White Sands Signal Agency
White Sands Proving Command
ATTN: SIGWS-PC-02
White Sands, N.M.

Consolidated Vultee Aircraft Corp.
Fort Worth Division
ATTN: C.R. Curnutt
Fort Worth, Texas
M/F Contract AF33(038)-21117

Textron American, Inc. Div.
Dalmo Victor Company
ATTN: Mr. Glen Walters
1414 El Camino Real
San Carlos, California
M/F Contract AF33(038)-30525

Dorne & Margolin
29 New York Ave.
Westbury
Long Island, New York
M/F Contract AF33(016)-2037

Raytheon Manufacturing Company
ATTN: Robert Borts
Wayland Laboratory, Wayland, Mass.

Douglas Aircraft Company, Inc.
Long Beach Plant
ATTN: J.C. Buckwalter
Long Beach 1, California
M/F Contract AF33(600)-25669

Boeing Airplane Company
ATTN: F. Bushman
7755 Marginal Way
Seattle, Washington
M/F Contract AF33(038)-31096

Chance-Vought Aircraft Division
United Aircraft Corporation
ATTN: Mr. F.N. Kickerman
THRU: BuAer Representative
Dallas, Texas

Consolidated-Vultee Aircraft Corp.
ATTN: Mr. R.E. Honer
P.O. Box 1950
San Diego 12, California
M/F Contract AF33(600)-26530

Grumman Aircraft Engineering Corp.
ATTN: J.S. Erickson,
Asst. Chief, Avionics Dept.
Bethpage
Long Island, New York
M/F Contract NO(as) 51-118

Hallicrafters Corporation
ATTN: Norman Foot
440 W. 5th Avenue
Chicago, Illinois
M/F Contract AF33(060)-26117

Hoffman Laboratories, Inc.
ATTN: S. Varian
Los Angeles, California
M/F Contract AF33(600)-17529

Hughes Aircraft Corporation
Division of Hughes Tool Company
ATTN: D. Adcock
Florence Avenue at Teale
Culver City, California
M/F Contract AF33(600)-27615
One copy each unless otherwise indicated

Illinois, University of
Head, Department of Elec. Eng.
ATTN: Dr. E.C. Jordan
Urbana, Illinois

Johns Hopkins University
Radiation Laboratory
ATTN: Librarian
1315 St. Paul Street
Baltimore, Maryland
M/F Contract AF33(616)-68

Electronics Research, Inc.
2300 N. New York Avenue
P.O. Box 327
Evansville 4, Indiana
M/F Contract AF33(616)-2113

Glenn L. Martin Company
Baltimore 3, Maryland
M/F Contract AF33(600)-21703

McDonnell Aircraft Corporation
ATTN: Engineering Library
Lambert Municipal Airport
St. Louis 21, Missouri
M/F Contract AF33(600)-8743

Michigan, University of
Aeronautical Research Center
ATTN: Dr. L. Cutrona
Willow Run Airport
Ypsilanti, Michigan
M/F Contract AF33(038)-21573

Massachusetts Institute of Tech.
ATTN: Prof. H.J. Zimmermann
Research Lab. of Electronics
Cambridge, Massachusetts
M/F Contract AF33(616)-2107

North American Aviation, Inc.
Aerophysics Laboratory
ATTN: Dr. J.A. Marsh
12214 Lakewood Boulevard
Downey, California
M/F Contract AF33(038)-18319

North American Aviation, Inc.
Los Angeles International Airport
ATTN: Mr. Dave Mason
Engineering Data Section
Los Angeles 45, California
M/F Contract AF33(038)18319

Northrop Aircraft Incorporated
ATTN: Northrop Library
Dept. 2L35
Hawthorne, California
M/F Contract AF33(600)-23893

Radioplane Company
Van Nuys, California
M/F Contract AF33(600)-22313

Lockheed Aircraft Corporation
ATTN: C.L. Johnson
P.C. Box 55
Burbank, California
M/F Noa(S)-52-763

Republic Aviation Corporation
ATTN: Engineering Library
Farmingdale
Long Island, New York
M/F Contract AF33(038)14810

Sperry Gyroscope Company
ATTN: Mr. B. Berkowitz
Great Neck
Long Island, New York
M/F Contract AF33(038)-14524

Temco Aircraft Corp.
ATTN: Mr. George Cramer
Garland, Texas
(Contract AF33(600)-21714)

Farnsworth Electronics Co.
ATTN: George Giffin
Ft. Wayne, Indiana
Marked: For Con. AF33(600)-25523

North American Aviation, Inc.
4300 E. Fifth Ave.
Columbus, Ohio
ATTN: Mr. James D. Leonard
Contract No(as) 54-323

Stanford Research Institute
Document Center
Menlo Park, California
ATTN: Mary Lou Fields, Acquisitions

Westinghouse Electric Corporation
Air Arm Division
ATTN: Mr. P.D. Newhouser
Development Engineering
Friendship Airport, Maryland
Contract AF33(600)-27852
DISTRIBUTION LIST (CONT.)  AF33(616)-3220

One copy each unless otherwise indicated

Ohio State Univ. Research Foundation
ATTN: Dr. W.C Tice
310 Administration Bldg.
Ohio State University
Columbus 10, Ohio
M/F Contract AF33(616)-3353

Air Force Development Field Representative
ATTN: Capt. Carl B. Ausfahl
Code 1010
Naval Research Laboratory
Washington 25, D.C.

Chief of Naval Research
Department of the Navy
ATTN: Mr. Harry Harrison
Code 427, Room 2604
Bldg. T-3
Washington 25, D.C.

Beech Aircraft Corporation
ATTN: Chief Engineer
600 E. Central Avenue
Wichita 1, Kansas
M/F Contract AF33(600)-20910

Land-Air Incorporated
Cheyenne Division
ATTN: Mr. R.J. Klessig
Chief Engineer
Cheyenne, Wyoming
M/F Contract AF33(600)-22964

Director, National Security Agency
RADE 1 GM, ATTN: Lt. Manning
Washington 25, D.C.

Melpar, Inc.
3000 Arlington Blvd.
Falls Church, Virginia
ATTN: K.S. Kelleher

Naval Air Missile Test Center
Point Mugu, California
ATTN: Antenna Section

Fairchild Engine & Airplane Corp.
Fairchild Airplane Division
ATTN: L. Fahnestock
Hagerstown, Maryland
M/F Contract AF33(038)-18499

Federal Telecommunications Lab.
ATTN: Mr. A. Kandoian
500 Washington Avenue
Nutley 10, New Jersey
M/F Contract AF33(616)-3071

Ryan Aeronautical Company
Lindbergh Drive
San Diego 12, California
M/F Contract W-33(038)-ac-21370

Republic Aviation Corporation
ATTN: Mr. Thatcher
Hicksville, Long Island, New York
M/F Contract AF18(600)-1602

General Electric Co.
French Road
Utica, New York
ATTN: Mr. Grimm, LMEED
M/F Contract AF33(600)-30632
One copy each unless otherwise indicated

Stanford Research Institute
Southern California Laboratories
ATTN: Document Librarian
320 Mission Street
South Pasadena, California
Contract AF19(604)-1296

Prof. J.R. Whinnery
Dept. of Electrical Engineering
University of California
Berkeley, California

Professor Morris Kline
Mathematics Research Group
New York University
45 Astor Place
New York, N.Y.

Prof. A.A. Oliner
Microwave Research Institute
Polytechnic Institute of Brooklyn
55 Johnson Street - Third Floor
Brooklyn, New York

Dr. C.H. Papas
Dept. of Electrical Engineering
California Institute of Technology
Pasadena, California

Electronics Research Laboratory
Stanford University
Stanford, California
ATTN: Dr. F.E. Terman

Radio Corporation of America
R.C.A. Laboratories Division
Princeton, New Jersey
ATTN: Librarian

Electrical Engineering Res. Lab.
University of Texas
Box 3026, University Station
Austin, Texas

Dr. Robert Hansen
8356 Chase Avenue
Los Angeles 45, California

Technical Library
Bell Telephone Laboratories
463 West St.
New York 14, N.Y.

Dr. R.E. Beam
Microwave Laboratory
Northwestern University
Evanston, Illinois

Department of Electrical Engineering
Cornell University
Ithaca, New York
ATTN: Dr. H.G. Booker

Applied Physics Laboratory
Johns Hopkins University
3621 Georgia Avenue
Silver Spring, Maryland

Exchange and Gift Division
The Library of Congress
Washington 25, D.C.

Ennis Kuhlman
c/o McDonnell Aircraft
P.O. Box 516
Lambert Municipal Airport
St. Louis 21, Mo.

Physical Science Lab.
New Mexico College of A and MA
State College, New Mexico
ATTN: R. Dressel

Technical Reports Collection
303 A Pierce Hall
Harvard University
Cambridge 38, Mass.
ATTN: Mrs. M.L. Cox, Librarian

Dr. R.H. DuHamel
Collins Radio Company
Cedar Rapids, Iowa

Dr. R.F. Hyneman
8725 Yorktown Avenue
Los Angeles 45, California